DSC 214
Topological Data Analysis
Topic 2: Simplicial Complexes

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Overview

- Simplicial complex
  - a specific type of topological space commonly used in practice to model data

- Notations

- Commonly used simplicial complexes from point cloud data (PCD)

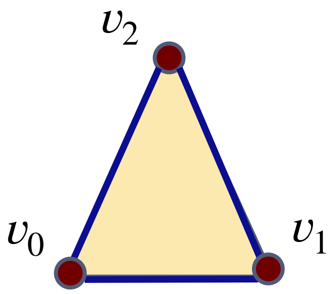
Introduction to Simplicial Complex

A (Geometric) Simplex

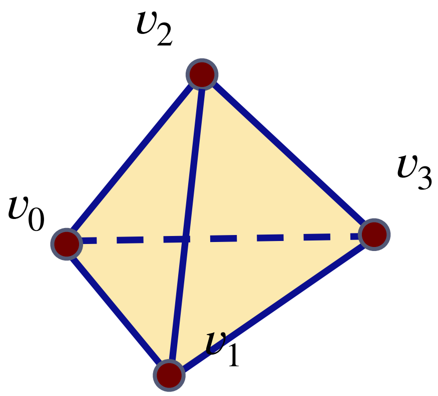
- Points $\left\{p_0, p_1, \ \dots, \ p_d\right\} \subset R^N$ are (affinely) independent
  - if vectors $v_i = p_i - p_0, \ i \in \left[0, d\right],$ are linearly independent
- Geometric *p*-simplex $\sigma = \{ \ v_0, v_1, \dots, \ v_p\}$
  - Convex combination of $p + 1$ *(affinely) independent* points in R^N
    - $\sigma = \{ \ \sum_{i=0}^{p} a_i v_i \mid a_i \geq 0, \sum a_i = 1 \ \}$
- Examples

0-simplex

1-simplex



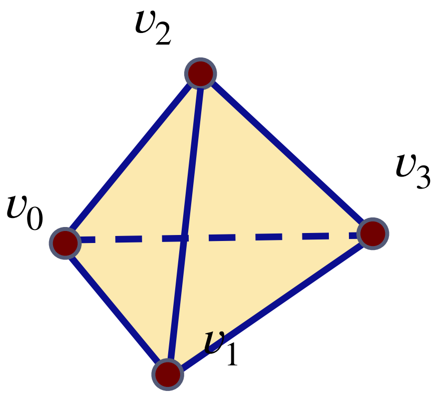
2-simplex



3-simplex

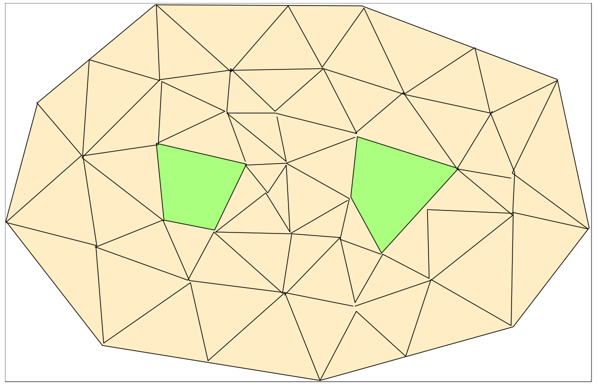
A (Geometric) Simplex

- Points $\left\{p_0, p_1, \ \ldots, \ p_d\right\} \subset R^N$ are (affinely) independent
  - if vectors $v_i = p_i - p_0, \ i \in \left[0, d\right],$ are linearly independent
- Geometric *p*-simplex $\sigma = \{ \ v_0, v_1, \ldots, \ v_p\}$
  - Convex combination of $p+1$ *affinely-independent* points in R^N
    - $\sigma = \{ \ \sum_{i=0}^{p} a_i v_i \mid a_i \geq 0, \ \sum a_i = 1 \ \}$
- Simplex τ formed by a subset of $\{ \ v_0, v_1, \ldots, \ v_p\}$ is called a face of σ, denoted by $\tau \subseteq \sigma$
  - τ is a proper face of σ if $\dim(\tau) = \dim(\sigma) - 1$
  - $bd\ (\sigma) =$ collection of all proper faces of σ
- For a d-simplex σ
  - $\sigma \cong B^d, \ bd(\sigma) \cong \mathbb{S}^{d-1}, \ int(\sigma) \cong \mathbb{R}^d$



Simplicial complex

- A geometric simplicial complex K
  - A collection of simplices such that
    - If $\sigma \in K$, then any fact $\tau \subseteq \sigma$ is also in K
    - If $\sigma \cap \sigma' \neq \emptyset$, then $\sigma \cap \sigma'$ is a face of both simplices.
  - $\dim(K)$ = highest dim of any simplex in K



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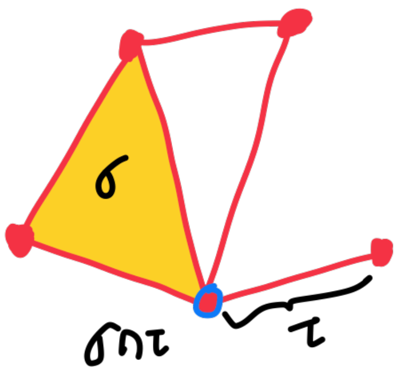

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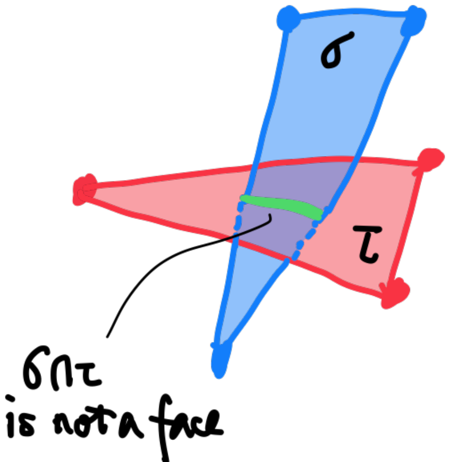
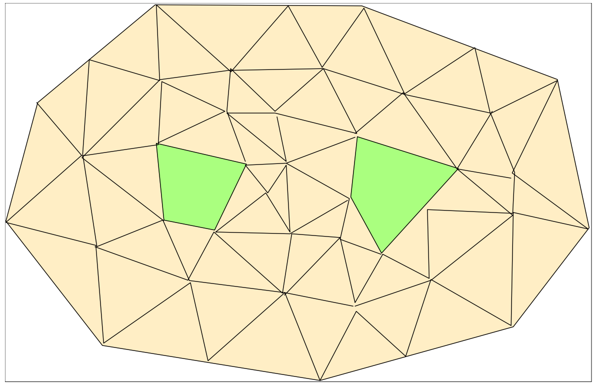


Simplicial complex



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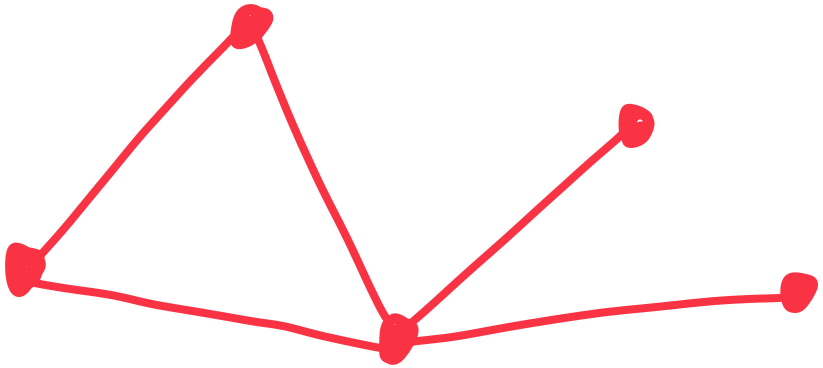
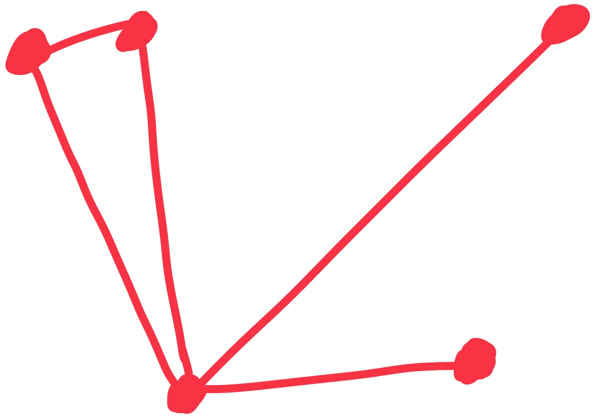
    - If $\sigma \in K$, then any fact $\tau \subseteq \sigma$ is also in K
    - If $\sigma \cap \sigma' \neq \emptyset$, then $\sigma \cap \sigma'$ is a face of both simplices.
  - $\dim(K)$ = highest dim of any simplex in K
- Subcomplex $L \subseteq K$ and L is a complex
- The p-skeleton of K consists of all simplices in K of dimension at most p
- Underlying space $|K|$ of K
  - is the pointwise union of all points in all simplices of K,
  - i.e, $|K| = \bigcup_{\sigma \in K} \{x \mid x \in \sigma\}$

- Geometric simplicial completes are nice for intuition / having a mental picture. But we are interested in topology





- Distinct geometrically but the same topologically (i.e., they are homeomorphic)
- A graph can be abstractly defined as $G = (V, E)$

Abstract simplicial complex

- An (abstract) p-simplex $\sigma = \{ v_0, v_1, \ldots, v_p\}$
  - a set of cardinality $p + 1$
  - A subset $\tau \subseteq \sigma$ is a face of σ
- An (abstract) simplicial complex $K = (V, \Sigma)$
  - A vertex set V
  - A collection Σ of simplices such that

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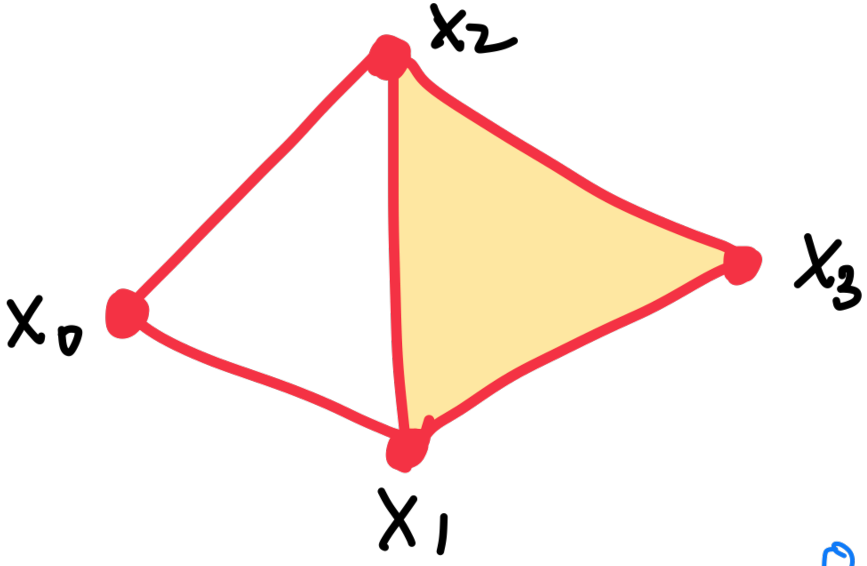
Abstract simplicial complex



$$V = \{x_0, x_1, \ldots, x_3\}$$

$$\Sigma = \left\{ \begin{array}{l} \{x_0\}, \{x_1\}, \{x_2\}, \{x_3\} \\ \{x_0, x_1\}, \{x_0, x_2\}, \{x_1, x_2\} \\ \{x_1, x_3\}, \{x_2, x_3\} \\ \{x_1, x_2, x_3\} \end{array} \right\}$$

0-simplices

1-simplices

2-simplex

Abstract Simplicial Complex

- **Geometric realization** of an abstract simplicial complex K
  - Is a geometric simplicial complex S whose associated abstract simplicial complex $(V(S), \Sigma(S))$ is the "same" as $(V(K), \Sigma(K))$

Theorem
Any abstract simplicial complex K has a geometric realization. Any two geometric realizations of K have homeomorphic underlying spaces

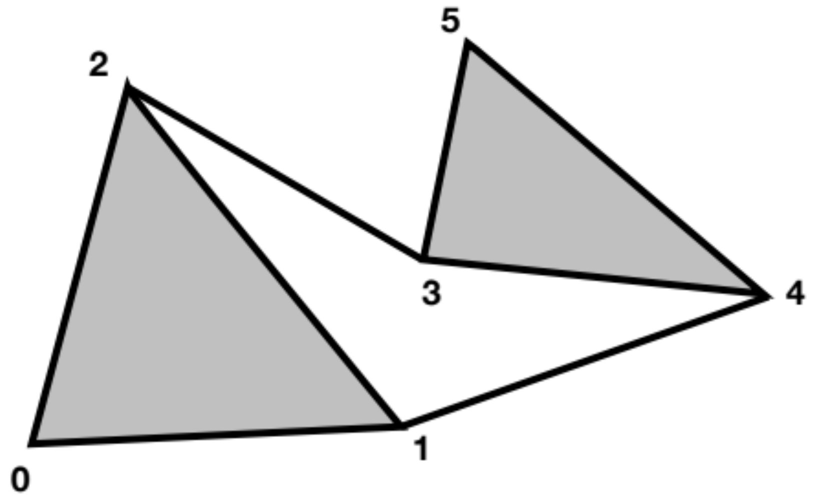
- We use $|K|$ to denote the underlying space of a geometric realization of K and call $|K|$ the underlying space of K.

Theorem

Any abstract simplicial complex K has a geometric realization. Any two geometric realizations of K are homeomorphic to each other.

