Big O notation:-
$f(n) = O(g(n))$ means there are positive constants c and n_0, such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0>0$

Ω notation:-
$f(n) = \Omega(g(n))$ means there are positive constants c and n_0, such that $0 \leq cg(n) \leq f(n)$ for all $n \geq n_0>0$

ϴ notation:-
$f(n) = \Theta(g(n))$ means there are positive constants c_1 , c_2 and n_0, such that $0<=c_1g(n) \leq f(n) \leq c_2g(n)$ for all $n \geq n_0>0$.

1. $2^{n+1} = O(2^n)$
Solution:-
Let f(n)=2^(n+1) and $g(n)=2^n$
$f(n)= 2*2^n = 2*g(n)$
With c=2 and n_0=1, we have $2\times2^n \geq 2^{n+1}$ for all $n\geq2$

Therefore,$2^{n+1} = O(2^n)$

2.$(n+1)^5 = O(n^5)$
Solution:- Let $f(n)=(n+1)^5$ and $g(n)=n^5$

$$
\begin{aligned}
f(n) &=(n+1)^5 \\
&=(n^4+ 4n^3 + 6n^2 + 4n + 1)\ (n+1) \\
&=(n^5 + 5n^4 + 10n^3 + 10n^2 + 5n +1) \\
&<=(n^5 + 5n^5 + 10n^5 + 10n^5 + 5n^5 + n^5) \qquad \text{(for } n_0=1) \\
&<= 32n^5
\end{aligned}
$$

So, for c=32 and n_0=1, $(n+1)^5 <= cn^5$

3. Let $p(n)=a_kn^k + a_{k-1}n^{k-1} + ... a_0$ be an asymptotically non-negative polynomialof degree k, k > 0 is a constant, prove that **$p(n) \in O(n^k)$**
Note:- This is the proof why lower order terms and multiplicative constants don't matter
Solution:- $p(n)=a_kn^k + a_{k-1}n^{k-1} + ... a_0$

$$
\begin{aligned}
&= a_kn^k (1 + (a_{k-1}/a_k)*(1/n) + (a_{k-2}/a_k)*(1/n^2) + ... + (a_0/a_k)*(1/n^k)) \\
&<=a_kn^k (1 + (|a_{k-1}|/a_k)*(1/n) + (|a_{k-2}|/a_k)*(1/n^2) + ... + (|a_0|/a_k)*(1/n^k)) \\
&<=a_kn^k (1 + (|a_{k-1}|/a_k)+ (|a_{k-2}|/a_k)+ .... + (|a_0|/a_k))
\end{aligned}
$$

For n_0=1 and $c=a_k (1 + (|a_{k-1}|/a_k)+ (|a_{k-2}|/a_k)+ .... + (|a_0|/a_k))$,
$p(n)<=c*n^k$
$=>p(n) \in O(n^k)$

4. $n^3=O (n^3 \log n)$
Solution:- Let $f(n)=n^3$ and $g(n)= n^3 \log n$

for n>=2, we can say $1<= \log_2 n$
$=> n^3 <= n^3 \log_2 n$
$=> n^3 <= c\ n^3 \log_2 n$, where c=1

So, for n_o=2 and c =1 , **$n^3=O (n^3 \log n)$**

5. max(f(n),g(n))=ϴ(f(n)+g(n))
Solution:-Let f(n) and g(n) be asymptotically nonnegative functions
By **ϴ** definition, we need to proof following inequations
(1) c_1 (f(n) + g(n)) <= max(f(n), g(n))
(2) max(f(n), g(n)) <= c_2 (f(n)+g(n))

(1) holds because max(f(n), g(n)) >= f(n) and max(f(n), g(n)) >= g(n). Thus we get max(f(n), g(n)) >= (f(n) + g(n))/2, obviously, we can choose c_1 = 1/2.

(2) holds because max(f(n), g(n)) <= f(n) + g(n) and c_2 >=1

6. $\sum_{i=1}^{n} i^2 = O(n^3)$

Solution:-
Let f(n)=$\sum_{i=1}^{n} i^2$ and g(n) = n^3
f(n)=$\sum_{i=1}^{n} i^2$
=n(n+1)(2n+1)/6
=$(2n^3 + 3n^2 + n)/6$
Using proof 3, we can prove that f(n)= $O(n^3)$

7. $\sum_{i=1}^{n} (i / 2^i) = O(1)$

Solution:-
Let S=$\sum_{i=1}^{n} (i / 2^i)$
S = $(1/2 + 2/2^2 + 3/2^3 +$ .... + $n/2^n)$
S/2 = $(1/2^2 + 2/2^3 +$ .... + $(n-1)/2^n + n/2^{n+1})$
S-S/2 = $(1/2 + 1/2^2 + 1/2^3 +$ .... + $1/2^n) - n/2^{n+1}$
For gemetric series, if n-> infinity and r<1 , the formula for sum is a/(1-r), where a is the first term in the series

S/2 = $(1/2)/(1 – 1/2) – n/2^{n+1}$
S= $2(1- n/2^{n+1})$

Since n>0 , $n/2^{n+1} > 0$
=> S= 2 * (something less than 1)
=> S< 2
=> S= O(1)