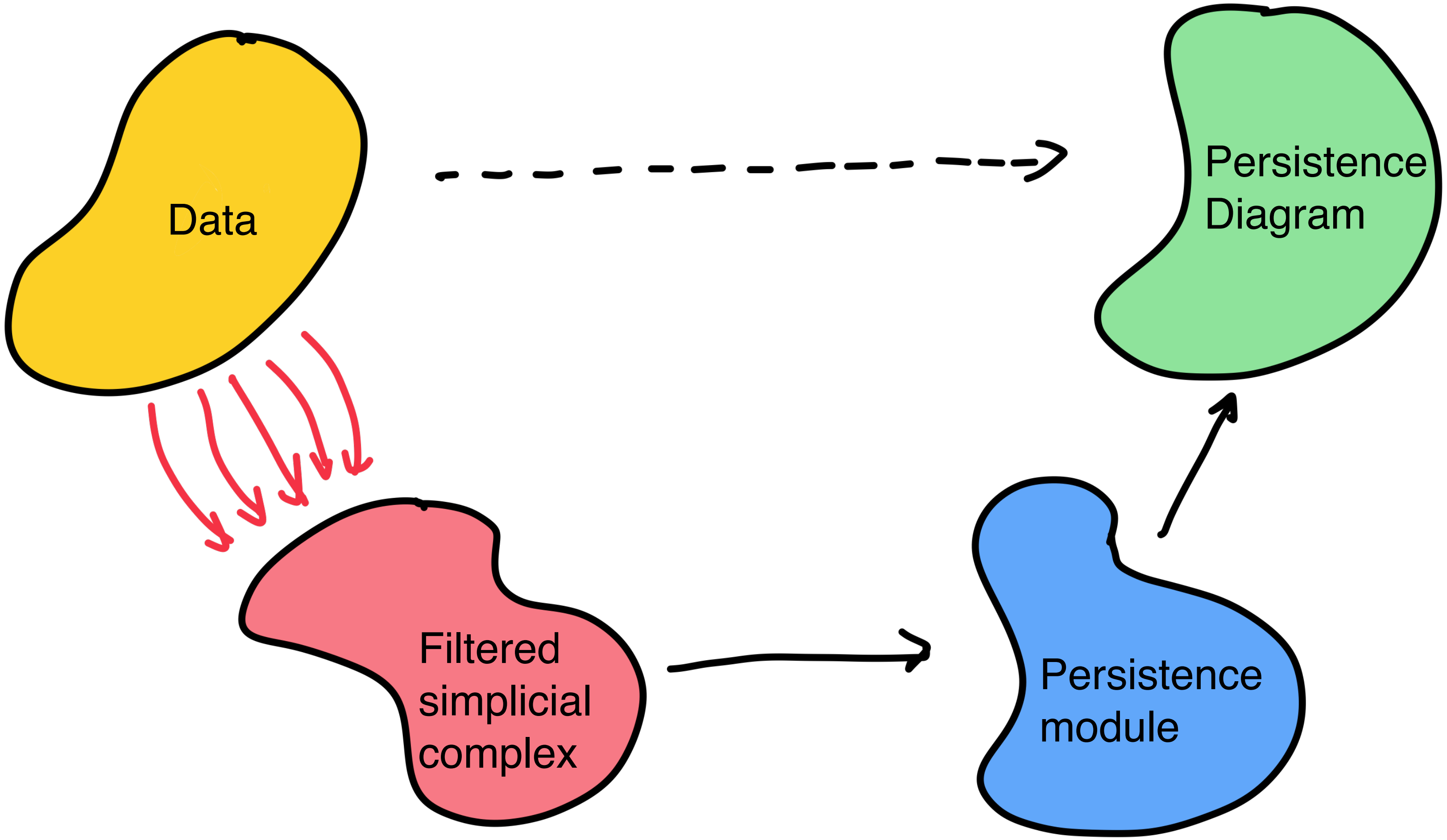
DSC 214
Topological Data Analysis

Topic 4-B: Persistent Homology for PCD and Functions

Instructor: Zhengchao Wan

Persistence-based Framework



Persistence-based Framework

- Persistence-based feature representation

Input data:
A collection of point sets, graphs, 2D/3D images, time-series , etc

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Persistence feature space:
A collection of persistence-based feature representations (e.g, vectors)

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Further data analysis:
Feed to data analysis pipelines, say SVM, CNN, etc

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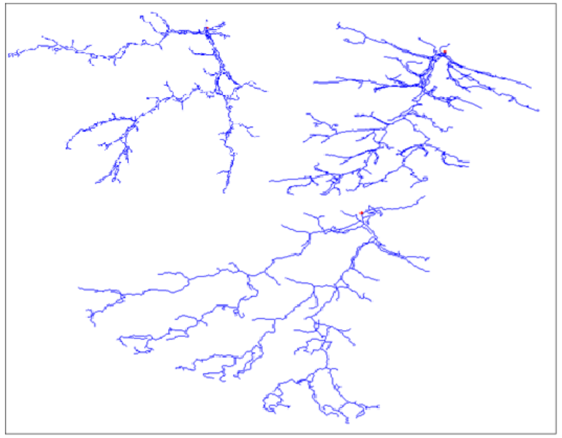

[Li, et al, W., PLOS One 2017]

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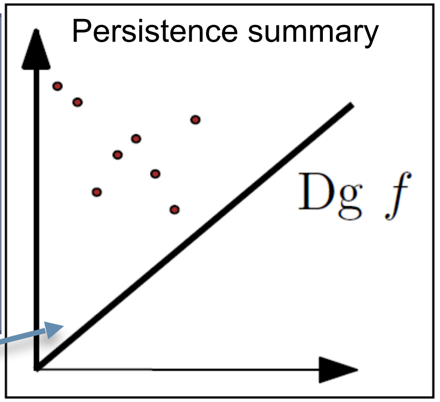

