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An Algorithm for Linear Convolution Based on Generalized Discrete Fourier Transform*

DAI Yin-yun, YI Hua†, YU Tao

(Department of Mathematics, Jinggangshan University, Ji'an, Jiangxi 343009)

Abstract: Linear convolutions can be converted to circular convolutions so that a fast transform with a convolution property can be used to implement the computation, which method is known as the FFT-based fast algorithm of linear convolution. In this paper, a novel proof of the computation of linear convolution based on Generalized Discrete Fourier Transform (GDFT) is constructed. Firstly, a relationship of linear convolution and circular convolution is thoroughly analyzed. Secondly, the computation of the linear convolution is translated to the multiplication of a special Toeplitz matrix and the signal. Lastly, this multiplication is accomplished by the inverse GDFT of the product of the GDFTs of the signal and the filter. Furthermore, a new method about the computation of complex linear convolution is constructed by using the GDFT with parameter -1, which is not considered in the previous literatures.

Keywords: generalized discrete Fourier transform; linear convolution; circular convolution; FFT

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1 Introduction

Linear convolution plays a fundamental role in scientific computation, such as the continuous wavelet transform[1], the multiplication algorithm for large integers, the multiplication of long polynomials[2].

Linear convolution can be implemented by using the definition of linear convolution, or the FFT-based method. In order to perform the classical FFT-based fast algorithm for linear convolution, the signal and filter must be appended with enough zeros so that their lengths satisfy the equivalent condition of linear convolution and circular

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†**Corresponding author:** H. Yi. E-mail address: 876145777@qq.com

convolution. In this paper, the relationship of linear convolution and circular convolution is analyzed. Then GDFT-based fast linear convolution method is constructed by using the properties of circular matrix and generalized discrete Fourier transform (GDFT). Some results about GDFT-based convolution have been reported[3,4]. However, the results of this paper is a generalization of the previous one. Additionally, a new method about the computation of complex linear convolution is constructed by using the GDFT with parameter -1, which is not considered in the previous literatures.

This paper is organized as follows. Section 2 gives preliminaries. Section 3 gives the construction of the computation of linear convolution based on GDFT.

2 Preliminaries and previous work

Definition 1 (Linear convolution)[5] The linear convolution of two discrete-time signals, $f=\{f[n]\}_{n=0}^{N_1-1}$ and $h=\{h[n]\}_{n=0}^{N_2-1}$ is defined as follows

$$y[n]=\sum_{p=\max\{0,n+1-N_2\}}^{\min\{n,N_1-1\}} f[p]h[n-p]:\overset{\text{def}}{=} f\star h[n],\quad n=0,1,2,\cdots,N_1+N_2-2. \tag{1}$$

Remark 1 The sum in (1) is over all the values of p which lead to legal subscripts for $f[p]$ and $h[n-p]$. Therefore, $\max\{0,n+1-N_2\}\le p\le \min\{n,N_1-1\}$.

Proposition 1[5] The linear convolution of f and h defined in (1) can be also written as

$$y=f\star h=Hf, \tag{2}$$

where $H\in R^{(N_1+N_2-1)\times N_1}$, and

$$H=\begin{bmatrix} h_0 & & & \\ h_1 & h_0 & & \\ \vdots & & \ddots & \\ h_{N_2-1} & & \ddots & \\ & \ddots & & h_0 \\ & & \ddots & h_1 \\ & & \ddots & \vdots \\ & & & h_{N_2-1} \end{bmatrix}. \tag{3}$$

Definition 2 (Convolution matrix)[5] The matrix in (3) is called convolution matrix generated by the vector h and f.

Definition 3 (Circulant matrix) An $N\times N$ circulant matrix C_H generated by a vector h takes the form [6,7],

$$C_H=\begin{bmatrix} h_0 & h_{N-1} & \cdots & h_2 & h_1 \\ h_1 & h_0 & h_{N-1} & & h_2 \\ \vdots & h_1 & h_0 & \ddots & \vdots \\ h_{N-2} & & \ddots & \ddots & h_{N-1} \\ h_{N-1} & h_{N-2} & \cdots & h_1 & h_0 \end{bmatrix}. \tag{4}$$

A circulant matrix is fully specified by its generator vector h, which appears as the first column of this matrix.

