CSC 226

Algorithms and Data Structures: II

Planar Graphs

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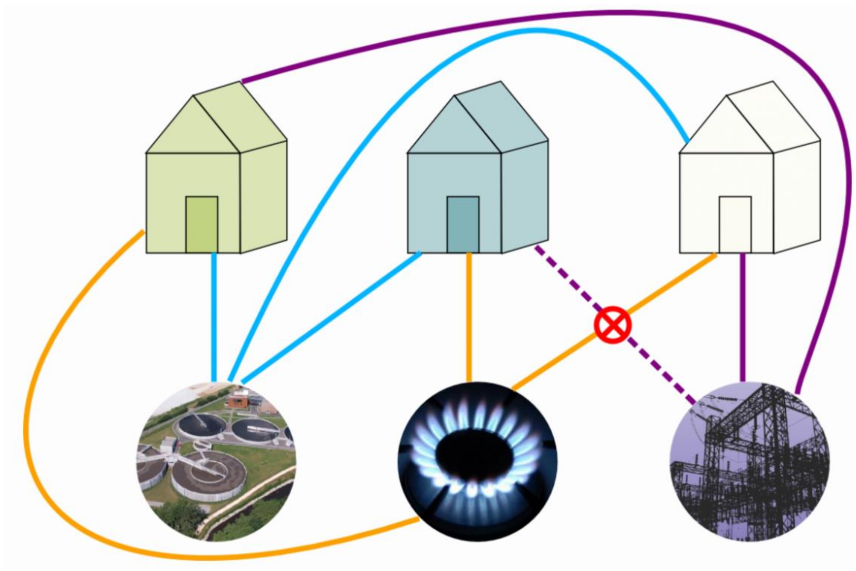
ECS 466

Planar Graphs



Graphs

- **Definition** – A ***simple undirected graph*** $G = (V, E)$, consists of a finite, nonempty set of ***vertices***, V, and $\mathrm{E} \subseteq V \times V$, where for each $u, v \in V$, with $u \neq v$, there is at most one edge $\{u, v\} \in E$.

- **Definition** – For each $v \in V$, the ***degree of v***, denoted $deg(v)$, is the number of edges in G incident upon v.

- **Theorem** – If $G = (V, E)$ is a graph with n vertices and m edges then,

$$\sum_{v \in V} \deg(v) = 2m$$

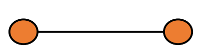
- **Proof** – Every edge contributes to the degree of exactly 2 distinct vertices.

Complete Graphs

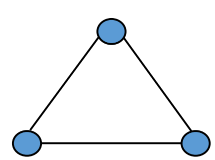
- **Definition** – Let V be a set of n vertices. The ***complete graph on V***, denoted K_n, is a simple undirected graph such that for all $u, v \in V$, where $u \neq v$, there is an edge $\{u, v\}$.

- **Examples** – Draw K_n for $n = 1,2,3,4$, and 5.

