

Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. They are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

Zero Sum Games: In this game, there are two players: a maximizer and a minimizer. We generally write the payoff matrix M in perspective of the maximizer, so every row corresponds to an action that the maximizer can take, every column corresponds to an action that the minimizer can take, and a positive entry corresponds to the maximizer winning. M is a n by m matrix, where n is the number of choices the maximizer has, and m is the number of choices the minimizer has.

A linear program that represents fixing the maximizer's choices to a probabilistic distribution where the maximizer has n choices, and the probability that the maximizer chooses choice i is p_i is the following:

$$\begin{aligned} & \max(z) \\ & M_{1,1}(p_1) + \cdots + M_{n,1}(p_n) \geq z \\ & M_{1,2}(p_1) + \cdots + M_{n,2}(p_n) \geq z \\ & \vdots \\ & M_{1,m}(p_1) + \cdots + M_{n,m}(p_n) \geq z \\ & p_1 + p_2 + \cdots + p_n = 1 \\ & p_1, p_2, \cdots, p_n \geq 0 \end{aligned}$$

The dual represents fixing the minimizers choices to a probabilistic distribution.

By strong duality, the optimal value of the game is the same if you fix the minimizer's distribution first or the maximizer's distribution first.

1 Zero-Sum Games Short Answer

- Suppose a zero-sum game has the following property: The payoff matrix M satisfies $M = -M^\top$. What is the expected payoff of the row player?
- True or False: If every entry in the payoff matrix is either 1 or -1 and the maximum number of 1s in any row is k , then for any row with less than k 1s, the row player's optimal strategy chooses this row with probability 0. Justify your answer.
- True or False: Let M_i denote the i th row of the payoff matrix. If $M_1 = \frac{M_2 + M_3}{2}$, then there is an optimal strategy for the row player that chooses row 1 with probability 0.

2 A Path and Edge Game

Edith and Paris are playing the following game: We have a unweighted undirected graph G , and a source and sink s, t . Edith picks an edge in G , and Paris picks a path from s to t in G . Edith wins if she picks an edge in Paris's path, and Paris wins if this does not happen.

Both players are allowed to use randomized strategies, i.e. they choose a distribution of edges/-paths respectively, and pick their edge/path from this distribution.

- (a) Suppose Edith announces her distribution over edges to Paris, and then Paris gets to pick a distribution over paths based on Edith's distribution. They then each sample from their distributions and decide who wins based on the results.

Briefly argue that if Paris knows Edith's distribution, there is always an optimal strategy for Paris where he deterministically picks a single path.

- (b) Consider two variants of the game:

- In Variant 1, Edith picks a distribution over edges and announces it to Paris. Paris then picks a single path.
- In Variant 2, Paris picks a distribution over paths and announces it to Edith. Edith then picks a single edge.

Fill in the blank: Regardless of how Paris plays, if Edith plays optimally, her chances of winning in Variant ___ are at least her chances of winning in the other Variant. (You don't need to formally justify your answer)

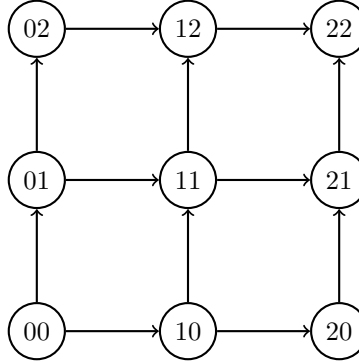
- (c) Show that in Variant 1, Edith has a strategy that wins with probability at least $1/C$ regardless of Paris's strategy, where C is the size of the minimum s - t cut in G .

- (d) Show that in Variant 2, Paris has a strategy such that Edith wins with probability at most $1/F$ regardless of Edith's strategy, where F is the size of the max flow s - t flow in G . (Hint: Any max flow can be decomposed into a set of paths p , where we push f_p flow on the path p , such that $f(e) = \sum_{p:e \in p} f_p$ and $\sum_p f_p = F$.)

- (e) Based on your answers to the previous parts, if both Edith and Paris play optimally, is Edith's chance of winning in Variant 1 greater than, equal or, or less than her chance of winning in Variant 2?

3 Multiplicative Weights

Consider the following simplified map of Berkeley. Due to traffic, the time it takes to traverse a given path can change each day. Specifically, the length of each edge in the network is a number between $[0, 1]$ that changes each day. The travel time for a path on a given day is the sum of the edges along the path.



For T days, both Tynan and Selina drive from node 00 to node 22.

To cope with the unpredictability of traffic, Selina builds a time machine and travels forward in time to determine the traffic on each edge on every day. Using this information, Selina picks the path that has the smallest total travel time over T days, and uses the same path each day.

Tynan wants to use the multiplicative weights update algorithm to pick a path each day. In particular, Tynan wants to ensure that the difference between his expected total travel time over T days and Selina's total travel time is at most $T/10000$. Assume that Tynan finds out the lengths of all the edges in the network, even those he did not drive on, at the end of each day.

- How many experts should Tynan use in the multiplicative weights algorithm?
- What are the experts?
- Given the weights maintained by the algorithm, how does Tynan pick a route on any given day?
- The regret bound for multiplicative weights is as follows:

Theorem. Assuming that all losses for the n experts are in the range $[0, 4]$, the worst possible regret of the multiplicative weights algorithm run for T steps is

$$R_T \leq 8\sqrt{T \ln n}$$

Use the regret bound to show that expected total travel time of Tynan is not more than $T/10000$ worse than that of Selina for large enough T .