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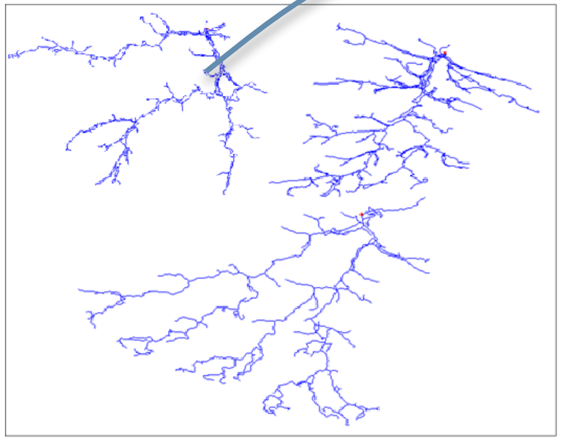

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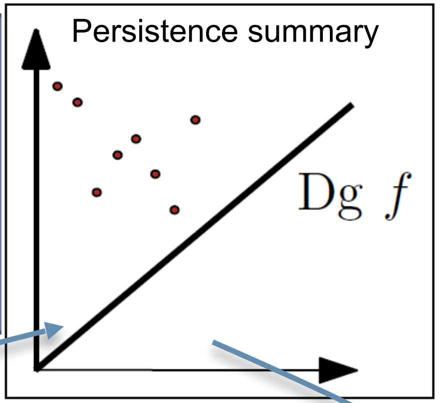

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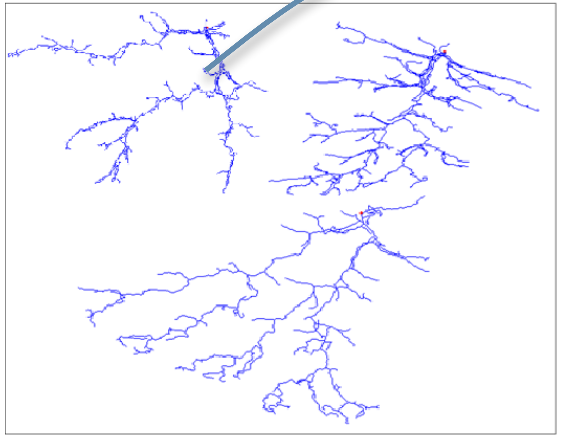

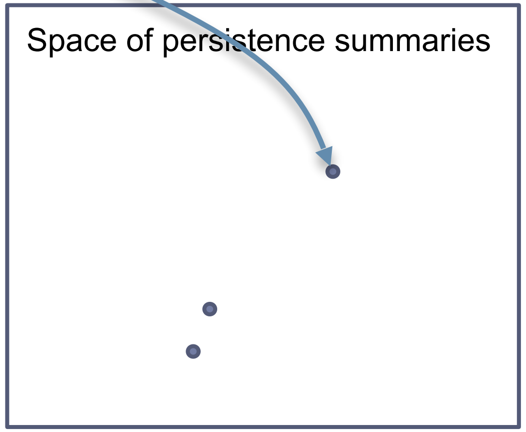

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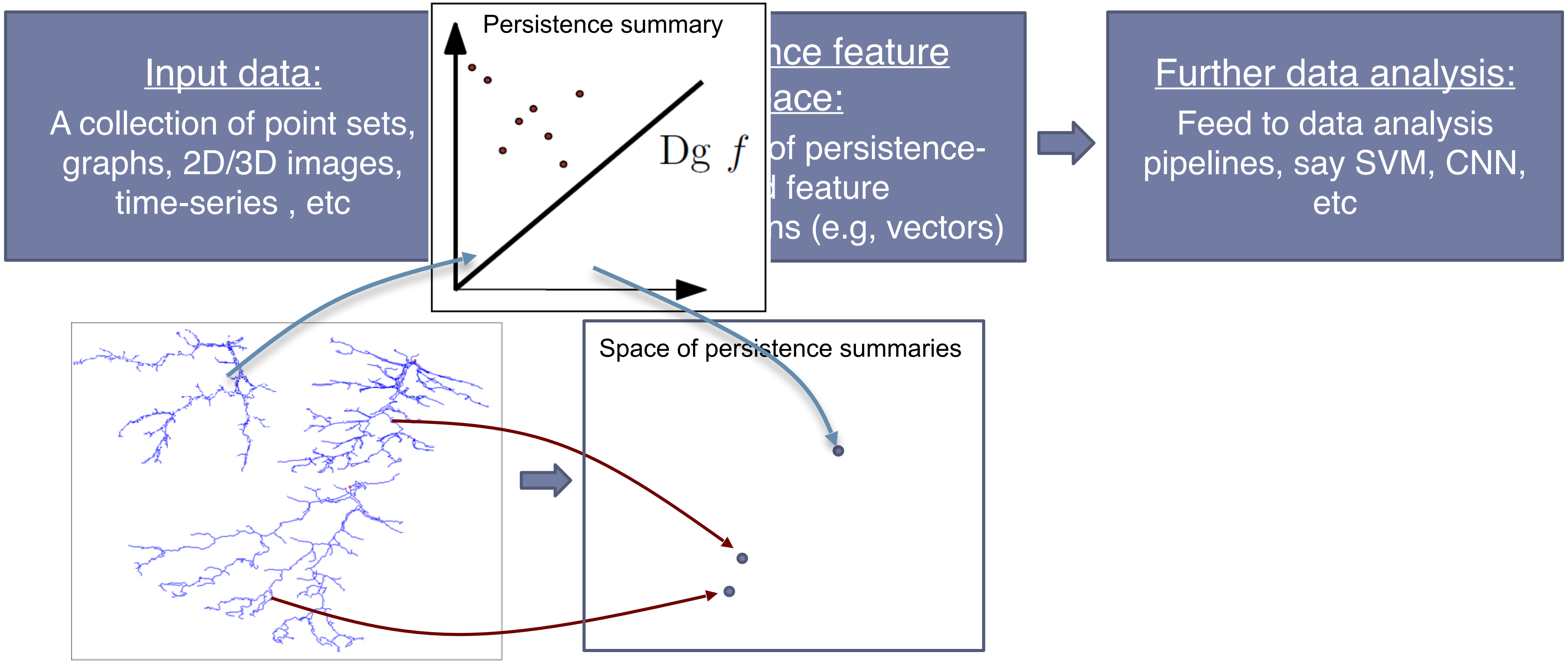