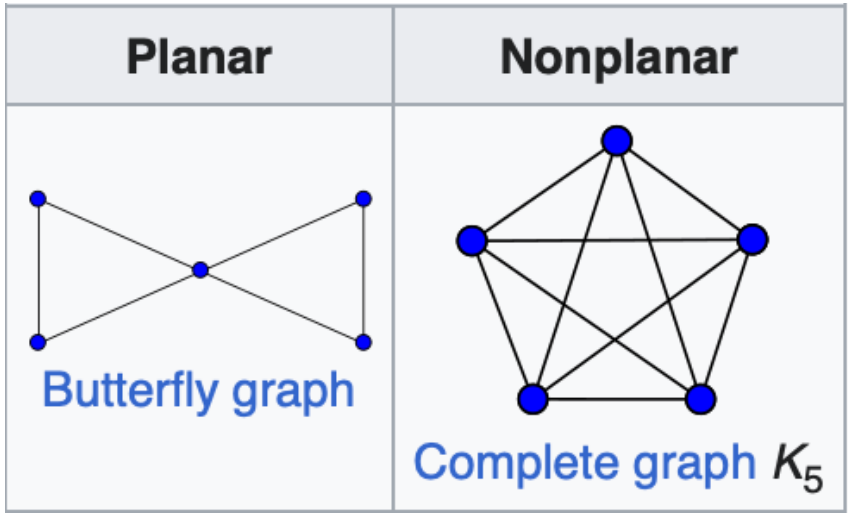
- If $V = \{v_0, \ldots, v_n\}$, embed V into $\mathbb{R}^{n+1}$ by $v_i \mapsto (0,\cdots,1\cdots,0) = e_i$
- For each simplex $\sigma = \{v_{i_0}, \ldots, v_{i_k}\}$, add geometric simplex $cnx\{e_{i_0}, \ldots, e_{i_k}\}$ to the realization

- The recipe in the proof is not efficient in terms of ambient dimension



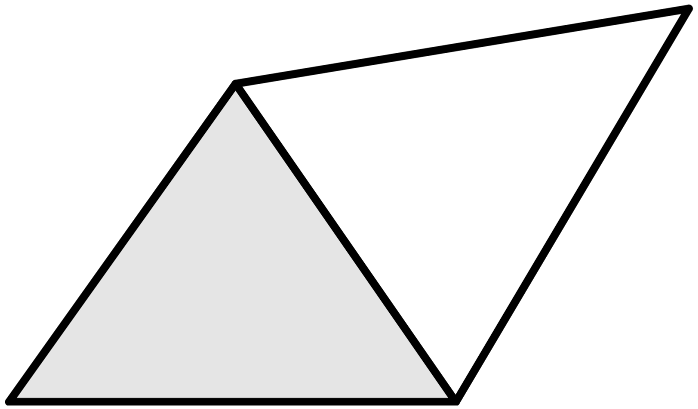
- Any finite d-dim abstract simplicial complex has a geometric realization in $\mathbb{R}^{2d+1}$

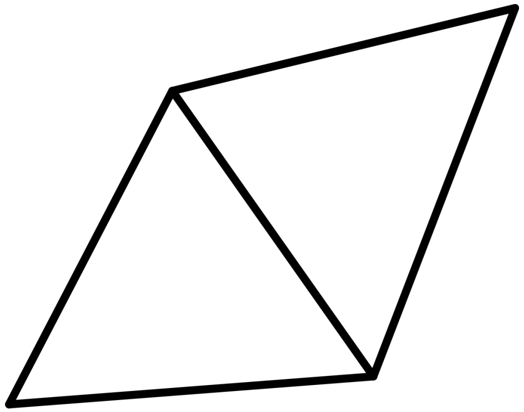
- The recipe in the proof is not efficient in terms of ambient dimension
- Any finite d-dim abstract simplicial complex has a geometric realization in $\mathbb{R}^{2d+1}$ but may not have a geometric realization in $\mathbb{R}^{2d}$
  - A graph (1-d simplicial complex) can be plotted in $\mathbb{R}^3$ but not necessarily in $\mathbb{R}^2$



Graphs and Simplicial Complexes

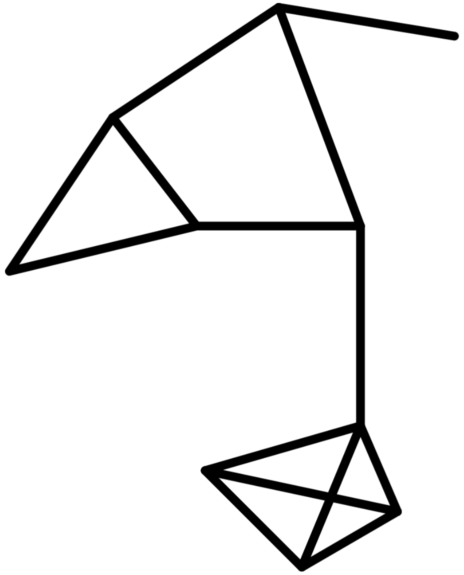
- Any simple graph (without double edge and self-loop) is a simplicial complex
- The 1-skeleton of a simplicial complex is a graph

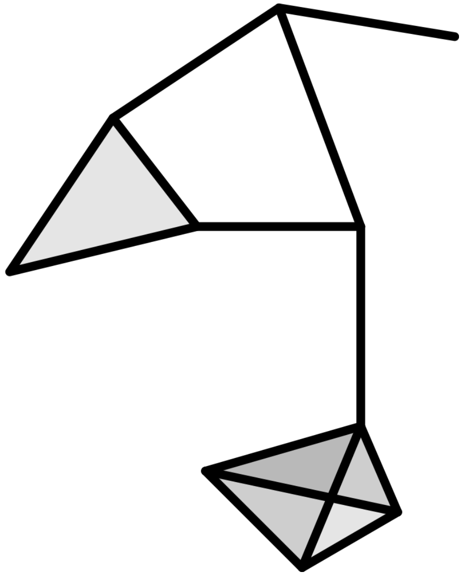




Graphs and Simplicial Complexes

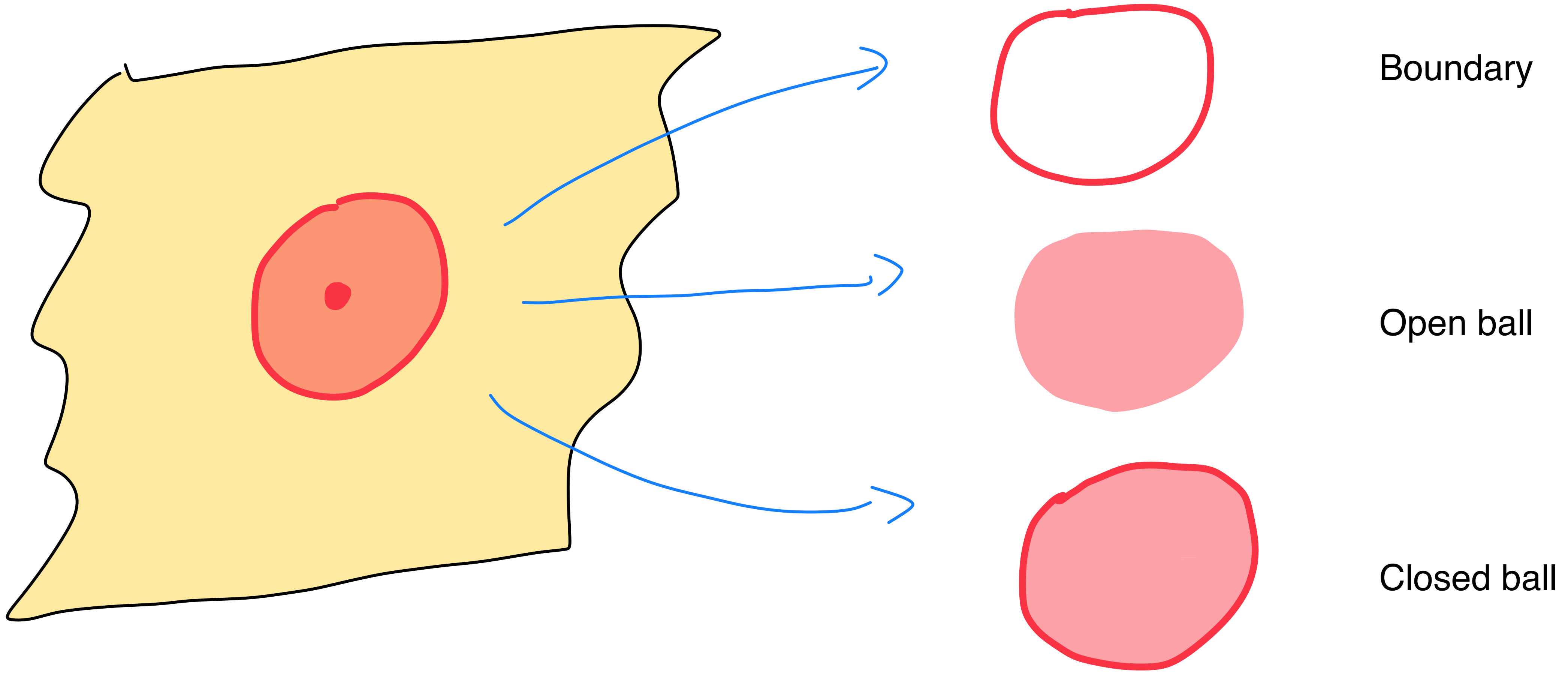
- Any simple graph (without double edge and self-loop) is a simplicial complex
- The 1-skeleton of a simplicial complex is a graph
- **Clique complex** induced by a graph





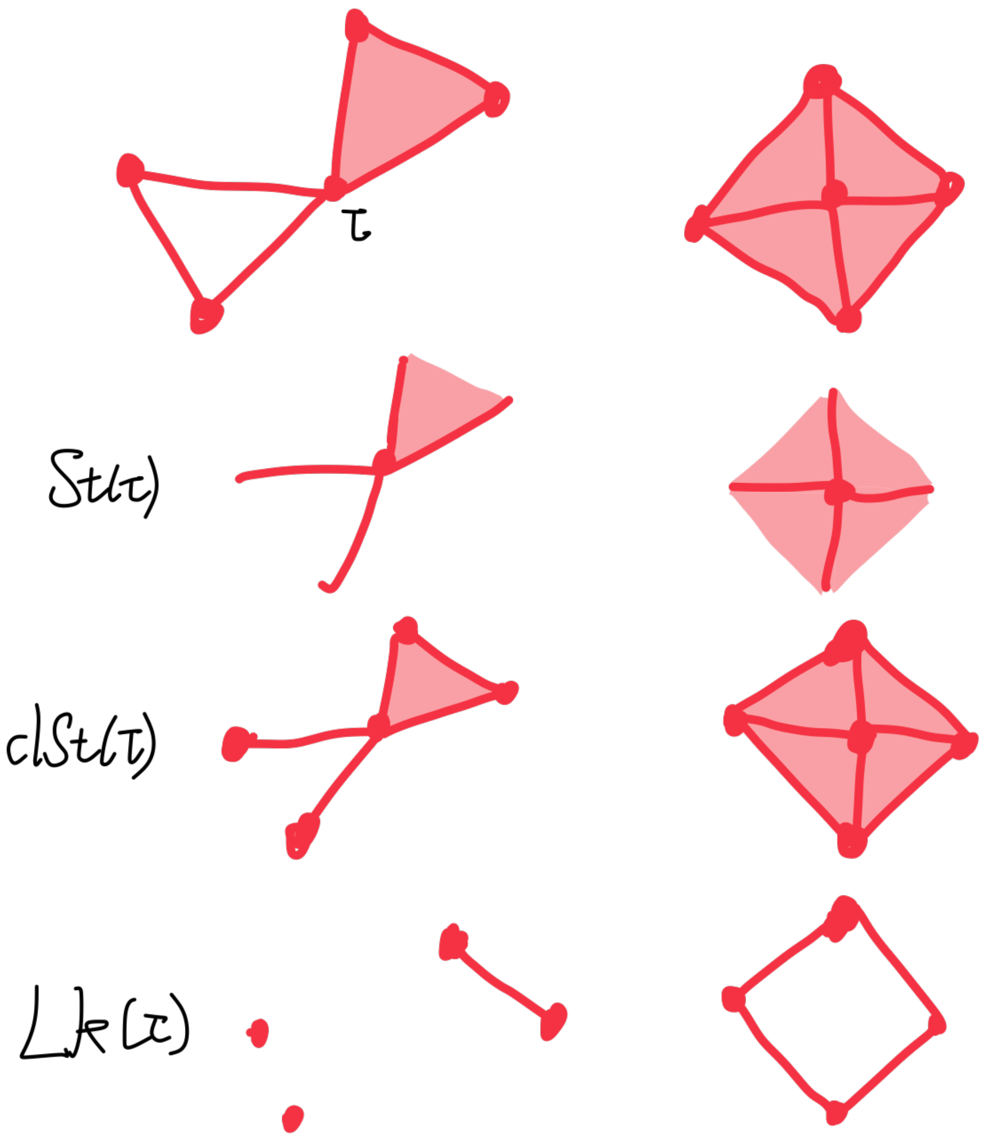
Some notions related to simplicial complexes

Star and links



Star and links

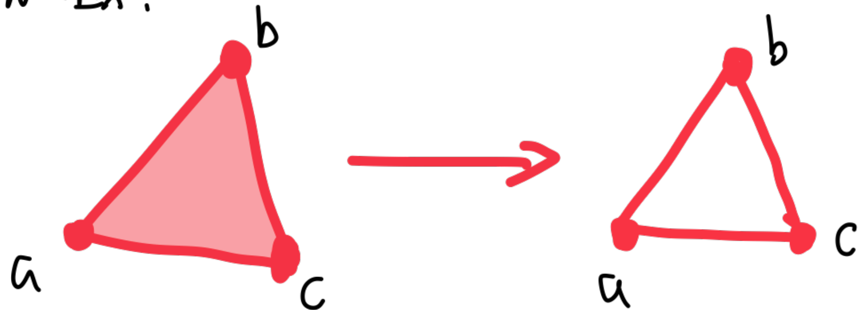
- Given a simplex $\tau \in K$
  - Star: $St(\tau) = \{\sigma \in K \mid \tau \subset \sigma\}$
  - Closed star: $cl\,St(\tau) = \bigcup_{\sigma \in St(\tau)} \{\, \sigma' \mid \sigma' \subset \sigma \,\}$
  - Link: $Lk(\tau) = \{\, \sigma \in cl\,St(\tau) \mid \sigma \cap \tau = \varnothing \,\}$



Simplicial map

- Intuitively, analogous to continuous maps between topological spaces
- Given simplicial complexes K and L
  - a function $f : V(K) \to V(L)$ is called a simplicial map if
    - for any $\sigma = \{p_0, \ldots, p_d\} \in \Sigma(K)$, $f(\sigma) = \left\{ f\left(p_0\right), \ \ldots, \ f\left(p_d\right) \right\}$ spans a simplex in L, i.e., $f(\sigma) \in \Sigma(L)$.
  - A simplicial map is also denoted $f : K \to L$

NON-EX:



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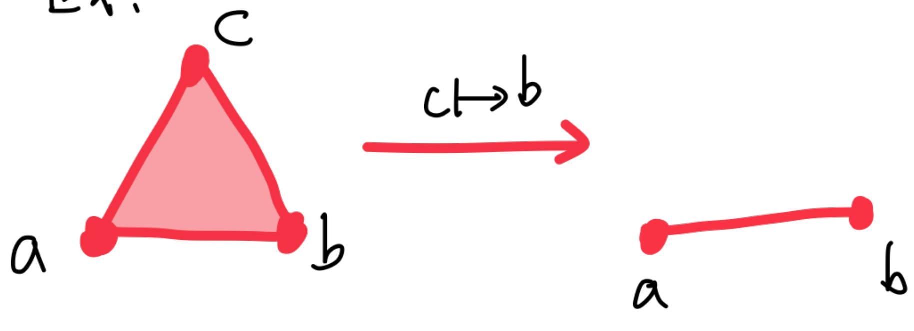

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  - A simplicial map is also denoted $f : K \to L$
- A simplicial map $f: K \to L$ is an **isomorphism**
  - if f is bijective and f^{-1} is a simplicial map

Simplicial map

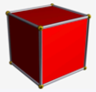
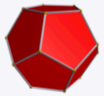
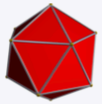
- A simplicial map $f: K \to L$ induces a natural continuous function $f': \left|K\right| \to \left|L\right|$
  - s.t $f'(x) = \sum_{i \in [0,d]} a_i f(p_i)$ for $x = \sum_{i \in [0,d]} a_i p_i \in \sigma = \{p_0, \ldots, p_d\}$

Simplicial map

- A simplicial map $f: K \to L$ induces a natural continuous function $f': |K| \to |L|$
  - s.t $f'(x) = \sum_{i \in [0,d]} a_i f(p_i)$ for $x = \sum_{i \in [0,d]} a_i p_i \in \sigma = \{p_0, \ldots, p_d\}$

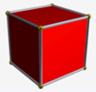
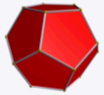
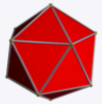
- Theorem:
  - An isomorphism $f: K \to L$ induces a **homeomorphism** $f': |K| \to |L|$

A topological invariant – Euler Characteristic

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V - E + F$
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

A topological invariant – Euler Characteristic

- For the surface of a polyhedron, the Euler Characteristic is defined as $\chi = V - E + F$.
- Euler's polyhedron formula:
  - $\chi = 2$ for surface of convex polyhedron

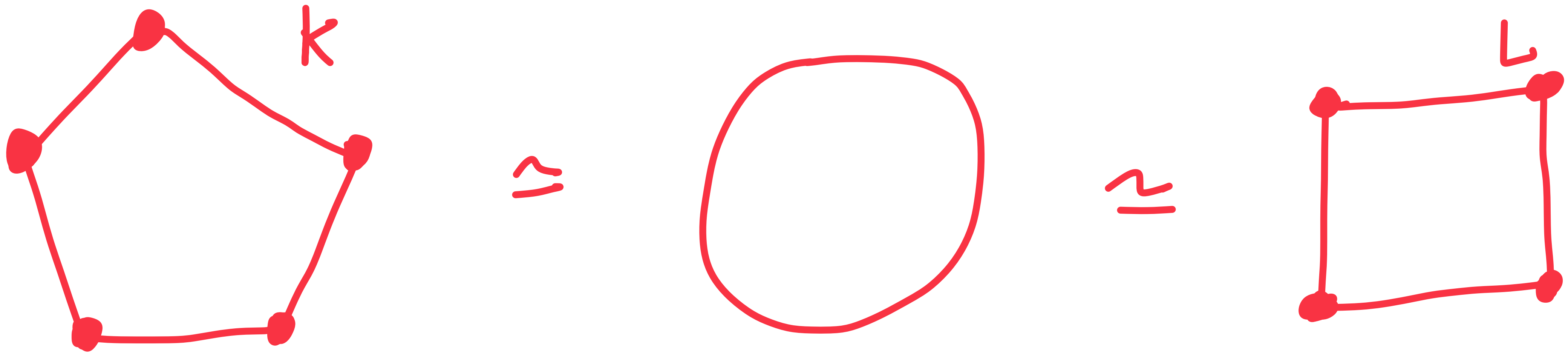
Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V - E + F$
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

A topological invariant – Euler Characteristic

A topological invariant – Euler Characteristic

- Given a d-dim simplicial complex K with n_i number of i-simplices
- the *Euler characteristic* of K is defined as:
  - $\chi(K) := \sum_{i=0} (-1)^i n_i$
- Euler characteristic is both a topological invariant and a homotopy invariant, meaning that it does not change under homeomorphism or homotopy equivalence.

