CSC 226

Algorithms and Data Structures: II
Kruskals Implementation

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ECS 466

Kruskal's Algorithm

```
Algorithm Kruskal
Input: a weighted connected graph G = (V, E)
Output: an MST T for G
Data structure: Disjoint sets (lists or union-find) DS;
    sorted weights priority queue A; and tree T

for each v ∈ V  do C(v) ← DS.insert(v) end // one cluster per vertex
for each (u,v) ∈ E  do A.insert((u,v)) end // sort edges by weight
T ← ∅
while T has fewer than n-1 edges do
   (u, v) ← A.deleteMin() // edge with smallest weight
   C(u)← DS.findCluster(u);
   C(v)← DS.findCluster(v);
   if C(u) ≠ C(v) then
      add edge (u, v) to T;
      DS.insert(DS.union(C(v), C(u))); // merge two clusters
   end
end
return T
```

Idea:

- Avoid sorting the edge weights by storing the edges in a heap …..

Building up a heap

- m standard insert-operations for a heap result in $O(m \log(m))$ time.
- Can we build up a heap for m given elements faster? Is $O(m)$ possible?

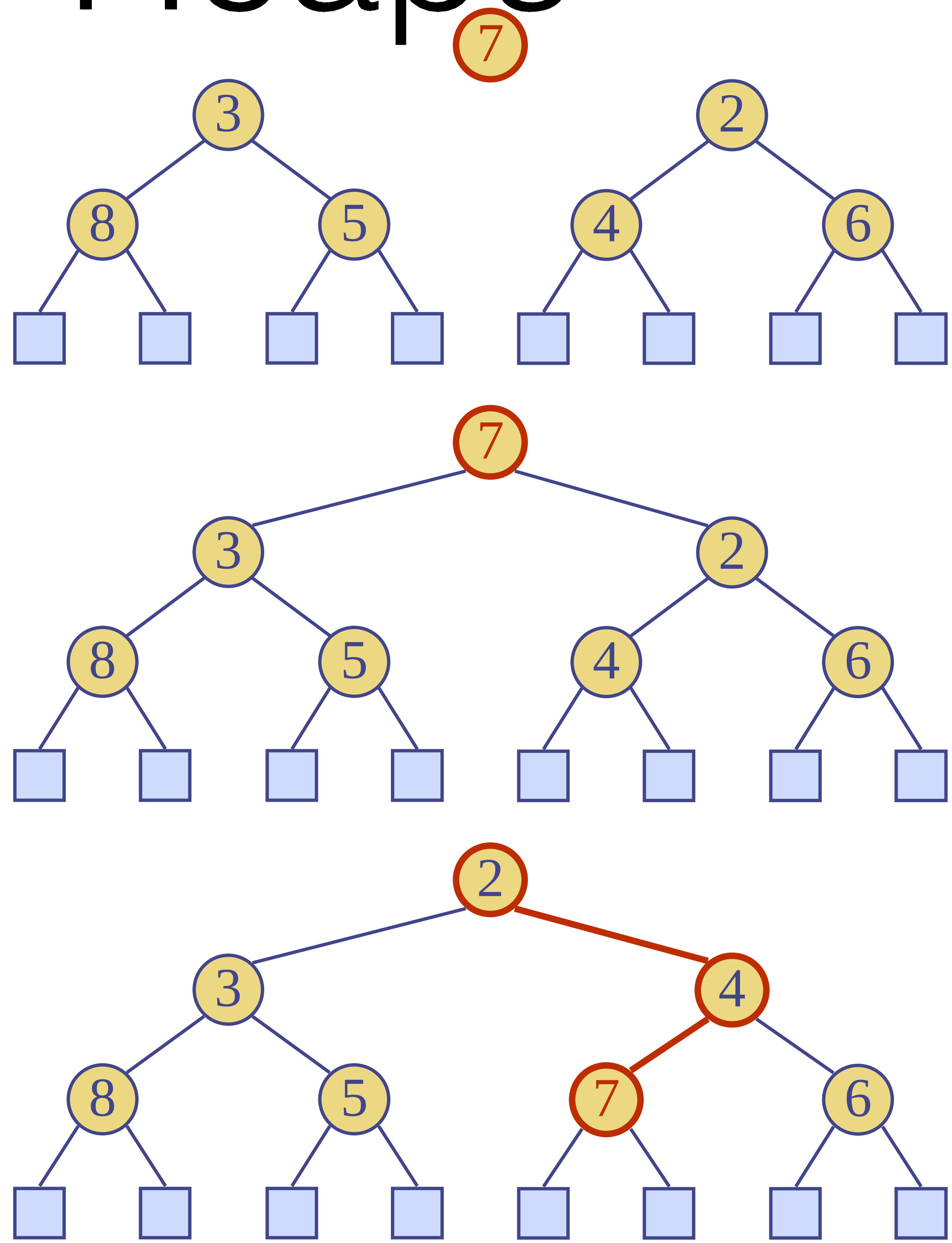
Bottom-Up Heap

```
Algorithm BottomUpHeap(S):
   Input: A list S storing m keys
   Output: A heap T storing the m keys

   if S is empty then
      return external node
   remove the first key, k, from S
   split S in half, lists S1 and S2
   T1 ← BottomUpHeap(S1)
   T2 ← BottomUpHeap(S2)
   T ← merge(k,T1,T2)
   DownHeap(T,root)
   return T
```

Merging Two Heaps

- We are given two heaps and a key *k*
- We create a new heap with the root node storing *k* and with the two heaps as subtrees
- We perform downheap to restore the heap-order property



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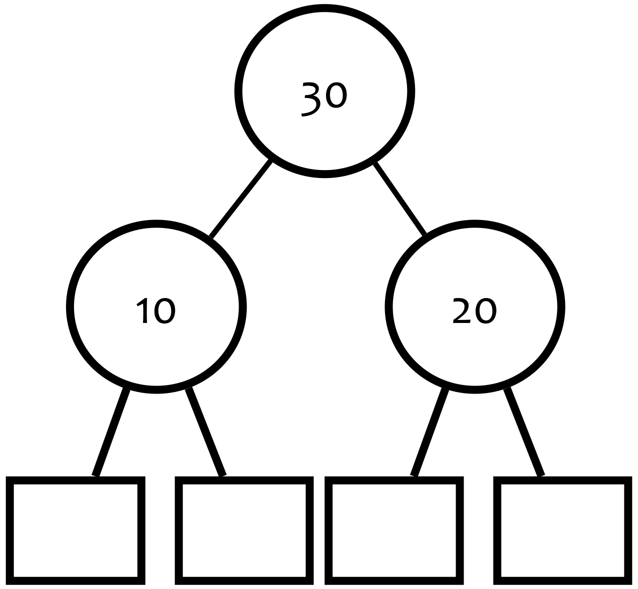

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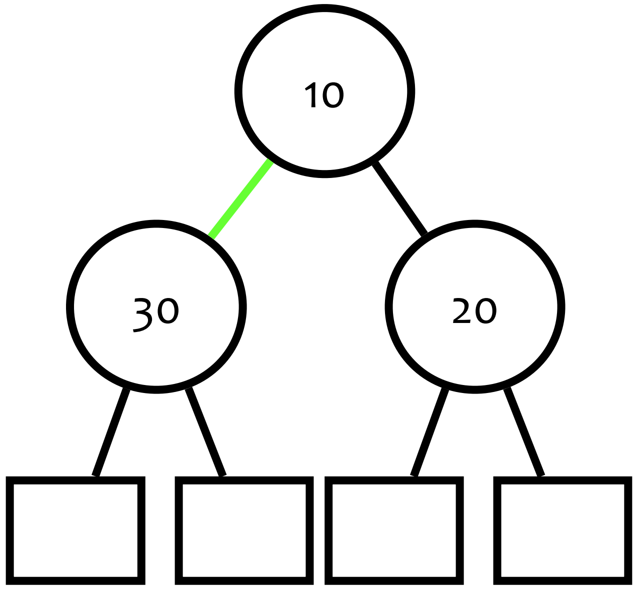

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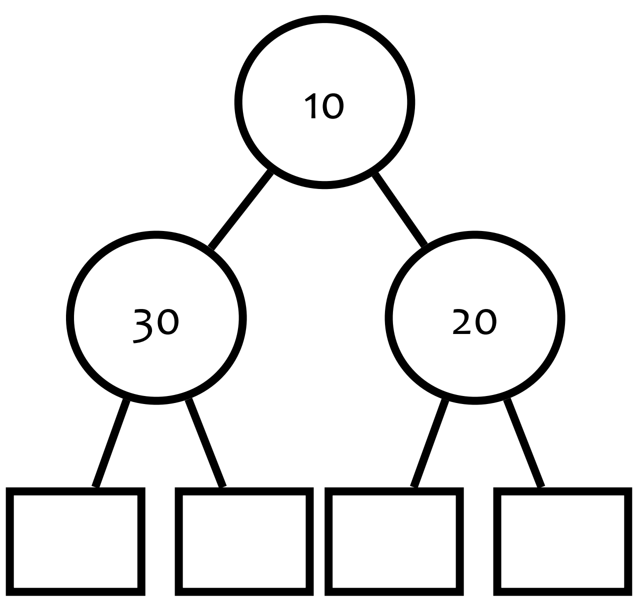

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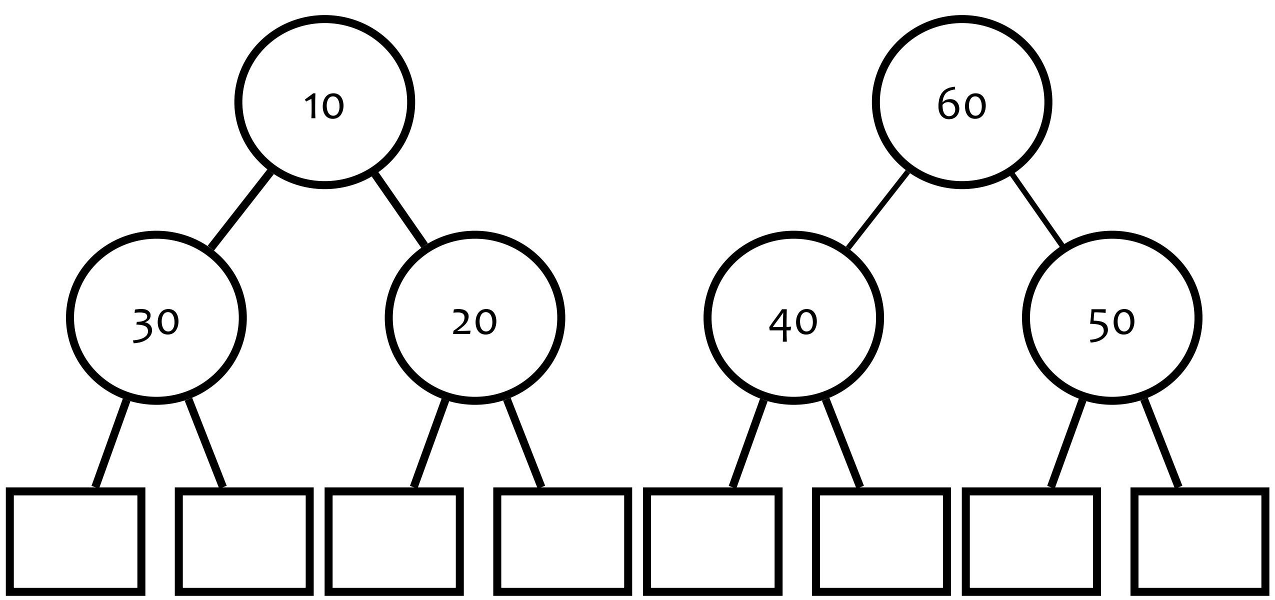

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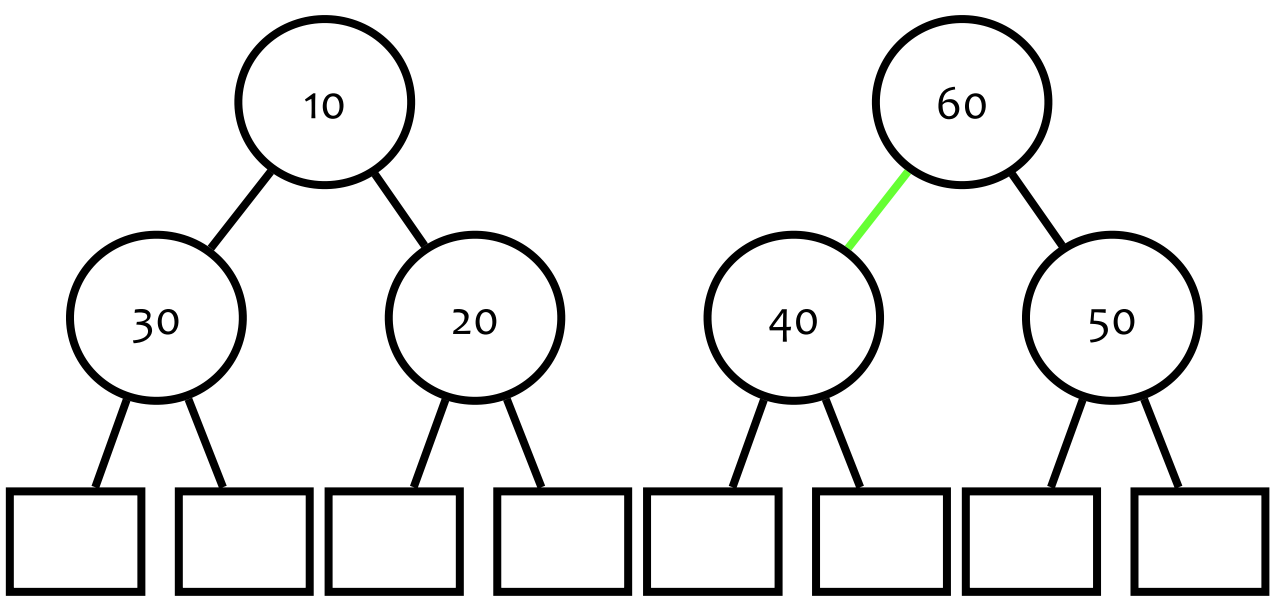

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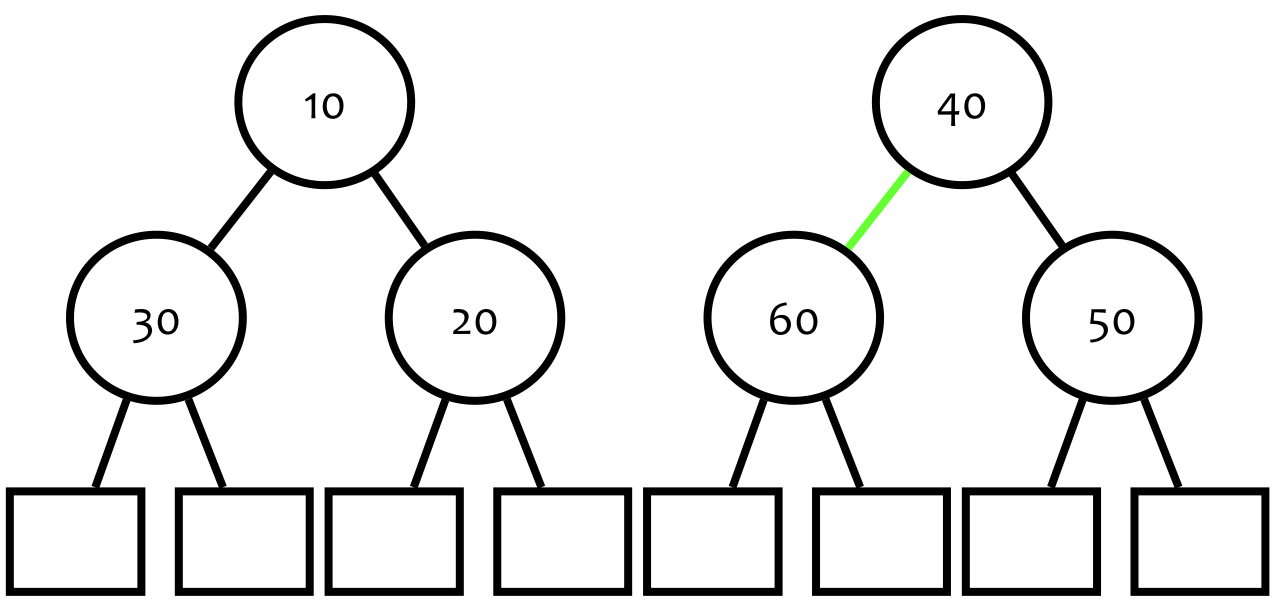

