CSC 226

Algorithms and Data Structures: II
Union Find

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ECS 466

Kruskal’s algorithm requires an efficient way of testing whether an edge creates a cycle with the edges already selected.

- The union-find data structure helps do this.

Kruskal's Algorithm

```
Algorithm Kruskal
Input: a weighted connected graph G = (V, E)
Output: an MST T for G
Data structure: Disjoint sets (lists or union-find) DS;
    sorted weights priority queue A; and tree T

for each v ∈ V  do C(v) ← DS.insert(v) end // one cluster per vertex
for each (u,v) ∈ E  do A.insert((u,v)) end // sort edges by weight
T ← ∅
while T has fewer than n-1 edges do
   (u, v) ← A.deleteMin() // edge with smallest weight
   C(u)← DS.findCluster(u);
   C(v)← DS.findCluster(v);
   if C(u) ≠ C(v) then
      add edge (u, v) to T;
      DS.insert(DS.union(C(v), C(u))); // merge two clusters
   end
end
return T
```

Disjoint-Set Data Structure

- Given a set of elements, it is often useful to break them up or partition them into a number of separate, non-overlapping sets
- A ***disjoint-set data structure*** is a data structure that keeps track of such a partitioning
- There are three useful operations on such a data structure:
  - ***Find***: Determine which set a particular element is in. Also useful for determining if two elements are in the same set.
  - ***Union***: Combine or merge two sets into a single set.
  - ***MakeSet***: Creates a set containing only a given element

http://en.wikipedia.org/wiki/Disjoint-set_data_structure

Disjoint Set Data Structure

- The universe consists of n elements, named $1, 2, \ldots, n$
- The ADT is a collection of sets of elements
- Each element is in exactly one set
  - Sets are disjoint
  - To start, each set contains one element
- Each set has a name
  - Which is the name of one of its elements
  - Any name of one of its elements will do

Disjoint Set Operations

- **find(elementName)**
  - Returns the name of the unique set that contains the given element
- **union(setName1, setName2)**
  - Merges two sets and replaces them with one

- **Time complexity analysis**
  - Involves analyzing the *amortized* worst-case running time over a sequence of f find and u union operations

Disjoint Set Implementation I

- Create a linked list for each set and choose the element at the head of the list as the representative
- ***MakeSet*** creates a list of one element
- ***Union*** simply appends two lists, a constant-time operation
- ***Find*** requires linear time (i.e., may search entire list)
- A sequence of m union-find operations takes time $O(mn)$.The amortized time per operation is $O(n)$.

How should we represent the sets?

- Each set is a rooted tree

- Each element of a set corresponds to a node in a tree

- *Canonical element* (= name of set) is the root of the tree

- The textbook has 3, increasingly better, ways of implementing this.
  1. quick-find
  2. quick-union
  3. weighted quick-union

Quick-find [eager approach]

Data structure.

if and only if

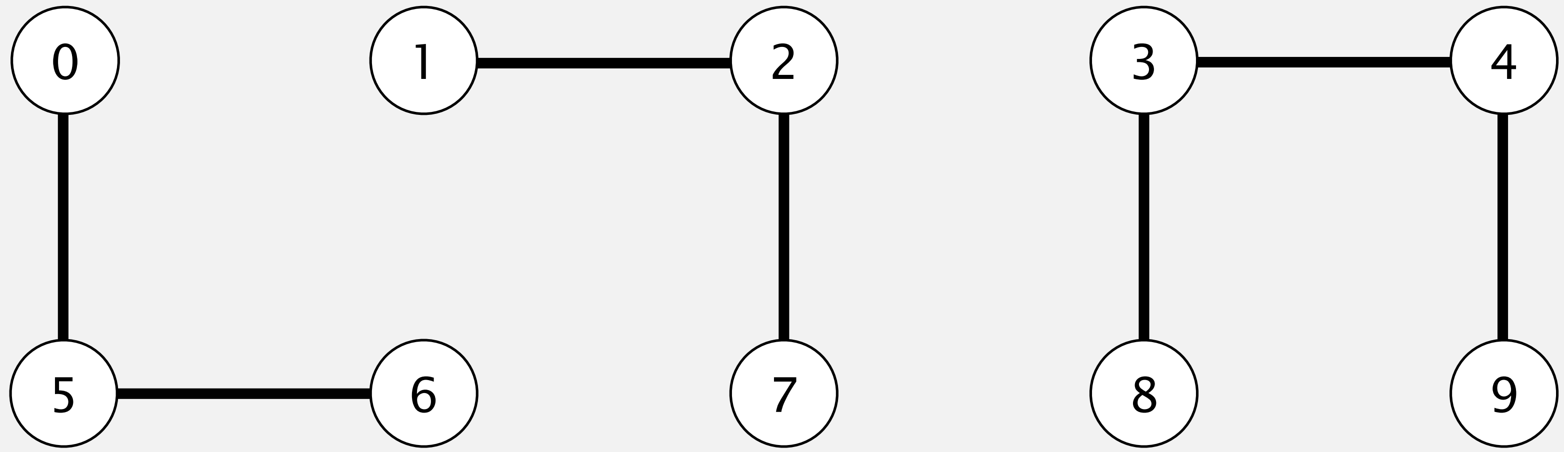
- Integer array `id[]` of length n.
- Interpretation: `id[p]` is the id of the component containing `p`.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8

0, 5 and 6 are connected

1, 2, and 7 are connected

3, 4, 8, and 9 are connected



Quick-find [eager approach]

Data structure.

- Integer array `id[]` of length n.
- Interpretation: `id[p]` is the id of the component containing `p`.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8

Find. What is the id of `p`?

Connected. Do `p` and `q` have the same id?

id[6] = 0; id[1] = 1

6 and 1 are not connected

Union. To merge components containing `p` and `q`, change all entries whose id equals `id[p]` to `id[q]`.

	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

after union of 6 and 1

problem: many values can change

Quick-find is too slow

Cost model. Number of array accesses (for read or write).

algorithm	initialize	union	find	connected
quick-find	N	N	1	1

order of growth of number of array accesses

quadratic

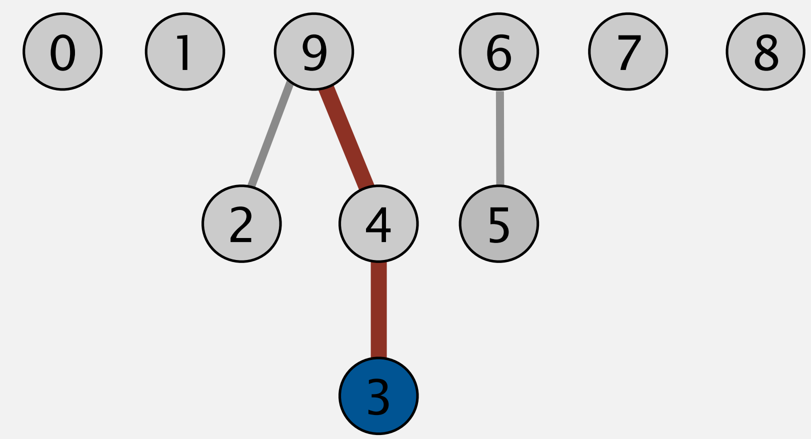
Union is too expensive. It takes N^2 array accesses to process a sequence of N union operations on N objects.

Data structure.

- Integer array `id[]` of length n.
- Interpretation: `id[i]` is parent of `i`.
- Root of `i` is `id[id[id[...id[i]...]]]`.

keep going until it doesn't change
(algorithm ensures no cycles)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



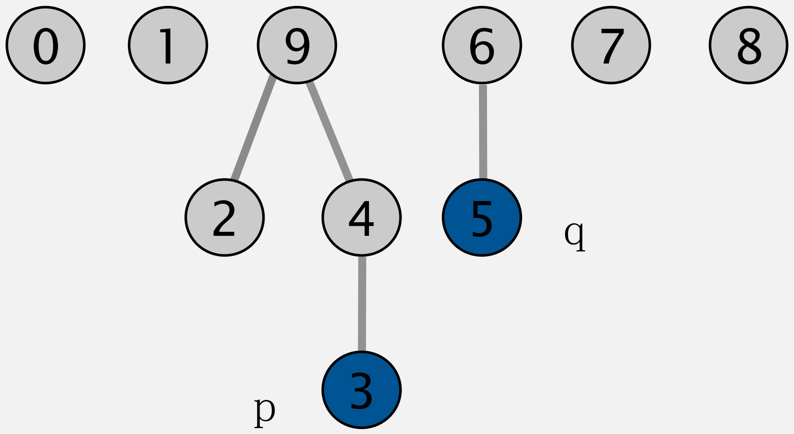
parent of 3 is 4
root of 3 is 9

Quick-union [lazy approach]

Data structure.

- Integer array `id[]` of length n.
- Interpretation: `id[i]` is parent of `i`.
- Root of `i` is `id[id[id[...id[i]...]]]`.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



root of 3 is 9

root of 5 is 6

3 and 5 are not connected

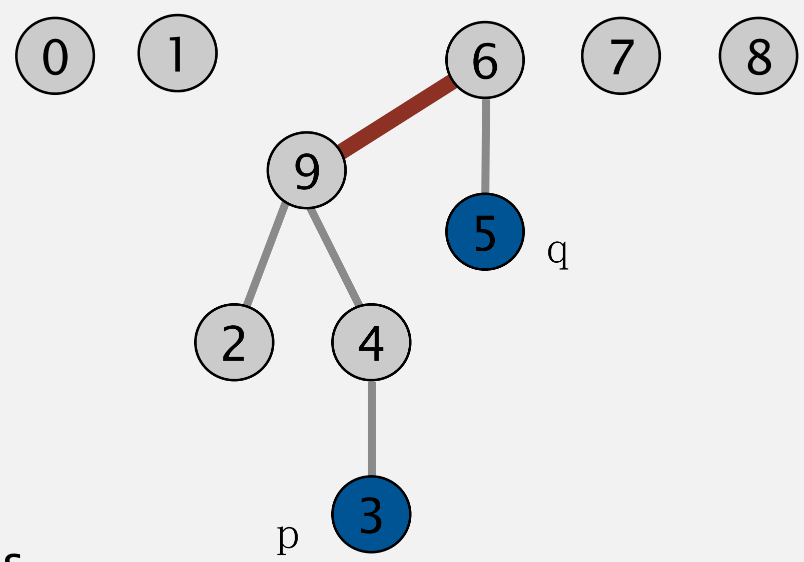
Find. What is the root of `p`?

Connected. Do `p` and `q` have the same root?

Union. To merge components containing `p` and `q`, set the id of `p`'s root to the id of `q`'s root.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	6

only one value changes



Quick-union is also too slow

Cost model. Number of array accesses (for read or write).

algorithm	initialize	union	find	connected
quick-find	N	N	1	1
quick-union	N	N †	N	N

← worst case

† includes cost of finding roots

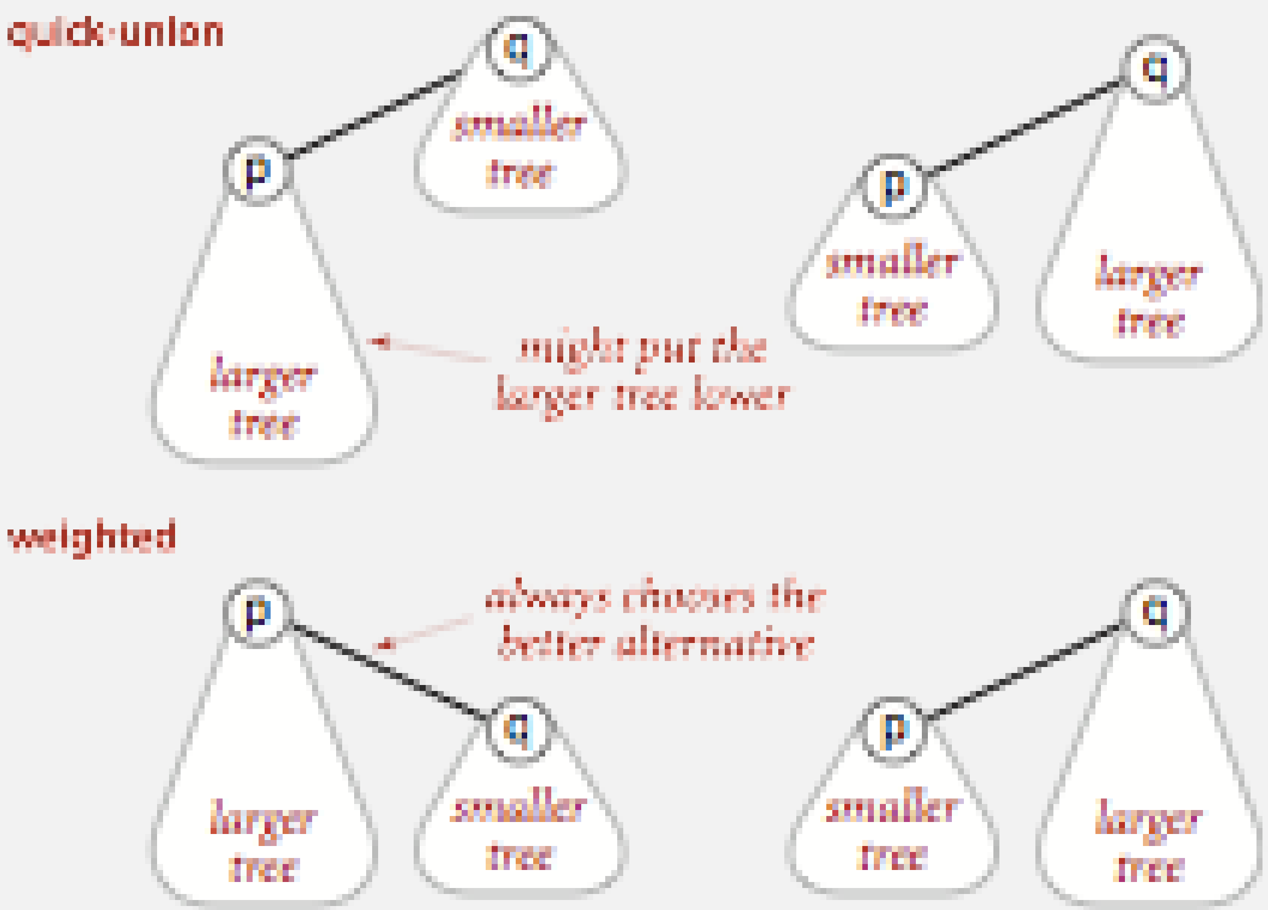
Quick-find defect.
- Union too expensive (N array accesses).