Definition 4 (Discrete Fourier transform)[5] The discrete Fourier transform for finite-length signal $f=\{f[k]\}_{k=0}^{N-1}$ is defined as follows

$$\mathcal{F}f=F_Nf, \tag{5}$$

where F_N, called DFT matrix is a $N\times N$ matrix, defined as

$$F_N(k,m)=e^{\frac{-i2\pi(k-1)(m-1)}{N}},\quad i=\sqrt{-1},\quad k,m=1,2,\cdots,N. \tag{6}$$

Then the inverse discrete Fourier transform is easily deduced as

$$\mathcal{F}^{-1}f=F_N^{-1}f. \tag{7}$$

Proposition 2[6,7] The circulant matrix C_H defined in (4) can be diagonalized by the DFT matrix F_N, namely

$$C_H=F_N^{-1}\mathrm{diag}\,(F_Nh)F_N, \tag{8}$$

where h is the first column of C_H, i.e., $h=\begin{bmatrix} h_0 & h_1 & \cdots & h_{N-1}\end{bmatrix}^{\mathrm{T}}$.

Definition 5 (Circular convolution)[5] The circular convolution of signal $\{f[k]\}_{k=0}^{N-1}$ with signal $\{h[k]\}_{k=0}^{N-1}$ is defined as a matrix vector multiplication as follows

$$y=f\circledast h=C_H\cdot f=\begin{bmatrix} h_0 & h_{N-1} & \cdots & h_2 & h_1 \\ h_1 & h_0 & h_{N-1} & & h_2 \\ \vdots & h_1 & h_0 & \ddots & \vdots \\ h_{N-2} & & \ddots & \ddots & h_{N-1} \\ h_{N-1} & h_{N-2} & \cdots & h_1 & h_0 \end{bmatrix}\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1}\end{bmatrix}, \tag{9}$$

where "$\circledast$" represents circular convolution.

Proposition 3 The computation of circular convolution can be implemented with FFT-based fast method. Specifically

$$y = f \circledast h = \mathcal{F}^{-1}\big((\mathcal{F}h).*(\mathcal{F}f)\big), \tag{10}$$

where ".*" means componentwise product of two vectors.

Proof The matrix involved in (9) is a circulant matrix. Thus, by using Proposition 2, we obtain

$$y = f \circledast h = C_H \cdot f = F_N^{-1}\mathrm{diag}(F_N h)F_N f = \mathcal{F}^{-1}\big((\mathcal{F}h).*(\mathcal{F}f)\big). \tag{11}$$

Proposition 4[8] Assume

$$f = [f_0, f_1, \cdots, f_{n-\beta-1}]^{\mathrm{T}}, \quad h = [h_0, h_1, \cdots, h_\beta]^{\mathrm{T}},$$

$$\tilde{f} = [f_0, f_1, \cdots, f_{n-\beta-1}, \underbrace{0, \overset{\beta}{\cdots}, 0}]^{\mathrm{T}}, \quad \tilde{h} = [h_0, h_1, \cdots, h_\beta, \underbrace{0, \overset{n-\beta-1}{\cdots}, 0}]^{\mathrm{T}},$$

then linear convolution $g = f \star h$ of f and h defined in (1) can be computed as circular convolution $\tilde{f} \circledast \tilde{h}$ of $\tilde{f}$ and $\tilde{h}$, namely,

$$g = f \star h = \tilde{f} \circledast \tilde{h} = \mathcal{F}^{-1}\big((\mathcal{F}\tilde{h}).*(\mathcal{F}\tilde{f})\big), \tag{12}$$

where ".*" means componentwise product of two vectors.

Definition 6 (Generalized discrete Fourier transform) The generalized discrete Fourier transform with parameter α for finite-length signal $\{f_n\}_{n=0}^{N-1}$ is defined as follows[3]

$$\mathcal{F}_\alpha f = \mathcal{F}(\mathcal{M}_\alpha f), \tag{13}$$

where $\mathcal{M}_\alpha f = \{f_n e^{\beta n}\}_{n=0}^{N-1}$, $\beta = \log(\alpha)/N$.

Remark 2 It is obvious that

$$\mathcal{F}_\alpha^{-1} = \mathcal{M}_\alpha^{-1}\mathcal{F}^{-1}. \tag{14}$$

Remark 3 If $\alpha = i = \sqrt{-1}$, then $\beta = \frac{\pi i}{2N}$. If $\alpha = -1 = e^{\pi i}$, then $\beta = \frac{\pi i}{N}$. The GDFT $\mathcal{F}_i$, $\mathcal{F}_{-1}$ with parameters i, -1, respectively will play an important role in the following part of this paper.

