# Breaking SIDH in Polynomial Time

Damien Robert[1,2](✉)

[1] INRIA Bordeaux-Sud-Ouest, 200 Avenue de la Vieille Tour,
33405 Talence Cedex, France
damien.robert@inria.fr
[2] Institut de Mathématiques de Bordeaux, 351 cours de la liberation,
33405 Talence Cedex, France

**Abstract.** We show that we can break SIDH in (classical) polynomial time, even with a random starting curve $E_0$.

## 1 Introduction

### 1.1 Result

We extend the recent attacks by [CD22, MM22] and prove that there exists a proven polynomial time attack on SIDH [JD11, DJP14]/SIKE [JAC+17], even with a random starting curve $E_0$.

Both papers had the independent beautiful idea to use isogenies between abelian surfaces (using [Kan97, Sect. 2]) to break a large class of parameters for SIDH. Namely, on a random starting curve $E_0$, if the degree of the secret isogenies are $N_A > N_B$, their attack essentially applies whenever $a := N_A - N_B$ is smooth. This is highly unlikely, however they use the fact that it is possible to tweak the parameters $N_A$ and $N_B$ to augment the probability of success (or reduce the smoothness bound on $a$), see Sect. 6. In the case where $\mathrm{End}(E_0)$ is known, [CD22] also have a (heuristic) polynomial time attack, essentially because one can use the endomorphism ring to compute an $a$-isogeny on $E_0$ even if $a$ is not smooth, see Sect. 5.

A natural idea is to work in higher dimensions to extend the range of parameters for which an attack is possible, even on a random curve $E_0$. We show in Sect. 2 that by going to dimension 8, it is possible to break in polynomial time all parameters for SIDH. The algorithm is deterministic, except for a randomized polynomial time precomputation (which does not depend on $E_0$) to decompose $N_A - N_B$ as a sum of four integer squares (see Remark 1).

From now and for the rest of this paper, we let $N_A > N_B$ be two coprime integers, and we assume that we are given their factorisations. We denote by $\ell_N$ the largest prime divisor of an integer $N$, and by $\ell_A$ the largest prime divisor of $N_A$, and $\ell_B$ the largest prime divisor of $N_B$.

**Theorem 1.** *Assume that we are given a decomposition $N_A - N_B = a_1^2 + a_2^2 + a_3^2 + a_4^2$ as a sum of four integer squares. Let $\phi_B : E_0 \to E_B$ be a $N_B$-isogeny*

© International Association for Cryptologic Research 2023
C. Hazay and M. Stam (Eds.): EUROCRYPT 2023, LNCS 14008, pp. 472–503, 2023.
https://doi.org/10.1007/978-3-031-30589-4_17

*defined over a finite field* $\mathbb{F}_q$*. Assume that* $E_0[N_A] \subset E_0(\mathbb{F}_q)$ *and that we are given a basis* $(P_1, P_2)$ *of* $E_0[N_A]$ *and the image of* $\phi_B$ *on this basis.*

*Then there is an algorithm* $\texttt{Eval}(E_0, E_B, P_1, P_2, \phi_B(P_1), \phi_B(P_2), P)$ *which returns* $\phi_B(P)$ *for any point* $P$ *of* $E_0(\mathbb{F}_q)$ *in* $\widetilde{O}(\ell_A^8 \log N_A)$ *arithmetic operations over* $\mathbb{F}_q$*.*

*In particular, if* $\operatorname{Ker}\phi_B \subset E_0[N_B](\mathbb{F}_q)$ *and we are given a basis* $R_1, R_2$ *of* $E_0[N_B](\mathbb{F}_q)$*, there is an algorithm*

$$\texttt{ComputeKernel}(E_0, E_B, P_1, P_2, R_1, R_2, \phi_B(P_1), \phi_B(P_2))$$

*which returns a generator for the kernel of* $\phi_B$ *in* $\widetilde{O}(\ell_A^8 \log N_A + \ell_B^{1/2} \log N_B)$ *arithmetic operations over* $\mathbb{F}_q$*.*

*Proof (Outline)* The full proof will be given in Sect. 2.

Notably, we will build in Sect. 2, Lemma 1 an explicit $N_A$-endomorphism[1] $F : E_0^4 \times E_B^4$ in dimension $g = 8$ (given by an $8 \times 8$ matrix) such that evaluating $F$ at $(P, P, P, P, Q, Q, Q, Q)$, for any $P \in E_0(\mathbb{F}_q), Q \in E_B(\mathbb{F}_q)$ allows to recover $\phi_B(P)$ and $\widetilde{\phi_B}(Q)$, where $\widetilde{\phi}_B$ is the dual (more precisely contragredient) isogeny. Furthermore the kernel of $F$ is described by 8 explicit rational generators which can be computed in time $O(\log N_A)$ by Lemma 2.

This reduces evaluating $\phi_B$ to evaluating the isogeny $F$ in dimension 8 on a point given generators of its kernel. As explained in Sect. 2, using the algorithm of [LR23], such an isogeny can be evaluated, via the naive algorithm to compute smooth isogenies, in time $O(\ell_A^8 \log N_A + \log^2 N_A)$. This cost can even be reduced to $\widetilde{O}(\ell_A^8 \log N_A)$ using the optimised computation of smooth isogenies of [DJP14, Sect. 4.2.2].

In particular, if $\operatorname{Ker}\phi_B \subset E_0[N_B](\mathbb{F}_q)$, and we are given a basis of $E_0[N_B](\mathbb{F}_q)$, we can evaluate $\phi_B$ on this basis by two calls to `Eval`, and then solve a DLP in a cyclic group of order $N_B$ for a cost of $\widetilde{O}(\log N_B \ell_B^{1/2})$ by Pohlig-Hellman's algorithm to recover a generator of $\operatorname{Ker}\phi_B$. We refer to Sect. 2 for more details and an alternative strategy to recover the kernel. □

*Remark 1.*
- The decomposition of $a$ as a sum of four squares is a precomputation step that only depends on $N_A$ and $N_B$. It can be done in random polynomial time $O(\log^2 a)$ binary operations by [RS86, PT18]. This is the only step of the algorithm which is not deterministic, we refer to [PT18, Sect. 5] for conjectural deterministic polynomial time algorithms.
- In the context of SIDH, $E_0$ and $E_B$ will be supersingular curves defined over $\mathbb{F}_q = \mathbb{F}_{p^2}$, the factorisations of $N_B$ and $N_A$ are known and we are given a basis of $E_0[N_A]$, $E_0[N_B]$ over $\mathbb{F}_q$, along with the evaluation of a $N_A$-isogeny $\phi_A$ on the basis of $E_0[N_B]$ and of a $N_B$-isogeny $\phi_B$ on the basis of $E_0[N_A]$, see Sect. 1.3. So we can apply Theorem 1 if $N_A > N_B$. Otherwise, if $N_B > N_A$ we will simply try to recover Alice's secret isogeny $\phi_A$ instead.

[1] We refer to Definition 1 for the definition of a $N$-isogeny $F : A \to B$ in higher dimension, if $A$ and $B$ are elliptic curves this simply means that $F$ is of degree $N$.

By considering the dual isogeny $\widetilde{F}$, we will also see in Sect. 6.4 that as in [QKL+21], in Theorem 1 it is also possible to directly reconstruct $\phi_B$ (with the same complexity) as long as $N_A^2 > N_B$.
- When $\ell_A = O(1)$, or even $\ell_A = O(\log\log N_A)$, the attack is thus "quasi-linear", i.e., in $\widetilde{O}(\log N_A)$ arithmetic operations in $\mathbb{F}_q$. So it is as efficient asymptotically as the key exchange itself (with a higher constant of course).
- The attack also breaks the TCSSI-security assumption of [DDF+21, Problem 3.2].
- Another contribution of this paper is to give a precise (but heuristic, see Heuristic 1) complexity bound for a dimension 4 attack: $\widetilde{O}(\log N_A \ell_A^4)$ arithmetic operations (after a precomputation), see Sect. 4. This precise complexity bound uses the fact mentioned above that we can also explicitly build a $N_A^2$-isogeny $F$ rather than just a $N_A$-isogeny. This gives more freedom for the tweaking of parameters needed for the dimension 4 attack.
- The method of Sects. 2 and 3 shows that the following powerful embedding lemma holds: for any $N$-isogeny $f : A \to B$ between abelian varieties of dimension $g$, and any $N' > N$, it is possible to efficiently embed $f$ as a matrix coefficient of a $N'$-isogeny $F$ in dimension $8g$ (or $4g$ or $2g$ in certain cases). This provides considerable flexibility at the cost of going up in dimension, and was used in [Rob22b] to show that an isogeny over a finite field always admits an efficient representation.

In this paper, if not specified our complexities should be understood as arithmetic complexities over the base field.

### 1.2 Outline

We prove Theorem 1 in Sect. 2. This Section is written to be short and self-contained, and since it applies in all cases, without requiring any parameter tweaks, the complexity analysis is straightforward. We recommend the reader, unless interested in the gory details of the dimension 2 and 4 attacks, to skip directly to this section.

For reasons stated in Remark 3, for practical attacks it would be more convenient to stay in lower dimension. We first describe a common framework encapsulating possible dimension $2g$ attacks in Sect. 3, before describing our dimension 4 attack in Sect. 4. We explain how the dimension 2 attacks of [CD22, MM22] fit into this common framework in Sect. 5. Parameter tweaks, needed for the dimensions 2 attack and the dimension 4 attack, are described in Sect. 6.

For this introduction, we give more context in Sect. 1.3 to explain how our attacks fit into the broad class of "torsion point attacks" in Sect. 1.4, and summarize in Sect. 1.5 the different complexities of the different dim 2, 4 and 8 attacks of [CD22, MM22, Rob22a].

### 1.3 Context

Supersingular Isogeny Diffie-Hellman (SIDH) is a post-quantum key exchange protocol initially proposed in [JD11] with further ameliorations (among many

other papers) in [DJP14, CLN16]. A standard transform gives a key encapsulation method SIKE (supersingular isogeny key encapsulation) [JAC+17], which was submitted to the NIST post-quantum competition and recently selected as an alternative candidate in the fourth round of the competition.

The key hardness problem of many isogeny based protocols is based on the difficulty of recovering a large degree isogeny $f : E \to E'$ between two ordinary or supersingular elliptic curves, the so-called *isogeny path problem*. To the best of our knowledge, without more information on $E$ and $E'$ (like an explicit representation of part of their endomorphism rings) this problem still has *exponential quantum security for supersingular curves*.

However, for the SIDH key exchange, Bob will reveal not only the codomain $E_B$ of his secret $N_B$-isogeny $\phi_B : E_0 \to E_B$ ($N_B$ a large smooth number) but also the action of $\phi_B$ on the $N_A$-*torsion* $E_0[N_A]$ for an integer $N_A$ prime to $N_B$, typically by revealing the image $Q_1 = \phi_B(P_1), Q_2 = \phi_B(P_2)$ of a basis $(P_1, P_2)$ of $E_0[N_A]$. This added information then allows Alice to pushforward her secret $N_A$ isogeny $\phi_A : E_0 \to E_A$ to $\phi'_A : E_B \to E_{AB}$, via $\operatorname{Ker} \phi'_A = \phi_B(\operatorname{Ker} \phi_A)$. Alice also reveals the action of her secret isogeny $\phi_A$ on $E_0[N_B]$, and then Bob can pushforward his secret $N_B$ isogeny to $\phi'_B : E_A \to E_{AB}$ via $\operatorname{Ker} \phi'_B = \phi_A(\operatorname{Ker} \phi_B)$. The codomain is the same since the maps $\phi'_B \circ \phi_A : E_0 \to E_A \to E_{AB}$ and $\phi'_A \circ \phi_B : E_0 \to E_B \to E_{AB}$ have the same kernel $\operatorname{Ker} \phi_A + \operatorname{Ker} \phi_B$:

$$\begin{array}{ccc} E_0 & \xrightarrow{\phi_B} & E_B \\ \downarrow{\scriptstyle \phi_A} & & \downarrow{\scriptstyle \phi'_A} \\ E_A & \xrightarrow{\phi'_B} & E_{AB} \end{array}$$

The supersingular curve $E_{AB}$ is then the common secret of Alice and Bob.

But as we will see, this is a *key weakness* that allows one to break the SIDH key exchange. This is worth emphasising: the work of [CD22, MM22, Rob22a] only breaks *SSI-T*, the supersingular isogeny with torsion problem, not the more general supersingular isogeny path problem. In particular, it does not apply to protocols like [CLM+18, DKL+20].

### 1.4 Torsion Points Attacks

Let us recall the setup. Eve wants to recover the secret $N_B$-isogeny $\phi_B$, and she knows the image of $\phi_B$ on a basis of $E_0[N_A]$. It has been well known that the publication of these so called torsion points could, for some parameters, reduce the security of the supersingular isogeny problem.

Petit in [Pet17] had the first key idea of the following "torsion points" attack: assume that the attacker Eve could somehow combine Bob's secret $N_B$-isogeny $\phi_B$ and/or its dual $\widetilde{\phi_B}$ with an isogeny $\alpha$ she controls into a $N_A$-isogeny $F : E_0 \to E'$. Eve knows the action of $\phi_B$ on $E_0[N_A]$ because Bob published it, and she also knows the action of the dual isogeny $\widetilde{\phi_B} : E_B \to E_0$ on $E_B[N_A]$. Indeed, if $(P_1, P_2)$ is a basis of $E_0[N_A]$, and $Q_1 = \phi_B(P_1)$, $Q_2 = \phi_B(P_2)$, then

$\widetilde{\phi}_B(Q_1) = N_B P_1, \widetilde{\phi}_B(Q_2) = N_B P_2$. Notice that $Q_1, Q_2$ is a basis of $E_B[N_A]$ since $N_A$ is prime to $N_B$.

Since she knows the action of $\alpha$ too because she controls it, she can recover the action of $F$ on (a basis of) $E_0[N_A]$. It is then easy for Eve to compute the kernel of $F$ using some linear algebra and discrete logarithms, see Lemma 4. These discrete logarithms are inexpensive because $N_A$ is assumed to be smooth.

From this kernel $\operatorname{Ker} F$, she can then evaluate $F$ on any point of $E_0$ via an isogeny algorithm, from which she can try to recover $\phi_B$ if extracting $\phi_B$ from $F$ is possible.

In his attack, Petit considers for $F$ an endomorphism of $E_0$ of the form $F = \widetilde{\phi}_B \circ \gamma \circ \phi_B + [d]$, where $\gamma$ is a trace 0 endomorphism (meaning that $\widetilde{\gamma} = -\gamma$) of degree $e$. Then it is easy to check that $F$ is a $(N_B^2 e + d^2)$-isogeny, so it remains to find parameters such that $N_B^2 e + d^2 = N_A$, and to construct a $\gamma$ of degree $e$. From the knowledge of $F$, it is not too hard to extract $\phi_B$.

