Question 1

$$S(x)=\begin{cases} S_0(x), & \text{If } 0 \leq x \leq \frac{2\pi}{3} \\ S_1(x), & \text{If } \frac{2\pi}{3} \leq x \leq \frac{4\pi}{3} \\ S_2(x), & \text{If } \frac{4\pi}{3} \leq x \leq 2\pi \end{cases}$$

$X_0 = 0 \quad f(0)=0$

$X_1 = \frac{2\pi}{3} \quad f(\frac{2\pi}{3}) = 0.75$

$X_2 = \frac{4\pi}{3} \quad f(\frac{4\pi}{3}) = 0.75$

$X_3 = 2\pi \quad f(2\pi) = 0$

$S_0(x) = a_0 + b_0(x-x_0) + c_0(x-x_0)^2 + d_0(x-x_0)^3$

$S_1(x) = a_1 + b_1(x-x_1) + c_1(x-x_1)^2 + d_1(x-x_1)^3$

$S_2(x) = a_2 + b_2(x-x_2) + c_2(x-x_2)^2 + d_2(x-x_2)^3$

$S_0(x) = a_0 + b_0(x-0) + c_0(x-0)^2 + d_0(x-0)^3$

$S_1(x) = a_1 + b_1(x-\frac{2\pi}{3}) + c_1(x-\frac{2\pi}{3})^2 + d_1(x-\frac{2\pi}{3})^3$

$S_2(x) = a_2 + b_2(x-\frac{4\pi}{3}) + c_2(x-\frac{4\pi}{3})^2 + d_2(x-\frac{4\pi}{3})^3$

By give condition:

① $f(0) = S_0(0) \Rightarrow a_0 = 0$

② $S_0(\frac{2\pi}{3}) = S_1(\frac{2\pi}{3}) = a_0 + b_0\frac{2\pi}{3} + c_0(\frac{2\pi}{3})^2 + d_0(\frac{2\pi}{3})^3 = a_1$

③ $S_0'(\frac{2\pi}{3}) = S_1'(\frac{2\pi}{3}) = b_0 + 2c_0\frac{2\pi}{3} + 3d_0(\frac{2\pi}{3})^2 = b_1$

④ $S_0''(\frac{3\pi}{3}) = S_1''(\frac{2\pi}{3}) = 2c_0 + 6d_0(\frac{2\pi}{3}) = 2c_1 \Rightarrow c_0 + 2\pi d_0 = c_1$

⑤ $S_1(\frac{4\pi}{3}) = S_2(\frac{4\pi}{3}) = a_1 + b_1(\frac{2\pi}{3}) + c_1(\frac{2\pi}{3})^2 + d_1(\frac{2\pi}{3})^3 = a_2$

⑥ $S_1'(\frac{4\pi}{3}) = S_2'(\frac{4\pi}{3}) = b_1 + 2c_1(\frac{2\pi}{3}) + 3d_1(\frac{2\pi}{3})^2 = b_2$

⑦ $S_1''(\frac{4\pi}{3}) = S_2''(\frac{4\pi}{3}) = 2c_1 + 6d_1(\frac{2\pi}{3}) = 2c_2 \Rightarrow c_1 + 2\pi d_1 = c_2$

⑧ $f(\frac{2\pi}{3}) = S_0(\frac{2\pi}{3}) \Rightarrow a_0 + \frac{2\pi}{3}b_0 + c_0(\frac{2\pi}{3})^2 + d_0(\frac{2\pi}{3})^3 = 0.75$

⑨ $f(\frac{2\pi}{3}) = S_1(\frac{2\pi}{3}) \Rightarrow a_1 = 0.75$

⑩ $f(\frac{4\pi}{3}) = S_1(\frac{4\pi}{3}) \Rightarrow a_0 + \frac{2\pi}{3}b_1 + c_1(\frac{2\pi}{3})^2 + (\frac{2\pi}{3})^3 d_1 = 0.75$

⑪ $f(\frac{4\pi}{3}) = S_2(\frac{4\pi}{3}) \Rightarrow a_2 = 0.75$

⑫ $f(2\pi) = S_2(2\pi) \Rightarrow a_2 + \frac{2\pi}{3}b_2 + c_2(\frac{2\pi}{3})^2 + (\frac{2\pi}{3})^3 d_2 = 0$

Question 2

Stack on setting Variable and function.