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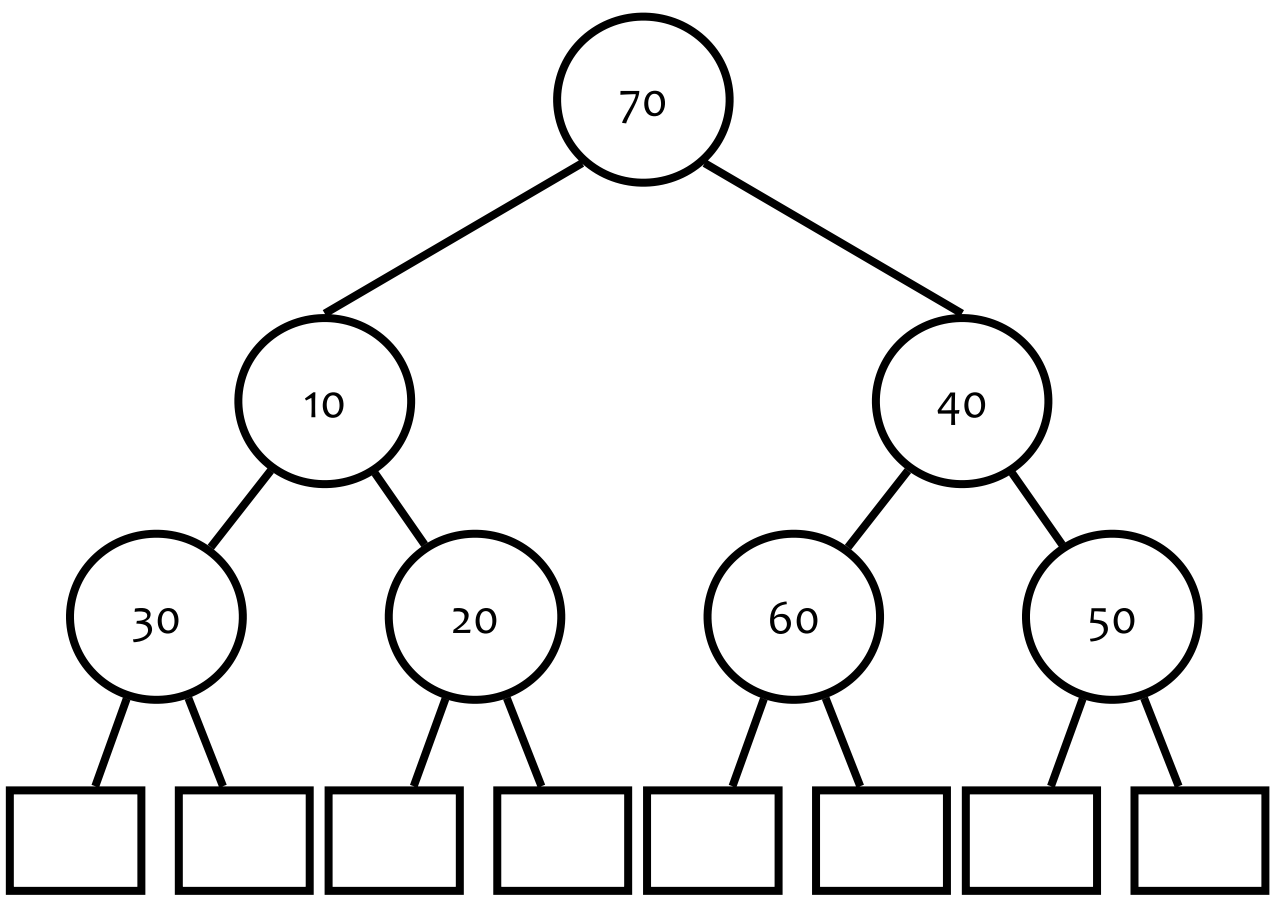

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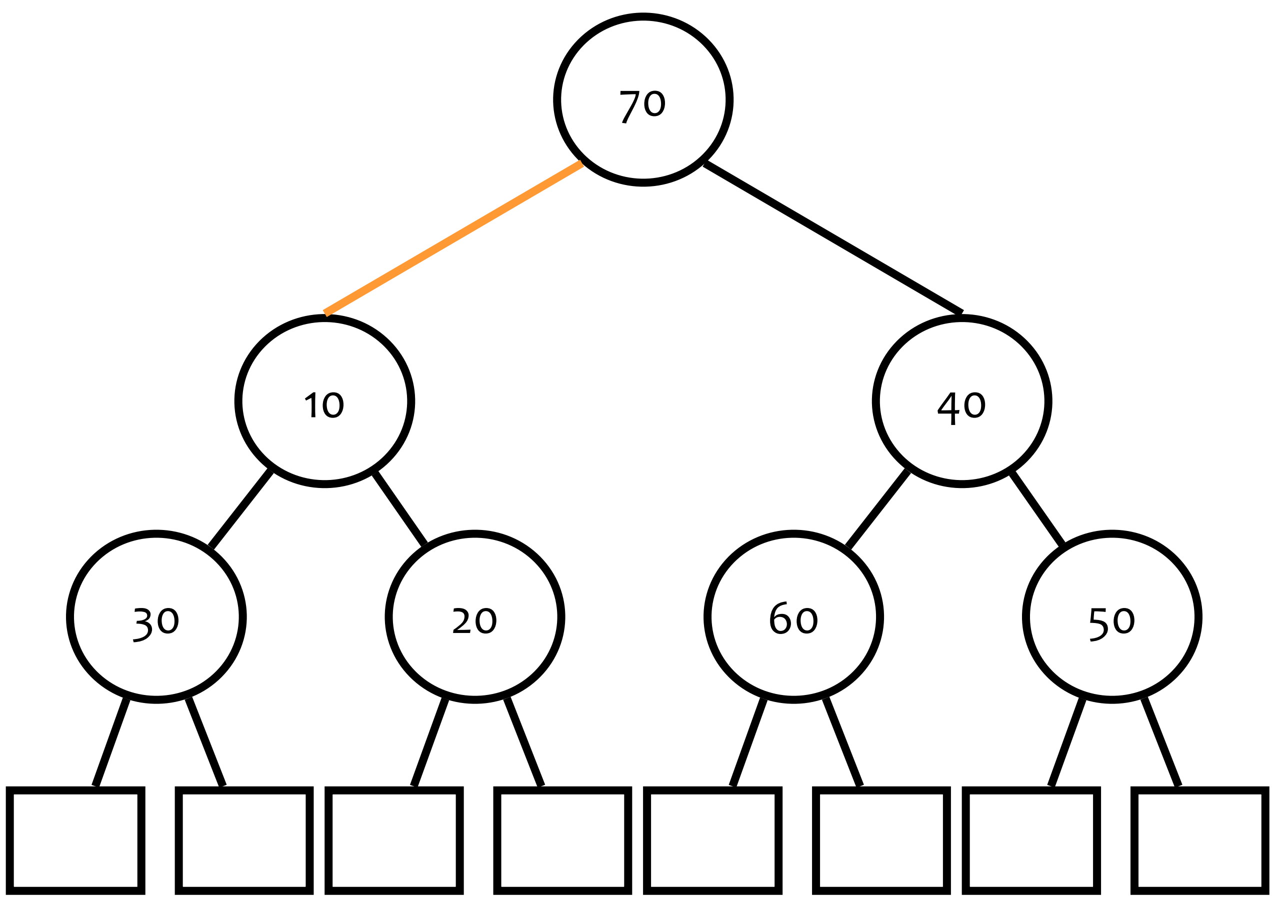

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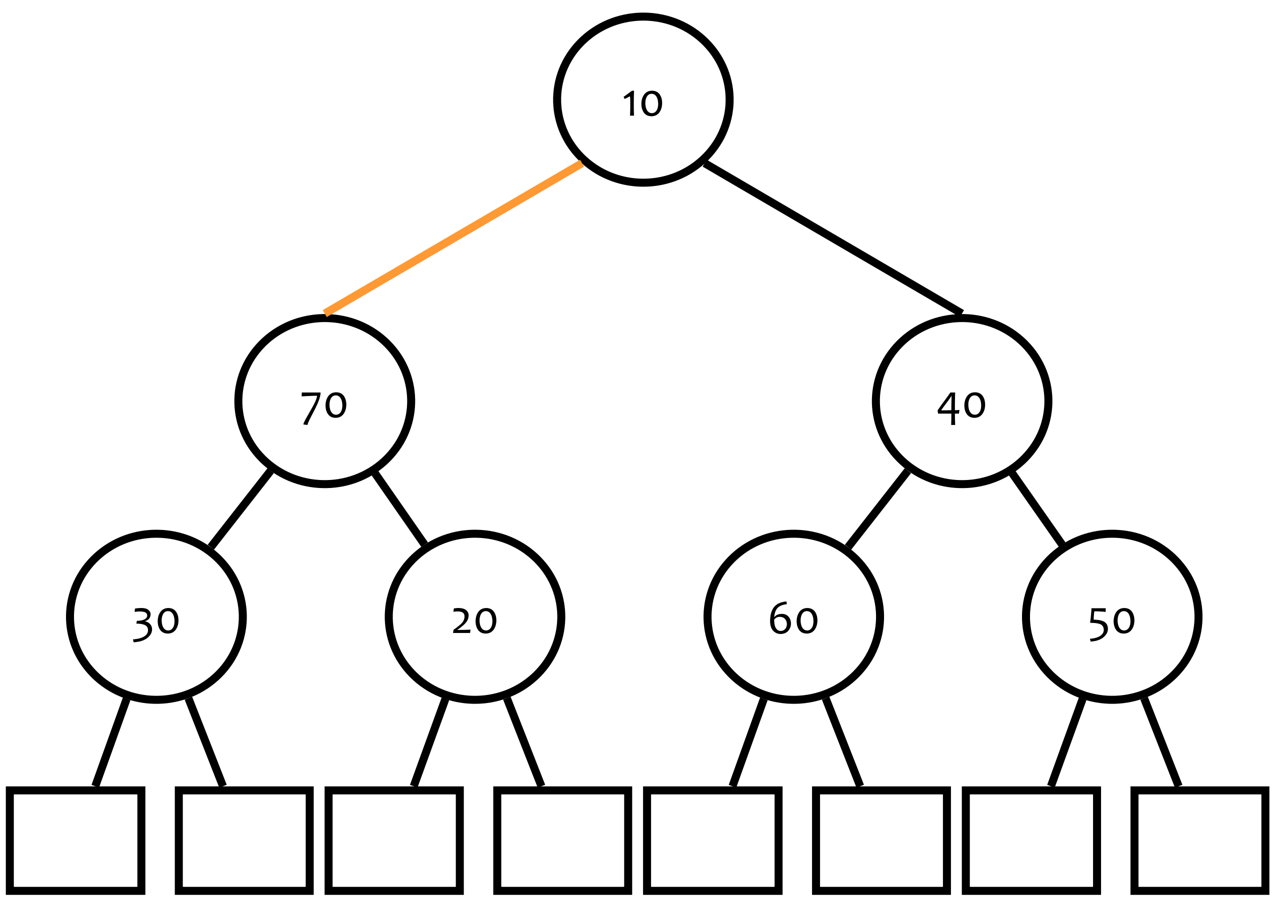

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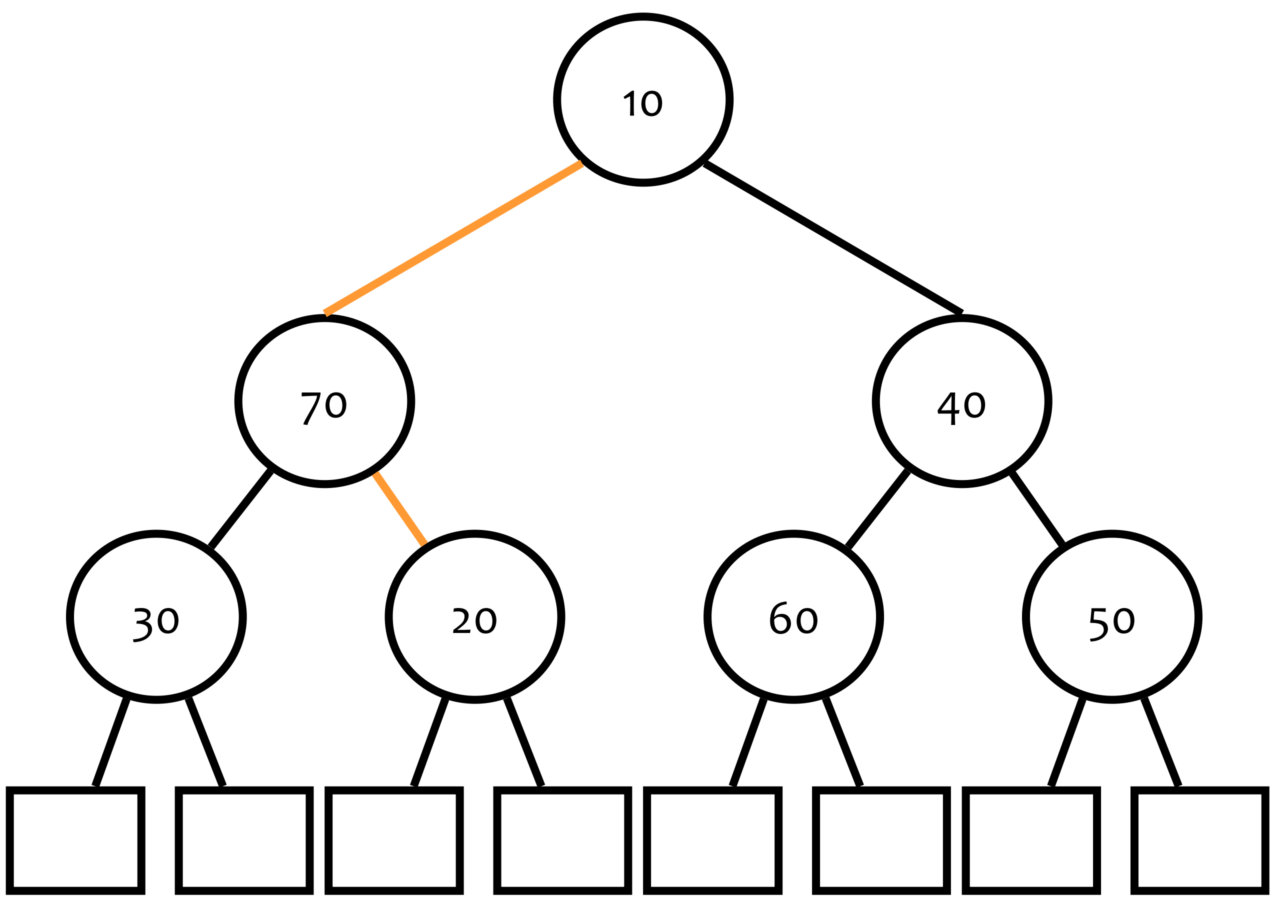

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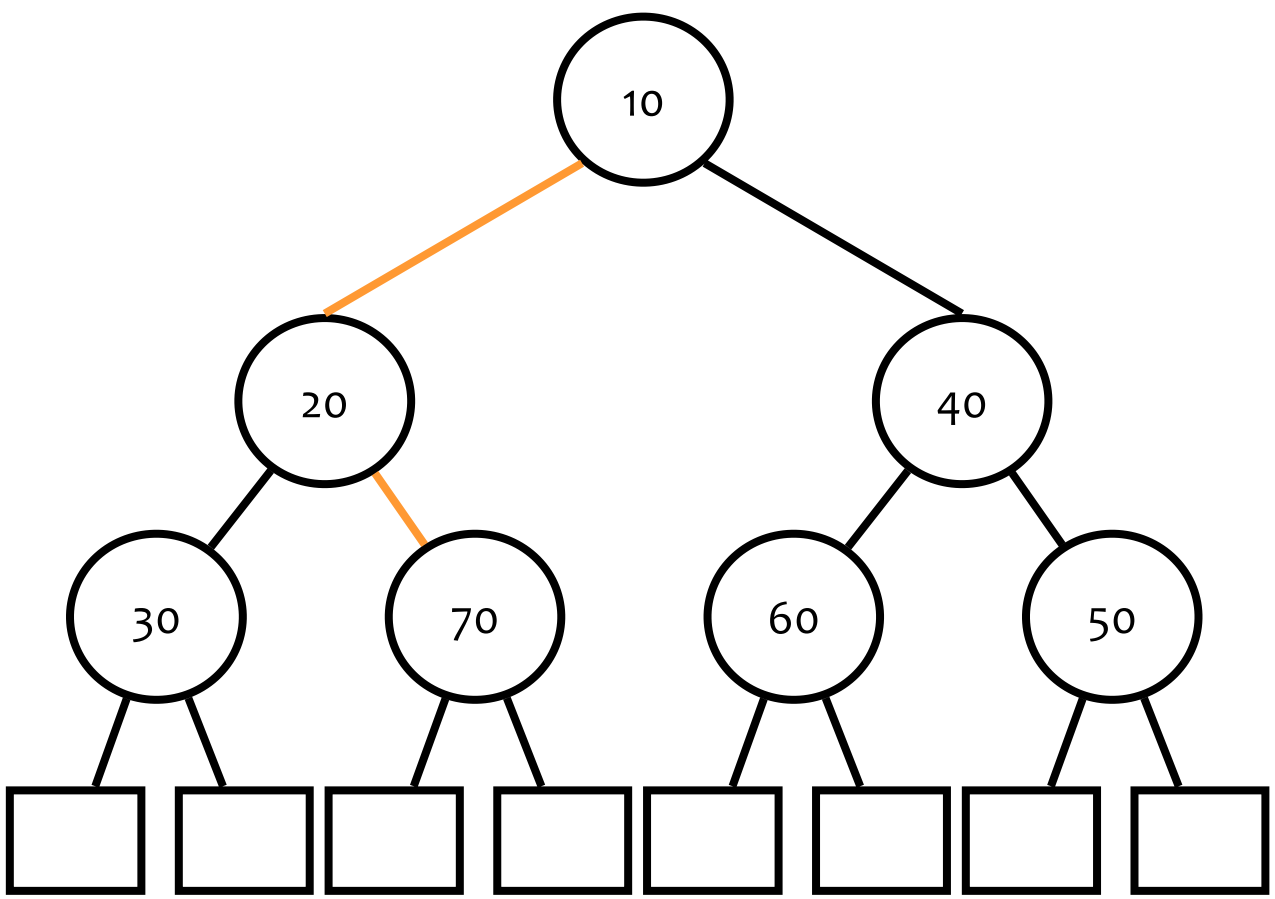