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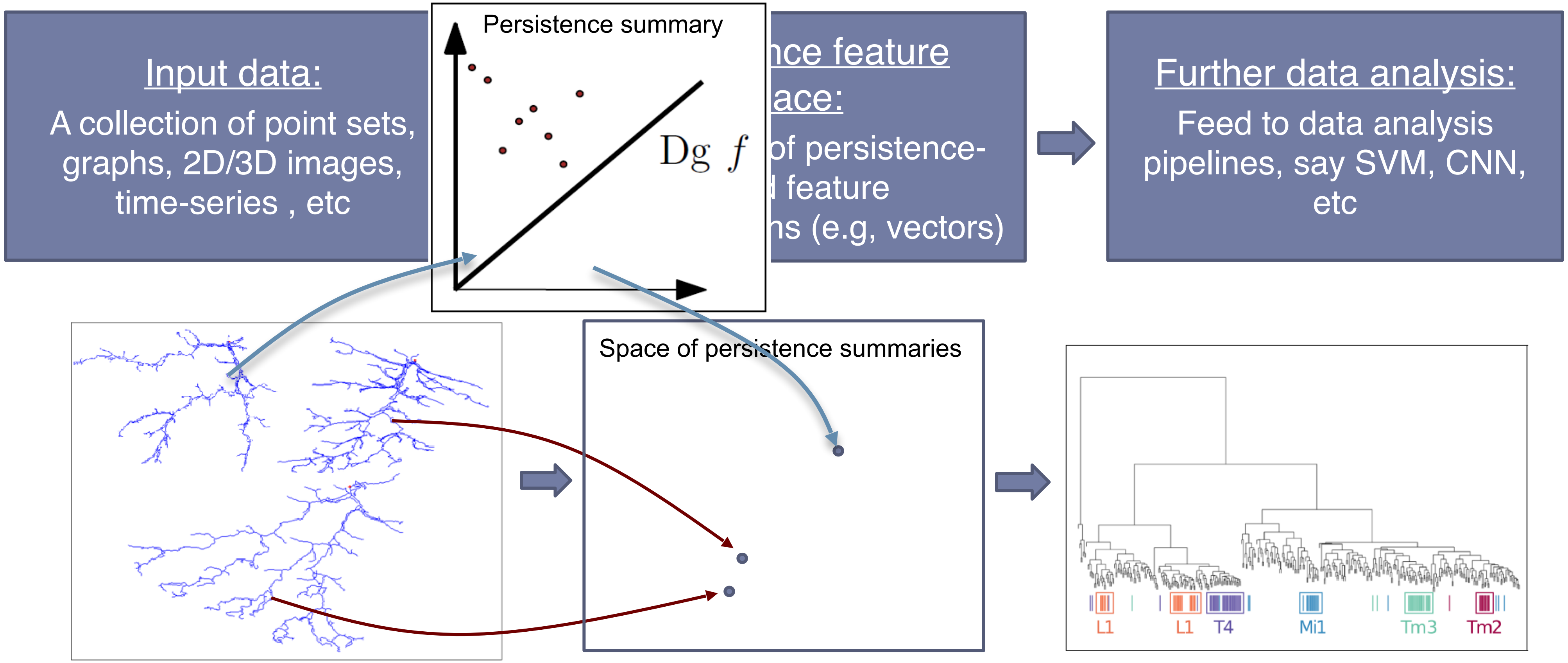
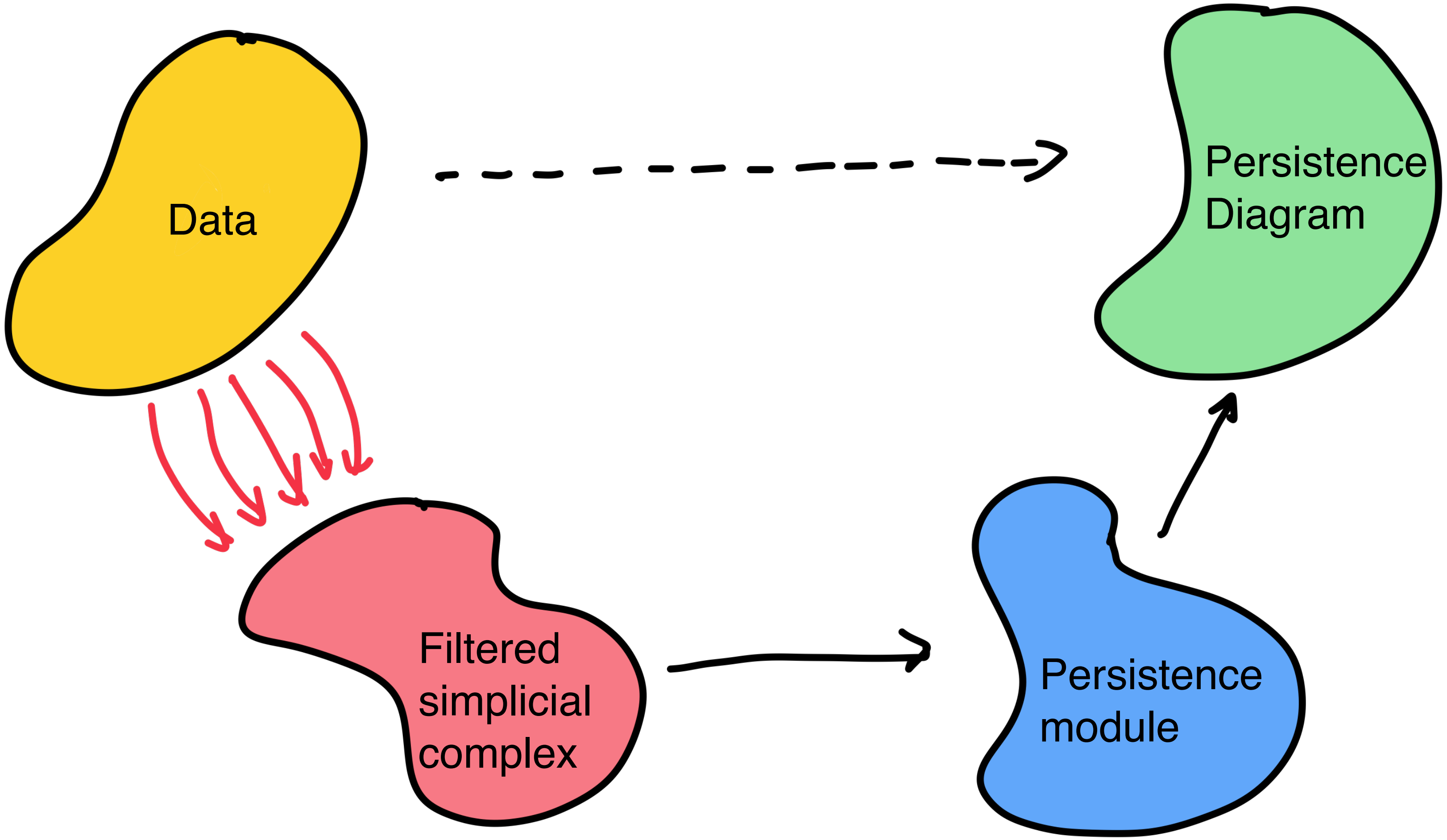


