Algorithms & Data Structures I
CSC 225

Ali Mashreghi

Fall 2018

Department of Computer Science, University of Victoria

A summary

	Comparison-based	Stable	In-place	Time Complexity
SELECTION-SORT	✔	✔	✔	$O(n^2)$

Good because the idea is very simple and intuitive!

A summary

	Comparison-based	Stable	In-place	Time Complexity
SELECTION-SORT	✔	✔	✔	$O(n^2)$
INSERTION-SORT	✔	✔	✔	$O(n^2)$

Efficient for small inputs of around 20 elements

A summary

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SELECTION-SORT	✔	✔	✔	$O(n^2)$
INSERTION-SORT	✔	✔	✔	$O(n^2)$
MERGE-SORT	✔	✔	❌	$O(n \log n)$

Stable and good time complexity

A summary

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MERGE-SORT	✔	✔		$O(n \log n)$
HEAP-SORT	✔	❌	✔	$O(n \log n)$

Good time complexity and in-place

A summary

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MERGE-SORT	✔	✔	✘	$O(n \log n)$
HEAP-SORT	✔	✘	✔	$O(n \log n)$

Exercise: show with an example that heap-sort is not stable? You have to show that original order of equal numbers is not preserved necessarily.

Quicksort

- Quicksort is an in-place comparison-based **randomized** algorithm with **expected running time** of $O(n \log n)$.

- To be able to analyze a randomized algorithm we need to know about **mathematical expectation**.

- But let's first clarify what it means for an algorithm to be randomized.

Randomized vs. deterministic

- A deterministic algorithm always takes the **same steps** on the same input.

- However, a randomized algorithm might take **different steps** even on the same input.

Randomized vs. deterministic

- Imagine we have n doors and behind one of them there is prize.

- Give an algorithm to find the prize.

Randomized vs. deterministic

- **Deterministic Algorithm 1:**
  Check the doors from left to right

Randomized vs. deterministic

- **Deterministic Algorithm 1:**
  Check the doors from left to right

- What if the prize is always behind the last door?

Randomized vs. deterministic

- **Deterministic Algorithm 2:**
  Check the doors from right to left

- What if the prize is always behind the first door?

Randomized vs. deterministic

- **Randomized Algorithm:**

    Flip a coin first:

        if it’s head check from left to right

        if it’s tail check from right to left

Randomized vs. deterministic

- We can show that unlike the deterministic algorithms that check all n doors in the worst case, the randomized algorithm is **expected** to check $\frac{n}{2}$ doors.

- Of course in this example we are saving only a constant factor which is not significant asymptotically; however, good randomized algorithms work very well even asymptotically.

Probability

- Probability is a measure to show how likely is an outcome in an **experiment.**

- Any probabilistic statement is associated with a **probability space S.**

- Each subset of the probability space is called an **event.**

- Each member of S is called an **elementary event.**

Example

- For example, **rolling a dice** is an experiment.
- The probability space is $\{1, 2, 3, 4, 5, 6\}$.
- $\{2,3, 5\}$ is an event
  means the event of getting **2 or 3 or 5** as the outcome.
- {1}, {2}, {3}, …, {6} are all elementary events.

Probability

- A probability function $\boldsymbol{Pr}$ associates a **real number** to each subset of S.

- We have $\Pr: \mathcal{P}(S) \to \mathbb{R}$ such that:
  1. For any event A, $0 \leq \Pr(A) \leq 1$
  2. $\Pr(S) = 1$

- $\mathcal{P}(S)$ is called **power set** of S, which is the set of all subsets of S.

Example

- In rolling a dice if we assume that each elementary event occurs with probability of $\frac{1}{6}$, we have

- $\Pr(\{1\}) = \cdots = \Pr(\{6\}) = \frac{1}{6}$

- $\Pr(\{4,5\}) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

- $\Pr(\{1,2,\ldots,6\}) = \frac{1}{6} + \frac{1}{6} + \ \ldots + \frac{1}{6} = 1$

Discrete Random variable

- A **random variable X is a function** that maps each elementary event to a real value, i.e. $X: S \to \mathbb{R}$
- A discrete random variable takes **only discrete values**.

