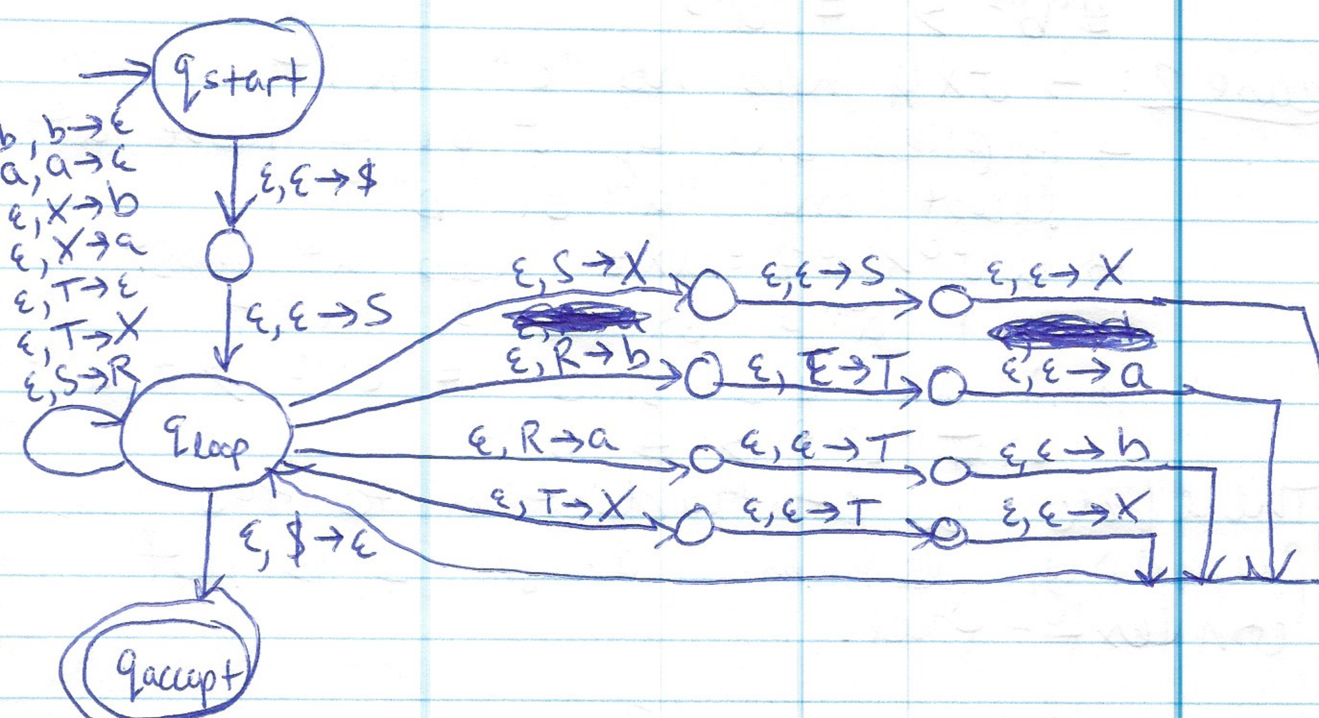
CSC 320 - ASSIGNMENT 4 - SOLUTIONS

1. Let $G = (\{R, S, T, X\}, \{a, b\}, P, S)$ where P is

$$
\begin{aligned}
S &\to XSX \mid R \\
R &\to aTb \mid bTa \\
T &\to XTX \mid X \mid \varepsilon \\
X &\to a \mid b
\end{aligned}
$$

Construct PDA $D = (Q, \{a, b\}, \{a, b, \$\}, \delta, q_{start}, \{q_{accept}\})$ with δ as follows,



2. Let $L = \{a^n b^n c^i \mid i \leq n\}$.

- Suppose L is context-free
- then $\exists p$, the pumping length by pumping lemma
- Let $s = a^p b^p c^p$, clearly $|s| = 3p \geq p$
- Then $s = uvxyz$ such that $|vxy| \leq p$, $v, y \neq \varepsilon$