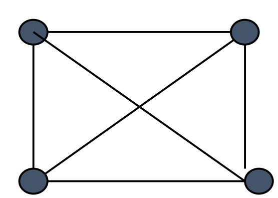
K_1



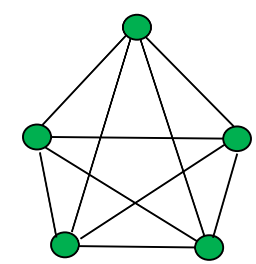
K_2



K_3



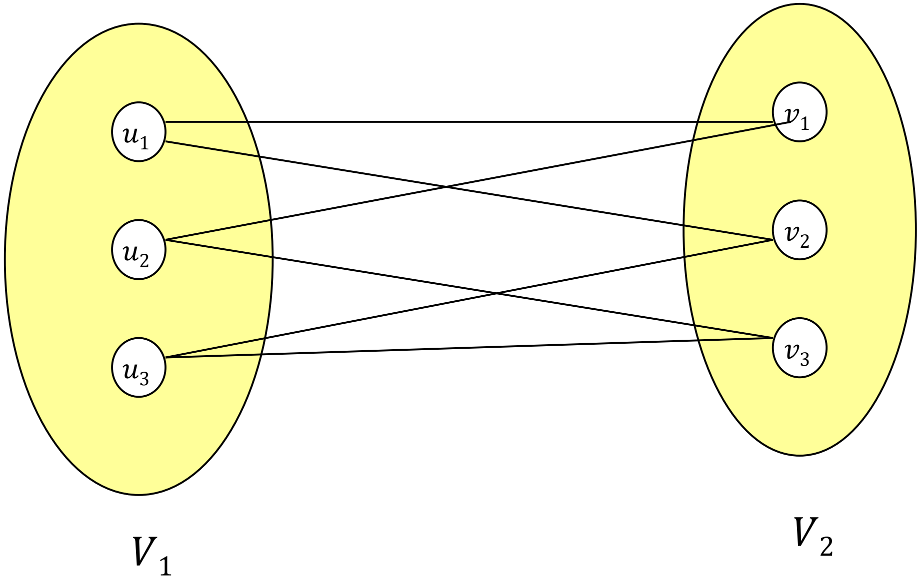
K_4



K_5

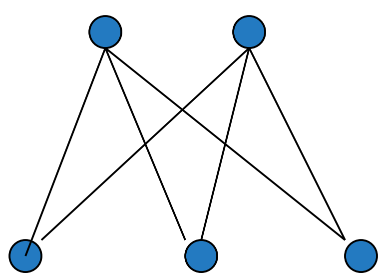
Bipartite Graphs

- **Definition** – A graph $G = (V, E)$ is ***bipartite*** if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ and for every edge $\{u, v\} \in E$, then $u \in V_1$ and $v \in V_2$.

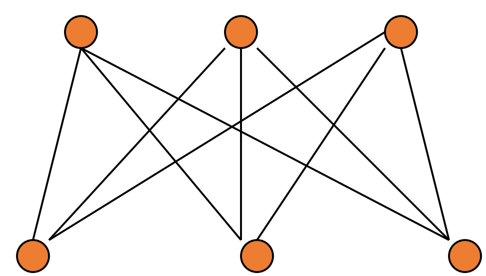


Complete Bipartite Graphs

- **Definition** – Let $G = (V, E)$ be a bipartite graph with $V = V_1 \cup V_2$ such that $|V_1| = m$ and $|V_2| = n$. Then, G is ***complete bipartite***, denoted $K_{m,n}$, if for each $u \in V_1$ and $v \in V_2$, there exists an edge $\{u, v\} \in E$.
- **Examples**



