Assortment Optimization with Consumer Search: Approximations and Applications

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Search behavior significantly influences consumer decision-making by facilitating information gathering, option comparison, and informed selections, ultimately enhancing the quality and value of their purchase determinations. In this paper, building upon the seminal work of Weitzman (1979), we examine a choice model wherein consumers of different types partake in a sequential search process before making purchase decisions. We then study a revenue-maximizing retailer's assortment recommendation decision under three scenarios: (1) the retailer possesses complete knowledge of individual consumer types, enabling the provision of tailored assortments; (2) the retailer lacks precise consumer type information but can offer personalized assortments while adhering to incentive-compatible constraints; (3) the retailer is constrained to providing non-personalized assortments. While all three problems are NP-hard to solve, we present a non-personalized heuristic that attains a tight 1/2 approximation ratio for each scenario. Notably, the heuristic can be easily derived solely using the purchase probability of each product when it is exclusively recommended to all consumers. Moreover, in terms of achievable approximation ratios, we establish that our heuristic surpasses all other non-personalized strategies in the first two scenarios and represents the optimal recommendation strategy in comparison to the revenue generated by a clairvoyant.

Additionally, our analysis yields a noteworthy by-product finding, demonstrating that under a mixed multinomial logit model with independent random product preference weight, a retailer's optimal revenue is approximately monotone submodular with respect to the set of products eligible for recommendation. Exploiting this property, we put forth computationally feasible constant-factor approximations for several practical applications: (1) a joint assortment optimization and customization problem, (2) a multi-stage non-overlapping assortment recommendation problem, (3) a joint information disclosure and assortment optimization problem, and (4) a sequential assortment selection problem.

Key words: Consumer Search, Personalization, Assortment Optimization, Approximation, Submodular

1. Introduction

In the dynamic and ever-evolving landscape of contemporary business, revenue management has attracted substantial scholarly interest and has arisen as an imperative field of study for organizations endeavoring to enhance their fiscal efficacy and sustain competitiveness. Efficiently executing revenue management necessitates the astute manipulation of pricing structures and product assortments. In order to attain this objective, organizations are required to delve deep into the psyche of their target audience, deciphering their preferences and consumption patterns. Such insight enables businesses to tailor their pricing and product offerings in a manner that not only maximizes revenue but also enhances consumer satisfaction and loyalty.

Over the years, numerous choice models grounded within the random utility framework have been proposed. For most of them, a recurring theme is the assumption that consumers can swiftly determine product value sans additional costs. However, this presupposition is not always realistic, especially when consumers grapple with products that are filled with complex details or information where the product utility is often ambiguous. This uncertainty stems from various factors: the overwhelming array of available choices, sophisticated marketing strategies that can sometimes obscure true product value, and the inherent variability in individual preferences. To navigate this terrain, consumers often turn to a *search* process to identify their optimal option. For instance, when buying electronics, consumers will study product details, read reviews, and sometimes reconsider based on feedback. This is not a snap judgment but a careful step-by-step process. During this process, to reduce the expenses linked to information search, consumers commonly evaluate products in a sequential manner, strategically adapting their subsequent decisions regarding whether to persist in their search or conclude it by making a purchase or opting for the outside option. Given the contemporary proliferation of information, it has become imperative to integrate consumer search behavior in elucidating their eventual purchase determinations.

Parallelly, the surge in online retail platforms has also amplified data collection capabilities. Every click, purchase, and even momentary pause on a webpage feeds into a vast data repository. This data revolution gave birth to the *personalized* assortment era. In contrast to personalized pricing which frequently encounters regulatory constraints, the implementation of personalized assortment is more pervasive. This is particularly evident in the realm of online retail, where the variability in consumers' disclosure of their demand information allows for the presentation of distinct product sets. Platforms now curate product offerings and even tailor promotions based on individual profiles. This seismic shift is evident in giants like Netflix and Alibaba, where personalization reigns supreme, optimizing inventory and forging lasting consumer bonds. However, the shift from non-personalized models to personalized assortment did not happen abruptly. How much a retailer knows about their consumers dictates their approach to showcasing products. In situations where the retailers have complete insight into individual consumer tastes, they use *personalization with precise type information*, fine-tuning product suggestions to align closely with what each user might want. When the retailers lack precise consumer-type information, they can lean on *personalization with incentive constraints*, where retailers craft strategies to encourage certain behaviors, such as offering new products to measure their appeal. While both personalization strategies have the potential to yield substantial increases in revenue for retailers, it is imperative to acknowledge that *personalization may also be unavailable*, for example, emanating from regulatory imperatives. Additionally, a pertinent factor contributing to this scenario may be a dearth of consumer receptiveness, marked by a lack of comprehension regarding the intricacy of the retailer's recommendation algorithm. Consequently, this deficiency may culminate in homogenous consumer behaviors, undifferentiated by personalized recommendations.

Given the aforementioned situation, our research endeavors to explore this intricate interplay between assortment recommendation, consumer search, and personalization. By weaving these threads together, we aim to provide a comprehensive framework that sheds light on the nuances of consumer decision-making in the digital age and offers actionable insights for retailers to navigate this complex landscape more effectively.

1.1. Model Summary and Main Results

In this paper, drawing upon Weitzman's theory (see, Weitzman 1979), we employ a sequential search model to elucidate the decision-making processes of consumers, acknowledging their heterogeneity in the perceived signals pertaining to each product's utility. Subsequently, we examine the assortment recommendation challenges for a revenue-maximizing retailer within three scenarios, differentiated by the retailer's ability to execute personalized strategies and her possession of accurate consumer-type information. The contributions of our work are summarized as follows.

First, we establish the revenue comparisons in the three scenarios. In contrast to an incentive-compatible personalization strategy, a personalization strategy incorporating accurate consumer-type information can lead to a remarkable revenue increase by a factor of 100% (Theorem 1(i)). This substantial revenue augmentation can also be attained through the implementation of an incentive-compatible personalized recommender system by the retailer, as opposed to a non-personalized approach (Theorem 1(ii)). Furthermore, it is also revealed that any recommendation strategy solely reliant on consumer-type information cannot consistently surpass half of the revenue generated by a clairvoyant, who possesses not only knowledge of the consumer's type but also their ex-post product utilities (Theorem 3).

Second, we prove the NP-hardness of the assortment optimization problems faced by the retailer in the three examined scenarios (Theorem 2). Subsequently, we establish the lower and upper bounds for the retailer's revenues. Leveraging the well-known prophet inequality and investigating a linear programming problem, we introduce a revenue-ordered non-personalized heuristic that attains a tight $1/2$ approximation ratio for each scenario (Theorems 4 and 5). The heuristic can be derived solely from the purchase probabilities associated with recommending each product exclusively to all consumers, thereby obviating the necessity and intricacy of calculating the revenue performance for any given assortment. Moreover, in terms of achievable approximation ratios, we find that our heuristic surpasses all other non-personalized strategies in the two personalization optimization problems and outperforms all other recommendation strategies when compared to a clairvoyant's revenue.

Third, our analysis yields a noteworthy by-product finding, demonstrating that under a mixed multinomial logit model with independent random product preference weight, a retailer's optimal revenue is approximately monotone submodular with respect to the set of products eligible for recommendation. Exploiting this property, we put forth computationally feasible constant-factor approximations for several practical applications: (1) a joint assortment optimization and customization problem, (2) a multi-stage non-overlapping assortment recommendation problem, (3) a joint information disclosure and assortment optimization problem, and (4) a sequential assortment selection problem. In particular, the first application complements the work of El Housni and Topaloglu (2023) by investigating a more general context. In the third application, the information strategy of our approximation has an intriguing property where the last-choice probability associated with each individual product will be maximized. The fourth application studies the MNL-prophet problem as studied in Goyal et al. (2023), with the main difference being that we allow a prophet to tailor assortment provision based on a consumer's type.

1.2. Literature Review

Our work is closely connected to several streams of literature: consumer search theory, assortment optimization, and personalization. In the following, we will provide a brief review for each stream respectively.

Consumer Search: Regarding consumer choice behavior, our work is related to the traditional consumer search theory. In the seminal paper of Weitzman (1979), the author studies the famous Pandora's problem, which has since been widely developed and applied to different scenarios. For example, Choi et al. (2018) examine price competition among different product owners, where the authors also find that the consumer's choice behavior in Pandora's problem aligns with a random utility model. Beyhaghi and Kleinberg (2019) assume the agent can choose a box without obligatory inspection and provide the first non-trivial approximation guarantees for this problem. Aouad et al. (2020), Gibbard (2022), and Gu and Wang (2022) consider a scenario where the agent may "partially" explore alternatives. Greminger (2022) and Brown and Uru (2023) consider consumers' sequential awareness of product sets. Chen (2022) employs search models to investigate postgraduate program applications. Aminian et al. (2023) study sequential search problems with fairness constraints. Alptekinoglu and Kosilova (2023) investigate a search model with exponentially distributed random shocks, and develop analytical tools to optimize prices for a given assortment of products. Similar to our focus, there are also several other works studying assortment optimization in the presence of consumer search. For example, Cachon et al. (2005) examine a framework wherein consumers may search for the outside option. Both Wang and Sahin (2018) and Derakhshan et al. (2022) investigate a two-stage consider-then-choose choice rule. Wang (2022a) delves into a model with two firms where upon visiting the first firm, the consumer observes the realized values of all products there, subsequently determining whether to pay a cost to visit the second firm. Complementing these studies, we allow consumers to evaluate each product individually in a sequential manner, alike to Weitzman (1979).

Assortment Optimization: Talluri and Van Ryzin (2004) study the assortment optimization problem under the well-known multinomial logit (MNL) model and show that the optimal assortment follows a nested-by-revenue property. A simple extension is to assume the existence of multiple consumer types, and the corresponding assortment optimization problem has been investigated in Bront et al. (2009), Méndez-Díaz et al. (2014), Rusmevichientong et al. (2014). We note that in our work, the choice probabilities under a non-personalized assortment can also be reduced to a mixed MNL, wherein product preference weights are random and independent. Recently, Goyal et al. (2023) study a sequential assortment selection problem, where the preference weight and the price of each product for an MNL consumer are uncertain before selection. One of our extensions expands on their work, with the main difference being that we allow a prophet to tailor assortment provision based on a consumer's type. Numerous works built upon other choice models include the nested logit model (e.g., Gallego and Topaloglu 2014), the MNL model with different behavioral effect (e.g., Cao et al. 2022, Gao et al. 2023b), the paired combinatorial logit model (e.g., Zhang et al. 2020), the Markov chain choice model (e.g., Blanchet et al. 2016), the exponomial choice model (e.g., Alptekinoğlu and Semple 2016), the click/attention-based choice model (e.g., Aouad et al. 2019, Gao et al. 2023a), the choice model with marginal distribution (e.g., Sun et al. 2020), the cascade model (e.g., Gao et al. 2022, Golrezaei et al. 2022, Liu et al. 2023), the multiple-purchase choice models (e.g., Feldman et al. 2021, Chen et al. 2023b), and the multiple-stage decision models (e.g., Feldman and Jiang 2021). Recently, incentive-compatible assortment optimization has also attracted scholars' attention. For example, Balseiro and Désir (2023) study an auction problem where each product owner has private information about its product attractiveness. Chen et al. (2020) study a position auction problem where sellers with private information compete for product ranking and information control. Ma (2023) considers an assortment recommendation strategy that depends on each consumer's report. In this paper, we also explore a scenario wherein personalization is contingent upon consumers' willingness to share their individual type information. Nevertheless, different from Ma (2023), the report of the consumers in our paper comprises signals of product utilities, instead of ordinal product rankings.