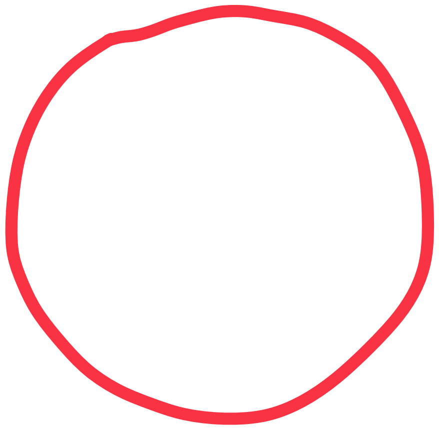
A topological invariant – Euler Characteristics

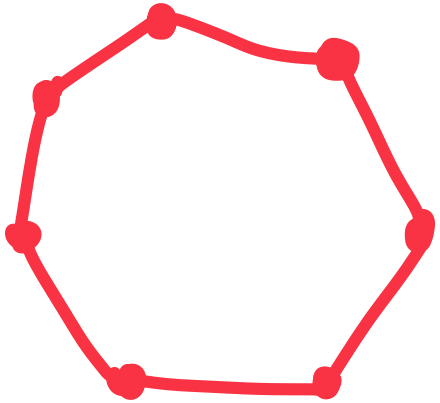


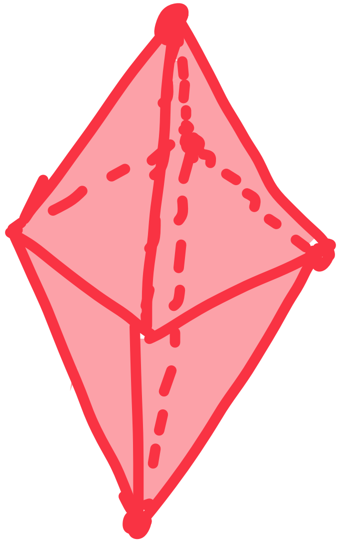
$\chi(K) = 5 - 5 = 0$ $\qquad$ $\chi(\mathbb{S}^1) = 0?$ $\qquad$ $\chi(L) = 4 - 4 = 0$

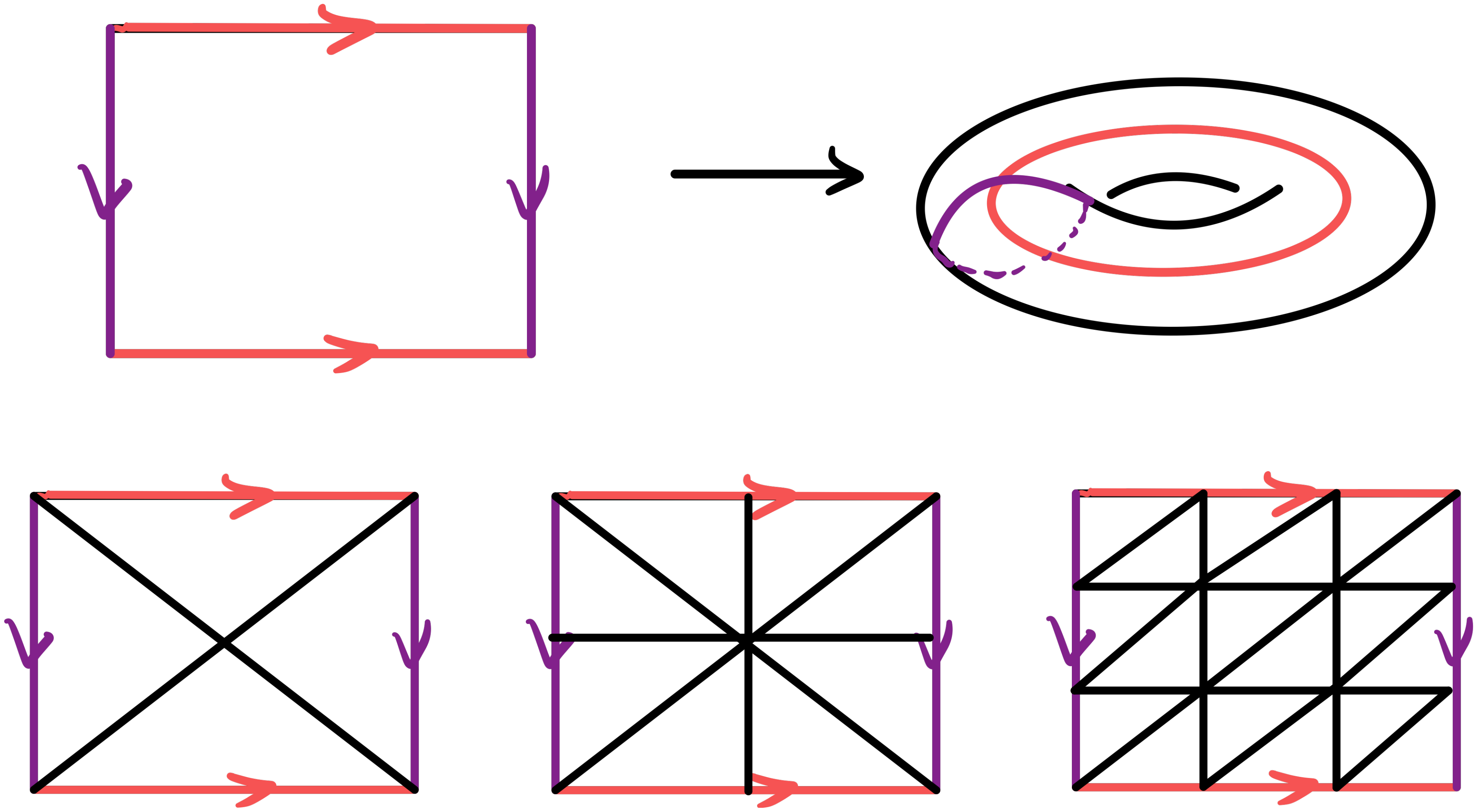
Triangulation of a manifold

- Given a manifold (with or without boundary) M, a simplicial complex K is a triangulation of M
  - if the underlying space $|K|$ of K is homeomorphic to M



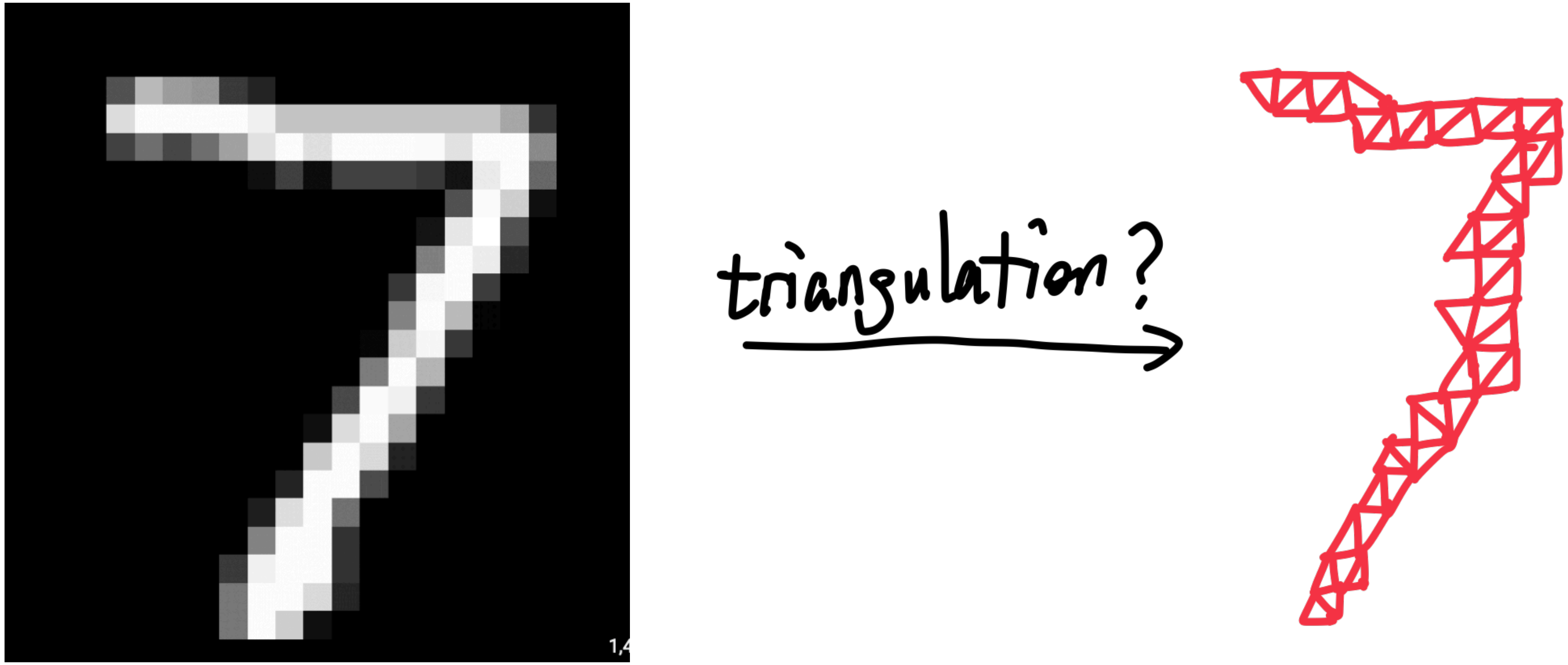






Other complexes

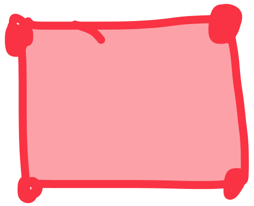
Image Data

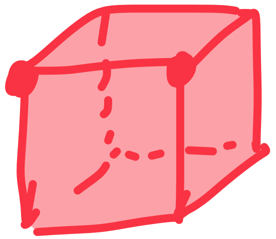


Cubical Complex

0-cube 

1-cube 

2-cube 

3-cube 

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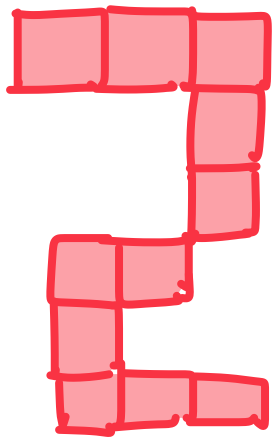
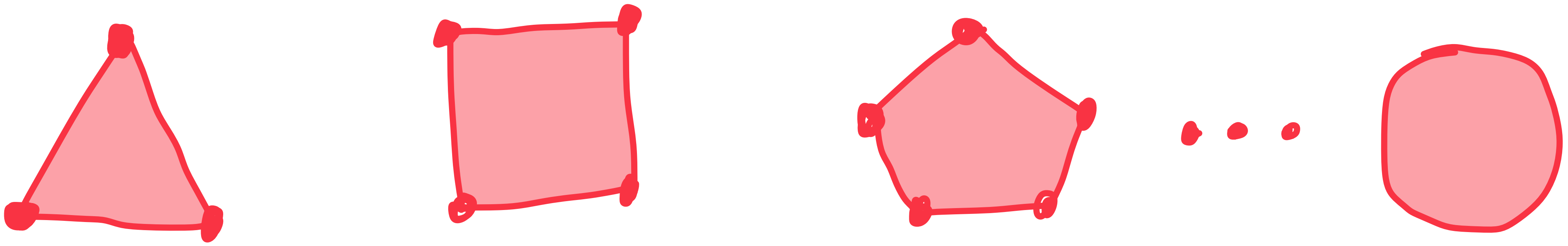
2-dim cubical complex



CW Complex



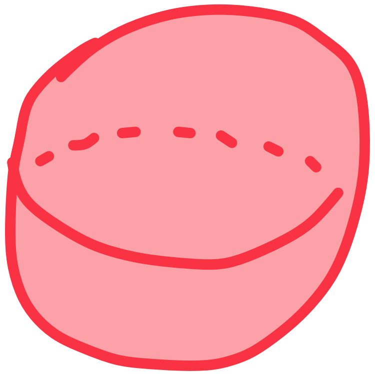
Triangle Rectangle Pentagon Disk

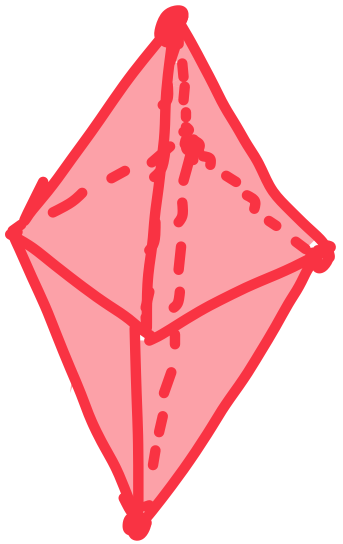
0-cell 1-cell 2-cell 3-cell K-cell

CW Complex

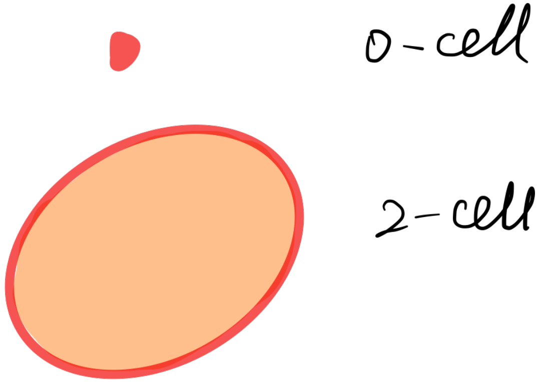
- A CW complex X is the union of a sequence of topological spaces
  - $\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \cdots$
  - Such that X_k is obtained from X_{k-1} by "gluing" k-cells $\{e_\alpha^k\}_\alpha$, each homeomorphic to $\mathbb{D}^k$, by continuous maps $\partial e_\alpha^k \to X_{k-1}$
  - Each X_k is called the k-skeleton of X