Quick-union defect.
- Trees can get tall.
- Find/connected too expensive (could be N array accesses).

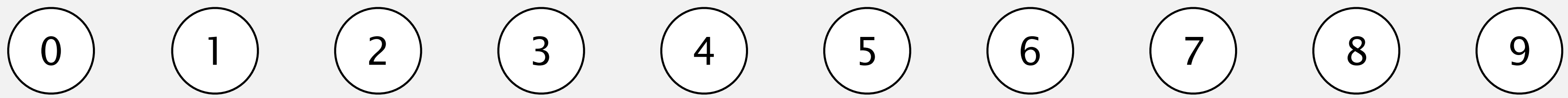
Weighted quick-union.

- Modify quick-union to avoid tall trees.
- Keep track of size of each tree (number of objects).
- Balance by linking root of smaller tree to root of larger tree.

reasonable alternatives:
union by height or "rank"



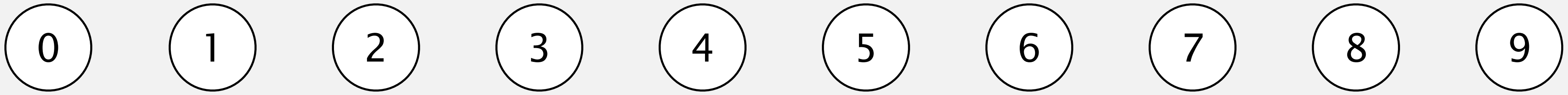
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

Weighted quick-union demo

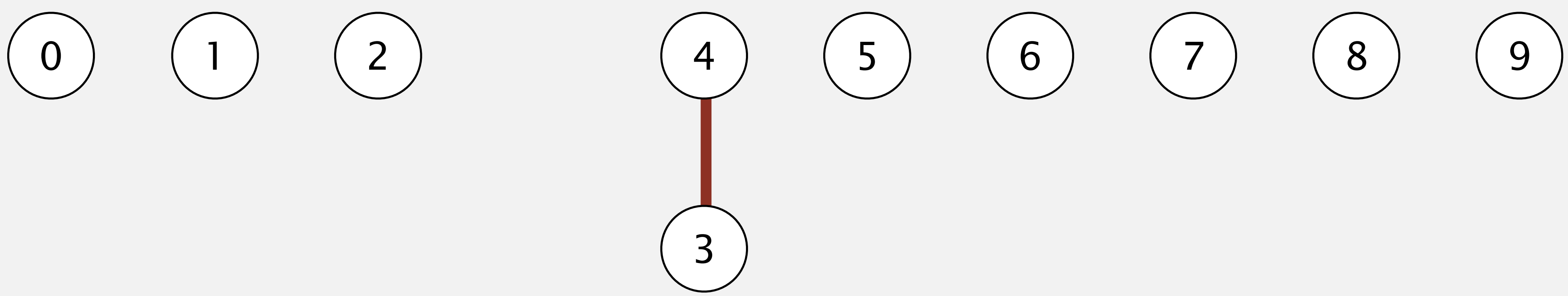
union(4, 3)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

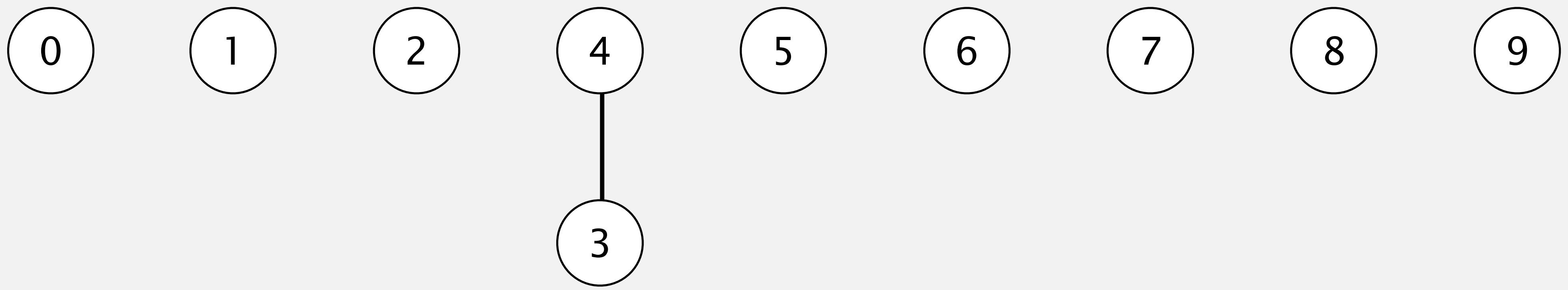
Weighted quick-union demo

union(4, 3)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	5	6	7	8	9

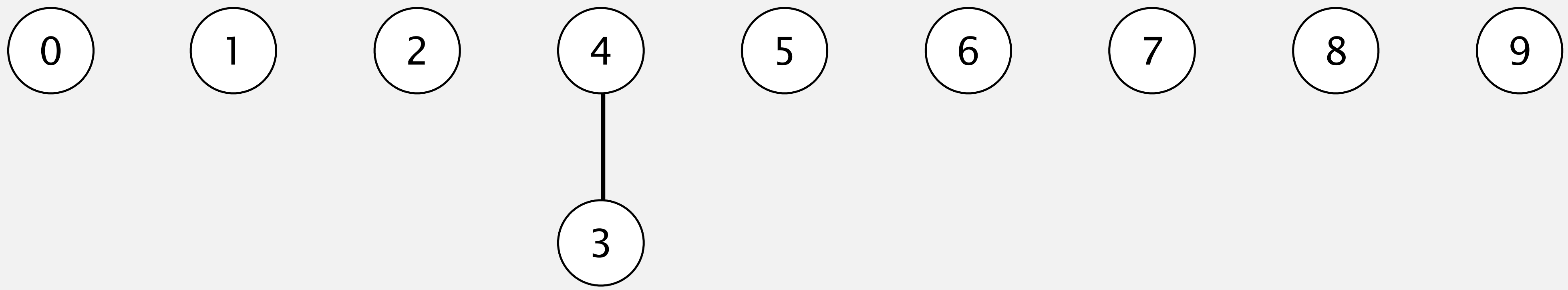
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	5	6	7	8	9

Weighted quick-union demo

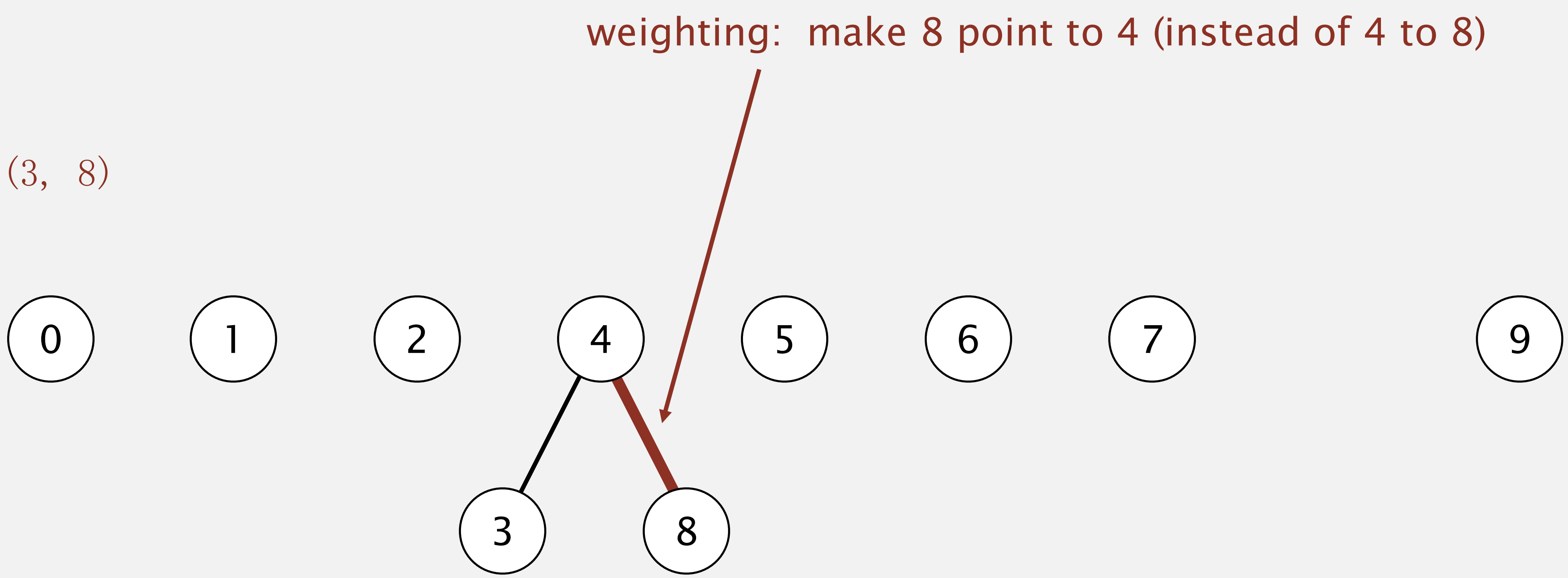
union(3, 8)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	5	6	7	8	9

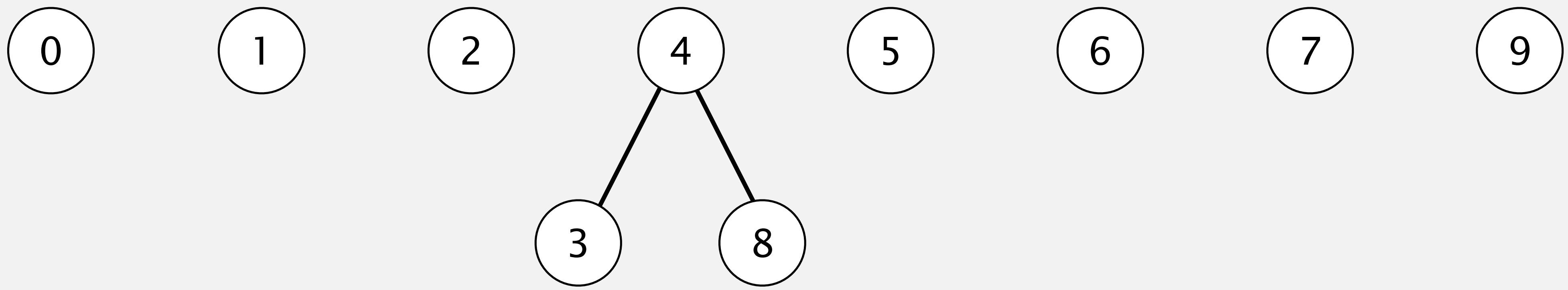
Weighted quick-union demo

union(3, 8)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	5	6	7	4	9

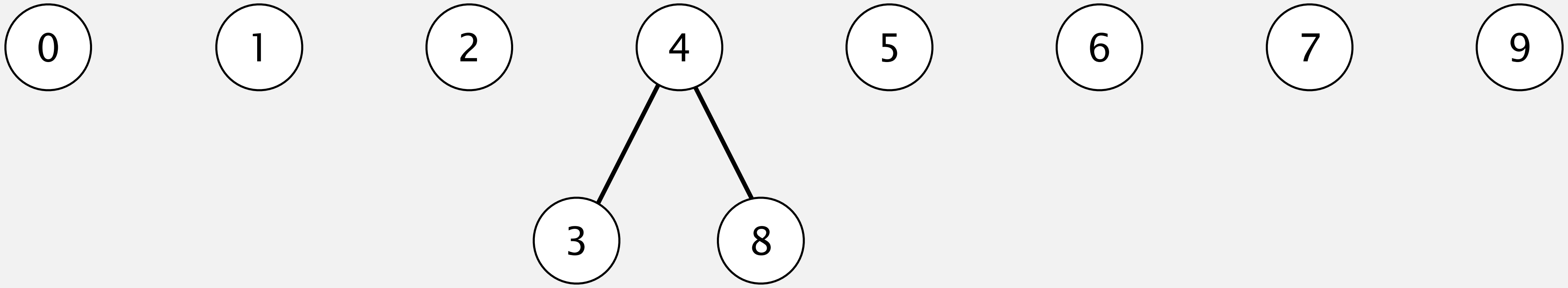
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	5	6	7	4	9