- Two cases, vxy contains a "c" or it doesn't.
- <u>case 1</u>: → vxy has a "c" in it
  - → implies vxy has no "a"'s
  - → one of v or y have at least one "b" or "c"
  - → uv^2xy^2z has at least $p+1$ "b"'s or $p+1$ "c"'s in it
  - → $\therefore uv^2xy^2z \notin L$ since # "c"'s > # "a"'s or # "b"'s > # "a"'s
- <u>case 2</u>: → vxy has no "c" in it
  - → implies one of v or y have at least one "a" or "b"
  - → $uv^0xy^0z = uxz$ has at most $p-1$ "a"'s or "b"'s in it
  - → $\therefore uxz \notin L$ since # "c"'s > # "a"'s or # "c"'s > # "b"'s
- Therefore a contradiction to the pumping lemma and L is not context-free.

3. Let $L = \{0^p \mid p \text{ is prime}\}$.
   - Suppose L is context-free, then by the pumping lemma
   - $\exists n$ the pumping length
   - Let $p \geq n+2$ be a prime number
   - Let $s = 0^p$, so $|s| \geq n$
   - Let $s = uvxyz$ such that $|vxy| \leq n$, $v,y \neq \varepsilon$
   - Let $|vy| = m \geq 1$, then $|uxz| = p-m$

- Now, consider $uv^{p-m}xy^{p-m}z$.
- $|uv^{p-m}xy^{p-m}z| = |uxz| + (p-m)|v| + (p-m)|y|$
$= (p-m) + (p-m)(|v|+|y|)$
$= (p-m) + (p-m)m$
$= (m+1)(p-m)$
- Thus, $uv^{p-m}xy^{p-m}z = 0^{(m+1)(p-m)}$ where $m \geq 1$, $\Rightarrow m+1 \geq 2$ and $p \geq n+2$, and $m \leq n$
- Implies that $p-m \geq 2$, thus $(m+1)(p-m)$ is not prime.
- Therefore $uv^{p-m}xy^{p-m}z \notin L$ and L is not context-free.

4. Let $L_1 = \{a^n b^{2n} c^m \mid n, m \geq 0\}$ and $L_2 = \{a^n b^m c^{2m} \mid n, m \geq 0\}$

(a) Let $G_1 = (\{S_1, A_1\}, \{a,b,c\}, P_1, S_1)$ and $G_2 = (\{S_2, A_2\}, \{a,b,c\}, P_2, S_2)$ where P_1 + P_2 are as follows:

<u>P_1</u>: $S_1 \rightarrow aS_1bbA_1 \mid A_1 \mid \varepsilon$
$A_1 \rightarrow cA_1 \mid \varepsilon$

<u>P_2</u>: $S_2 \rightarrow A_2bS_2cc \mid A_2 \mid \varepsilon$
$A_2 \rightarrow aA_2 \mid \varepsilon$

Therefore, L_1 + L_2 are context-free as G_1 + G_2 are context-free grammars that accept L_1 and L_2, respectively.

(b) $L_1 \cap L_2 = \{a^n b^{2n} c^{4n} \mid n \geq 0\}$.
- Assume $L_1 \cap L_2$ is context-free then
- $\exists p$ such that it satisfies the pumping lemma.