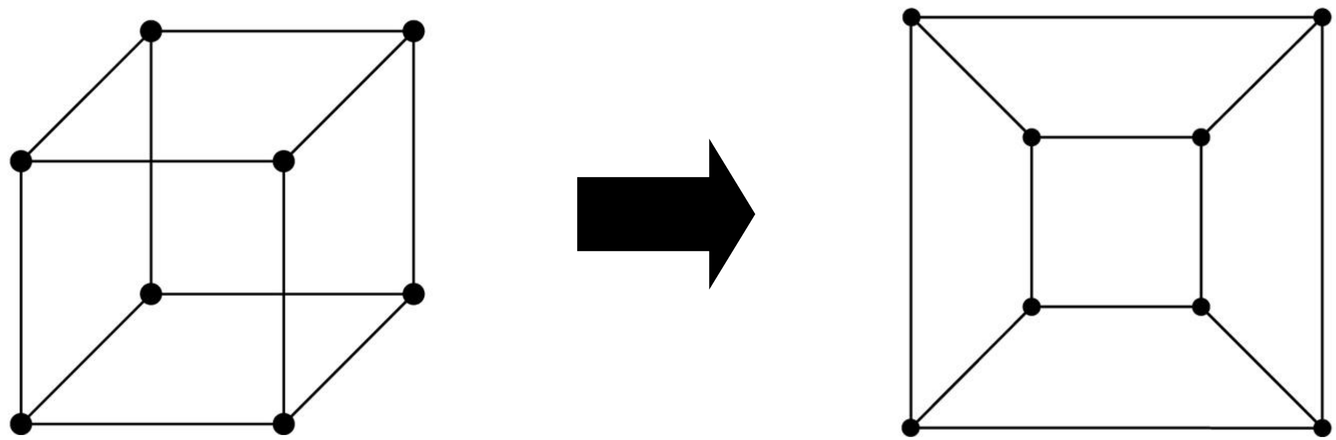
$K_{2,3}$



$K_{3,3}$

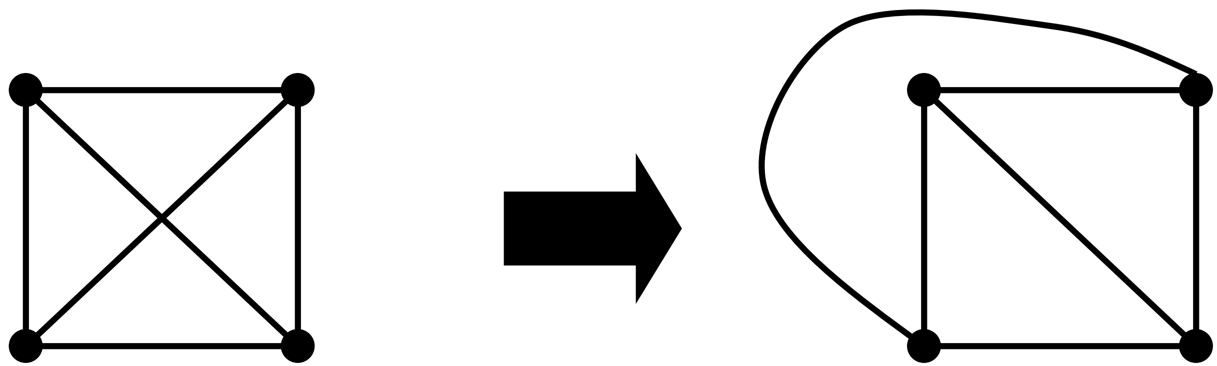
Planar Graphs

- **Definition** – A graph $G = (V, E)$ is called ***planar*** if G can be drawn in the plane with its edges intersecting only at vertices of G.
- Also called an ***embedding of G in the plane***.

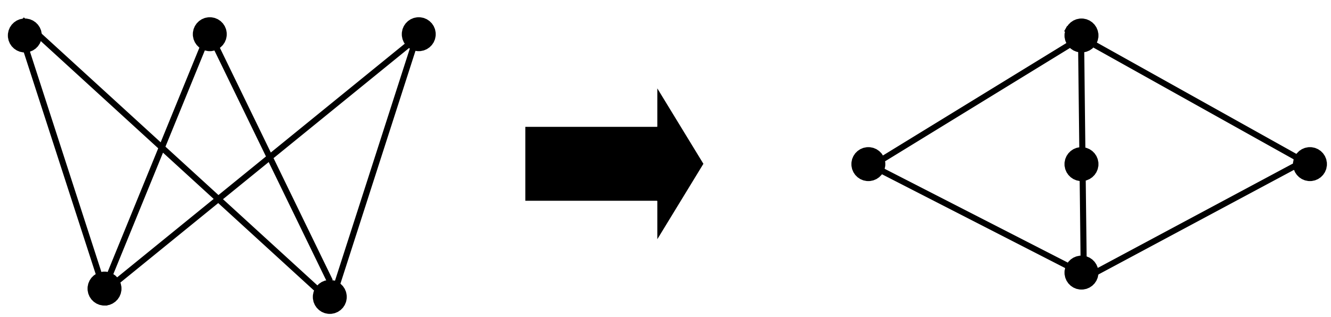


Planar Graphs?