Weighted quick-union demo

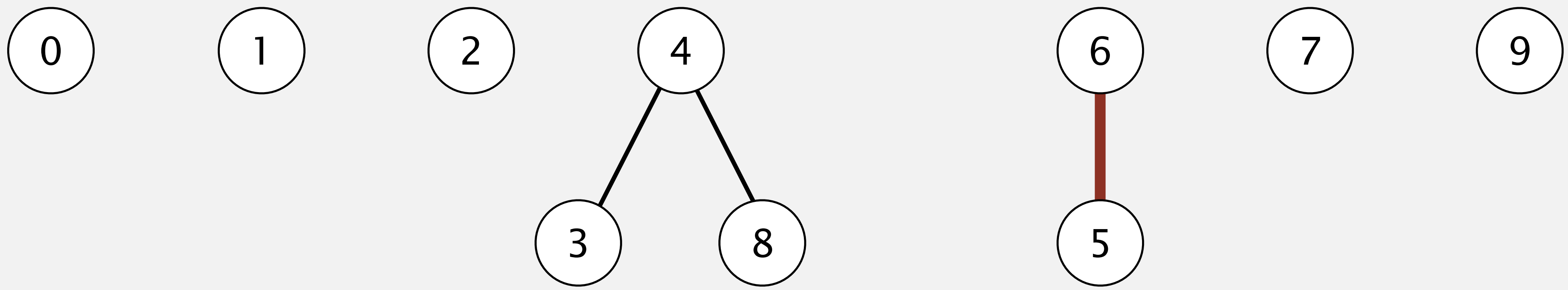
union(6, 5)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	5	6	7	4	9

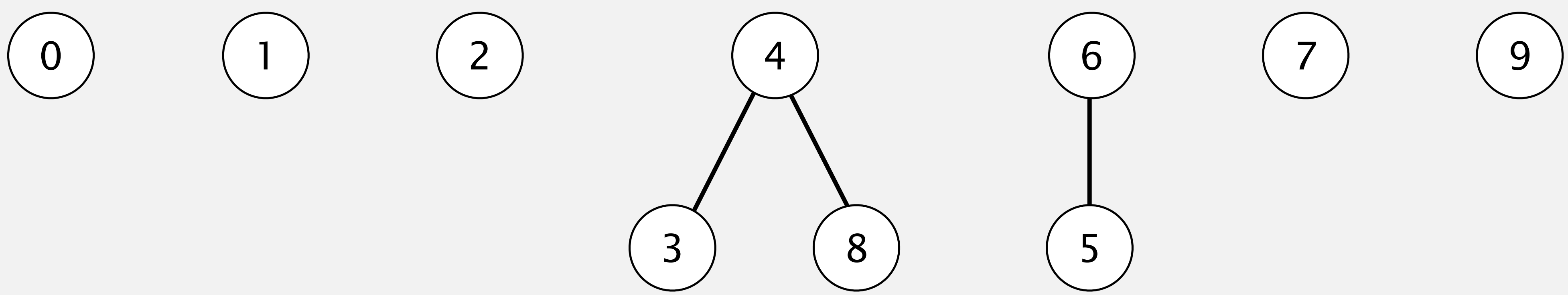
Weighted quick-union demo

union(6, 5)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	6	6	7	4	9

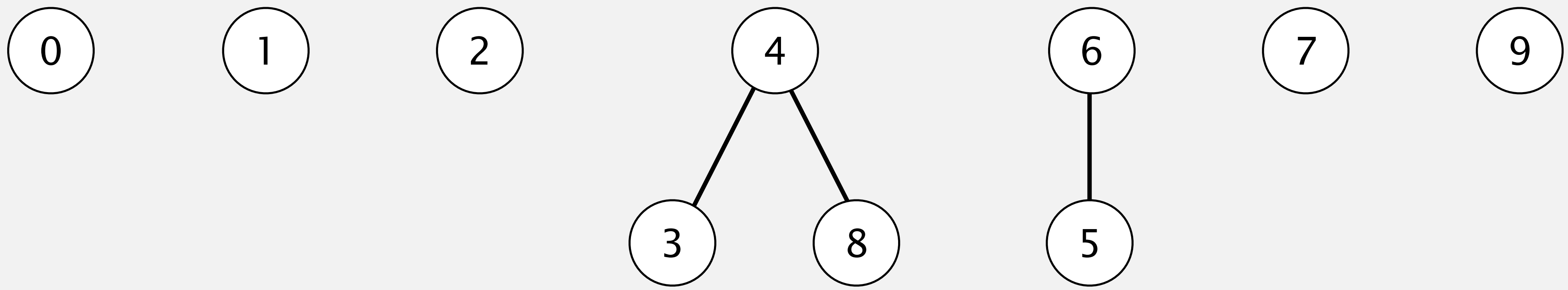
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	6	6	7	4	9

Weighted quick-union demo

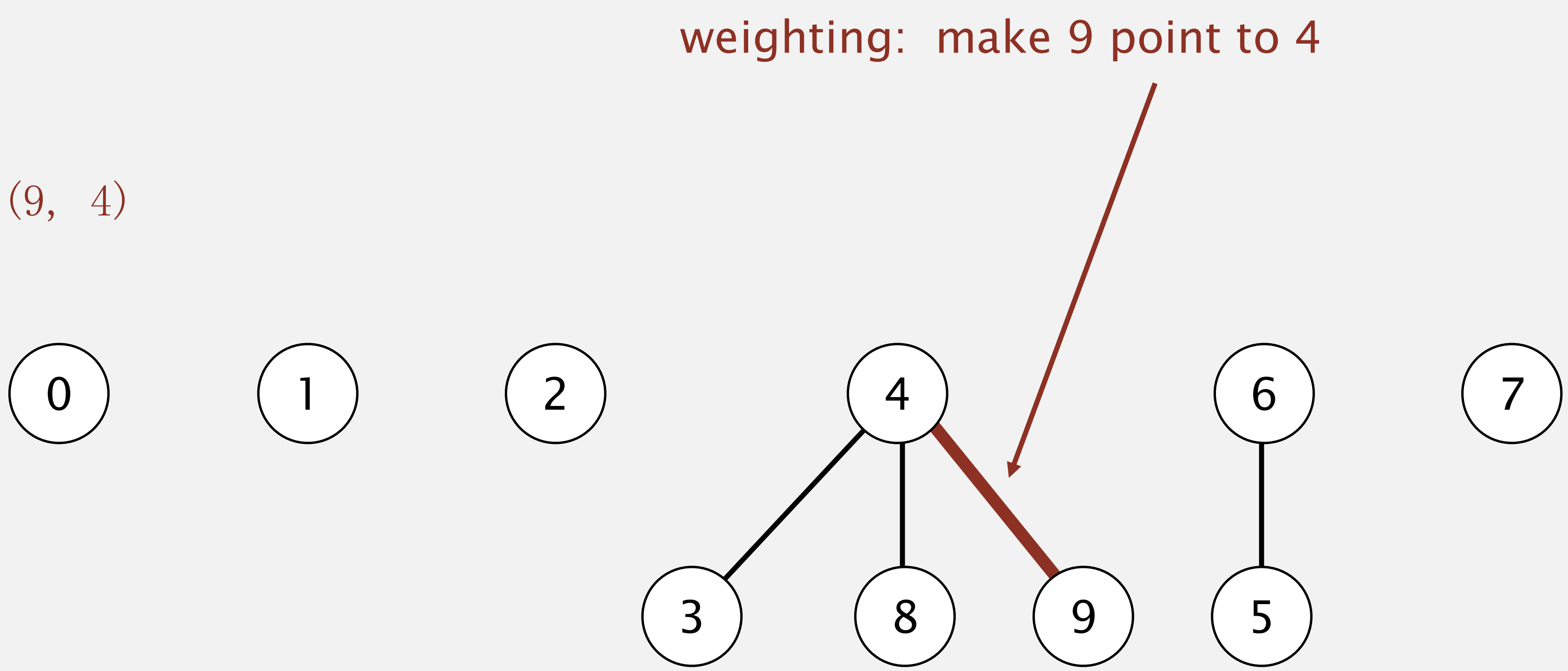
union(9, 4)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	6	6	7	4	9

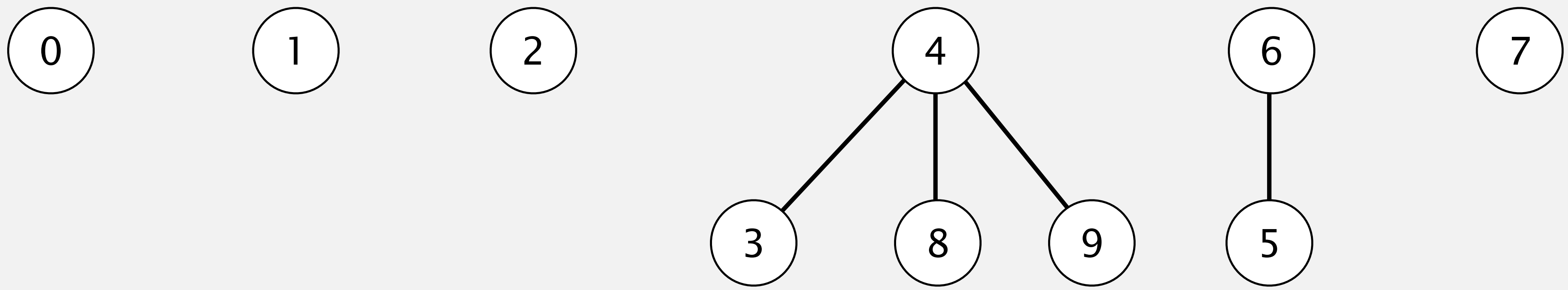
Weighted quick-union demo

union(9, 4)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	6	6	7	4	4

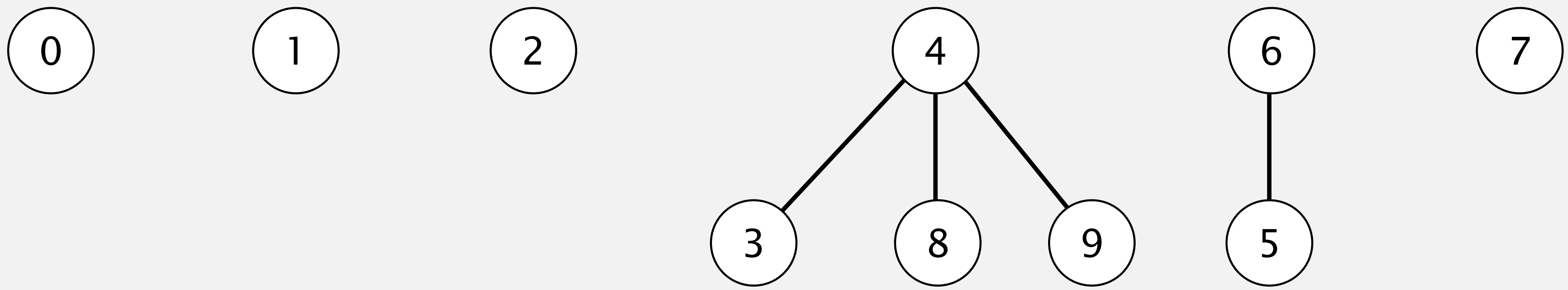
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	6	6	7	4	4

Weighted quick-union demo

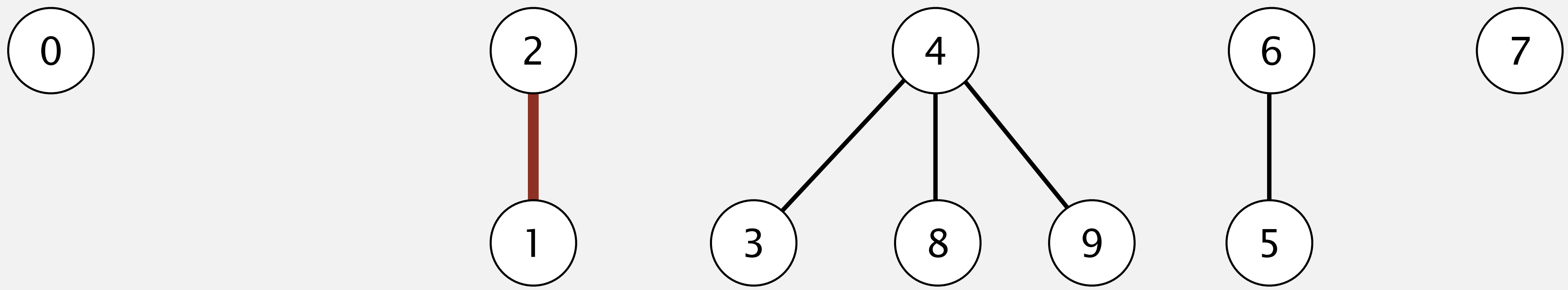
union(2, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	4	4	6	6	7	4	4