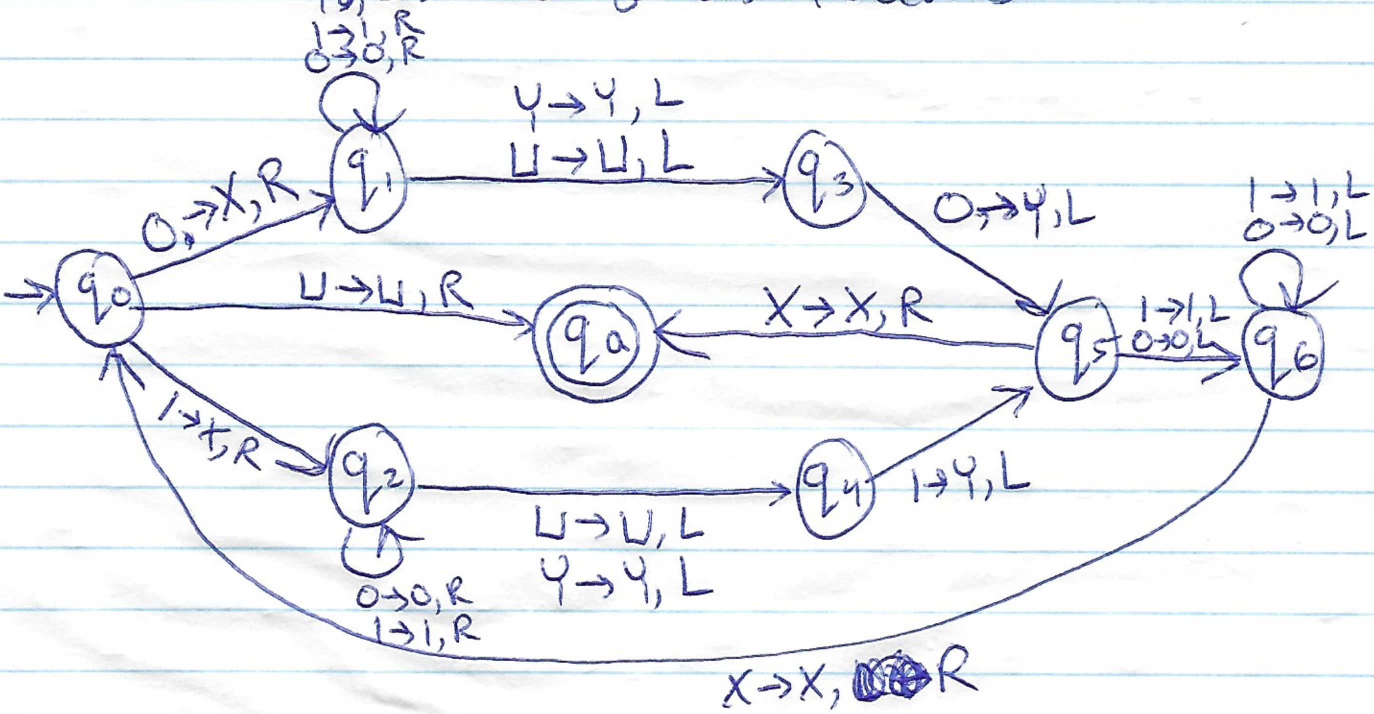
- Let $s = a^p b^{2p} c^{4p}$, where $|s| = 7p \geq p$
- Let $s = uvxyz$ such that $|vxy| \leq p$ and $v, y \neq \varepsilon$.
- Two cases, either $v\cancel{xy}$ contains a c or it doesn't
- <u>case 1</u>: → Suppose vxy contains ~~no~~ c
  → then v or ~~y~~ has at least one a or b
  → if an a, then uv^2xy^2z has at least $p+1$ a's and at most $2p$ a's
  → But $4(p+1) = 4p+4 > 4p$ so ratio of a's + c's is off
  → if a b, then uv^2xy^2z has at least $2p+1$ b's
  → But $2(2p+1) = 4p+2 > 4p$ so ratio of b's + c's ~~is~~ off.
  → Thus, $uv^2xy^2z \notin L_1 \cap L_2$
- <u>case 2</u>: → Suppose vxy contains a c
  → Then, vxy has no a's since $|vxy| \leq p$.
  → v or y has at least one b or c.
  → If a b, then uv^2xy^2z has at least $2p+1$ b's
  → But $2p < 2p+1$ and ratio of a's + b's is off
  → If a c, then uv^2xy^2z has at least $4p+1$ c's, but $4p < 4p+1$ and the ratio of a's + c's is off.
  → Thus, $uv^2xy^2z \notin L_1 \cap L_2$
- Therefore, by the pumping lemma, $L_1 \cap L_2$ is not context-free.

5. Let $L = \{ww^R \mid w \in \{0,1\}^*\}$. We will design a Turing Machine $M = (Q, \{0,1\}, \Gamma, \delta, q_0, q_a, q_r)$ with the following high-level description.

High-level:

1. Read the first symbol, 0 or 1, and mark it.
2. Move to the other end of the tape and read the last symbol
3. if it's the same mark it and move back to the beginning
4. if it's not, reject
5. repeat until all symbols are marked and accept.

Formal: Let $Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_a, q_r\}$, $\Gamma = \{a, b, X, Y, \sqcup\}$ and δ as follows



→ All other transitions go to q_r