CSC 226

Algorithms and Data Structures: II

MST

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ECS 466

Weighted Graphs

- A ***weighted graph*** is a graph model where we associate ***weights*** (or ***costs***) with each edge
- Minimum spanning trees
- Shortest Paths

Weighted edge API

Edge abstraction needed for weighted edges.

```
public class Edge implements Comparable<Edge>
```

	`Edge(int v, int w, double weight)`	*create a weighted edge v-w*
`int`	`either()`	*either endpoint*
`int`	`other(int v)`	*the endpoint that's not v*
`int`	`compareTo(Edge that)`	*compare this edge to that edge*
`double`	`weight()`	*the weight*
`String`	`toString()`	*string representation*

Idiom for processing an edge `e`: `int v = e.either(), w = e.other(v);`

Weighted edge: Java implementation

```
public class Edge implements Comparable<Edge>
{
   private final int v, w;
   private final double weight;

   public Edge(int v, int w, double weight)
   {
      this.v = v;
      this.w = w;
      this.weight = weight;
   }

   public int either()
   {  return v;  }

   public int other(int vertex)
   {
      if (vertex == v) return w;
      else return v;
   }

   public int compareTo(Edge that)
   {
      if      (this.weight < that.weight) return -1;
      else if (this.weight > that.weight) return +1;
      else                                return  0;
   }
}
```

constructor

either endpoint

other endpoint

compare edges by weight

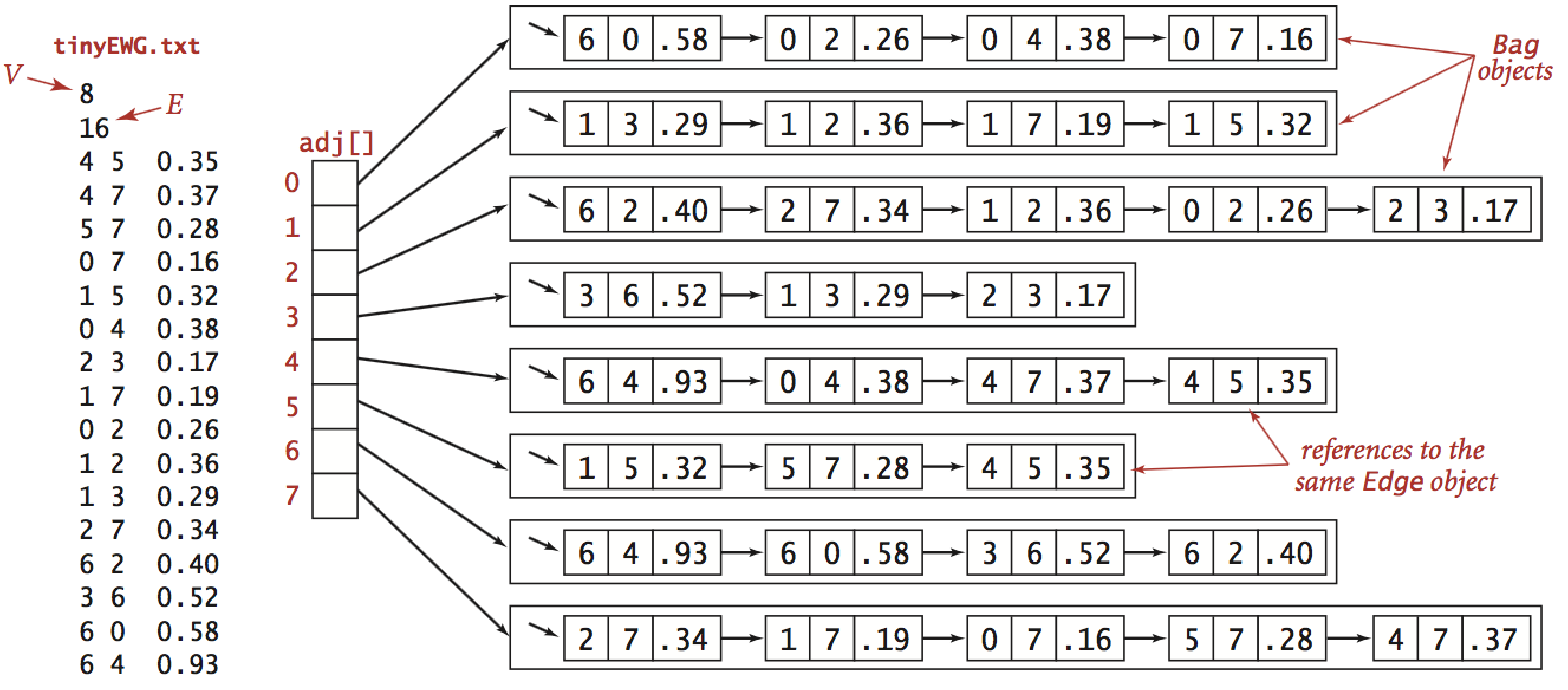
Edge-weighted graph API

public class EdgeWeightedGraph		
	EdgeWeightedGraph(int V)	*create an empty graph with V vertices*
	EdgeWeightedGraph(In in)	*create a graph from input stream*
void	addEdge(Edge e)	*add weighted edge e to this graph*
Iterable<Edge>	adj(int v)	*edges incident to v*
Iterable<Edge>	edges()	*all edges in this graph*
int	V()	*number of vertices*
int	E()	*number of edges*
String	toString()	*string representation*

Conventions. Allow self-loops and parallel edges.

Edge-weighted graph: adjacency-lists representation

Maintain vertex-indexed array of `Edge` lists.



Edge-weighted graph representation

Edge-weighted graph: adjacency-lists implementation

```
public class EdgeWeightedGraph
{
   private final int V;
   private final Bag<Edge>[] adj;

   public EdgeWeightedGraph(int V)
   {
      this.V = V;
      adj = (Bag<Edge>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<Edge>();
   }

   public void addEdge(Edge e)
   {
      int v = e.either(), w = e.other(v);
      adj[v].add(e);
      adj[w].add(e);
   }

   public Iterable<Edge> adj(int v)
   {  return adj[v];  }
}
```

same as `Graph`, but adjacency lists of `Edge`s instead of integers

constructor

add edge to both adjacency lists

Minimum Spanning Tree Definition

- *Input*: A weighted connected graph $G = (V, E)$ consisting of vertices (or nodes), V, and edges, E, with positive integer edge weights
- *Output*: A *minimum spanning tree* (*MST*) $T = (V, E_T)$, that is T is a connected subgraph of G ($E_T \subseteq E$) such that T is acyclic, and T is *lightest*

Definition

- Let $G = (V,E)$ be an undirected connected graph with edge weights that are *positive* integers
- $T = (V, E_T)$ is a minimum spanning tree for G if

  (1) T is a subgraph of G

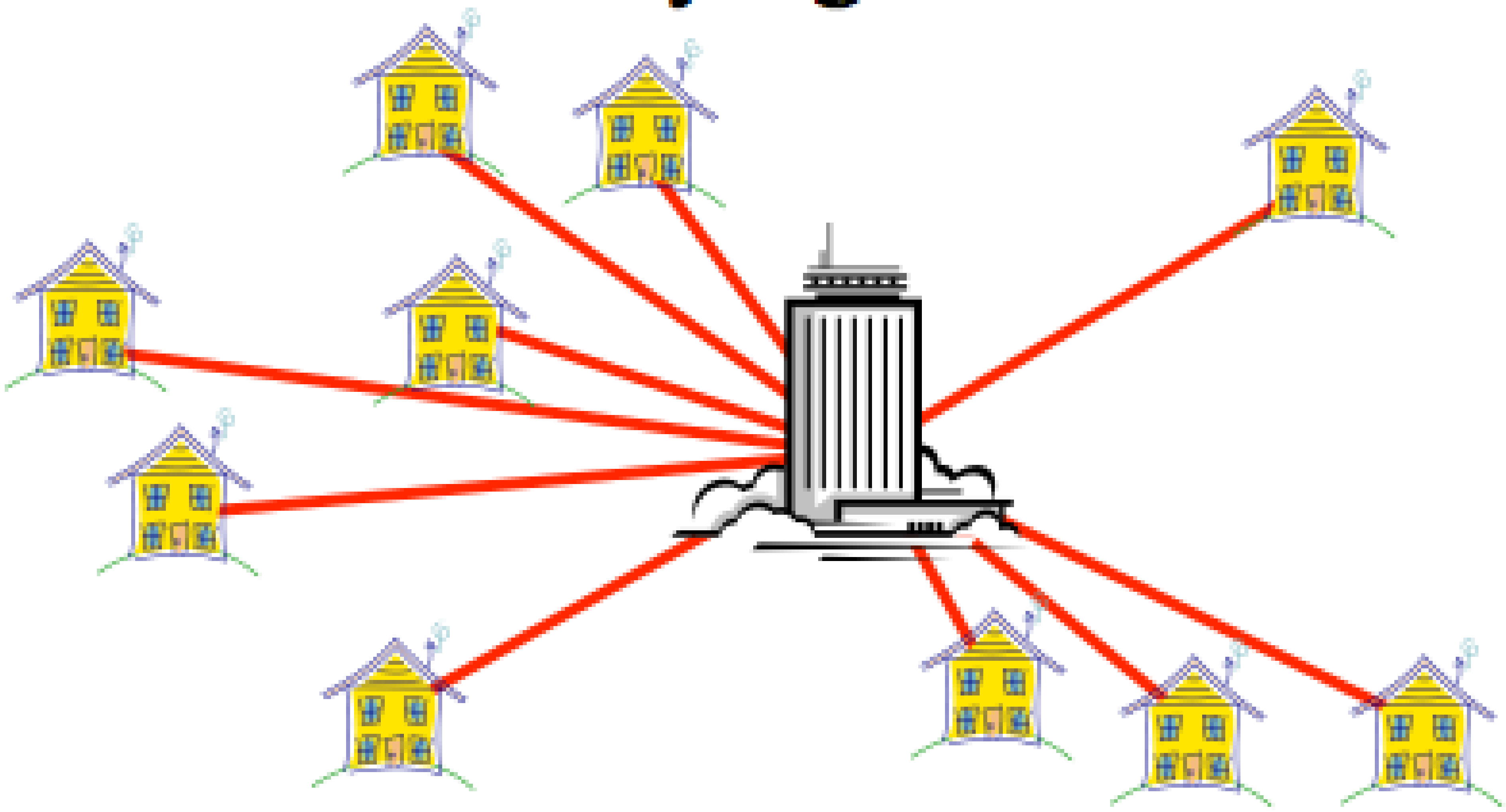
  (2) T is a tree

  (3) T is the *lightest* graph satisfying (1) and (2),

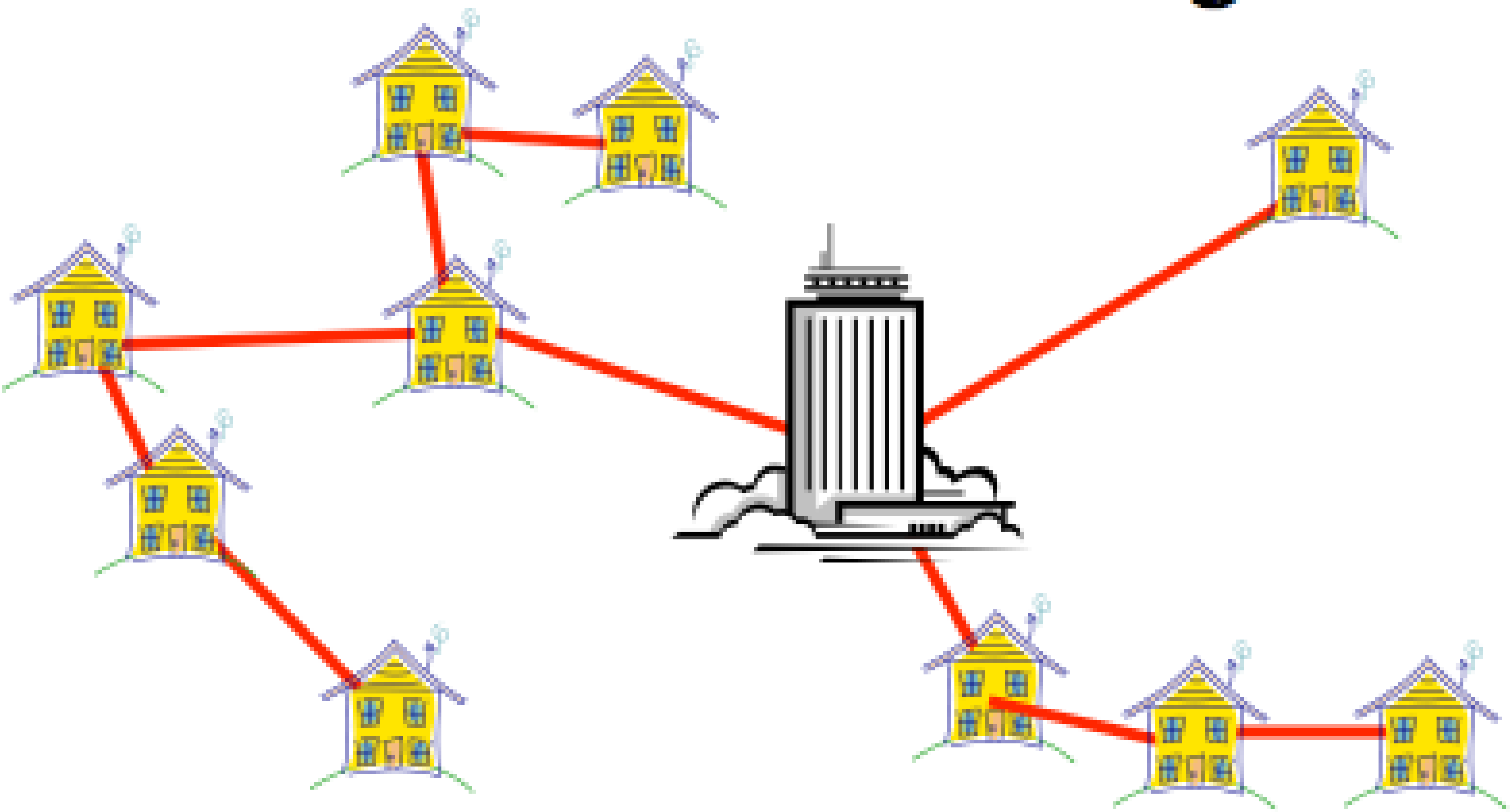
  i.e. , $$\sum_{e \in E_T} w(e) = \min_{\substack{T' \text{ is} \\ \text{spanning} \\ \text{tree for } G}} \sum_{e \in E_{T'}} w(e)$$