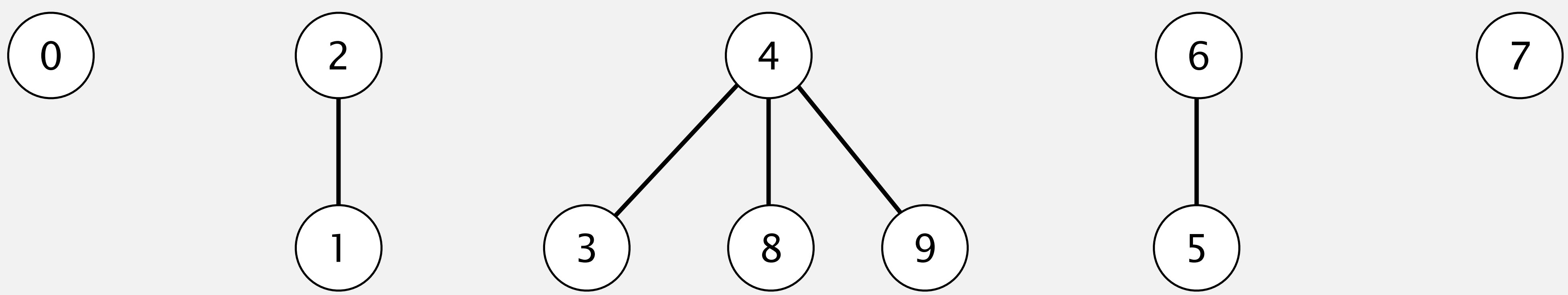
Weighted quick-union demo

union(2, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	0	2	2	4	4	6	6	7	4	4

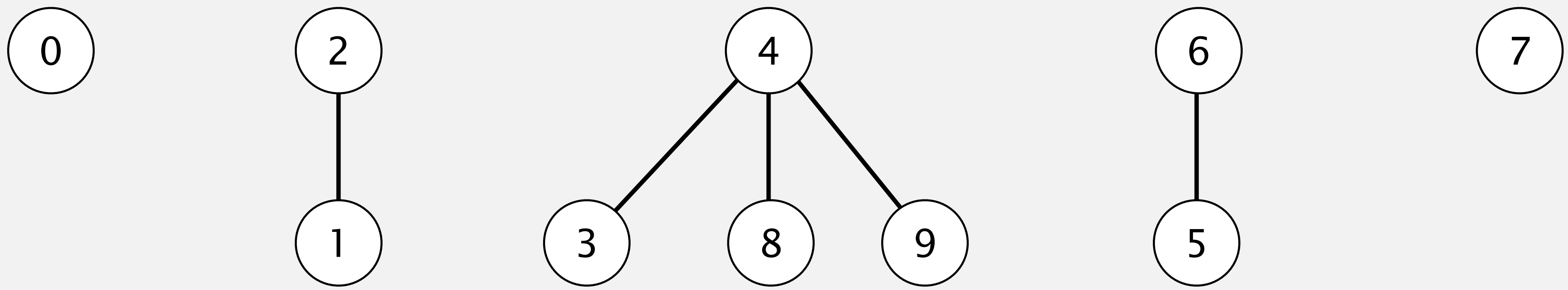
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	2	2	4	4	6	6	7	4	4

Weighted quick-union demo

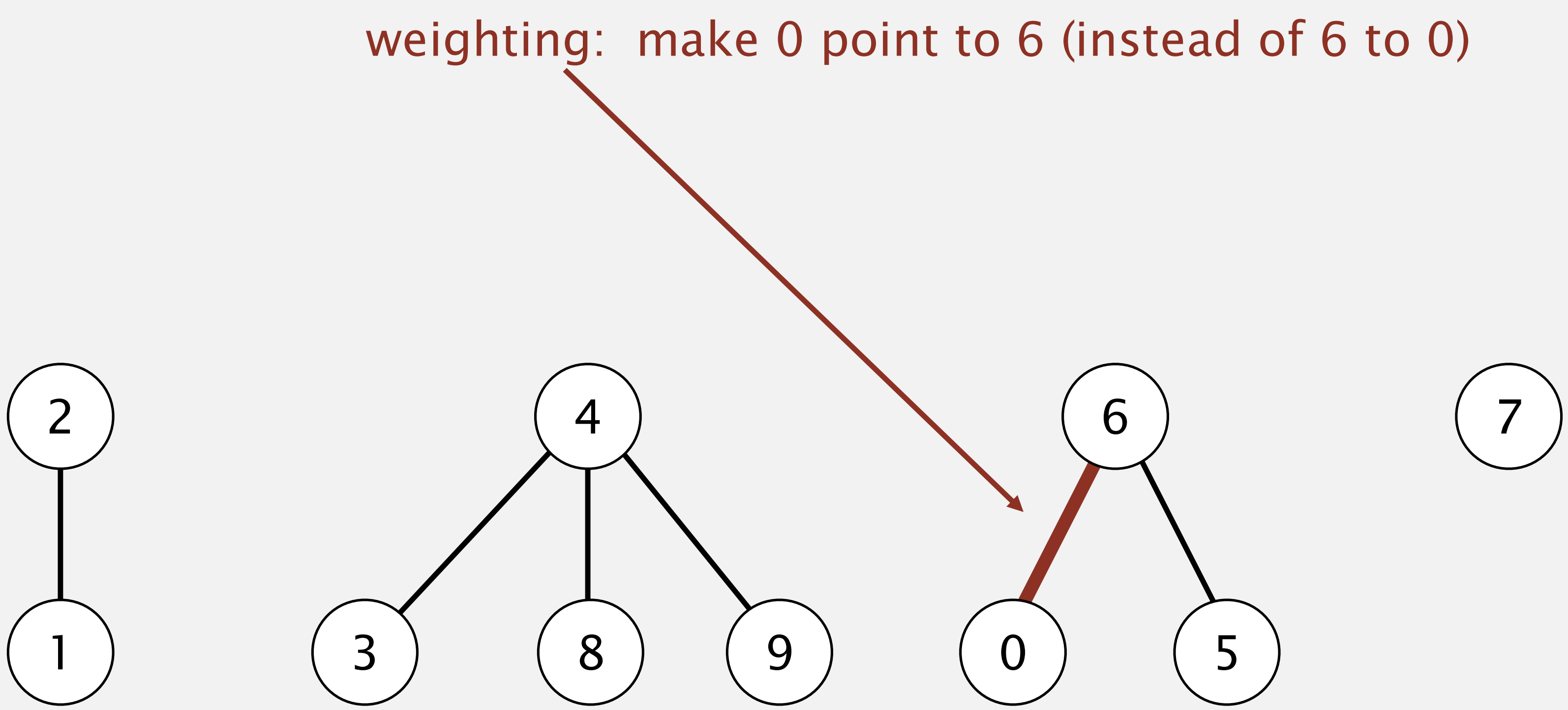
union(5, 0)



	0	1	2	3	4	5	6	7	8	9
id[]	0	2	2	4	4	6	6	7	4	4

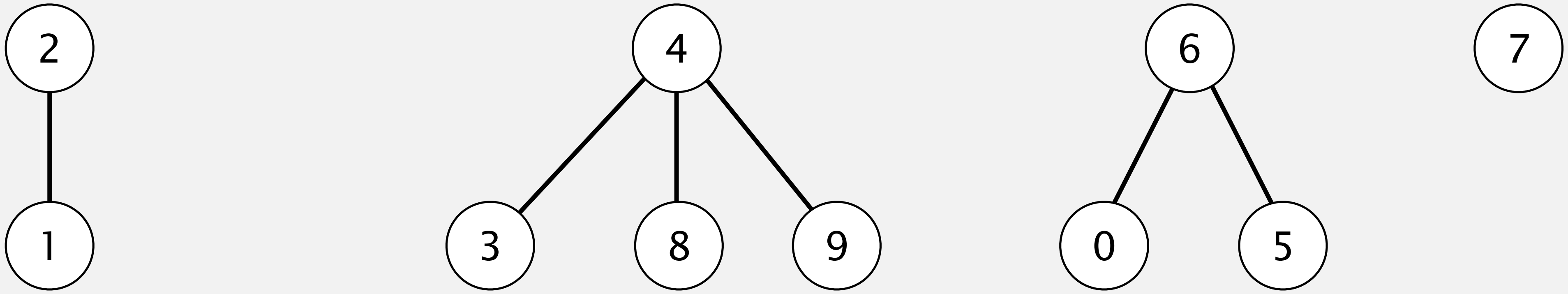
Weighted quick-union demo

union(5, 0)



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	2	4	4	6	6	7	4	4

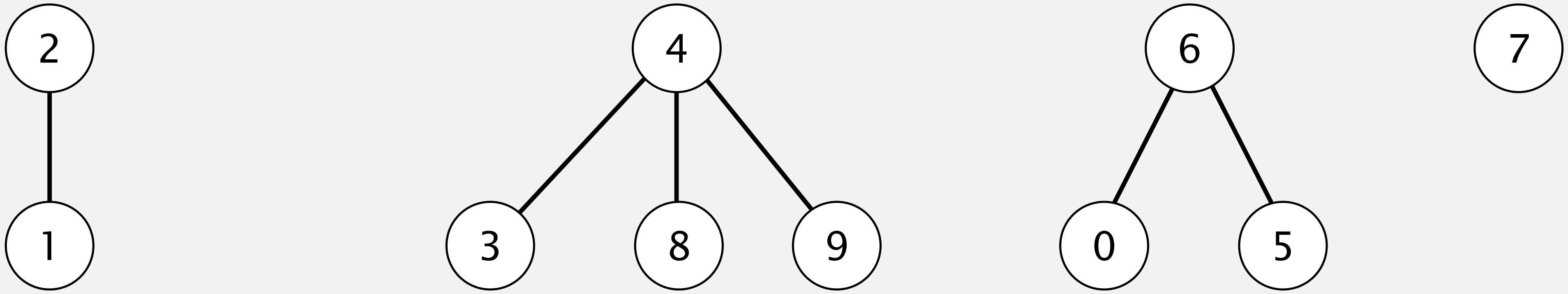
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	2	4	4	6	6	7	4	4

Weighted quick-union demo

union(7, 2)

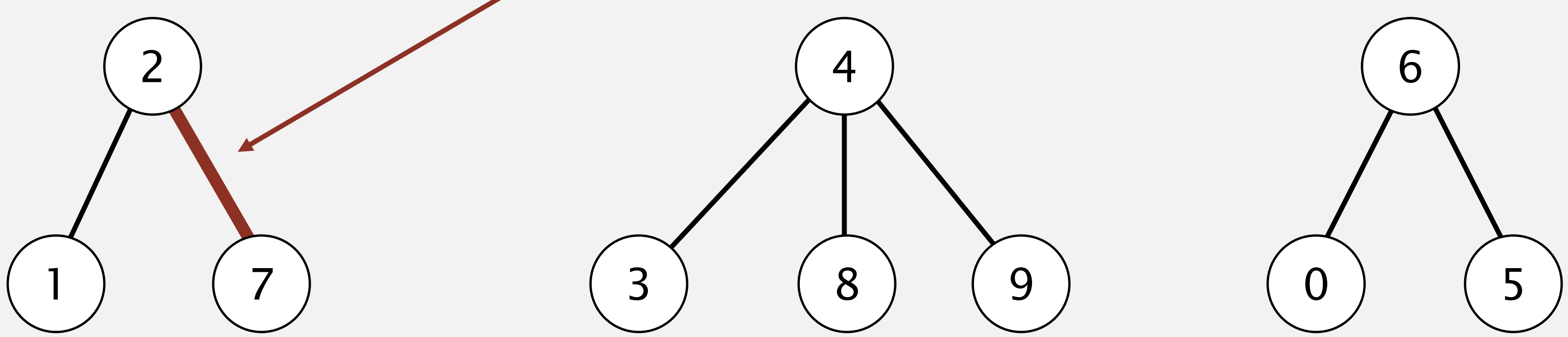


	0	1	2	3	4	5	6	7	8	9
id[]	6	2	2	4	4	6	6	7	4	4

Weighted quick-union demo

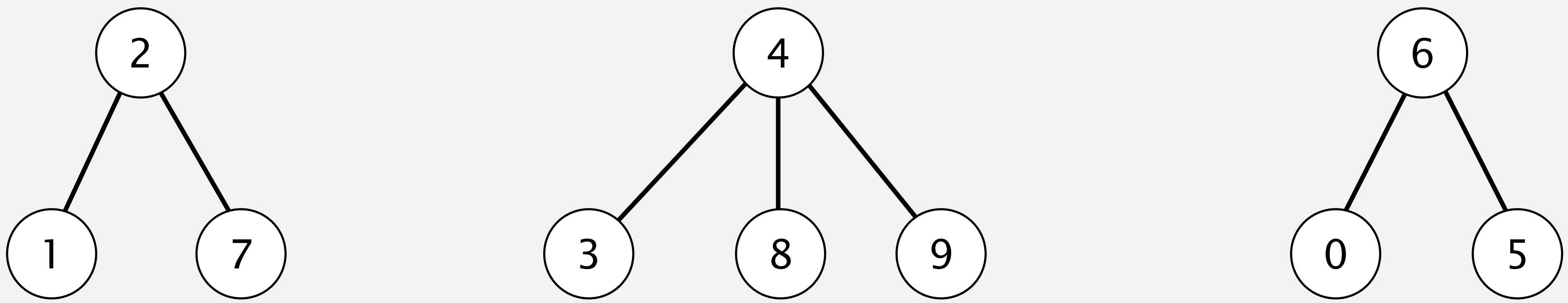
weighting: make 7 point to 2

union(7, 2)