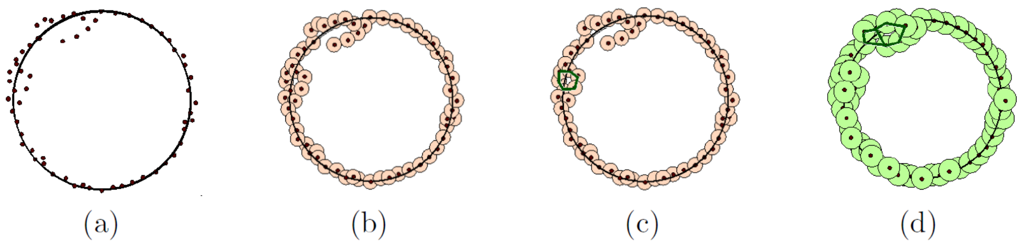
[Li, et al, W., PLOS One 2017]

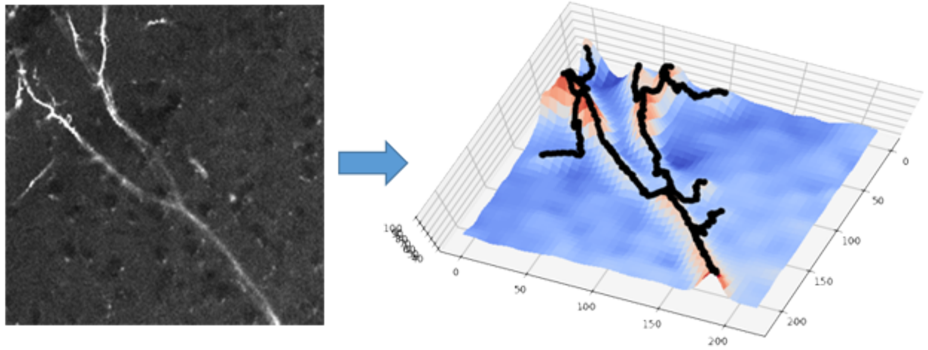
- Recently, many methods for mapping persistence diagrams to a finite vector space or a Hilbert space
    - Persistence landscapes
        - [Bubenik 2012]
    - Persistence scale space kernel
        - [Reininghause et al., 2014]
    - Persistence images
        - [Adams et al., 2015, 2017]
    - Persistence weighted Gaussian kernel
        - [Kusano et al., 2017]
    - Sliced Wasserstein kernel
        - [Carriere et al., 2017]
    - Persistence Fisher kernel
        - [Le and Yamada 2018]
    - ....

Persistence-based Framework



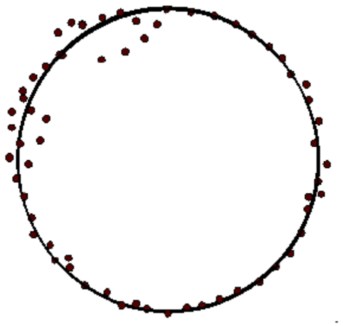
- how do we use persistent homology that introduced last time to different types of data?
- Two examples:
  - Point cloud data
  - Functions on triangulated spaces





Section 1:
PH for Point Cloud Data

Type I: Point Cloud Data



(a)

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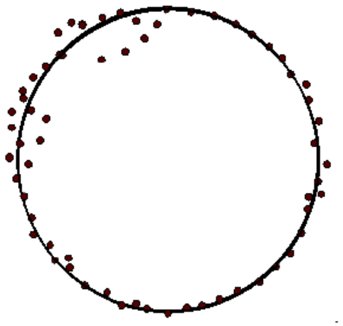

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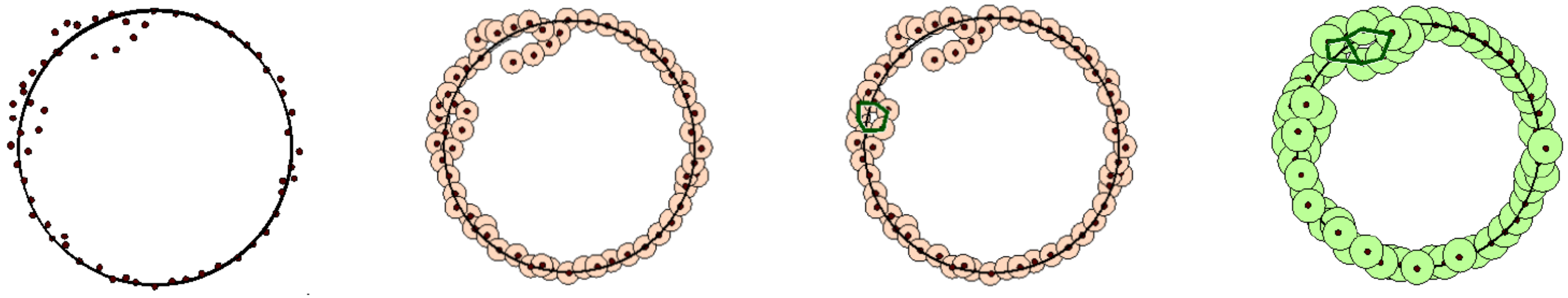
(a) (b) (c) (d)