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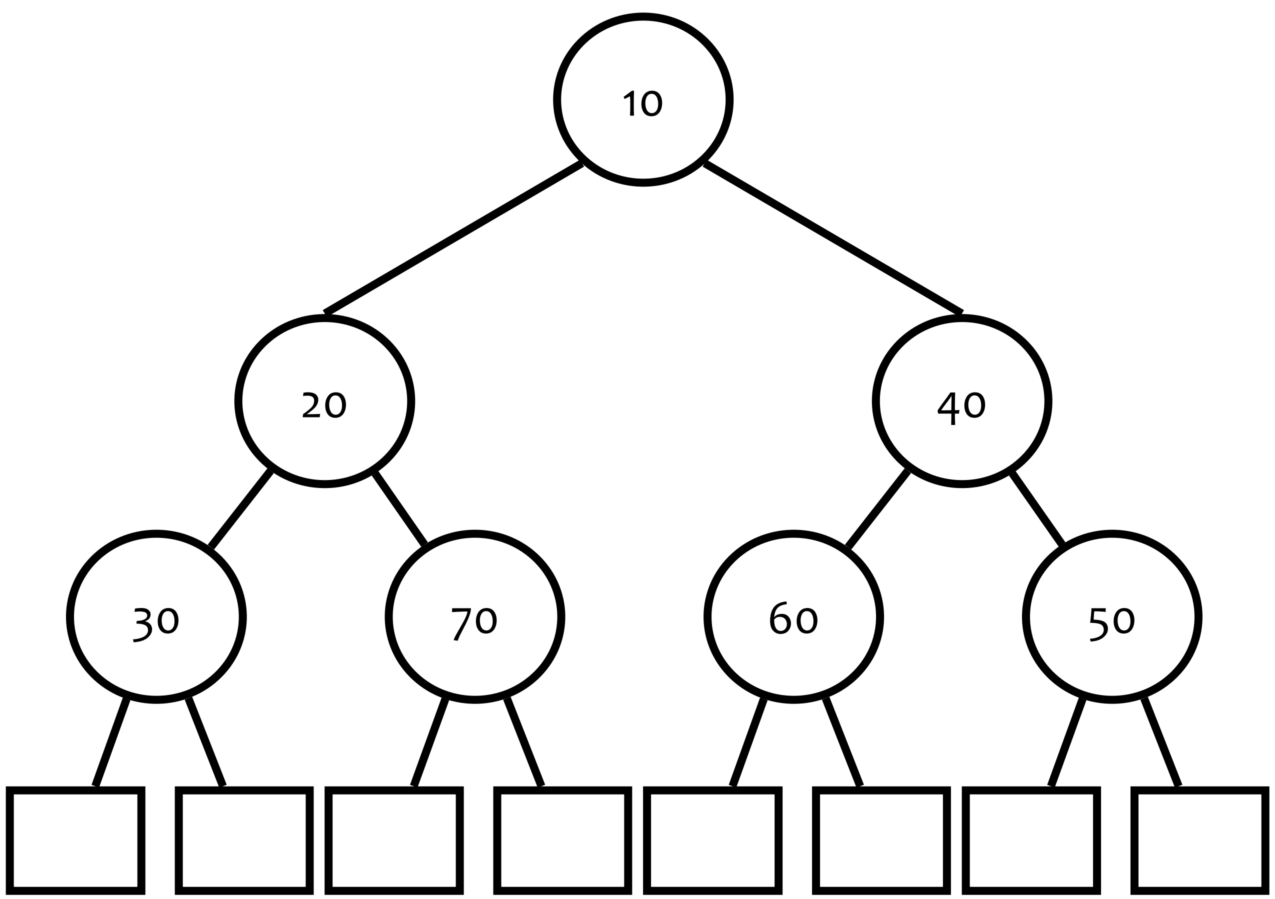

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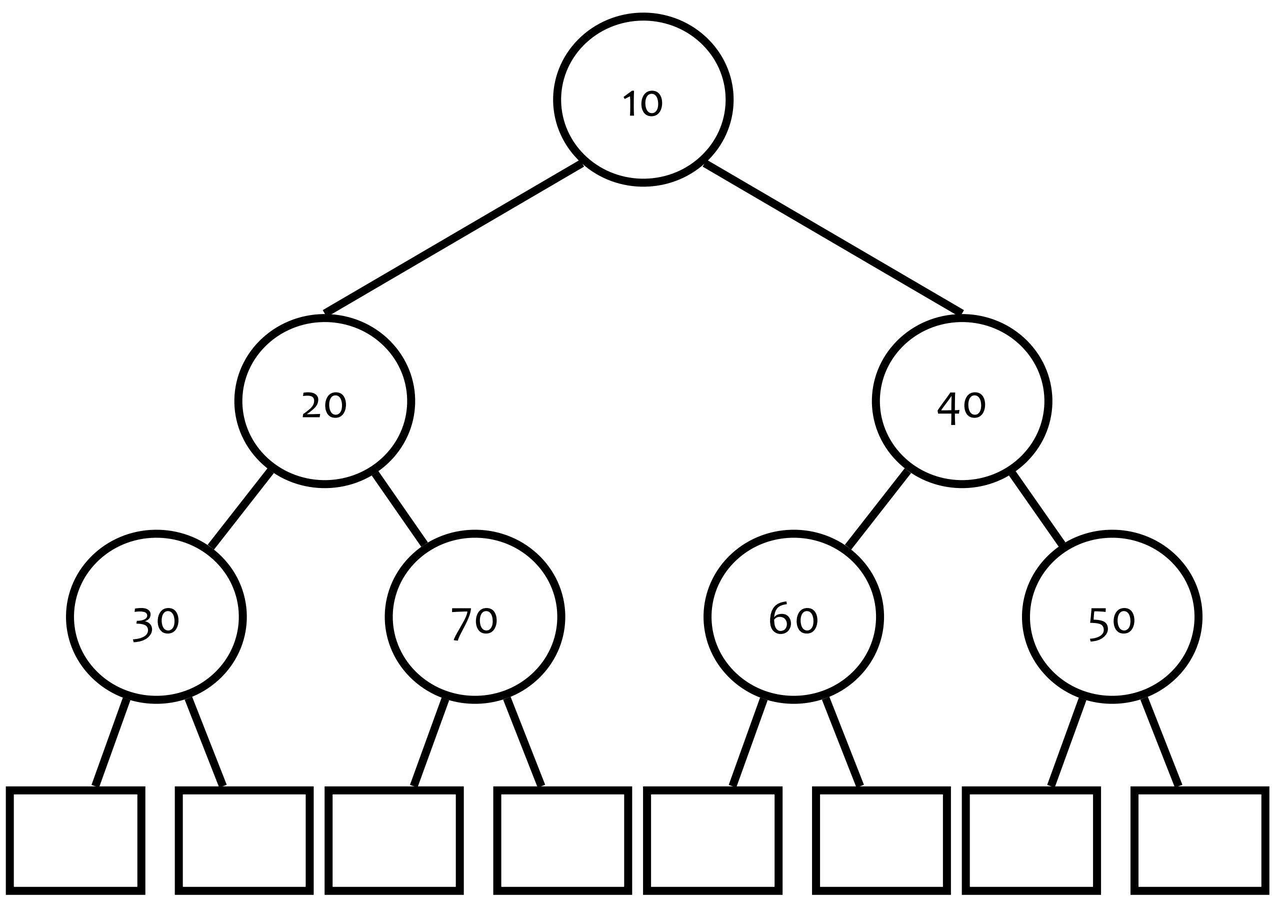

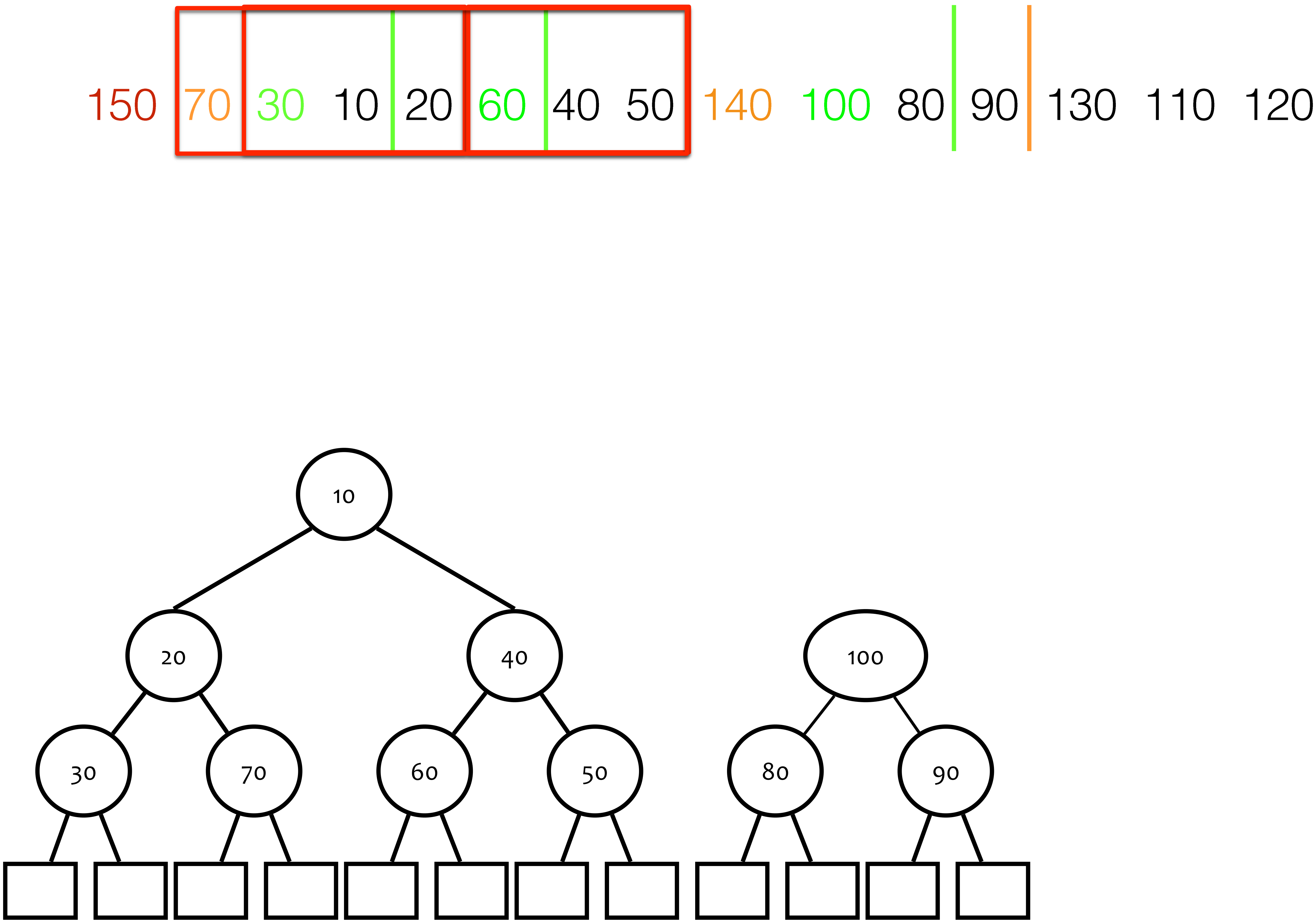
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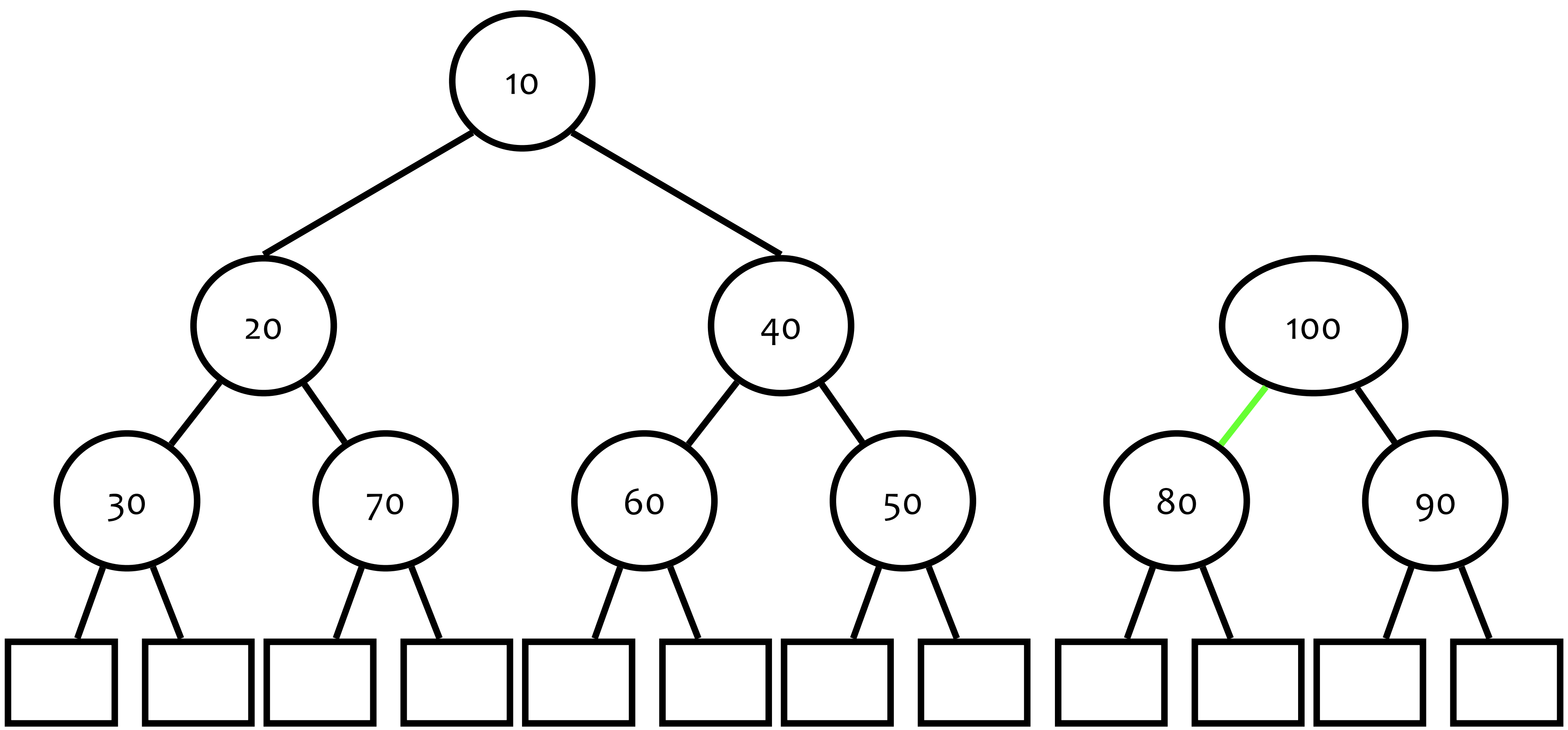

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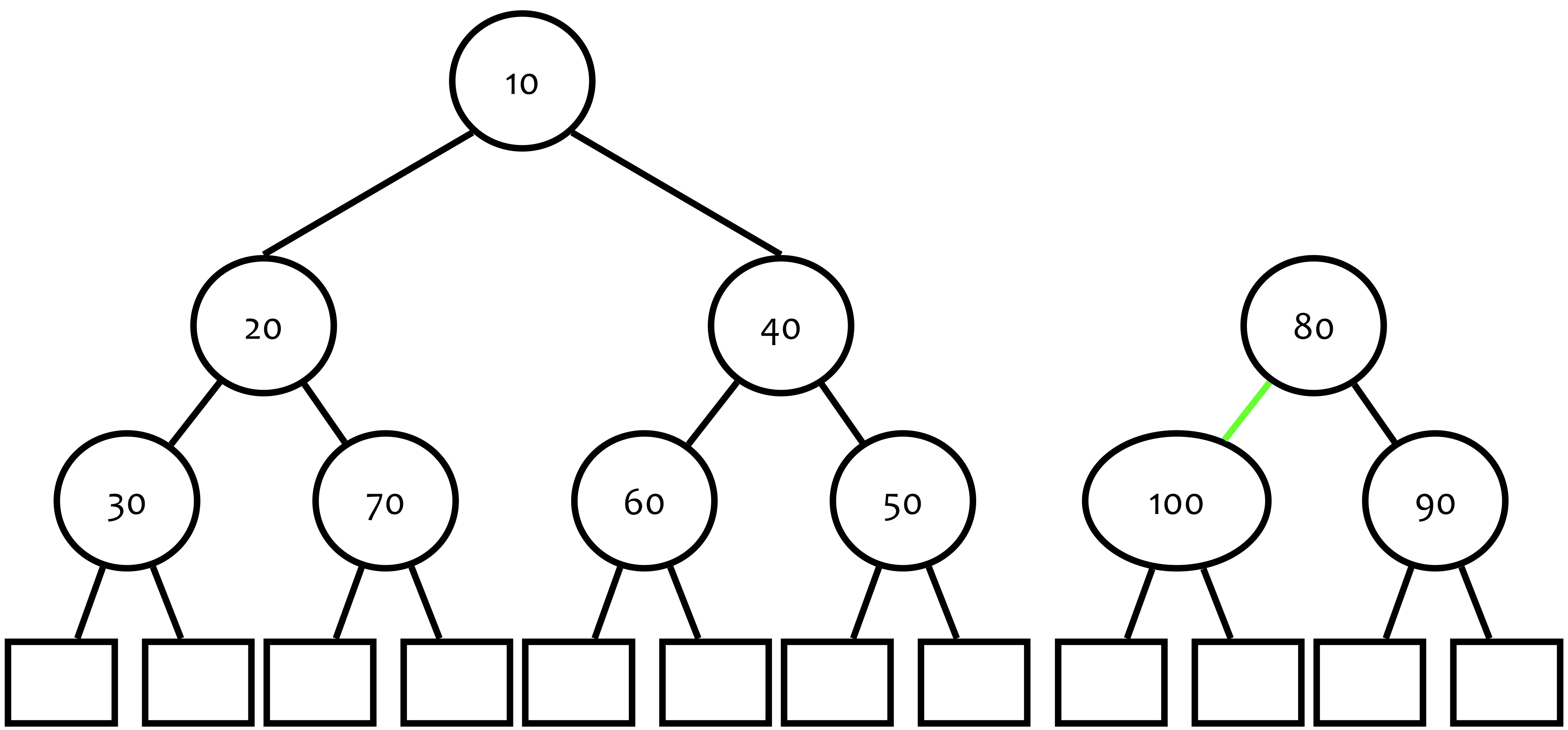