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	2	4	4	6	6	2	4	4

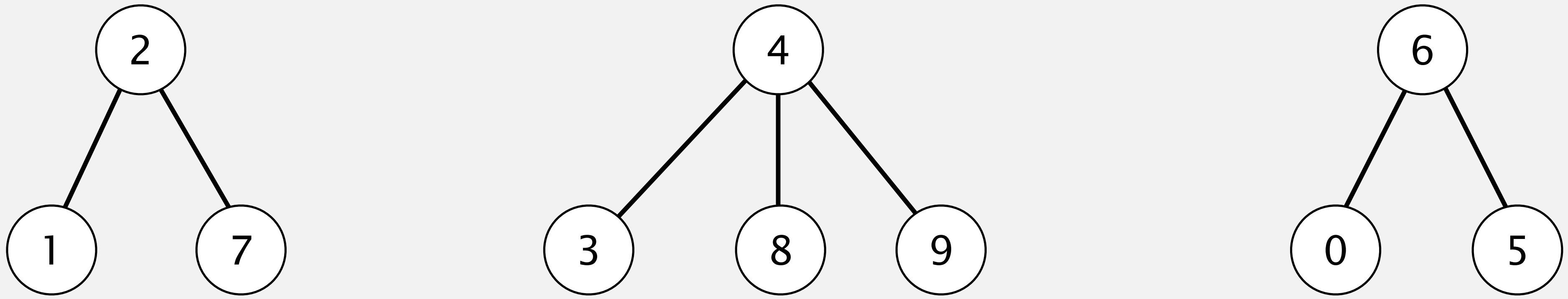
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	2	4	4	6	6	2	4	4

Weighted quick-union demo

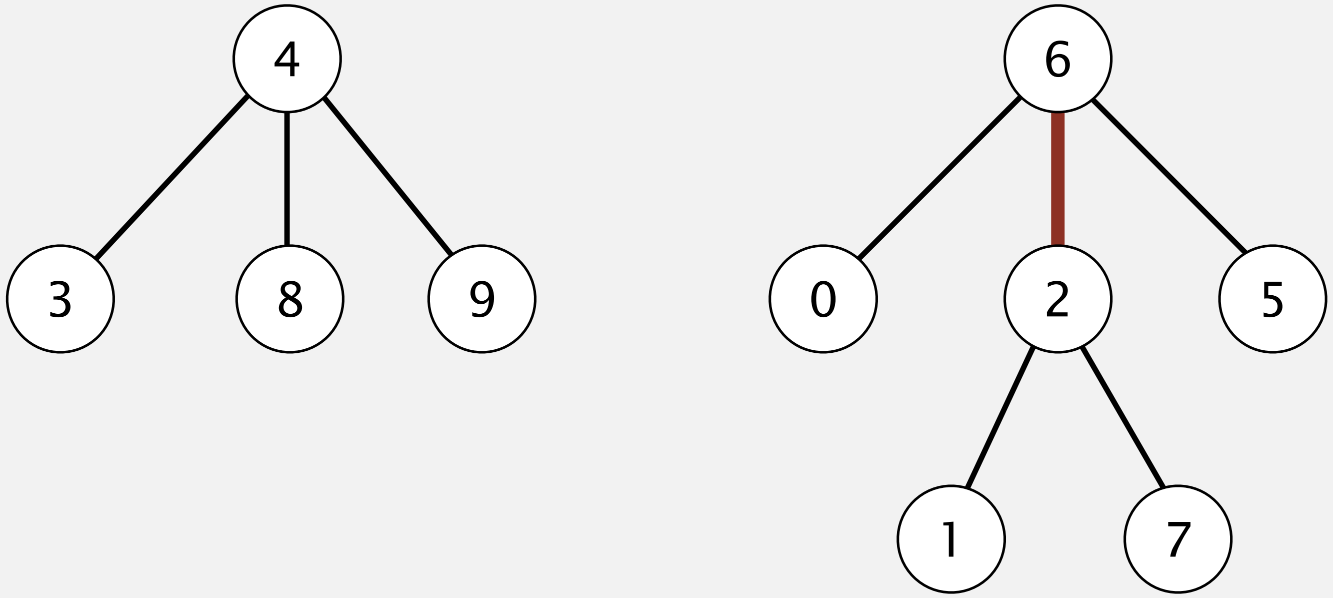
`union(6, 1)`



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	2	4	4	6	6	2	4	4

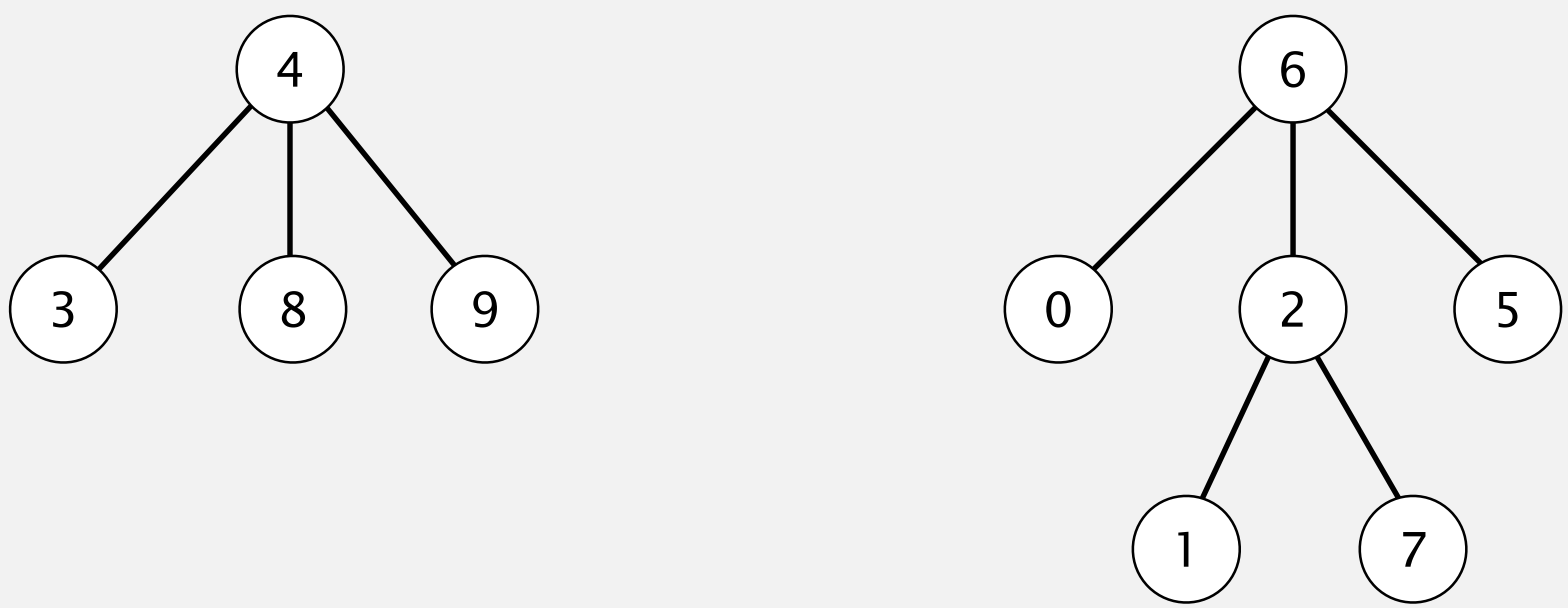
Weighted quick-union demo

union(6, 1)



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	4	6	6	2	4	4

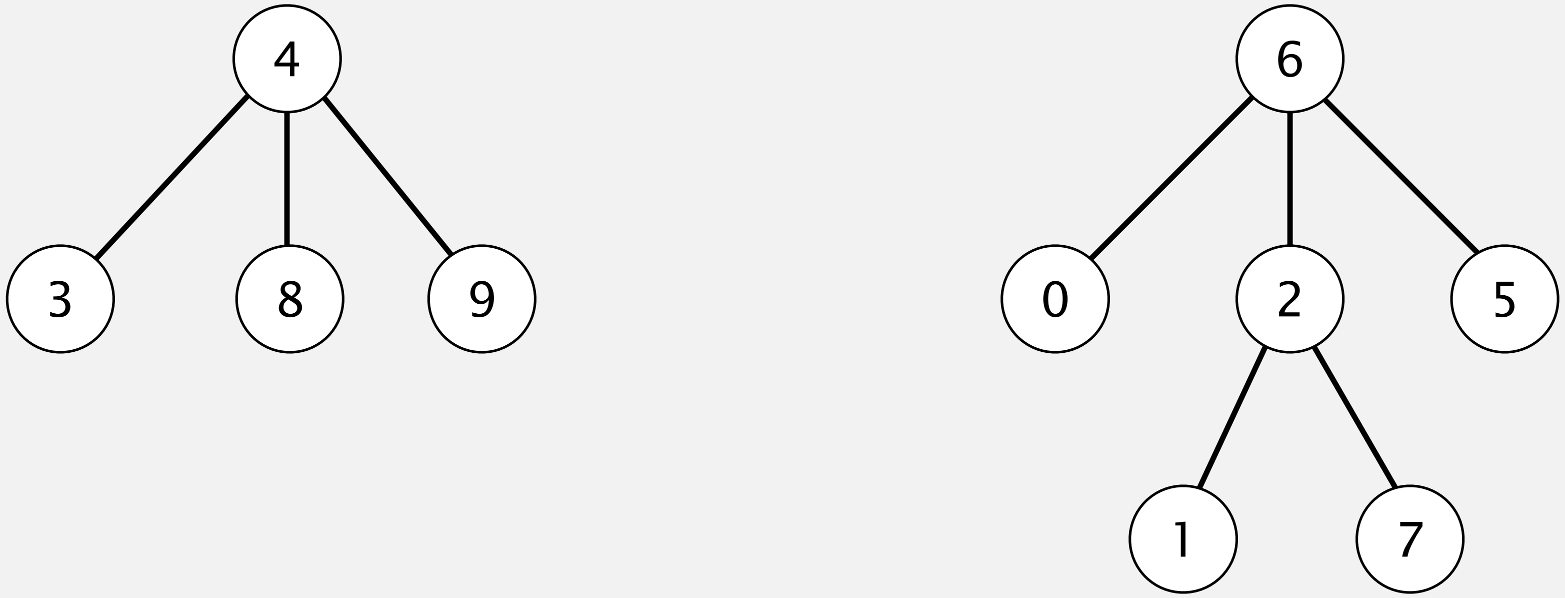
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	4	6	6	2	4	4