spanning tree

Problem: Laying Cable TV Wire

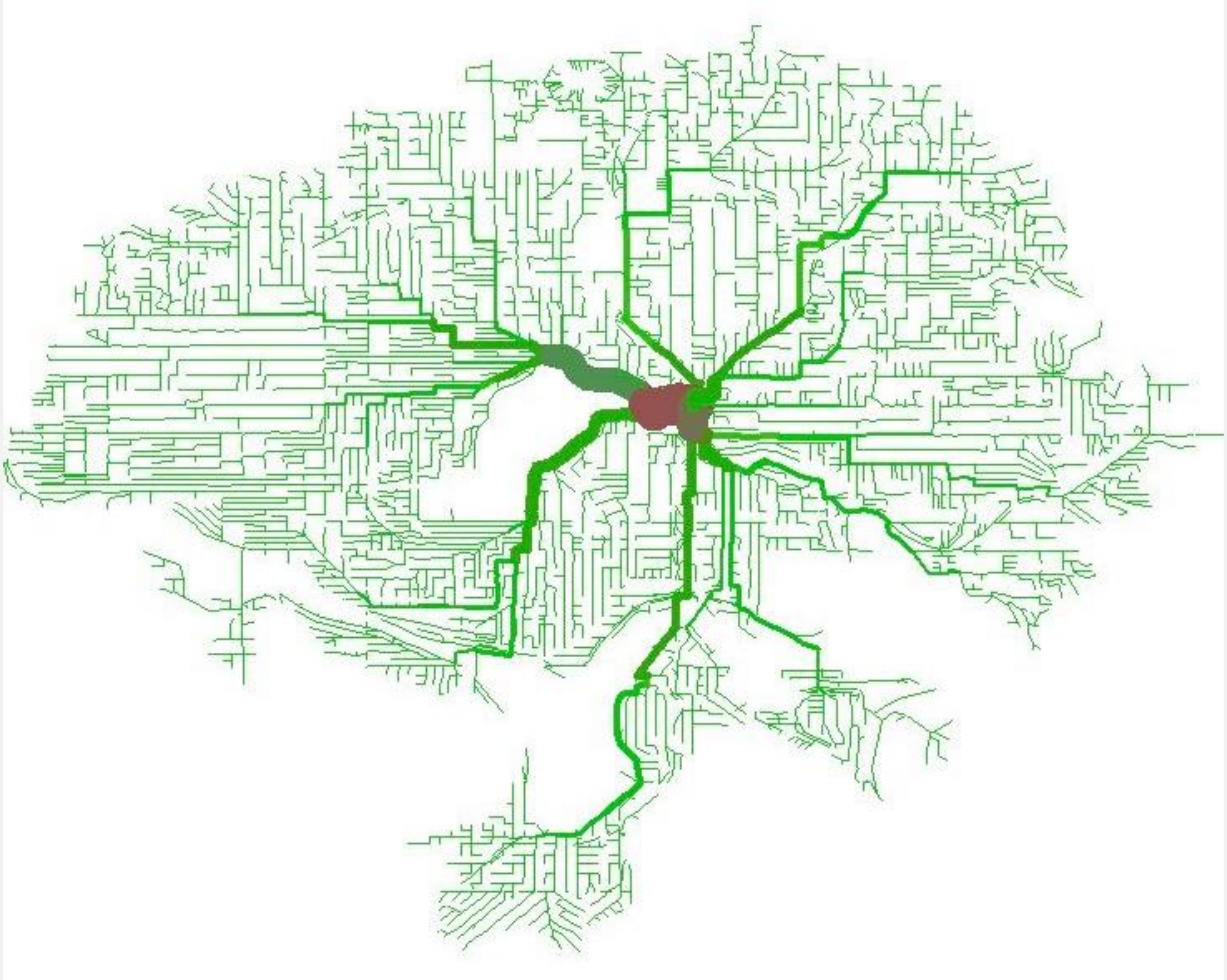


Minimize Wiring



Network design

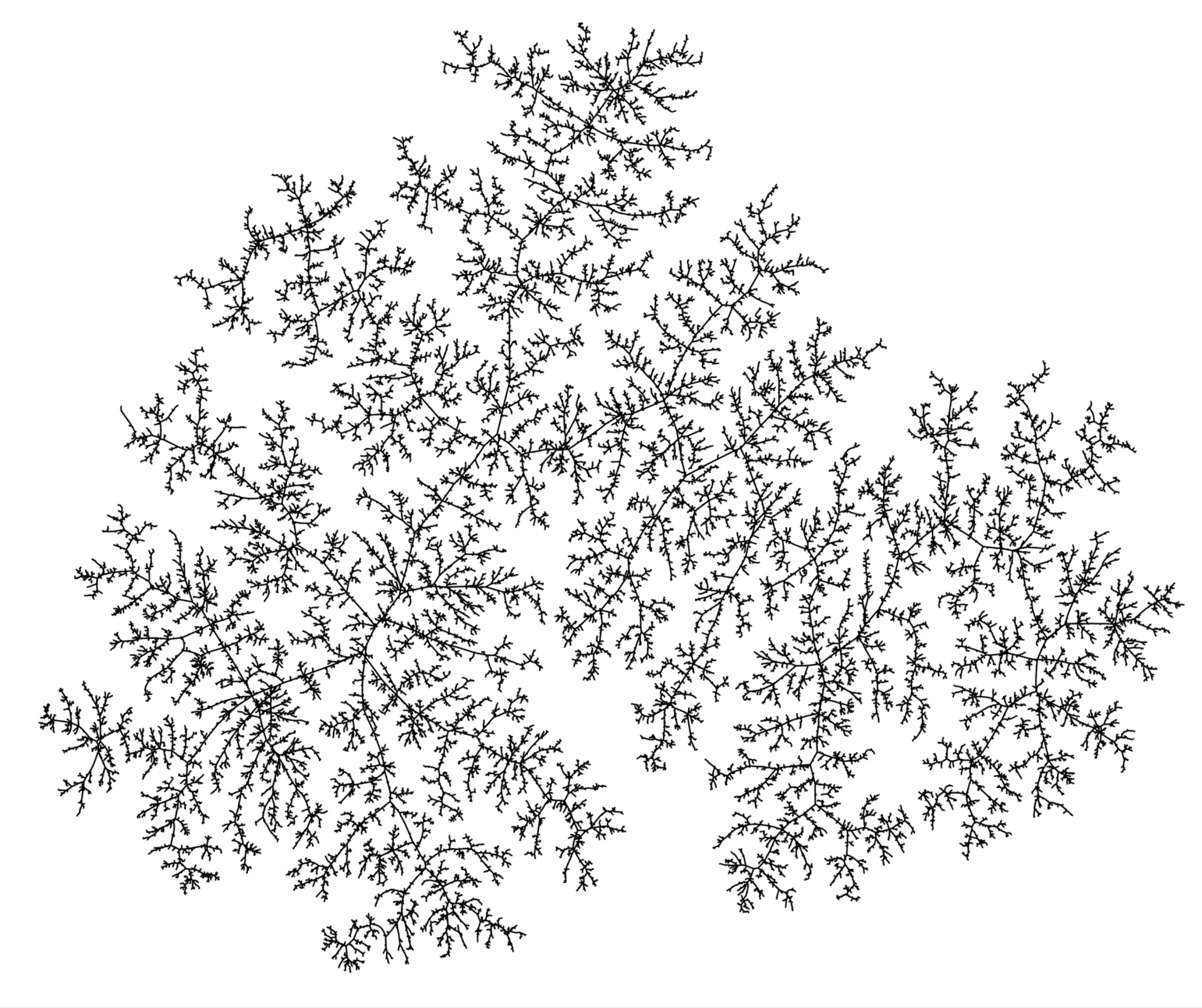
MST of bicycle routes in North Seattle



http://www.flickr.com/photos/ewedistrict/21980840

Models of nature

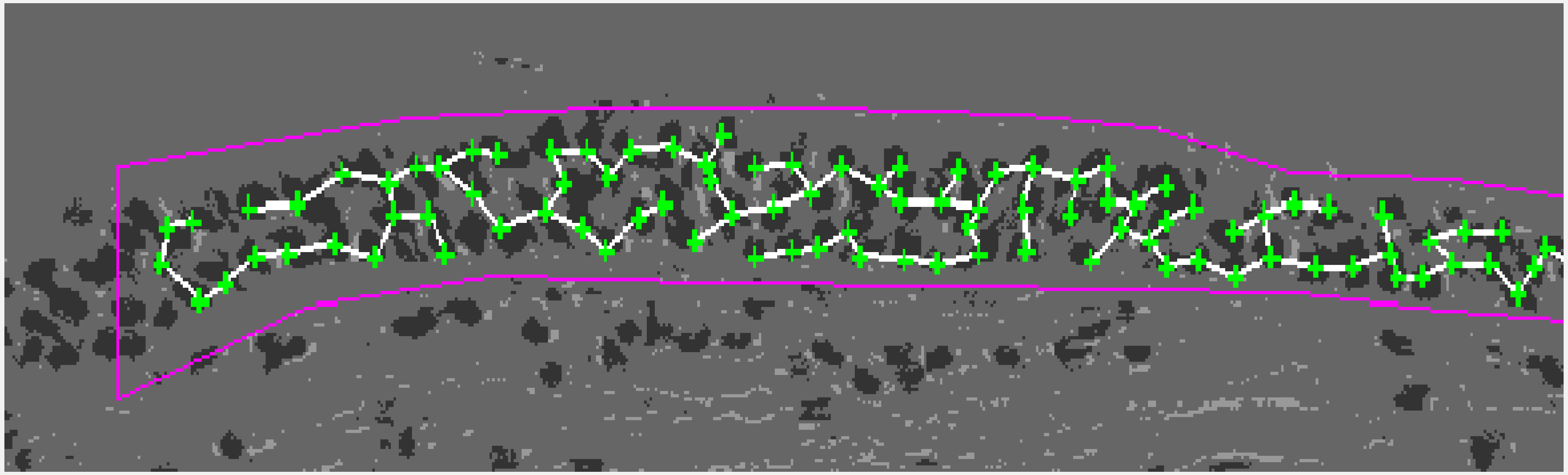
MST of random graph

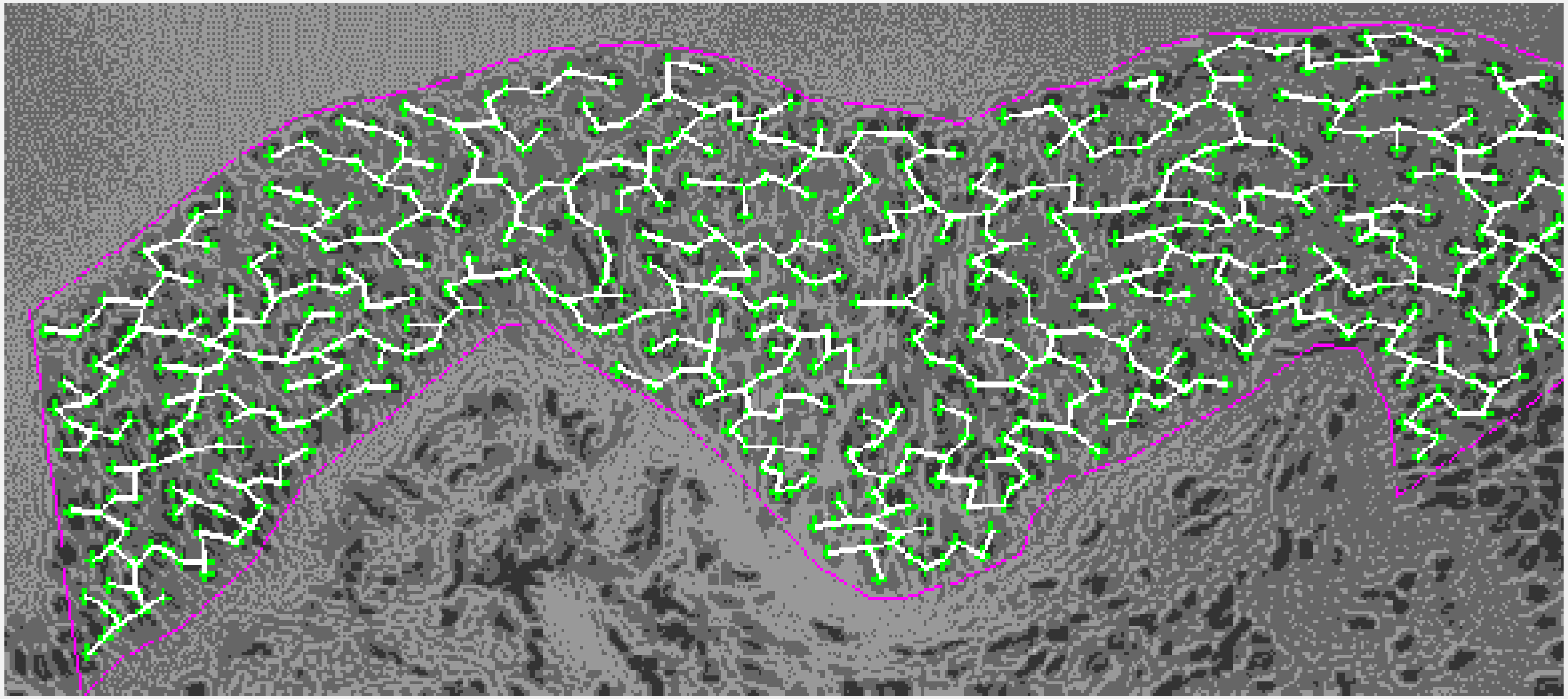


http://algo.inria.fr/broutin/gallery.html

Medical image processing

MST describes arrangement of nuclei in the epithelium for cancer research

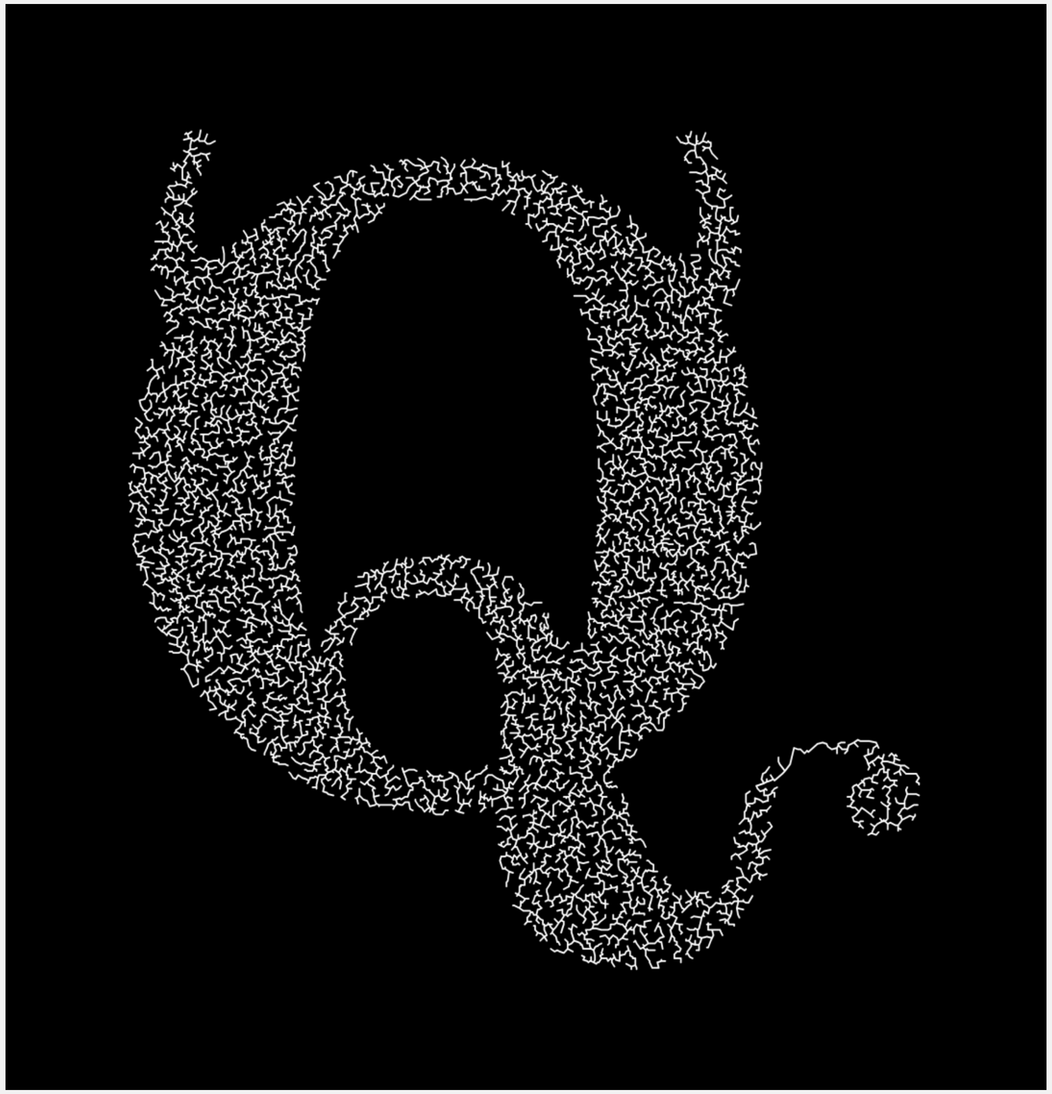




http://www.bccrc.ca/ci/ta01_archlevel.html

Dithering

MST dithering



http://www.flickr.com/photos/quasimondo/2695389651

A 1-bit image of the Statue of David, dithered with Floyd–Steinberg algorithm

MST is fundamental problem with diverse applications.

- Dithering.
- Cluster analysis.
- Max bottleneck paths.
- Real-time face verification.
- LDPC codes for error correction.
- Image registration with Renyi entropy.
- Find road networks in satellite and aerial imagery.
- Reducing data storage in sequencing amino acids in a protein.
- Model locality of particle interactions in turbulent fluid flows.
- Autoconfig protocol for Ethernet bridging to avoid cycles in a network.
- Approximation algorithms for NP-hard problems (e.g., TSP, Steiner tree).
- Network design (communication, electrical, hydraulic, computer, road).

http://www.ics.uci.edu/~eppstein/gina/mst.html

Determining MST by Brute Force

- Create all spanning trees
- Pick the lightest
- Not feasible!!
  - A complete graph (every pair of vertices is connected by an edge) has $|V|^{|V|-2}$ many spanning trees (Cayley's Formula [1889])

Reminder

Greedy Algorithm Design Technique

- Applied to optimization problems
  - an objective function is *minimized* or *maximized*
- Characterized by the *greedy-choice property:*
  - a global optimal configuration can be reached by a series of locally optimal choices
  - starting from a well-defined configuration, optimal choices are choices that are best from among the possibilities available at the time

Minimum Spanning Tree algorithms

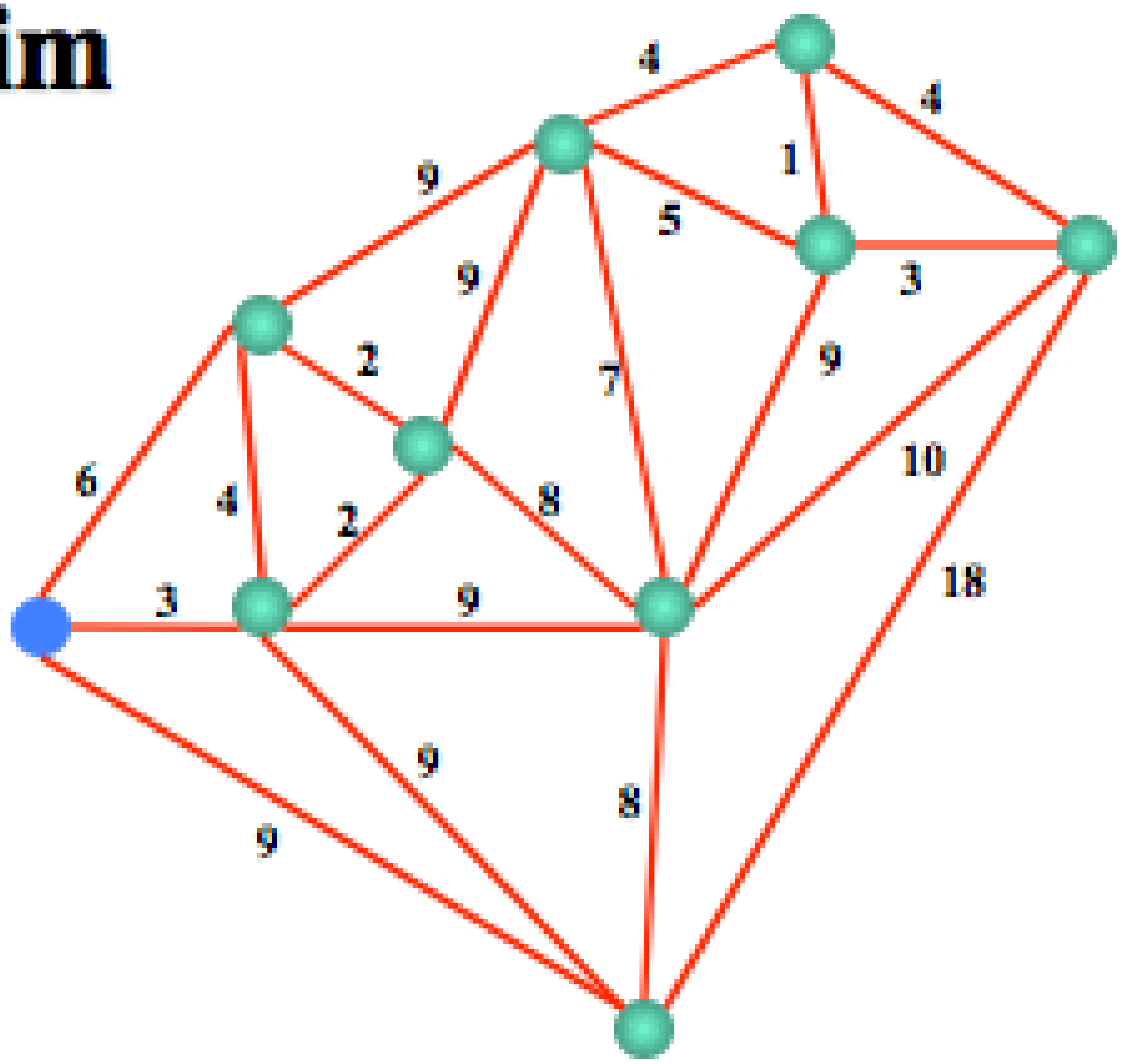
- 1926 Barůvka $O(m \log n)$
- 1930 Prim-Jarník's
  - 1930 Jarník
  - 1957 Dijkstra
  - 1959 Prim
  - 1964 with Heaps $O(m \log n)$
  - 1987 Fredman and Tarjan with Fibonacci Heaps $O(m+n \log n)$
- 1956 Kruskal's algorithm
  - 1956 Kruskal
  - 1974 Aho, Hopcroft and Ullman with Union-Find Disjoint Set $O(m \log n)$
- 1975 Yao $O(m \operatorname{loglog} n)$
- 1976 Cheriton and Tarjan $O(m \operatorname{loglog} n)$
- 1995 Karger, Klein and Tarjan Randomized MST based on Barůvka and Kruskal $O(m)$
- 2000 Chazelle $O(m\ \alpha(m,n))$

n: number of vertices
m: number of edges

Prim's Algorithm Idea

- Initialize tree with single chosen vertex
- Grow tree by finding lightest edge not yet in tree and connect it to tree; repeat until all vertices are in the tree
- *Example of greedy algorithm*