CW Complex



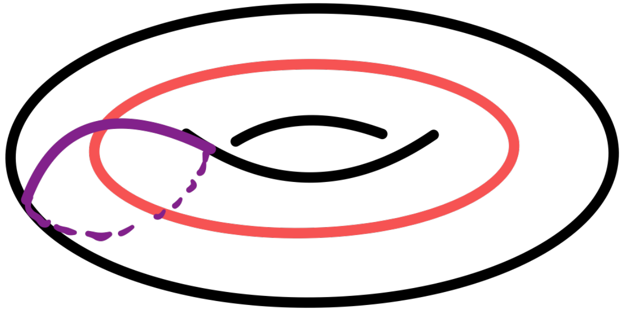


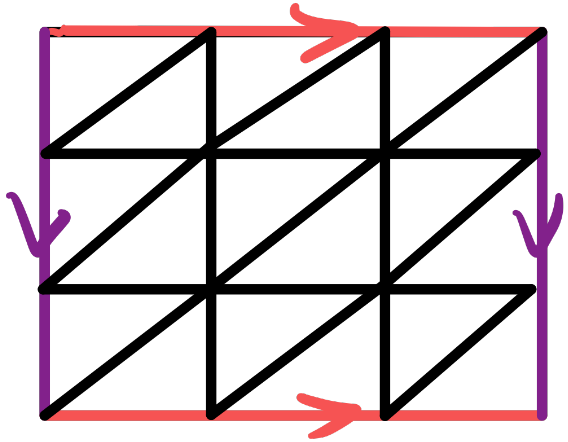
Triangulation of a sphere



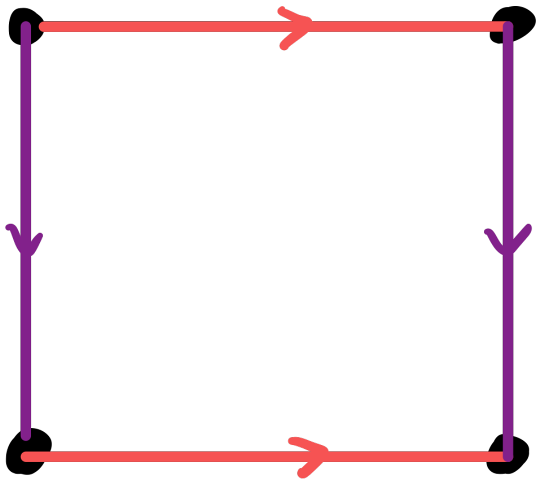
CW structure of a sphere

CW Complex



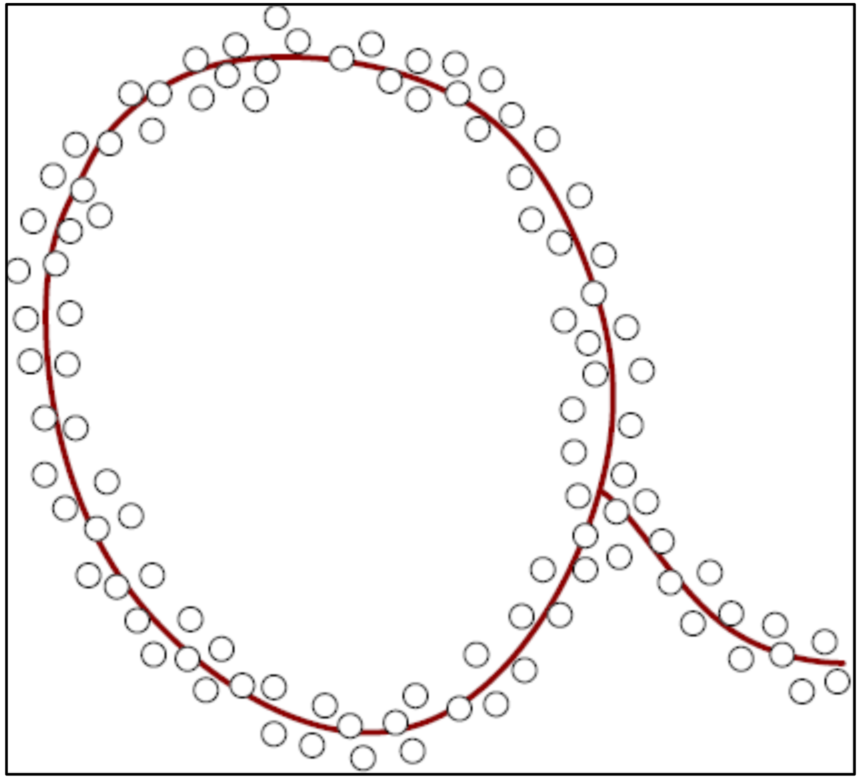


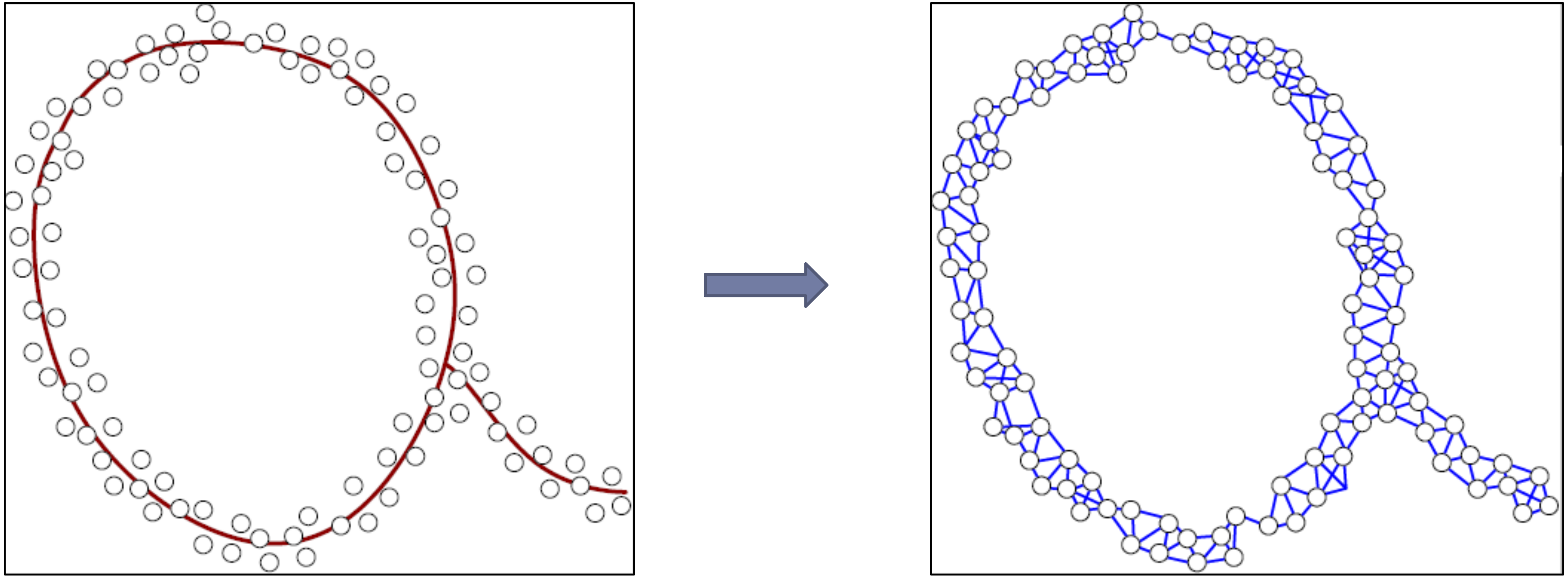
Triangulation of a torus

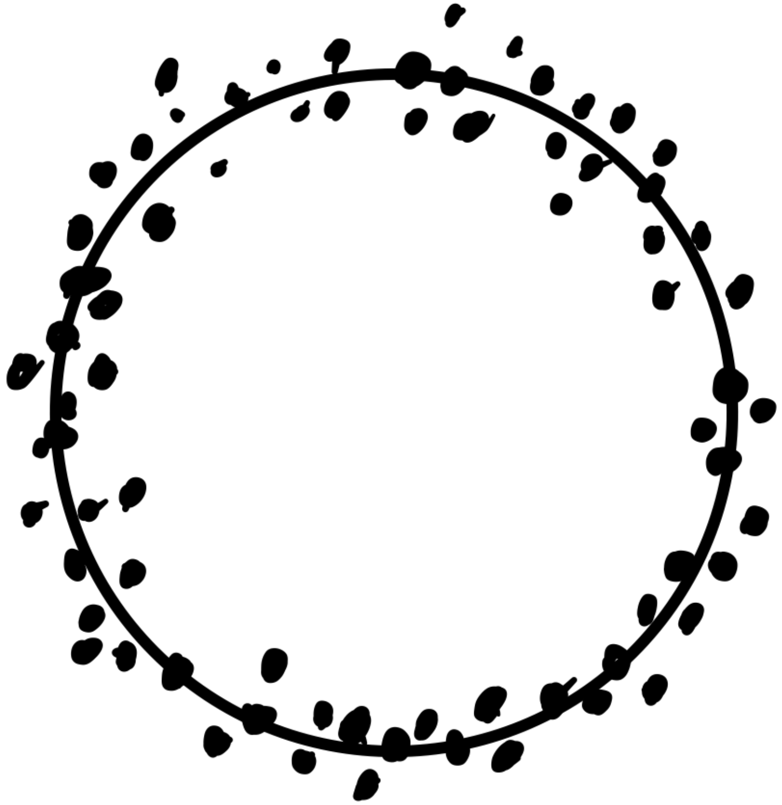


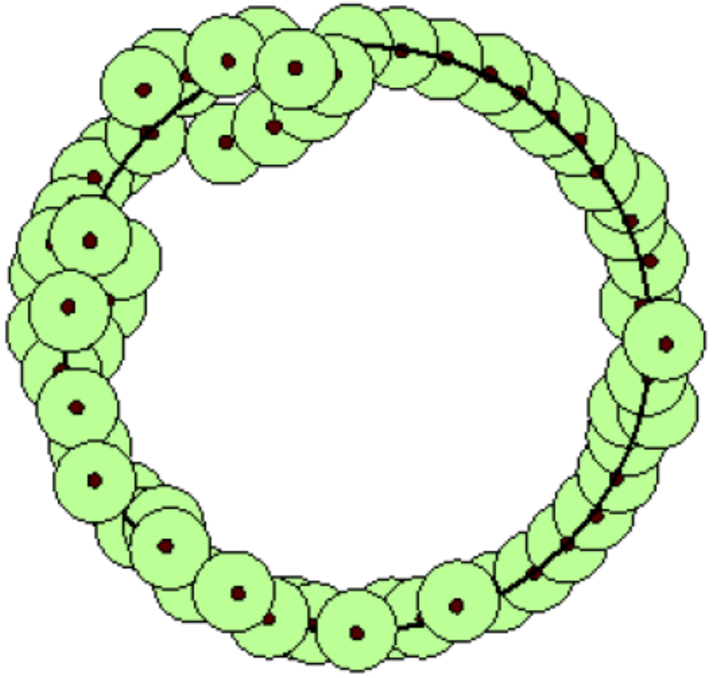
CW structure of a torus

Common Complexes



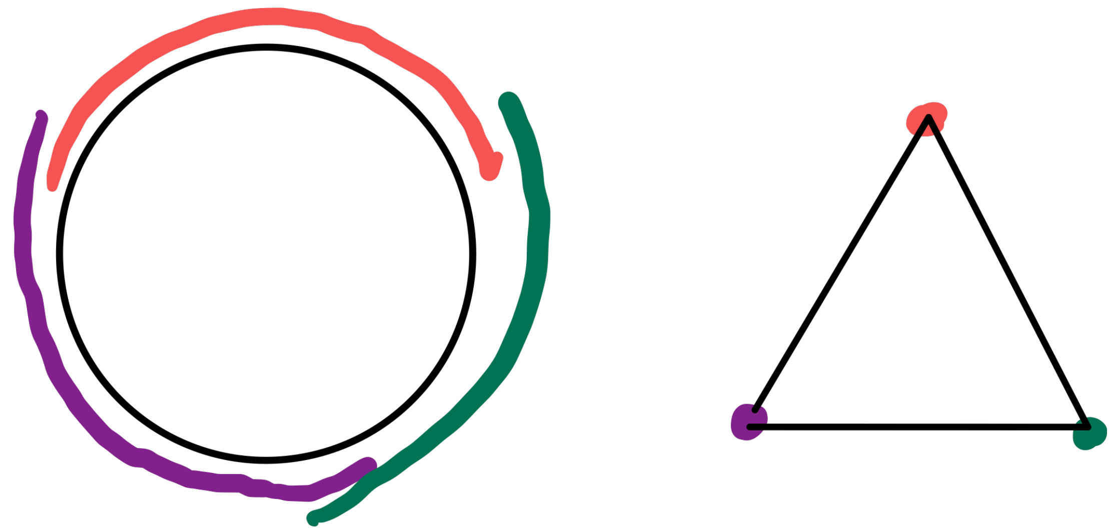






Nerves

- Given a finite collection of sets $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$, its nerve complex $Nrv(\mathscr{U})$ is a simplicial complex



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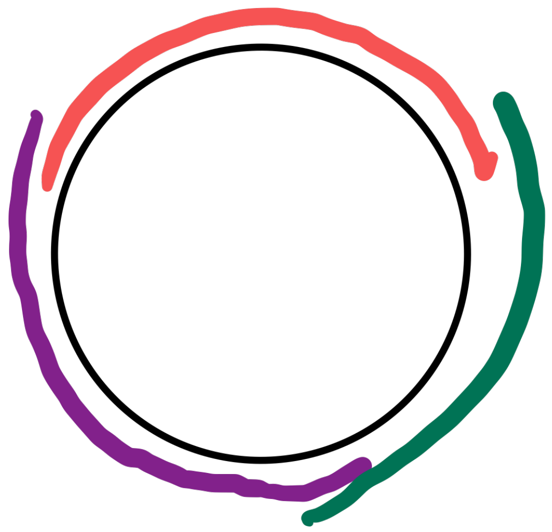
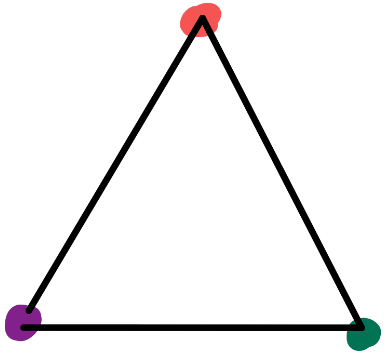
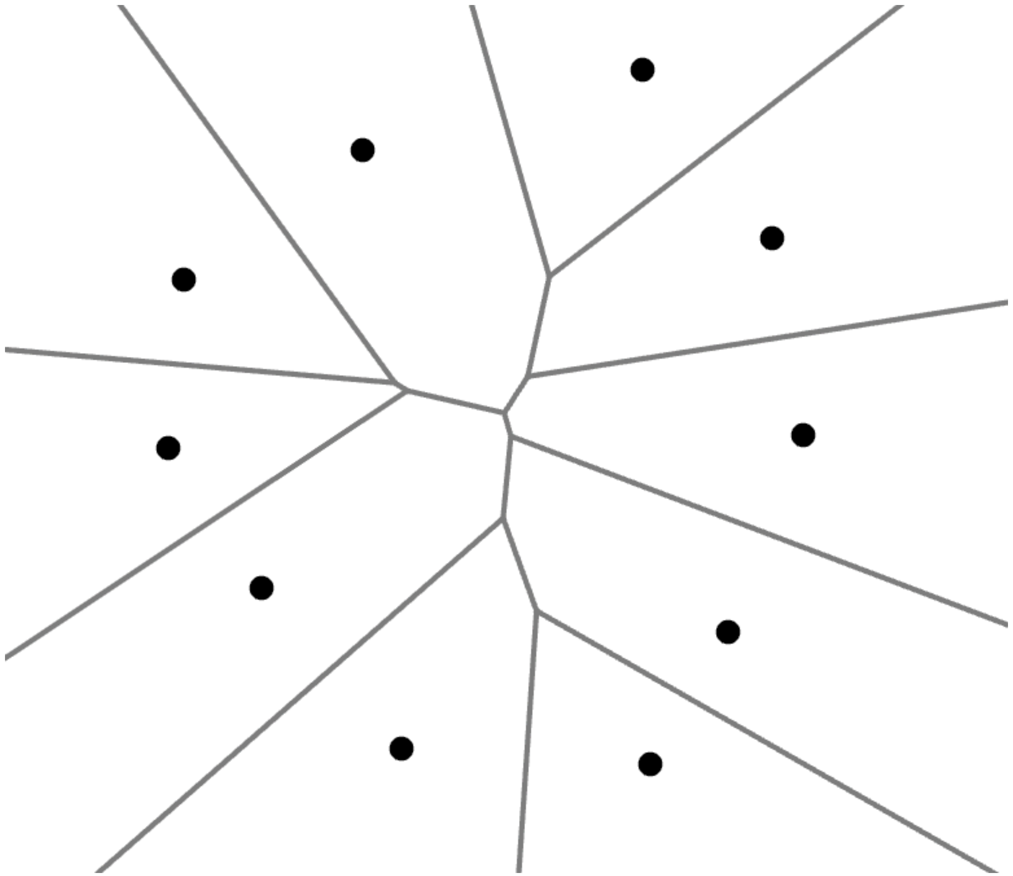
  - The vertex set $V = A$
  - $\{\alpha_0, \ldots, \alpha_k\} \in \Sigma$ iff $\cap_{i=0}^{k} U_{\alpha_i} \neq \emptyset$