Weighted quick-union demo

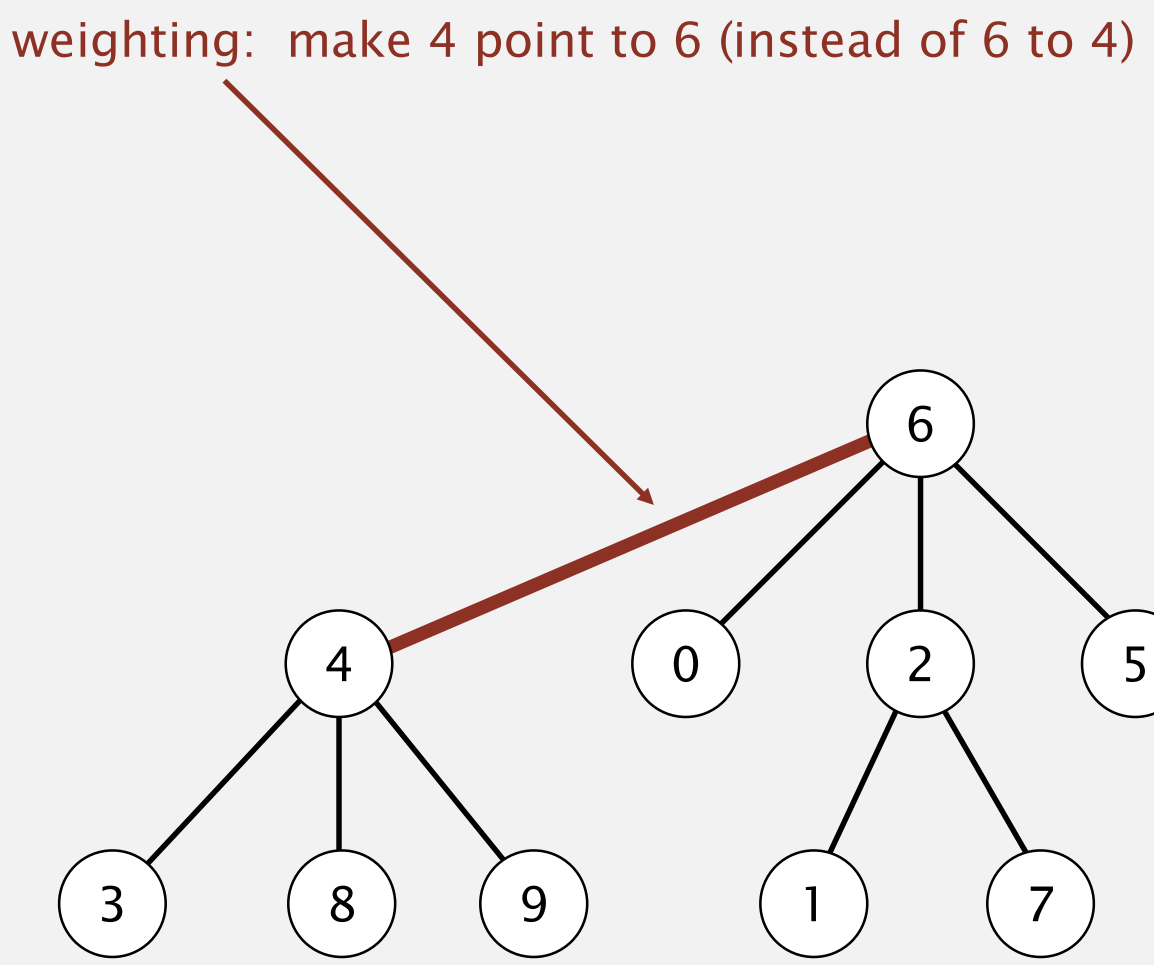
`union(7, 3)`



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	4	6	6	2	4	4

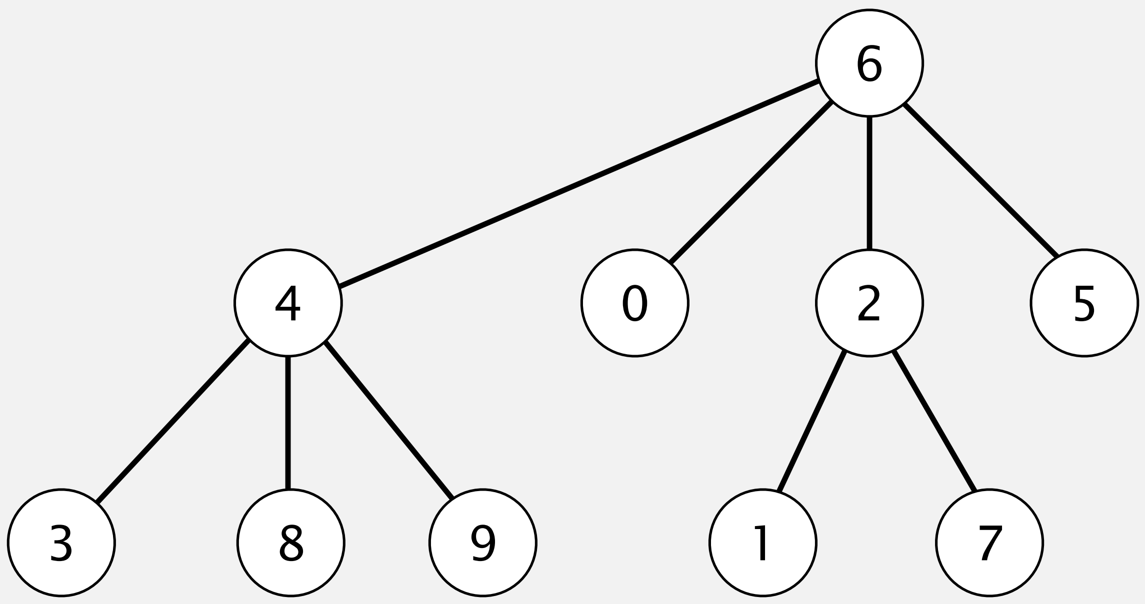
Weighted quick-union demo

union(7, 3)



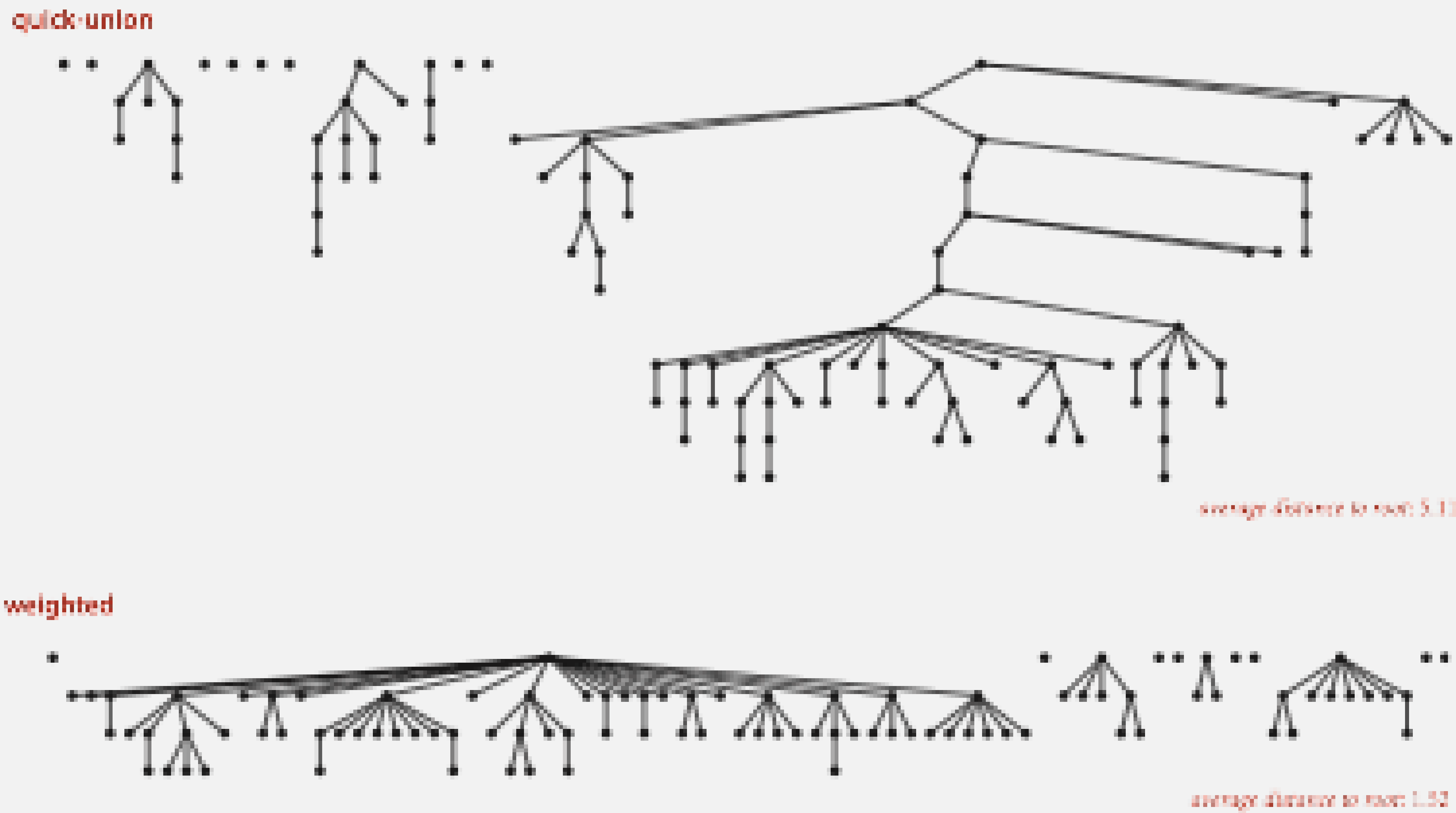
	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	6	6	6	2	4	4

Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	6	6	6	2	4	4

Quick-union and weighted quick-union example



Quick-union and weighted quick-union (100 sites, 88 union() operations)

Weighted quick-union: Java implementation

Data structure. Same as quick-union, but maintain extra array `sz[i]` to count number of objects in the tree rooted at `i`.

Find/connected. Identical to quick-union.

Union. Modify quick-union to:

- Link root of smaller tree to root of larger tree.
- Update the `sz[]` array.

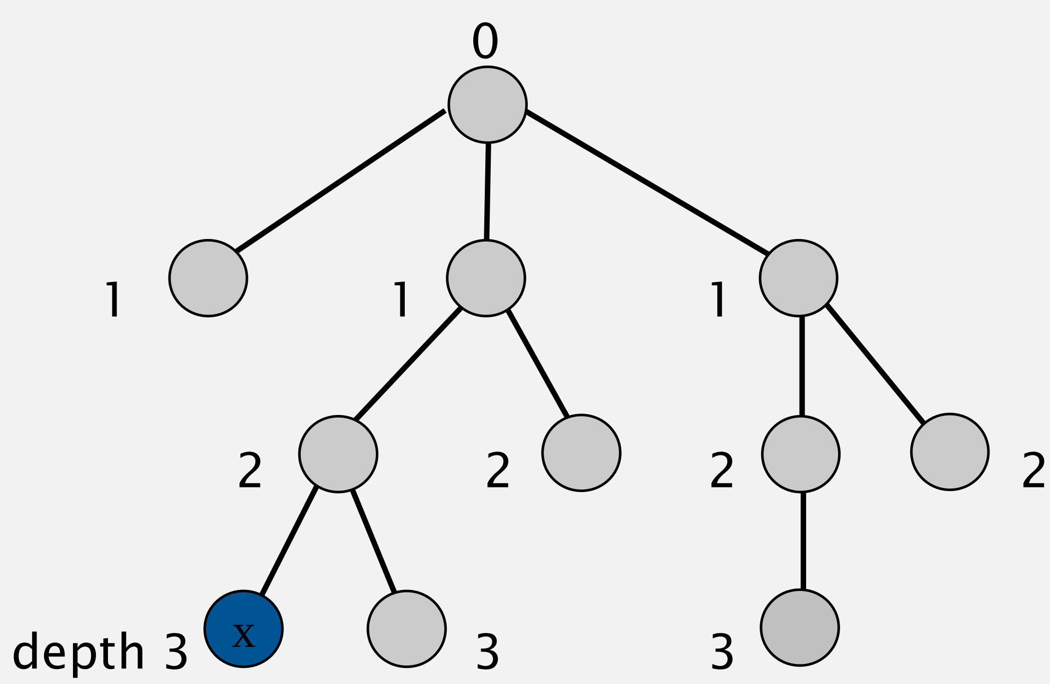
```
int i = find(p);
int j = find(q);
if (i == j) return;
if  (sz[i] < sz[j])
          { id[i] = j; sz[j] += sz[i]; }
else
          { id[j] = i; sz[i] += sz[j]; }
```

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given roots.

log = base-2 logarithm

Proposition. Depth of any node x is at most $\log n$.



N = 11

depth(x) = 3 ⩽ log n

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given roots.

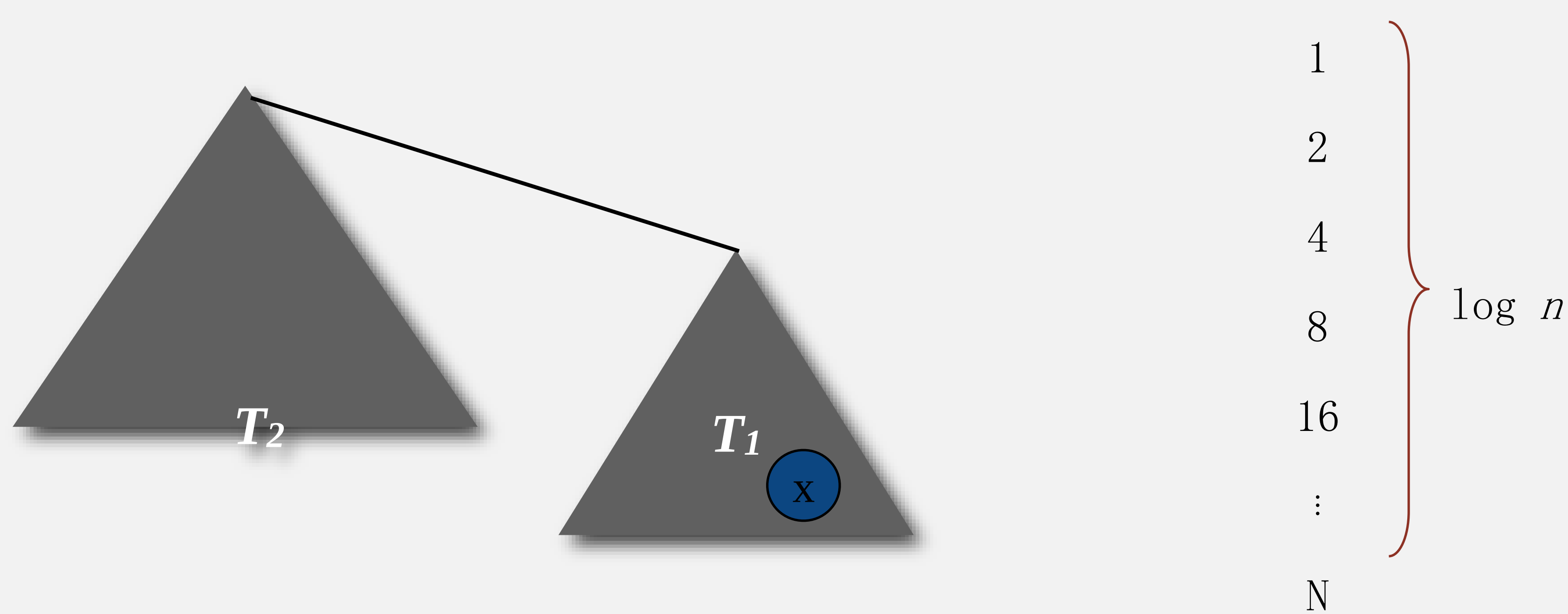
log = base-2 logarithm

Proposition. Depth of any node x is at most $\log n$.