3 Algorithm implementation

In order to perform the classical FFT-based algorithm for linear convolution, both the two signals f and h need to be added zeros so that the two appended signals are of same lengths according to Proposition 4. In this section, by using GDFT-based algorithm to perform linear convolution, none of the two signals needs to be added zeros if the lengths of the two signals are equal. If the lengths of the two signals are not equal, for example, N_1, the length of signal f, is larger than N_2, the length of signal h, then only h is needed to append zeros so that the length of the appended signal is N_1. For this moment, we can assume that the lengths of two signals involved in linear convolution are equal, i.e., $N_1 = N_2$. In fact, the Proposition 5 ensures the rationality of this assumption.

Proposition 5 Assume $f = \{f_n\}_{n=0}^{N_1-1}$, $h = \{h_n\}_{n=0}^{N_2-1}$, and $N_1 \neq N_2$. Without loss of generality, assume that $N_1 > N_2$. Let

$$\tilde{h} = [h_0, h_1, \cdots, h_{N_2-1}, \underbrace{0, \cdots, 0}^{N_1-N_2}]^{\mathrm{T}},$$

then

$$f \star \tilde{h}[n] = \begin{cases} f \star h[n], & \text{if } \ 0 \leq n \leq N_1 + N_2 - 2, \\ 0, & \text{if } \ N_1 + N_2 - 1 \leq n \leq 2N_1 - 2. \end{cases} \tag{15}$$

Proof The convolution matrix $\tilde{H} \in R^{(2N_1-1)\times N_1}$ generated by $\tilde{h}$ and f is

$$\tilde{H} = \begin{pmatrix} H \\ 0 \end{pmatrix}, \tag{16}$$

where $H \in R^{(N_1+N_2-1)\times N_1}$ is the convolution matrix generated by h and f, and $0 \in R^{(N_1-N_2)\times N_1}$ is a zero matrix. From Proposition 1, we have

$$f \star \tilde{h} = \tilde{H} f = \begin{pmatrix} Hf \\ 0 \end{pmatrix} = \begin{pmatrix} f \star h \\ 0 \end{pmatrix}.$$

If the lengths of f and h are equal, i.e., $N_1 = N_2 = N$, then the convolution matrix generated by h and f is

$$H = \begin{bmatrix} h_0 & & & \\ h_1 & h_0 & & \\ \vdots & \ddots & \ddots & \\ h_{N-1} & h_{N-2} & & h_0 \\ & h_{N-1} & & h_1 \\ & & \ddots & \\ & & & h_{N-1} \end{bmatrix}. \tag{17}$$

If define

$$L_H = \begin{bmatrix} h_0 & & & \\ h_1 & h_0 & & \\ \vdots & \ddots & \ddots & \\ h_{N-1} & h_{N-2} & & h_0 \end{bmatrix}, \tag{18}$$

and

$$U_H = \begin{bmatrix} 0 & h_{N-1} & h_{N-2} & \cdots & h_2 & h_1 \\ 0 & 0 & h_{N-1} & \cdots & h_3 & h_2 \\ 0 & 0 & 0 & \ddots & & h_3 \\ 0 & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \ddots & h_{N-1} \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}, \tag{19}$$

then

$$\begin{pmatrix} H \\ 0 \end{pmatrix} = \begin{pmatrix} L_H \\ U_H \end{pmatrix}, \tag{20}$$

where $0\in R^{1\times N}$. By Proposition 1, it follows that

$$\begin{pmatrix} f \star h \\ 0 \end{pmatrix} = \begin{pmatrix} Hf \\ 0 \end{pmatrix} = \begin{pmatrix} H \\ 0 \end{pmatrix} f = \begin{pmatrix} L_H f \\ U_H f \end{pmatrix}. \tag{21}$$

Therefore, the computation of linear convolution is equivalent to the computation of $L_H f$ and $U_H f$.

Remark 4 Note that the circulant matrix $C_H = L_H + U_H$. Define $C_H^\alpha = L_H + \alpha U_H$, i.e.,

$$C_H^\alpha = \begin{bmatrix} h_0 & \alpha h_{N-1} & \cdots & \alpha h_2 & \alpha h_1 \\ h_1 & h_0 & \alpha h_{N-1} & & \alpha h_2 \\ \vdots & h_1 & h_0 & \ddots & \vdots \\ h_{N-2} & & \ddots & \ddots & \alpha h_{N-1} \\ h_{N-1} & h_{N-2} & \cdots & h_1 & h_0 \end{bmatrix}, \tag{22}$$

where α is an undetermined parameter. Then, C_H^α is a Toeplitz matrix. Let $\alpha = i$ with $i = \sqrt{-1}$, the imaginary unit, then for real signals f and h, $L_H f$, $U_H f$ are just the real and imaginary parts of $C_H^\alpha f$, respectively. By noting (21), we know that the computation of linear convolution $f * h$ is solved as long as $C_H^\alpha f$ can be computed effectively.