Prim



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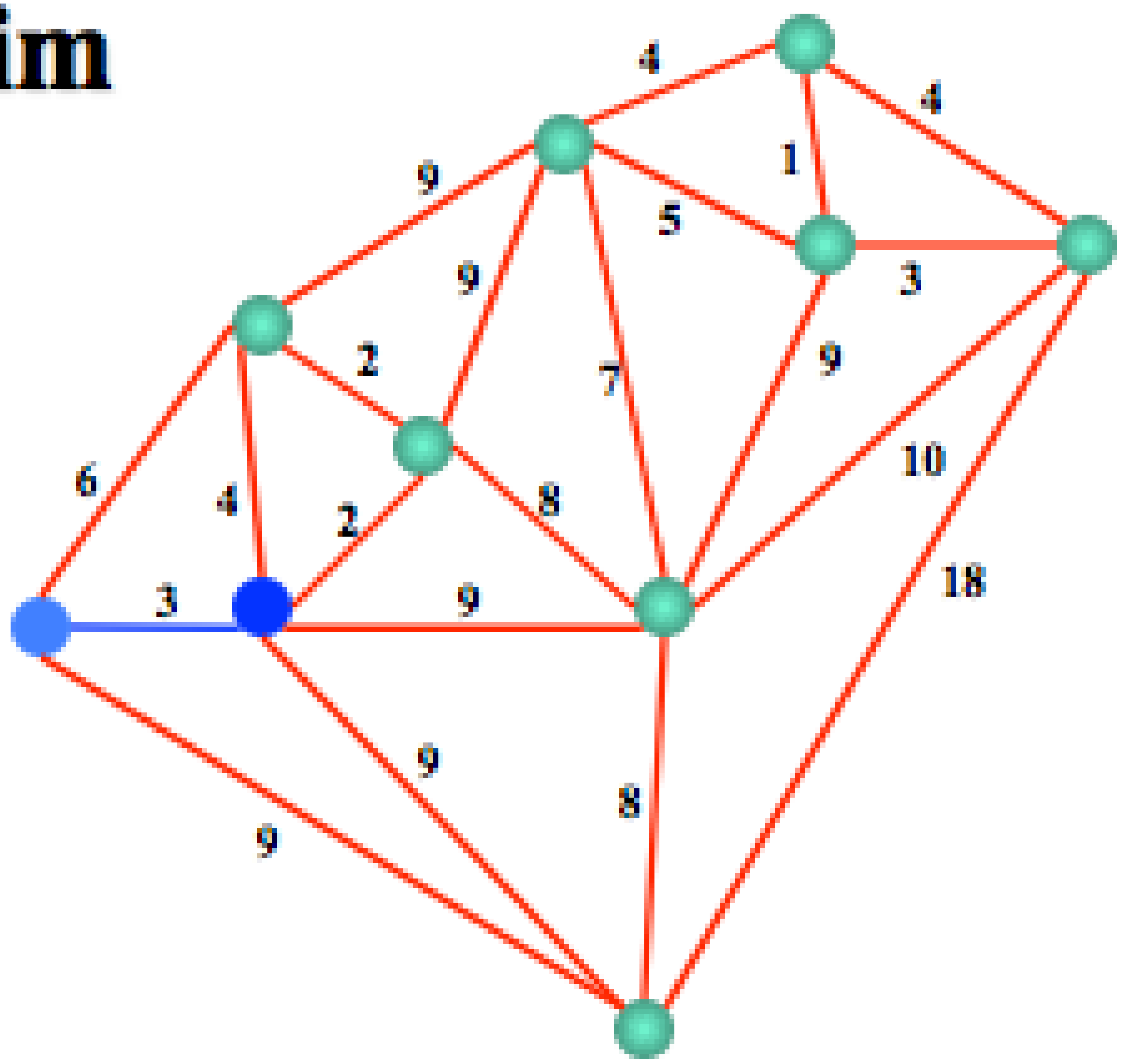

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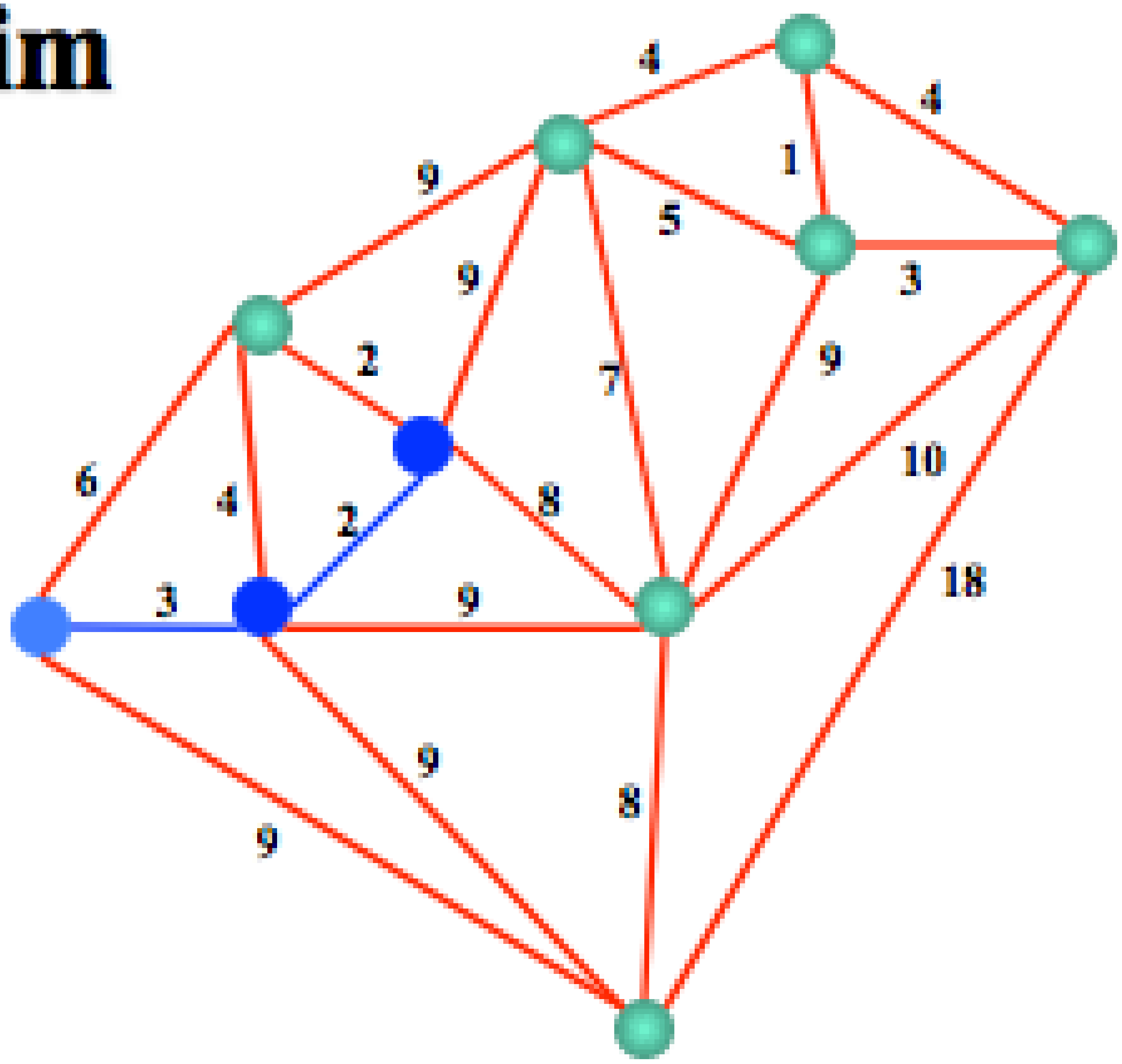

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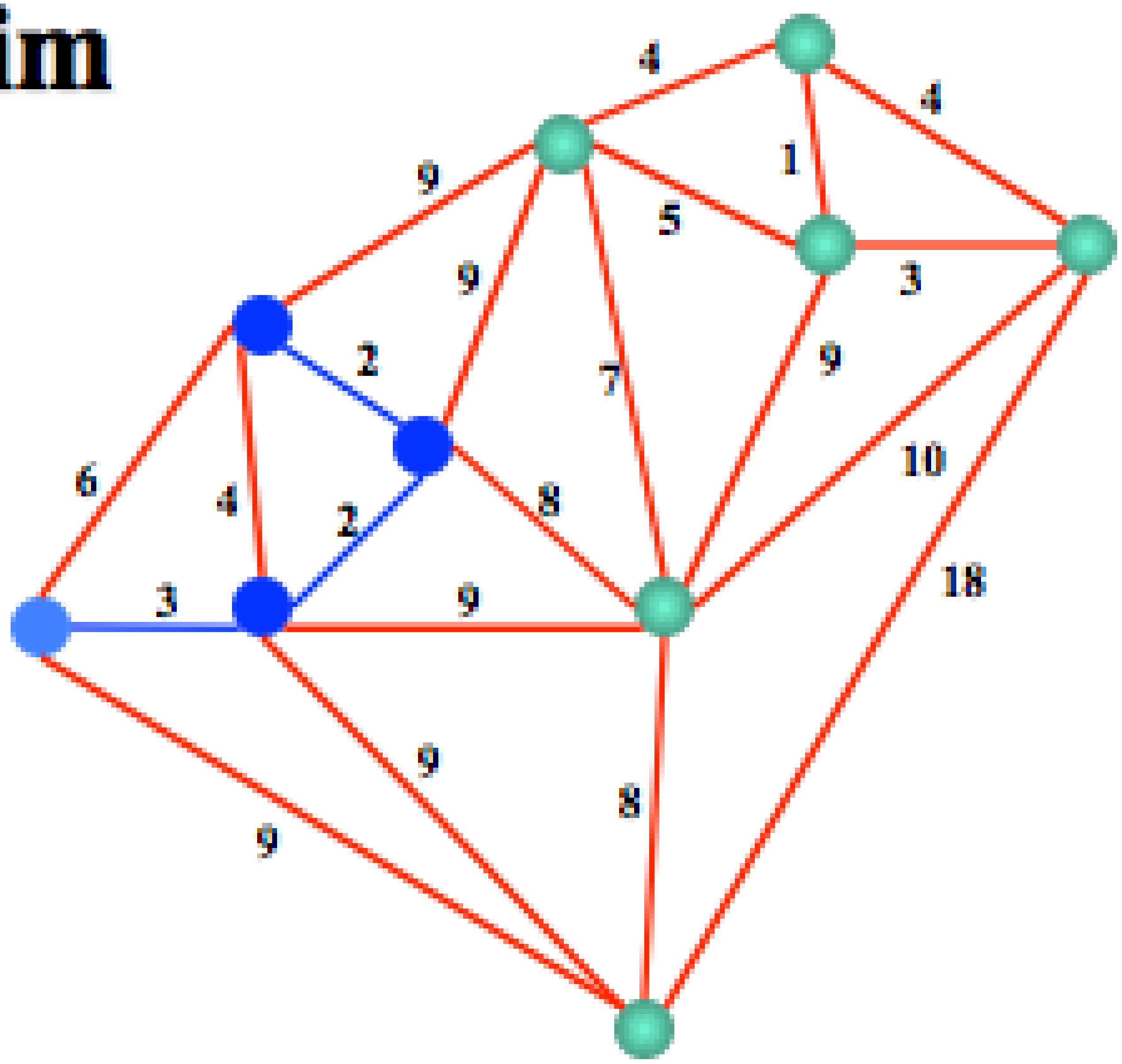

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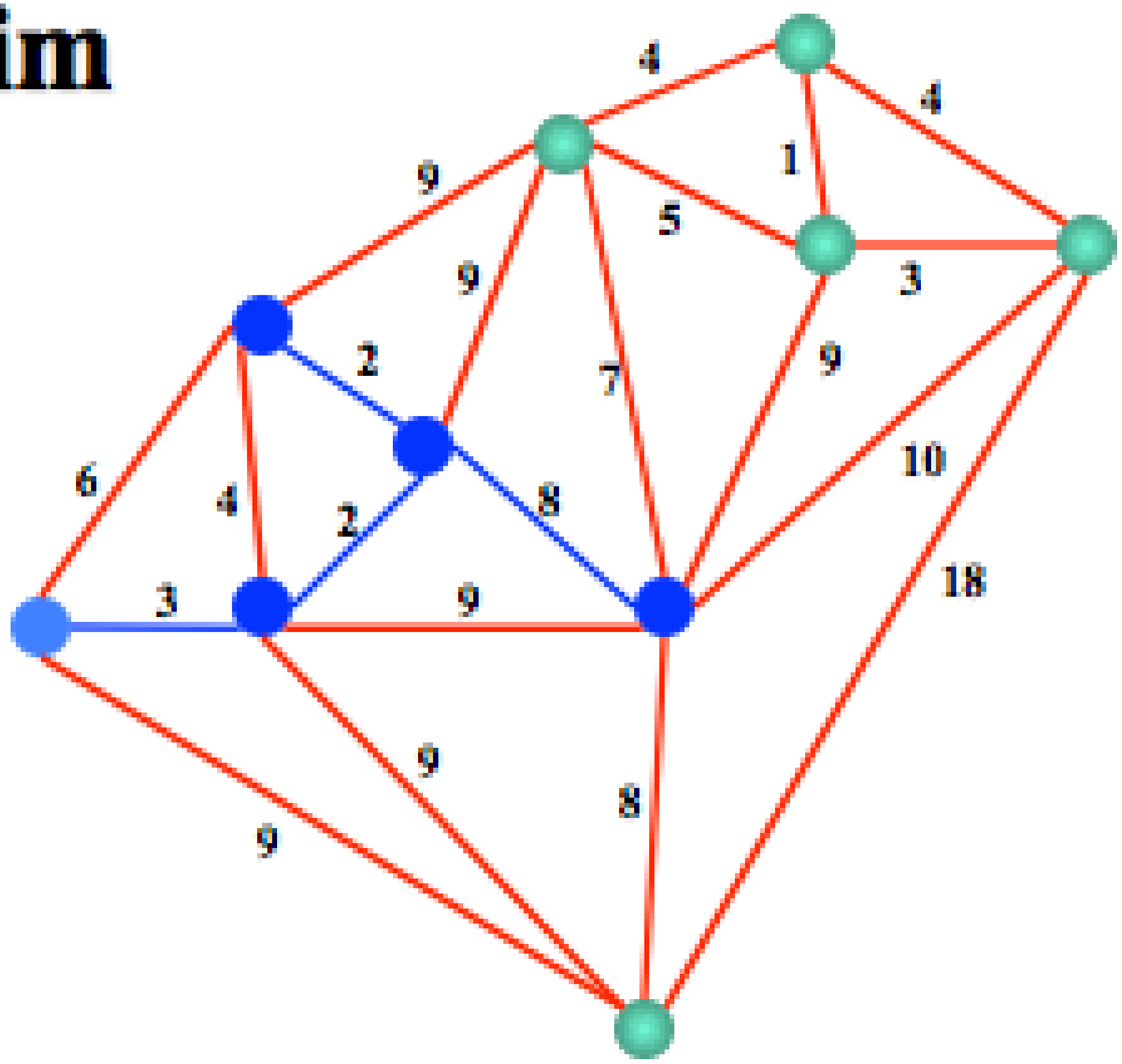

