Algorithms & Data Structures I
CSC 225

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Fall 2018

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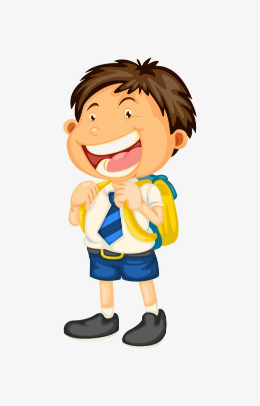
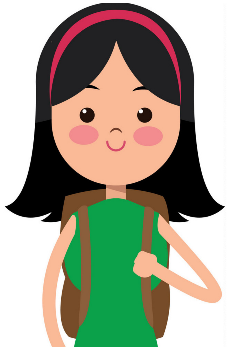
Induction

- Proof by induction is a very common technique to prove correctness of mathematical statements.

- **Proof by loop invariant** that we saw was actually based on induction.

Induction

- Say a teacher wants to hold an exam, and
- All n students are organized in a line

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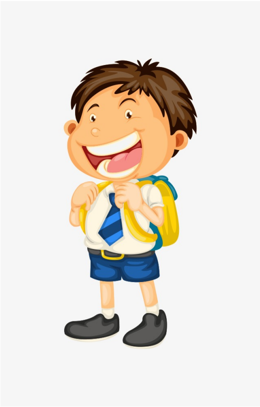
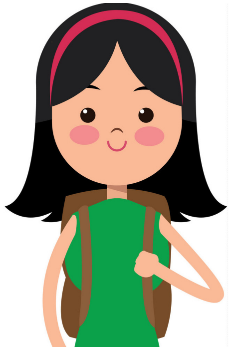

1 2 n-1 n

- Each student **promises** to take the exam only if **the person before them** takes it.

Induction

- How can we have all students to take the exam?

- Of course, we should somehow convince the first student to take the exam and then naturally everyone else will

- This is the idea behind proof by induction

Induction

- Let's say $S(n)$ is a statement about the natural number n.

- To show that $S(n)$ is always true we have to show that:

1. $S(1)$ is true; and
2. If $S(k)$ is true, then $S(k+1)$ is true.

Induction

- For example, prove that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Induction

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$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- The statement is true for $n = 1$, since

$$\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$$

- This is called the **base case**

Induction

- Now, assume that it is true for $n = k$ (this is called the **induction hypothesis**)

$$\sum_{i=1}^{k} i = 1 = \frac{k(k+1)}{2} \ (assumption)$$

Induction

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$$\sum_{i=1}^{k} i = 1 = \frac{k(k+1)}{2} \; (\mathit{assumption})$$

- We should show that when $n = k + 1$, we have

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2} \; (\mathit{to\ be\ proved})$$

Induction

$$\sum_{i=1}^{k+1} i = (k+1) + \sum_{i=1}^{k} i$$

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$$= (k+1) + \frac{k(k+1)}{2}$$

$$= (k+1)\left(1 + \frac{k}{2}\right) = \frac{(k+1)(k+2)}{2}$$

- This is what we get if we replace $n = k + 1$ in the formula. So, we're done!

Induction

- We also have **strong induction**

- Still the base case has to hold for strong induction

- However, in order to prove that some statement is true for $n = k + 1$, then we have to assume that the statement is true for all $n \leq k$ (not just $n = k$).

- So, our assumption is **stronger**