- **Examples** – Look at K_1 to K_5. Which are planar?
- $\boldsymbol{K_4}$:

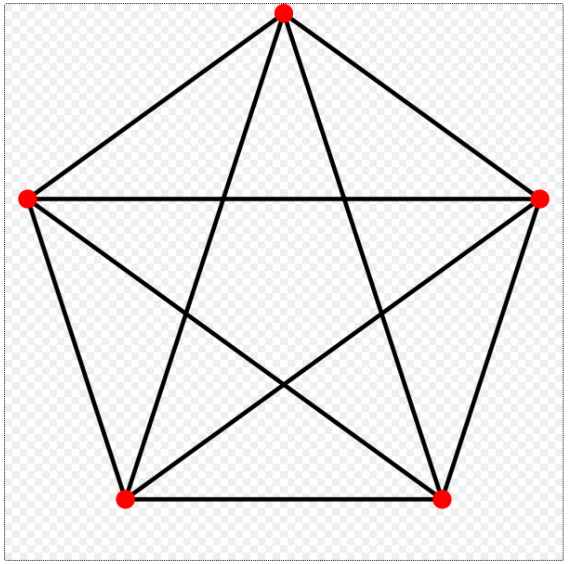


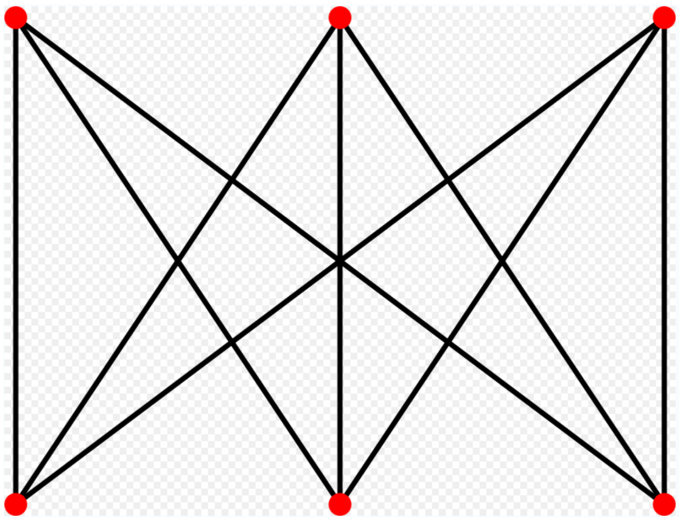
- **Examples** – Look at $K_{m,n}$ for $m, n = 1{,}2{,}3$. Which are planar?
- $\boldsymbol{K_{2,3}}$:



Nonplanar Graphs

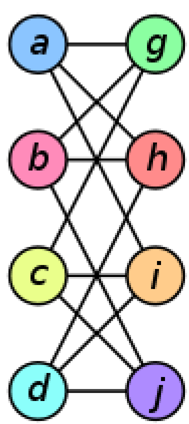
- K_5 and $K_{3,3}$ are not planar.



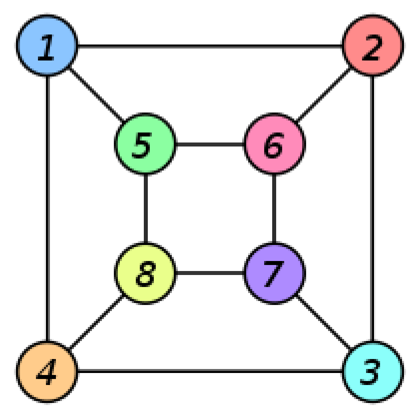


Subgraphs and Isomorphisms

- **Definition** – A graph $H = (V_H, E_H)$ is a ***subgraph*** of $G = (V_G, E_G)$ if and only if $V_H \subseteq V_G$ and $E_H \subseteq E_G$.
- **Definition** – Two graphs G_1 and G_2 are ***isomorphic*** if there exists a one-to-one function f from G_1 to G_2 such that $\{u, v\} \in E_1$ if and only if $\{f(u), f(v)\} \in E_2$.