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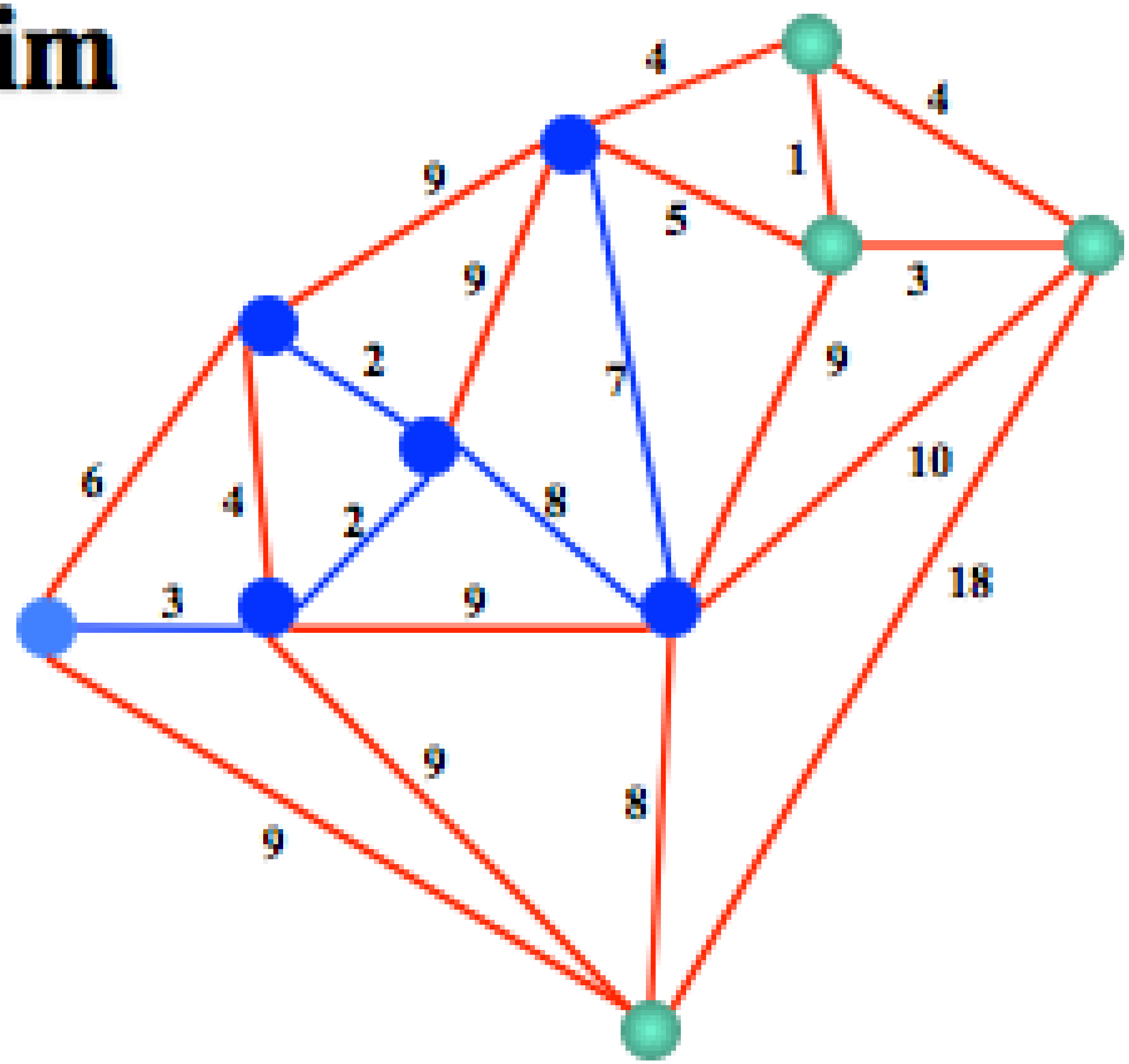

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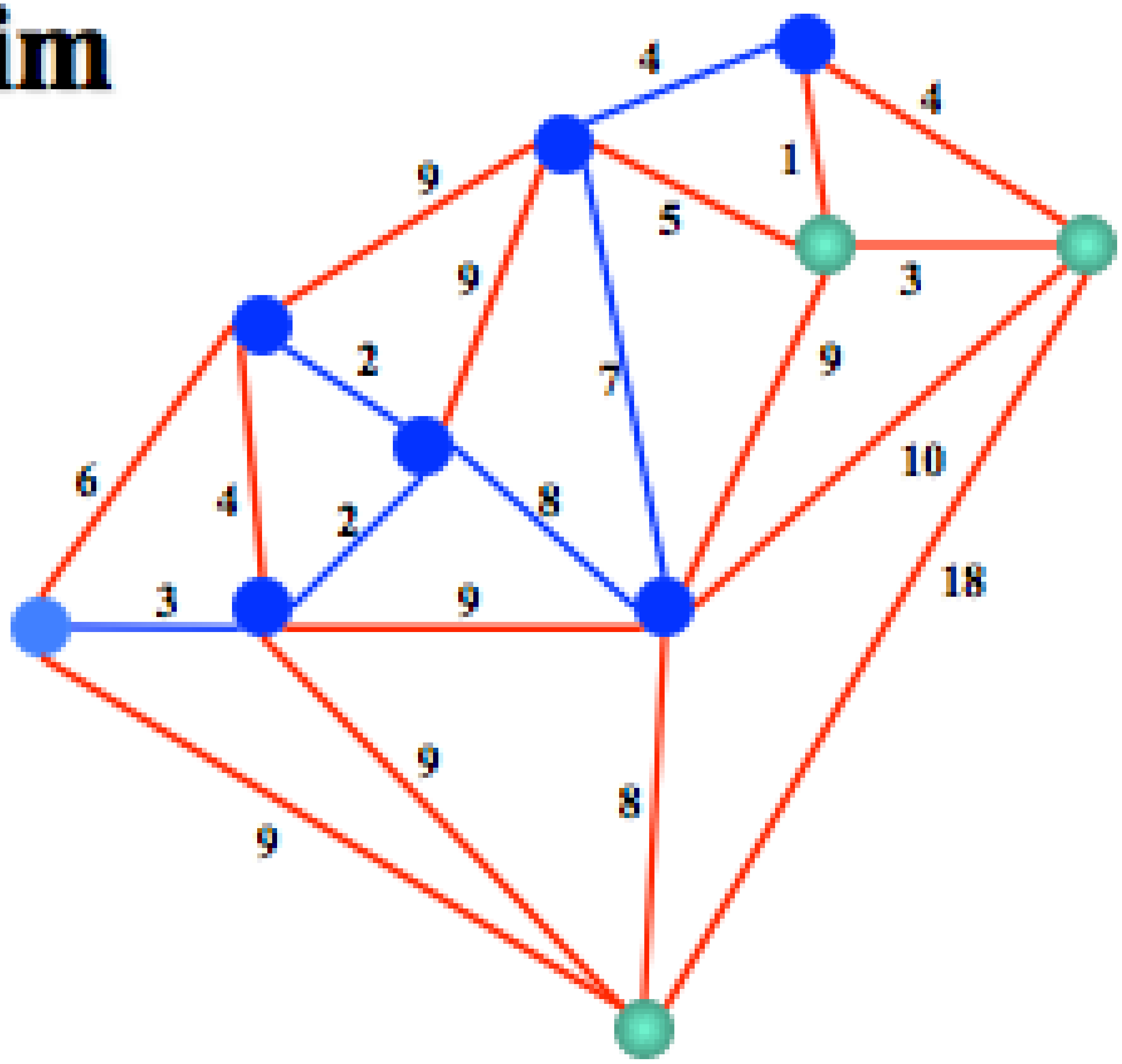

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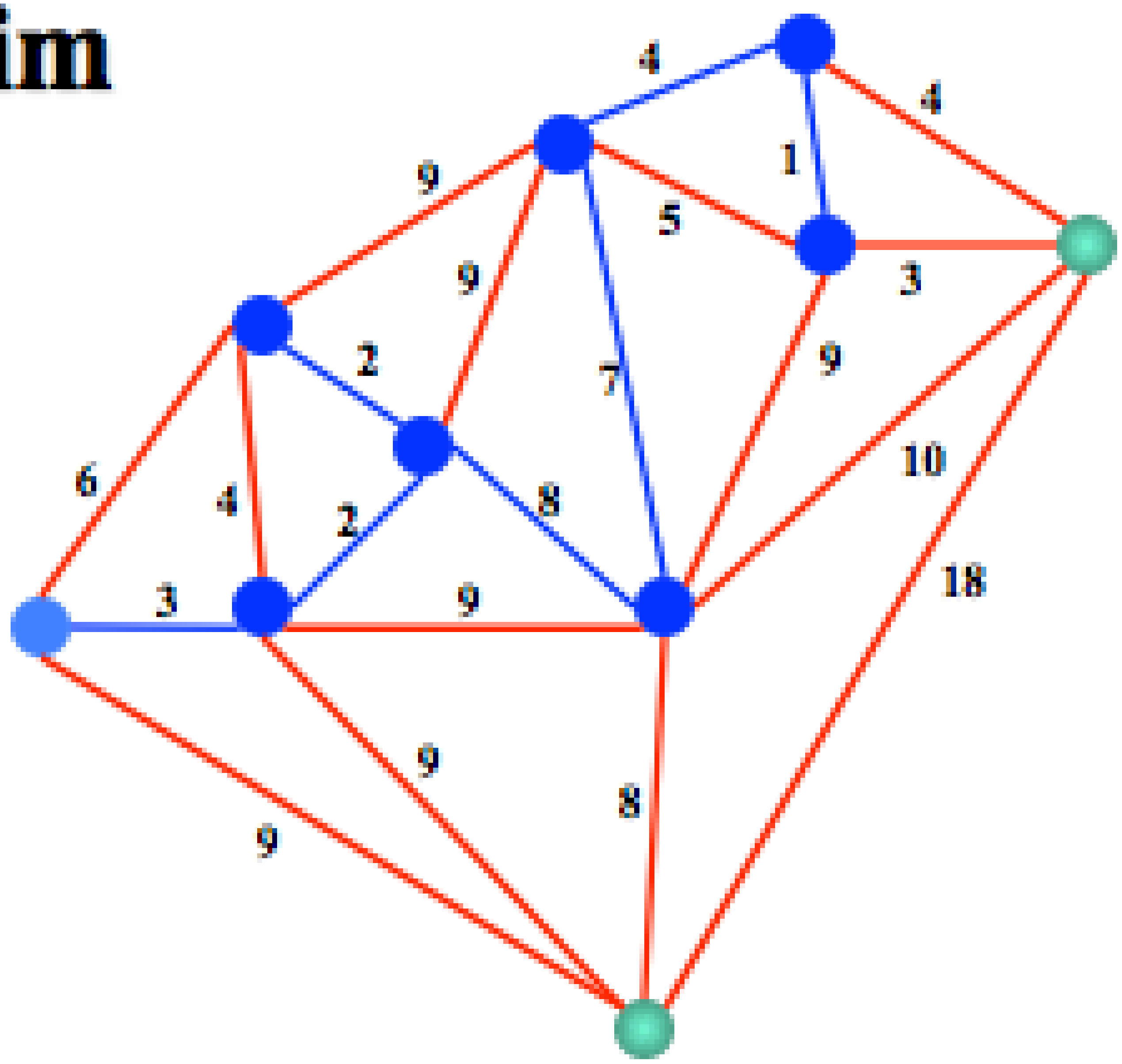

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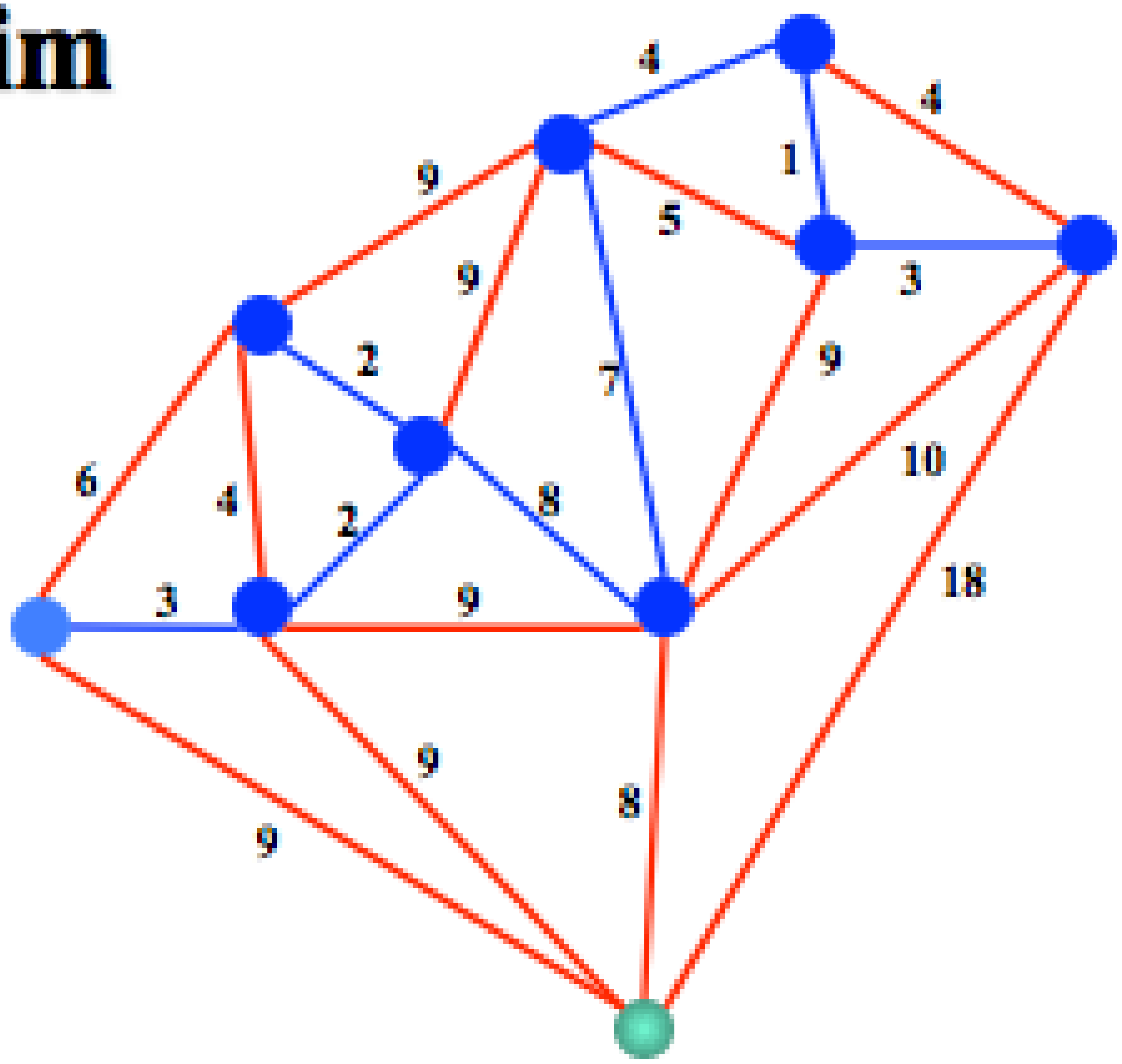

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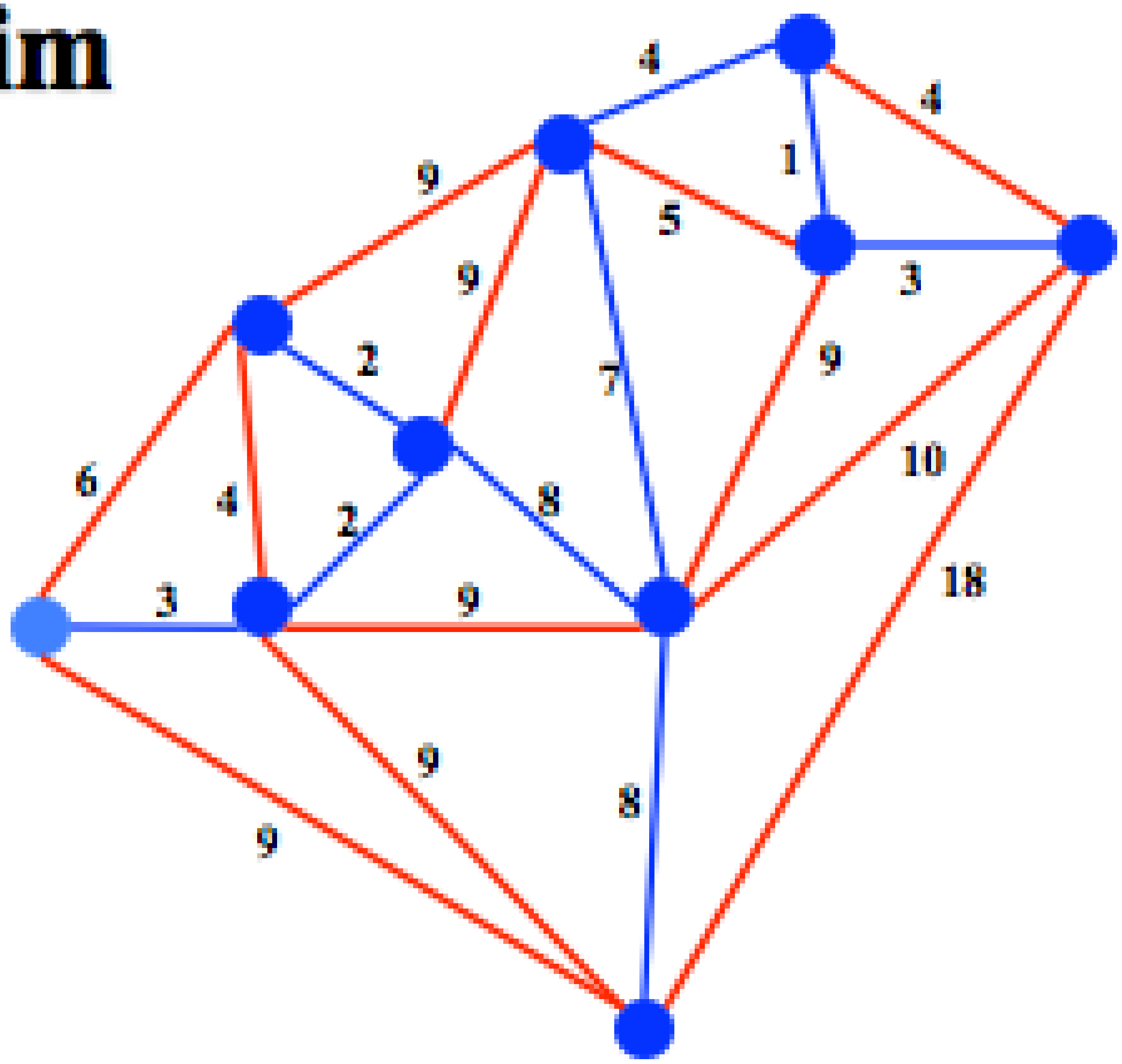