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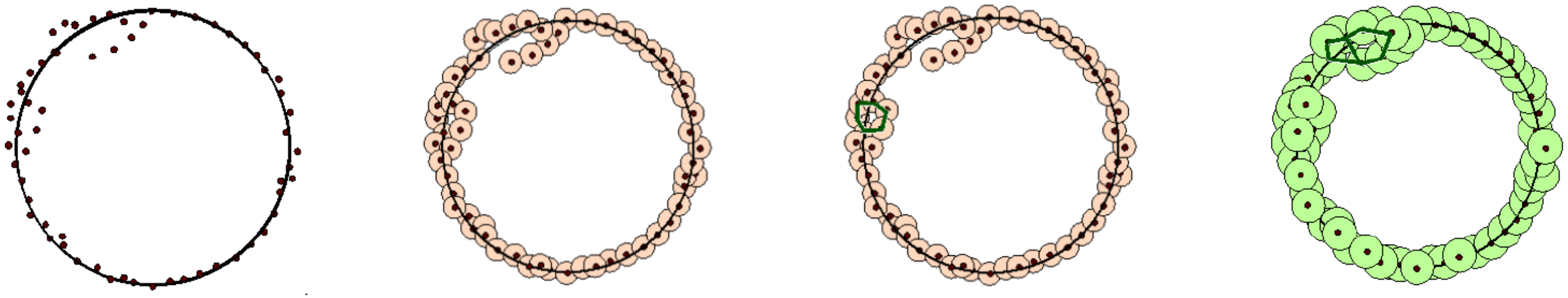
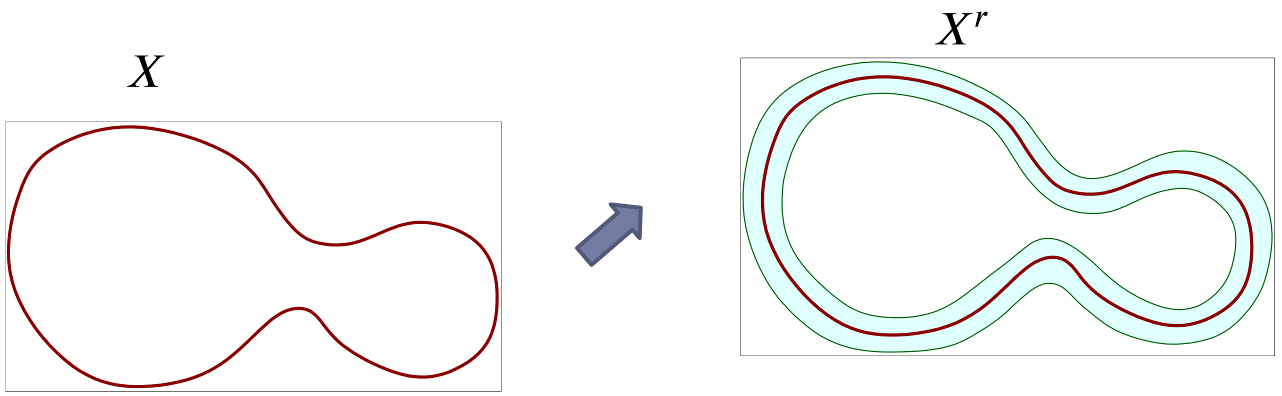


(a) (b) (c) (d)

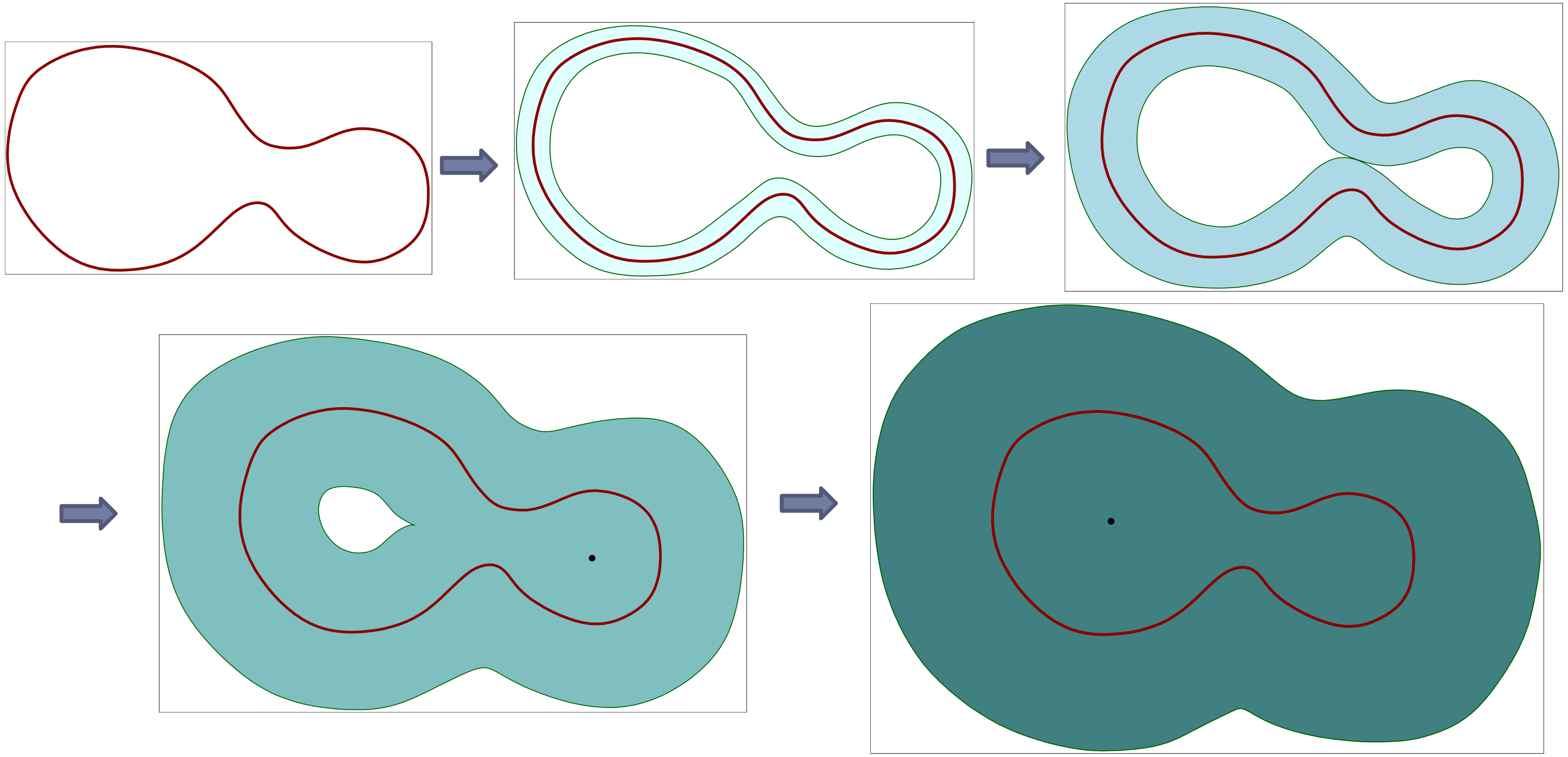
- Goal:
  - Construct certain family of simplicial complexes spanned by these input points P at multiple scales
  - Will mention two choices:
    - Čech complex and Rips complex induced by filtrations