Personalization: Among various forms of personalization, most of the related literature focused on personalized pricing (also called price discrimination). In the operations management community, there is a surge in research on how to practically and effectively implement personalized pricing strategies (e.g., Anderson and Dana Jr 2009, Aydin and Ziya 2009, Ban and Keskin 2021, Chen et al. 2022a) and when personalized pricing offers significant value (e.g., Huang et al. 2021, Elmachtoub et al. 2021, Chen et al. 2022b, Gallego and Berbeglia 2021). However, price discrimination tactics are generally perceived as an unfair practice that involves serious branding risks and potential consumer ill-will. Moreover, it is a matter of debate whether customized pricing is legal with respect to antitrust laws (e.g., Ramasastry 2005). To avoid these pitfalls, researchers, including Golrezaei et al. (2014), Bernstein et al. (2015), Gallego et al. (2016), Gallego and Berbeglia (2022), Chen et al. (2023a), recently began to study personalized assortment in the presence of a few consumer segments. The authors provide structural results about the optimal policy and develop heuristics with provable theoretical performance. Our study aligns with this stream of literature by investigating personalizations under Weitzman's model. We finally remark that a closely related study to ours is elucidated in Chawla et al. (2010). In their research, the authors demonstrate that, in diverse scenarios, a non-personalized pricing policy can attain a constant-factor approximation when juxtaposed with the optimal incentive-compatible personalized pricing strategy. Our contribution lies in extending this line of inquiry to the realm of personalized assortment recommendation strategies. Specifically, we establish that, akin to the findings in pricing strategies, a non-personalized recommendation policy can achieve a constant-factor approximation in diverse cases when compared to the optimal incentive-compatible personalized assortment recommendation strategy.

1.3. Roadmap

The remainder of the paper is organized as follows. Section 2 introduces the examined model and formulates the retailer's problems. Section 3 demonstrates the possible revenue gaps of the retailer across various scenarios and proposes a simple constant-factor approximation for the assortment optimization problems. In Section 4, we generalize our base model by incorporating randomness into the outside option. Section 5 departs from the focus on the Weitzman model and further analyzes some interesting implications of our analysis by studying several other applications. We conclude our paper in Section 6.

2. The Model

We consider an online revenue-maximizing retailer (or seller, she), who is faced with a representative unit-demand consumer (he). The ground product set is denoted as $\mathcal{N} = \{1, 2, \ldots, n\}$, where the products in this set are horizontally differentiated. Let r_i represent the revenue of product i. Without loss of generality, we assume a product with a smaller index has a higher product revenue, i.e., $r_1 \geq r_2 \geq \cdots \geq r_n$. For ease of exposition, throughout the paper, all random variables will be symbolized with a hat. The major notations used in this paper are summarized in Appendix A.

2.1. The Consumer

The representative consumer represents heterogeneous types. Each consumer's type is characterized by a n-element vector $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \ldots, \hat{\theta}_n) \in \Theta$, where $\hat{\theta}_i$ reflects his idiosyncratic preference for the particular product $i \in \mathcal{N}$. We assume that $\hat{\theta}_i$'s for different i are independent of each other. The probability distribution function of $\hat{\theta}_i$ is represented by $G_i(\cdot)$. Although the consumer knows his type, he is still unaware of the exact utility of each product. For example, part of product specifications or the usage experience shared by previous buyers may affect the consumer's utility of using a product, but this must be further scrutinized by the consumer, perhaps by visiting the retailer's website or examining the products in person. In other words, the consumer's type only represents a signal of his utility. We denote the consumer's random utility of product $i \in \mathcal{N}$ as $\hat{u}_i$. Given a type-$\boldsymbol{\theta}$ consumer, $\hat{u}_i$'s are independent of each other, and the probability distribution function of $\hat{u}_i$ is represented by $F_i(\cdot|\theta_i)$. We remark that the supports of $\hat{\theta}_i$ and $\hat{u}_i$ can be continuous or discrete. To fully ascertain the utility of a product, the consumer must incur an inspection cost, denoted as $s_i \geq 0$. (We note that our results remain unchanged even if the search cost is allowed to be random and satisfy some known distribution.) The inspection of a particular product is available only after it is recommended to the consumer. We refer to the non-purchase (or the outside option) as product 0. For ease of exposition, following the classic search literature (e.g., Weitzman 1979, Choi et al. 2018), in the base model, the non-purchase utility for each consumer is deterministic and normalized to 0; that is, $\hat{u}_0 \equiv 0$. However, in Section 4, we will consider the case where $\hat{u}_0$ is also random. Though simple, the base model encompasses various special cases. For instance, Weitzman (1979) delves into scenarios where all consumers are of the same type. Choi et al. (2018) investigate a distinct scenario with $\hat{u}_i = \hat{\theta}_i + \hat{z}_i$, where $\hat{z}_i$, independent of $\hat{\theta}_i$, represents the residual utility revealed to the consumer only after a costly inspection. Furthermore, a specific instance is explored in Wang (2022b), where, in the absence of search costs, $\hat{u}_i = \hat{\theta}_i$ adheres to a Gumbel distribution.

We now describe the consumer's decision process after being recommended an assortment. Assume the representative consumer's type is $\boldsymbol{\theta}$ and he is offered an assortment S. The consumer is risk-neutral and engages in a sequential search process to maximize the subsequent expected utility. Following each inspection, the consumer decides whether to continue searching, and if yes, which product from the assortment to search next; otherwise, he is at the discretion to select any inspected product or opt for the outside option. Notice that we are implicitly assuming that the consumer has the recourse of selecting any inspected product, which has been commonly adopted in the search literature (e.g., Weitzman 1979, Choi et al. 2018, Greminger 2022). Letting $T \subseteq S$ represent the set of products already inspected by the consumer upon the termination of his search, we can consequently express his ex-post utility as follows:

$$(\max_{i\in T} u_i^+) - \sum_{i\in T} s_i,$$

where $x^+ = \max\{0, x\}$ and u_i is the realization of $\hat{u}_i$ conditional on $\hat{\theta}_i = \theta_i$.

The aforementioned search process is typically complex, as demand for each product is influenced by multiple purchase paths, which may increase exponentially with the number of available products. Yet, following the famous index theorem proposed by Weitzman (1979) (see the details in Appendix B.1), many works, including Kleinberg et al. (2016) and Choi et al. (2018), demonstrate that the consumer's eventual decision rule aligns with a random utility model. More precisely, we have the following results.

Lemma 1. *Let $\kappa_i(\theta_i)$ be the value of x such that $s_i = \int_x^\infty (u-x)dF_i(u|\theta_i)$. Given a consumer of type $\boldsymbol{\theta}$ facing an assortment S,*

(a) (Theorem 1, Choi et al. 2018) The consumer's the eventual purchase probability of product $i \in S$ is $\mathbb{P}(i|S,\boldsymbol{\theta}) = \mathbb{E}_{\hat{\boldsymbol{u}}|\boldsymbol{\theta}}[\mathbb{I}(\min\{\hat{u}_i, \kappa_i(\theta_i)\} \geq \max_{j\in S}(\min\{\hat{u}_j, \kappa_j(\theta_j)\})^+)]$, where $\mathbb{I}$ is the indicator function.

(b) (Corollary 1, Choi et al. 2018) The consumer's expected utility can be expressed as $\mathcal{U}(S|\boldsymbol{\theta}) = \mathbb{E}_{\hat{\boldsymbol{u}}|\boldsymbol{\theta}}[\max_{j\in S}(\min\{\hat{u}_j, \kappa_j(\theta_j)\})^+]$.

2.2. The Retailer's Problems

We next turn to the expected revenue earned by the retailer. Let $\mathcal{R}(S|\boldsymbol{\theta})$ be the retailer's expected revenue when a specific consumer is recommended an assortment S. Lemma 1 implies that $\mathcal{R}(S|\boldsymbol{\theta}) = \sum_{i\in S} r_i \mathbb{P}(i|S,\boldsymbol{\theta})$. We shall examine three distinct scenarios predicated on the retailer's capacity to implement personalized strategies and her possession of precise consumer-type information. It is essential to underscore that our investigation is confined to the realm of personalized assortment recommendations. This strategic focus aligns with contemporary regulatory constraints that frequently preclude the implementation of personalized pricing schemes, as articulated in Article 22(1) of the General Data Protection Regulation. Moreover, brand reputation or consumer trust could be severely damaged by the adoption of discriminatory pricing strategies. Nonetheless, online retailers maintain the capability to offer varied product assortments contingent upon the extent of lucidity in consumers' disclosure of their demand-related information.

Personalization with Precise Type Information. In this scenario, the retailer possesses complete knowledge of individual consumer types. This configuration aligns with the premise that consumers engage in recurrent interactions with the retailer, leading to the accumulation of their pertinent information. The retailer is allowed to recommend personalized assortments contingent on the precise consumer type. Thus, we can express her problem as follows:

$$\mathcal{R}_{PP} = \mathbb{E}_{\hat{\boldsymbol{\theta}}}[\max_{S} \mathcal{R}(S|\hat{\boldsymbol{\theta}})], \tag{PP}$$

where the subscript 'PP' represents personalization with precise consumer-type information.

Personalization with Incentive Constraints. In this scenario, the retailer possesses knowledge solely regarding the probability distribution of $\hat{\boldsymbol{\theta}}$. However, she remains unaware of the precise type of each consumer. This configuration closely resembles real-world scenarios where retailers encounter challenges in amassing comprehensive data regarding consumer preferences, behaviors, and demographic attributes, thus hindering their ability to accurately categorize them into specific consumer types. In this regard, the feasibility of personalization is contingent upon consumers voluntarily disclosing the information of their type.

We introduce *a probability function* $\rho(\cdot|\cdot)$ to denote the retailer's recommendation strategy, where the inputs of ρ are a subset of $\mathcal{N}$ and an element in Θ, respectively. Given S and $\boldsymbol{\theta}$, $\rho(S|\boldsymbol{\theta})$ represents the probability that assortment S will be recommended to a consumer whose seller-inferred type is $\boldsymbol{\theta}$. Accordingly, $\sum_{S\subseteq\mathcal{N}} \rho(S|\boldsymbol{\theta}) = 1$ for any $\boldsymbol{\theta} \in \Theta$. In reality, the consumer base can be exceedingly vast. Consequently, $\rho(S|\boldsymbol{\theta})$ may equivalently be construed as the proportion of consumers that are recommended assortment S, within the consumer group who exhibit behavior consistent with a type-$\boldsymbol{\theta}$ consumer. The sequence of the game between the retailer and the consumer is summarized as follows:

(a) The retailer first designs a personalized recommender system, characterized by $\rho(\cdot|\cdot)$.

(b) The consumer observes his type and makes a report to the retailer. This reporting process incorporates how the consumer will communicate with the search engine about his demand information or whether the consumer will purposefully refrain from providing comprehensive historical data or deliberately introduce noise into their preferences.

(c) After the report, given that the consumer's seller-inferred type is $\boldsymbol{\theta}$, with probability $\rho(S|\boldsymbol{\theta})$, the consumer is recommended an assortment of S.

(d) Given the final recommendation, the consumer makes a corresponding choice decision as depicted in the previous subsection and the retailer collects a corresponding revenue.

By the revelation principle, we focus on the family of direct mechanisms in which the consumer truthfully reports his type. Correspondingly, the consumer's individual-rational (IR) constraints are $\sum_{S\subseteq\mathcal{N}}\rho(S|\boldsymbol{\theta})\mathcal{U}(S|\boldsymbol{\theta})\geq\mathcal{U}(\varnothing|\boldsymbol{\theta}),\forall\boldsymbol{\theta}\in\Theta$. As $\mathcal{U}(S|\boldsymbol{\theta})\geq\mathcal{U}(\varnothing|\boldsymbol{\theta})$ for any $S\subseteq\mathcal{N}$, this constraint will always be satisfied. The additional incentive-compatible (IC) constraints require that the consumer is always better off truthfully reporting his private type. Finally, we can express the retailer's problem as follows:

$$\begin{aligned}\mathcal{R}_{PI}=\max_{\rho}\quad & \mathbb{E}_{\hat{\boldsymbol{\theta}}}[\sum_{S\subseteq\mathcal{N}}\rho(S|\hat{\boldsymbol{\theta}})\mathcal{R}(S|\hat{\boldsymbol{\theta}})],\\ s.t.\quad & \sum_{S\subseteq\mathcal{N}}(\rho(S|\boldsymbol{\theta})-\rho(S|\tilde{\boldsymbol{\theta}}))\mathcal{U}(S|\boldsymbol{\theta})\geq 0,\quad\forall\boldsymbol{\theta},,\tilde{\boldsymbol{\theta}}\in\Theta,\end{aligned}\tag{PI}$$

where the subscript 'PI' represents personalization with incentive constraints. We emphasize here that the incentive constraints only incorporate the consumer's willingness to share his private information of $\boldsymbol{\theta}$, as the exact value of $\hat{u}_i$ can be revealed only after the recommendation. Furthermore, we note that a similar question is also explored in Ma (2023), where a consumer's report comprises ordinal product rankings, distinct from our analysis's utilization of product utility signals.