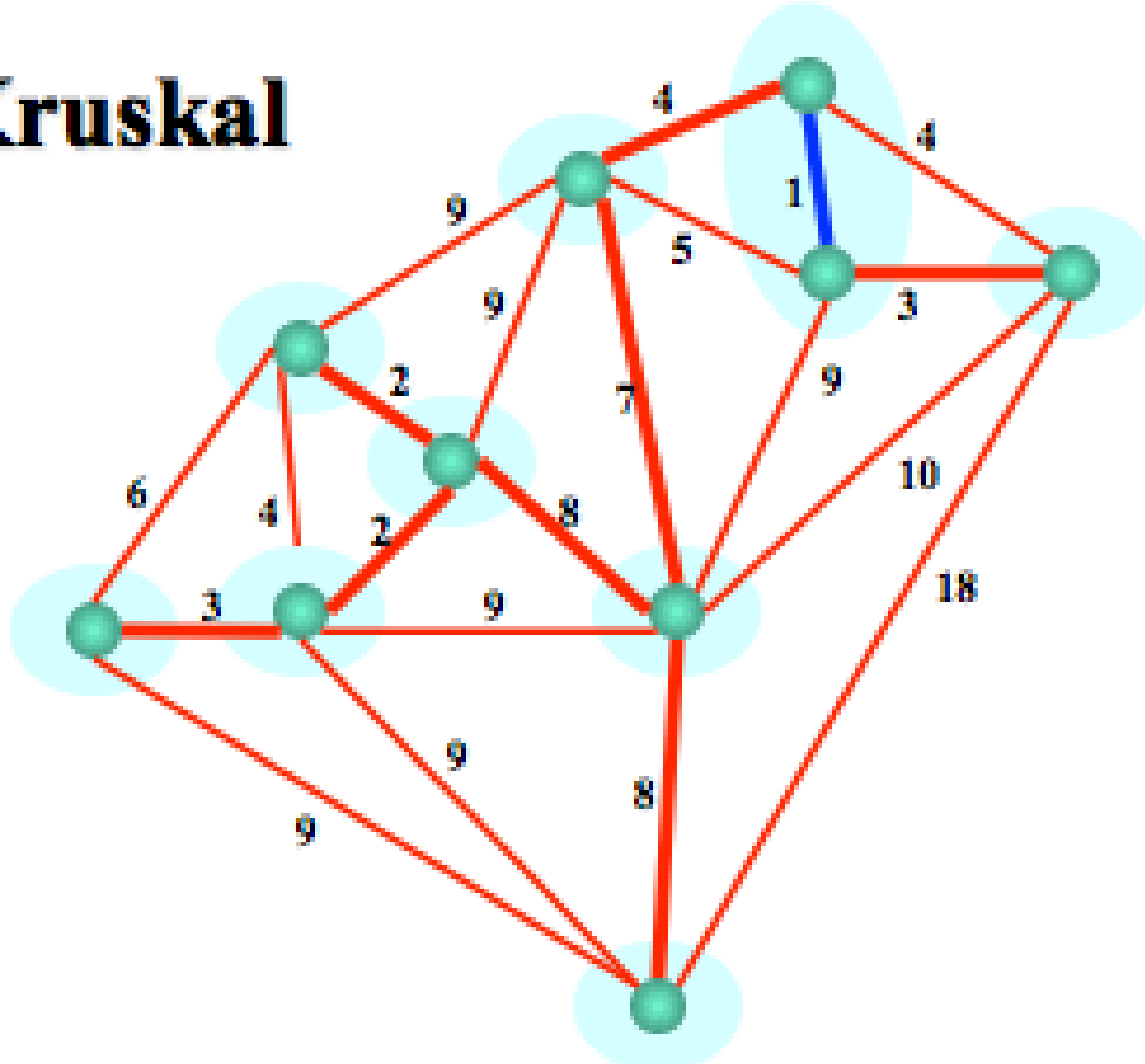
Prim’s Algorithm Idea

- Initialize tree with single chosen vertex
- Grow tree by finding lightest edge not yet in tree and connect it to tree; repeat until all vertices are in the tree
- *Example of greedy algorithm*

Kruskal's Algorithm
Idea

- Initialize a forest consisting of all nodes
- Pick a (non-selected) minimum weight edge and, if it connects two different trees of the forest, select it, otherwise discard it; repeat
- *Example of greedy algorithm*

Kruskal



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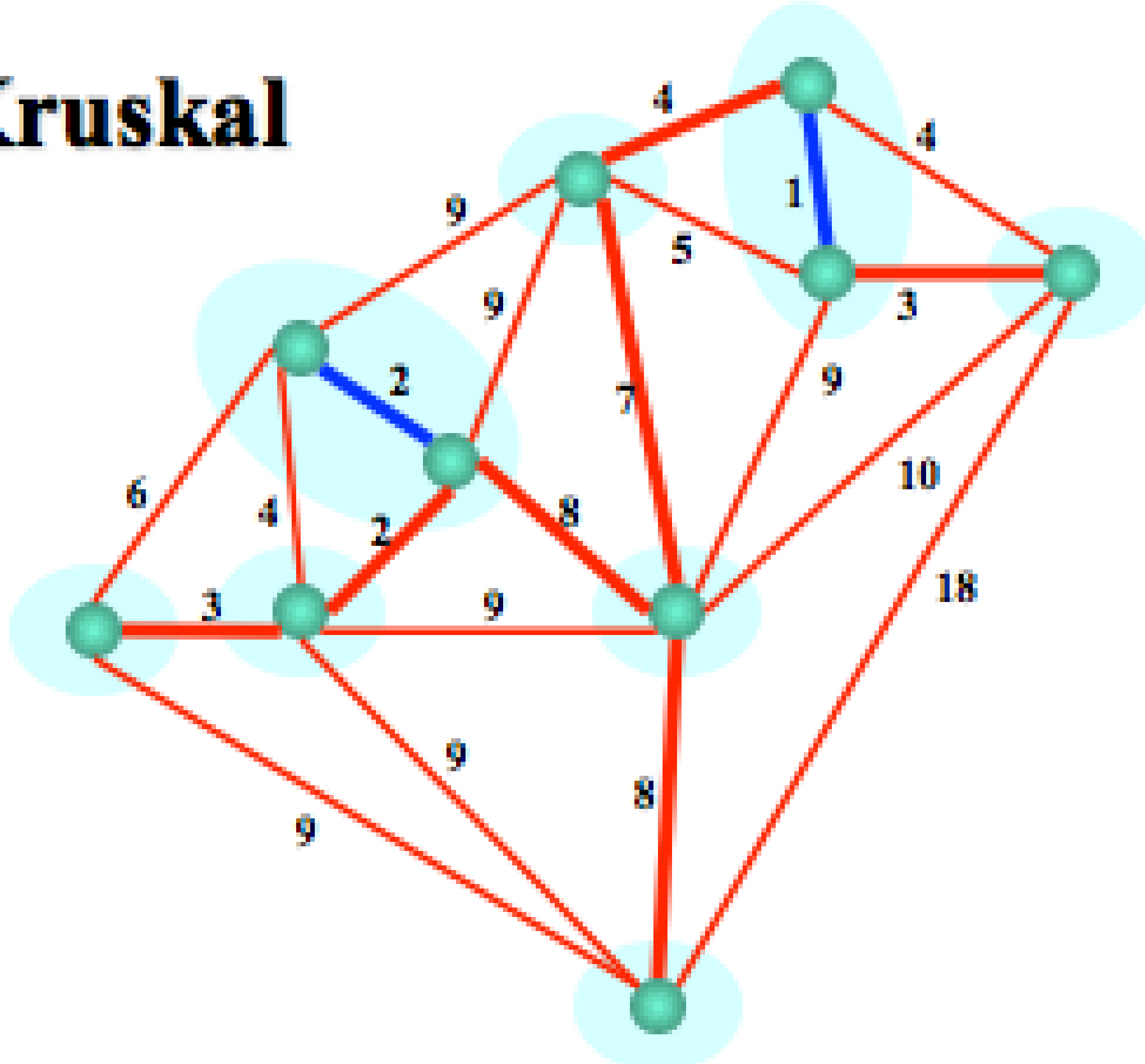
Kruskal



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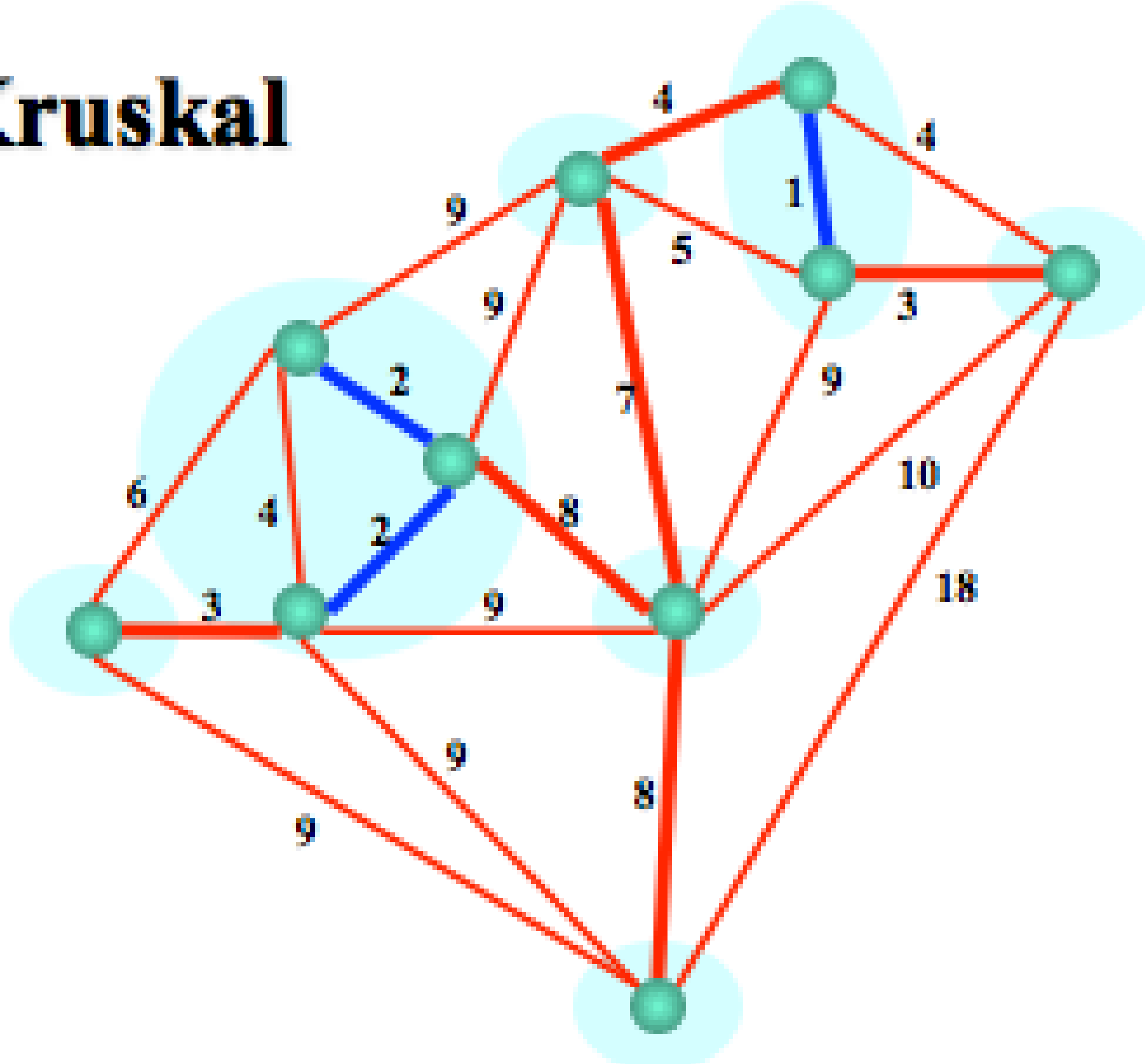

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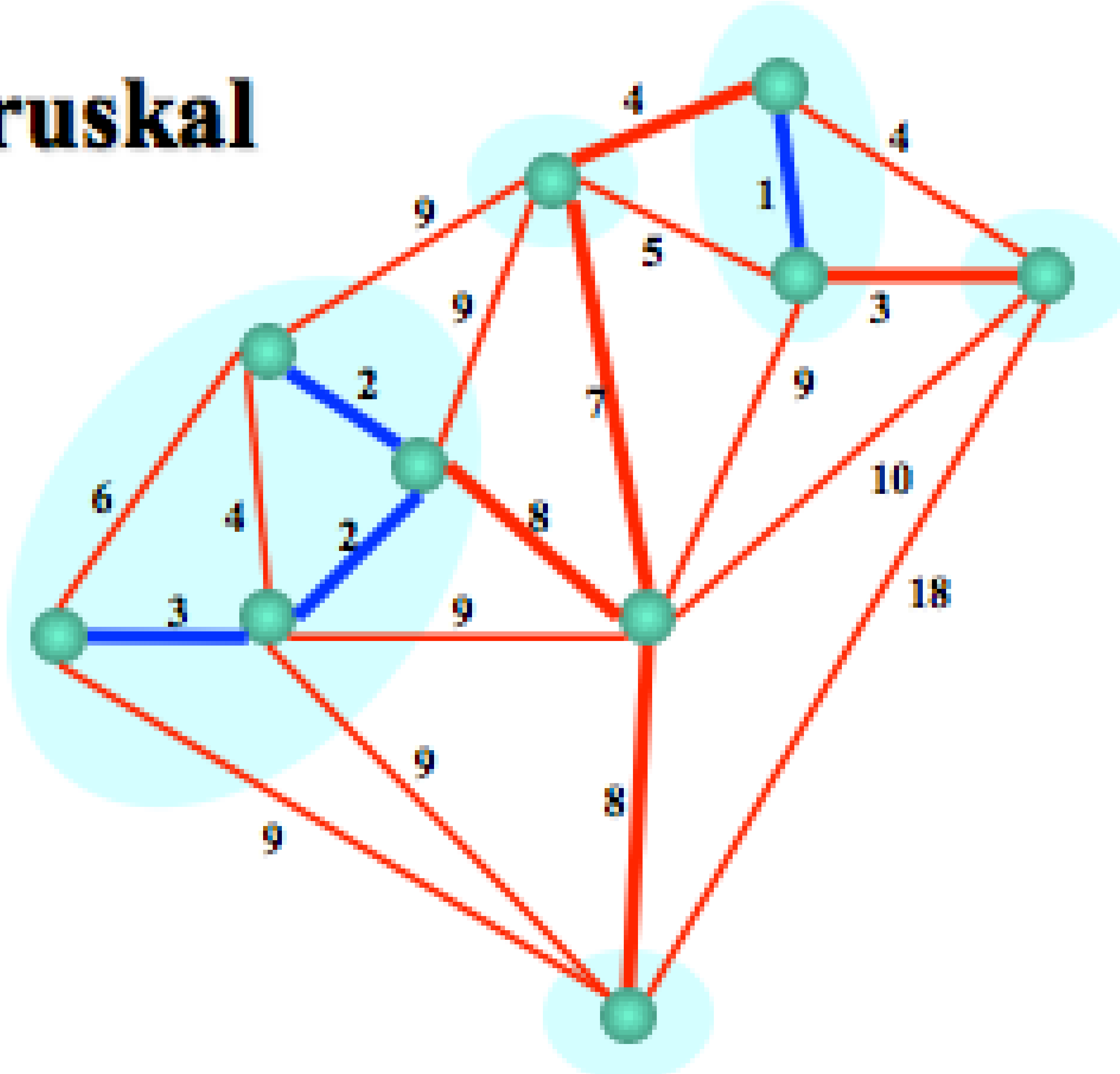

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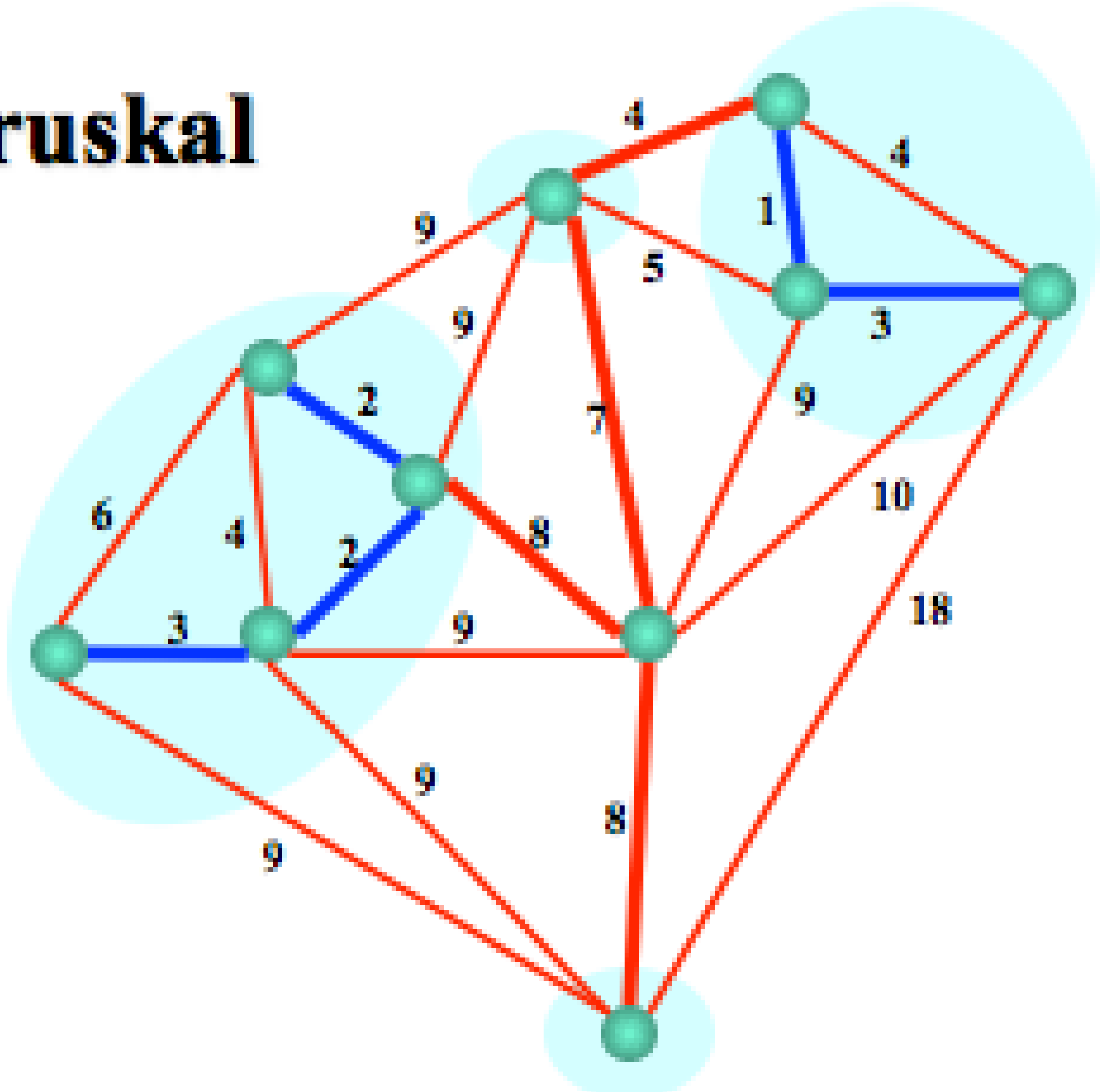