Example



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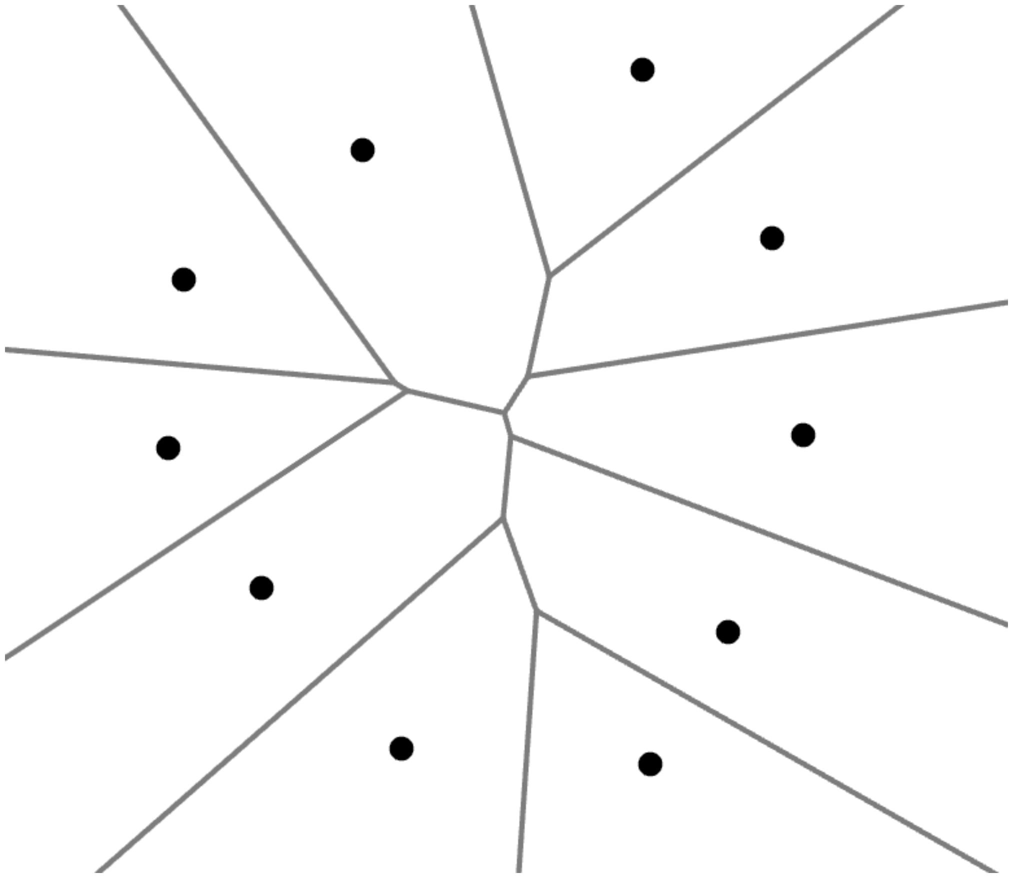

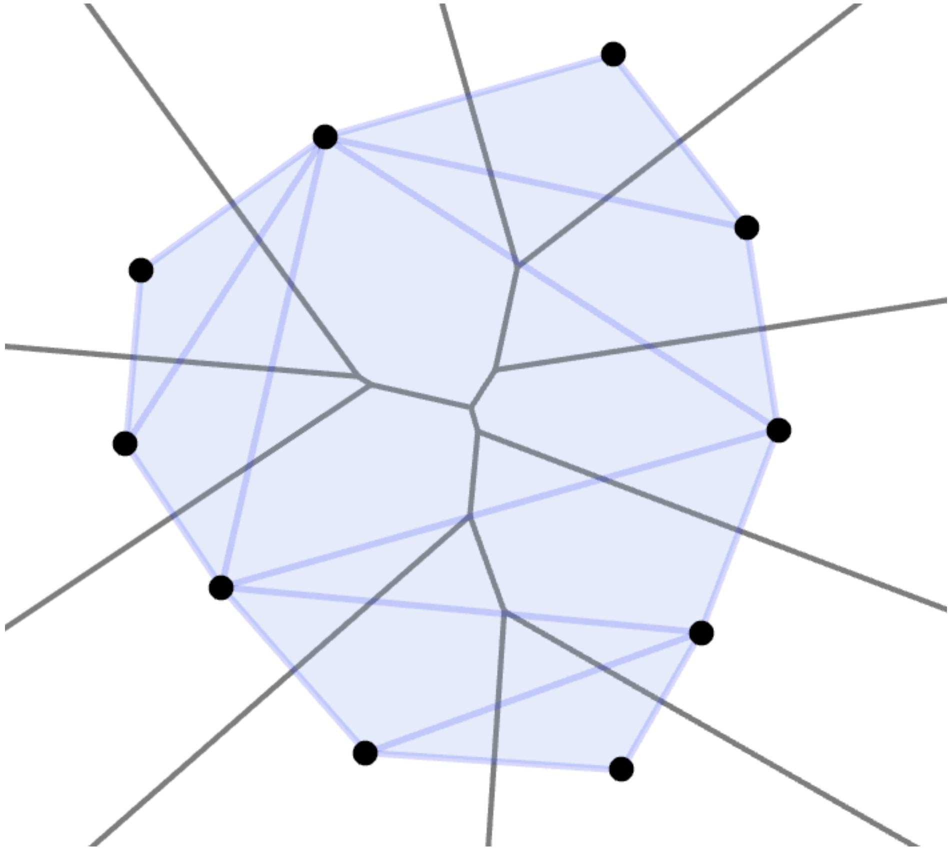

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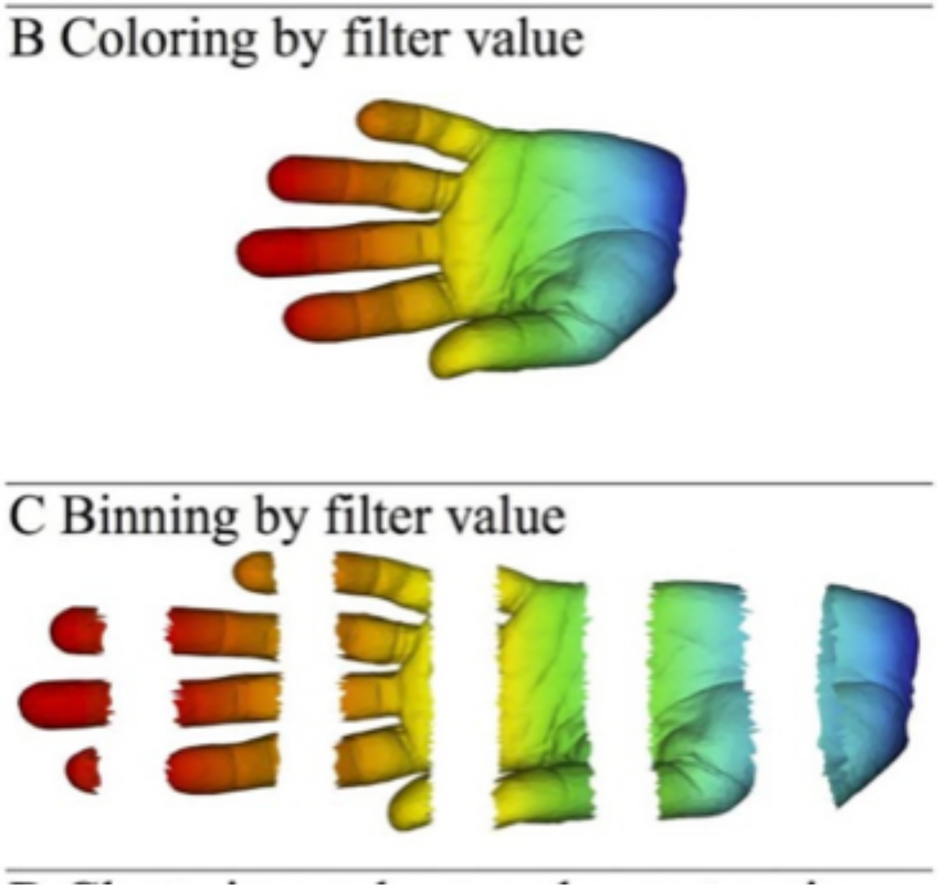
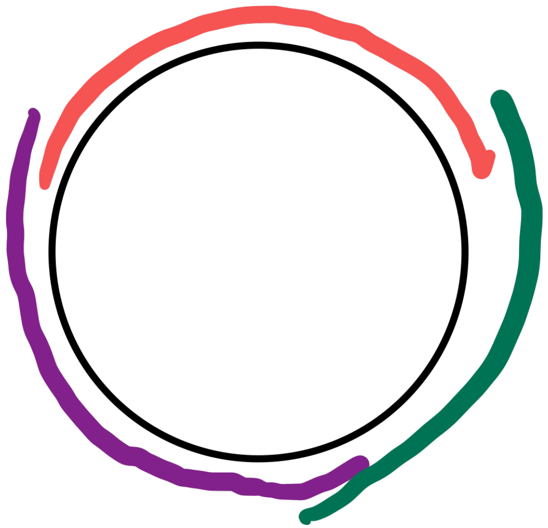
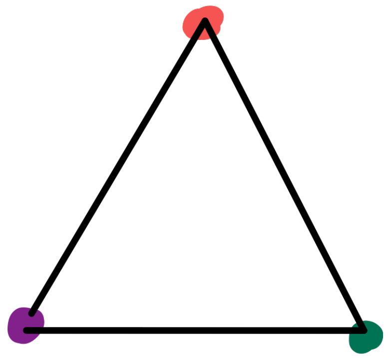
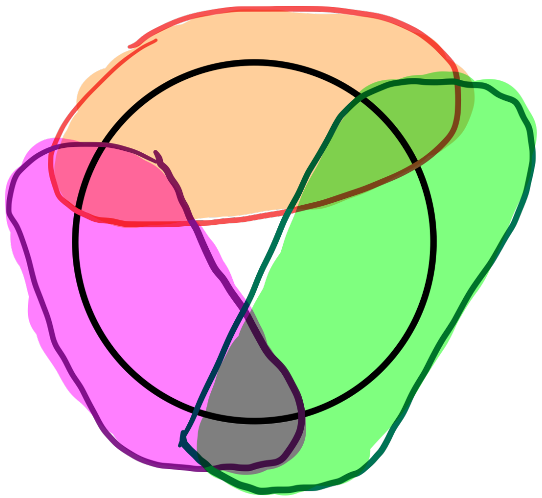


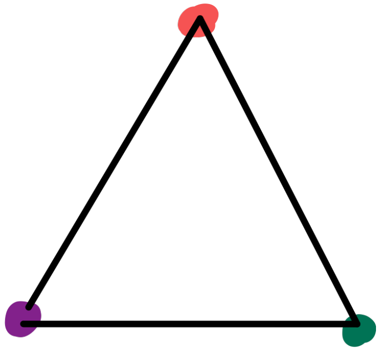
- Nerve Lemma (intrinsic):
  - Let $\mathscr{U}$ be an **open** cover of a metric space X such that $\cap_{i=1}^{k} U_{\alpha_i}$ is contractible for any finite elements in $\mathscr{U}$.
  - Then $|Nrv(\mathscr{U})| \simeq X$.

- Nerve Lemma (a simplified version):
  - Let $\mathscr{U}$ be a finite collection of **closed**, **convex** subsets in R^d. Then $|Nrv(\mathscr{U})| \simeq \cup_{\alpha \in A} U_\alpha \subset \mathbb{R}^d$.

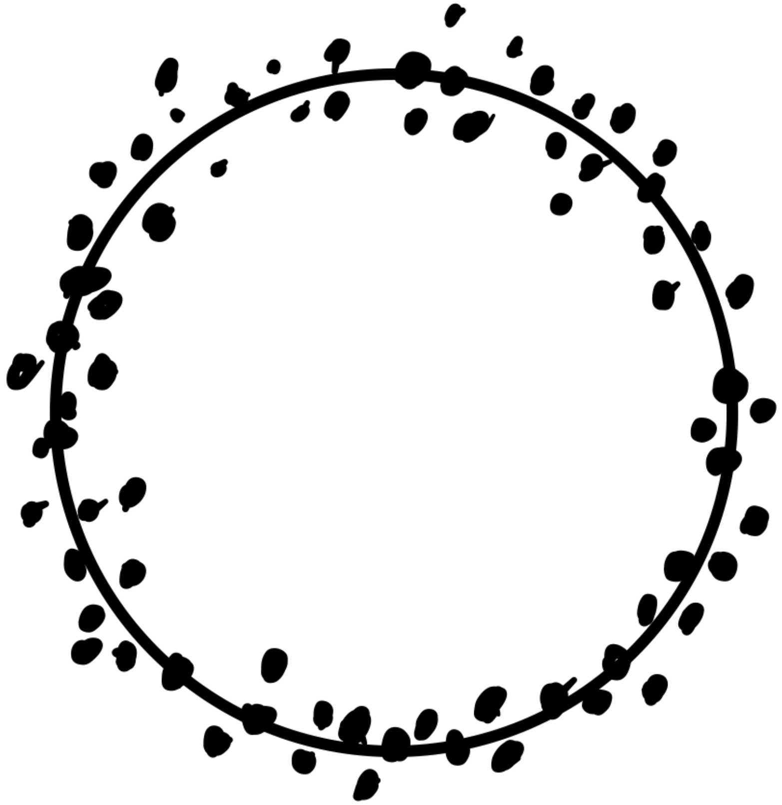


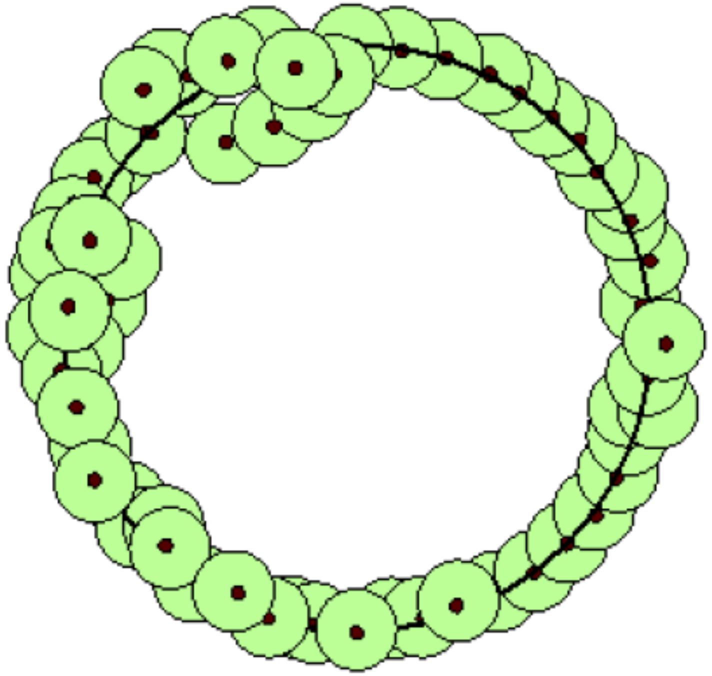






Čech complex





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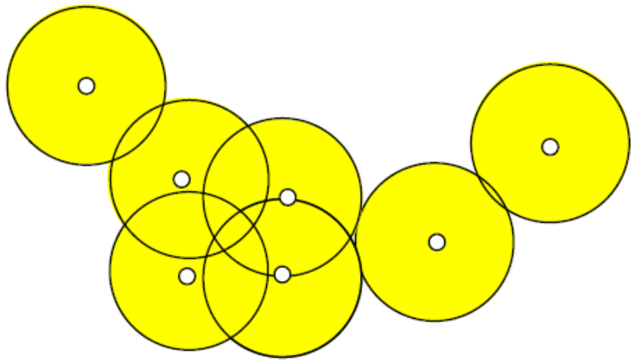