Non-Personalization. Lastly, we examine a scenario wherein the retailer encounters constraints precluding the customization of assortments tailored to individual consumers. These constraints may emanate from regulatory imperatives or a lack of consumer acumen regarding the intricacies of the retailer's recommendation algorithm, thereby resulting in uniform consumer behavior prior to the dissemination of recommendations. Accordingly, the retailer's problem becomes as follows:

$$\mathcal{R}_{NP}=\max_{S\subseteq\mathcal{N}}\mathbb{E}_{\hat{\boldsymbol{\theta}}}[\mathcal{R}(S|\hat{\boldsymbol{\theta}})],\tag{NP}$$

where the subscript 'NP' represents non-personalization. In this case, the same assortment is offered to all consumers.

3. Assortment Optimization

In this section, we show how to solve the retailer's revenue-maximizing assortment optimization problem for each of the three scenarios presented in the previous section.

3.1. Revenue Comparison

We first conduct a revenue comparison under the three examined scenarios.

LEMMA 2. $\mathcal{R}_{NP} \le \mathcal{R}_{PI} \le \mathcal{R}_{PP}$.

The result of the above lemma is quite intuitive. The first inequality is attributed to the fact that any non-personalized recommendation will naturally satisfy the incentive-compatible constraint. The second inequality is because of the absence of incentive constraints in Problem (PP). We are then motivated to investigate the magnitude of the revenue disparity across different scenarios. Addressing this question necessitates a meticulous calculation of the exact values of $\mathcal{R}_{PP}$, $\mathcal{R}_{PI}$ and $\mathcal{R}_{NP}$. We defer such an investigation for the general case to subsequent subsections within this section. Here, our attention is centered on the following illustrative example.

EXAMPLE 1. Let $\epsilon \in (0,1)$. Consider a retailer with two products, where $r_1 = 1/\epsilon$ and $r_2 = 1$. Assume that $s_i \equiv 0$ and $\hat{u}_i = \hat{\theta}_i$ for each $i \in \{1,2\}$; $\hat{\theta}_1$ is $\eta \in (0,1)$ with probability ϵ and -1 with probability $1-\epsilon$; $\hat{\theta}_2 = 1$ with probability one. □

We first consider the optimal personalized assortment with precise consumer-type information for the above example. One can easily show that it is optimal for the retailer to exclusively recommend product 1 when $\hat{\theta}_1 = \eta$, and product 2 otherwise. Thus, the resulting revenue for the retailer will be $2-\epsilon$. We next consider the optimal non-personalized assortment. Notice that as $\hat{u}_2 \equiv 1 > \hat{u}_1$, if product 2 is included in the assortment, the consumer will always purchase product 2, generating a revenue of 1 for the retailer. If only product 1 is recommended to the consumer, the consumer will buy it with a probability ϵ, resulting in expected revenue $r_1\epsilon = 1$. As a result, the optimal revenue of a non-personalized assortment is 1. We next investigate the scenario of finding an optimal incentive-compatible personalized assortment. Notice that there are only two consumer types, characterized by $\hat{\theta}_1 \in \{\eta, -1\}$. Moreover, after any report, the effect of recommending $\{1,2\}$ is the same as that of exclusively recommending product 2. Thus, the retailer's problem is reduced to determine the values of the following four quantities: $\rho(\{1\}|\theta_1 = \eta)$, $\rho(\{2\}|\theta_1 = \eta)$, $\rho(\{1\}|\theta_1 = -1)$, and $\rho(\{2\}|\theta_1 = -1)$. The retailer's optimization problem now becomes as follows:

$$\begin{aligned}
\max_{\rho} \quad & \rho(\{1\}|\theta_1 = \eta) + \epsilon\rho(\{2\}|\theta_1 = \eta) + (1-\epsilon)\rho(\{2\}|\theta_1 = -1) \\
s.t. \quad & \rho(\{1\}|\theta_1 = \eta)\eta + \rho(\{2\}|\theta_1 = \eta) \ge \rho(\{1\}|\theta_1 = -1)\eta + \rho(\{2\}|\theta_1 = -1), \\
& \rho(\{2\}|\theta_1 = -1) \ge \rho(\{2\}|\theta_1 = \eta), \\
& 0 \le \rho(\{1\}|\theta_1 = \eta), \rho(\{2\}|\theta_1 = \eta), \rho(\{1\}|\theta_1 = -1), \rho(\{2\}|\theta_1 = -1) \le 1, \\
& \rho(\{1\}|\theta_1 = \eta) + \rho(\{2\}|\theta_1 = \eta) \le 1, \\
& \rho(\{1\}|\theta_1 = -1) + \rho(\{2\}|\theta_1 = -1) \le 1,
\end{aligned}$$

where the first two constraints are the incentive constraints of a consumer with a type as $\theta_1 = \eta$ and $\theta_1 = -1$, respectively. The above problem is clearly a linear program, which can always be solved with exactness. When ϵ is very small (e.g., $< 10^{-5}$), one can show that at optimality, $\rho(\{1\}|\theta_1 = \eta) = 1$, $\rho(\{2\}|\theta_1 = -1) = \eta$, and $\rho(\{2\}|\theta_1 = \eta) = \rho(\{1\}|\theta_1 = -1) = 0$. The resulting revenue for the retailer will be $1 + (1-\epsilon)\eta$. These discussions lead to the following results.

THEOREM 1 **(Revenue Gap)**. *Under Example 1, we have that:*

1. *There exists an instance such that* $\mathcal{R}_{PP} = 2\mathcal{R}_{PI}$.
2. *There exists an instance such that* $\mathcal{R}_{PI} = 2\mathcal{R}_{NP}$.

The proof of the above result can be easily established by letting $\epsilon, \eta \to 0$ and $\epsilon \to 0, \eta \to 1$, respectively. This analysis reveals that in contrast to an incentive-compatible personalization strategy, a personalization strategy incorporating precise consumer-type information would yield a revenue enhancement by a factor of 100%. This substantial revenue augmentation can also be attained through the implementation of an incentive-compatible personalized recommender system, as opposed to a non-personalized approach. These findings underscore the potential imperative for revenue-maximizing retailers to either obtain consumer-type information or devise an incentive-compatible personalized recommender system.

3.2. Hardness Result and An Upper Bound

Next, we investigate the complexity of the retailer's problems. Consider the simple setting where $\hat{\theta}_i$ is constant for all i; that is, all consumers are of the same type. Then, Problems (PP), (PI) and (NP) are all equivalent, simplified to finding a deterministic assortment for all consumers. In the following theorem, we show that even under such a simple case, determining the optimal assortment recommendation strategy is computationally hard, where the proof is based on a reduction from the well-known NP-hard set partition problem.

THEOREM 2 **(Hardness Result)**. *Even when $\hat{\theta}_i$ is constant for all $i \in \mathcal{N}$, Problems (PP), (PI) and (NP) are NP-hard to solve.*

Due to the above result, recourse to approximation algorithms becomes a requisite strategy. To facilitate the analysis, we introduce the following problem,

$$\mathcal{R}_{PC}(\boldsymbol{\theta}, \boldsymbol{u}) = \max_{S \subseteq \mathcal{N}} \sum_{i \in S} r_i \mathbb{I}(\min\{u_i, \kappa_i(\theta_i)\} \geq \max_{j \in S} (\min\{u_j, \kappa_j(\theta_j)\})^+)\}. \tag{1}$$

The reason for using the subscript 'PC' will be discussed later. Define $\mathcal{R}_{PC} \triangleq \mathbb{E}_{\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{u}}}[\mathcal{R}_{PC}(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{u}})]$. Let $B_i(\hat{u}_i, \hat{\theta}_i) = \mathbb{I}(\min\{\hat{u}_i, \kappa_i(\hat{\theta}_i)\} \geq 0)$ be a Bernoulli random variable. Then if $B_i(u_i, \theta_i) = 0$, the indicator for product i in Problem (1) will never be 1. As a result, $\mathcal{R}_{PC}(\boldsymbol{\theta}, \boldsymbol{u}) \leq \max_{i \in \Gamma(\boldsymbol{\theta}, \boldsymbol{u})} r_i$, where $\Gamma(\boldsymbol{\theta}, \boldsymbol{u}) \triangleq \{i \in \mathcal{N} : B_i(u_i, \theta_i) = 1\}$. Thus, $\mathcal{R}_{PC}(\boldsymbol{\theta}, \boldsymbol{u})$ is upper bounded by the revenue of the lowest-index product (corresponding to the highest revenue product) that the consumer prefers over the no-purchase alternative. For any instance, the upper bound can be achieved if we set $S=\{i\}$ when $B_i(u_i,\theta_i)=1$ and $B_j(u_j,\theta_j)=0$ for all $j<i$. Consequently, taking expectations, we obtain

$$\mathcal{R}_{PC}=\sum_{i\in\mathcal{N}} r_i\mathbb{P}(B_i(\hat{u}_i,\hat{\theta}_i)=1, B_j(\hat{u}_j,\hat{\theta}_j)=0,\forall j<i)=\sum_{i\in\mathcal{N}} r_i q_i \prod_{j\in\mathcal{N}:j<i}(1-q_j),$$

where $q_i\triangleq\mathbb{P}(B_i(\hat{u}_i,\hat{\theta}_i)=1)$ represents the *last-choice* probability of product i, namely the purchase probability of this particular product when it is exclusively recommended to all consumers. Recalling the definition of $\mathcal{R}(S|\hat{\boldsymbol{\theta}})$, it is evident that $\mathcal{R}_{PP}=\mathbb{E}_{\hat{\boldsymbol{\theta}}}[\max_{S\subseteq\mathcal{N}}\mathcal{R}(S|\hat{\boldsymbol{\theta}})]\le\mathcal{R}_{PC}$.

Proposition 1 **(Upper Bound)**. $\mathcal{R}_{NP}\le\mathcal{R}_{PI}\le\mathcal{R}_{PP}\le\mathcal{R}_{PC}$.

The managerial implication of Proposition 1 is quite intuitive and borrows from the notion of a clairvoyant firm introduced in Gallego and Berbeglia (2022). The clairvoyant firm knows exactly $(\boldsymbol{\theta},\boldsymbol{u})$ for each arriving consumer, that is, not only the consumer's type but also the product utility accessible to the consumer solely through incurring a costly inspection. We posit that the clairvoyant can tailor the personalized recommendations contingent upon all the information she has. Notice that the clairvoyant knows $(\boldsymbol{\theta},\boldsymbol{u})$, while the consumer only knows the information of his type $\boldsymbol{\theta}$. Consequently, given an assortment, the consumer's decision-making process remains consistent with Weitzman's solution. By Lemma 1, recommending any non-empty set $S\subseteq\Gamma(\boldsymbol{\theta},\boldsymbol{u})$ will invariably lead to a purchase. Hence, to maximize the revenue, the clairvoyant should exclusively recommend the product $\arg\max_{i\in\Gamma(\boldsymbol{\theta},\boldsymbol{u})} r_i$ to a type-$\boldsymbol{\theta}$ consumer when his underlying product utilities are $\boldsymbol{u}$. Based on this discussion, the optimal expected revenue of the clairvoyant can be calculated to be the same as $\mathcal{R}_{PC}$, where 'PC' represents the personalization of a clairvoyant. Clearly, the clairvoyant's revenue will serve as an upper bound of the optimal revenue for a retailer who can tailor personalizations only by consumers' type information.