*Remark 2.* A variant is to "tweak" the parameters, in order to increase the range of susceptible parameters. For instance if we can find parameters such that $N_B^2 e + d^2 = uN_A$ with $u$ smooth, then $F$ will be a $uN_A$-isogeny. We only know its action on $E_0[N_A]$, so we cannot recover it directly. However $F$ is a composition $F_2 \circ F_1$ of a $N_A$-isogeny $F_1$ followed by an $u$-isogeny $F_2$, so we can at least recover $F_1$ and then try to brute force $F_2$. A similar strategy holds for higher dimensional attacks, we will describe more possible tweaks in Sect. 6.

This attack, while powerful, can only apply to unbalanced parameters (here $N_A > N_B^2$); and requires the knowledge of a non-trivial endomorphism of $E_0$. Further work, like [QKL+21], improves the range of parameters susceptible to these attacks, but still requires a non-trivial endomorphism.

For SIKE's NIST submission, such an endomorphism is easy to find because the starting curve $E_0 = E_{\mathrm{NIST}}$ is defined over $\mathbb{F}_p$. So in [Cos21], Costello argues that if this line of "torsion points" attacks is improved to reach the SIKE parameters submitted to NIST, a preventive measure would be to switch the starting elliptic curve $E_0$ to a "random" one, so that Eve has no prior information on its endomorphism ring. (This was not considered for SIKE's submission because it would involve either a trusted multipartite setup to build $E_0$ or for Alice to first walk a random path and publish a "random" $E_0$, hence adding some complexity to the key exchange.)

The second key breakthrough was in the recent attacks by [CD22, MM22] by Castryck–Decru and Maino–Martindale respectively (we refer to Sects. 1.5 and 5 for more details on these two articles). They both, independently, had the beautiful idea that it is possible to extend the range of parameters susceptible to "torsion points" attack by constructing a $N_A$-isogeny $F$ in dimension 2, on a product of two supersingular curves. Indeed, going up in dimension largely opens up the range of isogeny we can construct explicitly.

They exploit the following lemma, due to Kani in [Kan97] as part of his deep work on classifying covers $C \to E$ of elliptic curves by genus 2 curves: given a $N_B$-isogeny $\phi_B : E_0 \to E_B$ and an $a$-isogeny $\alpha : E_0 \to E'$, with $a$ prime to $N_B$, it is possible to build an explicit $(a + N_B)$-isogeny $F : E_0 \times E" \to E_B \times E'$

in dimension 2 (see Lemma 6 for a generalisation to dimension $g$). This means, assuming $N_A > N_B$, that Eve can break SIDH as long as she can find an isogeny from $E_0$ of degree $a = N_A - N_B$.

This is in particular the case whenever $a$ is smooth, and is the focus of Maino and Martindale's article (Castryck and Decru also consider this case briefly). While the probability to get a smooth $a$ is small, tweaking the parameters can increase it, and subsequent analysis by De Feo showed that this gives a (heuristic) subexponential $L(1/2)$ attack. In particular, torsion points attacks can apply even to "random curves"!

Castryck and Decru furthermore exploit the fact that for the NIST submission, the curve $E_0 = E_{\text{NIST}}$ is either $y^2 = x^3 + x$ or $y^2 = x^3 + 6x^2 + x$. It has an explicit endomorphism $2i$, hence it is easy to construct an $a$-isogeny $\alpha$ (which can be evaluated efficiently) whenever $a = a_1^2 + 4a_2^2$. In particular, they obtain a (heuristic) polynomial time attack for this specific $E_0$ (assuming the factorisation of $a$ is precomputed).

Our current work stems from the fact that it is easy to extend Kani's lemma to dimension $g$ abelian varieties (see Sect. 3). Namely, from an $a$-isogeny and a $N_B$-isogeny in dimension $g$ (with $a$ prime to $N_B$), we can build an explicit $(a+N_B)$-isogeny in dimension $2g$. We will apply this to the diagonal embedding of $\phi_B$ to $E_0^g \to E_B^g$, this is still an $N_B$-isogeny, so it remains to find an $a$-isogeny on $E_0^g$, where $a = N_A - N_B$. We then exploit that even if we do not know $\mathrm{End}(E_0)$, on $E_0^2$ we can always build endomorphisms of the form $\alpha = \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix}$, which give $(a_1^2+a_2^2)$-endomorphisms. Hence we get a dimension $2g$ attack, $g = 2$, whenever $a = a_1^2 + a_2^2$ (eventually after parameter tweaks).

The general case stems from the fact that an integer is always a sum of four squares: $a = a_1^2+a_2^2+a_3^2+a_4^2$, from which we can then build an $a$-endomorphism $\alpha$ on $E_0^4$ in dimension $g = 4$, hence get a dimension $2g = 8$ attack. The fact that there always exist $a$-endomorphisms on $A^4$ for any abelian variety $A$ and any integer $a$ was first used by Zarhin in [Zar74] to show that $A^4 \times \widehat{A}^4$ always has a principal polarisation, and is known as "Zarhin's trick" or the "quaternion trick".

We remark also that unlike the decomposition of $a$ as a sum of two squares, which requires its factorisation, the decomposition as a sum of four squares can be done in (random) polynomial time, see Remark 1. It is then easy to build by hand a $(N_B + a)$-endomorphism on $E_0^4 \times E_B^4$, we will see in Sect. 2 that $F = \begin{pmatrix} \alpha & \widetilde{\phi_B} \\ -\phi_B & \widetilde{\alpha} \end{pmatrix}$ fits.

As mentioned above, this endomorphism $F$ can be seen as a special case of the dimension $g$ generalisation in Sect. 3 of Kani's lemma to build isogenies on product of abelian varieties. But it can also be seen as a variant of Petit's endomorphism to higher dimension. Indeed, if $F_1$ is a $d_1$-endomorphism and $F_2$ is a $d_2$-endomorphism, then $F_1 + F_2$ is a $(d_1 + d_2)$-endomorphism whenever $\widetilde{F_1}F_2 = -\widetilde{F_2}F_1$. Our dimension 8 endomorphism is the case $F = F_1 + F_2$ with

$F_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \widetilde{\alpha} \end{pmatrix}$ an $a$-endomorphism and $F_2 = \begin{pmatrix} 0 & \widetilde{\phi}_B \\ -\phi_B & 0 \end{pmatrix}$, a $N_B$-endomorphism. Petit's endomorphism $F = \widetilde{\phi}_B \circ \gamma \circ \phi_B + [d]$ is the case where $F_1 = \widetilde{\phi}_B \circ \gamma \circ \phi_B$ is antisymmetric (i.e., of trace 0, i.e., $\widetilde{F_1} = -F_1$) and $F_2 = [d]$ is symmetric (i.e., $\widetilde{F_2} = F_2$), with $F_1 F_2 = F_2 F_1$.

### 1.5 Complexities of the Different Attacks

The article by Castryck and Decru was first posted on 2022-07-30, with only minor revisions since. As mentioned above, this article mainly focuses on the dimension 2 attack when $E_0 = E_{\mathrm{NIST}}$ is NIST's starting curve, i.e., contains the endomorphism $2i$. In this case they obtain a heuristic polynomial time algorithm (with no explicit bound).

The heuristic is due to two reasons. First in [CD22], Castryck and Decru guess a starting path for $\phi_B$ and use $F$ as an oracle to know if the guess was correct or not, then they iterate the process. The heuristic is then that if a wrong path is guessed, the codomain of $F$ will be a Jacobian of a superspecial curve rather than a product of two supersingular elliptic curves. Assuming heuristically that the codomain of $F$ for a wrong guess is uniform among all superspecial surfaces, the probability of a mistake is $\approx 1/p$, hence negligible. But, as first noticed by Maino and Martindale in [MM22], and also independently by Oudompheng [Oud22], Petit, and Wesolowski [Wes22b], the isogeny $F$ allows one to directly recover $\phi_B$. This gives a more direct attack (no need to guess many isogenies), and removes the first heuristic.

The second reason is that for their attack to work on the starting curve $E_0 = E_{\mathrm{NIST}}$, they need $a = N_A - N_B$ to be of the form $a = a_1^2 + 4a_2^2$. In this case they can build an $a$-isogeny $\alpha$ which can be evaluated in $O(\log a)$ arithmetic operations. For a uniform integer less than $x$, the probability to be decomposed in this form is roughly $1/\sqrt{\log x}$ (see Remark 9), so assuming that parameter tweaks behave like uniform integers, we may assume that we can tweak the parameters without increasing their size too much in such a way that the attack can apply. Also this decomposition (which is a precomputation) supposes access to a factorisation oracle; hence is in polynomial time only in the quantum model.

This second heuristic (and the need for factorisation) can be removed (under GRH) using work by Wesolowski [Wes22b] explaining how to directly build a $N_A - N_B$-isogeny $\alpha$ when $\mathrm{End}(E_0)$ is known. More precisely, Wesolowski builds an ideal $I_\alpha$ of norm $a$ which represents $\alpha$, and evaluating $\alpha$ on a point is done by using [FKM+22, Lemma 3.3]. Constructing this isogeny and then evaluating it on a point can be done in polynomial time, but there is no clear complexity bound as of yet. But the evaluation of $\alpha$ on a basis of $E_0[N_A]$ can be seen as a polynomial time precomputation, depending on $E_0$. Via this precomputation, the attack then reduces to evaluating a $N_A$-isogeny $F$ in dimension 2.

We mention also that Castryck and Decru implemented their attack in Magma (so far this is the only publicly available implemented attack), which showed that it was practical, breaking Microsoft's and the NIST submis-

sion parameters. The timings were then considerably improved in an open source reimplementation in Sage [POP+22], where Oudompheng implemented the direct isogeny recovery of [MM22] and the extended parameter tweaks of [Rob22a] (see Sect. 5).

The article by Maino and Martindale was posted on 2022-08-08, with a second major revision on 2022-08-25, fixing an error where their initial endomorphism candidate did not respect the product polarisations. The second version uses the correct matrix from [Rob22a, Oud22, Wes22b]. They focus on the case where $\mathrm{End}(E_0)$ is not known, which was also briefly investigated by Castryck and Decru. The first version does not contain a complexity estimate, but in the second version they use an analysis due to De Feo which shows that, using slightly more general parameter tweaks, they have a heuristic subexponential $L(1/2)$ attack. They then incorporated work by Panny, Pope and Wesolowski in their submission to eurocrypt [MMP+23].

This current article [Rob22a] was first posted on 2022-08-11 focusing mainly on the polynomial time dimension 8 attack (and explaining very briefly the dimension 4 attack). It was followed by revisions expanding on the dimension 4 attack and its complexity, and on giving a general dimension $2g$ framework.

At the time of its posting, [Rob22a] was the only one containing a precise complexity estimate, and the only available polynomial time attack (with or without random starting curve) with no heuristics. Due to the work of Wesolowski and De Feo mentioned above, and the improved parameters tweaks of Sect. 6, the current situation (as far as I am aware) is now as follows:

- When $E_0 = E_{\mathrm{NIST}}$ is NIST's starting curve, the attack of Castryck-Decru using the endomorphism $2i$ (as implemented in [POP+22]) is in heuristic polynomial time. We refer to Proposition 3 for a complexity analysis: We can find a decomposition $N_A = (b_1+4b_2^2)N_B/D+(a_1+4a_2^2)$ where $D$ is a divisor of $N_B$ heuristically of magnitude $\Theta(\log N_B)$ in $O(\log^3 N_A)$ binary operations for this precomputation step. The attack is then in $\widetilde{O}(D \log N_A \ell_A^2) = \widetilde{O}(\log^2 N_A \ell_A^2)$ arithmetic operations. We can reduce the magnitude of $D$ to $\Theta(\sqrt{\log N_B})$ (heuristically) at the price of doing $O(\sqrt{\log N_B})$ factorisation calls in the precomputation. The attack is then in $\widetilde{O}(\log^{1.5} N_A \ell_A^2)$ arithmetic operations.
  (In their version updated for the submission to eurocrypt, Castryck and Decru argue in [CD22, Sect. 10] that their attack is in heuristic subexponential time $L(1/4)$ when using only the endomorphism $2i$, and that they need to consider more general endomorphisms to obtain an heuristic polynomial time attack. This discrepancy with our analysis above comes from the fact that we use more general parameter tweaks.)
  Using [Wes22b], the dimension 2 attack can also apply to any elliptic curve with known endomorphism ring in proven polynomial time under GRH (but the exact degree has not been bounded yet). More precisely, after a polynomial time precomputation to construct the $a$-isogeny $\alpha$ and its action on a basis of $E_0[N_A]$, the attack is the same as in Theorem 1 except that $F$ is computed in dimension 2, hence its evaluation costs $\widetilde{O}(\log N_A \ell_A^2)$ arithmetic operations in $\mathbb{F}_q$, see Proposition 4.

- When $E_0$ is a "random" curve, the dimension 2 attack of Maino and Martindale (and also Castryck and Decru) is in (heuristic) subexponential time $L(1/2)$ [MM22].
  The dimension 4 attack of Sect. 4 is in heuristic polynomial time (because it needs parameter tweaks). The precomputation is very similar to the precomputation done for Castryck-Decru using the endomorphism $2i$ (because both attacks rely on decomposing an integer as a sum of two squares), except that in this case we can also build a $N_A^2$-isogeny with no added (asymptotic) cost by Sect. 6.4. Under Heuristic 1, the precomputation costs $O(\log^3 N_A)$ binary operations to find a decomposition $N_A^2 = (b_1^2 + 2b_2)^2 N_B + (a_1^2 + a_2^2)$, and then the attack is in $\widetilde{O}(\log N_A \ell_A^4)$ arithmetic operations by Proposition 2. We stress that for the dimension 4 attack the heuristic only concerns the average complexity of finding this decomposition of $N_A^2$ (provided it exists), not the attack itself.
  The dimension 8 attack of Sect. 2 is in proven polynomial time, and is in $\widetilde{O}(\log N_A \ell_A^8)$ arithmetic operations by Theorem 1. The precomputation step is the decomposition of $N_A - N_B$ as a sum of four squares and can be done in randomized $O(\log^2 N_A)$ binary operations.
  The dimension 8 (resp., 4) attack remains the only proven (resp., heuristic) polynomial time attack for a random curve $E_0$.
- When $\ell_A = O(1)$ (or even $O(\log\log N_A)$), the dimension 8, dimension 4, and if $\mathrm{End}(E)$ is known, the dimension 2 attacks, all have quasi-linear complexity of $\widetilde{O}(\log N_A)$ arithmetic operations.
  The constants involved will be larger for the higher dimensional attack, however the precomputation of the dimension 8 attack is faster than the precomputation of the dimension 2 attack. Furthermore, in dimension 2, when $E$ has known endomorphisms but is not $E_{\mathrm{NIST}}$, the precomputation step also depends on the starting curve $E_0$. An implementation is ongoing to compare timings.

### 1.6 Thanks

Many thanks are due to the persons who commented on the prior versions. Special thanks to Benjamin Wesolowski and Marco Streng. Thanks to the anonymous referees for numerous suggestions to improve the exposition of this paper.

This work was supported by the ANR `ANR-19-CE48-0008` project `Ciao`.