Pf. What causes the depth of object x to increase?

Increases by 1 when tree T_1 containing x is merged into another tree T_2.

- The size of the tree containing x at least doubles since $|T_2| \geq |T_1|$.
- Size of tree containing x can double at most $\log n$ times. Why?



Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given roots.

Proposition. Depth of any node x is at most $\log n$.

algorithm	initialize	union	find	connected
quick-find	n	n	1	1
quick-union	n	n †	n	n
weighted QU	n	log n †	log n	log n

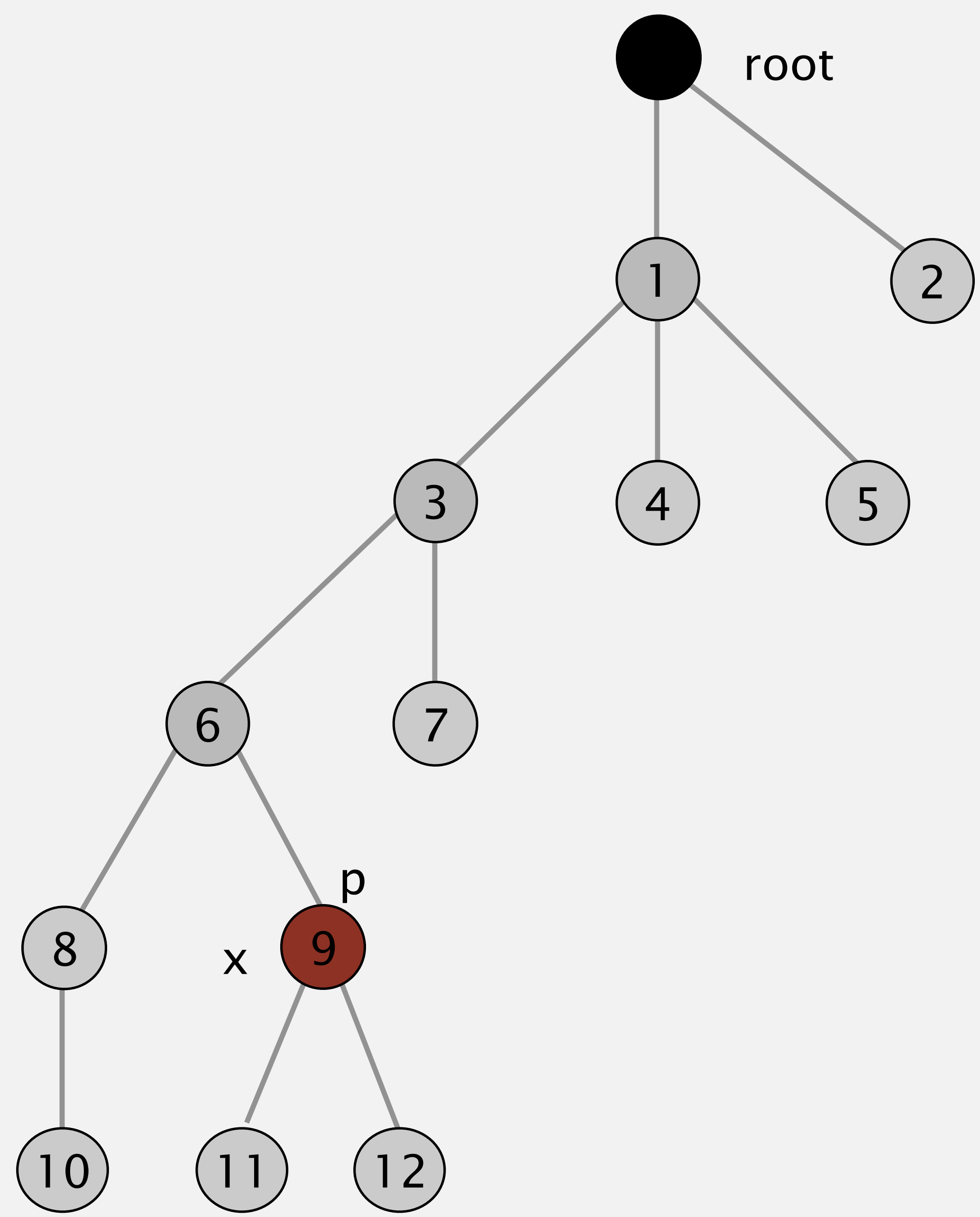
† includes cost of finding roots

Q. Stop at guaranteed acceptable performance?

A. No, easy to improve further.

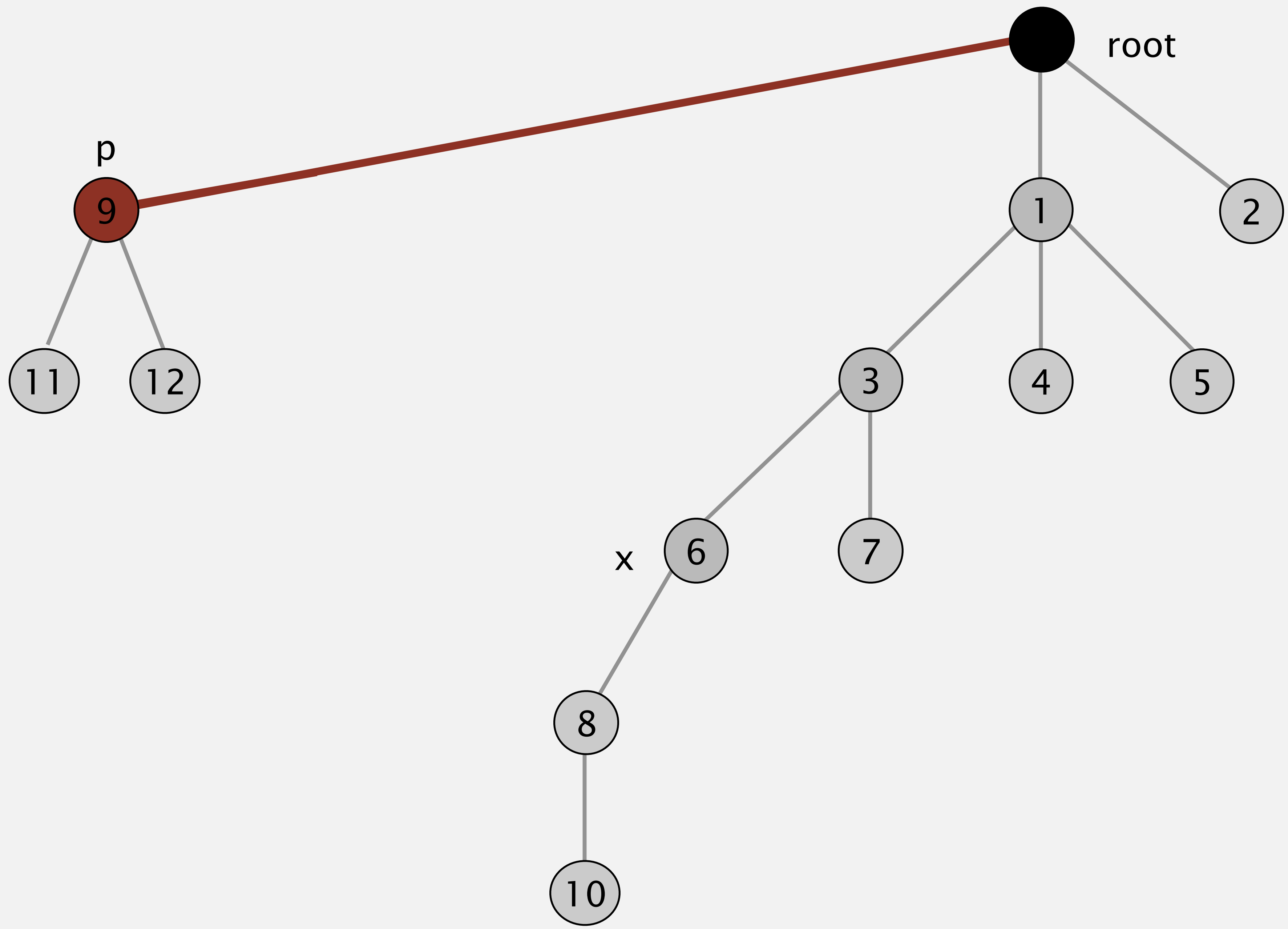
Improvement 2: path compression

Quick union with path compression. Just after computing the root of *p*, set the `id[]` of each examined node to point to that root.



Improvement 2: path compression

Quick union with path compression. Just after computing the root of p, set the id[] of each examined node to point to that root.



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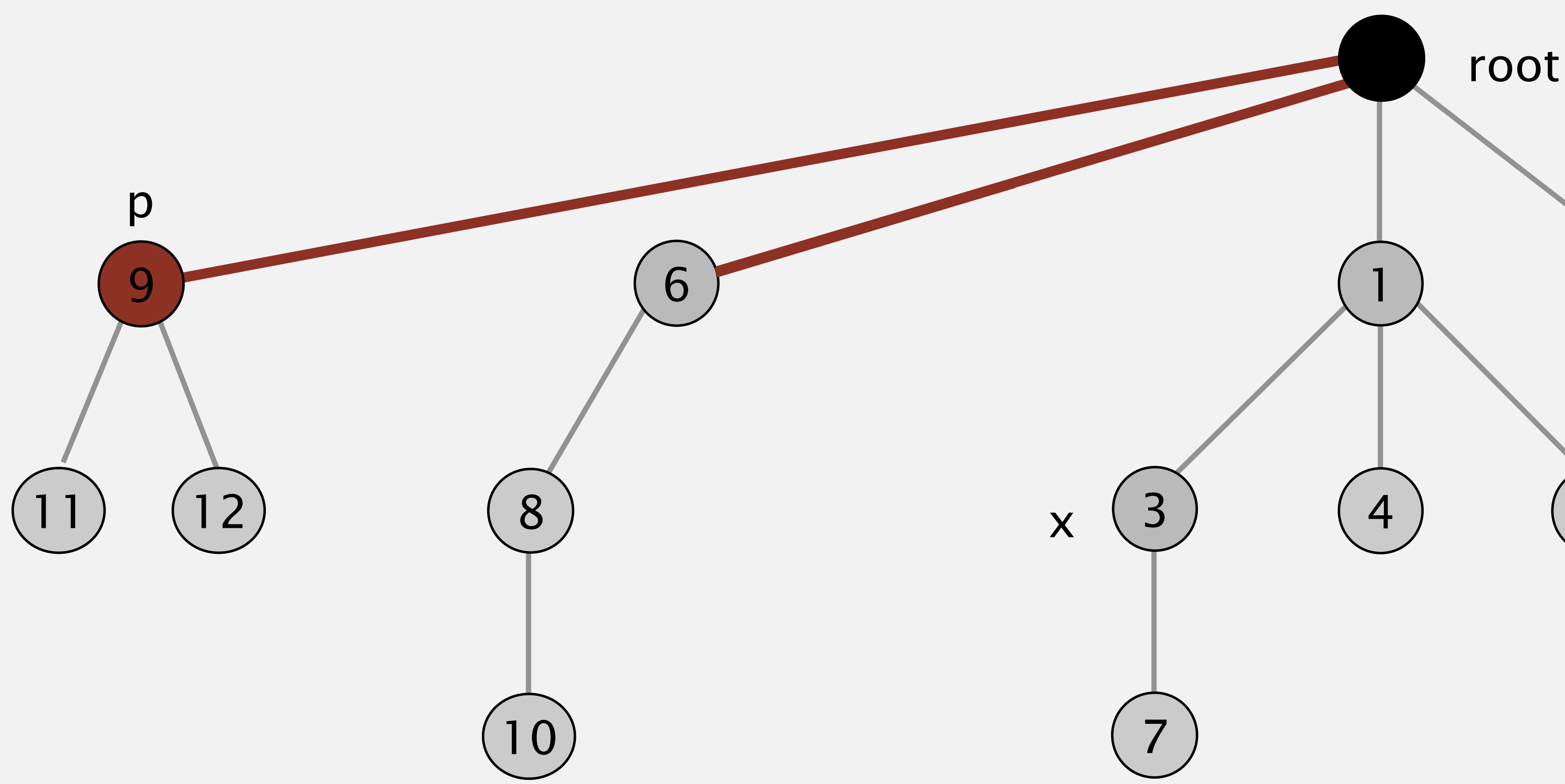