Offset – Union of Balls

- $X \subset R^d$: a compact subset of R^d
  - a hidden space of interests
- X^r: r-offset of X
  - $X^r = \left\{ y \in R^d \mid d\left(y,\ X\right) \le r \right\} = \bigcup_{x \in X} B(x,\ r)$



- Target filtration: $X^{\alpha_0} \subseteq X^{\alpha_1} \subseteq \cdots X^{\alpha} \subseteq \cdots$
- PH induced by this filtration provides a summary of X



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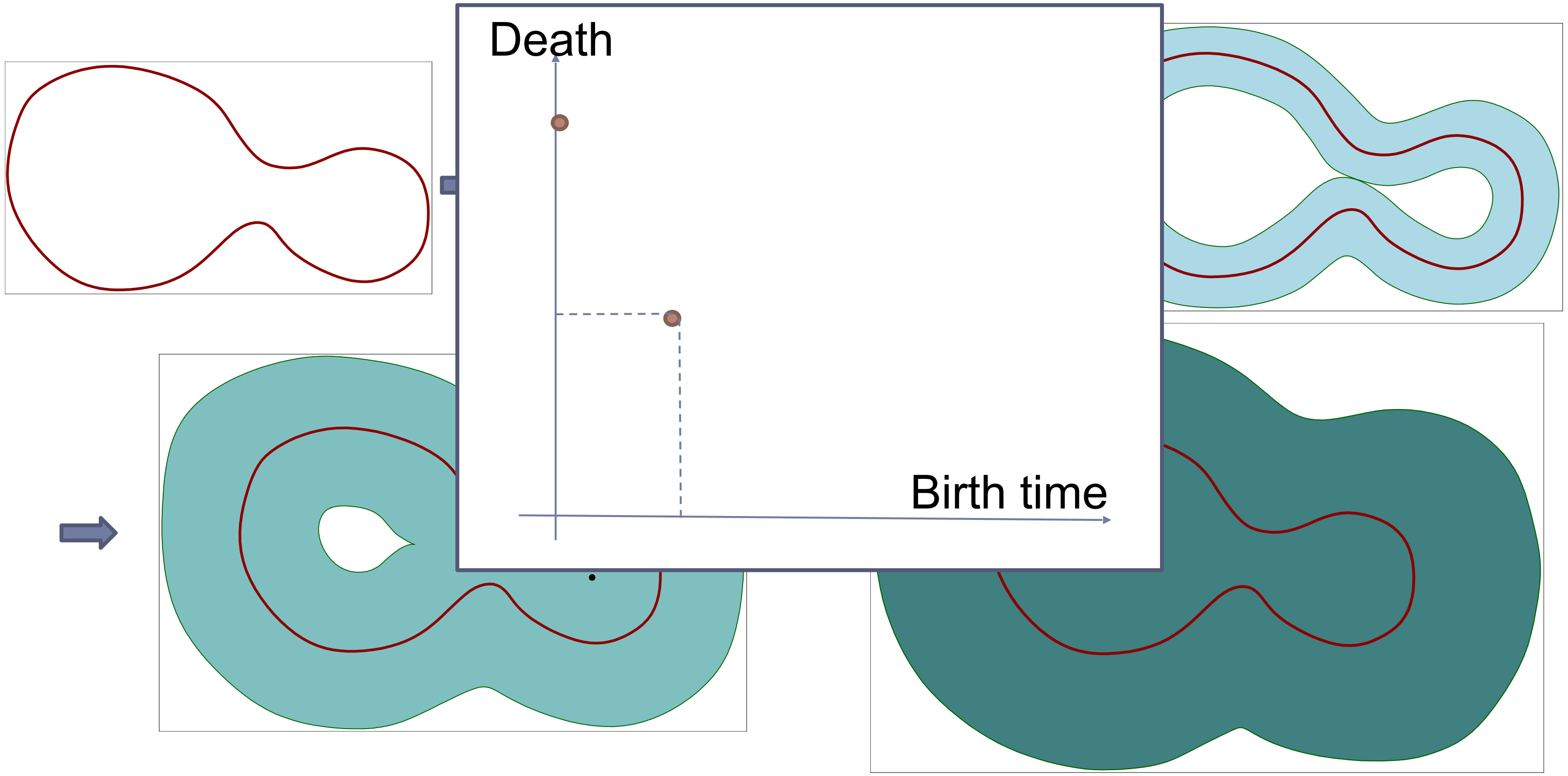
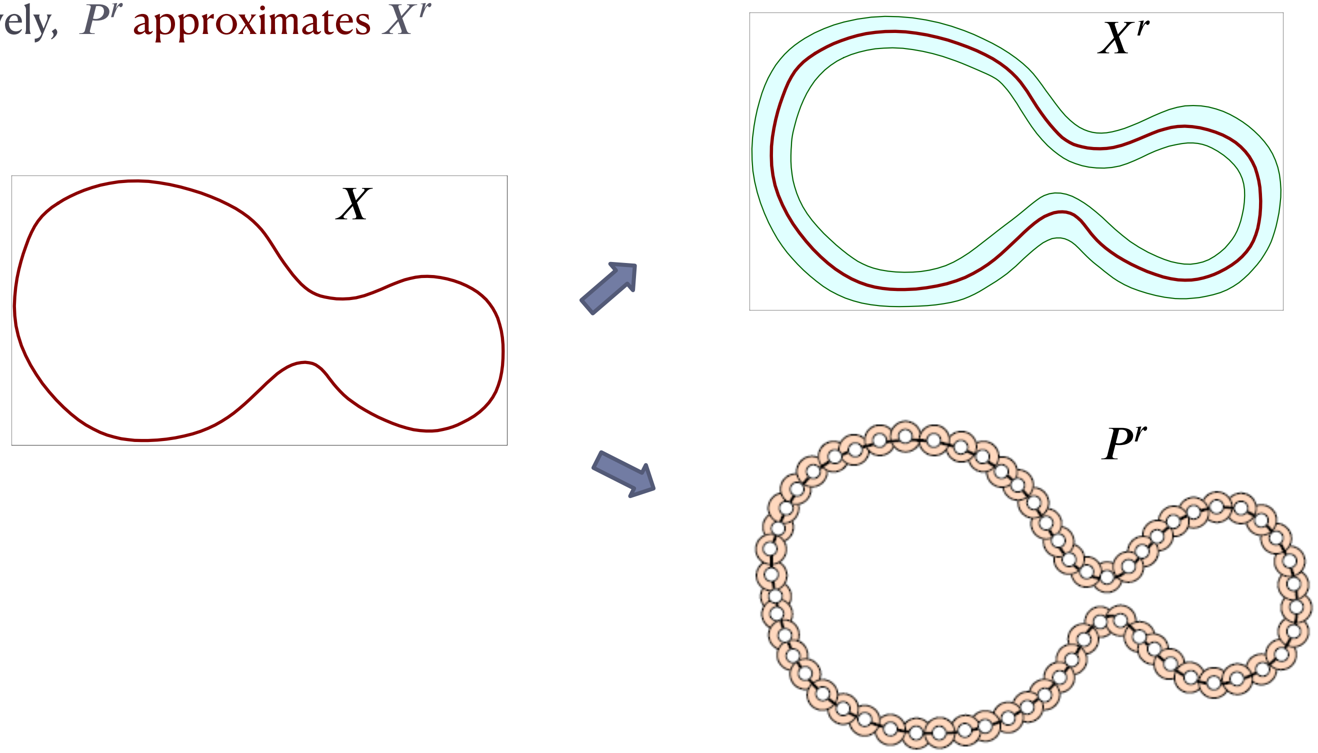