## 2 Dimension 8 Attack

Since $N_A > N_B$, write $N_A = N_B + a$ for a positive integer $a > 0$. It is harmless to suppose that $N_A$ is prime to $N_B$, otherwise if $d = \gcd(N_A, N_B)$, we could recover the kernel of a $d$-isogeny through which $\phi_B$ factors (since we know its action on $E_0[d] \subset E_0[N_A]$), so we could reduce to solving the problem with new coprime parameters $N'_A = N_A/d$, $N'_B = N_B/d$.

As $N_A$ is prime to $N_B$, $\gcd(N_A, a) = 1$. Let $M \in M_4(\mathbb{Z})$ be a $4 \times 4$ matrix such that $M^T M = a\,\mathrm{Id}$. Explicitly we write $a = a_1^2 + a_2^2 + a_3^2 + a_4^2$ and take

$$M = \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & a_4 & -a_3 \\ a_3 & -a_4 & a_1 & a_2 \\ a_4 & a_3 & -a_2 & a_1 \end{pmatrix},$$

the matrix of the multiplication of $a_1 + a_2 i + a_3 j + a_4 k$ in the standard quaternion algebra $\mathbb{Z}[i, j, k]$ $i^2 = j^2 = k^2 = -1, ij = k$. Let $\alpha_0$ be the endomorphism on $E_0^4$ represented by the matrix $M$. The dual (with respect to the product principal polarisation) $\widetilde{\alpha}_0$ of $\alpha_0$ is represented by the matrix $M^T$ (since integer multiplications are their own dual), so $\widetilde{\alpha}_0 \alpha_0 = a\,\mathrm{Id}$, hence $\alpha_0$ is an $a$-isogeny, which can be evaluated in $O(\log a)$ arithmetic operations. We let $\alpha_B$ be the endomorphism of $E_B^4$ given by the same matrix $M$, and by abuse of notation we denote by $\phi_B\,\mathrm{Id} : E_0^4 \to E_B^4$ the diagonal embedding of $\phi_B : E_0 \to E_B$. We remark that since $\alpha_0$ is given by an integral matrix, it commutes with $\phi_B$ in the sense that we have the equation: $\phi_B \alpha_0 = \alpha_B \phi_B$:

$$\begin{array}{ccc} E_0^4 & \xrightarrow{\phi_B\,\mathrm{Id}} & E_B^4 \\ \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_B} \\ E_0^4 & \xrightarrow{\phi_B\,\mathrm{Id}} & E_B^4 \end{array}$$

**Lemma 1.** *With the notations above, let* $F = \begin{pmatrix} \alpha_0 & \widetilde{\phi}_B\,\mathrm{Id} \\ -\phi_B\,\mathrm{Id} & \widetilde{\alpha}_B \end{pmatrix}$, *where* $\widetilde{\phi}_B$ *is the dual isogeny* $E_B \to E_0$ *of* $\phi_B$. *Then* $F$ *is a* $N_A$*-endomorphism on the* 8*-dimensional abelian variety* $X = E_0^4 \times E_B^4$.

*Proof.* This is a special case of Lemma 6 in Sect. 3.2 below. We give a direct proof: since the dual $\widetilde{F}$ of $F$ is given by $\widetilde{F} = \begin{pmatrix} \widetilde{\alpha}_0 & -\widetilde{\phi}_B\,\mathrm{Id} \\ \phi_B\,\mathrm{Id} & \alpha_B \end{pmatrix}$ by Lemma 3 in Sect. 3.1 below, we compute

$$\widetilde{F}F = F\widetilde{F} = \begin{pmatrix} N_B + a & 0 \\ 0 & N_B + a \end{pmatrix} = N_A\,\mathrm{Id}\,.$$

Hence $F$ is a $N_A$-isogeny on $X$ (with respect to the product polarisations). □

As in Sect. 1.4, since Bob reveals the action of $\phi_B$ on a basis of the $N_A$-torsion, the action of $F$ on the $N_A$-torsion is explicit, hence we can recover its kernel. We can also directly recover $\mathrm{Ker}\,F$ as follows:

**Lemma 2.** *Let* $(P_1, P_2)$ *be a basis of* $E_0[N_A]$. *The kernel of* $F$ *is given by the 8 generators* $(g_1, \ldots, g_8) = \{(\widetilde{\alpha}_0(P), (\phi_B\,\mathrm{Id})(P))\}$ *for* $P = (P_1, 0, 0, 0)$, $(P_2, 0, 0, 0)$, $(0, P_1, 0, 0)$, $(0, P_2, 0, 0)$, $(0, 0, P_1, 0)$, $(0, 0, P_2, 0)$, $(0, 0, 0, P_1)$, $(0, 0, 0, P_2)$. *These generators can be computed in* $O(\log a)$ *arithmetic operations in* $E_0(\mathbb{F}_q)$.

*Proof.* The kernel is given by the image of $\widetilde{F}$ on $X[N_A]$. Since $a$ is prime to $N_A$, by Lemma 6 in Sect. 3.2 below, $\operatorname{Ker} F$ is exactly the image of $\widetilde{F}$ on $E_0^4[N_A] \times 0$: $\operatorname{Ker} F = \{(\widetilde{\alpha}_0(P), (\phi_B \operatorname{Id})(P)) \mid P \in E_0^4[N_A]\}$. □

We can now prove Theorem 1.

*Proof (of Theorem 1).* Since we have generators of the kernel of $F$, we can compute $F$ (on any point $P \in X(\mathbb{F}_q)$) using an isogeny algorithm in dimension 8. We decompose the $N_A$-endomorphism $F$ as a chain of $\ell$-isogenies for $\ell$ the prime factors of $N_A$. If $\ell_A$ is the largest prime divisor of $N_A$, the complexity of the first $\ell_A$-isogeny computation will first be $\widetilde{O}(\log N_A)$ arithmetic operations in $A(\mathbb{F}_q)$ to compute the multiples $\frac{N_A}{\ell_A} g_i$, followed by the individual $\ell_A$-isogeny computations on $P$ and the $g_i$. These isogeny computations cost $O(\ell_A^8)$ operations over $\mathbb{F}_q$ using [LR23]. Since we compute a composition of at most $O(\log N_A)$ isogenies, the total cost of evaluating $F$ on $P$ is $O(\log^2 N_A + \log N_A \ell_A^8 \log \ell_A)$. This naive method uses $O(\log N_A)$ $\ell$-isogeny calls where $\ell \mid N_A$, and multiplications which cost $O(\log^2 N_A)$ in total. The optimised method of [DJP14, Sect. 4.2.2] shows that by increasing the number of isogeny calls to $\widetilde{O}(\log N_A)$, the multiplication cost can be reduced to $\widetilde{O}(\log N_A)$ multiplications by $\ell \mid N_A$. This optimised version thus costs $\widetilde{O}(\ell_A^8 \log N_A + \ell_A \log N_A) = \widetilde{O}(\ell_A^8 \log N_A)$. (Note that since a $\ell$-isogeny in dimension 8 is going to be much more expensive than a multiplication by $\ell$, for practical attacks it will be important to apply the optimised *weighted* strategy of [DJP14, Sect. 4.2.2] rather than their *balanced* strategy.)

Thus we can evaluate $F$ on any point of $X$, so we can evaluate $\phi_B$ or $\widetilde{\phi}_B$ on any point of $E_0$ (resp., $E_B$). This is enough to recover the kernel of $\phi_B$ on $E_0$, this is a special case of Lemma 5 and Remark 5 in Sect. 3.1 below. We can give a direct proof: if $\operatorname{Ker} \phi_B \subset E_0[N_B](\mathbb{F}_q)$ and we are given a basis $(R_1, R_2)$ of $E_0[N_B](\mathbb{F}_q)$ (we allow the possibility for $R_2$ to be 0 if $E_0[N_B](\mathbb{F}_q)$ is cyclic), we can compute $\phi_B(R_1), \phi_B(R_2)$ in two calls to the evaluation of $F$. We can then solve a DLP to recover a minimal linear relation between $\phi_B(R_1)$ and $\phi_B(R_2)$ from which we obtain a generator for the kernel of $\phi_B$. The DLP costs $\widetilde{O}(\ell_B^{1/2} \log N_B)$ arithmetic operations by Lemma 4 below.

We also remark that if $E_B[N_B]$ is rational, we have an alternative strategy to recover $\operatorname{Ker} \phi_B$. Indeed it is the image of $\widetilde{\phi}_B$ on $E_B[N_B]$. So if $(Q_1, Q_2)$ is a basis of $E_B[N_B]$, we compute $Q_i' = \widetilde{\phi}_B(Q_i)$ by evaluating $F$ on the point $(0, 0, 0, 0, Q_i, 0, 0, 0)$, and the kernel of $\phi_B$ is generated by whichever $Q_i'$ has order $N_B$. Checking the order costs $O(\log N_B \log\log N_B)$ operations in $E_0(\mathbb{F}_q)$ using a binary tree.

This concludes the complexity analysis of Theorem 1. □

*Remark 3.* The isogeny computations in [LR23, BCR10, Som21] use a (level $m = 4$ or $m = 2$) theta model of $X$, which we can compute as the (fourfold) product theta structure of the theta models of $E_0$ and $E_B$. It is also well known how to switch between the theta model and the Weierstrass model on an elliptic curve, and it is not hard to extend the conversion to the product of elliptic curves, since the product theta structure is given by the Segre embedding. The arithmetic on

the theta models can be done in $O(1)$ arithmetic operations in a $O(1)$-extension of $\mathbb{F}_q$ (if $8 \mid N_A N_B$ the theta model will already be rational). However the big $O()$ notation hides an exponential complexity in the dimension $g$. In dimension 8 and level $m = 4$, the theta model uses $2^{16}$ coordinates, so we would need in practice to switch to the *Kummer* model by working in level $m = 2$ which "only" requires $2^8$ coordinates. This is another reason why we would prefer to compute an endomorphism in dimension $g = 4$ rather than $g = 8$: in dimension 4 we would only need $2^8$ coordinates in level $m = 4$, or $2^4$ coordinates in level $m = 2$.

Finally, there is one technical difficulty when working with the theta model in level $m$: it involves choosing a level $m$ symmetric theta structure. Even if we start with a product theta structure on $E_0^4 \times E_B^4$, when computing $F$ we will not generally end with a product theta structure. So we need to correct the level $m$ structure we end up with by a symplectic action to get a product structure: this is important to project back to $E_0$. We can either try all of them (this is a $O(1)$ operation, but with a very big constant since we are in dimension 8). A much better strategy, if $N_A$ is prime to $m$, is to guess the image of $E_0[m]$ by $\phi_B$. Since we know the image of $\alpha$, if our guess is correct, this directly gives us the symplectic matrix we need to correct our theta structure with. So this greatly lowers the number of matrices we need to test. If $m$ is not prime to $N_A$, we need to guess the image of $E_0[mN_A]$ under $\phi_B$ instead; recall that we already know the image of $E_0[N_A]$, so we also have few guesses to make. In fact, if $m \mid N_A$ we can also use Sect. 6.4 to write $F$ as a $(N_A/m)$-isogeny followed by an $m$-isogeny, and in this case we have enough information to directly know how to glue the theta structures together in the middle.

*Remark 4.* It is immediate to generalise Theorem 1 to recover a $N_B$-isogeny $\phi_B$ between abelian varieties $E_0, E_B$ of dimension $g$. The attack reduces to computing one $N_A$-isogeny in dimension $8g$ (or eventually $4g$ or even $2g$ if the parameters allow for it).

The same proof as above holds; the complexity of evaluating the dimension $8g$ $N_A$-isogeny will be $\widetilde{O}(\log N_A \ell^{8g})$ arithmetic operations using [LR23] and the fast smooth isogeny computation of [DJP14, Sect. 4.2.2]. We can then recover $\operatorname{Ker} \phi_B$ using Lemma 5 below.

## 3 Description of the Dimension $2g$ Attack

In this section we generalize the construction of Sect. 2, which will be used in Sects. 4 and 5 to mount an attack in dimensions 4 and 2.

### 3.1 $N$-isogenies

**Definition 1.** *A $N$-isogeny $f : (A, \lambda_A) \to (B, \lambda_B)$ of principally polarised abelian varieties is an isogeny such that $f^*\lambda_B := \hat{f} \circ \lambda_B \circ f = N\lambda_A$, where $\hat{f} : \hat{B} \to \hat{A}$ is the dual isogeny. Letting $\widetilde{f} = \lambda_A^{-1}\hat{f}\lambda_B$ be the dual isogeny $\widetilde{f} : B \to A$ of $f$ with respect to the principal polarisations, this condition is equivalent to $\widetilde{f}f = N$.*

If $\Theta_B$ is a divisor associated to $\lambda_B$, then since $\lambda_B : P \mapsto t_P^*\Theta_B - \Theta_B \in \mathrm{Pic}^0(B) = \widehat{B}$ (where $t_P$ is the translation by $P$), we see that $f^*\lambda_B$ is the polarisation associated to $f^*\Theta_B$, so $f$ is a $N$-isogeny exactly when this polarisation is equal to $N\lambda_A$.

If $\Theta_A$ is a divisor associated to $\lambda_A$, sections of $m\Theta_A$ give coordinates on $A$ (if $m \geq 3$ we get a projective embedding by Lefschetz' theorem). Given a suitable model of $(A, m\Theta_A)$, a representation of the kernel $K = \mathrm{Ker}\, f$ of a $N$-isogeny $f$ (for instance coordinates for its generators), and the coordinates of a point $P \in A$, a $N$-isogeny algorithm will output a suitable model of $(B, m\Theta_B)$ and the coordinates of the image $f(P)$ in this model. For instance, the $N$-isogeny algorithm from [LR23] uses a theta model of level $m = 2$ or $m = 4$, and in dimension $g$ can compute the image of an $N$-isogeny in $O(N^g)$ arithmetic operations over the base field (where the theta model is defined).

Note that in general, for a $N$-isogeny algorithm, we only have the kernel $K$ and the source polarised abelian variety $(A, \Theta_A)$. We first need to check that the divisor $N\Theta_A$ descends through the isogeny $f : A \to B = A/K$. This implies that $K$ must be a subgroup of the kernel of the polarisation $N\lambda_A : A \to \hat{A}$ associated to $N\Theta_A$. And by descent theory [Mum66, Proposition 1 p. 291]; [Mum70, Theorem 2 p. 231], the descents of $N\Theta_A$ correspond exactly to level subgroups $\widetilde{K}$ of $K$ in Mumford's theta group $G(N\Theta_A)$. Hence $N\Theta_A$ descends if and only if $K$ is isotropic for the commutator pairing of $G(N\Theta_A)$ (and the descent $\Theta_B$ will be of degree one if and only if $K$ is maximal isotropic by a standard degree computation). Mumford proves in [Mum70, (5) p. 229] that this commutator pairing is yet another incarnation of the Weil pairing. So the descent condition is thus equivalent to $K$ being maximal isotropic for $e_{N,\Theta_A}$ in $A[N]$, as is well known (see e.g., [Kan97, Proposition 1.1]). Such a $K$ is usually the entry point of a $N$-isogeny algorithm.