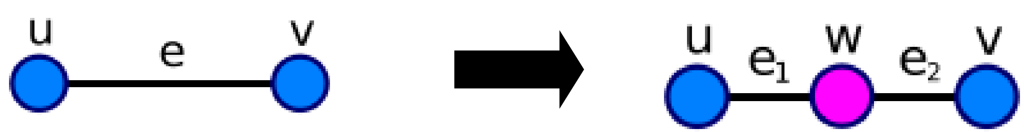
G_1



G_2

Elementary Subdivisions and Homeomorphisms

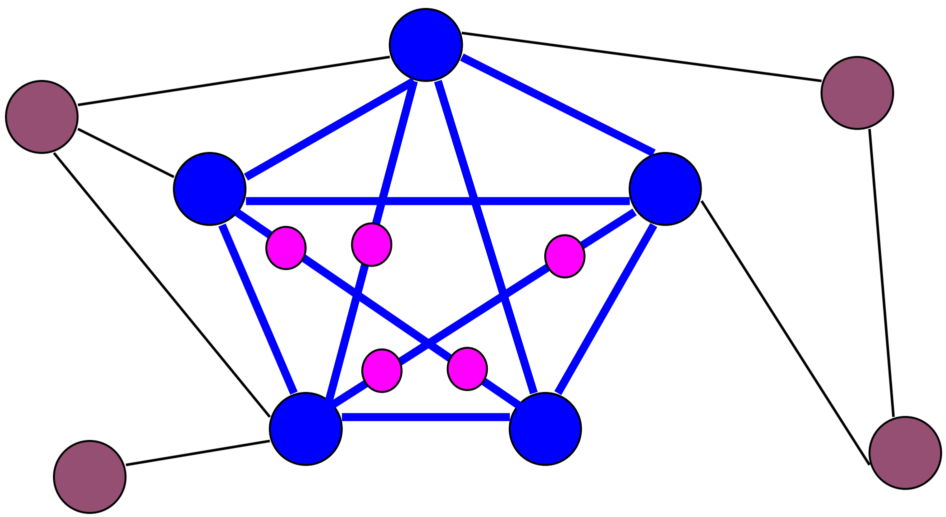
- **Definition** – An ***elementary subdivision*** of a graph $G = (V, E)$ is obtained by repeated operations of removing an edge $\{u, v\}$ from G and adding two new edges $\{u, w\}$ and $\{w, v\}$ such that $w \notin V$.



- **Definition** – Two graphs G_1 and G_2 are ***homeomorphic*** if they are *isomorphic* or they can both be obtained from the same graph H by a sequence of *elementary subdivisions*.

Kuratowski's Theorem

- **Theorem** – A graph is ***nonplanar*** if and only if it contains a ***subgraph*** that is ***homeomorphic*** to K_5 or $K_{3,3}$.



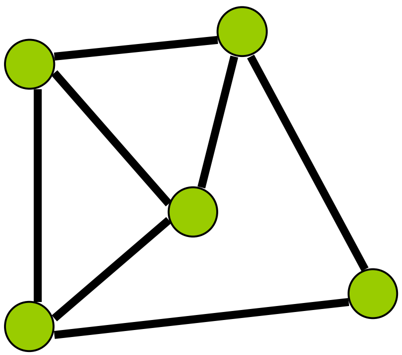
Euler's Formula

- **Theorem** – Let $G = (V, E)$ be a connected planar graph with $|V| = n$ and $|E| = m$. Let r be the number of *regions* (*faces*) determined by the embedding of G in the plane (including the unbounded face). Then,

$$n - m + r = 2$$

- **Proof** – By induction on m, the number of edges, where $m = n - 1$ is the base case.

- **Ex:**



$$\boldsymbol{n = 5}$$
$$\boldsymbol{m = 7}$$
$$\boldsymbol{r = 4}$$

Corollaries to Euler's Formula

- **Corollary** – Let $G = (V, E)$ be a planar graph with $|V| = n$, $|E| = m > 2$, and r faces. Then, $3r \leq 2m$ and $m \leq 3n - 6$.
- **Ex** – Show that K_5 is not planar.
- **Corollary** – Let $G = (V, E)$ be a planar graph with $|V| = n > 2$, $|E| = m$, and r faces. If all the cycles in G have length ≥ 4, then $4r \leq 2m$ and $m \leq 2n - 4$.
- **Ex** – Show that $K_{3,3}$ is not planar.
- **Corollary** – Let $G = (V, E)$ be a planar graph, then G has a vertex of degree at most 5. That is, there exists a $v \in V$ such that $\deg(v) \leq 5$.