In Theorem 3, we further establish the benefit of being a clairvoyant. The result also indicates the limitation of personalization. That is, even knowing each consumer's type, no personalization strategy can achieve a revenue that is always strictly larger than 1/2 of the clairvoyant's revenue.

Theorem 3 **(Benefit of Clairvoyance)**. *There exists an instance such that* $\mathcal{R}_{PC}=2\mathcal{R}_{PP}$.

3.3. A Constant-Factor Approximation

To approximate the retailer's problems, we turn to a non-personalized recommendation strategy. When assortment S is recommended to all consumers, the retailer's expected revenue is given by:

$$\mathcal{R}(S)=\mathbb{E}_{\hat{\boldsymbol{\theta}},\hat{\boldsymbol{u}}}[\sum_{i\in S} r_i\mathbb{I}(\min\{\hat{u}_i,\kappa_i(\hat{\theta}_i)\}\ge\max_{j\in S}(\min\{\hat{u}_j,\kappa_i(\hat{\theta}_j)\})^+)].$$

Recalling the definition of B_i, the expression above implies that if any, the consumer will always choose to purchase from the product set $\{i \in S : B_i(\hat{u}_i, \hat{\theta}_i) = 1\}$. Consequently, assuming the consumer always buys the largest-index product within this set, the resulting revenue will be a lower bound for $\mathcal{R}(S)$, and its expression is $\mathcal{R}_W(S|\boldsymbol{q},\boldsymbol{r}) \triangleq \sum_{i\in S} r_i q_i \prod_{j\in S:j>i}(1-q_j)$, where q_i is the last-choice probability of product i. Employing the well-established prophet inequality (e.g., Samuel-Cahn 1984), one can easily show that $\max_S \mathcal{R}_W(S|\boldsymbol{q},\boldsymbol{r}) \geq \mathcal{R}_B(\mathcal{N}|\boldsymbol{q},\boldsymbol{r})/2$, where $\mathcal{R}_B(S|\boldsymbol{q},\boldsymbol{r}) \triangleq \sum_{i\in S} r_i q_i \prod_{j\in S:j<i}(1-q_j)$. This inequality can also be found in studies of the cascade model (e.g., Gallego and Li 2017, Chen et al. 2021). The evident equivalence between $\mathcal{R}_B(\mathcal{N}|\boldsymbol{q},\boldsymbol{r})$ and $\mathcal{R}_{PC}$ leads to the following result.

Theorem 4 **(Heuristic Performance)**. *Define $S_* = \arg\max_S \mathcal{R}_W(S|\boldsymbol{q},\boldsymbol{r})$, where q_i is the last-choice probability of product i. Then,*

1. *S_* can be determined in $\mathcal{O}(n)$ iterations,*
2. *$\min_{i\in S_*} r_i > \max_{i\in\mathcal{N}\setminus S_*} r_i$,*
3. *$S_* = \{i \in \mathcal{N} : r_i \geq \mathcal{R}_W(S_*|\boldsymbol{q},\boldsymbol{r})\}$,*
4. *$\mathcal{R}(S_*) \geq \mathcal{R}_B(\mathcal{N}|\boldsymbol{q},\boldsymbol{r})/2 = \mathcal{R}_{PC}/2 \geq \mathcal{R}_{PP}/2 \geq \mathcal{R}_{PI}/2 \geq \mathcal{R}_{NP}/2$.*

Combined with Theorem 1, the above result implies that there is an instance such that $\mathcal{R}_{PC} \geq \mathcal{R}_{PI} = 2\mathcal{R}_{NP} \geq 2\mathcal{R}(S_*) \geq \mathcal{R}_{PC} \geq \mathcal{R}_{PP} \geq \mathcal{R}_{PI}$, where the second inequality evidently holds by the definition of $\mathcal{R}_{NP}$. Thus, the performance ratio of our heuristic is tight, compared to the optimal personalization with or without precise consumer-type information and the revenue of a clairvoyant. In Appendix C.4, we further show that there exists an instance such that $\mathcal{R}_{NP} = 2\mathcal{R}(S_*)$, implying the tightness of our heuristic for non-personalization.

The proposed heuristic demonstrates several aspects of superiority. First, as implied in Theorem 1, there does not exist a non-personalized recommendation strategy such that for any instance, its achieved revenue is always strictly larger than 1/2 of the optimal revenue attainable from personalization with or without precise consumer-type information. However, Theorem 4 suggests that our heuristic is already the best non-personalization strategy in terms of the achievable performance ratio compared to the optimal revenues from these two kinds of personalizations. Second, as suggested in Theorem 3, there does not exist a personalized recommendation strategy such that for any instance, its achieved revenue is always strictly larger than 1/2 of the clairvoyant's revenue. Similarly, Theorem 4 suggests that even contrasting with personalizations in the presence or absence of precise consumer-type information, our non-personalized heuristic is already the best recommendation strategy in terms of the achievable performance ratio compared to the clairvoyant's revenue. Third, our heuristic solely relies on the last-choice probability of each product, further solidifying its practicality and effectiveness. It is also pertinent to underline that since the heuristic is non-personalized, the privacy of consumers is also preserved to some degree.

Finally, it is imperative to acknowledge the existing limitations within our analysis. For example, we have omitted the consideration of a cardinality constraint concerning the assortment offered to each consumer type. Our focus adheres to scenarios where the retailer only owns a small set of products or is non-restrictive to display any number of products on her own website. However, when the cardinality constraint is imposed, in Appendix C.5, we show that both the ratios of $\mathcal{R}_{NP}/\mathcal{R}_{PI}$ and $\mathcal{R}_{PP}/\mathcal{R}_{PC}$ can be as small as k/n, where k is the number of products that the retailer can recommend to each consumer and n is the number of products the retailer has. The rationale behind the possibility of $\mathcal{R}_{NP}/\mathcal{R}_{PI}$ being k/n rests upon the distinction between personalized and non-personalized strategies. Under the personalized paradigm, the retailer retains the prerogative to recommend an unrestricted gamut of products, whereas the non-personalized strategy restricts the potential purchase pool to a fixed subset of products accessible to consumers. This observation underscores the benefits of personalized strategies or collecting consumer information from various channels, especially when the retailer confronts limitations on her capacity to recommend products. However, the possibility of $\mathcal{R}_{PP}/\mathcal{R}_{PC}$ being k/n also indicates that personalizations just based on consumer-type information are still very limited to extracting consumer surplus. This outcome can be ascribed to the difference in recommendation discrimination based on market segments (i.e., the type information) and the complete consumer information (including both the consumer's type and product utilities). In Appendix C.5, we show that there is a non-personalized strategy within polynomial time complexity, whose revenue is at least $k/(2n)$ of $\mathcal{R}_{PC}$. This result resonates with our main findings, wherein when the product number that the retailer has and that she can recommend to each consumer is comparable, non-personalization is already satisfactory even in comparison to a clairvoyant who knows both the consumer's type and his product utilities.

4. The Existence of Random Non-Purchase Utility

Now we consider the most concerning case where the non-purchase utility is also random. In particular, compared to the base model, we let each product's utility have an additional random term $\gamma\hat{\epsilon}_i$; that is, $\hat{u}_0^\gamma = \gamma\hat{\epsilon}_0$ and $\hat{u}_i^\gamma = \hat{u}_i + \gamma\hat{\epsilon}_i$ for $i \in \mathcal{N}$. The parameter $\gamma > 0$ represents kinds of consumer heterogeneity. As in Choi et al. (2018) and Gallego and Topaloglu (2019), $\hat{\epsilon}_i$'s, independent of $(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{u}})$, are assumed to be independent and identical Gumbel random variables with location parameter 0 and scale parameter 1. Still, the value of $\hat{u}_i$ is realized after a costly inspection. Then, the consumer's type is represented by the tuple $(\boldsymbol{\theta}, \boldsymbol{\epsilon})$, where $\boldsymbol{\epsilon}$ is the realization of $(\hat{\epsilon}_0, \dots, \hat{\epsilon}_n)$.

Employing similar analysis in Section 2, when a type-$(\boldsymbol{\theta}, \boldsymbol{\epsilon})$ consumer is recommended assortment S, the purchase probability of product $i \in S$ is given by $\mathbb{P}_\gamma(i|S, \boldsymbol{\theta}, \boldsymbol{\epsilon}) = \mathbb{E}_{\hat{\boldsymbol{u}}|\boldsymbol{\theta}}[\mathbb{I}(\min\{\hat{u}_i, \kappa_i(\theta_i)\} +$

$\gamma\epsilon_i - \gamma\epsilon_0 \geq \max_{j\in S}(\min\{\hat{u}_j, \kappa_j(\theta_j)\} + \gamma\epsilon_j - \gamma\epsilon_0)^+]$, where the function κ has the same definition as in Lemma 1. The consumer's expected utility can also be calculated as $\mathcal{U}_\gamma(S|\boldsymbol{\theta},\boldsymbol{\epsilon}) = \gamma\epsilon_0 + \mathbb{E}_{\hat{\boldsymbol{u}}|\boldsymbol{\theta}}[\max_{j\in S}(\min\{\hat{u}_j, \kappa_j(\theta_j)\} + \gamma\epsilon_j - \gamma\epsilon_0)^+]$. Letting $\mathcal{R}_\gamma(S|\boldsymbol{\theta},\boldsymbol{\epsilon}) \triangleq \sum_{i\in S} r_i \mathbb{P}_\gamma(i|S,\boldsymbol{\theta},\boldsymbol{\epsilon})$, the retailer's new problems are given by:

$$\mathcal{R}^\gamma_{PP} = \mathbb{E}_{\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\epsilon}}}[\max_{S\subseteq\mathcal{N}} \mathcal{R}_\gamma(S|\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\epsilon}})], \tag{R-PP}$$

$$\mathcal{R}^\gamma_{NP} = \max_{S\subseteq\mathcal{N}} \mathbb{E}_{\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\epsilon}}}[\mathcal{R}_\gamma(S|\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\epsilon}})]. \tag{R-NP}$$

$$\begin{aligned} \mathcal{R}^\gamma_{PI} = \max_{\rho} \quad & \mathbb{E}_{\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\epsilon}}}[\sum_{T\subseteq S} \rho(S|\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\epsilon}})\mathcal{R}_\gamma(S|\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\epsilon}})], \\ s.t. \quad & \sum_{S\subseteq\mathcal{N}} [\rho(S|\boldsymbol{\theta},\boldsymbol{\epsilon}) - \rho(S|\tilde{\boldsymbol{\theta}},\tilde{\boldsymbol{\epsilon}})]\mathcal{U}_\gamma(S|\boldsymbol{\theta},\boldsymbol{\epsilon}) \geq 0, \quad \forall \boldsymbol{\theta},\tilde{\boldsymbol{\theta}},\boldsymbol{\epsilon},\tilde{\boldsymbol{\epsilon}}, \end{aligned} \tag{R-PI}$$

Since the retailer's problems above are reduced to the base model when $\gamma \to 0$, approximations are still needed. For ease of exposition, in the rest of this section, we will use the notation $\hat{v}_i \triangleq \exp(\min\{\hat{u}_i, \kappa_i(\hat{\theta}_i)\}/\gamma)$; that is, $\gamma \ln \hat{v}_i = \min\{\hat{u}_i, \kappa_i(\hat{\theta}_i)\}$. Moreover, we note that the last-choice probability of product i in the generalized model now becomes as $q_i = \mathbb{E}_{\hat{\boldsymbol{u}},\hat{\boldsymbol{\theta}}}[\mathbb{I}(\min\{\hat{u}_i, \kappa_i(\hat{\theta}_i)\} + \gamma\epsilon_i \geq \gamma\epsilon_0] = \mathbb{E}[\hat{v}_i/(1+\hat{v}_i)]$.