Our current situation is different: we already have a target codomain $B$ with a polarisation $\lambda_B$, and we want $N\Theta_A$ to descend to $\lambda_B$, not just any other principal polarisation $\lambda'_B$ (of which there will be many, see Remark 7). So it does not suffice to check that $\mathrm{Ker}\, f$ is maximal isotropic for the Weil pairing, we want $f^*\Theta_B \simeq N\Theta_A$ (isomorphism up to algebraic equivalence), i.e., $\widetilde{f} \circ f = N$.

If this condition is satisfied, we know that $N\Theta_A$ descend, hence by the above discussion we automatically know that $\mathrm{Ker}\, f$ is maximal isotropic. Another way to see that without invoking descent theory is to use the fact that $\mathrm{Ker}\, f = \mathrm{Im}\, \widetilde{f} \mid B[N]$, and that since $\hat{f}$ is the dual of $f$ for the Weil pairings $e_{A,N}$ on $(A \times \hat{A})[N]$ and $e_{B,N}$ on $(B \times \hat{B})[N]$, then $\widetilde{f}$ is the dual of $f$ for the Weil pairings $e_{\lambda_A,N}$ on $(A \times A)[N]$ and $e_{\lambda_B,N}$ on $(B \times B)[N]$. In particular, if $x, y \in \mathrm{Ker}\, f$, $x = \widetilde{f}(x'), y = \widetilde{f}(y')$ for $x', y' \in B[N]$, so $e_{\lambda_A,N}(x, y) = e_{\lambda_A,N}(\widetilde{f}(x'), \widetilde{f}(y')) = e_{\lambda_B,N}(x', f \circ \widetilde{f}(y')) = e_{\lambda_B,N}(x', Ny') = 1$.

We need the following standard Lemma:

**Lemma 3.** *If* $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (A, \lambda_A) \times (B, \lambda_B) \to (C, \lambda_C) \times (D, \lambda_D)$*, then for the product polarisations on* $A \times B$ *and* $C \times D$*,* $\widetilde{F} = \begin{pmatrix} \widetilde{a} & \widetilde{c} \\ \widetilde{b} & \widetilde{d} \end{pmatrix}$.

*Proof.* Recall that we have a canonical isomorphism $\hat{A} \simeq \mathrm{Pic}^0(A)$, and that under this isomorphism the dual of $f$ is given by $\hat{f} = f^*$. This shows that $\hat{F} : \hat{C} \times \hat{D} \to \hat{A} \times \hat{B}$ is given by $\hat{F} = \begin{pmatrix} \hat{a} & \hat{c} \\ \hat{b} & \hat{d} \end{pmatrix}$ (see e.g., [EGM12, Proposition 11.28]). Since the product polarisations act component by component by definition (see e.g., the proof of [BL04, Corollary 5.3.6] or the proof of [Kan16, Proposition 61]), we then get that $\widetilde{F} = \begin{pmatrix} \widetilde{a} & \widetilde{c} \\ \widetilde{b} & \widetilde{d} \end{pmatrix}$. □

We will also use the fact that once we have evaluated an isogeny on a basis of the $N$-torsion it is easy to evaluate it on any other $N$-torsion point:

**Lemma 4.** *Let* $f : A \to B$ *be an isogeny between abelian varieties. Assume that the* $N$*-torsion of* $A$ *is rational and that we are given a basis* $(P_1, \dots P_{2g})$ *of it. Then given the evaluation* $f(P_i)$ *of all* $P_i$*, it is possible to evaluate* $f$ *on a point* $P \in A[N]$ *in time* $\widetilde{O}(\log N \ell_N^{1/2})$ *arithmetic operations, where* $\ell_N$ *denotes the largest prime divisor of* $N$*.*

*Proof.* Given a point $P \in A[N]$, we can evaluate the Weil pairing $e_N(P, P_i)$ in $O(\log N)$ arithmetic operations (this assumes we work over a model which can compute the Weil pairing; this will be the case in the theta model by [LR10, LR15]).

From the Weil pairing matrix of the $e_N(P_i, P_j)$, we can first do $O(g^2)$ discrete logarithm computations from a $N$-th root of unity $\zeta$ to get a matrix with coefficients in $\mathbb{Z}/N\mathbb{Z}$. By linear algebra over $\mathbb{Z}/N\mathbb{Z}$, it is easy to compute a symplectic basis $(a_1, \dots, a_g, a'_1, \dots a'_{2g})$ of the $N$-torsion, along with the values of $f$ on this basis. Using a naive linear algebra algorithm, this can be done in $O(g^3 \log N)$. The dominant cost will be the discrete logarithms.

The Pohlig-Hellman algorithm [PH78] has complexity $O(E \log N \ell_N^{1/2})$ operations in $A$, where if $N = \prod \ell_i^{e_i}$, $E = \sum e_i$. The iterative version of Pohlig-Hellman's algorithm which increases the current exponent $e$ in the $\ell_i$-discrete logarithm by 1 at each step, can be replaced by a Newton like version which double the precision. This faster variant, described in [Sho09, Sect. 11.2.3], has complexity[2] $\widetilde{O}(\log N \ell_N^{1/2})$.

Given the symplectic basis, one can decompose a point $P$ in this basis by $O(g)$ calls to the Weil pairing and discrete logarithms. Evaluating $f(P)$ can thus be done in $\widetilde{O}(\log N \ell_N^{1/2})$. If $P = \sum_{i=1}^g \lambda_i a_i + \lambda'_i a'_i$, then $f(P) = \sum_{i=1}^g (\lambda_i f(a_i) + \lambda'_i f(a'_i))$. □

[2] Since we use the Weil pairing to reduce to DLPs over $\mathbb{F}_q^*$, the index calculus method gives an algorithm subexponential in $\ell_N$ rather than in $\widetilde{O}(\ell_N^{1/2})$. But in our applications $\ell_N$ will be small, so the generic algorithm will be faster in our case.

We can use Lemma 4 to recover the kernel of a $N$-isogeny given its evaluation on a basis of the $N$-torsion.

**Lemma 5.** *Let $g$ be a fixed integer, and $f : A \to B$ be a $N$-isogeny in dimension $g$ and $\widetilde{f} : B \to A$ the contragredient isogeny. Assume that we are given a rational basis $P_1, \ldots, P_{2g}$ and $Q_1, \ldots, Q_{2g}$ of $A[N]$ and $B[N]$ respectively, and either the images of the basis $P_i$ by $f$ or the images of the basis $Q_i$ by $\widetilde{f}$. Then it is possible to recover a basis of* $\operatorname{Ker} f$ *in $\widetilde{O}(\log N \ell_N^{1/2})$ arithmetic operations.*

*If* $\operatorname{Ker} f$ *is of rank $g$ and we are given the image of the basis $Q_i$ by $\widetilde{f}$, it is also possible to recover a basis of* $\operatorname{Ker} f$ *in $\widetilde{O}(\log N)$ arithmetic operations.*

*Proof.* Assume first we are given the images $f(P_i)$. Since $f$ is an isogeny, $\operatorname{Ker} f \subset A[N]$. Since we are given a rational basis of $B[N]$, we can first transform this into a symplectic basis $(b_1, \ldots, b_g, b'_1, \ldots b'_g)$ as in the proof of Lemma 4. We can express $f(P_i)$ in this basis using the Weil pairing and discrete logarithms, and solve a linear system over $\mathbb{Z}/N\mathbb{Z}$. The discrete logarithms will dominate the complexity analysis and cost $\widetilde{O}(\log N \ell_N^{1/2})$. We remark that in this situation, we do not require a full rational basis of $A[N]$, we just need that $\operatorname{Ker} f \subset A[N](\mathbb{F}_q)$ and to have a basis of $A[N](\mathbb{F}_q)$.

If we are given the images $\widetilde{f}(Q_i)$, then since $f$ is a $N$-isogeny, $\operatorname{Ker} f = \operatorname{Im} \widetilde{f} \mid B[N]$, so the $\widetilde{f}(Q_i)$ generate $\operatorname{Ker} f$. Like above, using discrete logarithms (via the Weil pairing), we can then extract a basis of $\operatorname{Ker} f$ by linear algebra.

If $\operatorname{Ker} f$ is of rank $g$, we can also find a basis of $\operatorname{Ker} f$ by finding a subset of $g$ points of the basis $Q_i$ such that the $\widetilde{f}(Q_i)$ generate the full kernel. We write the $2g \times 2g$ Weil pairing matrix of the $P_i$ with the $\widetilde{f}(Q_i)$, and we look for a $2g \times g$ submatrix that generates the full image. This reduces to finding a $g \times g$ submatrix whose determinant $\delta$ is of primitive order $N$.

If $\omega(N)$ is the number of distinct prime divisors of $N$, checking if $\delta^{N/\ell} \neq 1$ for each prime $\ell \mid N$ costs $O(\omega(N) \log N)$ arithmetic operations. This can be improved to $O(\log N \log \log N)$ using a binary tree. Note however that this last method has a complexity exponential in $g$. □

*Remark 5.* Let $f : A \to B$ be a $N$-isogeny between abelian varieties in dimension $g$ whose kernel if of rank $g$. If $\operatorname{Ker} f \subset A(\mathbb{F}_q)$ and we are given the image of $f$ on a basis of $A[N](\mathbb{F}_q)$, but we are not given a basis of $B[N]$, we can no longer reduce to DLPs over $\mathbb{F}_q^*$ via the Weil pairing so we need to use a multidimensional DLP. By [Sut11], we can recover a basis of $\operatorname{Ker} f$ in $\widetilde{O}(\log N \ell_N^{g/2})$.

Likewise, if $\operatorname{Ker} f = \widetilde{f}(B[N](\mathbb{F}_q))$ (this holds if $B[N] \subset B(\mathbb{F}_q)$ or more generally if the $N$-Tate pairing is trivial on $\operatorname{Ker} f \times \operatorname{Ker} f$), and we are given the image of $\widetilde{f}$ on a basis of $B[N](\mathbb{F}_q)$, we obtain generators of $\operatorname{Ker} f$ from which we can extract a basis in $\widetilde{O}(\log N \ell_N^{g/2})$ by [Sut11].

### 3.2 Isogeny Diamonds

The endomorphism $F$ of Sect. 2 is a particular case of a construction due to Kani for $g = 1$ [Kan97, Sect. 2, Proof of Theorem 2.3], which generalises immediately to $g > 1$.

We define a $(d_1, d_2)$-isogeny diamond as a decomposition of a $d_1 d_2$-isogeny $f : A \to B$ between principally polarised abelian varieties of dimension $g$ into two different decompositions $f = f_1' \circ f_1 = f_2' \circ f_2$ where $f_1$ is a $d_1$-isogeny and $f_2$ is a $d_2$-isogeny. Then $f_1'$ will be a $d_2$-isogeny and $f_2'$ a $d_1$-isogeny:

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "f_2"] & A_1 \arrow[d, "f_1'"] \\ A_2 \arrow[r, "f_2'"] & B \end{tikzcd}$$

**Lemma 6 (Kani).** *Let $f = f_1' \circ f_1 = f_2' \circ f_2$ be a $(d_1, d_2)$-isogeny diamond as above. Then $F = \begin{pmatrix} f_1 & \widetilde{f_1'} \\ -f_2 & \widetilde{f_2'} \end{pmatrix}$ is a $d$-isogeny $F : A \times B \to A_1 \times A_2$ where $d = d_1 + d_2$.*

*Its kernel is given by the image of $\widetilde{F} = \begin{pmatrix} \widetilde{f_1} & -\widetilde{f_2} \\ f_1' & f_2' \end{pmatrix}$ on $(A_1 \times A_2)[d]$. If $d_1$ is prime to $d_2$, we also have $\operatorname{Ker} F = \{(\widetilde{f_1}(P), f_1'(P)) \mid P \in A_1[d]\}$, the kernel is thus of rank $2g$.*

*Proof.* We check, using Lemma 3, that $\widetilde{F}F = d\,\mathrm{Id}$. Furthermore if $d_1$ is prime to $d_2$, then the restriction of $\widetilde{F}$ to $A_1[d] \times \{0\}$ is injective, hence its image spans the full kernel since $\#A_1[d] = d^{2g}$. □

The matrix $F$ from Sect. 2 is a special case of Lemma 6 where $A = E_0^4, B = E_B^4$ and $F$ is actually an endomorphism.

### 3.3 Description of the Attack

Write $N_A = N_B + a$, $a > 0$. Suppose that we can find an explicit $a$-isogeny $\alpha_0 : E_0^g \to X_0$. Then we can consider the following pushout:

$$\begin{tikzcd} E_0^g \arrow[r, "\phi_B"] \arrow[d, "\alpha_0"] & E_B^g \arrow[d, "\alpha_B"] \\ X_0 \arrow[r, "\phi_B'"] & X_B \end{tikzcd}$$

Hence we have the following isogeny diamond

$$\begin{tikzcd} X_0 \arrow[r, "\widetilde{\alpha_0}"] \arrow[d, "\phi_B'"] & E_0^g \arrow[d, "\phi_B"] \\ X_B \arrow[r, "\widetilde{\alpha_B}"] & E_B^g \end{tikzcd}$$

so by Lemma 6, $F = \begin{pmatrix} \widetilde{\alpha}_0 & \widetilde{\phi}_B \\ -\phi'_B & \alpha_B \end{pmatrix}$ is a $N_A$-isogeny $F : X_0 \times E_B^g \to E_0^g \times X_B$. In particular, $\operatorname{Ker} F$ is the image of $\widetilde{F}$ on $(E_0^g \times X_B)[N_A]$. Since $a$ is prime to $N_B$, it is also the image of $\widetilde{F}$ on $E_0^g[N_A] \times 0$: $\operatorname{Ker} F = \{(\alpha_0(P), \phi_B(P)) \mid P \in E_0^g[N_A]\}$. In particular, we don't need to build $X_B$, we will recover it when evaluating $F$. Evaluating $F$ gives the evaluation of $\widetilde{\phi}_B$ which we can use to recover the kernel of $\phi_B$.

Notice that if $\alpha_0 : E_0 \to E'$ is an $a$-isogeny, then $\operatorname{diag}(\alpha_0) : E_0^g \to X_0 := E'^g$ is also an $a$-isogeny. So on our product of elliptic curves, we can always compose or precompose with smooth isogenies, see Sect. 6.2.

To increase the parameters susceptible to this attack, we can also postcompose and precompose $\phi_B : E_0^g \to E_B^g$ by isogenies $\beta_1$, $\beta_2$. Write $N_A = bN_B + a$, $a, b > 0$; eventually applying the parameter tweaks of Sect. 6. Note that since $N_A$ is coprime to $N_B$, then dividing by $\gcd(N_A, a, b)$ if necessary, we may assume that $N_A, a, b$ are coprime. Write $b = b_1 b_2$, and suppose that we can find an explicit $b_1$-isogeny $\beta_1 : E_0^g \to Y_0$, a $b_2$-isogeny $\beta_2 : E_B^g \to Y_B$, and an $a$-isogeny $\alpha_0 : E_0^g \to X_0$. Let $\gamma = \beta_2 \circ \phi_B \circ \widetilde{\beta_1} : Y_0 \to Y_B$, it is a $bN_B$-isogeny. Consider the following pushouts,

$$\begin{tikzcd}
Y_0 \arrow[d, "\alpha'_0"] & E_0^g \arrow[l, "\beta_1"'] \arrow[r, "\phi_B"] \arrow[d, "\alpha_0"] & E_B^g \arrow[r, "\beta_2"] \arrow[d, "\alpha_B"] & Y_B \arrow[d, "\alpha'_B"] \\
Z_0 & X_0 \arrow[l, "\beta'_1"] \arrow[r, "\phi'_B"] & X_B \arrow[r, "\beta'_2"] & Z_B
\end{tikzcd}$$

since $a$ is prime to $bN_B$, $\gamma' = \beta'_2 \circ \phi'_B \circ \widetilde{\beta'_1} : Z_0 \to Z_B$ is a $N_B b$-isogeny and $\alpha'_0, \alpha'_B$ are $a$-isogenies.