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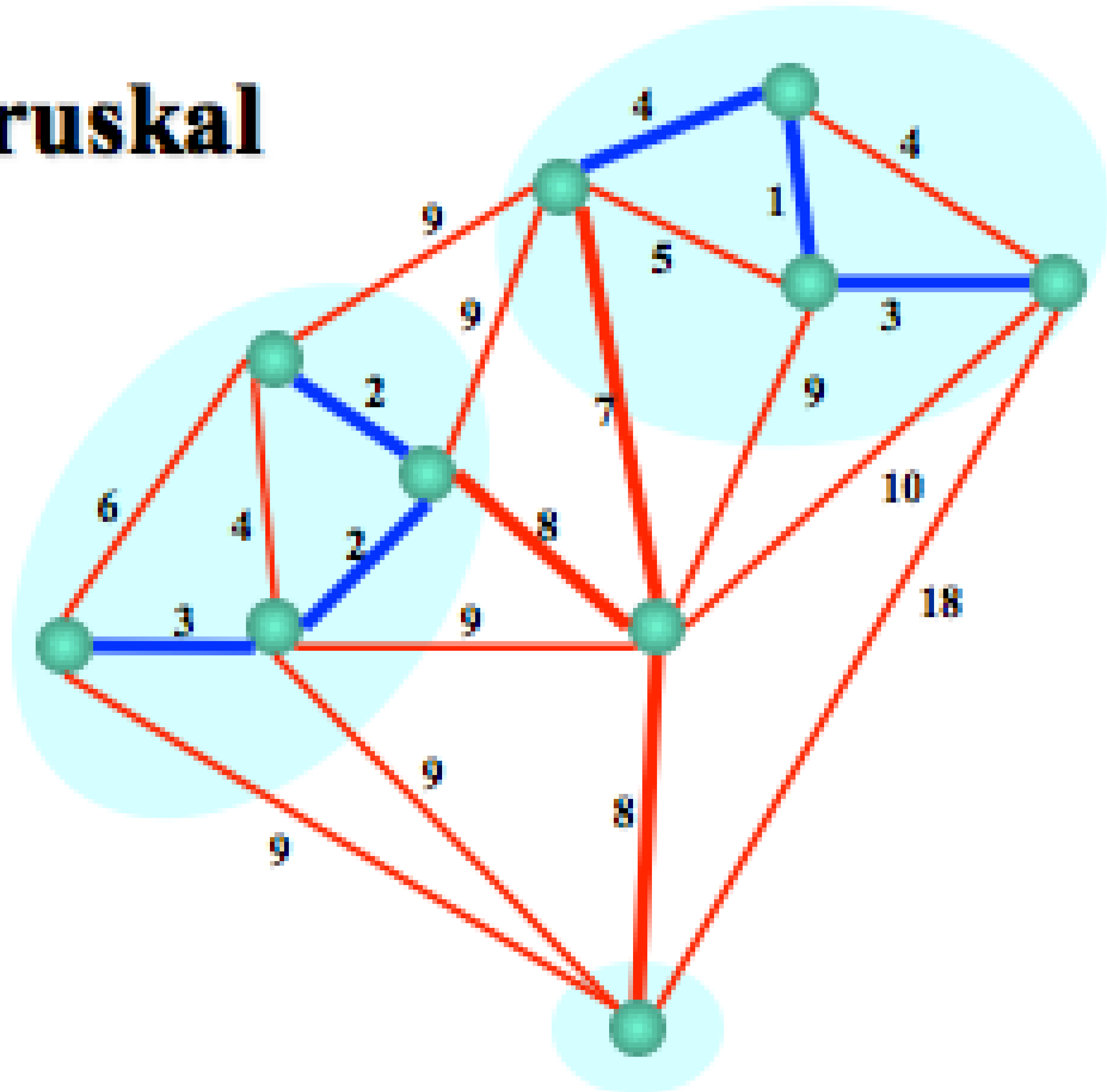

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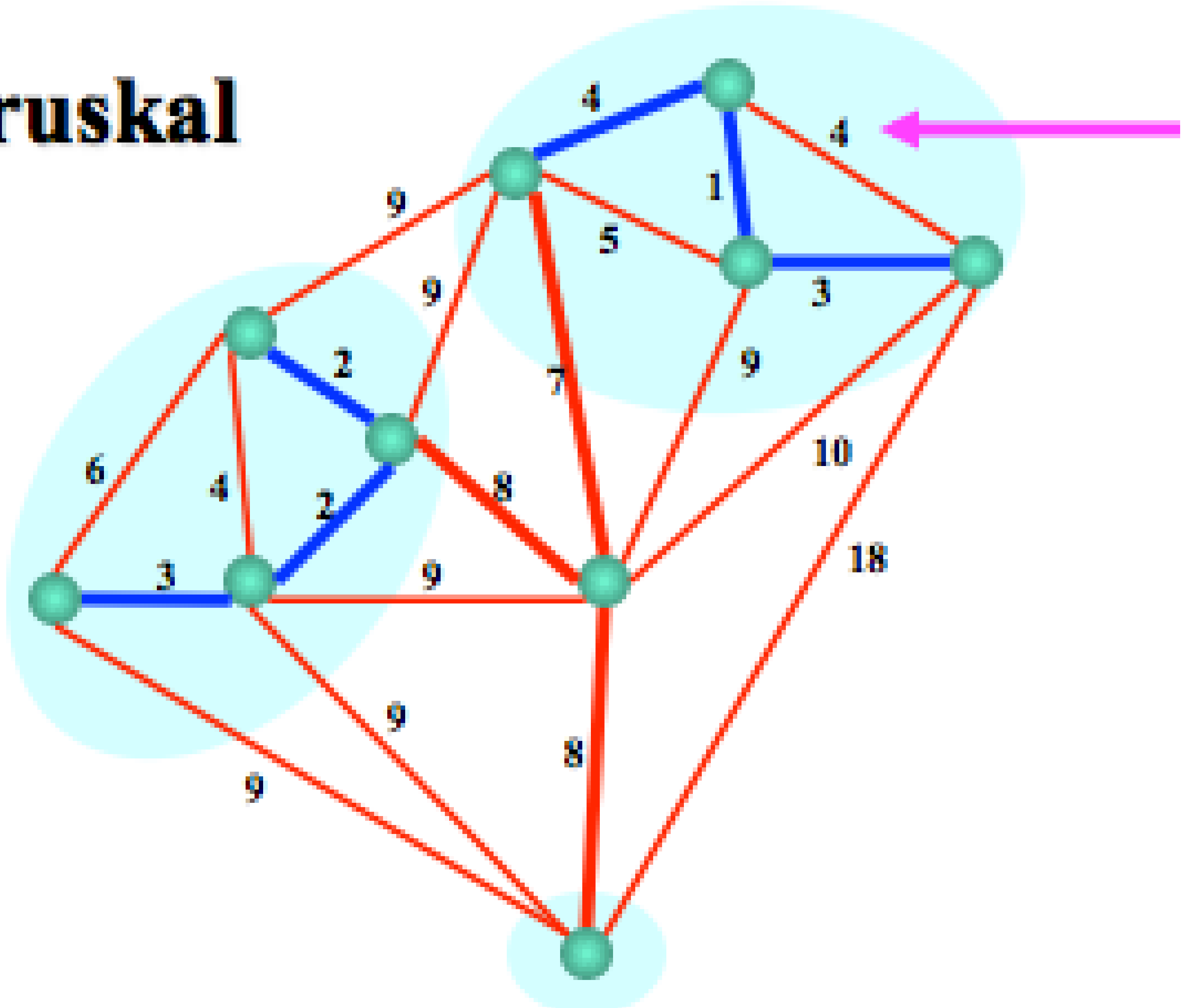

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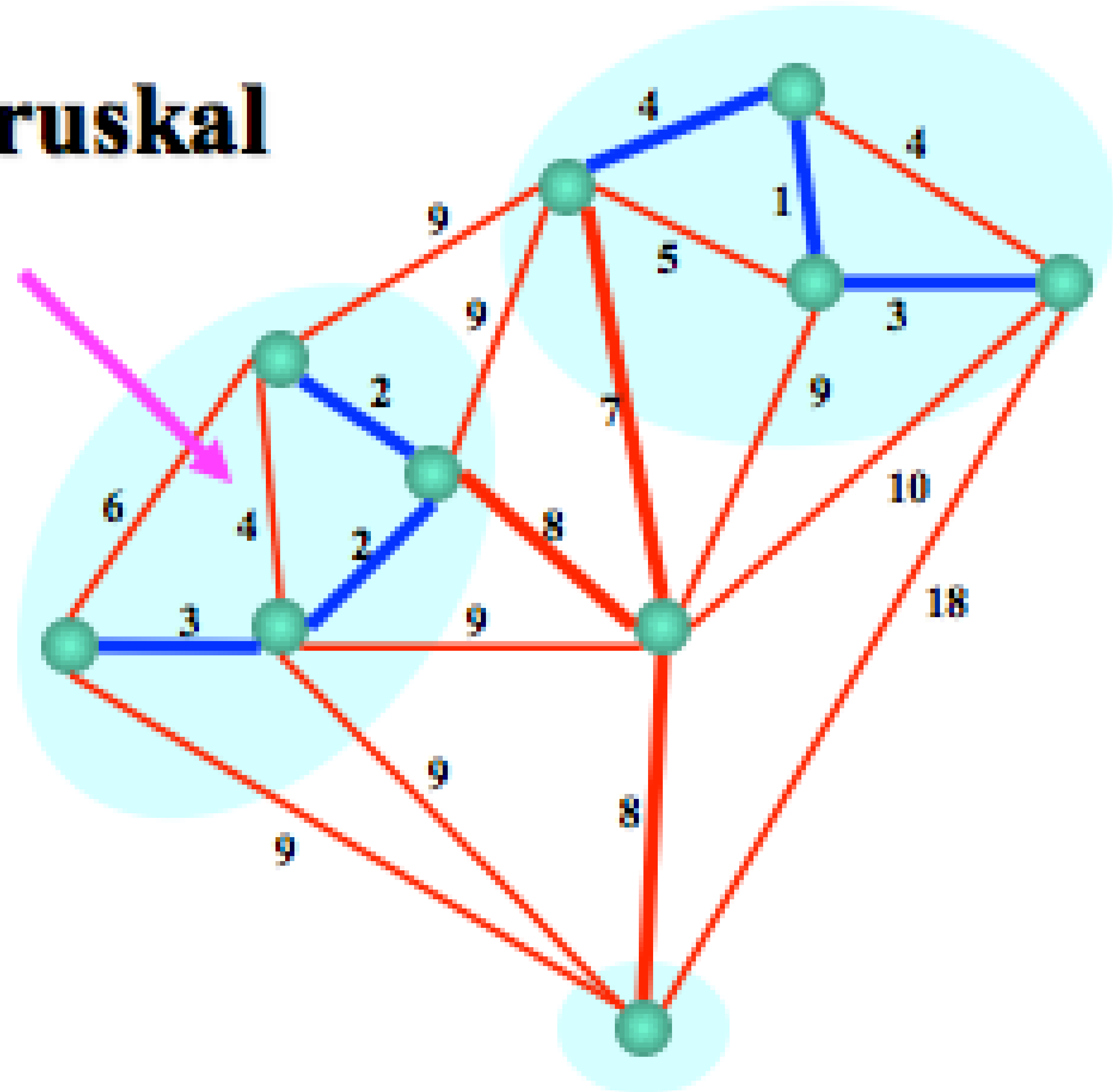

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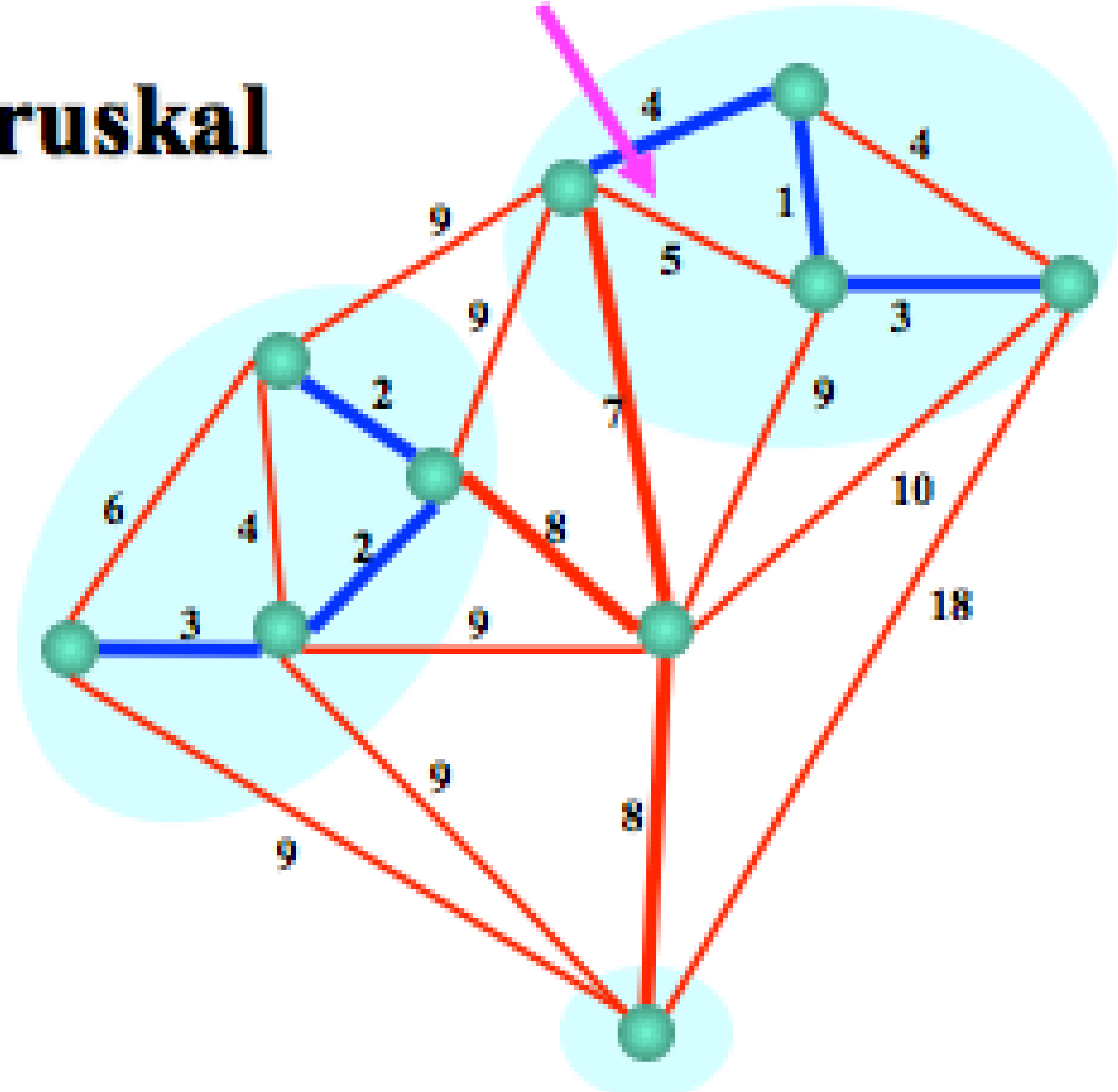

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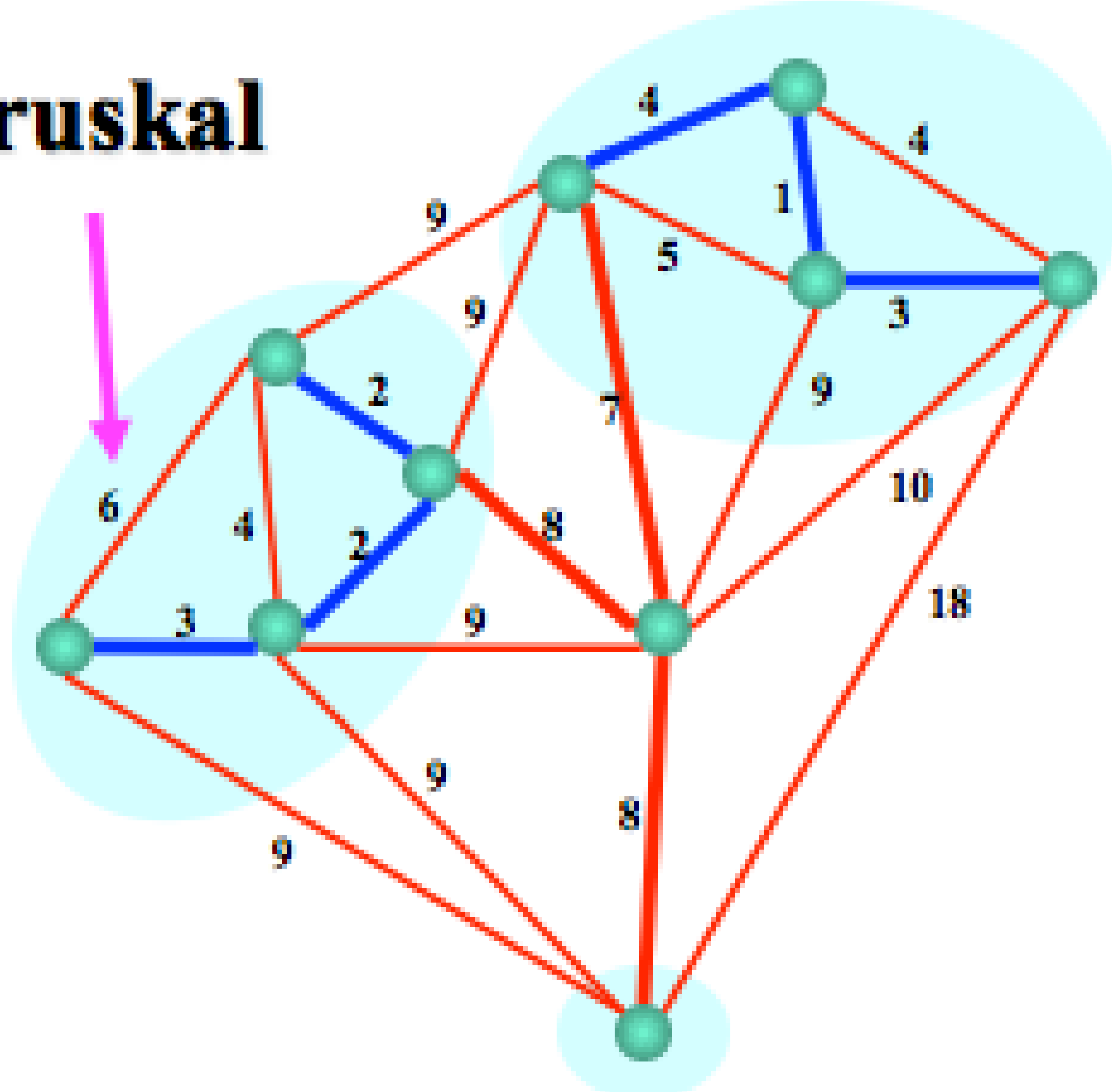