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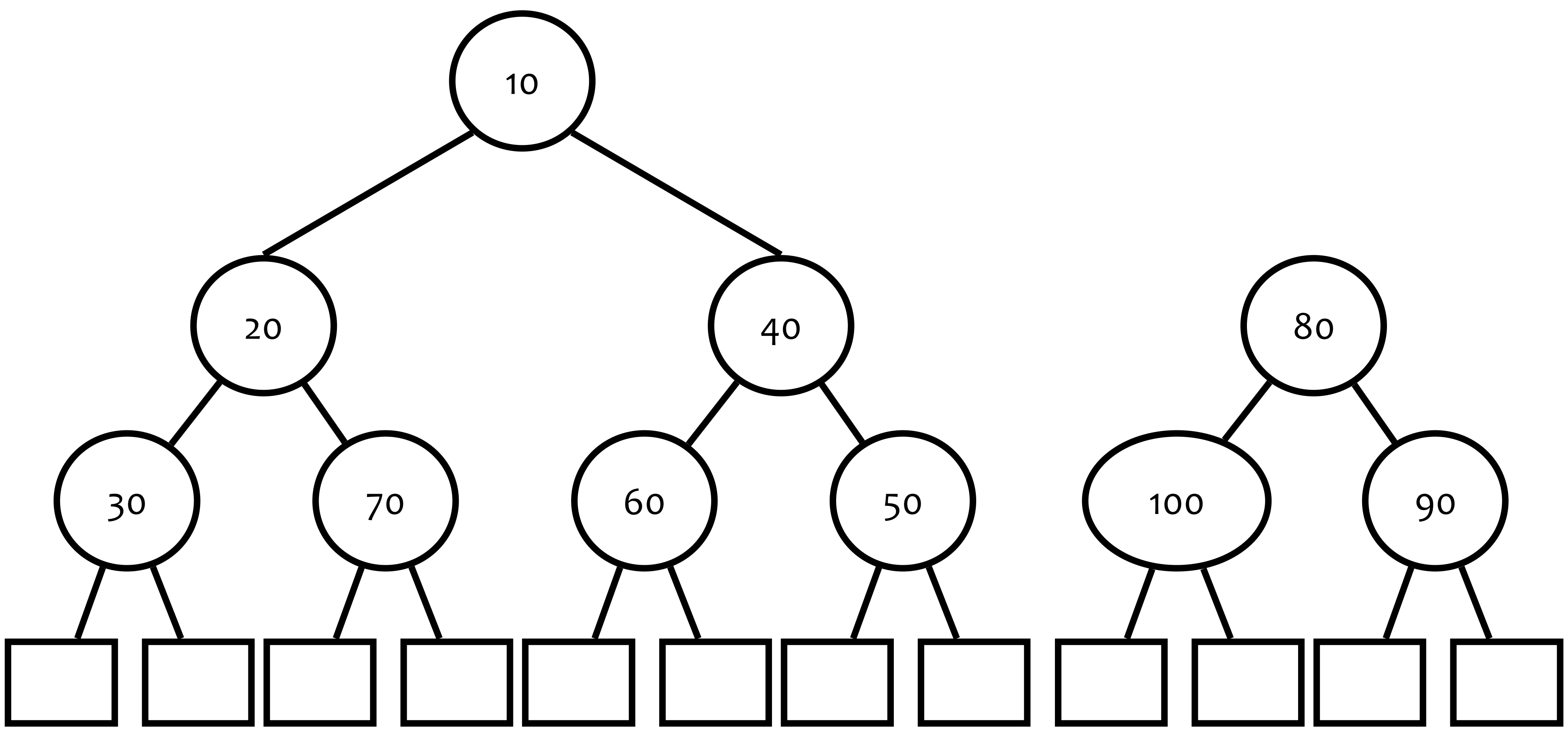

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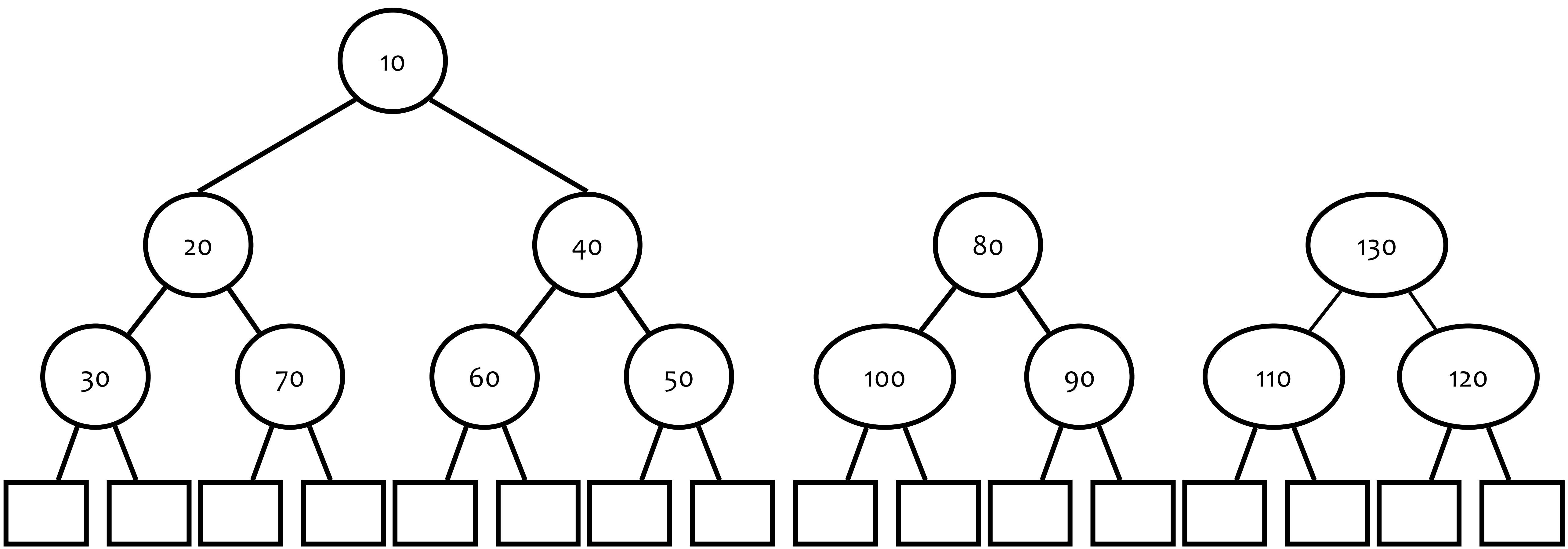

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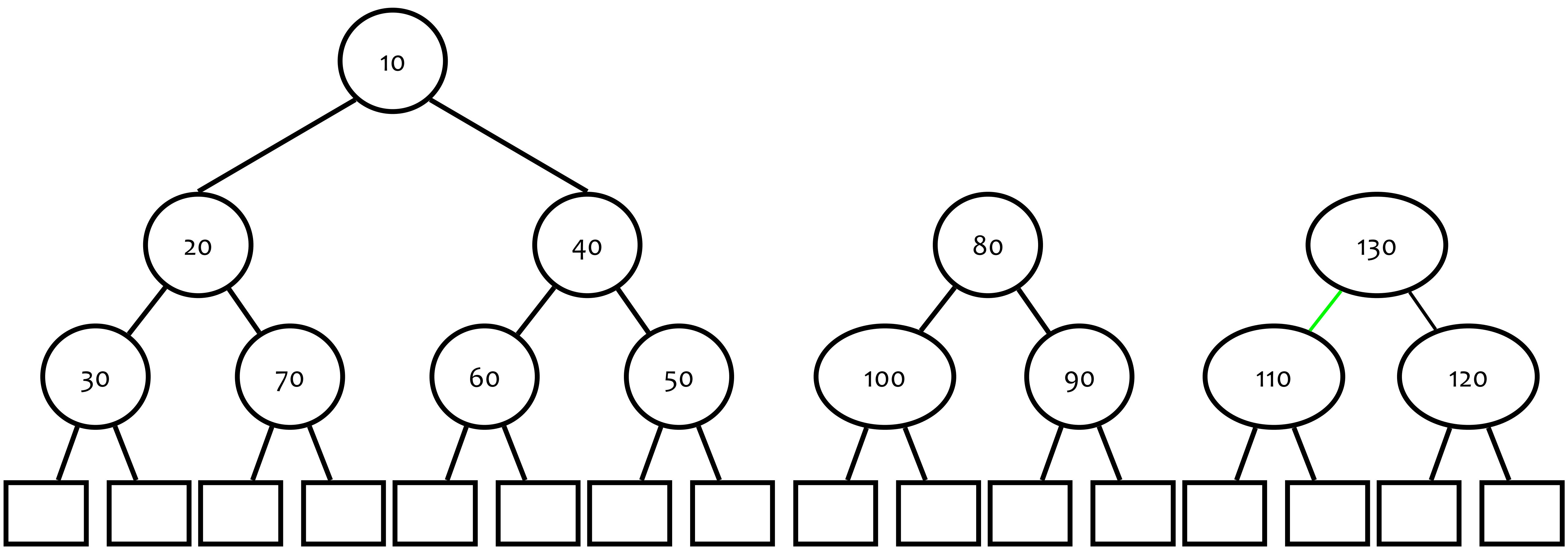

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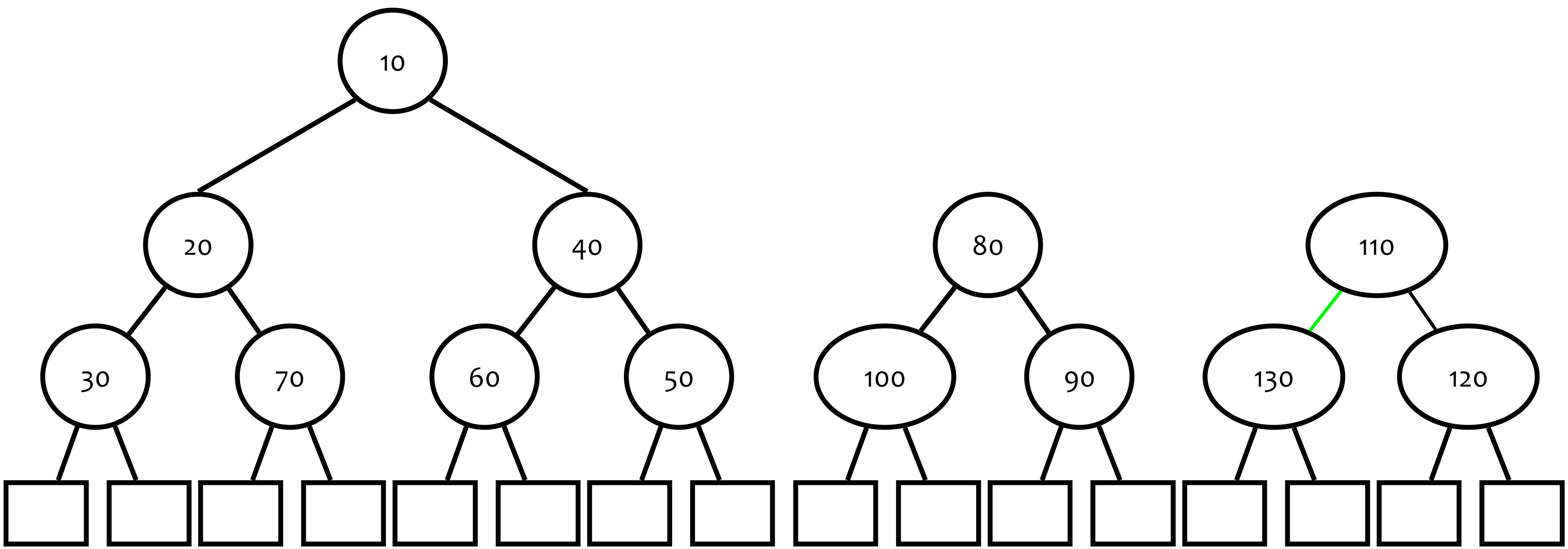