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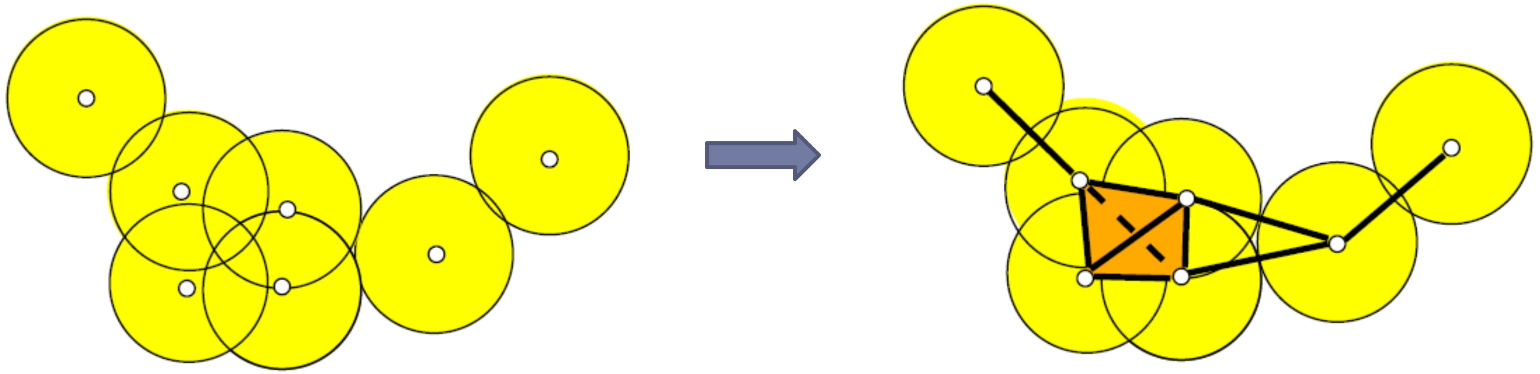

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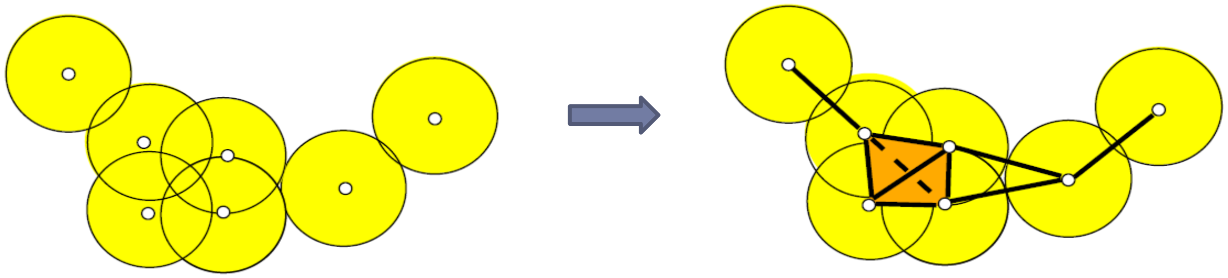

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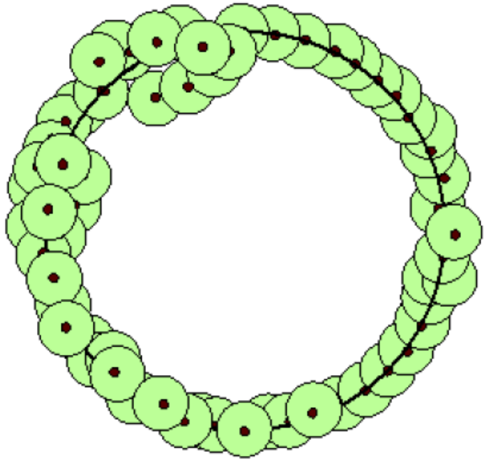
- Given a set of points P
  - approximating a hidden domain M
  - $U^r(P) = \bigcup_{p \in P} B(p, r)$ approximates M
  - $C^r(P)$ approximates $U^r(P)$



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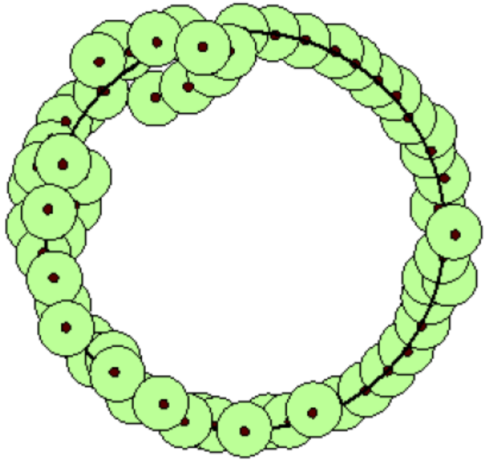
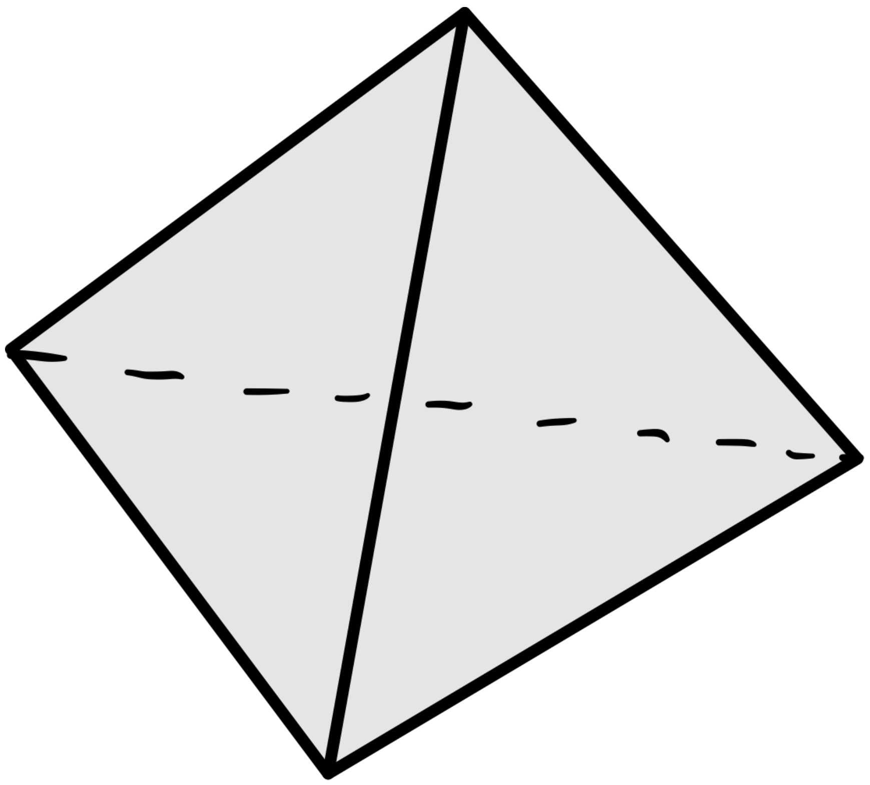


More on Čech

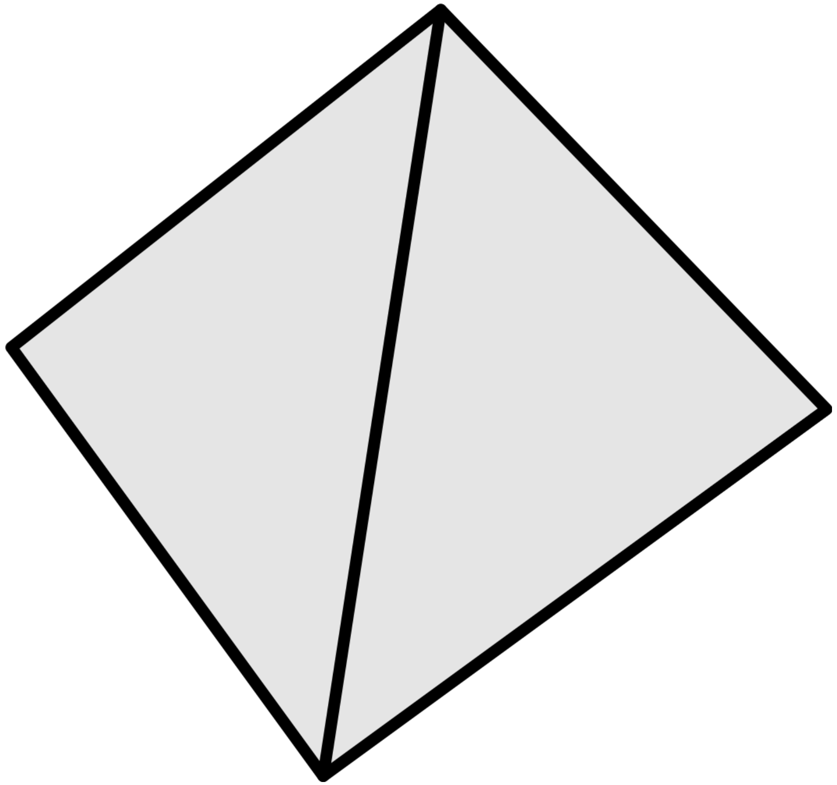
See Demo by Henry Adams

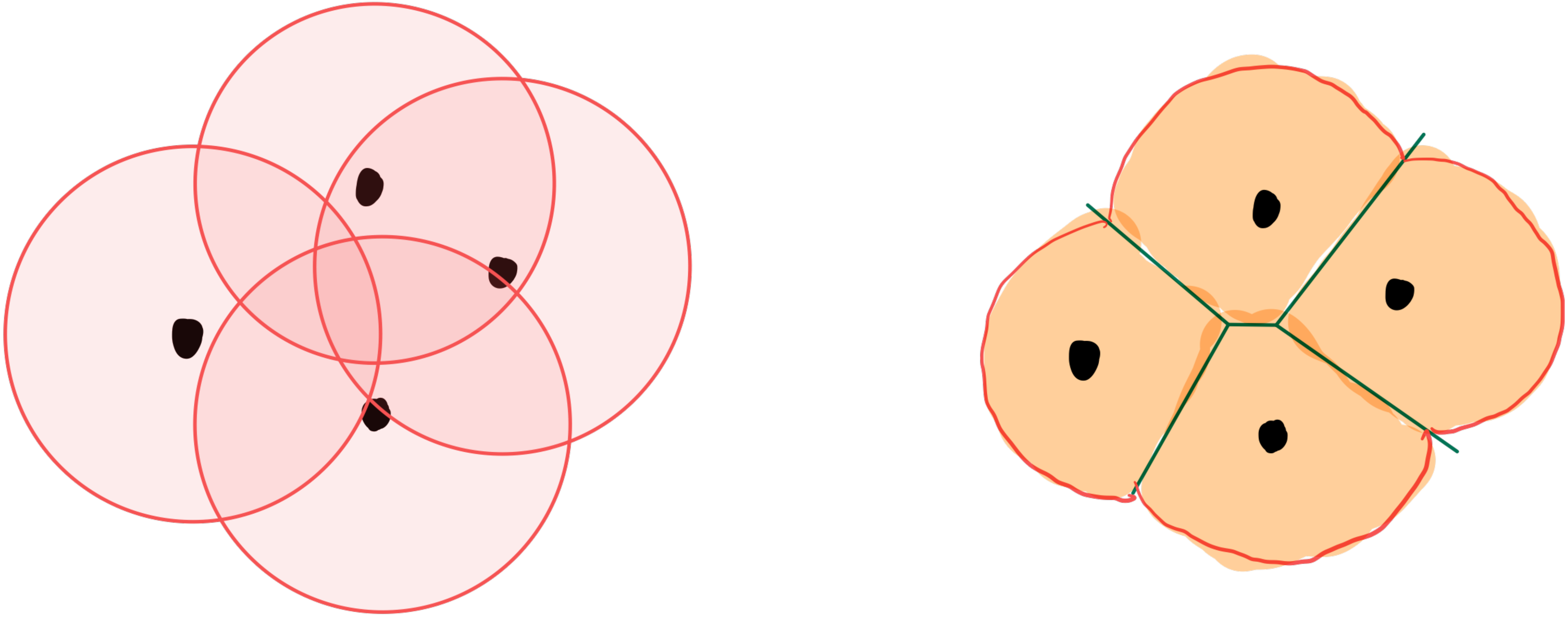
- Given a set of points $P \subset R^d$
  - $C^r(P)$ could have simplex of dimension larger than d
    - In particular, $C^\infty(P)$ is the same as n-simplex.
  - often only d-skeleton of $C^r(P)$ is needed
    - as $U^r(P)$ has trivial topology beyond dimension d
- $C^r(P)$ can be huge!! When r is large enough, there exists $O(2^n)$ many simplices!