Lemma 1 Assume that $\{z_n\}_{n=0}^{n=N-1} = C_H^\alpha f$, then $z = \mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha h. * \mathcal{F}_\alpha f)$.

Proof Let $\beta = \frac{1}{N}\log(\alpha)$. Because $\{z_n\}_{n=0}^{n=N-1} = C_H^\alpha f$, therefore

$$\begin{aligned} z_n &= \sum_{m=0}^{n} f(m)h(n-m) + \sum_{m=n+1}^{N-1} f(m)h(N+n-m)\alpha \\ &= \sum_{m=0}^{n} f(m)h(n-m) + \sum_{m=n+1}^{N-1} f(m)h(N+n-m)e^{\beta N}, \\ z_n e^{\beta n} &= \sum_{m=0}^{n} f(m)e^{\beta m}h(n-m)e^{\beta(n-m)} \\ &\quad + \sum_{m=n+1}^{N-1} f(m)e^{\beta m}h(N+n-m)e^{\beta(N+n-m)}. \end{aligned}$$

From the definition of $\mathcal{M}_\alpha$, it is seen that

$$\mathcal{M}_\alpha z_n = \sum_{m=0}^{n} \mathcal{M}_\alpha f(m)\mathcal{M}_\alpha h(n-m) + \sum_{m=n+1}^{N-1} \mathcal{M}_\alpha f(m)\mathcal{M}_\alpha h(N+n-m),$$

therefore

$$\begin{bmatrix} \mathcal{M}_\alpha z_0 \\ \mathcal{M}_\alpha z_1 \\ \vdots \\ \mathcal{M}_\alpha z_{N-1} \end{bmatrix} = C \begin{bmatrix} \mathcal{M}_\alpha f_0 \\ \mathcal{M}_\alpha f_1 \\ \vdots \\ \mathcal{M}_\alpha f_{N-1} \end{bmatrix},$$

where

$$C = \begin{bmatrix} \mathcal{M}_\alpha h_0 & \mathcal{M}_\alpha h_{N-1} & \cdots & \mathcal{M}_\alpha h_2 & \mathcal{M}_\alpha h_1 \\ \mathcal{M}_\alpha h_1 & \mathcal{M}_\alpha h_0 & \mathcal{M}_\alpha h_{N-1} & & \mathcal{M}_\alpha h_2 \\ \vdots & \mathcal{M}_\alpha h_1 & \mathcal{M}_\alpha h_0 & \ddots & \vdots \\ \mathcal{M}_\alpha h_{N-2} & & \ddots & \ddots & \mathcal{M}_\alpha h_{N-1} \\ \mathcal{M}_\alpha h_{N-1} & \mathcal{M}_\alpha h_{N-2} & \cdots & \mathcal{M}_\alpha h_1 & \mathcal{M}_\alpha h_0 \end{bmatrix}.$$

From (9), it is seen that

$$\mathcal{M}_\alpha z = \mathcal{M}_\alpha f \circledast \mathcal{M}_\alpha h.$$

From Proposition 3, we have

$$\mathcal{M}_\alpha z = \mathcal{F}^{-1}(\mathcal{F}\mathcal{M}_\alpha h. * \mathcal{F}\mathcal{M}_\alpha f).$$

Thus, $\mathcal{F}\mathcal{M}_\alpha z = \mathcal{F}\mathcal{M}_\alpha h. * \mathcal{F}\mathcal{M}_\alpha f$. From the definition of GDFT, we have

$$\mathcal{F}_\alpha z = \mathcal{F}_\alpha h. * \mathcal{F}_\alpha f.$$

Thus $z = \mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha h. * \mathcal{F}_\alpha f)$.