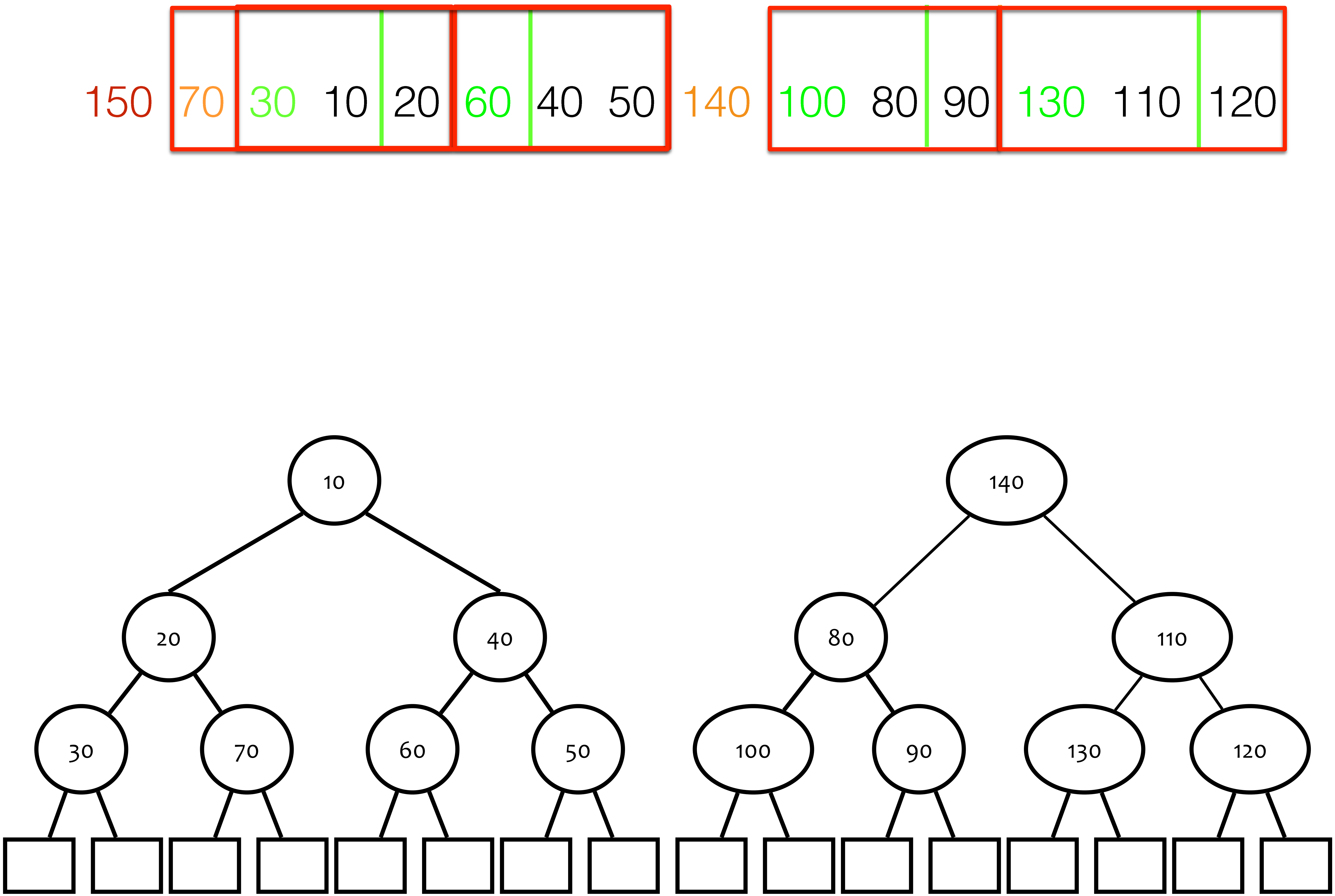
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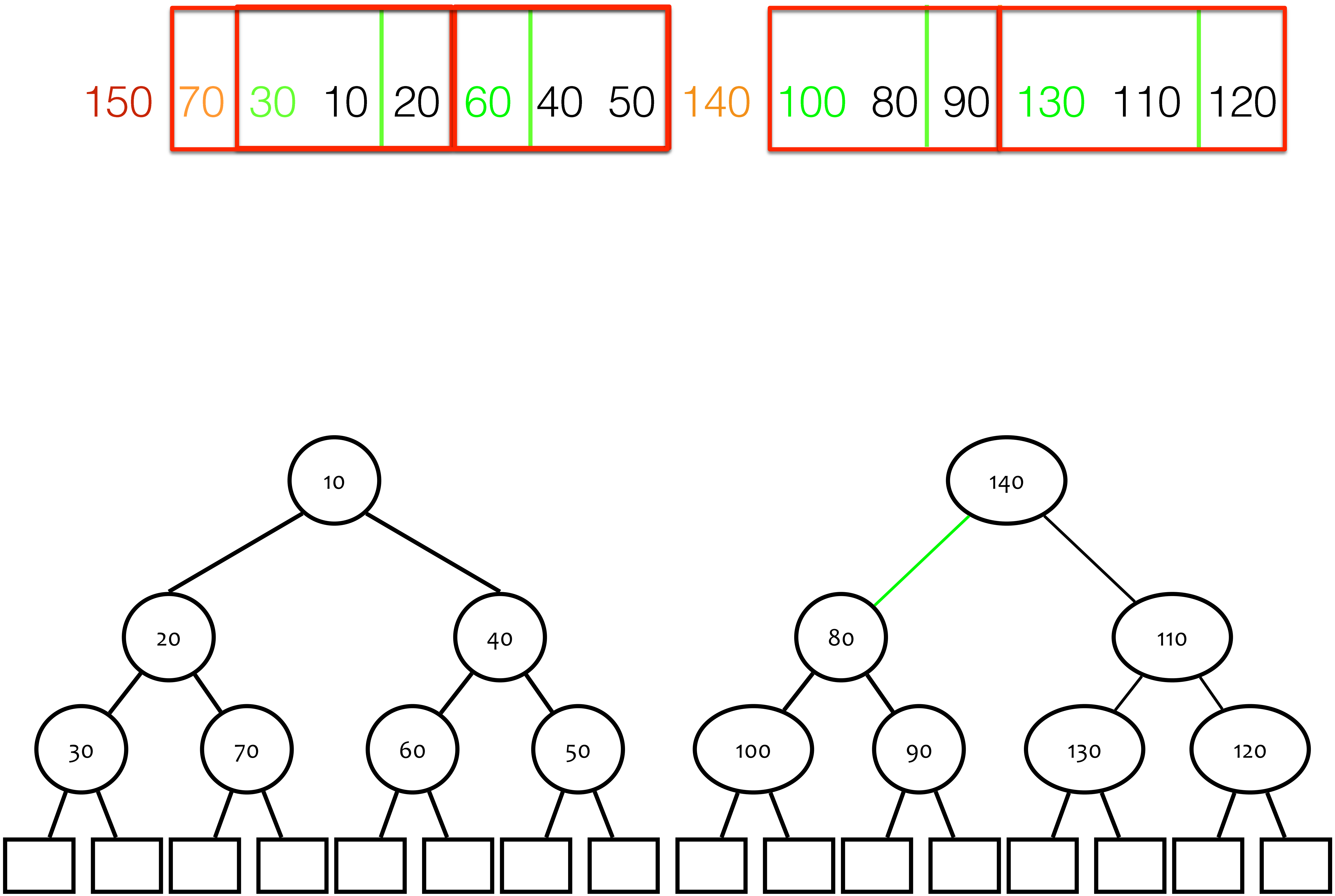
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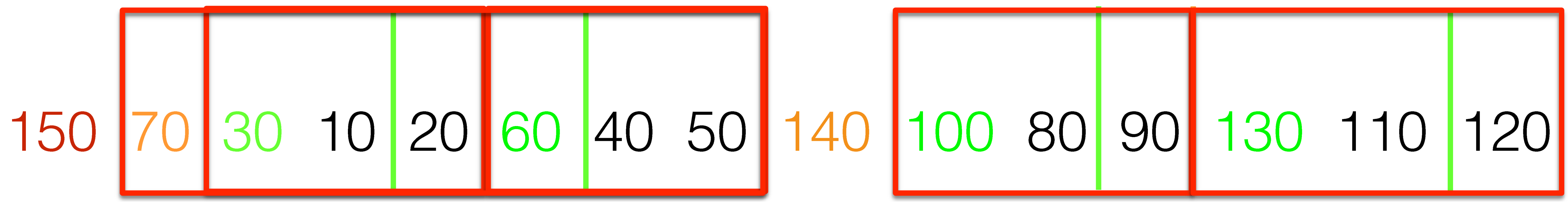
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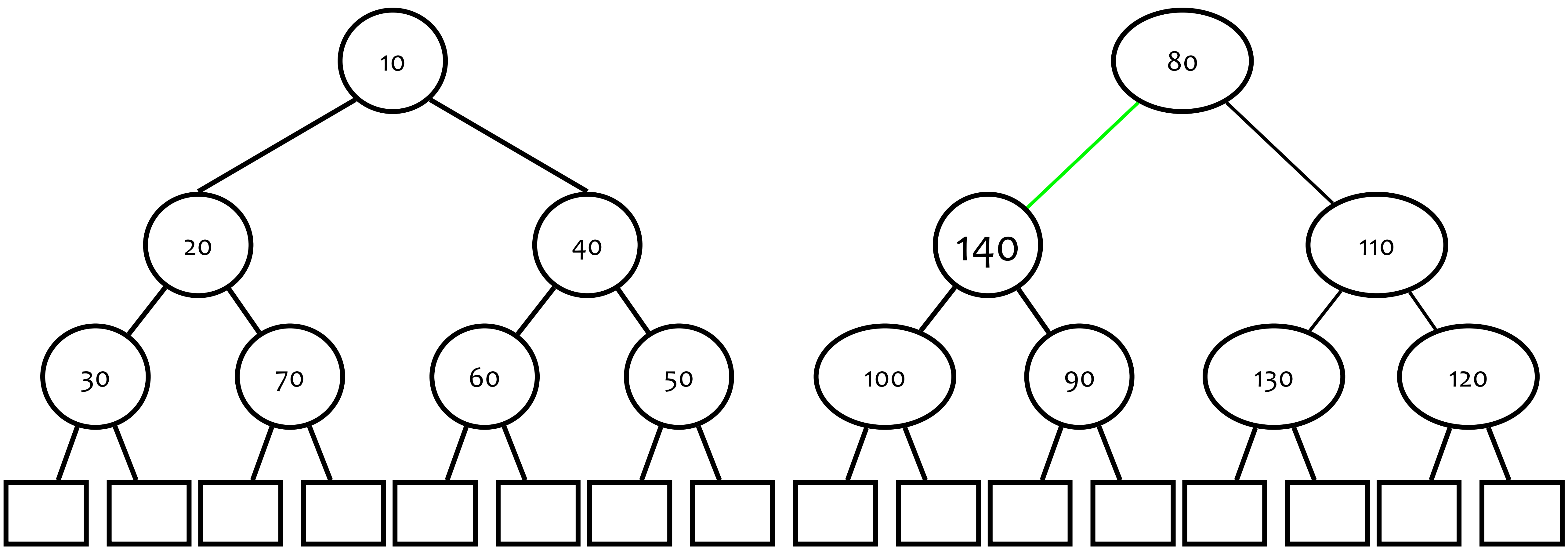
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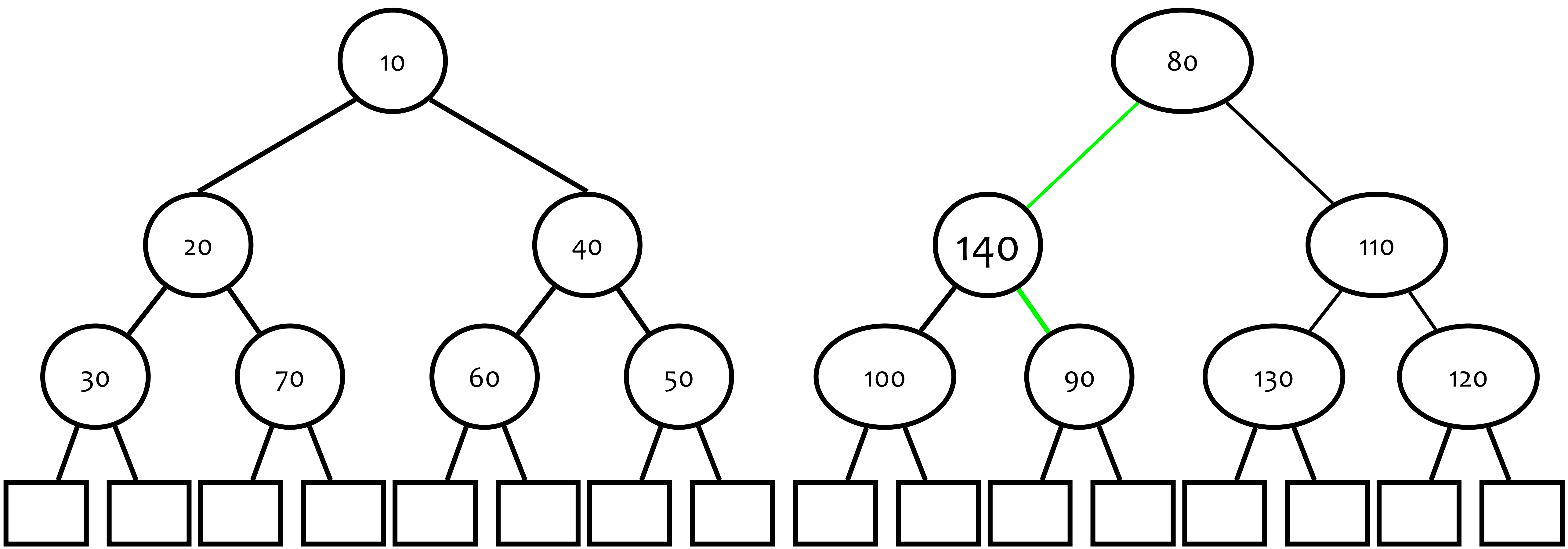

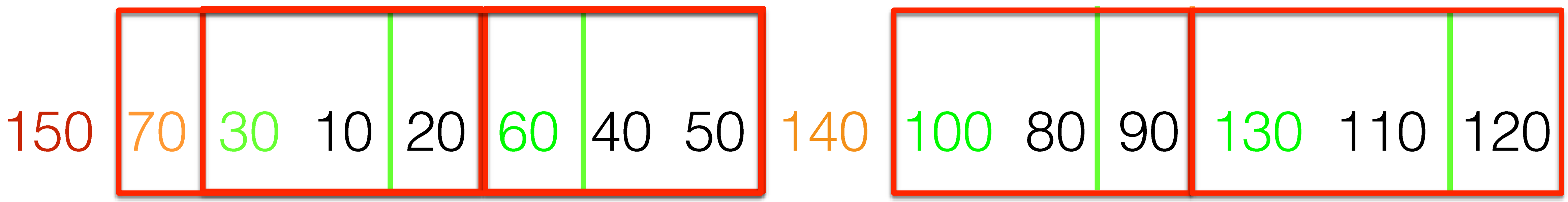
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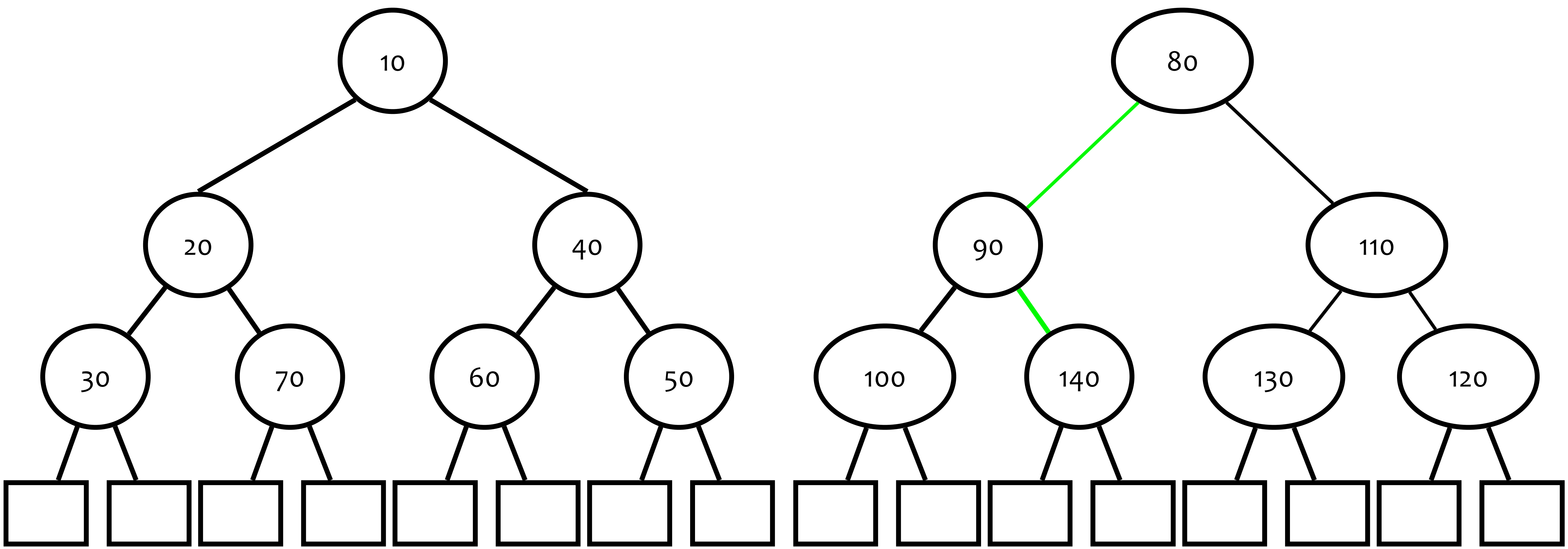
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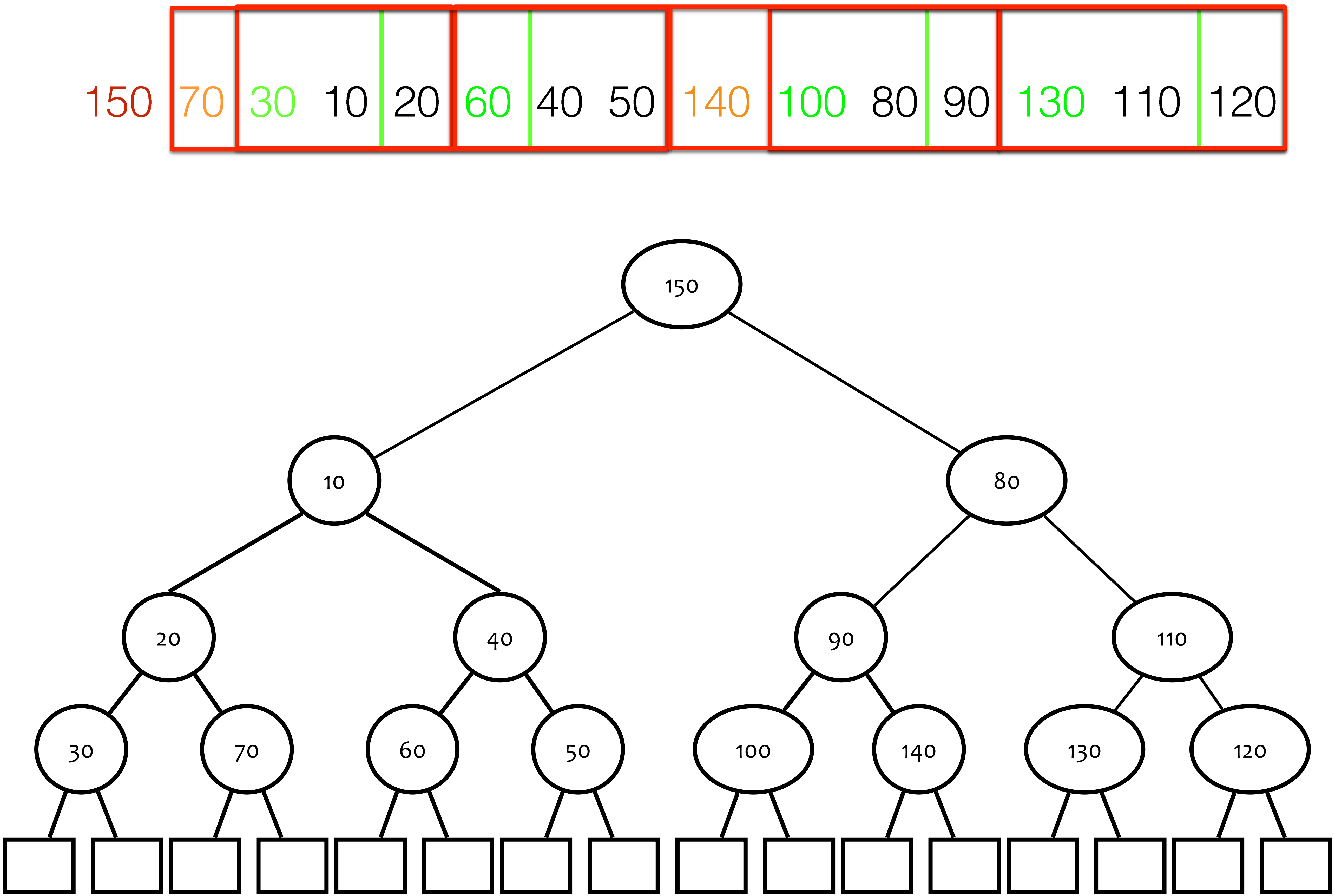