Alpha complex



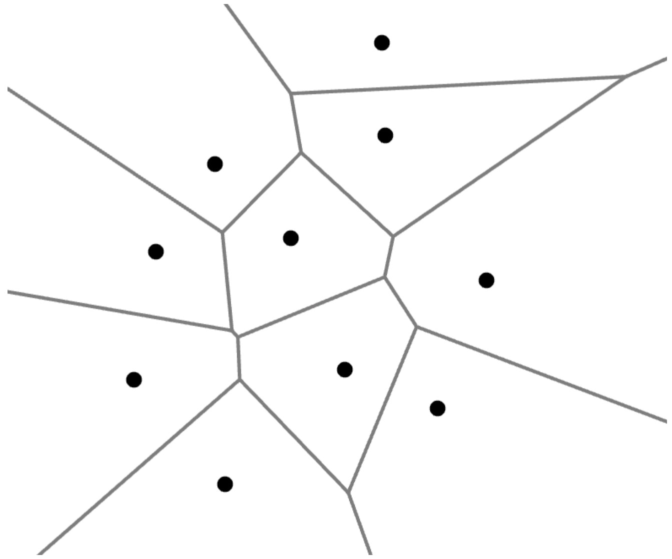
$\simeq$





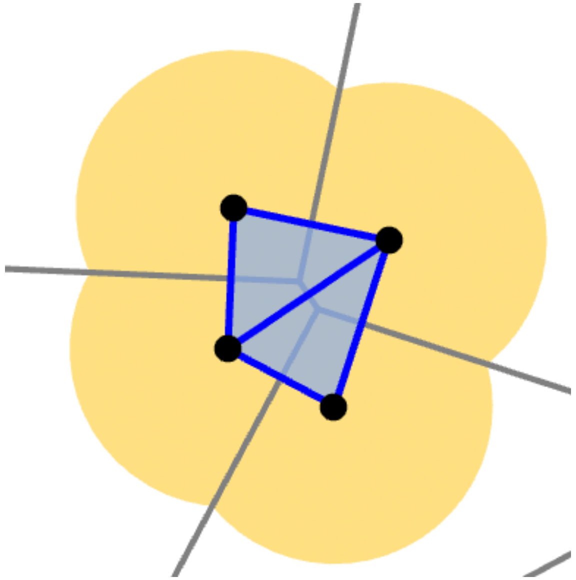
Voronoi Diagram

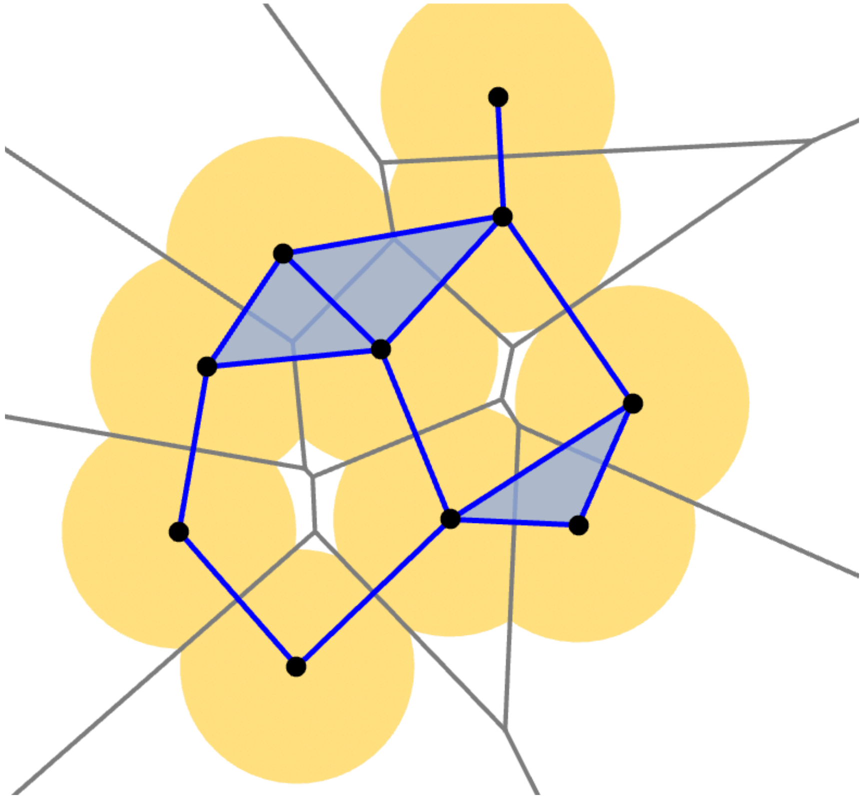
- Given a finite set $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$, the **Voronoi cell** of p_i is
  - $Vor(p_i) = \{x \in \mathbb{R}^d \mid \|x - p_i\| \leq \|x - p_j\|, \forall j \neq i\}$
- The **Voronoi Diagram** of P is the collection of all Voronoi cells.



Alpha complex

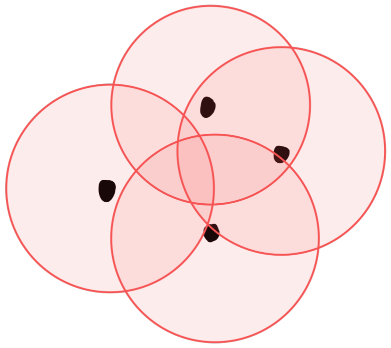
- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$
- Given a real value $r > 0$, the *Alpha complex* $Del^r(P)$ is the nerve of the set $\{B(p_i, r) \cap Vor(p_i)\}_{i=1}^{n}$

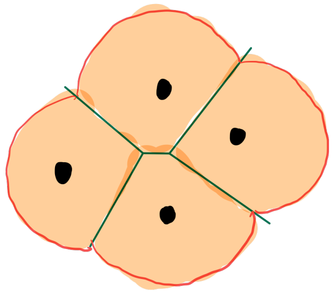




Alpha complex vs Čech complex

- $Del^r(P) \subset C^r(P)$
- $|Del^\infty(P)| = O(n^{\frac{d}{2}})$ whereas $|C^\infty(P)| = O(2^n)$
- $\dim Del^r(P) \leq d$ for generic P





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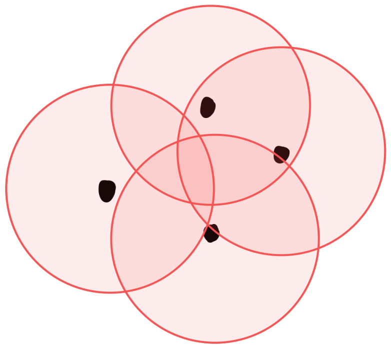
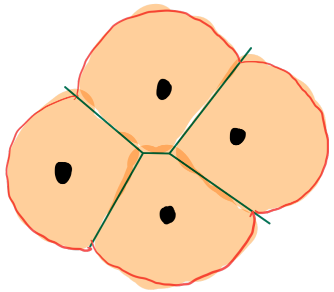


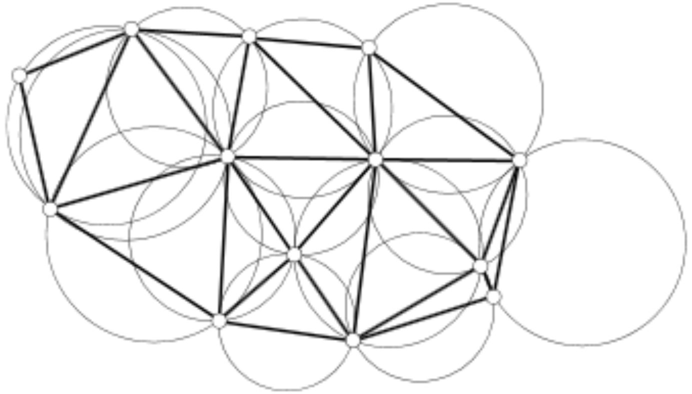


- Proposition:
  - $Del^r(P) \simeq C^r(P) \simeq \cup_p B(p, r)$, i.e, $C^r(P)$ and $Del^r(P)$ are homotopy equivalent.

Delaunay Complex

- $Del^{\infty}(P)$ is called the **Delaunay complex** of P, denote by $Del(P)$
  - $Del(P) = Nrv(\left\{ Vor\left(p\right) \mid p \in P \right\})$
- Delaunay complex $Del(P)$
  - A simplex $\sigma = \left[p_{i_0}, \; p_{i_1}, \; \ldots, \; p_{i_k}\right]$ is in $Del(P)$ if and only if
    - There exists a ball B whose boundary contains vertices of σ, and that the interior of B contains no other point from P.

See Demo by Ondrej Draganov



Delaunay Complex

- Many beautiful properties
  - Connection to Voronoi diagram: given $p \in P$
    - Voronoi cell of p is $Vor(p) := \{x \in R^d \mid d(x, p) = d(x, P)\}$
  - If points from R^d are in generic positions, then a geometric simplicial complex in R^d
- Foundation for surface reconstruction and meshing in 3D
  - *[Dey, Curve and Surface Reconstruction, 2006],*
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- However,
  - Computationally very expensive

  in **high dimensions**

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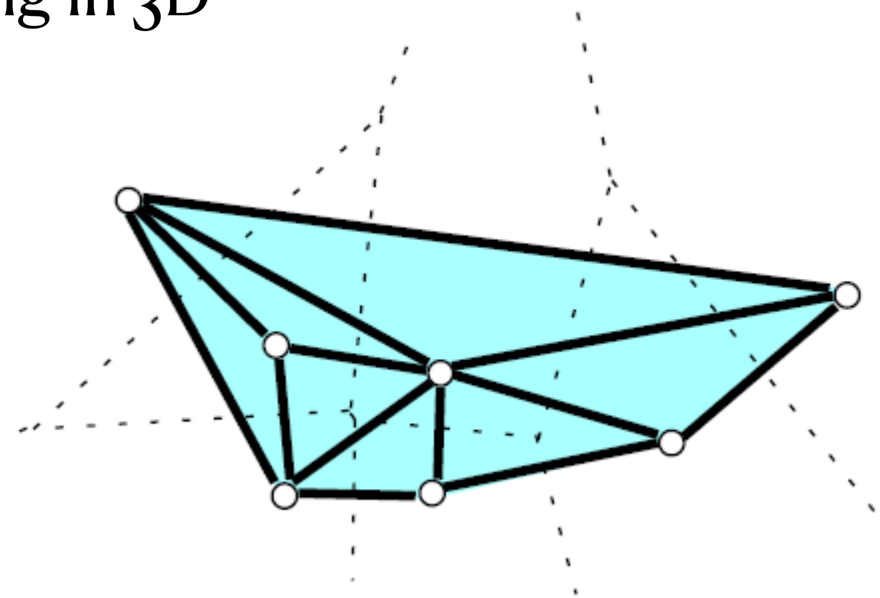

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Vietoris Rips complex

Vietoris-Rips (Rips) Complex

- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$
- Given a real value $r > 0$, the *Vietoris-Rips (Rips) complex* $Rips^r(P)$ is:
  - $\left\{(p_{i_0}, \ldots, p_{i_k}) \,|\, B(p_{i_l}, r) \cap B(p_{i_j}, r) \neq \emptyset, \forall l, j = 0, \ldots, k\right\}.$
- More generally for P in a metric space (X, d):
  - $Rips^r(P) = \left\{(p_{i_0}, \ldots, p_{i_k}) \,|\, d(p_{i_l}, p_{i_j}) \leq 2r, \forall l, j = 0, \ldots, k\right\}.$

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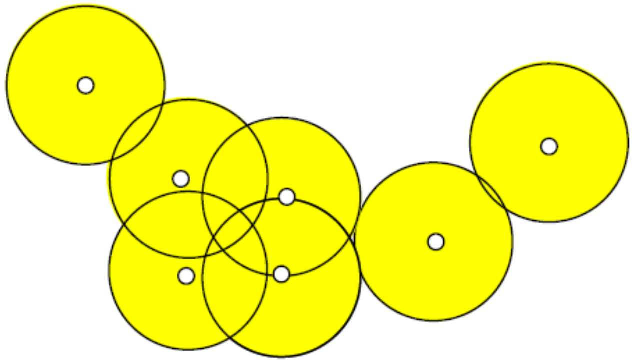

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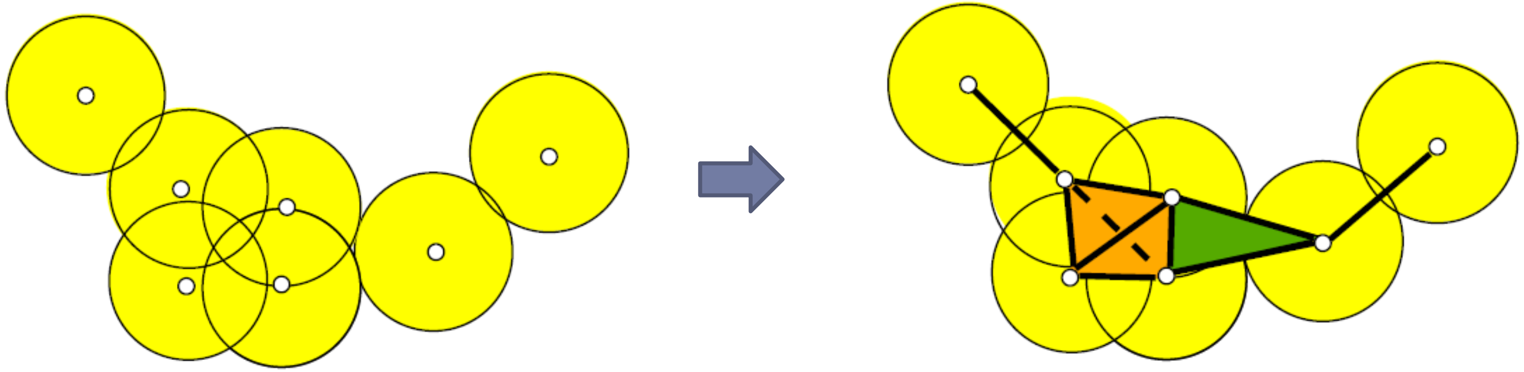

Vietoris-Rips (Rips) Complex

- The 1-skeleton of $Rips^r(P)$ is the $2r$ neighborhood graph of P, i.e., $\{p_i, p_j\} \in E$ if $d(p_i, p_j) \leq 2r$
  - Same for Čech
- $Rips^r(P)$ is the clique complex of its 1-skeleton
  - If $\{\{p_{i_k}, p_{i_l}\}\}_{k \neq l \in 0,\ldots,m}$ are edges,
  - then $d(p_{i_k}, p_{i_l}) \leq 2r$ for $k \neq l \in 0, \ldots, m$
  - Hence $\{p_{i_0}, \ldots, p_{i_m}\} \in Rips^r(P)$

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- $Rips^r(P)$ is the clique complex of its 1-skeleton
- Computing $Rips^r(P)$ reduces to computing the $2r$ neighborhood graph and finding its clique complex

Rips vs Čech

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Rips vs Čech

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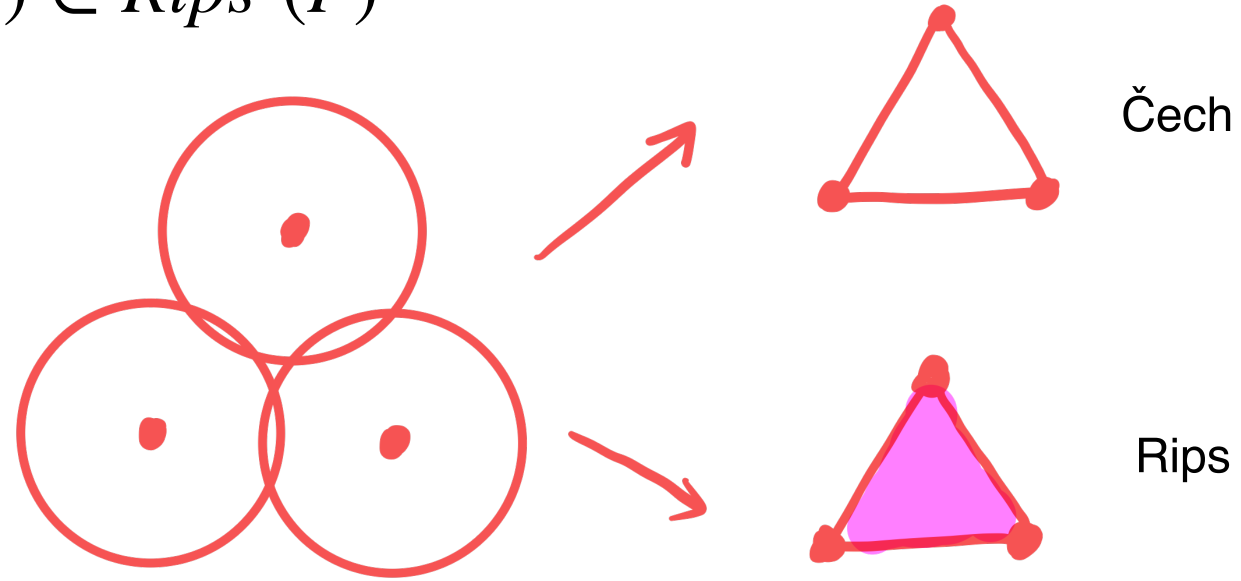