We thus have the following isogeny diamond

$$\begin{tikzcd}
Z_0 \arrow[r, "\widetilde{\alpha'_0}"] \arrow[d, "\gamma'"] & Y_0 \arrow[d, "\gamma"] \\
Z_B \arrow[r, "\widetilde{\alpha'_B}"] & Y_B
\end{tikzcd}$$

so by Lemma 6, $F = \begin{pmatrix} \widetilde{\alpha'_0} & \widetilde{\gamma} \\ -\gamma' & \alpha'_B \end{pmatrix}$ is a $N_A$-isogeny $F : Z_0 \times Y_B \to Y_0 \times Z_B$. In particular, $\operatorname{Ker} F$ is the image of $\widetilde{F}$ on $(Y_0 \times Z_B)[N_A]$. Since $a$ is prime to $bN_B$, it is also the image of $\widetilde{F}$ on $Y_0[N_A] \times 0$: $\operatorname{Ker} F = \{(\alpha'_0(P), \gamma(P)) \mid P \in Y_0[N_A]\}$. Note that as before, this means that we don't need to construct $Z_B$ explicitly, however in this case we need to construct the pushout $Z_0$.

This allows one to compute $F$ as a smooth $N_A$-isogeny of dimension $2g$ in time $O(\log^2 N_A + \log N_A \ell_A^{2g})$ by [LR23] or even $\widetilde{O}(\log N_A \ell_A^{2g})$ via the fast isogeny decomposition of [DJP14, Sect. 4.2.2]. We can hence evaluate $F$ on the $N_A$-torsion to recover the kernel of $\widetilde{F}$, which allows us to evaluate $\widetilde{F}$ too. In particular, we can compute $\gamma = \beta_2 \circ \phi_B \circ \widetilde{\beta_1}$ on any point of $Y_0$. It remains

to recover $\phi_B$ from $\gamma$. Applying $\widetilde{\beta_2}$ and $\beta_1$, we can always recover $b\phi_B$, hence we may recover $\phi_B$ whenever $b$ is prime to $N_B$. Otherwise, we at least recover a $(N_B/\gcd(b, N_B))$-isogeny through which $\phi_B$ factors, and we iterate, which is possible as long as $\gcd(b, N_B) < N_B$. Alternatively, since $F$ gives us the evaluation of $\widetilde{\gamma}$, we can recover $b\widetilde{\phi_B}$ by the same method, which is also enough to give the kernel of $\phi_B$ as long as $b$ is prime to $N_B$.

In summary we have reduced recovering $\phi_B$ to evaluating the isogeny $F$ in dimension $2g$:

**Theorem 2.** *Let $\phi_B : E_0 \to E_B$ be a $N_B$-isogeny defined over a finite field $\mathbb{F}_q$. Assume that $E_0[N_A] \subset E_0(\mathbb{F}_q)$ and that we are given a basis $(P_1, P_2)$ of $E_0[N_A]$ and the image of $\phi_B$ on this basis.*

*Suppose that we can find $a, b > 0$ such that $N_A = bN_B + a$, with $a, b, N_a$ coprime, $b = b_1 b_2$, and an $b_1$-isogeny $\beta_1 : E_0^g \to Y_0$, a $b_2$-isogeny $\beta_2 : E_B^g \to Y_B$, and an $a$-isogeny $\alpha_0 : E_0^g \to X_0$. Assume furthermore for simplicity that $\gcd(b, N_B) = 1$ (or is small). Let $T$ be a bound on the arithmetic operations required to evaluate $\beta_1, \beta_2$ (and their duals) and the pushout $\alpha'$ of $\alpha$ and $\beta_1$ on a basis of the $N_A$-torsion of $E_0^g, E_B^g, Y_0$ respectively. Then, there is an algorithm to evaluate $\phi_B$ on any point $P \in E_0(\mathbb{F}_q)$(resp. $\widetilde{\phi}_B$ on any point in $E_B(\mathbb{F}_q)$) in $O(\ell_A^{2g} \log N_A + \log^2 N_A + T)$ arithmetic operations in $\mathbb{F}_q$, or even in $\widetilde{O}(\ell_A^{2g} \log N_A + T)$.*

*Remark 6.* In the situation of Theorem 2, we will see in Sect. 6 ways to tweak the parameters $N_A, N_B$ to improve our range of parameters which can be decomposed as in the Theorem. Since we can evaluate $\phi_B$ and $\widetilde{\phi}_B$, we can use Lemma 5 and Remark 5 to recover a generator of its kernel.

We leave to the reader the case where we have an isogeny $\alpha_B : E_B^g \to X_B$ constructed from $E_B$ instead of the isogeny $\alpha_0$. Note that, using discrete logarithms if needed, we only need to evaluate $\alpha_0, \beta_1, \beta_2$ on a basis of the $N_A$-torsion of their respective domains. It is thus better to build the isogenies from $E_0^g$ rather than from $E_B^g$, indeed for $\alpha'$ and $\beta_1$ these evaluations can then be seen as a precomputation (involving the parameters and $E_0$).

*Remark 7.* In dimension 8, the domain (and codomain) of $F$ is a product of supersingular elliptic curves, so is a superspecial abelian variety. The same is true for the isogeny $F$ in dimension $2g$: since $F$ is a $N_A$-isogeny with $N_A$ prime to the characteristic of the base field, $F$, or its decomposition into a product of $\ell$-isogenies, preserves the $a$-number of the intermediate abelian varieties. Hence they have $a$-number equal to $2g$, so they are still superspecial. By a theorem due to Deligne, Ogus and Shioda [Shi79, Theorem 3.5], they are all isomorphic (without the polarisation!) to $E_0^{2g}$. So in the decomposition of $F$ we always stay on the same abelian variety $E_0^{2g}$, except that we gradually change its polarisation. For instance in the dimension 2 attack, we start with a product polarisation but the intermediate polarisations will generically be indecomposable, hence correspond to Jacobians of genus 2 hyperelliptic superspecial curves.

## 4 Dimension 4 Attack

In dimension 2, we can always write an $a$-endomorphism on $E_0^2$ whenever $a = a_1^2 + a_2^2$. So using Sect. 3, we can do a dimension 4 attack whenever we can find $a, b > 0$ such that $N_A = bN_B + a$ and both $a$ and $b$ are a sum of two squares. Note that unlike the decomposition of $a$ as a sum of four squares from Sect. 2, these decompositions into a sum of two squares requires the factorisation of $a, b$. To increase our probability of success, we can also tweak the parameters $N_A$ and $N_B$ as explained in Sect. 6.

*Remark 8.* Since we can always prolong $\alpha$ and $\beta$ by isogenies of smooth degree using Sect. 6.2, we can consider the more general decompositions: $N_A = (b_1^2 + b_2^2)eN_B + (a_1^2 + a_2^2)f$ with $e, f$ sufficiently smooth. But smooth integers are of negligible density compared to sum of two squares, so for simplicity we focus only on the case $e = f = 1$ here.

**Theorem 3.** *Let $\phi_B : E_0 \to E_B$ be a $N_B$-isogeny defined over a finite field $\mathbb{F}_q$. Assume that $E_0[N_A] \subset E_0(\mathbb{F}_q)$ and that we are given a basis $(P_1, P_2)$ of $E_0[N_A]$ and the image of $\phi_B$ on this basis.*

*Suppose that we can find $a, b > 0$ such that $N_A = bN_B + a$ with $N_A, a, b$ coprime and $a$, $b$ can be written as a sum of two squares: $a = a_1^2 + a_2^2$, $b = b_1^2 + b_2^2$. Assume furthermore for simplicity that $\gcd(b, N_B)$ has its odd prime divisors congruent to 1 modulo 4, and if $2 \mid \gcd(b, N_B)$ then $4 \nmid b$.*

*Then, given the decomposition of $a$ and $b$ as these sums of two squares (e.g., given their factorisations), we can evaluate $\phi_B$ on any point $P \in E_0(\mathbb{F}_q)$ in time $O(\ell_A^4 \log \ell_A \log N_A + \log^2 N_A)$ arithmetic operations in $\mathbb{F}_q$, or even $\widetilde{O}(\log N_A \ell_A^4)$ with the fast variant of smooth isogeny computation.*

As in Remark 6, we can use Lemma 5 and Remark 5 to recover a generator of $\operatorname{Ker} \phi_B$.

*Proof.* Write $\alpha = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}$, $\beta = \begin{pmatrix} b_1 & -b_2 \\ b_2 & b_1 \end{pmatrix}$. These matrices can be interpreted as endomorphisms $\alpha_0$ of $E_0^2$ or $\alpha_B$ of $E_B^2$ and commute with $\phi_B \operatorname{Id}$: $\beta_B \phi_B \operatorname{Id} = \phi_B \operatorname{Id} \beta_0, \alpha_B \phi_B \operatorname{Id} = \phi_B \operatorname{Id} \alpha_0$. Furthermore, $\widetilde{\alpha}\alpha = (a_1^2 + a_2^2) \operatorname{Id}$, so $\alpha$ is an $a$-endomorphism, and similarly $\beta$ is a $b$-endomorphism:

$$\begin{array}{ccc} E_0^2 & \xrightarrow{\alpha_0} & E_0^2 \\ \downarrow{\scriptstyle \phi_B \beta} & & \downarrow{\scriptstyle \phi_B \beta} \\ E_B^2 & \xrightarrow{\alpha_B} & E_B^2 \end{array}$$

We can now apply Theorem 2. We can also check directly using Lemma 6 or a direct computation, that $F = \begin{pmatrix} \alpha_0 & \widetilde{\phi_B \operatorname{Id}} \widetilde{\beta_B} \\ -\beta_B \phi_B \operatorname{Id} & \widetilde{\alpha_B} \end{pmatrix}$ is a $N_A$-endomorphism of $E_0^2 \times E_B^2$ with $N_A = a + bN_B$. Its kernel is given by $\operatorname{Ker} F =$

$\{(\widetilde{\alpha_0}(P), \beta_B\phi_B \operatorname{Id}(P)) \mid P \in E_0^2[N_A]\}$. We can thus evaluate $F$, hence evaluate $\beta_B\phi_B \operatorname{Id} = \phi_B \operatorname{Id} \beta_0$ on any point in $E_0^2(\mathbb{F}_q)$ in $O(\log^2 N_A + \log N_A \ell_A^4)$ arithmetic operations over $\mathbb{F}_q$ by [LR23].

In this situation we can recover more than just $b\phi_B$. Indeed from the matrix $\beta_B\phi_B \operatorname{Id}$ we can directly recover $b_1\phi_B$ and $b_2\phi_B$; so if $b' = \gcd(b_1, b_2)$, we can recover $b'\phi_B$ in $O(\log b)$ arithmetic operations on $E_B$. This means that we can recover the kernel of a $N_B/\gcd(N_B, b')$-isogeny $E_0 \to E'_B$ through which $\phi_B$ factors. If $\gcd(N_B, b') = 1$ we have directly recovered $\phi_B$, otherwise we iterate the process, which is possible as long as $\gcd(N_B, b') < N_B$.

Under the hypothesis of Theorem 3, we have $\gcd(N_B, b') = 1$ by Remark 9 below, so we can directly recover $\phi_B$. □

*Remark 9 (Sum of two squares).* To decompose a number $b$ as a sum of two squares $b = b_1^2 + b_2^2$ is the same as finding a factorisation $b = (b_1 + ib_2)(b_1 - ib_2) = \beta\overline{\beta}$ in the Gaussian integers $\mathbb{Z}[i]$. The order $\mathbb{Z}[i] \subset \mathbb{Q}(i)$ is of discriminant $-4$, so it is the maximal order, and it is euclidean, hence is principal. The prime $(2) = ((1+i)(1-i)) = \big((1+i)^2\big)$ is ramified, and the other integer primes are unramified. By the quadratic reciprocity law, when $p$ is an odd prime, $-1$ is a square modulo $p$ if and only if $p \equiv 1 \pmod 4$. Hence when $p \equiv 1 \pmod 4$ it splits in $\mathbb{Z}[i]$, otherwise when $p \equiv 3 \pmod 4$ it stays inert. In particular, $p$ is a sum of two squares if and only if $p = 2$ or $p \equiv 1 \pmod 4$.

We deduce that $b$ is a sum of two squares if and only if all odd primes $p \equiv 3 \pmod 4$ dividing $b$ have even exponent $v_p(b)$. Also, $\gcd(b_1, b_2) \mid \gcd(\beta, \overline{\beta}) \mid 2\gcd(b_1, b_2)$. Therefore, if $b = b_1^2 + b_2^2$, $\gcd(b_1, b_2) = 2^{\lfloor v_2(b)/2 \rfloor} \times \prod_{p \mid b, p \equiv 3 \pmod 4} p^{v_p(b)/2}$. In particular, $b$ admits a primitive representation as a sum of two squares if and only if the odd prime divisors of $b$ are all congruent to 1 modulo 4 and $4 \nmid b$. We will call such a sum $b = b_1^2 + b_2^2$ with $\gcd(b_1, b_2) = 1$ a primitive sum of two squares. More generally, if the odd prime divisors of $\gcd(b, N_B)$ are congruent to 1 modulo 4, and either $2 \nmid N_B$ or $4 \nmid b$, we can find a decomposition $b = b_1^2 + b_2^2$ such that $\gcd(b_1, b_2, N_B) = 1$.

In Sect. 5, we will need decompositions of the form $b = b_1^2 + 4b_2^2$. Such a decomposition exists if $\beta \in \mathbb{Z}[2i]$, which is a suborder of $\mathbb{Z}[i]$ of index 2. So $b$ admits such a decomposition if and only if it can be written a sum of two squares and $v_2(b)$ is even.

Furthermore, the number of integers less than $x$ that can be written as a sum of two squares is given by the asymptotic behaviour of the $L$-function $L(s) = (1 - \frac{1}{2^s})^{-1} \prod_{p \equiv 1 \pmod 4} (1 - \frac{1}{p^s})^{-1} \prod_{p \equiv 3 \pmod 4} (1 - \frac{1}{p^{2s}})^{-1}$ at $s = 1$. By Perron's formula, it is equivalent to $Cx/\sqrt{\log x}$ [LeV12, Volume 2, pp. 260–263], where $C \approx 0.7642$ is the Landau-Ramanujan constant. Adapting the proof, the same asymptotic bound holds for the number of integers that are a primitive sum of two squares (resp., of the form $b_1^2 + 4b_2^2$) via the $L$-function $L(s) = (1 + \frac{1}{2^s}) \prod_{p \equiv 1 \pmod 4} (1 - \frac{1}{p^s})^{-1}$ (resp., $L(s) = (1 - \frac{1}{2^{2s}})^{-1} \prod_{p \equiv 1 \pmod 4} (1 - \frac{1}{p^s})^{-1} \prod_{p \equiv 3 \pmod 4} (1 - \frac{1}{p^{2s}})^{-1}$ ), except with a different constant $C \approx 0.49$ (resp., $C \approx 0.57$).