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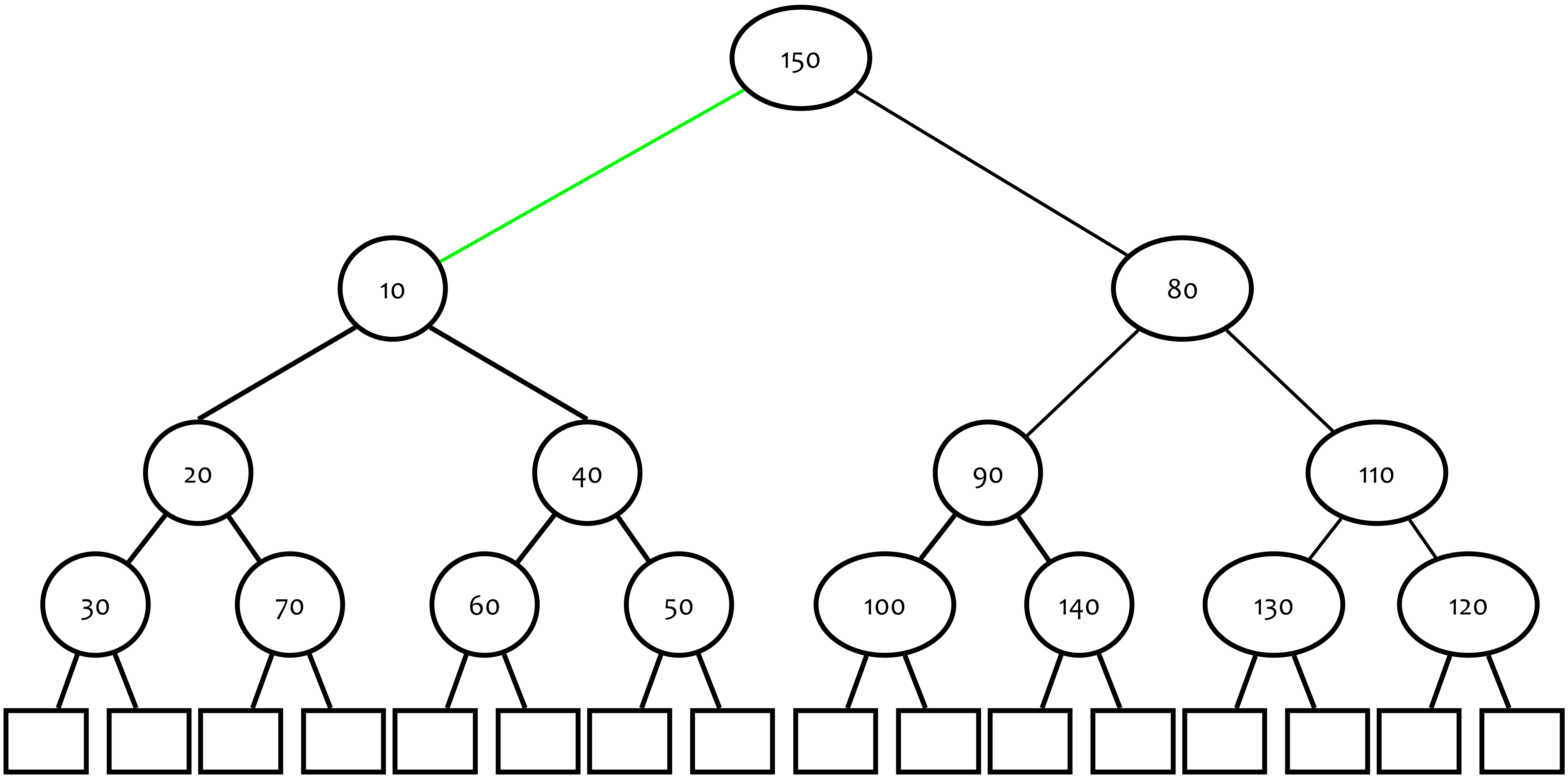

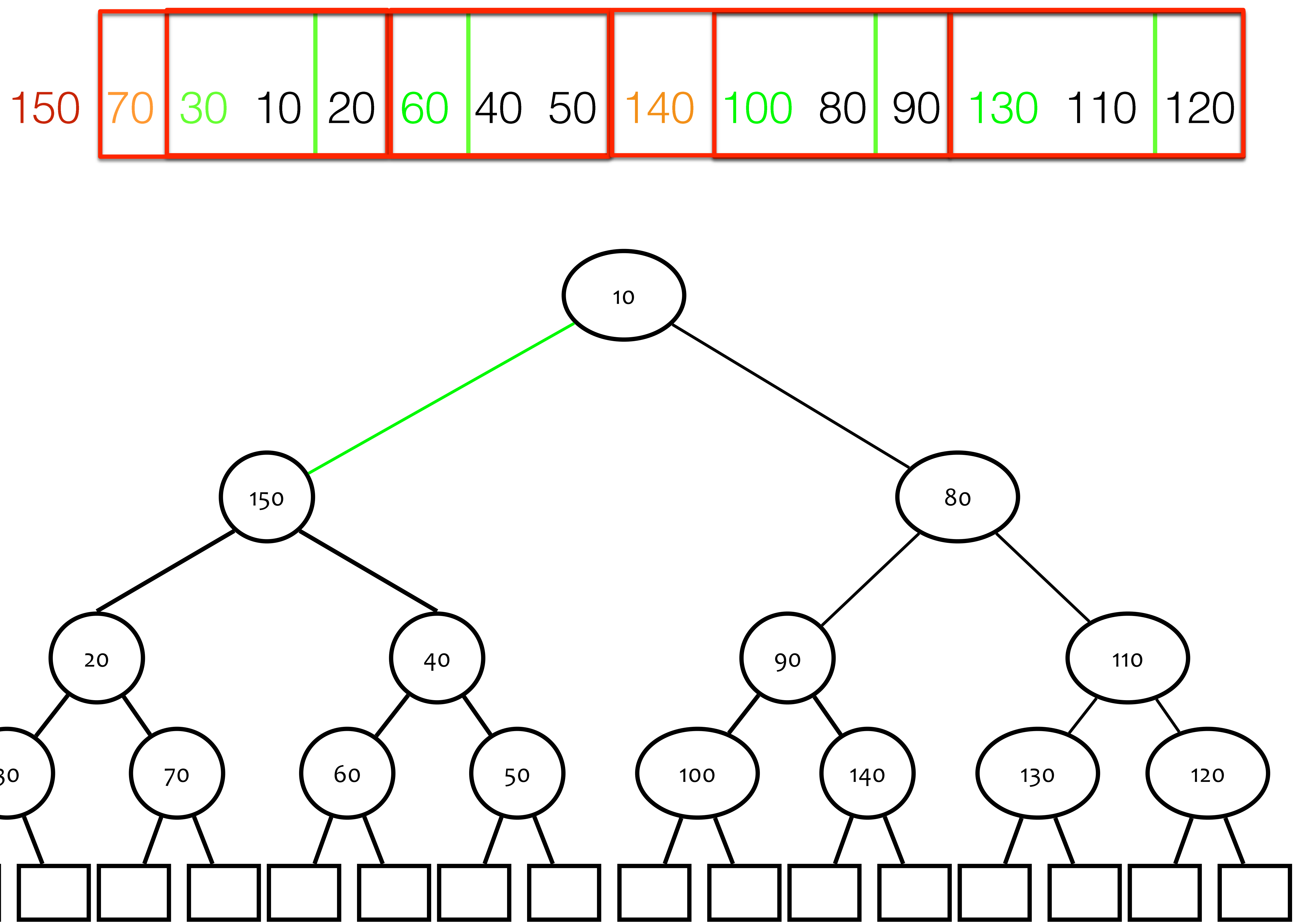
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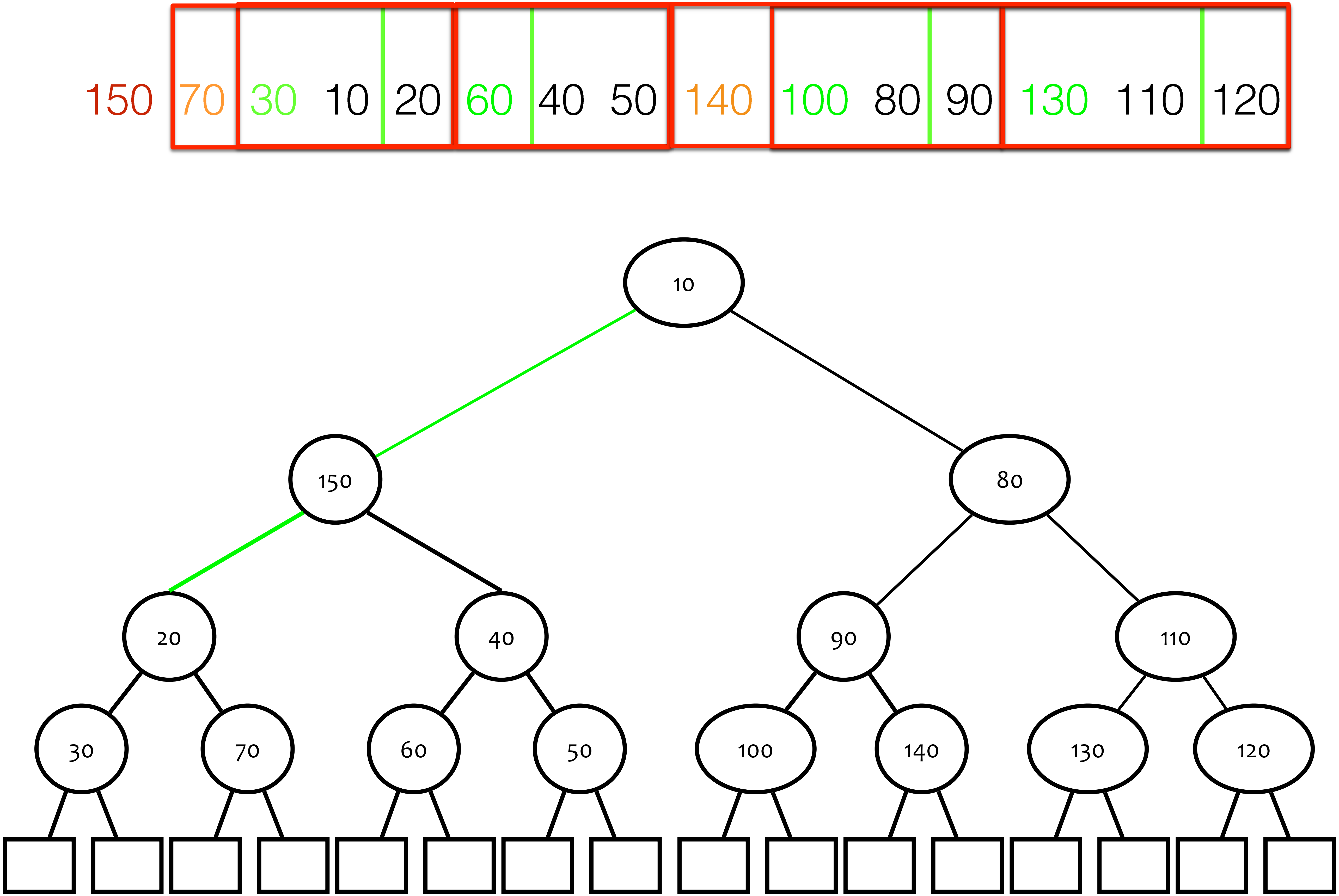
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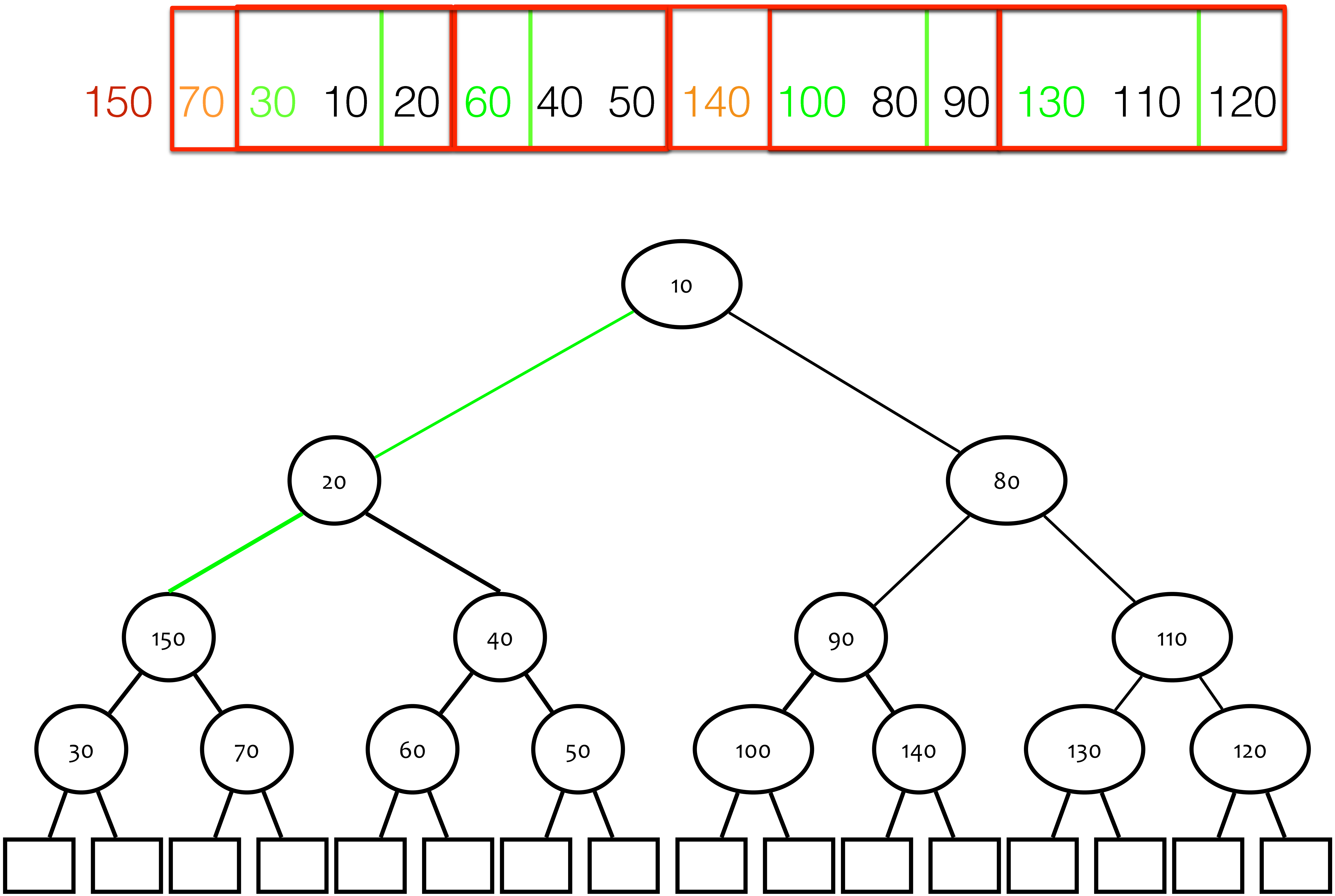
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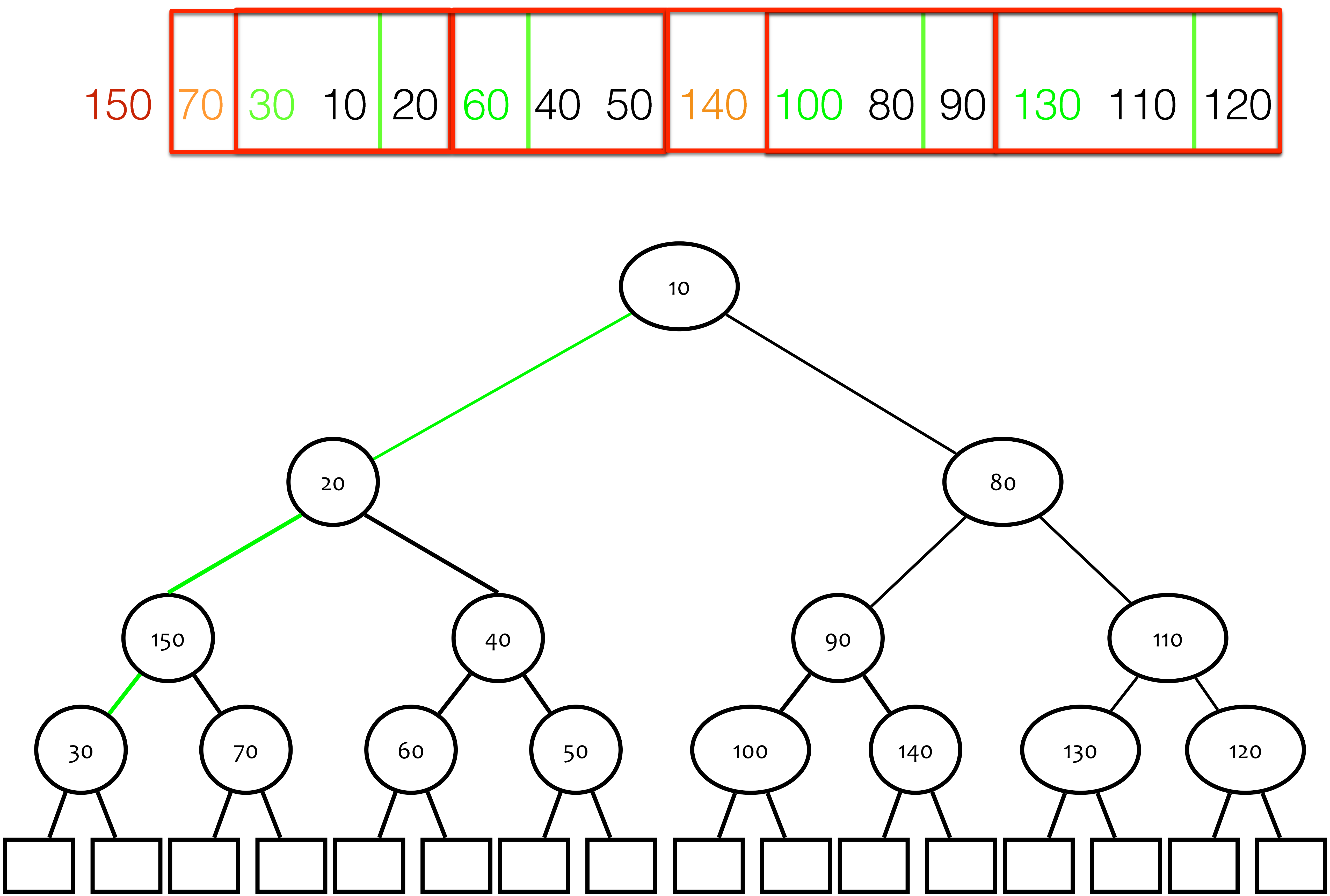