Union of Balls

- Target filtration: $X^{r_0} \subseteq X^{r_1} \subseteq \cdots X^{r} \subseteq \cdots$
- Instead of X, we are only given PCD P
  - $P^r = \bigcup_{p \in P} B(p, r)$
  - Intuitively, P^r approximates X^r



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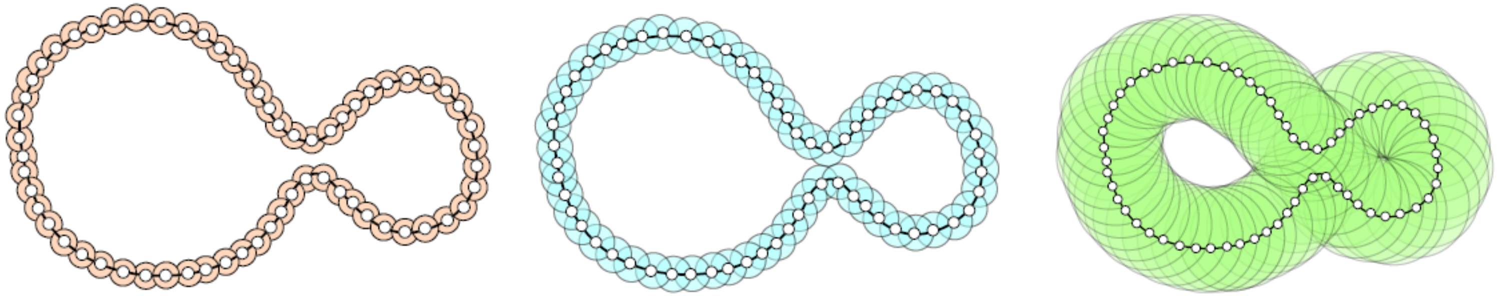