4.1. An Upper Bound

We first establish an upper bound of the retailer's revenues. Similar to the analysis in Section 3.2, we introduce the following quantity:

$$\mathcal{R}^\gamma_{PC}(\boldsymbol{v},\boldsymbol{\epsilon}) \triangleq \max_{S\subseteq\mathcal{N}} \sum_{i\in S} r_i \mathbb{I}(\gamma\ln v_i + \gamma\epsilon_i - \gamma\epsilon_0 \geq \max_{j\in S}(\gamma\ln v_j + \gamma\epsilon_j - \gamma\epsilon_0)^+).$$

Define $\mathcal{R}^\gamma_{PC} \triangleq \mathbb{E}_{\hat{\boldsymbol{v}},\hat{\boldsymbol{\epsilon}}}[\mathcal{R}^\gamma_{PC}(\hat{\boldsymbol{v}},\hat{\boldsymbol{\epsilon}})]$, whose closed-form expression can be derived as follows:

$$\begin{aligned} \mathcal{R}^\gamma_{PC} &= \sum_{i\in\mathcal{N}} r_i \mathbb{E}_{\hat{\boldsymbol{v}}}\left[\mathbb{E}_{\hat{\boldsymbol{\epsilon}}}[\mathbb{I}(\gamma\ln\hat{v}_i + \gamma\hat{\epsilon}_i - \gamma\hat{\epsilon}_0 \geq 0, \max_{j\in\mathcal{N}:j<i} \gamma\ln\hat{v}_j + \gamma\hat{\epsilon}_j - \gamma\hat{\epsilon}_0 < 0]\right] \\ &= \sum_{i\in\mathcal{N}} r_i \mathbb{E}_{\hat{\boldsymbol{v}}}\left[\frac{1}{1+\sum_{j\in\mathcal{N}:j<i}\hat{v}_j}\frac{\hat{v}_i}{1+\sum_{j\in\mathcal{N}:j\leq i}\hat{v}_j}\right] \\ &= \sum_{i\in\mathcal{N}} r_i \mathbb{E}_{\hat{\boldsymbol{v}}}\left[\frac{\sum_{j\in\mathcal{N}:j\leq i}\hat{v}_j}{1+\sum_{j\in\mathcal{N}:j\leq i}\hat{v}_j} - \frac{\sum_{j\in\mathcal{N}:j<i}\hat{v}_j}{1+\sum_{j\in\mathcal{N}:j<i}\hat{v}_j}\right] \\ &= \sum_{i=1}^n (r_i - r_{i+1})\mathbb{E}_{\hat{\boldsymbol{v}}}\left[\frac{\sum_{j\in\mathcal{N}:j\leq i}\hat{v}_j}{1+\sum_{j\in\mathcal{N}:j\leq i}\hat{v}_j}\right] \end{aligned} \tag{2}$$

where r_{n+1}=0 and in the second equality, we adopt a similar probability calculation method to that in Gao et al. (2021). A similar argument in Section 3.2 concludes that $\mathcal{R}^\gamma_{PC}$ represents the clairvoyant's revenue and $\mathcal{R}^\gamma_{NP} \leq \mathcal{R}^\gamma_{PI} \leq \mathcal{R}^\gamma_{PP} \leq \mathcal{R}^\gamma_{PC}$.

Motivated by the analysis in the base model, the following proposition further establishes an upper bound solely based on the last-choice probability.

Proposition 2 **(Upper Bound)**. *$\mathcal{R}^{\gamma}_{NP} \leq \mathcal{R}^{\gamma}_{PI} \leq \mathcal{R}^{\gamma}_{PP} \leq \mathcal{R}^{\gamma}_{PC} \leq \mathcal{R}_B(\mathcal{N}|\boldsymbol{q},\boldsymbol{r})$, where q_i represents the last-choice probability of product i.*

The last inequality in the above proposition suggests that in the generalized model, $\mathcal{R}^{\gamma}_{PC}$ may not equal to $\mathcal{R}_B(\mathcal{N}|\boldsymbol{q},\boldsymbol{r})$ as in the base model. The rationale is as follows. Due to the existence of $\hat{\epsilon}_0$, dissatisfaction with one product implies a greater utility from the outside option, thus diminishing the likelihood of other products being satisfactory. Mathematically, it means that $\mathbb{P}(\max_{j\in S}\ln\hat{v}_j + \hat{\epsilon}_j < \hat{\epsilon}_0) = 1 - \mathbb{E}_{\hat{\boldsymbol{v}}}[\sum_{j\in S}\hat{v}_j/(1+\sum_{j\in S}\hat{v}_j)] \geq \prod_{j\in S}\mathbb{P}(\ln\hat{v}_j + \hat{\epsilon}_j < \hat{\epsilon}_0) = \prod_{j\in S}(1-q_j)$. Then, based on the last equality in (2), $\mathcal{R}^{\gamma}_{PC} \leq \sum_{i=1}^{n}(r_i - r_{i+1})(1-\prod_{j\leq i}(1-q_j)) = \mathcal{R}_B(\mathcal{N}|\boldsymbol{q},\boldsymbol{r})$.

4.2. A Constant-Factor Approximation

The approximation for our generalized model is still a non-personalized strategy. Consider that all consumers are recommended the same assortment S. We use $\mathbb{P}_{\gamma}(i|S)$ to denote the eventual probability of purchasing $i \in S$. Then, according to the discussion at the beginning of this section,

$$\mathbb{P}_{\gamma}(i|S) = \mathbb{E}_{\hat{\boldsymbol{v}},\hat{\boldsymbol{\epsilon}}}[\mathbb{I}(\gamma\ln\hat{v}_i + \gamma\hat{\epsilon}_i - \gamma\hat{\epsilon}_0 \geq \max_{j\in S}(\gamma\ln\hat{v}_j + \gamma\hat{\epsilon}_j - \gamma\hat{\epsilon}_0)^{+}] = \mathbb{E}_{\hat{\boldsymbol{v}}}[\frac{\hat{v}_i}{1+\sum_{j\in S}\hat{v}_j}]. \tag{3}$$

In Appendix B.1, we illustrate that the underlying decision rule behind the above choice probability encompasses special cases such as the click-based MNL model (e.g., Aouad et al. 2019), the double logit model (e.g., Compiani et al. 2021), and the cascade model (e.g., Najafi et al. 2019). Moreover, this choice probability form is clearly a special case of the mixed MNL model, wherein product preference weights are random and independent. We note that this explicit elucidation based on the search process may enable the augmentation of parameter estimation techniques by utilizing observed behavioral data, such as what products the consumers have searched for. In Appendix B.1, we also propose a variant of the simple cascade model to mitigate potential challenges in subsequent empirical exploration of Weitzman's model.

Based on (3), the retailer's revenue from a non-personalized assortment S is given by:

$$\mathcal{R}_{\gamma}(S) \triangleq \mathbb{E}_{\hat{\boldsymbol{v}}}[\frac{\sum_{i\in S}\hat{v}_i r_i}{1+\sum_{j\in S}\hat{v}_j}]. \tag{4}$$

Motivated by the connection between $\mathcal{R}^{\gamma}_{PC}$ and $\mathcal{R}_B(\cdot)$ established in Proposition 2, it is natural to think whether the same heuristic proposed in the base model still serves as an approximation for the retailer's problems under our generalized model. Recall that for the base model, the proof of the heuristic's performance is mainly based on the observation that $\mathcal{R}(S) \geq \mathcal{R}_W(S|\boldsymbol{q},\boldsymbol{r})$ for any S, where q_i is the last-choice probability of product i. Unfortunately, in our generalized model, due to the existence of $\hat{\epsilon}_0$, $\mathcal{R}_{\gamma}(S)$ may be strictly smaller than $\mathcal{R}_W(S|\boldsymbol{q},\boldsymbol{r})$. To elucidate, consider the following example: $S = \{1,2\}$, $\hat{v}_1 \equiv \hat{v}_2 \equiv 1$, and $r_1 = r_2 = 1$. In this particular scenario, the last-choice probabilities are $q_1 = \mathbb{E}[\mathbb{I}(\gamma\ln v_1 + \gamma\hat{\epsilon}_1 > \gamma\hat{\epsilon}_0)] = 1/2$ and $q_2 = \mathbb{E}[\mathbb{I}(\gamma\ln v_1 + \gamma\hat{\epsilon}_2 > \gamma\hat{\epsilon}_0)] = 1/2$, respectively. One can show that $\mathcal{R}_\gamma(S) = \mathbb{P}(\max_{i=1,2}\hat{\epsilon}_i > \hat{\epsilon}_0) = 2/3$ and $\mathcal{R}_W(S|\boldsymbol{q},\boldsymbol{r}) = q_1 + (1-q_1)q_2 = (1-1/2)+1/4 = 3/4$.

To address the above issue while still utilizing the property $\max_S \mathcal{R}_W(S|\boldsymbol{q},\boldsymbol{r}) \geq \mathcal{R}_B(\mathcal{N}|\boldsymbol{q},\boldsymbol{r})/2$, our first attempt is to lower bound the relative gap between $R_\gamma(S)$ and $\mathcal{R}_W(S|\boldsymbol{q},\boldsymbol{r})$.

PROPOSITION 3 **(Lower Bound)**. *Let q_i represent the last-choice probability of product i. Then for any S,*

$$\mathcal{R}_\gamma(S) \gtrsim 0.77\mathcal{R}_W(S|\boldsymbol{q},\boldsymbol{r}).$$

Moreover, there exists an instance such that the equality holds.

Based on the above result, similar to the proof of Theorem 4, we have $\mathcal{R}_\gamma(S_*) \gtrsim 0.385\mathcal{R}^\gamma_{PC} \gtrsim 0.385\mathcal{R}^\gamma_{PP} \gtrsim 0.385\mathcal{R}^\gamma_{PI} \gtrsim 0.385\mathcal{R}^\gamma_{NP}$, where $S_* = \arg\max_S \mathcal{R}_W(S|\boldsymbol{q},\boldsymbol{r})$. Nonetheless, when γ approaches 0, as Theorem 4 implies, the performance guarantee of the same heuristic S_* is actually 1/2. This observation motivates us to tighten the performance gap subsequently.

By Theorem 4, $S_* = \{i \in \mathcal{N} : r_i \geq \mathcal{R}_W(S_*|\boldsymbol{q},\boldsymbol{r})\}$, then $\mathcal{R}_B(\mathcal{N}|\boldsymbol{q},\boldsymbol{r}) = \sum_{i\in\mathcal{N}} r_i q_i \prod_{j:j<i}(1-q_j) \leq \mathcal{R}_B(S_*|\boldsymbol{q},\boldsymbol{r}) + \prod_{j\in S_*}(1-q_j)\mathcal{R}_W(S_*|\boldsymbol{q},\boldsymbol{r})$. Moreover, since S_* is revenue-ordered, assuming $S_* = \{1,\cdots,m\}$, the lower bound of $\mathcal{R}_\gamma(S_*)/\mathcal{R}_B(\mathcal{N}|\boldsymbol{q},\boldsymbol{r})$ can be derived by solving the following problem:

$$\begin{aligned}
\min_{\hat{\boldsymbol{v}},\boldsymbol{r},\boldsymbol{q},m}\quad & \underbrace{\mathbb{E}_{\hat{\boldsymbol{v}}}\Big[\frac{\sum_{i\leq m}\hat{v}_i r_i}{1+\sum_{j\leq m}\hat{v}_j}\Big]}_{\mathcal{R}_\gamma(S_*)} \Big/ \Bigg(\underbrace{\sum_{i\leq m} r_i q_i \prod_{j\leq m: j<i}(1-q_j)}_{\mathcal{R}_B(S_*|\boldsymbol{q},\boldsymbol{r})} + \prod_{j\leq m}(1-q_j)\underbrace{\sum_{i\leq m} r_i q_i \prod_{j\leq m: j>i}(1-q_j)}_{\mathcal{R}_W(S_*|\boldsymbol{q},\boldsymbol{r})}\Bigg) \\
s.t.\quad & r_1 \geq \cdots \geq r_m \geq \underbrace{\sum_{i\leq m} r_i q_i \prod_{j\leq m: j>i}(1-q_j)}_{\mathcal{R}_W(S_*|\boldsymbol{q},\boldsymbol{r})} \geq 0, \\
& q_i = \mathbb{E}[\hat{v}_i/(1+\hat{v}_i)], \forall i \leq m,
\end{aligned} \tag{5}$$

where the second last inequality of the first constraint ensures that for any $i \leq m$, $i \in S_*$. We note that in the above formulation, to fix $S_* = \{1,\cdots,m\}$, one should also let $r_{m+1} \leq \mathcal{R}_W(S_*|\boldsymbol{q},\boldsymbol{r})$. However, as r_{m+1} does not appear in the objective function, we can simply delete this constraint without changing the optimization problem.