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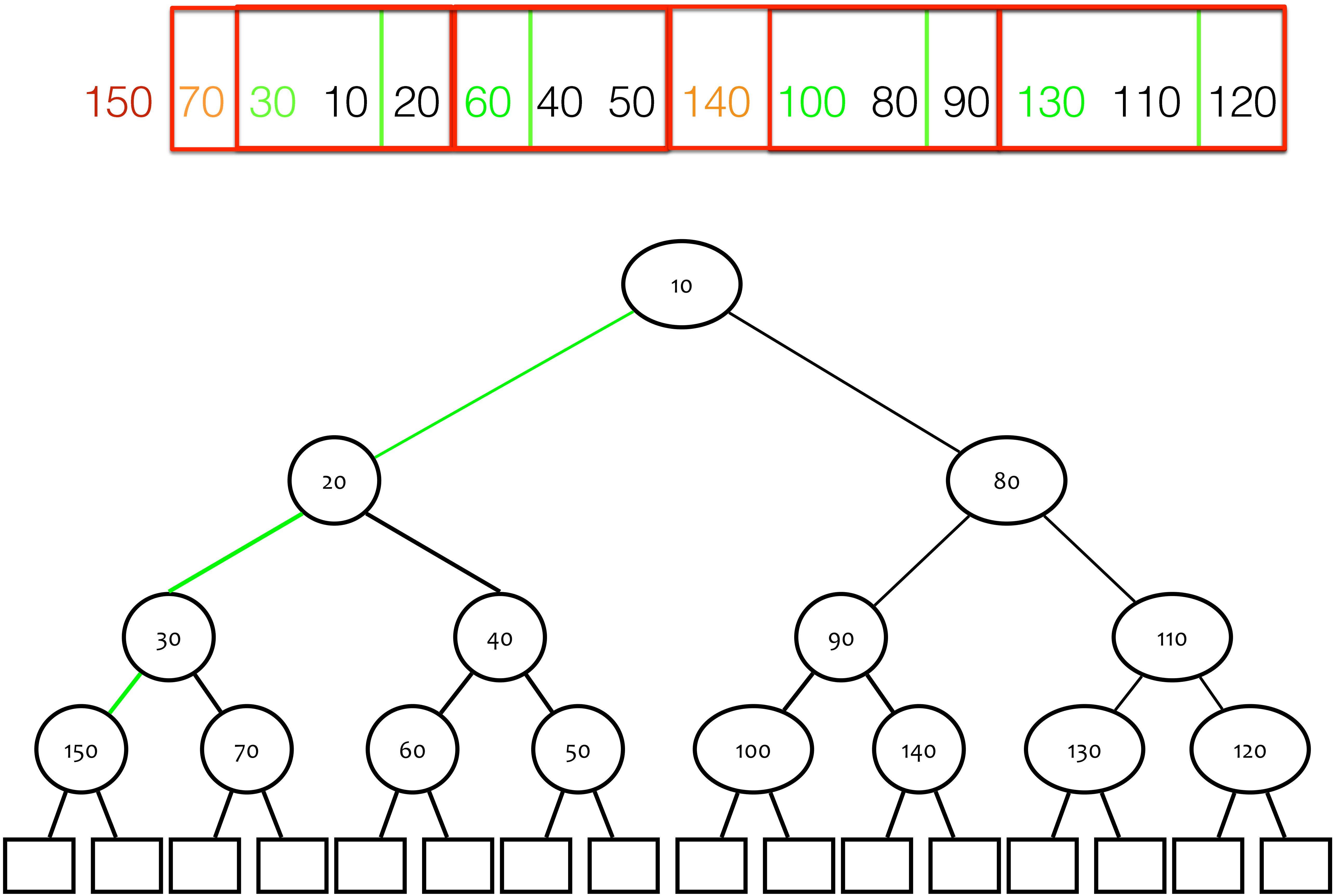
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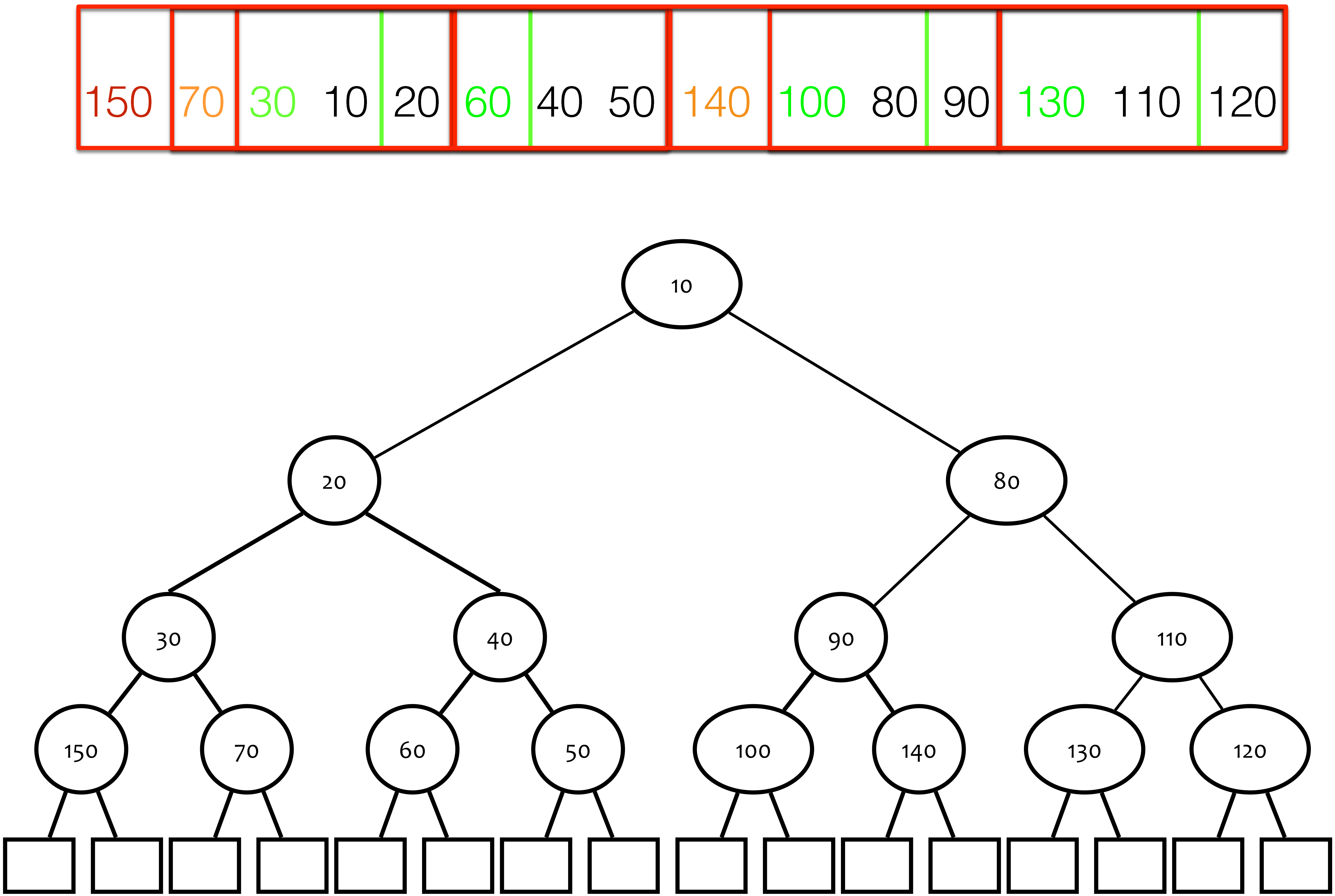
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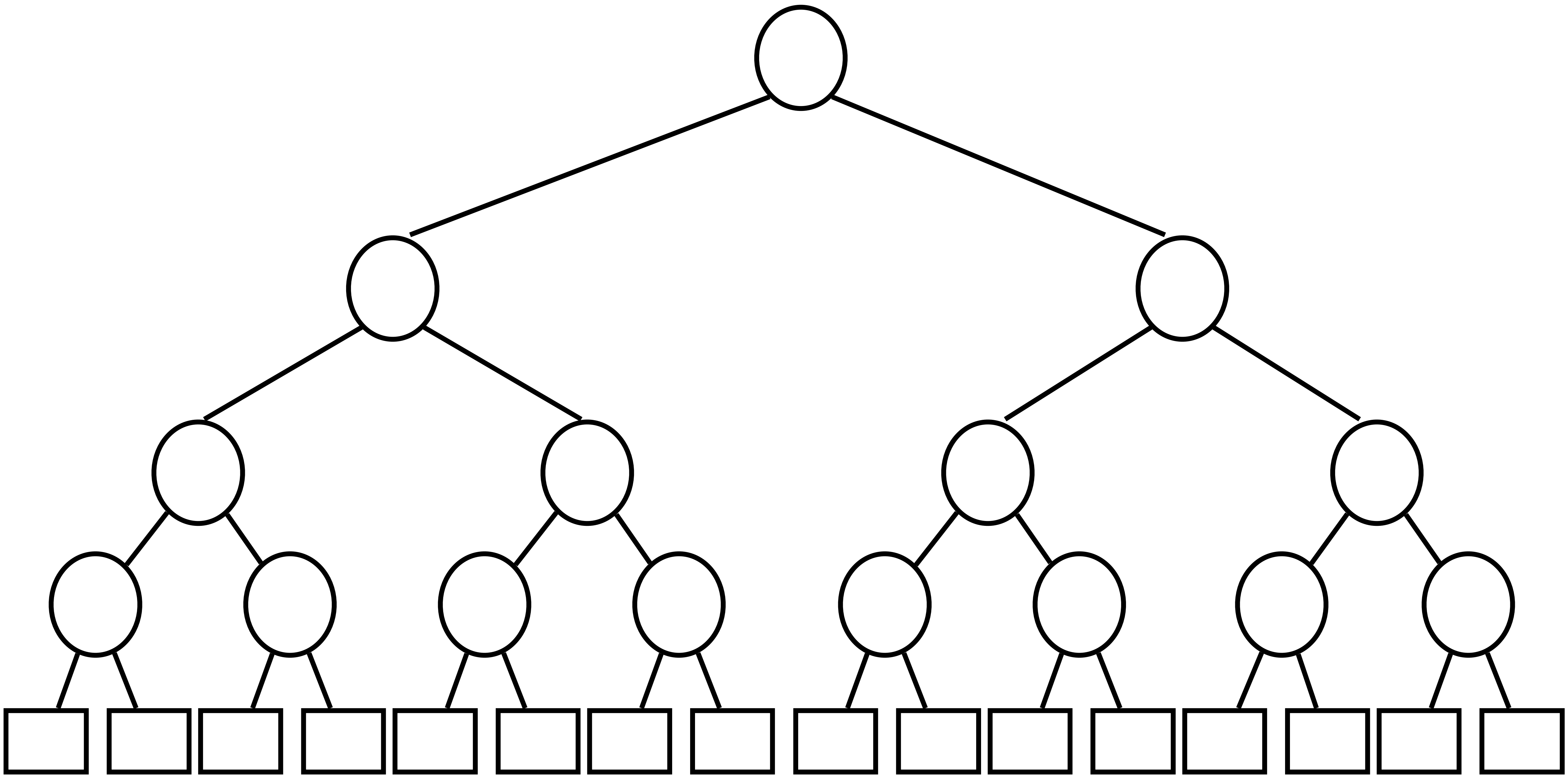
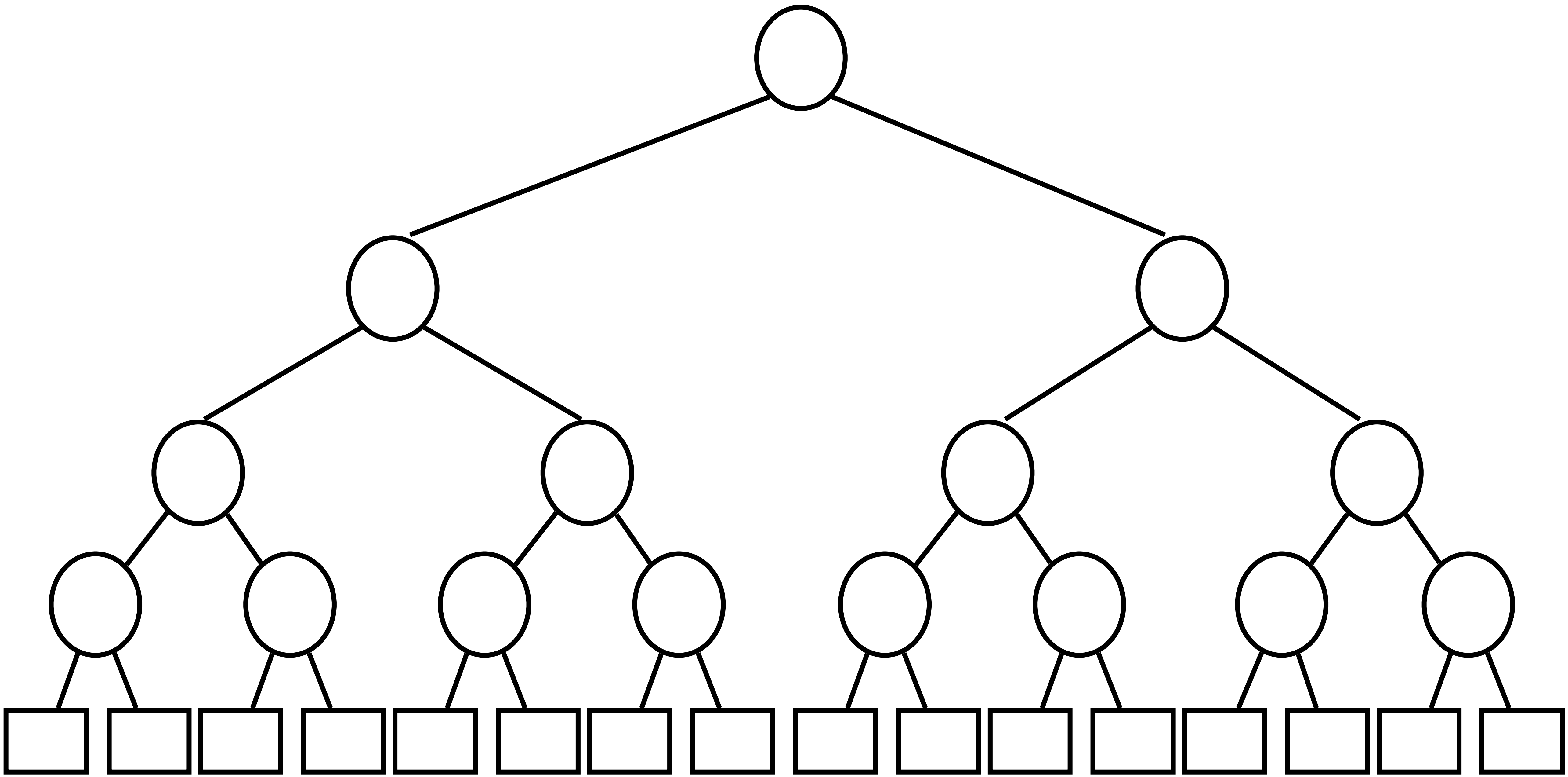
Did we really insert all m elements in $O(m)$ time??



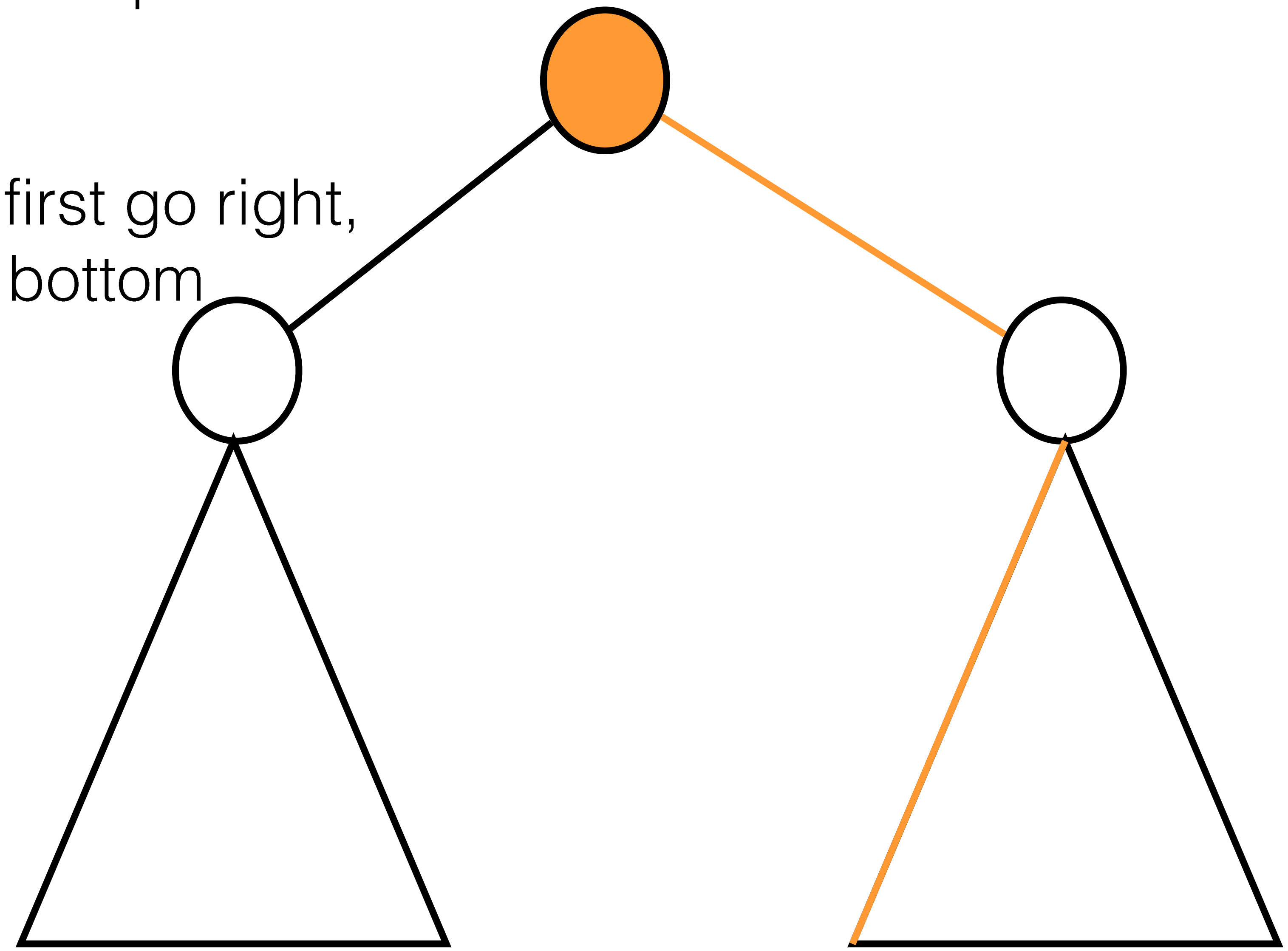
We show: max# bubble down ops < # heap edges