Question3

1.) To determine the language form of Intapolating polynomial $P(x)$

$$P(x) = \sum_{i=1}^{n} P_j(x) \quad \text{where } P_j(n) = f(n_j)\left(\prod_{k=1, k\neq j}^{n} \frac{x - x_k}{x_j - x_k}\right)$$

Let $f(x_i) = y_j$

So $P_1(x) = y_1 \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$

where $x_1 = 0 \quad x_2 = 2h \quad x_3 = 3h$

$$P_1(x) = y_1\left(\frac{(x-2h)(x-3h)}{(0-2h)(0-3h)}\right) = y_1\left(\frac{(x-2h)(x-3h)}{6h^2}\right)$$

$$\therefore P_1(x) = y_1\left(\frac{(x-2h)(x-3h)}{6h^2}\right)$$

$$P_2(x) = y_2\left(\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}\right) = y_2\left(\frac{(x-0)(x-3h)}{(2h-0)(2h-3h)}\right) = y_2\left(\frac{x(x-3h)}{-2h^2}\right)$$

$$\therefore P_2(x) = -y_2 \frac{x(x-3h)}{2h^2}$$

$$P_3(x) = y_3\left(\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_1)}\right) = y_3\left(\frac{(x-0)(x-2h)}{(3h-0)(3h-2h)}\right) = y_2\left(\frac{x(x-2h)}{3h^2}\right)$$

$$\therefore P_3(x) = y_3 \frac{x(x-2h)}{3h^2}$$

$$\therefore P(x) = y_1 \frac{(x-2h)(x-3h)}{6h^2} - y_2 \frac{x(x-3h)}{2h^2} + y_3 \frac{x(x-2h)}{3h^2}$$

2. $a_0 f(0) + a_1 f(2h) + a_2 f(3h)$

$$I = \int_0^{3h} f(x)dx \Rightarrow \sum_{i=1}^{n} P_i(x)\,dx$$

consider

$$\int_0^{3h} P_1(x)dx = y_1 \int_0^{3h} \frac{(x-2h)(x-3h)}{6h^2} dx$$

$$= y_1 \int_0^{3h} \frac{x^2}{6h^2} - \frac{5hx}{6h^2} + \frac{6h^2}{6h^2}\, dx$$

$$= y_1 \int_0^{3h} \frac{x^2}{6h^2} - \frac{5hx}{6h^2} + 1\, dx$$

$$= y_1\left(\frac{x^3}{18h^2} - \frac{5h}{6h^2}\left(\frac{x^2}{2}\right) + x \right|_0^{3h}$$

$$= y_1\left(\frac{27h^3}{18h^2} - \frac{5}{6h}\cdot\frac{9h^2}{2} + 3h\right)$$

$$= y_1\left(\frac{3}{2}h - \frac{15h}{4} + 3h\right)$$

$$= y_1\left(\frac{6h}{4} - \frac{15h}{4} + \frac{12h}{4}\right)$$

$$= y_1 \cdot \frac{3}{4}h$$

$$\int_0^{3h} P_2(x) = -y_2 \int_0^{3h} \frac{x(x-3h)}{2h^2}\,dx = -y_2 \int_0^{3h} \left[\frac{x^2}{2h^2} - \frac{3hx}{2h^2}\right] dx$$
$$= -y_2 \left[\frac{x^3}{6h^2} - \frac{3hx^2}{4h^2}\right]\Big|_0^{3h}$$
$$= -y_2 \left[\frac{27h^3}{6h^2} - \frac{27h^3}{4h^2}\right]$$
$$= -y_2 \left(\frac{9h}{2} - \frac{27h}{4}\right)$$
$$= -y_2 \left(\frac{18h}{4} - \frac{27h}{4}\right)$$
$$= -y_2 \left(-\frac{9h}{4}\right)$$
$$= y_2 \cdot \frac{9h}{4}$$

$$\int_0^{3h} P_3(x) = y_3 \int_0^{3h} \frac{x(x-2h)}{3h^2}\,dx = y_3 \int_0^{3h} \frac{x^2}{3h^2} - \frac{2x}{3h}\,dx$$
$$= y_3 \left(\frac{x^3}{9h^2} - \frac{x^2}{3h}\right)\Big|_0^{3h}$$
$$= y_3 \left(\frac{27h^3}{9h^3} - \frac{9h^2}{3h}\right)$$
$$= y_3 (3h - 3h) = 0$$

$$\therefore \int_0^{3h} f(x)\,dx = \frac{3h}{4} f(0) + \frac{9h}{4} f(2h) + 0 \cdot f(3h)$$

3. $\int_0^{0.36} f(x)\,dx$ if we set $h = 0.12$ so $0.36 = 3 \cdot 0.12 = 3h$

$$\int_0^{0.36} f(x)\,dx = \frac{3 \cdot 0.12}{4} \cdot 0.5 + \frac{9 \cdot 0.12}{4} \cdot 0.50727$$
$$= 0.045 + 0.1369629$$
$$= 0.1819629$$

$$RE = \frac{0.1819695 - 0.1819629}{0.1819629} \approx 0.00003627113 \approx 0.003627113\,\%$$

The RE is below tolerance (1%)