In order to solve Problem (5), we first vary the values of $\boldsymbol{r}$ while keeping the other parameters constant. As the objective function is a linear fractional form of $\boldsymbol{r}$, then without altering the optimization problem, we can always scale $\boldsymbol{r}$ such that the denominator of the objective is always 1. As a result, given other parameters, Problem (5) can be reformulated as follows:

$$\begin{aligned}
\min_{\boldsymbol{r}}\quad & \mathbb{E}_{\hat{\boldsymbol{v}}}\Big[\frac{\sum_{i\leq m}\hat{v}_i r_i}{1+\sum_{j\leq m}\hat{v}_j}\Big] \\
\text{s.t.}\quad & \sum_{i\leq m} r_i q_i \prod_{j<i}(1-q_j) + \prod_{j\leq m}(1-q_j)\sum_{i\leq m} r_i q_i \prod_{j\leq m: j>i}(1-q_j) = 1, \\
& r_1 \geq \cdots \geq r_m \geq \sum_{i\leq m} r_i q_i \prod_{j\leq m: j>i}(1-q_j) \geq 0.
\end{aligned}$$

It can be easily seen that the above problem belongs to a linear programming class. Moreover, combined with $r_1 \geq \cdots \geq r_m \geq 0$, the first constraint implies that r_i must be finite for each i and $r_1 > 0$. The positive value of r_1 leads to $\sum_{i\leq m} r_i q_i \prod_{j\leq m:j>i}(1-q_j) > 0$. As a result, by the optimality of extreme points, one optimal solution to the above problem will cause at most one inequality (except the last one) in the second constraint to be non-binding. In other words, there is an optimal solution satisfying $r_1 = \cdots = r_k$ and $r_{k+1} = \cdots = r_m = \sum_{i\leq m} r_i q_i \prod_{j\leq m:j>i}(1-q_j) = r_1 Q$ for some $k \leq m$, where $Q = 1 - \prod_{j=1}^{k}(1-q_j)$. Given this property, we can also show that $\sum_{i\leq m} r_i q_i \prod_{j<i}(1-q_j) + \prod_{j\leq m}(1-q_j)\sum_{i\leq m} r_i q_i \prod_{j\leq m:j>i}(1-q_j) = r_1 Q + r_1 Q(1-Q) = r_1 Q(2-Q)$. Consequently, Problem (5) is reduced to the following problem:

$$
\begin{aligned}
\min_{\hat{\boldsymbol{v}},\boldsymbol{q},Q,m,r_1\geq 0,k\leq m} \quad & \mathbb{E}_{\hat{\boldsymbol{v}}}\Big[\frac{r_1\sum_{i\leq k}\hat{v}_i + r_1 Q\sum_{i\leq m:i>k}\hat{v}_i}{1+\sum_{j\leq m}\hat{v}_j}\Big], \\
s.t. \quad & r_1 Q(2-Q) = 1, \\
& Q = 1 - \prod_{j=1}^{k}(1-q_j), \\
& q_i = \mathbb{E}[\hat{v}_i/(1+\hat{v}_i)], \forall i \leq m,
\end{aligned}
$$

By using Jensen's inequality, we can show that the optimal objective of the above problem is not less than 1/2. Thus, we reach the following theorem.

THEOREM 5 **(Heuristic Performance)**. *Define $S_* = \arg\max_S \mathcal{R}_W(S|\boldsymbol{q},\boldsymbol{r})$, where q_i is the last-choice probability of product i. Then,*

$$
\mathcal{R}_\gamma(S_*) \geq \mathcal{R}_B(\mathcal{N}|\boldsymbol{q},\boldsymbol{r})/2 \geq \mathcal{R}_{PC}^\gamma/2 \geq \mathcal{R}_{PP}^\gamma/2 \geq \mathcal{R}_{PI}^\gamma/2 \geq \mathcal{R}_{NP}^\gamma/2.
$$

As the generalized model reduces to the base model when γ approaches 0, S_* provides a tight performance bound for Problems (R-PP), (R-PI) and (R-NP). Despite the superiority of S_* mentioned in Section 3.3, we would like to additionally comment on its remarkable computational efficacy. To elaborate, suppose for each i, $\hat{v}_i$ can take m distinct values. In accordance with its definition, the computational effort to execute our heuristic is delineated by a time complexity of $\mathcal{O}(nm)$, as it only needs to calculate the last-choice probability of each product. Moreover, although our heuristic suggests a satisfactory performance of a revenue-ordered heuristic, it is essential to underscore that it is impractical to find the optimal revenue-ordered candidate by brutely comparing their performance. This is because even under the click-based MNL model (i.e., one special case of our model with $m = 2$), a fully polynomial time approximation scheme will be required to calculate the revenue for any given assortment (see the discussion in Aouad et al. 2019). Consequently, knowing model parameters, our heuristic exhibits exceptional characteristics by virtue of its capacity to obviate the necessity and intricacy of calculating revenue performance across various assortments.

When model parameters are not known, irrespective of the functional form of $(\boldsymbol{F}, \boldsymbol{G})$ and even the value of $\boldsymbol{s}$, in Appendix B.5, we devise an online learning algorithm that achieves $\mathcal{O}(\log T)$ regret relative to the best revenue-ordered heuristic.

5. IM-MNL: Approximate Submodularity and Its Applications

In this section, we depart from the focus on the Weitzman model and further analyze some interesting implications of our analysis. To begin with, we consider a new choice model, where the utility of product i is $\ln \hat{v}_i + \hat{\epsilon}_i$ and the non-purchase utility is $\hat{\epsilon}_0$. We assume that any two variables in $(\hat{\epsilon}_0, \cdots, \hat{\epsilon}_n, \hat{v}_1, \cdots, \hat{v}_n)$ are independent of each other and each $\hat{\epsilon}_i$ is a Gumbel random variable with location parameter 0 and scale parameter 1. Then, given an assortment S, if the consumer always chooses the product with the largest utility, the eventual choice probability will be

$$\mathbb{P}^{\mathrm{IM}}(i|S) = \mathbb{E}_{\hat{\boldsymbol{v}}}[\frac{\hat{v}_i}{1 + \sum_{j \in S} \hat{v}_j}], \forall i \in S,$$

where the superscript 'IM' represents a mixed MNL model with independent random product preference weights. For ease of exposition, we will refer to it as the independent mixed MNL model (IM-MNL for short). Similar to the study of Gallego et al. (2023), we also introduce the *satisfying* IM-MNL model: given a ranking π, the consumers will sequentially evaluate the products according to this ranking and purchase the first evaluated product with a utility larger than the outside option. Accordingly, the resulting choice probability of product i will be

$$\begin{aligned}\mathbb{P}^{\mathrm{IM}}_{\pi}(i|S) =& \mathbb{E}_{\hat{\boldsymbol{v}}}[\frac{\sum_{j \in S: \pi_j \leq \pi_i} \hat{v}_j}{1 + \sum_{j \in S: \pi_j \leq \pi_i} \hat{v}_j}] - \mathbb{E}_{\hat{\boldsymbol{v}}}[\frac{\sum_{j \in S: \pi_j < \pi_i} \hat{v}_j}{1 + \sum_{j \in S: \pi_j < \pi_i} \hat{v}_j}] \\ =& \mathbb{E}_{\hat{\boldsymbol{v}}}[\frac{1}{1 + \sum_{j \in S: \pi_j < \pi_i} \hat{v}_j} \frac{\hat{v}_i}{1 + \sum_{j \in S: \pi_j \leq \pi_i} \hat{v}_j}], \forall i \in S,\end{aligned}$$

where π_i is the ranking of product i in π and a smaller π_i indicates an earlier evaluation. Let $\mathcal{R}^{\mathrm{IM}}(S) \triangleq \sum_{i \in S} r_i \mathbb{P}^{\mathrm{IM}}(i|S)$ and $\mathcal{R}^{\mathrm{IM}}_{\pi}(S) \triangleq \sum_{i \in S} r_i \mathbb{P}^{\mathrm{IM}}_{\pi}(i|S)$. An evident observation is that $\arg\max_{\pi} \mathcal{R}^{\mathrm{IM}}_{\pi}(S)$ will be a ranking such that $\pi_i < \pi_j$ if $r_i > r_j$; that is, the consumer prioritizes the evaluation of a product with a higher revenue. Furthermore, Proposition 3 and Theorem 5 in the previous section imply the following result.

Theorem 6 **(Approximate Submodularity)**. *Define $q_i = \mathbb{E}[\hat{v}_i/(1 + \hat{v}_i)]$ as the last-choice probability of product i. Then for any S,*

1. $\max_{\pi} \mathcal{R}^{IM}_{\pi}(S) \geq \mathcal{R}^{IM}(S) \gtrsim 0.77 \mathcal{R}_W(S|\boldsymbol{q}, \boldsymbol{r})$.
2. $\mathcal{R}_B(S|\boldsymbol{q}, \boldsymbol{r}) \geq \max_{\pi, T \subseteq S} \mathcal{R}^{IM}_{\pi}(T) \geq \max_{T \subseteq S} \mathcal{R}^{IM}(T) \geq \mathcal{R}^{IM}(T_*) \geq \mathcal{R}_B(S|\boldsymbol{q}, \boldsymbol{r})/2$, *where* $T_* = \arg\max_{T \subseteq S} \mathcal{R}_W(T|\boldsymbol{q}, \boldsymbol{r})$,

By definition, $\mathcal{R}_B(\cdot|\boldsymbol{q},\boldsymbol{r})$ can be easily proven to be a monotone submodular set function, thus the above result implies that the seller's optimal revenue under either the IM-MNL or the satisfying IM-MNL is approximately submodular with respect to the set of products eligible for recommendation. In the rest of this section, we will show several applications of the above result.

5.1. Joint Assortment Optimization and Customization

Our first application generalizes the setup of El Housni and Topaloglu (2023), where retailers commit to a selection of products before the start of the selling season, but can potentially customize the displayed assortment for each consumer type.

Consider m consumer types, where the proportion of type j is λ_j. A consumer of a certain type makes a choice among the products offered to her according to the IM-MNL model. Specially, denote the preference weight $\hat{v}_i^j$ that consumer type j attaches to product i. Then, given that we offer the set of products S_j to a consumer of type j, she purchases product $i \in S_j$ with probability $\mathbb{E}[\hat{v}_i^j/(1+\sum_{l\in S_j}\hat{v}_l^j)]$. We denote the retailer's revenue as r_i^j if product i is purchased by a type-j consumer. We next investigate a two-stage optimization problem faced by the retailer. In particular, in the first stage (e.g., before the sales season), the retailer has to select a subset $S\subseteq\mathcal{N}$ with a cardinality constraint $|S|\le K$. In the second stage (e.g., during the sales season), the retailer observes the consumer type and offers her a personalized assortment, which is a subset of the products in S carried initially. Then, the customized assortment problem can be formulated as follows:

$$\max_{S\subseteq\mathcal{N}:|S|\le K}\sum_{j=1}^m \lambda_j \max_{S_j\subseteq S}\mathbb{E}[\frac{\sum_{i\in S_j}\hat{v}_i^j r_i^j}{1+\sum_{l\in S_j}\hat{v}_l^j}]. \tag{6}$$

When $\hat{v}_i^j \equiv v_i^j$ and $r_i^j = r_i$ are constant for any i and j, the above problem is reduced to the base model examined in El Housni and Topaloglu (2023). Theorem 2.1 of El Housni and Topaloglu (2023) demonstrates that Problem (6) under such a simple parameter configuration is already NP-hard to approximate within a factor better than $(1-1/e)$. The authors have devised a fully polynomial-time approximation scheme. For the same question, Udwani (2023) proposes an algorithm with time complexity $\mathcal{O}(n\log(K)/\epsilon)$ and an approximation ratio $0.5-\epsilon$ for any $\epsilon\in(0,1)$. In the following, we show how to achieve a constant-factor approximation for the general case where $\hat{v}_i^j$ is random and r_i^j depends both on the consumer type and the product.