Proof idea

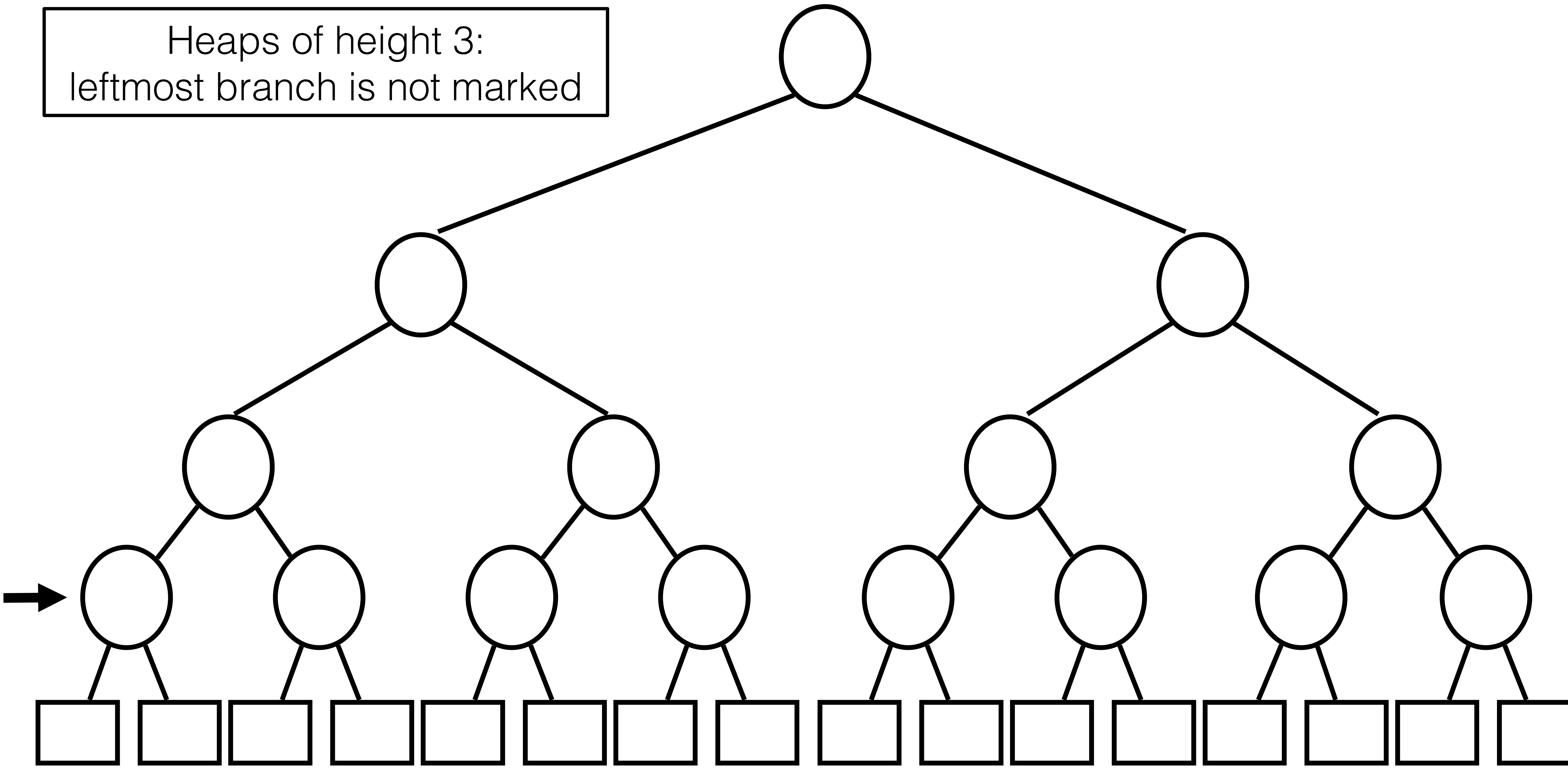
For each new node joining two heaps: mark path of maximum number of bubble-down operations

For marking: first go right, then left until bottom



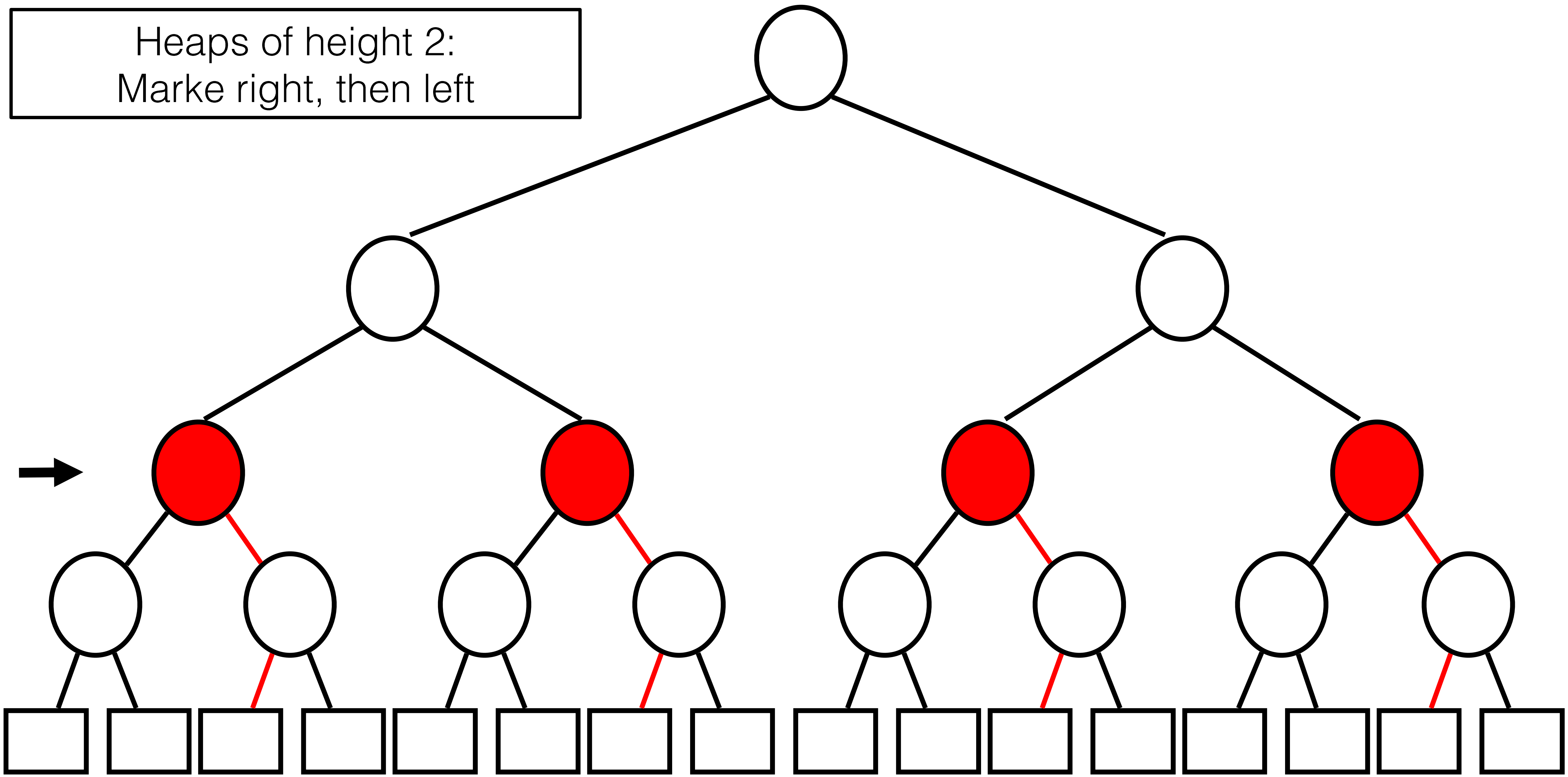
For each new node joining two heaps: mark path with maximum number of bubble-down operations

Heaps of height 3:
leftmost branch is not marked



For each new node joining two heaps: mark path with maximum number of bubble-down operations

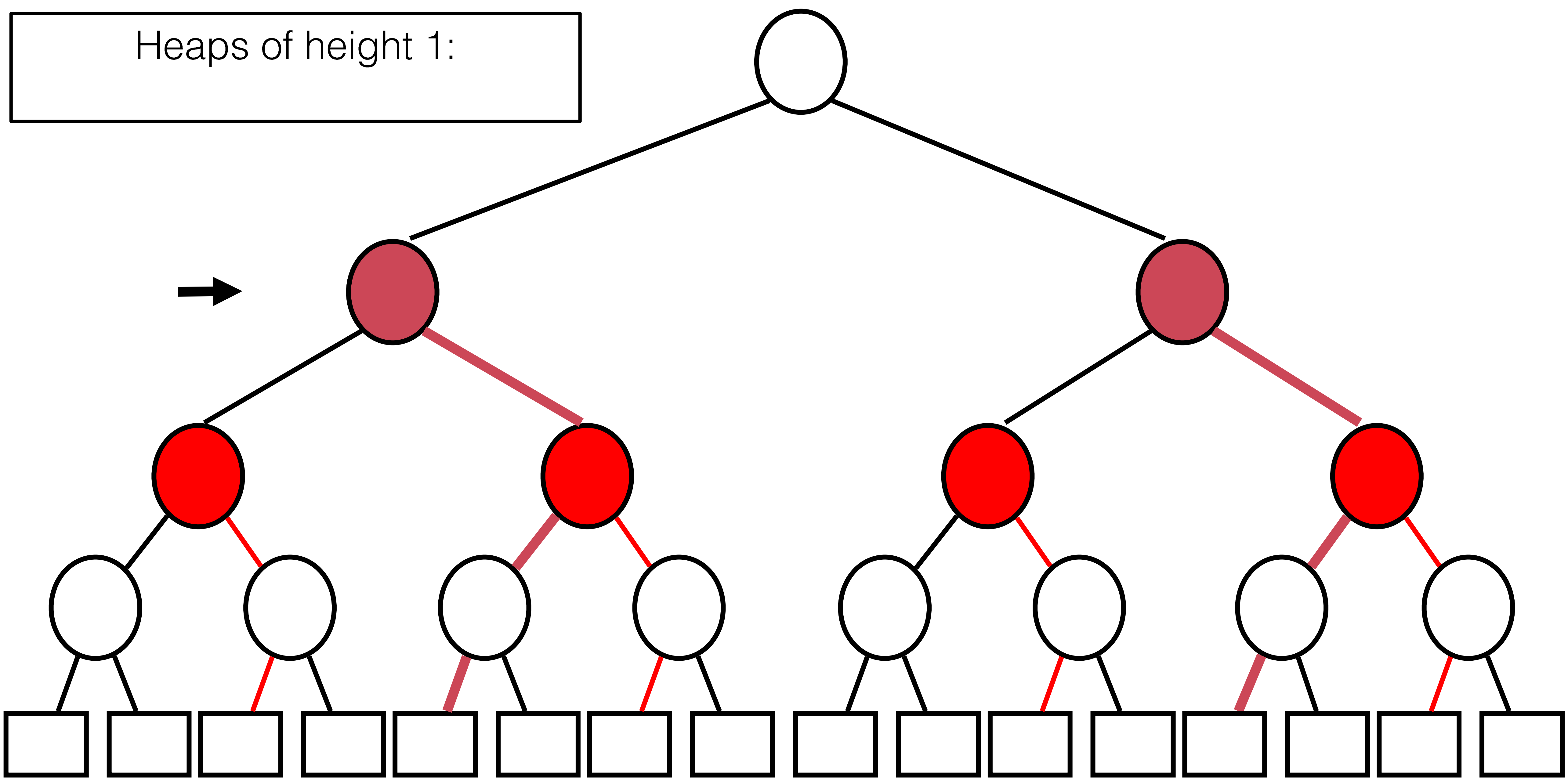
Heaps of height 2:
Marke right, then left



For each height-2 heap, leftmost branch not marked

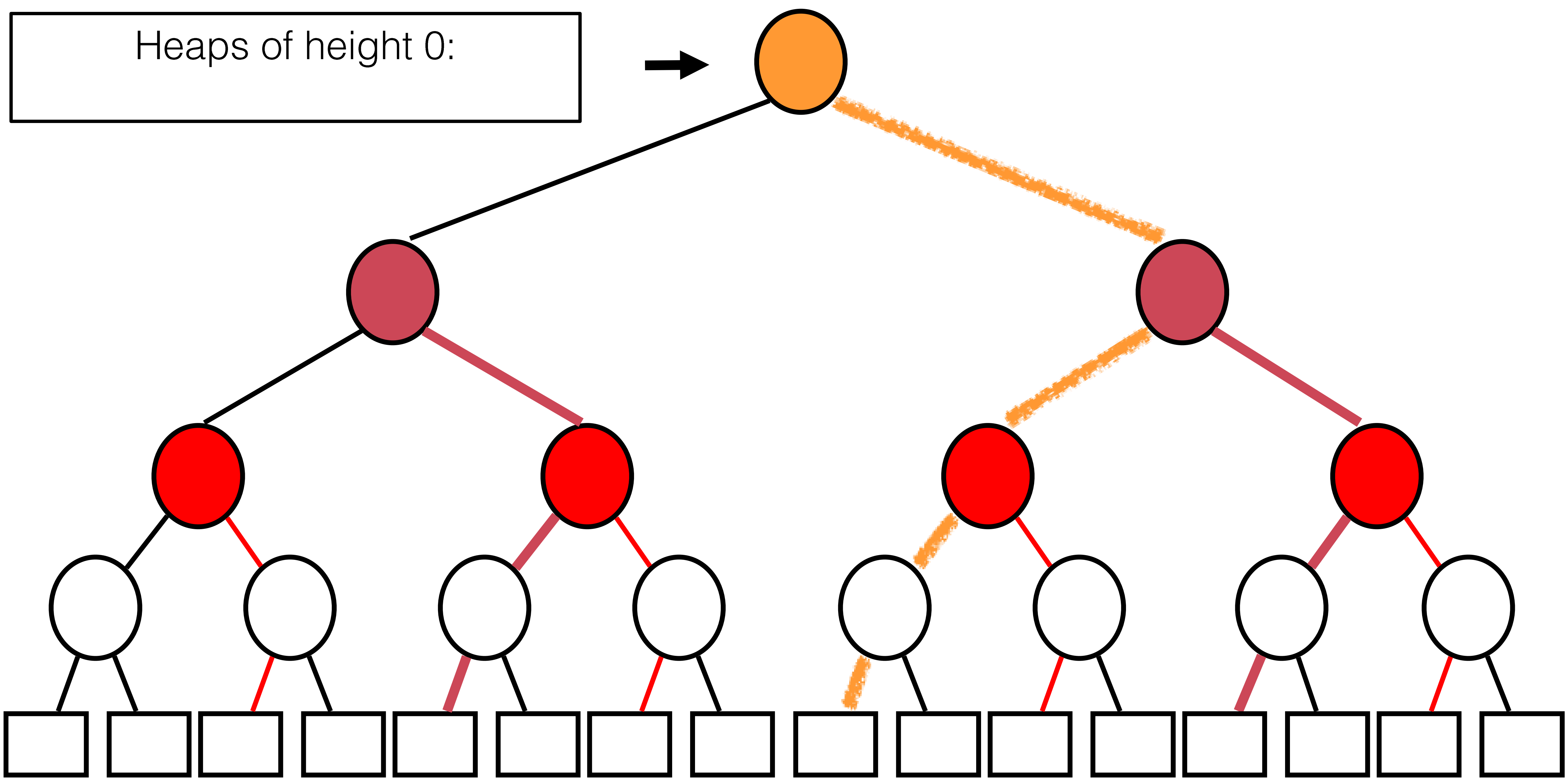
For each new node joining two heaps: mark path with maximum number of bubble-down operations

Heaps of height 1:



For each height-3 heap, leftmost branch not marked

For each new node joining two heaps: mark path with maximum number of bubble-down operations



For height-4 heap, leftmost branch not marked

Inductive argument: marking procedure will never mark all edges in heap, since the leftmost branch is never marked

- Note: leftmost branch in height-h heap: not marked
- When joining 2 heaps of height h to heap of height $h + 1$: new edges to be marked are
  - edge joining new node and right heap of height h, and
  - edges on left path in the right heap of height h
- We conclude: leftmost branch in height $(h+1)$ heap is not marked

Build Heap In-place

Algorithm buildHeap(A,n):
  for $i \leftarrow \lfloor n/2 \rfloor$ to 1 do
    downHeap(A,i)

Algorithm downHeap(A,i):
  $l \leftarrow 2i$
  $r \leftarrow 2i + 1$
  if $l \leq n \wedge A[l] < A[i]$ then
    $min \leftarrow l$
  else
    $min \leftarrow i$
  if $r \leq n \wedge A[r] < A[min]$ then
    $min \leftarrow r$
  if $i \neq min$ then
    swap(i,min)
    downHeap(A,min)