Theorem 1[3] For real signals $\{f[k]\}_{k=0}^{N-1}$ and $\{h[k]\}_{k=0}^{N-1}$, assume $\alpha = i$, the imaginary unit, thus, the last element of $\mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha h. * \mathcal{F}_\alpha f)$ is a real number, and furthermore

$$\begin{pmatrix} f \star h \\ 0 \end{pmatrix} = \begin{pmatrix} \mathrm{real}\,(\mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha h. * \mathcal{F}_\alpha f)) \\ \mathrm{imag}\,(\mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha h. * \mathcal{F}_\alpha f)) \end{pmatrix}, \tag{23}$$

where "real, imag" mean the real and imaginary part of a complex number. Specifically, the linear convolution of f and h is obtained by concatenating the real parts and the first $N-1$ imaginary parts of $\mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha h. * \mathcal{F}_\alpha f)$.

Proof By Lemma 1 and (22), noting $\alpha = i$, we have

$$\mathrm{real}\left(\mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha h. * \mathcal{F}_\alpha f)\right) = \mathrm{real}\,(z) = L_H f,$$

$$\mathrm{imag}\left(\mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha h. * \mathcal{F}_\alpha f)\right) = \mathrm{imag}\,(z) = U_H f.$$

Then (23) is obtained by (21).

Theorem 1 has been given in [3] with an abstract proof based on the generalized poisson summation formula. The proof of this paper, based on the properties of matrix, is more elementary, specific and profound.

Theorem 2 For complex signals $\{f[k]\}_{k=0}^{N-1}$ and $\{h[k]\}_{k=0}^{N-1}$, let $z_{-1} = \mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha h. * \mathcal{F}_\alpha f)$ with $\alpha = -1$ and $z = \mathcal{F}^{-1}(\mathcal{F}h. * \mathcal{F}f)$, then, the first N samples of the linear convolution are obtained by $\frac{z_{-1}(n)+z(n)}{2}$ and the remaining samples by $\frac{z(n)-z_{-1}(n)}{2}$.

Proof From (18), (19) and (22), it follows that

$$\frac{C_H^1 f + C_H^{-1} f}{2} = L_H f, \quad \frac{C_H^1 f - C_H^{-1} f}{2} = U_H f.$$

By Lemma 1, we have

$$\frac{z_{-1}(n) + z(n)}{2} = L_H f, \quad \frac{z(n) - z_{-1}(n)}{2} = U_H f.$$

Then the proof is completed by (21).

References:

[1] Yi H, Chen Z Q, Cao Y H. High precision computation of Morlet wavelet transform for multi-period analysis of climate data[J]. Journal of Information and Computational Science, 2014, 11(17): 6369-6385

[2] Fürer M. Faster integer multiplication[J]. SIAM Journal on Computing, 2009, 39(3): 979-1005

[3] Martinez J, Heusdens R, Hendriks R C. A generalized poisson summation formula and its application to fast linear convolution[J]. Signal Processing Letters, IEEE, 2011, 18(9): 501-504

[4] Martinez J, Heusdens R, Hendriks R C. A generalized fourier domain: signal processing framework and applications[J]. Signal Processing, 2013, 93(5): 1259-1267

[5] Burrus C S S, Parks T W. DFT/FFT and Convolution Algorithms: Theory and Implementation[M]. New York: John Wiley & Sons, 1991

[6] Jin X Q. Preconditioning Techniques for Toeplitz Systems[M]. Beijing: Higher Education Press, 2010

[7] Nagy J G, Plemmons R J, Torgersen T C. Iterative image restoration using approximate inverse preconditioning[J]. IEEE Transactions on Image Processing, 1996, 5(7): 1151-1162

[8] Yi H, Xin S X, Yin J F. A class of algorithms for continuous wavelet transform based on the circulant matrix[J]. Algorithms, 2018, 11(3): 1-17

基于广义傅里叶变换的线性卷积算法

戴银云, 易 华, 余 涛

(井冈山大学数理学院，江西 吉安 343009)

摘 要: 线性卷积可以转化为循环卷积，循环卷积可以转化为频域的乘法，从而线性卷积可以采用基于 FFT (快速 Fourier 变换) 的方法进行计算．本文给出了一种基于广义离散 Fourier 变换的线性卷积计算方法．本文首先分析了线性卷积和循环卷积的关系．然后，线性卷积的计算转化成一个特殊的 Toeplitz 矩阵与向量的乘积．然后，通过利用信号和滤波器的广义离散 Fourier 变换以及反变换，推导了这个乘积的快速算法．另外，本文推导方法还可以得到基于参数为 -1 的广义离散 Fourier 变换计算线性卷积的方法．

关键词: 广义离散傅里叶变换；线性卷积；循环卷积；FFT