### 4.1 Parameter Selection

In order to find parameters such that we may apply Theorem 3, a first idea is the following. We search, using Sect. 6, for parameters $a, b$ such that $eN_A = bN_B/D + a$, where $e$ is an integer, $D$ is some divisor of $N_B$ (that we will want as small as possible), and $a$, $b$ primitive sum of two squares. Since $N_A > N_B$, there are $O(eD)$ possible choices for $b$, among which $\Omega(eD/\sqrt{\log eD})$ will be a primitive sum of two squares by Remark 9. We thus have $\Omega(eD/\sqrt{\log eD})$ candidates for $a$. If we make the *heuristic assumption* that these $a$ behave like a random integer between 0 and $N_A$, the probability to find an $a$ that is a sum of two squares is $\Omega(1/\sqrt{\log N_A})$ by Remark 9. Hence we need to take $eD = \widetilde{O}(\sqrt{\log N_A})$. There are $O(D)$ candidate $D$-isogenies through which $\phi_B$ may factor, and we need to apply Theorem 3 to each of these candidates. Likewise, there are $O(e^3)$ possibilities to guess the image of $\phi_B$ on the $N_A e$-torsion (and this does not even take into account the cost of finding the $eN_A$-torsion which possibly lives in an extension of $\mathbb{F}_q$). Thus it appears that for the tweaking of parameters, it is preferable to use $e = 1$, $D = \widetilde{O}(\sqrt{\log N_A})$. So these parameter tweaks will lose a factor $O(D)$ in the final arithmetic complexity of the attack.

However, for the dimension 4 attack, we will see that by using Sect. 6.4 we can actually set $e = N_A$ without extra cost (asymptotically).

The question remains of the cost of the precomputation of the parameters $a, b$. We can directly iterate through sums of two squares for $b$, but checking if $a$ is a sum of two squares requires its factorisation. Here we can use a trick from [Wes22a]: we restrict to the case $a$ is a prime congruent to 1 modulo 4. This only requires a primality test, hence is much less expensive. However the probability that $a$ is a prime (congruent to 1 modulo 4) will only be (heuristically) $\Omega(1/\log N_A)$, so this strategy will require larger parameters $eD$. Luckily, for the dimension 4 attack we can take $e = N_A$ as we have seen, which is more than large enough.

Reframing the above discussion, we need the following heuristic:

**Heuristic 1.** – *Let $N_1 > N_2$ be two coprime integers, with $N_2$ and $N_1/N_2$ sufficiently large. Then if $b$ is uniform amongst the numbers $x < N_1/N_2$ that are sum of two squares (resp., a primitive sum of two squares, resp., of the form $u^2 + 4v^2$), the probability that $a = N_1 - bN_2$ is a sum of two squares (resp., a primitive sum of two squares, resp., of the form $u^2 + 4v^2$) is $\Omega(1/\sqrt{\log N_1})$.*

– *Under the same assumptions, if $b$ is uniform amongst the numbers $x < N_1/N_2$ that are sum of two squares (resp., a primitive sum of two squares, resp., of the form $u^2 + 4v^2$), the probability that $a = N_1 - bN_2$ is* prime *and a sum of two squares is $\Omega(1/\log N_1)$.*

*Motivation.* The motivation behind this heuristic is that the $a$ we get will behave like a uniform integer between 1 and $N_1$. The density of sum of two squares (resp., a primitive sum of two squares, resp., of the form $u^2 + 4v^2$) less than $N_1$ is equal asymptotically to $C/\sqrt{\log N_1}$, where $C$ depends on the exact form

we want. Likewise, the density of primes congruent to 1 less than $N_1$ is equivalent asymptotically to $C/\log N_1$ by the prime number theorem and Dirichlet's theorem on arithmetic progressions. □

This heuristic allows us to derive the following complexity cost of the precomputation step.

**Proposition 1.** *Assume Heuristic 1 is true. Let $N_1 > N_2$ be two coprime integers, with $N_2$ sufficiently large. If $\epsilon > 0$, then there is a constant $C_\epsilon$ such that if $N_1/N_2 > C_\epsilon \log^{1/2} N_1$, we can find with probability $> 1-\epsilon$ a decomposition $N_1 = bN_2 + a$ where $a, b$ are sum of two squares (resp., a primitive sum of two squares, resp., of the form $u^2 + 4v^2$). This decomposition requires on average $O(\sqrt{\log N_1})$ factorisation calls and $O(\log^{2.5} N_A)$ binary operations.*

*If $N_1/N_2 > C_\epsilon \log N_1$, we can find such a decomposition on average $O(\log N_1)$ tests of primality. It will cost on average $O(\log^3 N_1)$ binary operations.*

*Proof.* By Heuristic 1, we need to sample $\Omega(\log^{1/2} N_1)$ $b$ of the form $b_1^2 + b_2^2$ to find an $a$ which is also a sum of two squares, or $\Omega(\log N_1)$ if we also want $a$ prime. The same also holds for the other decomposition, only the constant in the $\Omega$ changes.

We first look at the complexity analysis of the second case. Testing the primality of $a$ via the Miller-Rabin pseudo-primality test [Mil76,Rab80] costs $O(\log^2 a)$, and we have the same average complexity to find an integer $z$ such that $z^2 = -1 \pmod a$ (this is more or less equivalent to the Miller-Rabin pseudo-primality test). From $z$ and $a$, a continued fraction expansion allows one to decompose $a$ as a sum of two squares, so given $z$, the decomposition $a = a_1^2 + a_2^2$ can be done in time $O(\log^2 a)$ by the Euclidean algorithm (it is well known that the complexity can be improved to $\widetilde{O}(\log a)$, see e.g., [BCG+17, Sect. 6.3]) for a total complexity of $O(\log^2 a)$ on average to test the primality of $a$ and write it as a sum of two squares.

For the first case, we need to factor $a$ to see if it can be written as a sum of two squares. Given the prime factors of $a$, we can use the method above to find the decomposition of $a$ into irreducible factors in the Gaussian integers $\mathbb{Z}[i]$, so we can also decompose $a$ as a sum of two squares in time $O(\log^2 a)$. □

**Proposition 2.** *Assume Heuristic 1 is true. The precomputation step of the dimension 4 attack takes average time $O(\log^3 N_A)$ binary operations to find a decomposition $N_A^2 = (b_1^2 + b_2^2)N_B + a_1^2 + a_2^2$. Once this decomposition is found, the dimension 4 attack can be done in $\widetilde{O}(\log N_A \ell_A^4)$ arithmetic operations.*

*Proof.* By Heuristic 1, we can find $e \mid N_A$ such that $eN_A = (b_1^2+b_2^2)N_B+(a_1^2+a_2^2)$ with $b_1, b_2$ coprime. This precomputation costs $\widetilde{O}(\log^3 N_A)$ by Proposition 1. We can now construct an $eN_A$-endomorphism $F : X \to X$ where $X = E_0^2 \times E_B^2$ as in Theorem 3. We only know its action on $X[N_A]$, but by considering $\widetilde{F}$, we can explicitly decompose $F$ as $F = F_2 \circ F_1$ where $F_1$ is a $N_A$-isogeny and $F_2$ an $e$-isogeny, see Sect. 6.4. This decomposition costs $\widetilde{O}(\log N_A + \log e\ell_A^4)$ to compute (more precisely: to recover the domain of $F_2$ and its kernel), and evaluating $F$ via this decomposition costs $\widetilde{O}(\log N_A \ell_A^4)$. □

## 5 Dimension 2 Attack

We briefly describe how the dimension 2 attacks, due to [CD22,MM22], fit into the general framework of Sect. 3.

Write $N_A = bN_B + a$. To apply Sect. 3 for $g = 1$, we need to construct an $a$-isogeny $\alpha = \alpha_0 : E_0 \to X_0$ and a $b$-isogeny $\beta : E_0 \to Y_0$ (or $\beta : E_B \to Y_B$) to get the push-out square:

$$\begin{tikzcd} Y_0 \arrow[d, "\alpha_0'"] & E_0 \arrow[l, "\beta"'] \arrow[r, "\phi_B"] \arrow[d, "\alpha_0"] & E_B \arrow[d, "\alpha_B"] \\ Z_0 & X_0 \arrow[l, "\beta'"] \arrow[r, "\phi_B'"] & X_B \end{tikzcd}$$

The corresponding isogeny diamond

$$\begin{tikzcd} Z_0 \arrow[r, "\widetilde{\alpha_0'}"] \arrow[d, "\phi_B' \circ \widetilde{\beta'}"] & Y_0 \arrow[d, "\phi_B \circ \widetilde{\beta}"] \\ X_B \arrow[r, "\widetilde{\alpha_B}"] & E_B \end{tikzcd}$$

shows that $F = \begin{pmatrix} \widetilde{\alpha_0'} & \beta \circ \widetilde{\phi_B} \\ -\phi_B' \circ \widetilde{\beta'} & \alpha_B \end{pmatrix}$ is a $N_A$-isogeny $F : Z_0 \times E_B \to Y_0 \times X_B$ by Lemma 6.

If we don't assume that $\mathrm{End}(E_0)$ is known, we can only construct an $a$-endomorphism whenever $a$ is a square: if $a = a_1^2$ we take the $a$-endomorphism $[a_1]$. More generally, since it is also easy to construct isogenies of smooth degree starting from $E_0$ or $E_B$ (see Sect. 6.2), the framework of Sect. 3 shows that the attack applies whenever $N_A = b_1^2 e N_B + a_1^2 f$ where $e, f$ are sufficiently smooth. This is essentially the attack of [MM22]; in the first version they only looked at $N_A - N_B$ smooth (and tweaking of parameters), but to get a subexponential complexity they needed to look at the more general $N_A = eN_B + f$ case, which was already considered in [CD22] (squares are of negligible density compared to smooth numbers, so we can forget about them).

As mentioned in Sect. 1.5, in [CD22] the authors use the matrix $F$ as an oracle attack, which requires many isogeny guesses compared to the direct isogeny recovery of [MM22]. However, they also use the fact that for the parameters of SIKE submitted to NIST (or the Microsoft challenge [Cos21]), $E_0$ has a known endomorphism $\gamma = 2i$, so $\mathrm{End}(E_0) \supset \mathbb{Z}[2i]$. Hence we can construct an explicit $a$-endomorphism $\alpha$ on $E_0$ whenever $a = a_1^2 + 4a_2^2$, which is possible whenever all primes $p$ such that $p \equiv 3 \mod 4$ or $p = 2$ are of even exponent in $a$ by Remark 9. By Sect. 3, prolonging by isogenies of smooth degrees if necessary, for this starting curve $E_0$ the attack holds whenever $N_A = (b_1^2 + 4b_2^2)eN_B + (a_1^2 + 4a_2^2)f$. Otherwise, one needs to do some guesses, as in Sect. 6. In [CD22], the authors only look at $N_A = N_B + (a_1^2 + 4a_2^2)f$, but in [POP+22], Oudompheng, inspired by an earlier version of this paper describing the dimension 4 attack,

implemented the more general formula above. This bumped down the time to solve the SIKEp217 challenge from 9 to 2 s and SIKEp964 instances from more than one hour to thirty seconds.

The discussion of Sect. 4.1 shows:

**Proposition 3.** *Assume Heuristic 1 is true and assume that $E_0$ has known endomorphism $\gamma = 2i$. The dimension 2 attack has, after a precomputation step involving $O(\sqrt{\log N_A})$ factorisations and $O(1)$ calls to $\gamma$, complexity $\widetilde{O}(\log^{1.5} N_A \ell_A^2)$ arithmetic operations.*

*Alternatively, we can dispense with factorisations in the precomputation step at the cost of increasing the complexity of the attack: still under Heuristic 1, after a precomputation step costing $O(\log^3 N_A)$ binary operations and $O(1)$ calls to $\gamma$, the dimension 2 attack has complexity $\widetilde{O}(\log^2 N_A \ell_A^2)$ arithmetic operations.*

*Proof.* We proceed as in the proof of Proposition 2. In Proposition 1, we require $a$, $b$ to decompose as $a = a_1^2 + 4a_2^2$ and $b = b_1^2 + 4b_2^2$. To find such $a$ and $b$, we look for relations $N_A = bN_B/D + a$ where $D$ is a divisor of $N_B$. When we look for $a$ a sum of two squares in Proposition 1, we can take $D = \Theta(\sqrt{\log N_A})$, if we require furthermore that $a$ is prime to decrease the precomputation cost, then we need $D = \Theta(\log N_A))$. We assume implicitly that it is possible to find a divisor $D$ of $N_B$ of this magnitude, this will be the case if $N_B$ is sufficiently smooth.

Also, since the endomorphisms $\alpha$ and $\beta$ are built from $\gamma$, the evaluation cost of these endomorphisms will depend on the cost of evaluating $\gamma$. But we only need to evaluate $\alpha, \beta$ on $N_A$-torsion points, so we may consider the computation of $\gamma$ on a basis of $E_0[N_A]$ to be a precomputation (depending on $E_0$). Evaluating $\alpha$ and $\beta$ then takes $\widetilde{O}(\log N_A \ell_A^{1/2})$ by Lemma 4. When $E_0 = E_{\text{NIST}}$, the evaluation of $\gamma$ is done in $O(1)$, so evaluating $\alpha$ and $\beta$ can be done directly in $O(\log N_A)$.

Once these precomputations are done, the evaluation of $F$ takes time $\widetilde{O}(\log N_A \ell_A^2)$ arithmetic operations. We need to multiply this complexity by $O(D)$, the number of isogenies we need to guess. □

When $E_0 \neq E_{\text{NIST}}$ has known endomorphisms, Castryck and Decru use [KLP+14, LB20] to build a path from $E_{\text{NIST}}$ to $E_0$. This allows them to pushforward the $a$-isogeny $\alpha_{\text{NIST}}$ from $E_{\text{NIST}}$ to an $a$-isogeny $\alpha$ on $E_0$ using the methods of [GPS17, GPS20, DKL+20]. This time, evaluating $\alpha$ on rational points can only be done in polynomial time. But since the attack only needs the action of $\alpha$ on the $N_A$-torsion, it is sufficient to evaluate $\alpha$ on a basis of $E_0[N_A]$. This can be seen as a precomputation, which in this case involves not only the parameters $N_A, N_B$ but also the starting curve $E_0$. The remaining evaluations on points of $N_A$-torsion can then be done in $\widetilde{O}(\log N_A \ell_A^{1/2})$ by Lemma 4.