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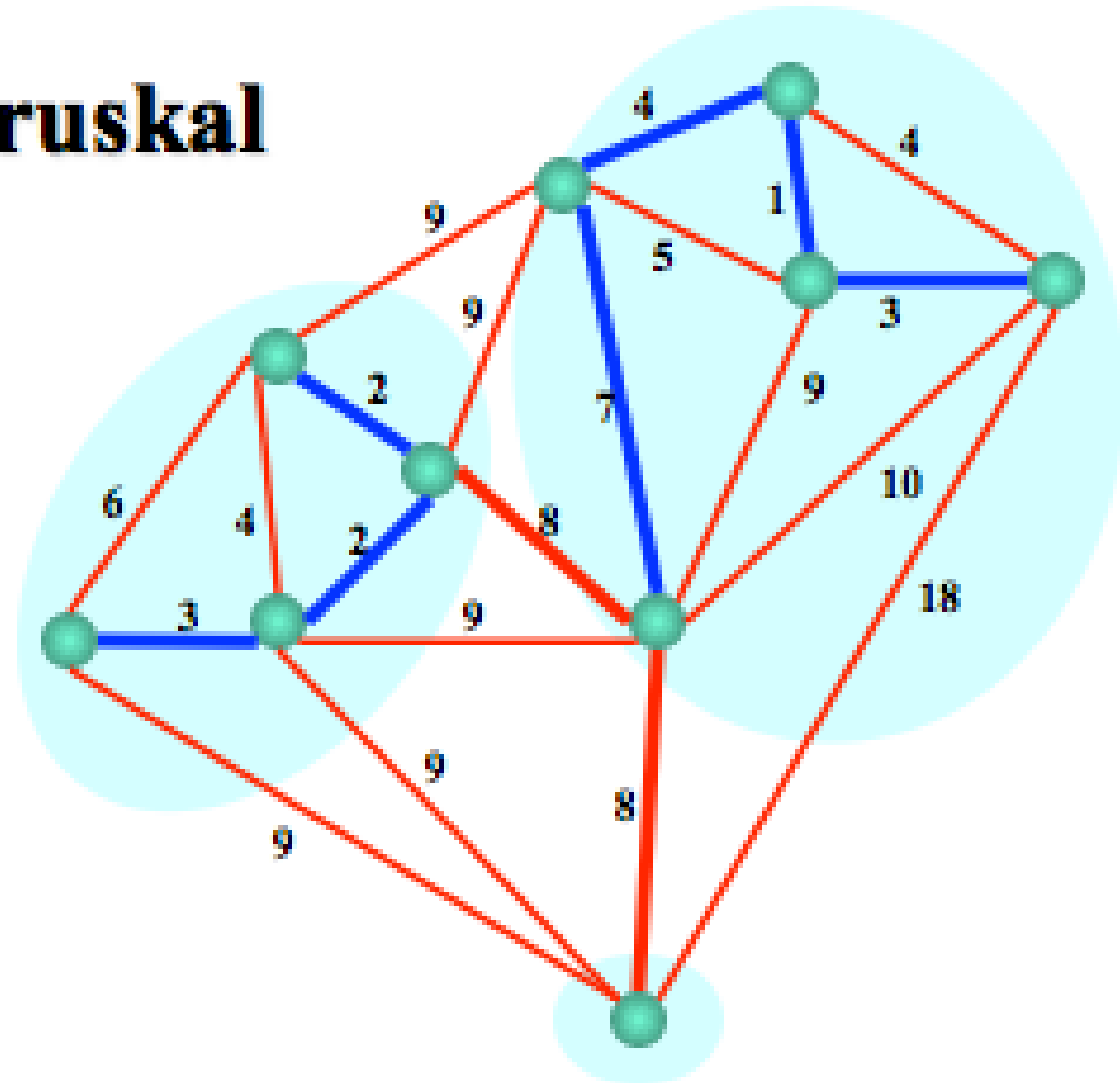

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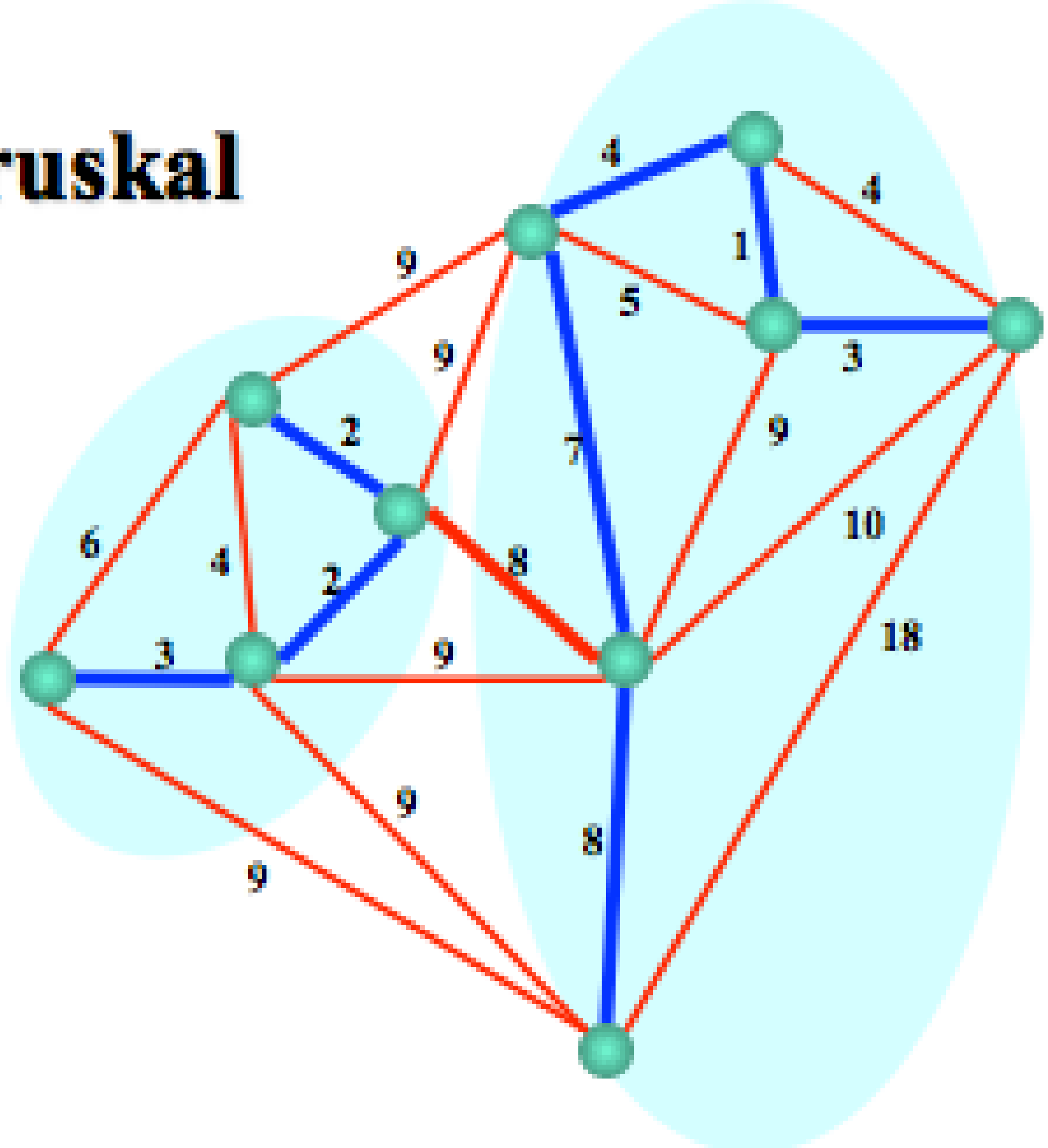

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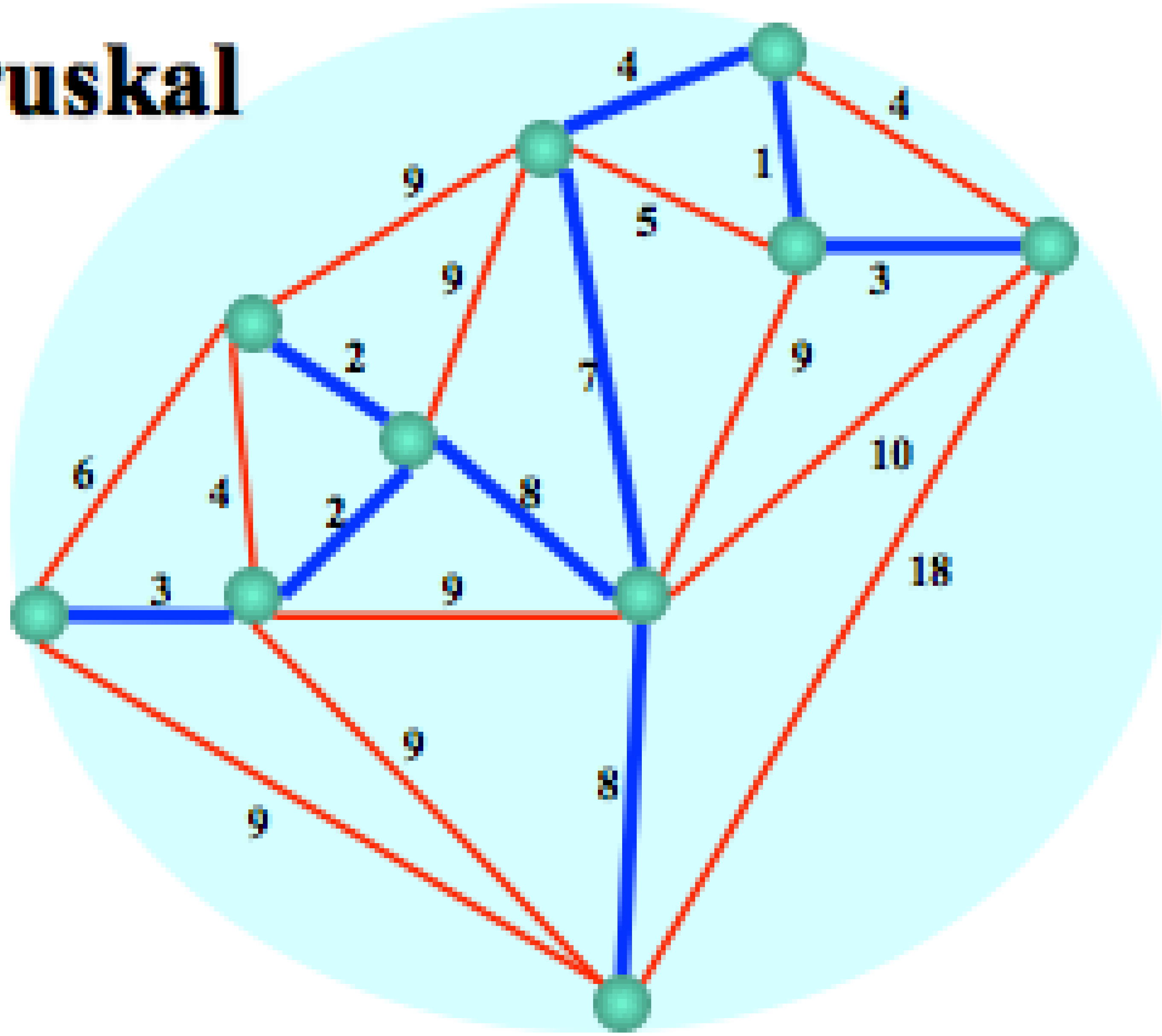

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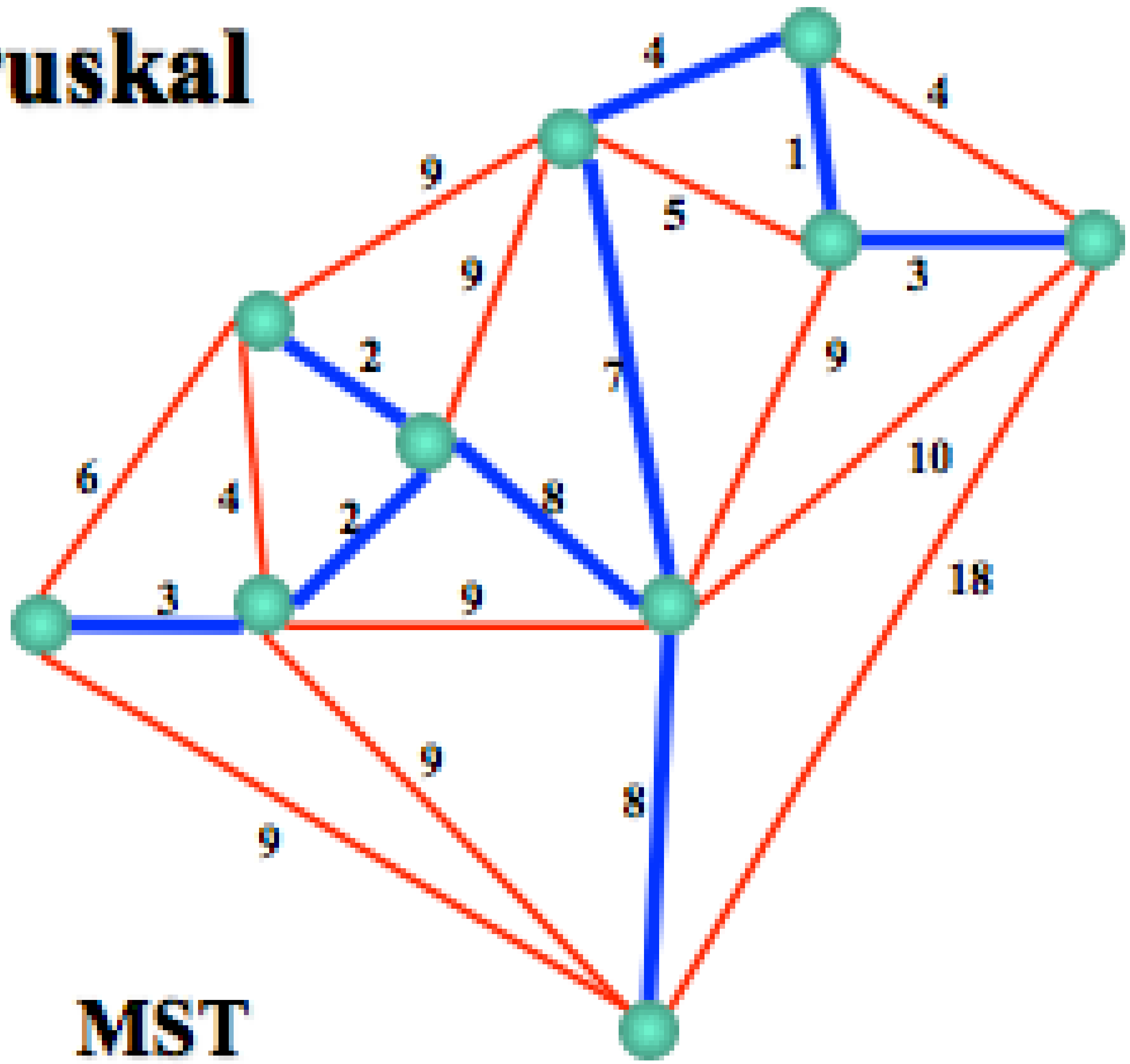