Linear time algorithms for embedding:

Hopcroft & Tarjan, '74

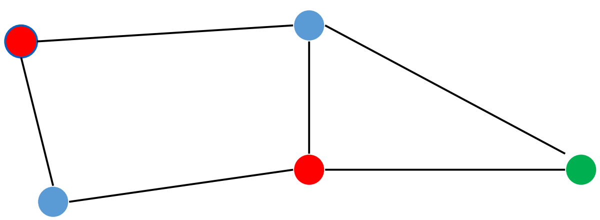
Booth and Lueker, '76

Boyer & Myrvold, '01

OPEN: Find a really simple $O(n)$ or maybe $O(n \log n)$ algorithm.

Graph Colouring Problem

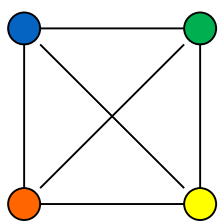
- **Definition** – A ***colouring*** of a graph G assigns a colour to each vertex (or edge, or face, etc.) of G, with the restriction that two adjacent vertices never have the same colour.
  - Can 2-colour a graph using DFS or BFS if 2-colourable.



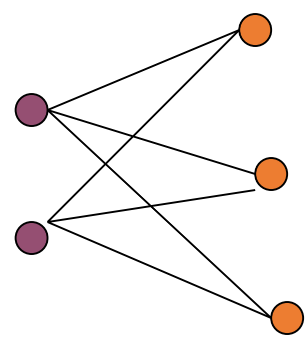
- **Definition** – The ***chromatic number*** of G, $\chi(G)$, is the minimum number of colours needed to colour G.

Basic Theorems

- **Theorem** – The chromatic number of K_n is n.
- **Theorem** – The chromatic number of $K_{m,n}$ is 2.