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Discrete Random variable

- A random variable X is a function that maps each elementary event to a real value, i.e. $X: S \rightarrow \mathbb{R}$
- A discrete random variable takes only discrete values.

- **Question:** how is this different from probability function $\Pr: \mathcal{P}(S) \rightarrow \mathbb{R}$?
- **Answer:** Prob. function says how likely an outcome is, but a random variable says what is the value if an outcome occurs.

Discrete Random variable

- Example of rolling a dice
- Say X is the random variable **showing the dice's outcome**.

- If you get a 3 $\rightarrow X = 3$
- If you get a 6 $\rightarrow X = 6$
- …

- However the probabilities of these events are the same.

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- We can also define events based on X:
- $\Pr(X = 3) = \frac{1}{6}$

Discrete Random variable

- Example of rolling a dice
- Say X is the random variable **showing the dice's outcome**.

- We can also define events based on X:
- $\Pr(X = 3) = \frac{1}{6}$
- $\Pr(X \leq 2.5) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$ (X has to be either 1 or 2)

Expected value

- A basic characteristic of a random variable is its **expectation**.

$$E[X] = \sum_i i \cdot \Pr(X = i)$$

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Expected value

- A basic characteristic of a random variable is its **expectation**.

$$E[X] = \sum_i i \cdot \Pr(X = i)$$

- If X is the outcome of a dice, then
- $E[X] = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \ \ldots + 6 \times \frac{1}{6} = 3.5$

Expected value

- Expected value doesn’t mean which outcome is most likely.

- Intuitively, it means if we keep repeating the experiment **many many times** what is the value that we are going to see on average.

Linearity of expectation

- An awesome property of the expected value is linearity
- For ***any*** collection of random variables $X_1, X_2, \ldots, X_n$, if we have:
- $Y = X_1 + X_2 + \cdots + X_n$, then
- $$\begin{aligned} E[Y] &= E[X_1 + X_2 + \cdots + X_n] \\ &= E[X_1] + E[X_2] + \cdots + E[X_n] \end{aligned}$$

Other properties

- Expected value of non-random quantity is the quantity itself, e.g. $E[-3] = -3$

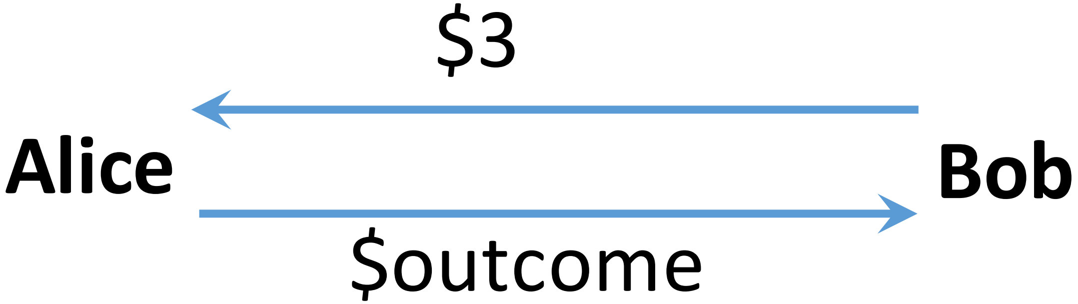
- $E[cX] = cE[X]$ for any constant c.

A simple game

- Say Alice and Bob are playing a game with a dice

- **Rules:**

1. Each round Bob has to pay \$3 to play the game
2. In each round Alice throws the dice and pays Bob as much as the outcome of the dice



A simple game

- Let X the random variable **representing Bob's profit**

- **Question:** How much profit can Bob expect to make in one round?