MST

Sorted edge weights

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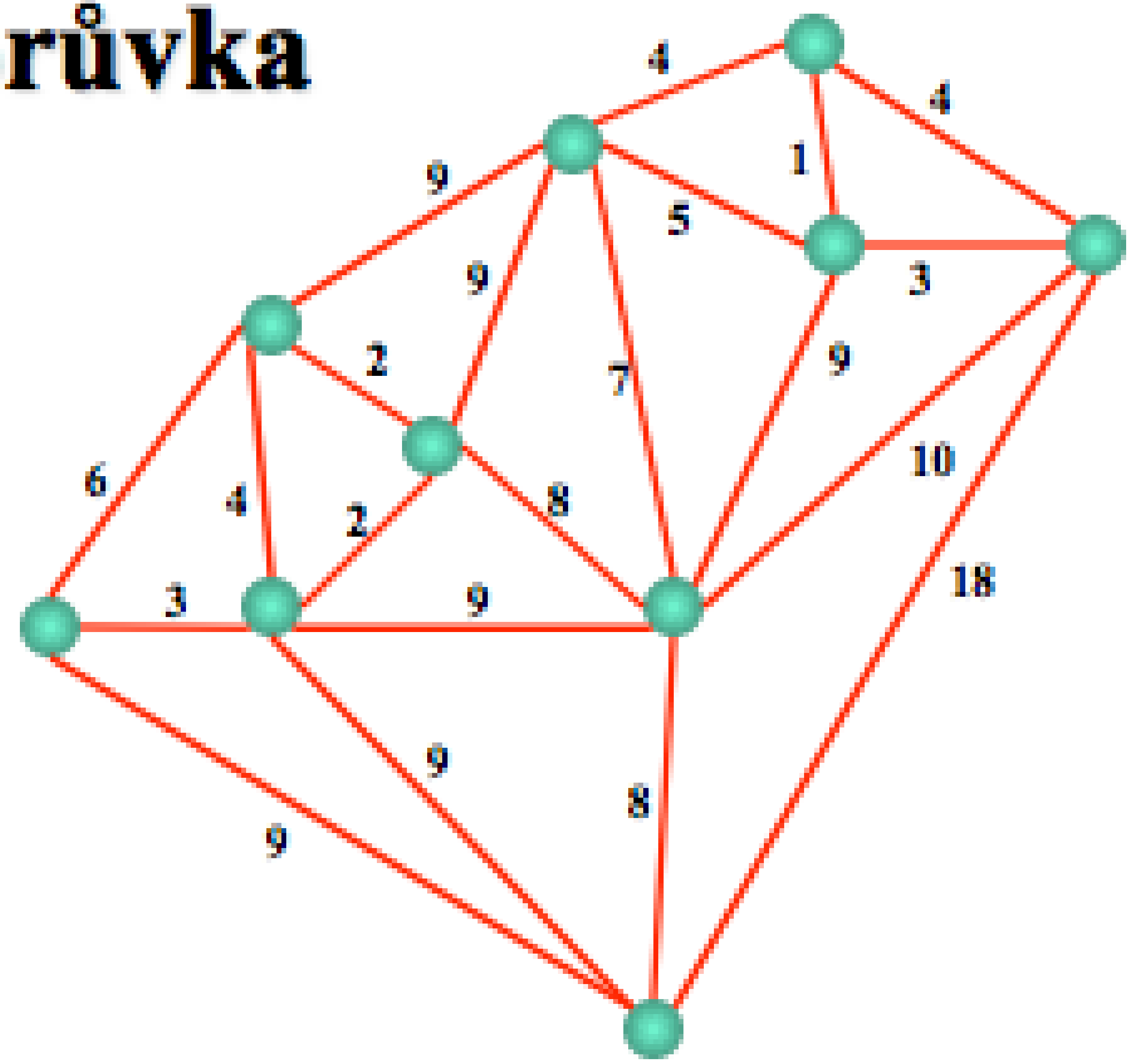
Kruskal’s Algorithm Idea

- Initialize a forest consisting of all nodes
- Pick a (non-selected) minimum weight edge and, if it connects two different trees of the forest, select it, otherwise discard it; repeat
- *Example of greedy algorithm*

Borůvka's Algorithm Idea

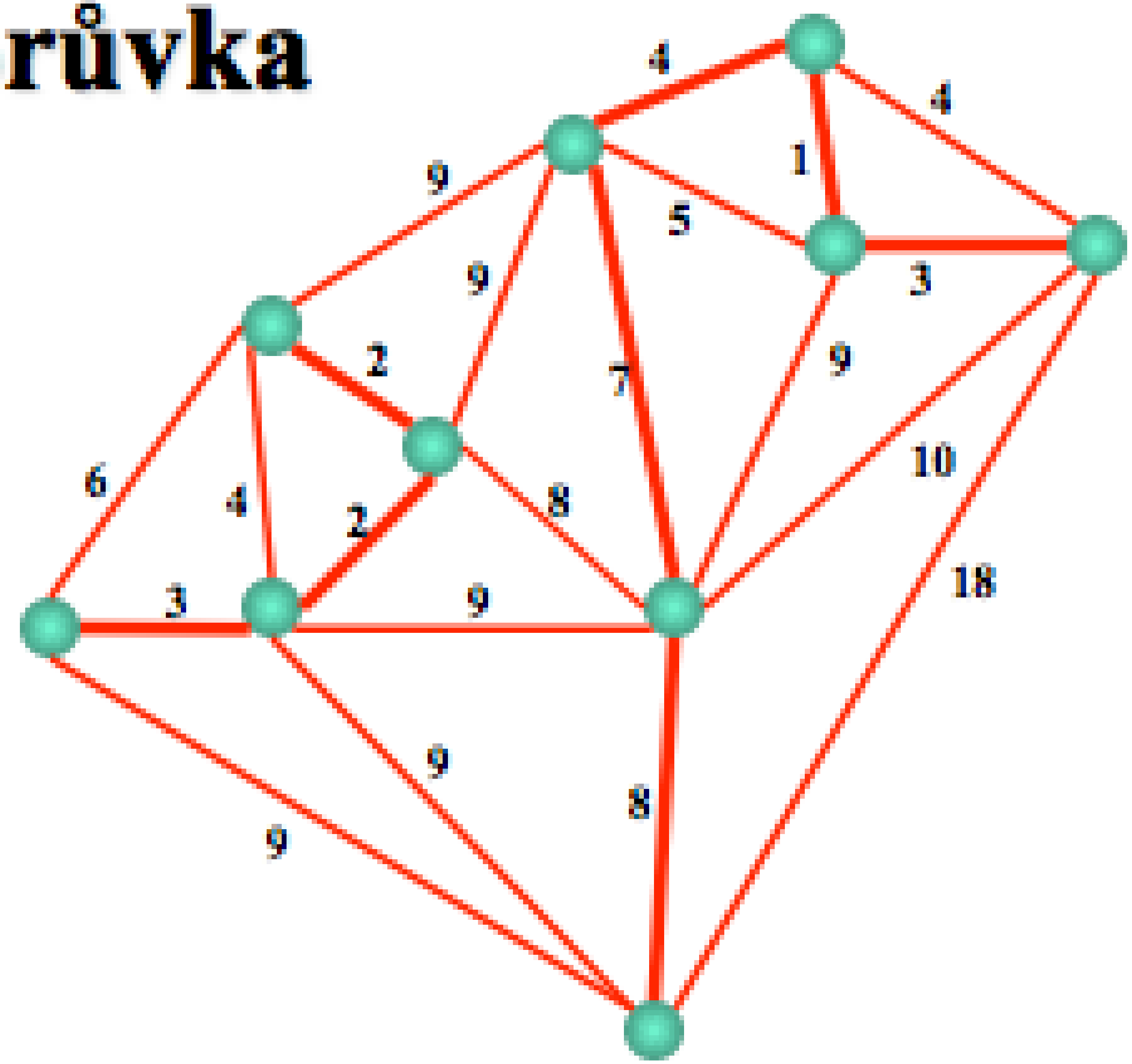
- Often assume every edge has a unique weight.
- Initially, each vertex is considered a separate component.
- The algorithm merges disjoint components as follows; repeating the step until only one component exists.
- In each step, every component is merged with some other using the cheapest outgoing edge of the given component.

Borůvka

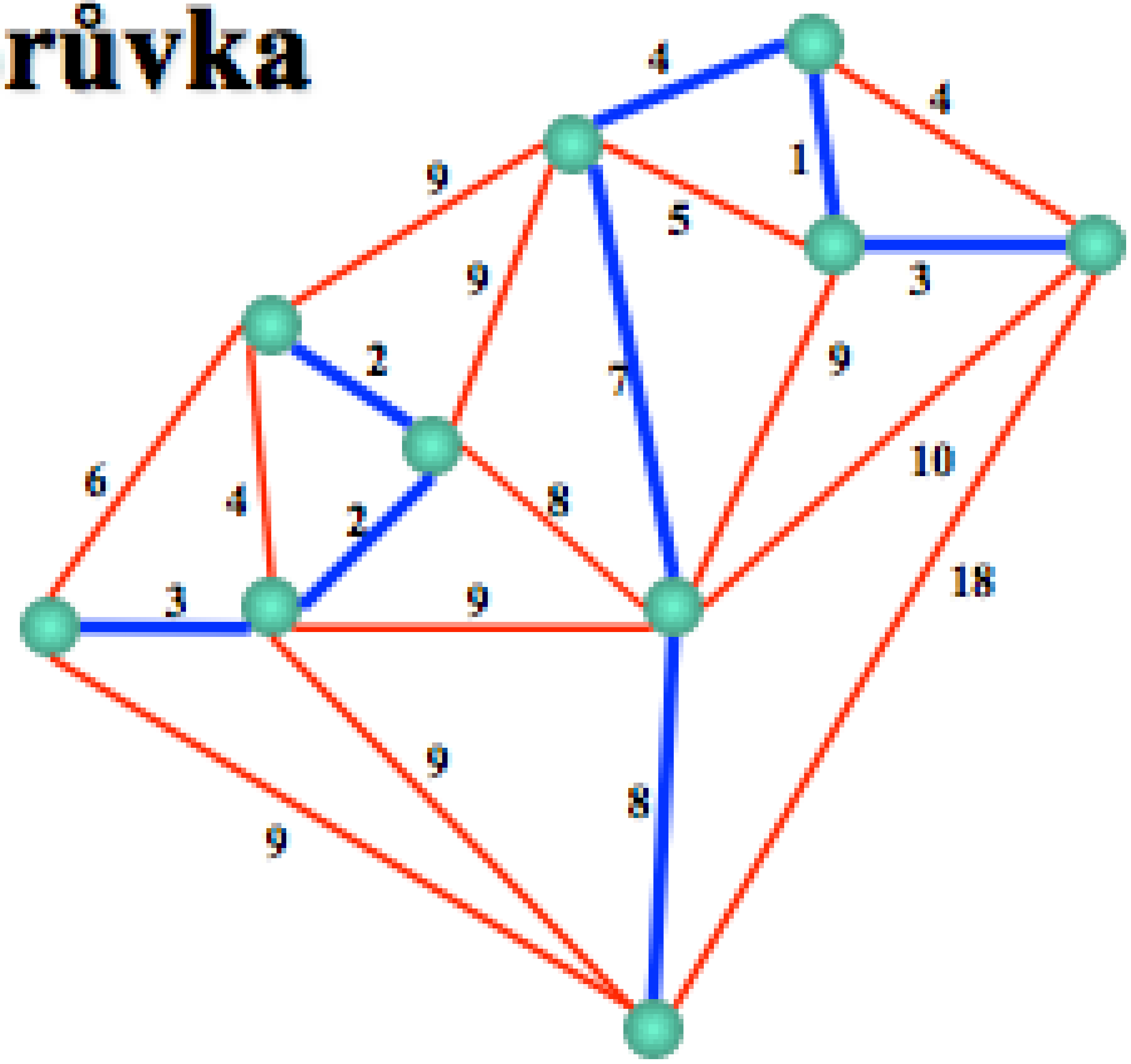


Every vertex is a tree

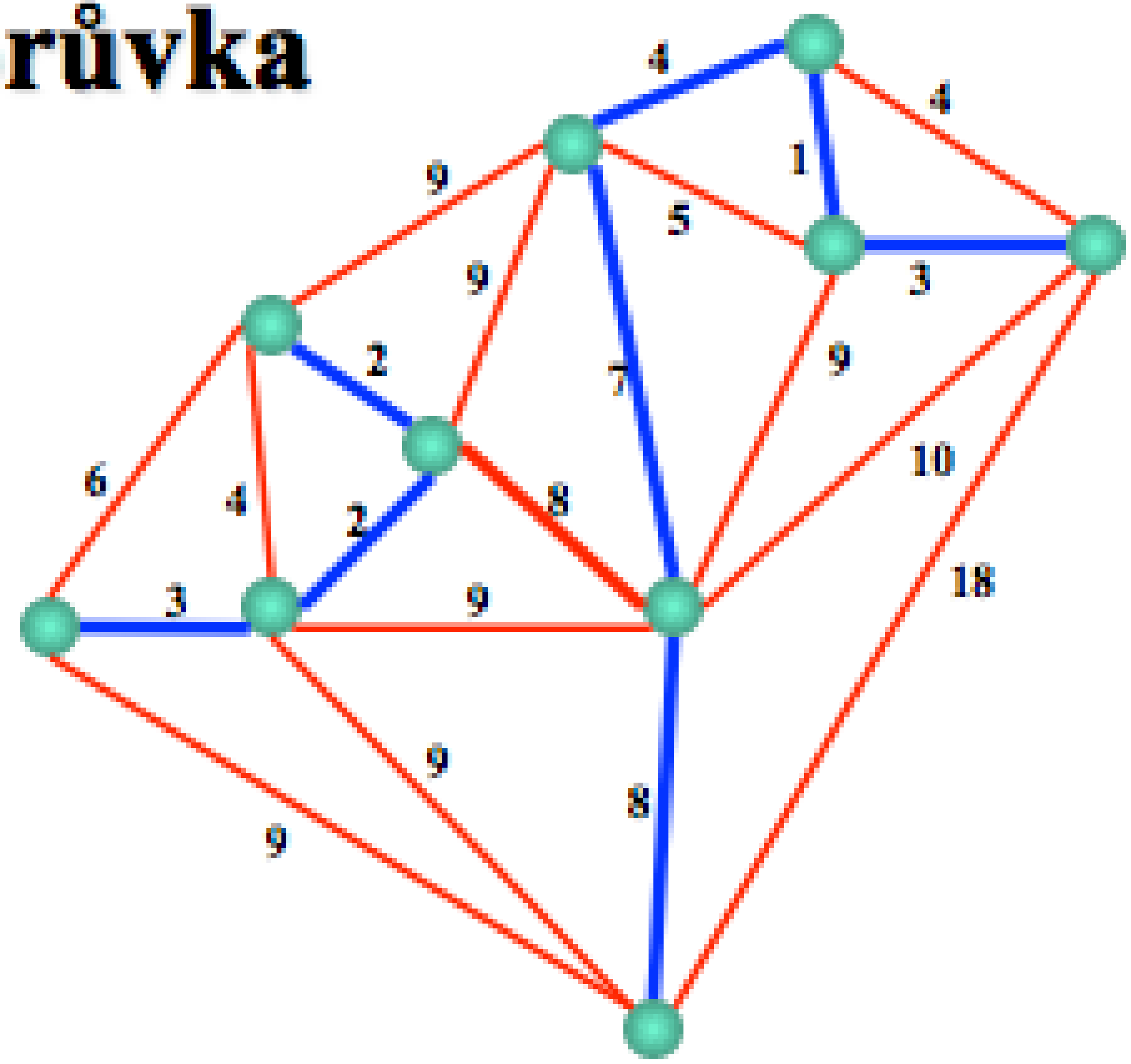
Borůvka



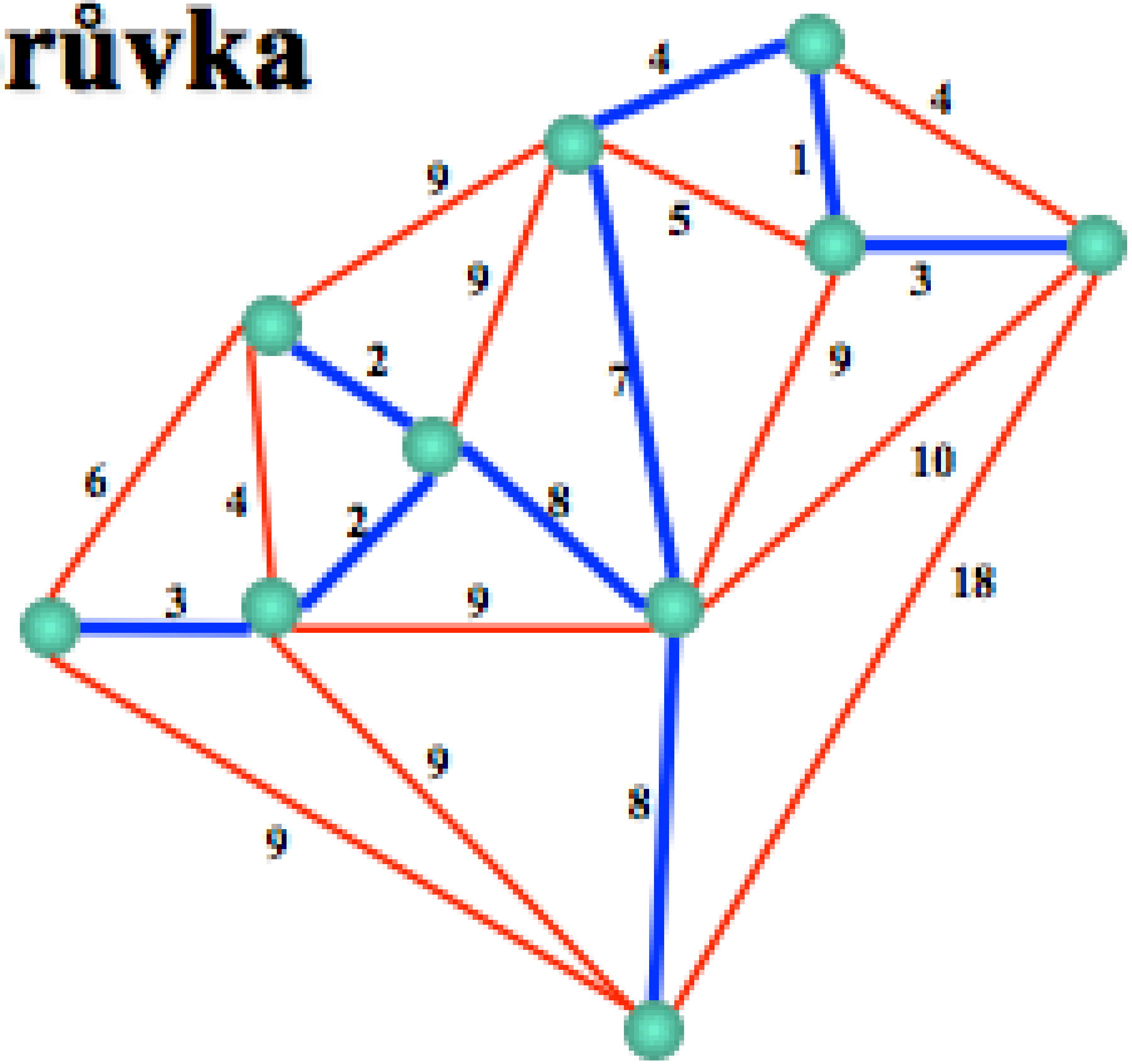
Borůvka



Borůvka



Borůvka



Borůvka's Algorithm Idea

- Initially, each vertex is considered a separate component.
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To come

- Why do these algorithm ideas work (and produce correct MSTs)?
- How do we implement these algorithms efficiently? What are good data structures?