We first define $q_i^j = \mathbb{E}[\hat{v}_i^j/(1+\hat{v}_i^j)]$ as the last-choice probability of product i when it is offered to consumers of type j. Then, part two of Theorem 6 implies that the optimal objective of Problem (6) is upper bounded by the following problem:

$$\max_{S\subseteq\mathcal{N}:|S|\le K}\sum_{j=1}^m \lambda_j \mathcal{R}_B(S|\boldsymbol{q}^j,\boldsymbol{r}^j).$$

By the property of $\mathcal{R}_B$, the objective function in the above problem is a monotone submodular set function. Then, based on the existing studies on submodular maximization (e.g., Nemhauser et al. 1978), there is an $(1-1/e)$-approximation with time complexity $\mathcal{O}(nK)$ for the above problem. We denote this approximation heuristic as $S_\dagger$ and define $S_\dagger^j = \arg\max_{S\subseteq S_\dagger} \mathcal{R}_W(S|\boldsymbol{q}^j, \boldsymbol{r}^j)$. Then, the last inequality of part two in Theorem 6 leads to the following result.

PROPOSITION 4. *Choosing $S_\dagger$ in the first stage and recommending $S_\dagger^k$ for consumers of type j in the second stage will yield revenue that is at least $(1-1/e)/2$ of the optimal objective of Problem (6).*

To test the performance of our heuristic, we conduct some numerical experiments in Appendix B.4. We note that a similar argument will also ensure the existence of a $(1-1/e)/2$ approximation if the assortment that the retailer selects in the first stage must be subject to a knapsack or Matroid constraint (e.g., Calinescu et al. 2011).

Furthermore, similar to the extension in El Housni and Topaloglu (2023), one can also consider a two-stage assortment planning problem with cardinality constraint on the assortment offered to each consumer type. More precisely, after choosing the assortment in the first state, in the second stage, the retailer needs to determine a subset $S_j \subseteq S$ with $|S_j| \leq K_j$ to recommend to a type-j consumer. The retailer's problem now becomes as follows:

$$\max_{S\subseteq\mathcal{N}:|S|\leq K} \sum_{j=1}^m \lambda_j \max_{S_j\subseteq S:|S_j|\leq K_j} \mathbb{E}[\frac{\sum_{i\in S_j} \hat{v}_i^j r_i^j}{1+\sum_{l\in S_j} \hat{v}_l^j}].$$

Then, one can also find an approximation by first solving

$$\max_{S\subseteq\mathcal{N}:|S|\leq K} \sum_{j=1}^m \lambda_j \max_{S_j\subseteq S:|S_j|\leq K_j} \mathcal{R}_B^j(S_j|\boldsymbol{q}^j, \boldsymbol{r}^j).$$

Combined with Theorem 6, the study of Balkanski et al. (2016) on the above two-stage submodular maximization problem then guarantees the existence of a $(1-1/e-\epsilon)/2$-approximation for the retailer's problem for any $\epsilon \in (0,1)$.

5.2. Multi-Stage Non-Overlapping Assortment Optimization

The practice of providing non-overlapping assortments is commonly found in e-commerce platforms and online marketplaces. For example, company-oriented takeout platforms need to do so in order to reduce clients' aversion to repeated consumption of the same menu. Moreover, platforms that specialize in flash sales and limited-time offers frequently provide non-overlapping assortments for short durations. Examples include websites like Gilt, Rue La La, and HauteLook, where products are available for a limited time, creating a sense of urgency for customers to make purchases.

We assume there are T days in total. Product i purchased on day t will generate revenue r_i^t for the seller, where the dependence of the revenue on t incorporates factors such as discounting. We assume the consumer's decision rule follows the IM-MNL model. Then, the seller's revenue on day t can be calculated as $\mathcal{R}_t(S_t) = \mathbb{E}[(\sum_{i\in S_t}\hat{v}_i^t r_i^t)/(1+\sum_{l\in S_t}\hat{v}_l^t)]$, where S_t is the assortment offered to the consumers on day t. The seller's problem can be formulated as follows:

$$\begin{aligned} \max_{S_1\subseteq\mathcal{N},\cdots,S_T\subseteq\mathcal{N}} \quad & \sum_{t=1}^{T}\mathcal{R}_t(S_t) \\ s.t. \quad & S_t\cap S_{t'}=\varnothing, \forall t\neq t' \end{aligned} \tag{7}$$

Based on the set-partition problem, we can easily prove that solving the above problem is NP-hard.

LEMMA 3. *Even when $\hat{v}_i^t \equiv v_i$ and $r_i^t = r$ are constant for any i and t, solving Problem (7) is NP-hard.*

Next, we show how to derive an approximation. We introduce $q_i^t = \mathbb{E}[\hat{v}_i^t/(1+\hat{v}_i^t)]$ as the last-choice probability of product i when it is offered to consumers on day t. Then, part two of Theorem 6 implies that the optimal objective of Problem (7) is upper bounded by the following problem:

$$\begin{aligned} \max_{S_1\subseteq\mathcal{N},\cdots,S_T\subseteq\mathcal{N}} \quad & \sum_{t=1}^{T}\mathcal{R}_B(S_t|\boldsymbol{q}^t,\boldsymbol{r}^t) \\ s.t. \quad & S_t\cap S_{t'}=\varnothing, \forall t\neq t'. \end{aligned}$$

By the property of $\mathcal{R}_B$, the objective function of the above problem is clearly monotone T-submodular (see the definition in Ohsaka and Yoshida 2015). Theorem 3.1 of Ohsaka and Yoshida (2015) suggests that for the above problem, there exists a feasible solution $(S_1^\dagger,\cdots,S_T^\dagger)$ that can be found in polynomial time and provide a 1/2-approximation. We denote $S_t^* = \arg\max_{S\subseteq S_t^\dagger}\mathcal{R}_W(S|\boldsymbol{q}^t,\boldsymbol{r}^t)$. Then, according to the last inequality of part two in Theorem 6, we have the following result.

PROPOSITION 5. *$(S_1^*,\cdots,S_T^*)$ is feasible for Problem (7) and provides a 1/4-approximation.*

5.3. Joint Information Disclosure and Assortment Optimization

Determining an information disclosure strategy for each product after providing an assortment is a crucial aspect of retail and marketing. The way information is presented to consumers can significantly impact consumers' purchasing decisions. For example, when introducing a new product to the market, providing comprehensive and compelling details can generate interest and excitement among consumers, potentially driving sales. Some products may also require additional education to help consumers understand their uses, benefits, or technical specifications. Sellers may need to provide detailed information through product guides, tutorials, or other educational content to assist consumers in making informed choices. On the other hand, sellers may strategically hide partial information to guide consumer focus toward specific products or categories.

We denote the set of available information strategies for product i as Φ_i. Each specific strategy $\phi_i \in \Phi_i$ represents one where (1) whether or not to display some product attributes to consumers, or (2) adopting some persuading strategy to shift the consumer's perception value over a product. Given ϕ, we assume that the consumers' decision rule follows an IM-MNL model, where the consumer's random preference weight for product i is $\hat{v}_i^{\phi_i}$. As a result, the seller's problem can be formulated as follows:

$$\max_{S\subseteq\mathcal{N},\boldsymbol{\phi}} \quad \mathbb{E}\left[\frac{\sum_{i\in S} r_i \hat{v}_i^{\phi_i}}{1+\sum_{i\in S}\hat{v}_i^{\phi_i}}\right] \tag{8}$$

For ease of understanding, we provide the following example to explain one possible micro-level impact process of information on consumers' decision-making and show that determining a proper information strategy is nontrivial and crucial for revenue generation.

Example 2. The consumer's utility of the outside option is $\hat{\epsilon}_0$ and his utility of purchasing product i is $\ln\hat{v}_i + \hat{\epsilon}_i$. Assume that any two random variables in $\{\hat{\epsilon}_0,\cdots,\hat{\epsilon}_n,\hat{v}_1,\cdots,\hat{v}_n\}$ are independent of each other and each $\hat{\epsilon}_i$, known to the consumer, follows a Gumbel distribution with location parameter 0 and scale parameter 1. Denote $\mu_i = \exp(\mathbb{E}[\ln\hat{v}_i])$. The seller's information strategy for each product is binary; that is, whether or not to display the product description. Given that information is disclosed, the consumer will know the exact values of $\hat{v}_i$; otherwise, the consumer will think the utility of buying product i is $\ln\mu_i + \epsilon_i$. Let T_1 (T_2) be the set of products offered to the consumers with (without) information disclosure. Then, the purchase probability of product $i \in T_1\cup T_2$ from a risk-neutral utility-maximizing consumer can be derived as follows:

$$\mathbb{E}[\frac{\mathbb{I}(i\in T_1)\hat{v}_i + \mathbb{I}(i\in T_2)\mu_i}{1+\sum_{j\in T_1}\hat{v}_j + \sum_{j\in T_2}\mu_j}].$$

One can easily see that the seller's problem in this example is a special case of Problem (8). The following two instances further demonstrate the importance of properly choosing the information strategy for revenue generation.

- Suppose that there is only one product, where $\ln\hat{v}_1 \in (-\infty, 0)$ with the same probability. If not disclosing information, Then, the consumer's expected utility for this product will be negatively infinite. Thus, any consumer will not purchase. Conversely, if we disclose the information, once the consumer observes $\ln\hat{v}_1 = 0$, there is a probability that the consumer will purchase.

- Let $\eta > 0$. Consider only one product, where $\ln \hat{v}_1$ is e^η with probability $\eta/(\eta + e^\eta)$ and $\ln \hat{v}_1$ is $-\eta$ with probability $e^\eta/(\eta + e^\eta)$. If not disclosing information, the purchase probability of product 1 will be $1/2$. If disclosing information, the purchase probability will be
$$\frac{\eta}{\eta + e^\eta} \frac{\exp(e^\eta)}{1 + \exp(e^\eta)} + \frac{e^\eta}{\eta + e^\eta} \frac{\exp(-\eta)}{1 + \exp(-\eta)}.$$
Letting $\eta \to \infty$, the quantity above will be 0. □

Next, we will give a constant-factor approximation for Problem (8). We first define $q_i^{\phi_i} = \mathbb{E}[\hat{v}_i^{\phi_i}/(1 + \hat{v}_i^{\phi_i})]$. Then, part two of Theorem 6 implies that the optimal objective of Problem (8) is upper bounded by the following problem:

$$\max_{\boldsymbol{\phi}} \mathcal{R}_B(\mathcal{N} | \boldsymbol{q}^{\boldsymbol{\phi}}, \boldsymbol{r}).$$

By the definition of $\mathcal{R}_B$, the optimal solution for the above problem can be easily derived as $\phi_i^* = \arg\max_{\phi_i \in \Phi_i} q_i^{\phi_i}$. We denote $S_* = \arg\max_S \mathcal{R}_W(S | \boldsymbol{q}^{\boldsymbol{\phi}^*}, \boldsymbol{r})$. Then, the last inequality of part two in Theorem 6 leads to the following result.

PROPOSITION 6. *Providing assortment S_* with information strategy $\boldsymbol{\phi}^*$ will provide a 1/2-approximation for Problem (8).*

The implication of the above result is far-reaching. In particular, our heuristic admits a simple information structure where each product's last-choice probability will be maximized.

