UNIVERSITY OF VICTORIA
FINAL EXAM DECEMBER 2020
COMPUTER SCIENCE 349A - NUMERICAL ANALYSIS: I - A01, A02
CRN 10794, 10795

NAME: ______________________________ STUDENT NO. ________________________

INSTRUCTOR: Rich Little

. DURATION: 180 minutes

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THIS QUESTION PAPER HAS 11 PAGES INCLUDING THIS COVER PAGE.

NOTES: (0) OPEN BOOK EXAM; ONLY BASIC CALCULATORS ARE ALLOWED,
(1) THERE ARE A TOTAL OF 100 MARKS,
(2) YOU MAY NOT USE THE INTERNET FOR ANY PURPOSE OTHER THAN THE COURSE WEBSITE ON CONNEX WITH THE EXCEPTION OF AN ONLINE PDF TOOL IF NECESSARY

Question	Possible marks	Actual marks
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

1. (a) [2 points] The IEEE standard for single precision floating point representation uses 23 bits for the mantissa of a normalized floating-point number, where the lead 1 is not stored in the word (that is, it is implied.) What is the *machine epsilon* for this representation?

   (b) [2 points] Convert the base-8 number 3.14_8 to base-10.

   (c) [2 points] What is the distance between any two consecutive numbers in the interval from 125_{10} to 625_{10} when using a precision $k = 4$, base 5 floating-point number system? The answer should be in decimal.

   (d) [4 points] Evaluate and interpret the condition number for $f(x) = e^{-x}$ for $x = 10$.

2. (a) [5 points] Determine the fourth order ($n = 4$) Taylor series expansion for $f(x) = \cos(x - \pi)$ expanded about $a = \pi$, (do not include the remainder term.)

   (b) [2 points] Use the polynomial approximation in (a) to derive a polynomial approximation to the function $y(x) = \frac{1-\cos(x-\pi)}{(x-\pi)^2}$ that is accurate for x close to π.

   (c) [3 points] Use the polynomial approximation in (b) to show that the problem of computing $y(3.15159) \approx 0.499996$ is well-conditioned. Note, let $\pi = 3.14159$.

3. Consider the function $f(x) = \sin(x - \sqrt{3}) - x + \sqrt{3}$.

   (a) [4 points] Let $x_l = 1$ and $x_u = 2$ and run one iteration of the Bisection method for approximating the root of $f(x)$ and determine x_l and x_u for the second iteration.

   (b) [3 points] If Newton's method is used to compute an approximation to a zero of $f(x)$ using the initial approximation $x_0 = 1$, convergence is obtained to the zero $x_t = \sqrt{3}$. If this computation is carried out, what is the order of convergence? Justify your answer

   (c) [3 points] Determine the multiplicty of the of the zero $x_t = \sqrt{3}$.

4. (a) [6 points] Consider the following polynomial: $f(x) = (x-2)(x^3+2x^2+1)$ which has true root $x_t = 2$. What is the approximate deflated polynomial $Q(x)$ if $x_0 = 3$? What is the actual deflated polynomial $\hat{f}(x)$?

   (b) [4 points] Derive a new version of Horner's algorithm for evaluating the following cubic polynomial at x_0.

$$P(x) = a_0 + a_1(x-t_1) + a_2(x-t_1)(x-t_2) + a_3(x-t_1)(x-t_2)(x-t_3)$$

5. (a) [8 points] Consider the following system of linear equations $Ax = b$.

$$\begin{aligned} -x_1 + x_2 + 4x_3 &= -2 \\ 2x_1 + x_2 + 5x_3 &= 4 \\ x_1 + x_2 + 6x_3 &= 6 \end{aligned}$$

Specify the augmented matrix for this system of equations and use Gaussian Elimination with partial pivoting (no scaling) to compute the solution vector x. Show all of your work.

(b) [2 points] Calculate the determinant of A using the final augmented matrix $[A|b]$ from above (i.e the one in upper triangular form.)

6. Consider the following piecewise cubic polynomial

$$S(x) = \begin{cases} S_0 = 2 - x - x^3, & 0 \le x \le 1 \\ S_1 = a + b(x-1) + c(x-1)^2 + d(x-1)^3, & 1 \le x \le 2 \end{cases}$$

(a) [8 points] Determine the values of a, b, c, and d so that $S(x)$ is a natural cubic spline.

(b) [2 points] Determine the values of a, b, c, and d so that $S(x)$ is a clamped cubic spline with $S'(2) = 2$. You do not need to recalculate any values that are the same from part (a).

7. (a) [4 points] Give the Lagrange form of the interpolating polynomial of degree 2 for the function $f(x)$ at $x = -h$, 0, and h.

   (b) [6 points] Integrate the above polynomial from $x = -h$ to $x = h$ in order to derive a quadrature formula for approximating $\int_{-h}^{h} f(x)dx$.

8. (a) [6 points] Use the order $n = 3$ Taylor approximations, with error term, expanded about $a = x_0$, for both $f(x_0 + h)$, and $f(x_0 + 2h)$ to derive the following formula.

$$f''(x_0) \approx \frac{f(x_0) - 2f(x_0 + h) + f(x_0 + 2h)}{h^2}$$

(b) [2 points] In big-Oh notation what is the order of the error for this method in terms of step size h?

(c) [2 points] Use the above formula on the data below with $h = 0.15$ to compute an approximation to $f''(1.35)$.

x	$f(x)$
1.35	0.2466
1.45	0.2869
1.50	0.3033
1.55	0.3173
1.65	0.3393

9. (a) [6 points] Derive the iterative formula for the Taylor method of order $n = 2$ for approximating the solution of the initial value problem

$$y'(x) = -xy^2, \text{ with } y(2) = 1.$$

   (b) [1 point] What is the order of the global truncation error of this method?

   (c) [3 points] Use the Taylor formula in (a) to compute an approximation to the solution of the above differential equation at $x = 2.1$ using $h = 0.1$.

10. Consider an initial-value problem $y'(x) = 1 + \frac{y}{x}$, with $y(1) = 2$. If the solution to this problem is approximated using the second-order Runge-Kutta method called the Midpoint Method with a fixed step size of $h = 0.01$ on $[1, 1.10]$, then the following results are obtained:

x_i	y_i	$y(x_i)$
1.00	2.0	2.0
1.01	2.0300498	2.0300498
$\vdots$	$\vdots$	$\vdots$
1.08	2.2431173	2.2431179
1.09	2.2739330	2.2739337
1.10	2.3048404	2.3048412

(a) [2 points] What is the **global** truncation error at $x = 1.09$? (use 8 significant figures)

(b) [8 points] What is the **local** truncation error at $x = 1.09$? (use 8 significant figures)

END