K_4

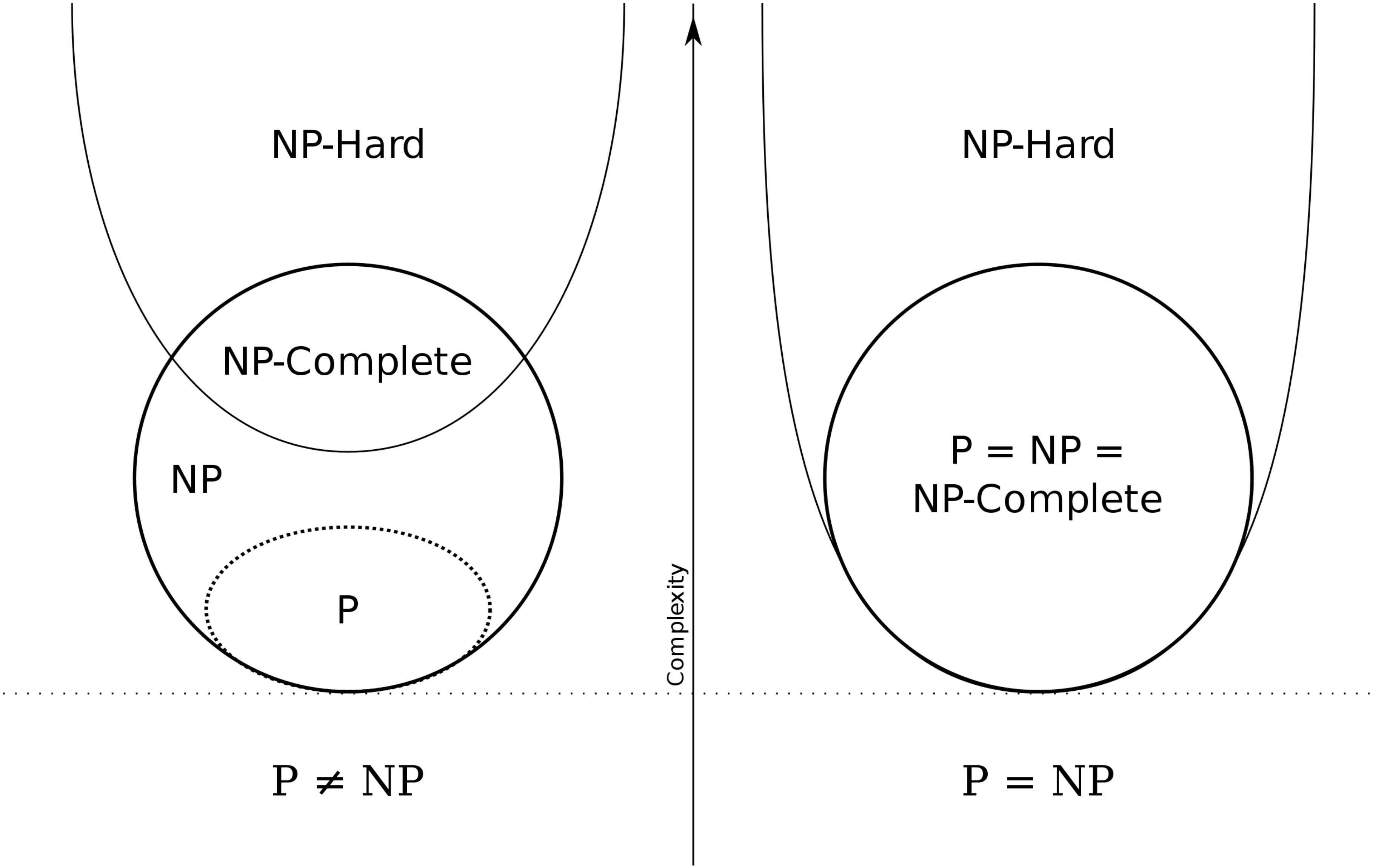


$K_{2,3}$

- **Theorem** – If all $v \in V$ of graph $G = (V, E)$ have degree $\leq D$, then G can be coloured with $D + 1$ colours.

P versus NP

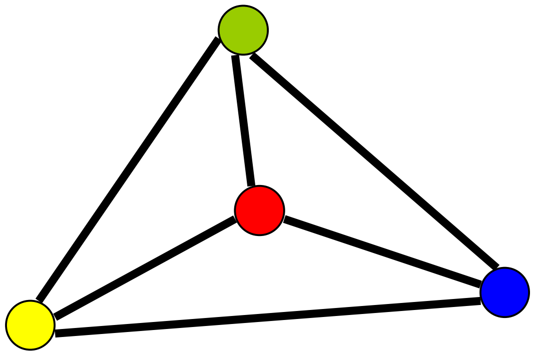
- **P** = Set of problems solvable in deterministic polynomial time.
- **NP** = Set of problems that are solvable in nondeterministic polynomial time.
  - i.e. Verifiable in deterministic polynomial time
- **NP-Hard** = Set of problems, H, such that for every $C \in NP$, C can be reduced to H in polynomial time.
- **NP-Complete** = Set of problems that are NP-Hard and are verifiable in polynomial time.
  - i.e. Intersection of NP and NP-Hard.


NP-Hard
NP-Complete
NP
P
P ≠ NP
Complexity
NP-Hard
P = NP =
NP-Complete
P = NP

Computing Colours

- Unfortunately, in general, graph colouring problems require exponential time to solve.
  - Either calculating a chromatic number or determining if a graph is k-colourable for some k.
- Most versions of these questions are either NP-complete or NP-Hard.
- But, Planar graphs are much easier to colour.

The Four-Colour Theorem



- According to the four-color-theorem of Appel and Haken the chromatic number of a planar graph is at most 4.
- This bound cannot be

  improved.
- Proof is quite complicated.
- Actually, depends on a program to exhaustively search multiple cases
  - Originally 1,936 cases (1976)
- Much simpler proof – 6-Colour Planar Graphs
  - Planar graphs have at least one vertex with degree < 6
  - If all $v \in V$ have degree $\leq D$, then G can be coloured with $D + 1$ colours.