Recall also from Sect. 1.5 that [Wes22b] gives a method to construct an $a$-isogeny in proven polynomial time on any supersingular elliptic curve with known endomorphism ring. This isogeny can also be evaluated in polynomial time. Applying this to $a = N_A - N_B$, computing this $a$-endomorphism $\alpha$ and its evaluation on a basis $E_0[N_A]$ can be seen as a precomputation, and then we have a direct isogeny recovery without parameter tweaks as in Sect. 2, except we only need to compute isogenies in dimension 2 rather than 8.

**Proposition 4 (Wesolowski).** *If* $\mathrm{End}(E_0)$ *is known, after a polynomial time precomputation to compute an a-isogeny* $\alpha$ *and its action on the* $N_A$*-torsion, the dimension* 2 *attack has complexity* $\tilde{O}(\log N_A \ell_A^2)$ *arithmetic operations.*

Unfortunately, it is not clear what is the exact bound on the precomputation step of Wesolowski's approach.

Finally, we mention that for the isogeny computations in dimension 2, since any principally polarised surface is either a Jacobian or a product of two elliptic curves, one can also use the Jacobian model of [CE14] (which can be extended to the case of product of elliptic curves), rather than the theta model of [LR23].

## 6 Parameter Tweaks

We recall the decomposition of the parameters we need for the different attacks from the generic framework of Sect. 3:

- In dimension 8, or in dimension 2 when $\mathrm{End}(E_0)$ has known endomorphism ring (using [Wes22b]), no tweaks!
- In dimension 4, we need a decomposition $N_A = e(b_1^2 + b_2^2)N_B + f(a_1^2 + a_2^2)$, $e, f$ sufficiently smooth. For the dimension 2 attack of [CD22] where $\mathrm{End}(E_0)$ has endomorphism $2i$, we need the very similar decomposition $N_A = (b_1^2 + 4b_2^2)eN_B + (a_1^2 + 4a_2^2)f$.
- For [MM22], in dimension 2 when $\mathrm{End}(E_0)$ is not known, we need $N_A = eN_B + f$ with $e, f$ sufficiently smooth.

These decompositions rely on the fact that we can build isogenies of smooth degree on $E_0$ and $E_B$; we detail that complexity in Sect. 6.2.

We can furthermore tweak the parameters $N_A$ and $N_B$ as follows, as in the strategies of [CD22, MM22]. In the following, we assume that we are in the context of SIDH, so $E_0, E_B$ are supersingular elliptic curves defined over $\mathbb{F}_q$ with $q = p^2$.

1. We can replace $N_A$ by $N'_A = N_A/d_A$ where $d_A$ any divisor of $N_A$.
2. We can replace $N_B$ by $N_B/d_B$, where $d_B$ is a small divisor of $N_B$. This requires guessing the first $d_B$-isogeny step of $\phi_B$, and we have $O(d_B)$ guesses.
3. We can replace $N_A$ by $N'_A = eN_A$ where $e$ is a small integer prime to $N_B$. This means that we will construct $F$ a $(N'_A = eN_A)$-isogeny in dimension $2g$, but we only know its action on the $N_A$-torsion. To evaluate $F$ (e.g., to recover its kernel), we need to know its action on the $N'_A$-torsion. For a general $e$, we explain possible strategies in Sect. 6.3, strategies which can be much improved when $e \mid N_A$, see Sect. 6.4.

The rest of this section is devoted to determining the complexity of these tweaks.

### 6.1 Constructing a Basis of the $e$-torsion of $E$

We look at the complexity of building a basis of the $e$-torsion on $E$.

**Lemma 7.** *Let $E/\mathbb{F}_q$ be a supersingular elliptic curve, and $k$ the degree of the smallest extension where $E[\ell] \subset E(\mathbb{F}_{q^k})$. We can find a basis of the $e$-torsion in randomized time $\widetilde{O}(k^2 \log^2 q) = O(e^2 \log^2 q)$ operations.*

*Proof.* By the group structure theorem of supersingular elliptic curves, since $\pi_{q^k} = (-p)^k$ where $\pi_{q^k}$ is the Frobenius of $E/\mathbb{F}_{q^k}$, $E(\mathbb{F}_{q^k}) \simeq \mathbb{Z}/((-p)^k - 1) \oplus \mathbb{Z}/((-p)^k - 1)$. Hence the smallest extension of $\mathbb{F}_q$ where the $e$-torsion points of $E$ live is of degree $k$, the order of $-p$ modulo $e$, so $k = O(e)$. Sampling a basis of the $e$-torsion of $E$ can be done by constructing the field $\mathbb{F}_{q^k}$, sampling random points in $E(\mathbb{F}_{q^k})$, multiplying by the cofactor $\frac{(-p)^k-1}{e}$ and then checking if we have a basis using the Weil pairing. The construction of $\mathbb{F}_{q^k}$ costs $\widetilde{O}(k^2 \log q + k \log^2 q)$ using [Sho94] or $\widetilde{O}(k \log^5 q)$ using [CL13]. The dominant cost will be the sampling phase, which costs $O(k \log q)$ arithmetic operations in $\mathbb{F}_{q^k}$. In total we get $\widetilde{O}(k^2 \log^2 q) = O(e^2 \log^2 q)$ operations. □

### 6.2 Building a Smooth Isogeny on a Supersingular Elliptic Curve $E/\mathbb{F}_{p^2}$

We want to build a smooth isogeny of degree $e$. We can build it as a composition of $O(\log e)$ $\ell$-isogenies, for primes $\ell \mid e$. If $\ell \mid N_A N_B$, since we have access to a rational $N_A$ and $N_B$ torsion basis, we can simply use it to sample an element of order $\ell$ in time $O(\min(\log N_A, \log N_B))$ arithmetic operations, and the isogeny can then be computed in time $\widetilde{O}(\sqrt{\ell})$ arithmetic operations using `sqrtVelu` [BDL+20].

We now detail the general case.

**Lemma 8.** *Let $E/\mathbb{F}_q$ be a supersingular elliptic curve. We can recover the kernel of a $\ell$-isogeny with domain $E$ in $\widetilde{O}(\ell^2 \log q + \ell \log^2 q)$ arithmetic operations.*

*Proof.* Since $\pi_q = [-p]$, all cyclic kernels of order $\ell$ of $E$ are rational, and their generators live in an extension of degree at most $k = O(\ell)$, the order of $-p$ modulo $\ell$. We can construct $\mathbb{F}_{q^k}$ then sample a generator (any non zero point $P$ of $\ell$-torsion) in $O(k^2 \log^2 q)$ operations as in Sect. 6.1, then compute the isogeny using Vélu's formula [Vél71] or the `sqrtVelu` algorithm [BDL+20] in time $O(\ell k \log q)$ (resp., $\widetilde{O}(\ell^{1/2} k \log q)$) for a total cost of $\widetilde{O}(k^2 \log^2 q + \ell^{1/2} k \log q) = \widetilde{O}(\ell^2 \log^2 q)$.

An alternative is to compute and factor the $\ell$-division polynomial $\psi_\ell$. It is of degree $O(\ell^2)$ and can be computed in time $\widetilde{O}(\ell^2 \log q)$ via the recurrence formula. Furthermore, all points of $\ell$-torsion live in the same extension of degree $k$. If $\ell$ is odd and $P \in E[\ell]$, $x_P$ will live in the same extension as $P$ unless $k$ is even, in which case $\pi_q^{k/2} P = -P$ so $x_P$ lives in an extension of degree $k/2$. This shows that the factors of $\psi_\ell$ are all of the same degree $k$ if $k$ is odd or $k/2$ if $k$ is even. We can then skip the distinct degree factorisation phase, hence

compute a factorisation of $\psi_\ell$ in time $\widetilde{O}(\ell^2 \log^2 q)$ by [VS92]. Any factor $Q$ of $\psi_f$ then gives us a construction of $\mathbb{F}_{q^k}$ and of a point of $\ell$-torsion $P$ in $E(\mathbb{F}_{q^k})$ via, if $E : y^2 = h(x)$, $P = \big(x \mod Q(x), y \mod (y^2 - h(x), Q(x))\big)$. Note that the polynomial $y^2 - h(x)$ splits in $\mathbb{F}_q[x]/Q(x)$ if $\deg Q = k$, otherwise it is irreducible, $\deg Q = k/2$ and it allows one to construct $\mathbb{F}_{q^k}$ as a degree 2 tower over $\mathbb{F}_{q^{k/2}} = \mathbb{F}_q[x]/Q(x)$. We can then apply Vélu or `sqrtVelu` to $P$ as above, for a total cost of $\widetilde{O}(\ell^2 \log^2 q)$.

A third method is to construct a $\ell$-isogeny using the $\ell$-modular polynomial $\phi_\ell$ (and its derivative), as in the SEA algorithm [Sch95]. We can evaluate this modular polynomial in time $\widetilde{O}(\ell^2 \log q)$ by an easy adaptation of [Kie20] (see [Rob21, Remark 5.3.9]), then recover a root in time $\widetilde{O}(\ell \log^2 q)$. Recovering the isogeny can then be done in quasi-linear time by solving a differential equation [BMS+08, Rob21, Sect. 4.7.1]. This reduces the complexity to $\widetilde{O}(\ell^2 \log q + \ell \log^2 q)$ operations. □

### 6.3 Recovering a $N_A e$-isogeny from Its Action on the $N_A$-torsion

We have a $N_A e$-isogeny $F$ in dimension $2g$, that Eve built from the secret isogeny $\phi_B : E_0 \to E_B$ and some auxiliary isogeny she controls. She wants to recover $F$ in order to retrieve $\phi_B$ from it.

One way to do that is to guess the action of $\phi_B$ on the $eN_A$-torsion of $E_0$. This requires one to compute a basis of the $eN_A$-torsion on $E_0$, as described in Sect. 6.1, possibly taking an extension of degree $k$, and then guessing the images of $\phi_B$ on the $N_A e$-torsion. Note that since the $N_A$-torsion is rational by assumption, we have $k = O(e)$. Guessing the image of $\phi_B$ on this basis involves $O(e^3)$-tries, using the compatibility of $\phi_B$ with the Weil pairing and the known image of the $N_A$-torsion.

An alternative strategy, when the codomain $Y$ of $F : X \to Y$ is known, is as follows: since $F$ is a $(N'_A = eN_A)$-isogeny, and we know the action of $\phi_B$ on the $N_A$-torsion, we can still recover $\operatorname{Ker} F \cap X[N_A]$. So taking a maximal isotropic subgroup of $\operatorname{Ker} F \cap X[N_A]$ for the Weil pairing $e_{N_A}$ (for the $F$ we build in Sect. 3, this intersection is already maximal isotropic), we can thus recover $F_1$ in a decomposition $F = F_2 \circ F_1$, with $F_1$ a $N_A$-isogeny and $F_2$ an $e$-isogeny. Then we can try to bruteforce $F_2$ by an $e$-isogeny search in dimension $2g$.

### 6.4 Recovering a $N_A^2$-isogeny from Its Action on the $N_A$-torsion

When $F : X \to Y$ is a $N_A e$-isogeny with $e \mid N_A$, and the action of $F$ on $X[N_A]$ is known, then by using the dual $\widetilde{F}$ there is a much better strategy to recover $F$ than in Sect. 6.3. This is the same strategy used in [QKL+21] when $F$ is an endomorphism of elliptic curves.

**Lemma 9.** *Let $F : X \to Y$ be a $Ne$-isogeny between principally polarised abelian varieties in dimension $g$, whose kernel has rank $g$. Assume that we are given a basis of $X[N]$, $Y[N]$ over $\mathbb{F}_q$ along with the image of $F$ on this basis of*

*$X[N]$, and that $e \mid N$. Then we can decompose $F = F_2 \circ F_1$ with $F_1 : X \to X_1$ a $N$-isogeny and $F_2 : X_1 \to Y$ an $e$-isogeny. Furthermore, we can compute a basis of the kernels of $F$, $\widetilde{F}$, $F_1$ and $\widetilde{F_2}$ in $\widetilde{O}(\log N\ell_N^{1/2})$; and a basis of the kernel of $F_2$ in $\widetilde{O}(\log N\ell_N^{1/2})$ along with $2g$ evaluations of $\widetilde{F_2}$. Once the kernels of $F_1$ and $F_2$ are computed, we can evaluate $F$ on any point in $\widetilde{O}(\log N\ell_N^g)$ arithmetic operations.*

*Proof.* Since $K = \operatorname{Ker} F$ is of rank $g$, it admits a symplectic complement $K'$: $X[eN_A] = K \oplus K'$, and $\operatorname{Ker} \widetilde{F} = F(X[eN_A]) = F(K')$. Decompose $F = F_2 \circ F_1$, $F_1 : X \to X_1$, $F_2 : X_1 \to Y$, with $\operatorname{Ker} F_1 = \operatorname{Ker} F \cap X[N_A] = K[N_A]$. Then we have $\operatorname{Ker} \widetilde{F_2} = \operatorname{Im} F_2 \mid X_1[e] = \operatorname{Im} F \mid X[e] = \operatorname{Ker} \widetilde{F} \cap Y[e] = F(K')[e] = F(K'[e])$ (indeed $\operatorname{Im} F \mid X[e] \subset \operatorname{Im} F_2 \mid X_1[e]$ but they have the same cardinality $e^{2g}$ since the kernel is of rank $2g$, so we have equality). So we can build $F_1$ from $X$ through its kernel $\operatorname{Ker} F \cap X[N_A]$ (which is maximal isotropic of rank $2g$ in $X[N_A]$), build $\widetilde{F_2}$ from $Y$ through its kernel $\operatorname{Im} F \mid X[e]$, then compute $\operatorname{Ker} F_2 = \operatorname{Im} \widetilde{F_2} \mid Y[e]$ to recover $F_2$, hence $F = F_2 \circ F_1$. We can recover these kernels via DLPs as in Lemma 4. We also notice that evaluating $\widetilde{F_2}$ takes $\widetilde{O}(\log e\ell_e^g)$ arithmetic operations.

Once we have the kernels of $F_1$ and $F_2$ evaluate $F$ by an isogeny algorithm. □

*Example 1.* Note that the isogeny $F$ in dimension $2g$ constructed in Sect. 3 has its kernel of rank $2g$. In particular this strategy applies for the attacks in dimension 4 of Sect. 4 and in dimension 8 of Sect. 2.