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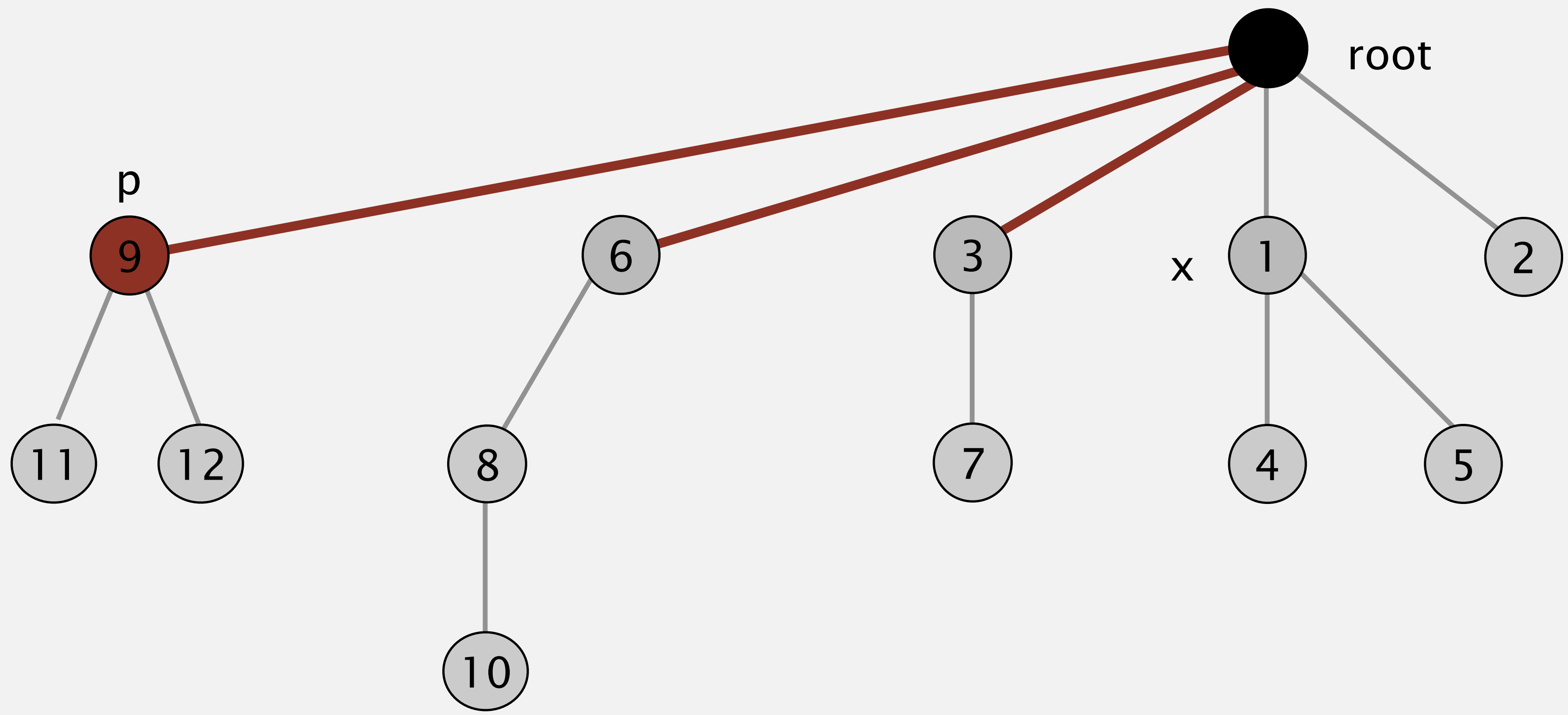

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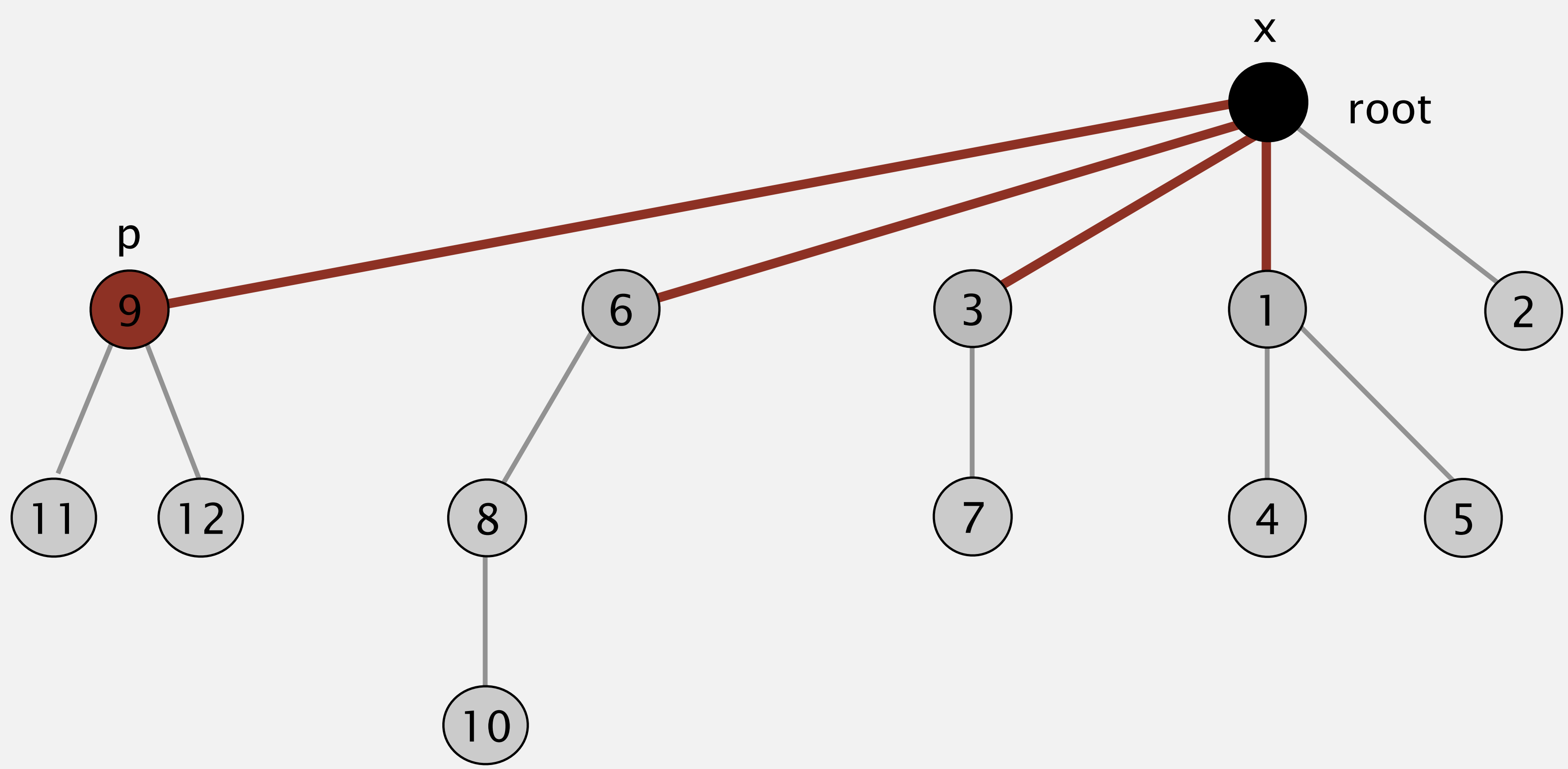

Bottom line. Now, find() has the side effect of compressing the tree.

Path compression: Java implementation

Two-pass implementation: add second loop to `find()` to set the `id[]` of each examined node to the root.

Simpler one-pass variant (path halving): Make every other node in path point to its grandparent.

```
public int find(int i)
{
   while (i != id[i])
   {
      id[i] = id[id[i]];
      i = id[i];
   }
   return i;
}
```

only one extra line of code !

In practice. No reason not to! Keeps tree almost completely flat.

Proposition. [Hopcroft-Ulman, Tarjan] Starting from an empty data structure, any sequence of m union–find ops on n objects takes $O(n + m \log^* n)$ array accesses.

- Analysis can be improved to $n + m\,\alpha(n)$.
- Simple algorithm with fascinating mathematics.

n	log* n
1	0
2	1
4	2
16	3
65536	4
2^{65536}	5

iterated log function

Linear-time algorithm for m union-find ops on n objects?

- Cost within constant factor of reading in the data.
- In theory, WQUPC is not quite linear.
- In practice, WQUPC is linear.

Amazing fact. [Fredman-Saks] No linear-time algorithm exists.

in "cell-probe" model of computation

log*(n): The iterated logarithm

- log*(n) = min{i : $t(i) \geq n$} is the inverse of the *tower-of-twos function* $t(i)$ where

$$t(i) = \begin{cases} 1 & \text{if } i = 0 \\ 2^{t(i-1)} & \text{if } i \geq 1 \end{cases}$$

$\alpha(n)$: inverse of Ackerman function $A(n)$

$\alpha(n) = \min\{m : A(m) \geq n\}$ where $A(n) = A_n(n)$ and

$$A_i(n) = \begin{cases} 2n & \text{for } i = 0 \text{ and } n \geq 0 \\ A_{i-1}(2) & \text{for } i \geq 1 \text{ and } n = 1 \\ A_{i-1}(A_i(n-1)) & \text{for } i \geq 1 \text{ and } n \geq 2 \end{cases}$$

$\alpha(n)$: inverse Ackerman function

$$A(2) = A_2(2) = A_1(A_2(1)) = A_1(A_1(2))$$
$$= A_1(A_0(A_1(1))) = A_1(A_0(2^2))$$
$$= A_1(2 \cdot 2^2) = A_1(2^3) = A_0(A_1(2^3 - 1))$$
$$= A_0(A_0(A_1(2^3 - 2))) = A_0(A_0(A_0(A_1(2^3 - 3))))$$
$$= A_0(A_0(A_0(A_0(A_1(2^3 - 4)))))$$
$$= A_0(A_0(A_0(A_0(A_0(A_1(2^3 - 5))))))$$
$$= A_0(A_0(A_0(A_0(A_0(A_0(A_1(2^3 - 6)))))))$$
$$= A_0(A_0(A_0(A_0(A_0(A_0(A_0(A_1(2^3 - 7))))))))$$

$\alpha(n)$: inverse Ackerman function

$$= A_0(A_0(A_0(A_0(A_0(A_0(A_0(A_1(1))))))))$$

$$= A_0(A_0(A_0(A_0(A_0(A_0(A_0(2^2)))))))$$

$$= A_0(A_0(A_0(A_0(A_0(A_0(2^3))))))$$

$$= A_0(A_0(A_0(A_0(A_0(2^4)))))$$

$$= A_0(A_0(A_0(A_0(2^5))))$$

$$= A_0(A_0(A_0(2^6))) = A_0(A_0(2^7))$$

$$= A_0(2^8) = 2^9$$

Key point. Weighted quick union (and/or path compression) makes it possible to solve problems that could not otherwise be addressed.

algorithm	worst-case time
quick-find	$O(m\, n)$
quick-union	$O(m\, n)$
weighted QU	$O(n + m \log n)$
QU + path compression	$O(n + m \log n)$
weighted QU + path compression	$O(n + m \log^* n)$

order of growth for m union-find operations on a set of n objects

Ex. [10^9 unions and finds with 10^9 objects]

- WQUPC reduces time from 30 years to 6 seconds.
- Supercomputer won't help much; good algorithm enables solution.

Kruskal's algorithm: Java implementation

```
public class KruskalMST
{
   private Queue<Edge> mst = new Queue<Edge>();

   public KruskalMST(EdgeWeightedGraph G)
   {
      MinPQ<Edge> pq = new MinPQ<Edge>(G.edges());      // build priority queue (or sort)

      UF uf = new UF(G.V());
      while (!pq.isEmpty() && mst.size() < G.V()-1)
      {
         Edge e = pq.delMin();                          // greedily add edges to MST
         int v = e.either(), w = e.other(v);
         if (!uf.connected(v, w))                       // edge v-w does not create cycle
         {
            uf.union(v, w);                             // merge sets
            mst.enqueue(e);                             // add edge to MST
         }
      }
   }

   public Iterable<Edge> edges()
   {  return mst;  }
}
```