CSC 226

Algorithms and Data Structures: II

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ECS 516

Fundamental Principles of Counting

The Rule of Sum

If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of $m + n$ ways.

Fundamental Principles of Counting

The Rule of Product

If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks can be performed simultaneously, then performing both tasks can be accomplished in any one of $m \cdot n$ ways.

Example 1

I have 5 distinct red cards and 4 distinct black cards

a) How many ways can I choose 1 card?
b) How many ways can I choose 1 red card then 1 black card?
c) How many ways can I choose 2 cards?

Fundamental Principles of Counting

Permutation

Application of the Rule of Products when counting linear arrangements of distinct objects.

Example 2

I have 5 distinct red cards and 4 distinct black cards

a) How many arrangements of the red cards?
b) How many arrangements of the black cards?
c) How many ways of arranging all the red followed by all the black cards?
d) How many arrangements of all the cards?

Example 3

Consider a full standard deck of 52 distinct cards

a) How many arrangements of a standard deck of cards are there? That is, what is the total number of possible shuffles?

b) How many ways can I arrange 5 cards from the deck? That is, how many permutations of 5 cards from 52?

Fundamental Principles of Counting

Permutations

In general, the number of permutations of size r from n distinct objects, where $0 \leq r \leq n$, is given by

$$P(n,r) = \frac{n!}{(n-r)!}$$

- Note: $P(n,0) = \frac{n!}{n!} = 1$ and $P(n,n) = \frac{n!}{0!} = n!$

Example 3 Revisited

Consider a full standard deck of 52 distinct cards

c) How many distinct poker hands exist? That is, how many ways can you pull 5 cards from 52 where order doesn't matter?

Fundamental Principles of Counting

Combinations

In general, the number of combinations of r objects from n distinct objects, where $0 \leq r \leq n$, is given by

$$\binom{n}{r} = C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!\,(n-r)!}$$

- Note: $C(n,0) = \frac{n!}{0!n!} = 1$ and $C(n,n) = \frac{n!}{n!0!} = 1$

Poker Hand Rankings

Example 3 Revisited

Consider a full standard deck of 52 distinct cards

d) How many royal flushes exist?

e) How many straight flushes?

f) 4 of a kind?

g) Full house?

h) …

Fundamental Principles of Counting

The Binomial Theorem

If x and y are variables and n a positive integer, then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- Proof: Consider $(x+y)^n = \underbrace{(x+y)\cdots(x+y)}_{n\ times}$.

  For any $0 \leq k \leq n$, the number of combinations of k x's is $\binom{n}{k}$.

Example 4

What is the coefficient of x^5y^2 in the expansion of $(2x-3y)^7$?

Answer:

$$\binom{7}{5}(2)^5(-3)^2 = 21 \cdot 32 \cdot 9 = 6048$$

Example 5

7 people go for lunch. Each person has an option of one of a cheeseburger (c), hot dog (h), taco (t), or fish sandwich (f). How many possible orders are there?

- Possibilities:
  1. ccccccc
  2. chhttff
  3. hhhffff
  4. …

Fundamental Principles of Counting

Combinations with Repetition

In general, taking n distinct objects, with repetition, taken r at a time can be done in

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!\,(n-1)!}$$

ways.

Example 6

A donut shop has 20 distinct types of donuts with at least 12 of each type in the store. How many ways can you select a dozen (12) donuts?

Example 7

Determine all the integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 7$$

where $x_i \geq 0$ for all $i = 1,2,3,4$.

The Pigeonhole Principle

If m pigeons occupy n pigeonholes and $m > n$, then at least one pigeonhole has two or more pigeons roosting in it.

Examples

1. If I draw 14 cards from a standard deck of 52, will there be a pair?
2. Given a text file with 500,000 words of size 4 or less, is it possible that the words are distinct?
3. While on a 4-week vacation I will drink at least 1 beer a day but no more than 40 total. Will there be a span of consecutive days in which I drink exactly 15 beers?

Posets

Cartesian Product

For sets A and B, the Cartesian product (or cross product) of A and B is

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Relation

For sets A and B, any subset of $A \times B$ is called a relation from A to B.

Posets

Partial Order

A relation $\Re$ on a set A is called a <u>partial order</u> if $\Re$ is

i. Reflexive: For all $x \in A, (x, x) \in \Re$

ii. Antisymmetric: For all $x, y \in A$, if $(x, y), (y, x) \in \Re$, then $x = y$.

iii. Transitive: For all $x, y, z \in A$, if $(x, y), (y, z) \in \Re$, then $(x, z) \in \Re$.

Posets

Partially Ordered Set

Let A be a set and $\Re$ a relation on A. The pair $(A, \Re)$ is called a <u>partially ordered set</u> (or <u>poset</u>) if $\Re$ on A is a partial order.

Hasse Diagram

If $\Re$ is a partial order on A, we construct a <u>Hasse diagram</u> for $\Re$ on A by connecting x "up" to y if and only if $(x, y) \in \Re$ and there are no other $z \in A$ such that $(x, z) \in \Re$ or $(z, y) \in \Re$.

Examples

4. Let $A = \{1,2,3,4\}$. Define $\Re$ on A by

$$\Re = \{(x, y): x, y \in A \text{ and } x|y\}$$

Draw a relation diagram and a Hasse diagram for $\Re$.

5. Consider the power set, $\mathrm{P}(A)$, where $A = \{1,2,3\}$. Draw the Hasse diagram to illustrate the subset relation.

Posets

Total Order

If $(A, \mathfrak{R})$ is a poset, it is a total order if for all $x, y \in A$, either $(x, y) \in \mathfrak{R}$ or $(y, x) \in \mathfrak{R}$.