Let us detail this case: in these examples, the endomorphism $F$ of $E_0^g \times E_B^g$ is always of the form $F = \begin{pmatrix} \alpha_0 & \widetilde{\beta}\widetilde{\phi_B} \\ -\phi_B\beta & \widetilde{\alpha_B} \end{pmatrix} \operatorname{Id}$ with $\alpha_0$ an $a$-endomorphism of $E_0^g$, $\beta$ a $b$-endomorphism of $E_0^g$, and $\alpha_B$ the $a$-endomorphism of $E_B^g$ making the diagram commute:

$$\begin{array}{ccc} E_0^g & \xrightarrow{\phi_B\beta} & E_B^g \\ \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_B} \\ E_0^g & \xrightarrow{\phi_B\beta} & E_B^g \end{array}$$

We also have $a, b, N_A$ coprime to each other. In particular, $\operatorname{Ker} F = \{(\widetilde{\alpha_0}(P), (\phi_B\beta)(P)) \mid P \in E_0^g[eN_A]\}$, and $\operatorname{Ker} \widetilde{F} = \{(\alpha_0(P), (-\phi_B\beta)(P)) \mid P \in E_0^g[eN_A]\}$ are of rank $g$. We decompose $F = F_2 \circ F_1$, where $\operatorname{Ker} F_1 = \operatorname{Ker} F[N_A] = \{(\widetilde{\alpha_0}(P), (\phi_B\beta)(P)) \mid P \in E_0^g[N_A]\}$, and $\operatorname{Ker} \widetilde{F_2} = \operatorname{Ker} \widetilde{F}[e] = \{(\alpha_0(P), (-\phi_B\beta)(P)) \mid P \in E_0^g[e]\}$. Since we know the image of $\phi_B$ on a basis of $E_0[N_A]$, we know the image of $\phi_B$ on a basis of $E_0[e]$ via $O(\log(N_A/e))$ arithmetic operations. So we can recover the image of $\phi_B\beta$ on this basis in $\widetilde{O}(\log N_A\ell_A^{1/2})$ and $O(1)$ evaluations of $\beta$ by Lemma 4. We also need $O(1)$ calls to $\alpha_0$.

In these examples, the endomorphisms $\beta$ and $\alpha_0$ can be evaluated in time $O(\log N_A)$, so the kernel of $F_1$ and of $\widetilde{F_2}$ can be computed in time $\widetilde{O}(\log N_A\ell_A^{1/2})$. A linear complement of $\operatorname{Ker} \widetilde{F_2}$ is given by $0 \times E_B^g[e]$. Indeed it is of rank $g$ and

cardinality $q^{2g}$, and if $x = (0, Q) \in \operatorname{Ker} \widetilde{F_2}$, then $Q = -\phi_B \beta(P)$ for a $P \in E_0^g[e]$ such that $\alpha_0 P = 0$. But this implies $aP = 0$, hence $P = 0$ since $a$ is prime to $e \mid N_A$, so $Q = 0$. So $\operatorname{Ker} F_2 = \widetilde{F_2}(0 \times E_B^g[e])$ can be recovered in $2g$ calls to the evaluation of the $e$-isogeny $\widetilde{F_2}$.

The total cost to recover the domain of $F_2$ and a basis of its kernel is thus $\widetilde{O}(\log N_A \ell_A^{1/2} + \log e \ell_e^{2g}) = \widetilde{O}(\log N_A \ell_A^{2g})$.

Unfortunately, this strategy does not work for the dimension 2 attack of Sect. 5, because (with the notations of this Section), $X_B$ is constructed as a pushout, and we only obtain it when we compute the codomain of $F$. But this means that if $F$ is a $N_A^2$-isogeny, there is no easy way to obtain $\operatorname{Ker} \widetilde{F}[N_A]$, hence split $F$ as a product of two $N_A$-isogenies, without first computing $F$ fully.

## 7 Open Problem

By Theorem 1 and Remark 1, we have a new toolbox for recovering an $N_B$-isogeny $f : A \to B$ given its action on the $N_A$-torsion as long as $N_A^2 \geq N_B$ and $N_A$ is sufficiently smooth. This tool allows one to break SIDH efficiently in all cases. Can it also be used to build new isogeny based cryptosystems?

## References


[BDL+20] Bernstein, D., De Feo, L., Leroux, A., Smith, B.: Faster computation of isogenies of large prime degree. In: Algorithmic Number Theory Symposium (ANTS XIV), vol. 4.1, pp. 39–55. Mathematical Sciences Publishers (2020). arXiv:2003.10118. https://msp.org/obs/2020/4/p04.xhtml

[BL04] Birkenhake, C., Lange, H.: Complex Abelian Varieties, vol. 302. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], pp. xii+635. Springer, Cham (2004). ISBN: 3-540-20488-1, https://doi.org/10.1007/978-3-662-06307-1

[BCR10] Bisson, G., Cosset, R., Robert, D.: AVIsogenies. Magma package devoted to the computation of isogenies between abelian varieties (2010). https://www.math.u-bordeaux.fr/damienrobert/avisogenies/. Free software (LGPLv2+), Registered to APP (Reference IDDN.FR.001.440011.000.R.P.2010.000.10000). Latest version 0.7, Released on 13 Mar 2021

[BMS+08] Bostan, A., Morain, F., Salvy, B., Schost, E.: Fast algorithms for computing isogenies between elliptic curves. Math. Comput. **77**(263), 1755–1778 (2008)

[BCG+17] Bostan, A., et al.: Algorithmes efficaces en calcul formel (2017). https://hal.inria.fr/hal-01431717/document

[CD22] Castryck, W., Decru, T.: An efficient key recovery attack on SIDH (preliminary version). Cryptology ePrint Archive, Paper 2022/975 (2022). https://eprint.iacr.org/2022/975

[CLM+18] Castryck, W., Lange, T., Martindale, C., Panny, L., Renes, J.: CSIDH: an efficient post-quantum commutative group action. In: Peyrin, T., Galbraith, S. (eds.) ASIACRYPT 2018. LNCS, vol. 11274, pp. 395–427. Springer, Cham (2018). https://doi.org/10.1007/978-3-030-03332-3_15

[Cos21] Costello, C.: The case for SIKE: a decade of the supersingular isogeny problem. In: Cryptology ePrint Archive (2021)

[CLN16] Costello, C., Longa, P., Naehrig, M.: Efficient algorithms for supersingular isogeny Diffie-Hellman. In: Robshaw, M., Katz, J. (eds.) CRYPTO 2016. LNCS, vol. 9814, pp. 572–601. Springer, Heidelberg (2016). https://doi.org/10.1007/978-3-662-53018-4_21, https://ecc2017.cs.ru.nl/slides/ecc2017-costello.pdf

[CE14] Couveignes, J.-M., Ezome, T.: Computing functions on Jacobians and their quotients. LMS J. Comput. Math. **18**(1), 555–577 (2014). arXiv:1409.0481

[CL13] Couveignes, J.-M., Lercier, R.: Fast construction of irreducible polynomials over finite fields. Israel J. Math. **194**(1), 77–105 (2013)

[DDF+21] De Feo, L., et al.: Séta: supersingular encryption from Torsion attacks. In: Tibouchi, M., Wang, H. (eds.) ASIACRYPT 2021. LNCS, vol. 13093, pp. 249–278. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-92068-5_9

[DJP14] De Feo, L., Jao, D., Plût, J.: Towards quantum-resistant cryptosystems from supersingular elliptic curve isogenies. J. Math. Cryptol. **8**(3), 209–247 (2014)

[DKL+20] De Feo, L., Kohel, D., Leroux, A., Petit, C., Wesolowski, B.: SQISign: compact post-quantum signatures from quaternions and isogenies. In: Moriai, S., Wang, H. (eds.) ASIACRYPT 2020. LNCS, vol. 12491, pp. 64–93. Springer, Cham (2020). https://doi.org/10.1007/978-3-030-64837-4_3

[EGM12] Edixhoven, B., van der Geer, G., Moonen, B.: Abelian Varieties. Book Project (2012). http://van-der-geer.nl/gerard/AV.pdf

[FKM+22] Fouotsa, T.B., Kutas, P., Merz, S.P., Ti, Y.B.: On the isogeny problem with torsion point information. In: Hanaoka, G., Shikata, J., Watanabe, Y. (eds.) Public-Key Cryptography, PKC 2022. LNCS, vol. 13177, pp. 142–161. Springer, Cham (2022). https://doi.org/10.1007/978-3-030-97121-2_6

[GPS17] Galbraith, S.D., Petit, C., Silva, J.: Identification protocols and signature schemes based on supersingular isogeny problems. In: Takagi, T., Peyrin, T. (eds.) ASIACRYPT 2017. LNCS, vol. 10624, pp. 3–33. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-70694-8_1

[GPS20] Galbraith, S.D., Petit, C., Silva, J.: Identification protocols and signature schemes based on supersingular isogeny problems. J. Cryptol. **33**(1), 130–175 (2020)

[JAC+17] Jao, D., et al.: SIKE: supersingular isogeny key encapsulation (2017). https://sike.org/

[JD11] Jao, D., De Feo, L.: Towards quantum-resistant cryptosystems from supersingular elliptic curve isogenies. In: Yang, B.-Y. (ed.) PQCrypto 2011. LNCS, vol. 7071, pp. 19–34. Springer, Heidelberg (2011). https://doi.org/10.1007/978-3-642-25405-5_2

[Kan97] Kani, E.: The number of curves of genus two with elliptic differentials. J. für die reine und angewandte Mathematik **485**, 93–122 (1997)

[Kan16] Kani, E.: The moduli spaces of Jacobians isomorphic to a product of two elliptic curves. Collectanea mathematica **67**(1), 21–54 (2016)

[Kie20] Kieffer, J.: Evaluating modular polynomials in genus 2 (2020). arXiv: 2010.10094 [math.NT]. HAL: hal-02971326

[KLP+14] Kohel, D., Lauter, K., Petit, C., Tignol, J.-P.: On the quaternion isogeny path problem. LMS J. Comput. Math. **17**(A), 418–432 (2014)

[LeV12] LeVeque, W.J.: Topics in Number Theory, vol. I and II. Courier Corporation, New York (2012)

[LB20] Love, J., Boneh, D.: Supersingular curves with small noninteger endomorphisms. In: Open Book Series (ANTS XIV), vol. 4(1), pp. 7–22 (2020). https://msp.org/obs/2020/4/p02.xhtml

[LR10] Lubicz, D., Robert, D.: Efficient pairing computation with theta functions. In: Hanrot, G., Morain, F., Thomé, E. (eds.) ANTS 2010. LNCS, vol. 6197, pp. 251–269. Springer, Heidelberg (2010). https://doi.org/10.1007/978-3-642-14518-6_21

[LR15] Lubicz, D., Robert, D.: A generalisation of Miller's algorithm and applications to pairing computations on abelian varieties. J. Symb. Comput. **67**, 68–92 (2015). https://doi.org/10.1016/j.jsc.2014.08.001

[LR23] Lubicz, D., Robert, D.:. Fast change of level and applications to isogenies. Res. Number Theory (ANTS XV Conf.) **9**(1), 7 (2023). https://doi.org/10.1007/s40993-022-00407-9

[MM22] Maino, L., Martindale, C.: An attack on SIDH with arbitrary starting curve. Cryptology ePrint Archive, Paper 2022/1026 (2022). https://eprint.iacr.org/2022/1026

[MMP+23] Maino, L., Martindale, C., Panny, L., Pope, G., Wesolowski, B.: A direct key recovery on SIDH. In: Eurocrypt. Springer, Cham (2023)

[Mil76] Miller, G.L.: Riemann's hypothesis and tests for primality. J. Comput. Syst. Sci. **13**(3), 300–317 (1976)

[Mum66] Mumford, D.: On the equations defining abelian varieties. I. Invent. Math. **1**, 287–354 (1966)

[Mum70] Mumford, D.: Abelian Varieties. Tata Institute of Fundamental Research Studies in Mathematics, vol. 5, pp. viii+242. Published for the Tata Institute of Fundamental Research, Bombay (1970)

[Oud22] Oudompheng, R.: A note on implementing direct isogeny determination in the Castryck-Decru SIKE attack, August 2022. http://www.normalesup.org/oudomphe/textes/202208-castryckdecru-shortcut.pdf

[Pet17] Petit, C.: Faster algorithms for isogeny problems using torsion point images. In: Takagi, T., Peyrin, T. (eds.) ASIACRYPT 2017. LNCS, vol. 10625, pp. 330–353. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-70697-9_12

[PH78] Pohlig, S., Hellman, M.: An improved algorithm for computing logarithms over GF(p) and its cryptographic significance (Corresp.). IEEE Trans. Inf. Theory **24**(1), 106–110 (1978)

[PT18] Pollack, P., Treviño, E.: Finding the four squares in Lagrange's theorem. Integers **18**, A15 (2018)

[POP+22] Pope, G., Oudompheng, R., Panny, L., et al.: Castryck-Decru Key Recovery Attack on SIDH, August 2022. https://github.com/jack4818/Castryck-Decru-SageMath

[QKL+21] de Quehen, V., et al.: Improved torsion-point attacks on SIDH variants. In: Malkin, T., Peikert, C. (eds.) CRYPTO 2021. LNCS, vol. 12827, pp. 432–470. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-84252-9_15

[Rab80] Rabin, M.O.: Probabilistic algorithm for testing primality. J. Number Theory **12**(1), 128–138 (1980)

[RS86] Rabin, M.O., Shallit, J.O.: Randomized algorithms in number theory. Commun. Pure Appl. Math. **39**(S1), S239–S256 (1986)

[Rob21] Robert, D.: Efficient algorithms for abelian varieties and their moduli spaces. HDR thesis. Université Bordeaux, June 2021. http://www.normalesup.org/robert/pro/publications/academic/hdr.pdf. Slides: 2021-06-HDR-Bordeaux.pdf (1h, Bordeaux)

[Rob22a] Robert, D.: Breaking SIDH in polynomial time, August 2022
[Rob22b] Robert, D.: Evaluating isogenies in polylogarithmic time, August 2022
[Sch95] Schoof, R.: Counting points on elliptic curves over finite fields. J. Théor. Nombres Bordeaux **7**(1), 219–254 (1995)
[Shi79] Shioda, T.: Supersingular K3 surfaces. In: Lønsted, K. (ed.) Algebraic Geometry. LNM, vol. 732, pp. 564–591. Springer, Heidelberg (1979). https://doi.org/10.1007/BFb0066664
[Sho94] Shoup, V.: Fast construction of irreducible polynomials over finite fields. J. Symb. Comput. **17**(5), 371–391 (1994)
[Sho09] Shoup, V.: A Computational Introduction to Number Theory and Algebra. Cambridge University Press, Cambridge (2009)
[Som21] Somoza, A.: thetAV. Sage package devoted to the computation with abelian varieties with theta functions, rewrite of the AVIsogenies magma package (2021). https://gitlab.inria.fr/roberdam/avisogenies/-/tree/sage
[Sut11] Sutherland, A.: Structure computation and discrete logarithms in finite abelian p-groups. Math. Comput. **80**(273), 477–500 (2011)
[Vél71] Vélu, J.: Isogénies entre courbes elliptiques. Compte Rendu Académie Sciences Paris Série A-B **273**, A238–A241 (1971)
[VS92] Von Zur Gathen, J., Shoup, V.: Computing Frobenius maps and factoring polynomials. Comput. Complexity **2**(3), 187–224 (1992)
[Wes22a] Wesolowski, B.: The supersingular isogeny path and endomorphism ring problems are equivalent. In: 2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS), pp. 1100–1111. IEEE (2022)
[Wes22b] Wesolowski, B.: Understanding and improving the Castryck-Decru attack on SIDH, August 2022. https://www.dropbox.com/s/pmv3lrsg1gayl13/attacksidh.pdf?dl=0
[Zar74] Zarhin, J.G.: A remark on endomorphisms of abelian varieties over function fields of finite characteristic. Math. USSR-Izvestiya **8**(3), 477 (1974)