5.4. Sequential Assortment Selection Under Uncertainty

Our last application investigates the MNL-prophet problem as studied in Goyal et al. (2023), with the main difference being that we also allow a prophet to tailor assortment provision based on a consumer's type. The setup is as follows. Each consumer's type is characterized by a vector $(\hat{\epsilon}_0, \cdots, \hat{\epsilon}_n)$, where $\hat{\epsilon}_i$ represents the consumer's horizontal preference over product i. Different $\hat{\epsilon}_i$'s are independent Gumbel random variables with location parameter 0 and scale parameter 1. Each product is associated with two parameters $(\hat{v}_i, \hat{r}_i)$ that are mutually independent of other items and are jointly drawn from some known distribution. The two parameters of each product represent its preference weight and revenue, respectively. For ease of presentation, we let $v_0 = 1$ and $r_0 = 0$. Given that assortment S with product parameters $\{(v_i, r_i)\}_{i \in S}$ is offered a type-$\boldsymbol{\epsilon}$ consumer, product $i \in S$ will be purchased if and only if $\ln v_i + \epsilon_i \geq \max_{j \in S \cup \{0\}} \ln v_j + \epsilon_j$.

The seller, consistently unknown of the consumer's type, observes the realizations of $(\hat{v}_i, \hat{r}_i)$'s in an arbitrary and unknown order and – upon observing the parameters – decides irrevocably whether to accept it in the final assortment offered to the consumer, or skip it and move to the next. The seller's goal is to maximize the expected total revenue from the finally constructed assortment. To measure the performance of any online heuristic, we compare it to the optimal expected revenue of the following two kinds of prophets:

- For a prophet who knows each product's parameters and the consumer's type, her optimal expected revenue can be expressed as $\mathcal{R}_{\text{P-F}} \triangleq \mathbb{E}_{\hat{\boldsymbol{v}},\hat{\boldsymbol{r}},\hat{\boldsymbol{\epsilon}}}[\max_S \mathcal{R}(S|\hat{\boldsymbol{v}},\hat{\boldsymbol{r}},\hat{\boldsymbol{\epsilon}})]$, where the subscript P-F represents a prophet with full information and $\mathcal{R}(S|\boldsymbol{v},\boldsymbol{r},\boldsymbol{\epsilon}) = \sum_{i\in S} r_i \mathbb{I}(\ln v_i + \epsilon_i \geq \max_{j\in S\cup\{0\}} \ln v_j + \epsilon_j)$. After some simple algebra, one can prove that

$$\mathcal{R}_{\text{P-F}} = \mathbb{E}_{\hat{\boldsymbol{v}},\hat{\boldsymbol{r}}}\left[\sum_{i\in\mathcal{N}} \frac{\hat{v}_i \hat{r}_i}{(1+\sum_{j:\hat{r}_j>\hat{r}_i}\hat{v}_j)(1+\sum_{j:\hat{r}_j\geq\hat{r}_i}\hat{v}_j)}\right].$$

- For a prophet who only knows each product's parameters, her optimal expected revenue can be expressed as $\mathcal{R}_{\text{P-P}} \triangleq \mathbb{E}_{\hat{\boldsymbol{v}},\hat{\boldsymbol{r}}}[\max_S \mathbb{E}_{\hat{\boldsymbol{\epsilon}}}[\mathcal{R}(S|\hat{\boldsymbol{v}},\hat{\boldsymbol{r}},\hat{\boldsymbol{\epsilon}})]]$, where the subscript P-P represents a prophet with partial information. After some simple algebra, one can prove that

$$\mathcal{R}_{\text{P-P}} = \mathbb{E}_{\hat{\boldsymbol{v}},\hat{\boldsymbol{r}}}[\max_S \frac{\sum_{i\in S}\hat{v}_i\hat{r}_i}{1+\sum_{j\in S}\hat{v}_j}].$$

Clearly, the performance of any online policy will be (weakly) smaller than $\mathcal{R}_{\text{P-P}}$ and $\mathcal{R}_{\text{P-P}} \leq \mathcal{R}_{\text{P-F}}$. Theorem 3.2 of Goyal et al. (2023) demonstrates that accepting every arriving item with revenue greater than or equal to $\mathcal{R}_{\text{P-P}}/2$ will guarantee that the seller's expected revenue is at least $\mathcal{R}_{\text{P-P}}/2$. Mathematically, it means that

$$\mathbb{E}_{\hat{\boldsymbol{v}},\hat{\boldsymbol{r}}}[\frac{\sum_{i:\hat{r}_i\geq\mathcal{R}_{\text{P-P}}/2}\hat{v}_i\hat{r}_i}{1+\sum_{j:\hat{r}_j\geq\mathcal{R}_{\text{P-P}}/2}\hat{v}_j}] \geq \frac{\mathcal{R}_{\text{P-P}}}{2}.$$

In the following proposition, we further establish that this policy also leads to revenue that is comparable to the prophet who knows both each product's parameters and the consumer's type.

Proposition 7. *Accepting every arriving item with revenue greater than or equal to $\mathcal{R}_{P\text{-}P}/2$ will guarantee that the seller's expected revenue is at least $\mathcal{R}_{P\text{-}F}/4$. Moreover, there exists an instance where product revenues are not random and the seller's expected revenue from this policy is exactly $\mathcal{R}_{P\text{-}F}/4$.*

Despite the excellent performance of the policy proposed in Goyal et al. (2023), we note that it may be computationally hard to compute the exact value of $\mathcal{R}_{\text{P-P}}$ to execute the policy. The reason can be found in the discussion after Theorem 5. In the rest of this section, we aim to find a computationally effective policy also with a provable performance guarantee. To begin with, we first consider the case where each product's revenue is fixed.

Corollary 1. *Assume that each product's revenue is fixed; that is, $\hat{r}_i \equiv r_i$. Denote $q_i = \mathbb{E}[\hat{v}_i/(1+\hat{v}_i)]$ and $r_* = \max_S \mathcal{R}_W(S|\boldsymbol{q},\boldsymbol{r})$. Then, accepting every arriving item with revenue greater than or equal to r_* will guarantee that the seller's expected revenue is at least $\mathcal{R}_{P\text{-}F}/2$. Moreover, there exists an instance such that the seller's expected revenue from this policy is exactly $\mathcal{R}_{P\text{-}F}/2$.*

The above result is a direct corollary of part two in Theorem 6. The proposed heuristic demonstrates remarkable computational efficacy as we only need to calculate the last-choice probability of each product.

We next turn to the general case where product revenues are also random. To facilitate the analysis, we denote the support of $\hat{r}_i$ as $\Omega_i = \{r_i^1, r_i^2, \cdots\}$, where $r_i^k > r_i^{k+1}$ for any k. Let $q_i(r_i) = \mathbb{E}[\hat{v}_i/(1+\hat{v}_i)|\hat{r}_i = r_i]$ for $r_i \in \Omega_i$. Assume further that $\hat{R}_i^k = r_i^k$ with probability $q_i(r_i^k)$ and $\hat{R}_i^k = 0$ with probability $1 - q_i(r_i^k)$. We denote $\tau = \mathbb{E}_{\hat{\boldsymbol{r}}}[\max_{i\in\mathcal{N}} \sum_k \mathbb{I}(\hat{r}_i = r_i^k)\hat{R}_i^k]/2$. Then, we have the following result.

Proposition 8. *Accepting every arriving item with revenue greater than or equal to τ will guarantee that the seller's expected revenue is at least $0.385\mathcal{R}_{P\text{-}F}$.*

We first comment on the computational efficiency of our heuristic. Suppose that for each product i, $(\hat{v}_i, \hat{r}_i)$ can take at most m different values. Then, the number of values that $\max_{i\in\mathcal{N}} \sum_k \mathbb{I}(\hat{r}_i = r_i^k)\hat{R}_i$ can take will be nm. As a result, the threshold τ in our policy can be computed in $\mathcal{O}(nm)$ iterations. The proof sketch of the above result is as follows. First, based on the first inequality in part two of Theorem 6, we establish that $\mathcal{R}_{\text{P-F}} \le \mathbb{E}_{\hat{\boldsymbol{r}}}[\mathcal{R}_B(\mathcal{N}|\boldsymbol{q}(\hat{\boldsymbol{r}}), \hat{\boldsymbol{r}})] = \mathbb{E}_{\hat{\boldsymbol{r}}}[\max_{i\in\mathcal{N}} \sum_k \mathbb{I}(\hat{r}_i = r_i^k)\hat{R}_i^k] = 2\tau$. Then, based on a proof of the well-known prophet inequality, we have $\mathbb{E}_{\hat{\boldsymbol{r}}}[g_\tau(\{\sum_k \mathbb{I}(\hat{r}_i = r_i^k)\hat{R}_i^k\})] \ge \tau$, where $g_\tau(\{x_i\}) = \min_{i:x_i\ge\tau} x_i$ if $\max_i x_i \ge \tau$ and $g_\tau(\{x_i\}) = 0$ otherwise. Lastly, we establish that $\mathbb{E}_{\hat{\boldsymbol{r}}}[g_\tau(\{\sum_k \mathbb{I}(\hat{r}_i = r_i^k)\hat{R}_i^k\})] = \mathbb{E}_{\hat{\boldsymbol{r}}}[\mathcal{R}_W(\{i \in \mathcal{N} : \hat{r}_i \ge \tau\})|\boldsymbol{q}(\hat{\boldsymbol{r}}), \hat{\boldsymbol{r}})]$. As a result, part one of Theorem 6 implies our result; that is,

$$\mathbb{E}_{\hat{\boldsymbol{v}},\hat{\boldsymbol{r}}}[\frac{\sum_{i:\hat{r}_i\ge\tau} \hat{v}_i\hat{r}_i}{1+\sum_{j:\hat{r}_j\ge\tau} \hat{v}_j}] \gtrsim 0.77\mathbb{E}_{\hat{\boldsymbol{r}}}[\mathcal{R}_W(\{i\in\mathcal{N} : \hat{r}_i \ge \tau\})|\boldsymbol{q}(\hat{\boldsymbol{r}}), \hat{\boldsymbol{r}})] \gtrsim 0.385\mathcal{R}_{\text{P-F}}.$$

In Appendix B.2, we have further conducted some numerical experiments to compare our heuristic with that proposed in Goyal et al. (2023). We find that our heuristic demonstrates superior performance in most of the constructed instances.

6. Conclusion

Consumer search plays a crucial role in their decision-making. In this work, building upon Weitzman (1979), we investigate a retailer's decisions under the choice model wherein consumers of different types engage in a step-by-step search process to decide what to buy. Three scenarios are subsequently investigated, differentiated by the retailer's ability to execute personalized strategies and possession of accurate consumer-type information. Despite the inherent computational complexity of these problems, we proposed a simple heuristic with a proven performance guarantee for each scenario, which demonstrates remarkable efficiency and excels in multiple facets concerning achievable performance ratios. To prove the results, our approach is innovative and solving a linear programming problem.

Additionally, our analysis unveils the approximate submodularity inherent in the retailer's revenue function under the proposed IM-MNL model. Based on this result, (1) we complement El Housni and Topaloglu (2023) by investigating a more general context, (2) we put forth a computationally feasible constant-factor approximation for a multi-stage non-overlapping assortment recommendation problem, (3) we demonstrate that maximizing the last-choice probability of each product can provide a provable performance guarantee for a joint information disclosure and assortment optimization problem, (4) we study the MNL-prophet problem as studied in Goyal et al. (2023), and provide a simpler threshold policy that can achieve a better approximation ratio when compared to a prophet who knows both each product's parameters and the consumer's type.

Looking forward, two areas beckon deeper exploration. Firstly, we emphasize the imperative of investigating personalized pricing within the context of the search model. In stark contrast to the conventional assumption that consumers possess complete knowledge of product utilities prior to reporting their preferences, a search model posits that consumers must undergo a resource-intensive process to accurately ascertain the utilities of products. Secondly, the approximate submodularity we find has paved the way for the exploration of various extensions and applications, such as how dynamic information control strategies enable retailers to optimize their inventory levels.

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