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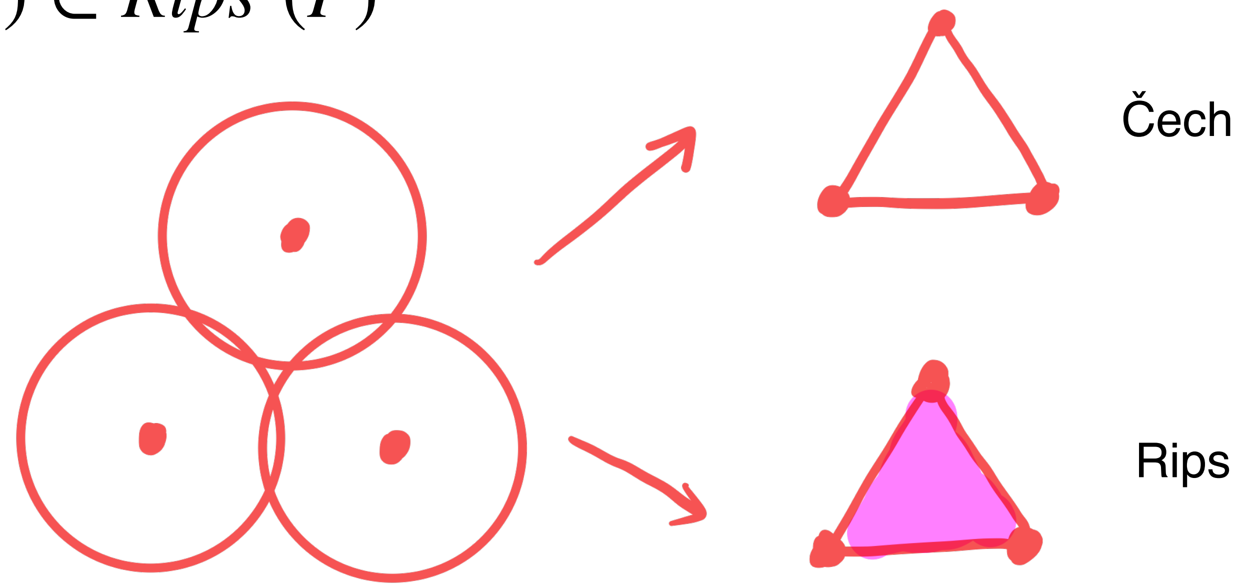

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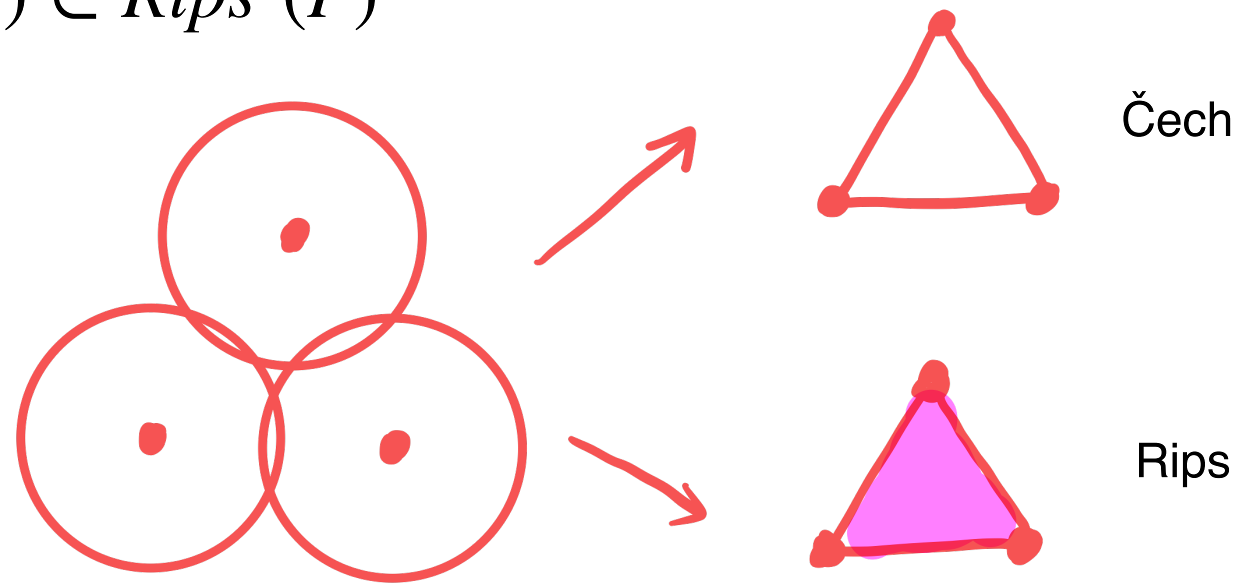

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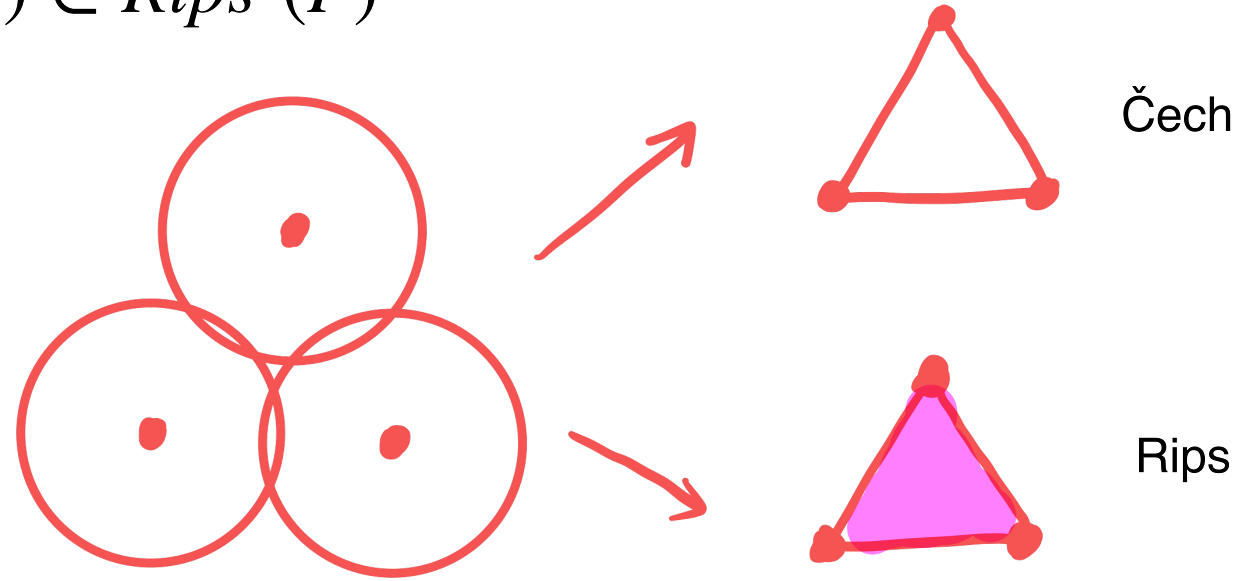

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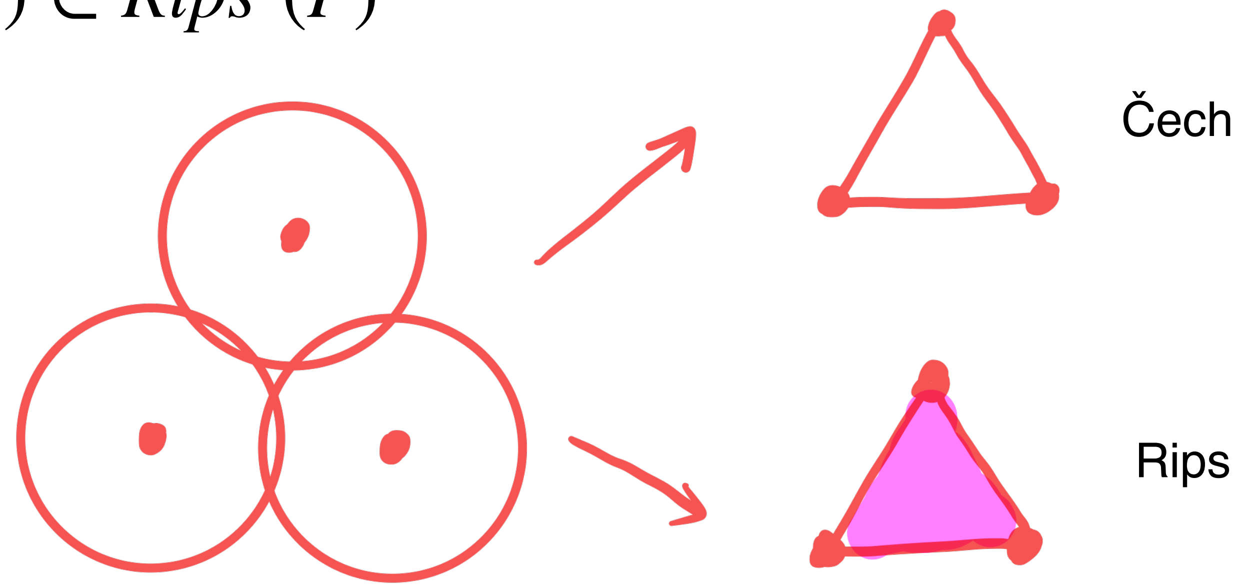

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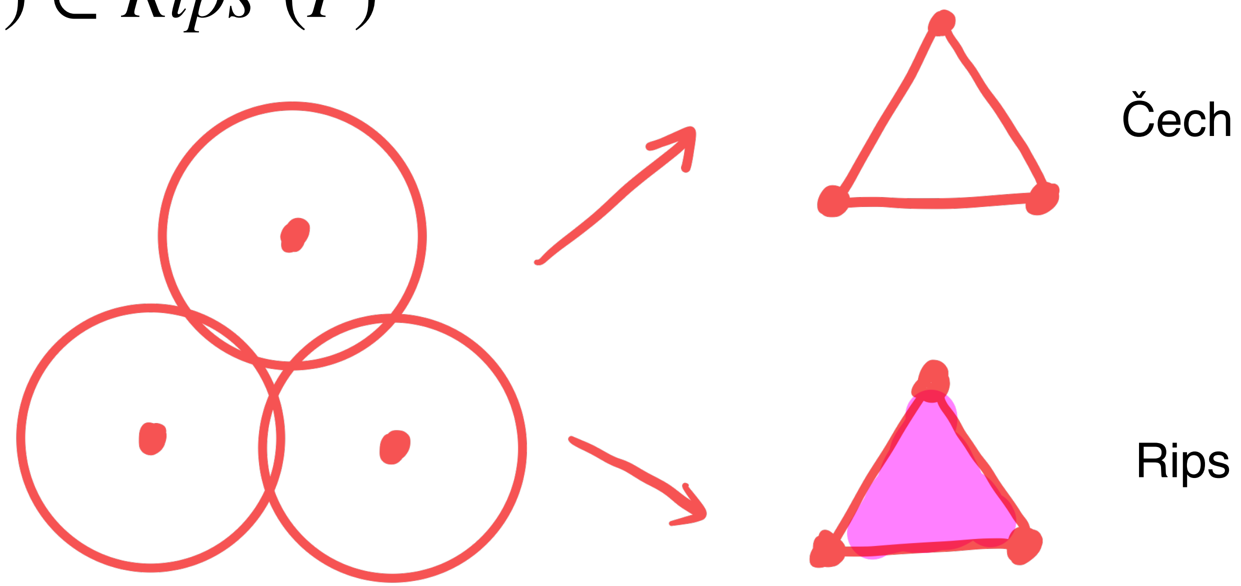

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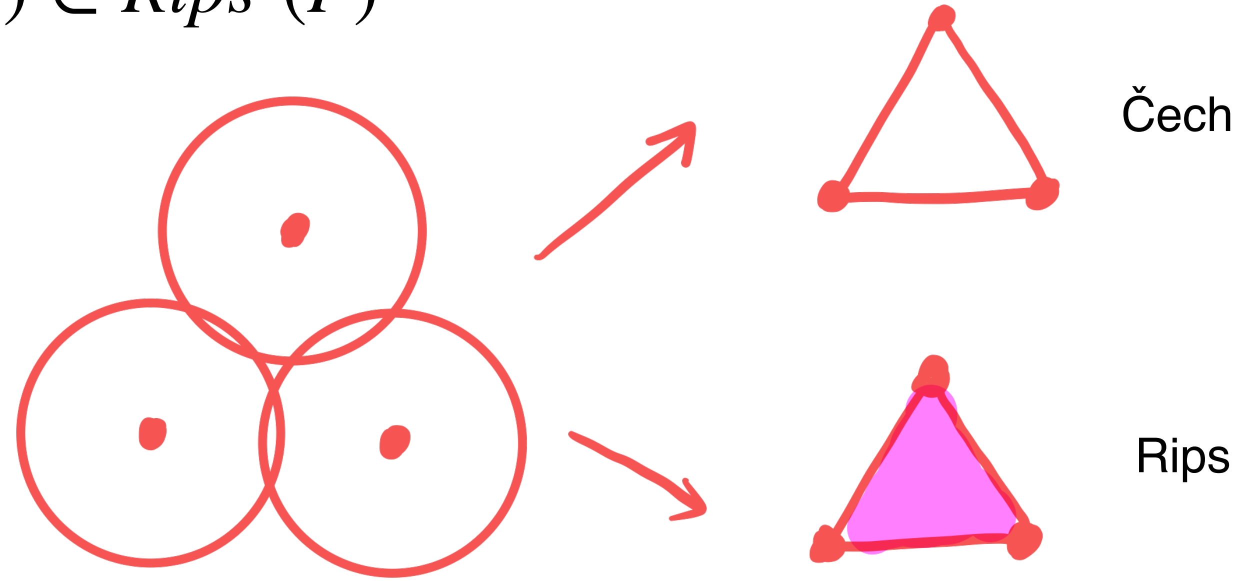

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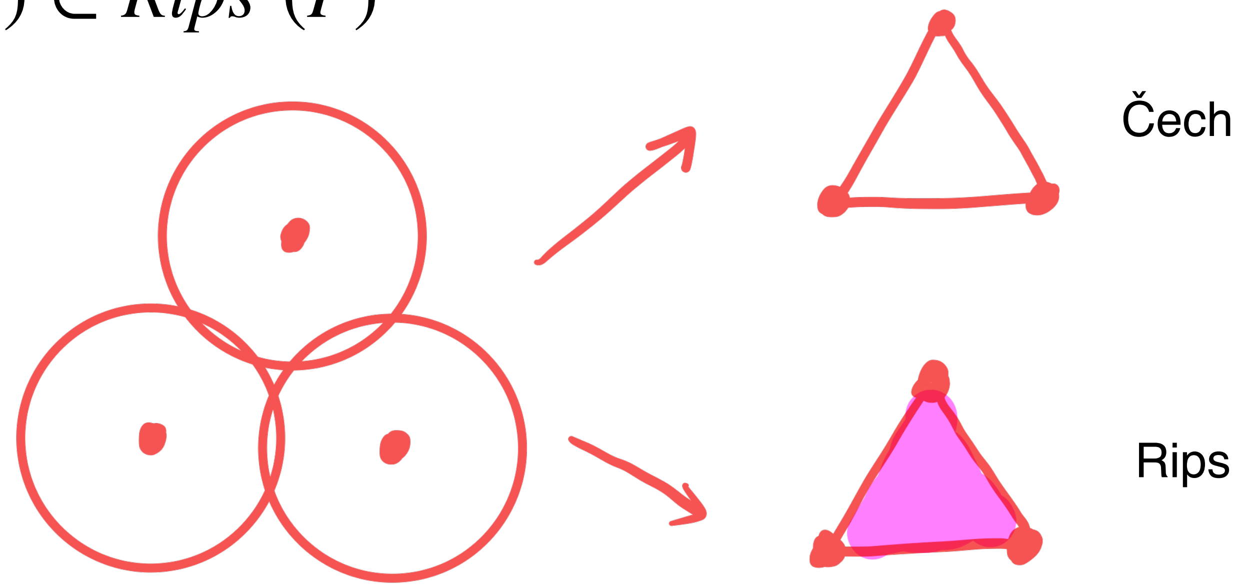

- $C^r(P) \subset Rips^r(P) \subset C^{2r}(P)$
  - $C^r(P) \subseteq Rips^r(P) \subseteq C^{\sqrt{2}r}(P)$ when $P \subseteq \mathbb{R}^d$

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  - Many optimized packages for Rips complex

FIN