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- **Question:** How much profit can Bob expect to make in one round?
- **Answer:** Say Y is the dice's outcome.

$$E[X] = E[-3 + Y]$$
$$= E[-3] + E[Y] = -3 + 3.5 = \ \$0.5$$

A simple game

- Let X the random variable representing Bob's profit

- **Question:** How much profit can Bob expect to make in **ten rounds**?
- **Answer:** We can define $X_1, X_2, \ldots, X_{10}$ for each round the Bob plays. Because of the linearity of expectation, Bob's overall expected profit is \$5.

Let's change the game

- Say Alice and Bob are playing a game with a dice
- **Rules:**

Bob pays \$5 to play	Alice pays • \$2 on odd numbers • \$4 on a 2 • \$6 on a 4 • \$12 on a 6

Let's change the game

- If you are Bob you are **doomed!**

Let's change the game

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- Let X be Bob's profit, and Y be what Alice pays. In each round,
- $E[Y] = \frac{1}{2}2 + \frac{1}{6}4 + \frac{1}{6}6 + \frac{1}{6}12 = 4.66$

Let’s change the game

- If you are Bob you are **doomed!**

- Let X be Bob’s profit, and Y be what Alice pays. In each round,
- $E[Y] = \frac{1}{2}2 + \frac{1}{6}4 + \frac{1}{6}6 + \frac{1}{6}12 = 4.66$
- $E[X] = -5 + E[Y] = -0.34\$$

Expected value

- In algorithms usually the random variable is **the running time of the algorithm**.

- A nice thing about expected value is that if for some variable X, $E[X] = a$, then most of the time we can prove that with high probability X is close to a.

- If interested to know more about this, take a look at Chernoff bounds for example.

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- **Problem:** Let’s say we have an unbiased coin, how many times do we expect to flip it until we get the **first head**?

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- Let X be the random variable **representing the answer**.

- $\Pr(X = 1) = \frac{1}{2}$

One last problem

- **Problem:** Let's say we have an unbiased coin, how many times do we expect to flip it until we get the **first head**?
- Let X be the random variable **representing the answer**.
- $\Pr(X = 2) = \underbrace{\frac{1}{2}}_{\text{tail}} \times \underbrace{\frac{1}{2}}_{\text{head}} = \frac{1}{4}$

One last problem

- **Problem:** Let's say we have an unbiased coin, how many times do we expect to flip it until we get the **first head**?

- Let X be the random variable **representing the answer**.

- $\Pr(X = 3) = \underbrace{\frac{1}{2} \times \frac{1}{2}}_{\text{tail}} \times \underbrace{\frac{1}{2}}_{\text{head}} = \frac{1}{8}$

One last problem

- **Problem:** Let's say we have an unbiased coin, how many times do we expect to flip it until we get the **first head**?

- Let X be the random variable **representing the answer**.

- $$\Pr(X = k) = \underbrace{\frac{1}{2} \times \frac{1}{2} \times \cdots}_{\text{k-1 tails}} \times \underbrace{\frac{1}{2}}_{\text{head}} = \left(\frac{1}{2}\right)^{k-1} \times \frac{1}{2} = \frac{1}{2^k}$$

One last problem

- So,

$$E[X] = \sum_{k=1}^{\infty} k \Pr(X = k) = \sum_{k=1}^{\infty} \frac{k}{2^k}$$

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$$E[X] = \sum_{k=1}^{\infty} \frac{k}{2^k}$$

- We know that this sum is equal to 2 at infinity.
- Therefore, we expect 2 flips until we get the first head.

One last problem

- So,

$$E[X] = \sum_{k=1}^{\infty} \frac{k}{2^k}$$

- We know that this sum is equal to 2 at infinity.
- Therefore, we expect 2 flips until we get the first head.

- In general, if the success probability in a Bernoulli trial is p, we expect $\frac{1}{p}$ trials to see the first success.