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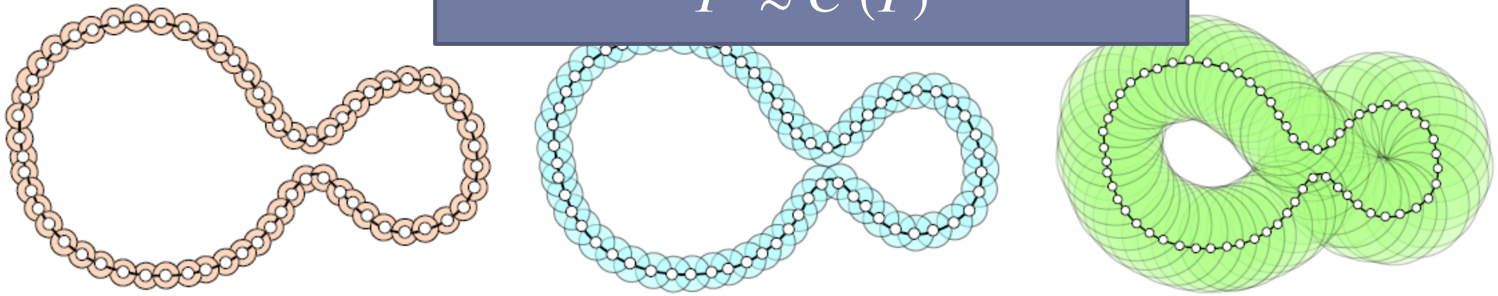
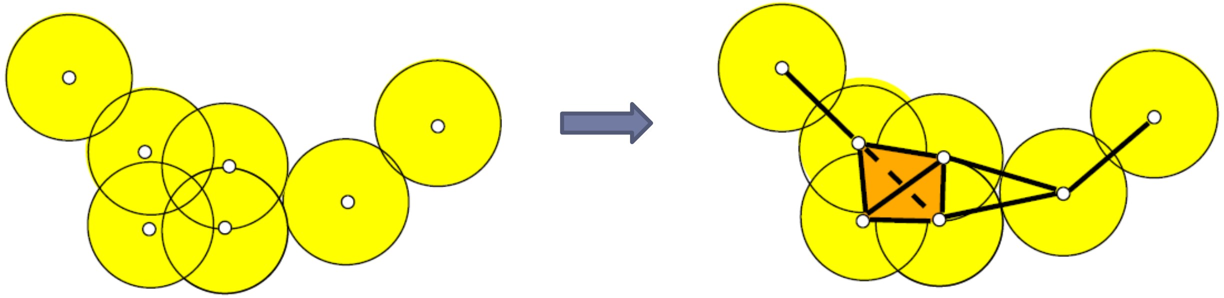
By Nerve Lemma,
$P^r \approx C^r(P)$



Recall: Čech Complex

- Given a set of points $P = \{ p_1, p_2, \ldots, p_n \} \subset R^d$
- Given a real value $r > 0$, the *Čech complex* $C^r(P)$ is the nerve of the set $\left\{ B(p_i, r) \right\}_{i \in [1,n]}$
  - i.e, $\sigma = \left\{ p_{i_0}, \ \ldots, p_{i_s} \right\} \in C^r(P)$ iff $\bigcap_{j \in [0,s]} B\left(p_{i_j}, \ r\right) \neq \emptyset$
- The definition can be extended to a finite sample P of a metric space.



Nerves

- Given a finite set F, its nerve complex $Nrv(F)$ is
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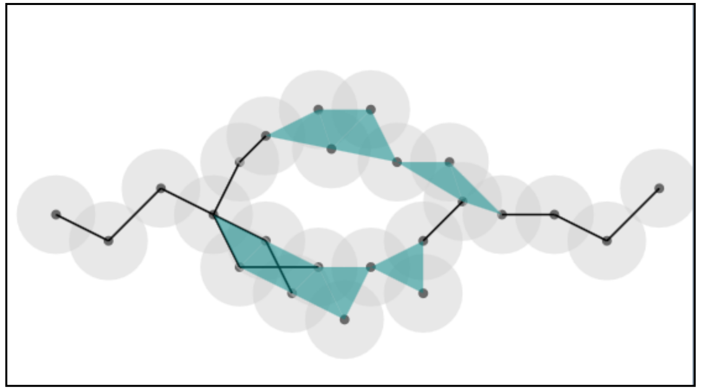
- Hence Čech complex $C^r(P)$
  - is the nerve of $F = \left\{ B(p, r) \mid p \in P \right\}$
  - i.e, $C^r(P) = Nrv(F)$

Nerve Lemma

- Nerve Lemma (a simplified version):
  - Let $\mathscr{U}$ be a finite collection of closed, convex subsets in R^d. Then $|Nrv(\mathscr{U})| \simeq \cup_{\alpha\in A} U_\alpha \subset \mathbb{R}^d$.



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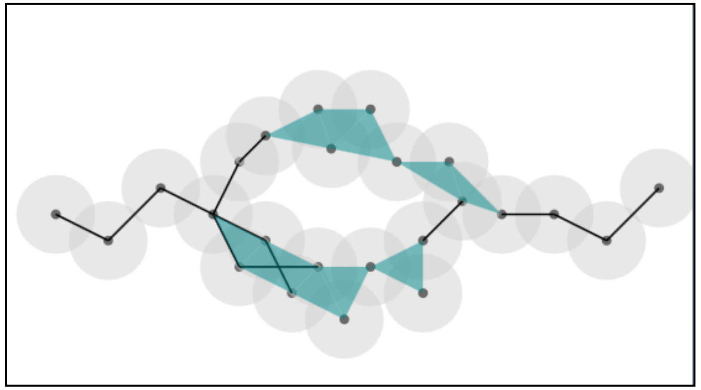
- Corollary:
  - $|C^r(P)| \simeq \cup_{p \in P} B(p, r)$, i.e, $|C^r(P)|$ is homotopy equivalent to the union of r-balls around points in P



Persistent Homology Inference from PCD

- Input:
  - A set of points $P \subseteq R^d$ sampled on/around X
- Question:
  - How to approximate the persistence module induced by F_X?

Target filtration (F_X):	$X^{r_0} \subseteq X^{r_1} \subseteq \cdots X^{r} \subseteq \cdots$
Intermediate filtration:	$P^{r_0} \subseteq P^{r_1} \subseteq \cdots P^{r} \subseteq \cdots$

$\cong$ By Nerve Lemma

Čech filtration ($\mathscr{C}_X$):	$C^{r_0} \subseteq C^{r_1} \subseteq \cdots C^{r} \subseteq \cdots$

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Intermedi

The approximation of their persistence diagrams can be made precise by using the so-called interleaving distance.

By Nerve Lemma

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Recall: Rips Complex

- Given a set of points $P = \{ p_1, p_2, \ldots, p_n \} \subset R^d$
- Given a real value $r > 0$, the *Vietoris-Rips (Rips) complex* $R^r(P)$ is:
  - $\{ \left(p_{i_0}, p_{i_1}, \ldots, p_{i_k} \right) \mid B_r\left(p_{i_l}\right) \cap B_r\left(p_{i_j}\right) \neq \emptyset, \ \forall\, l, \ j \in [0, k\,] \}.$

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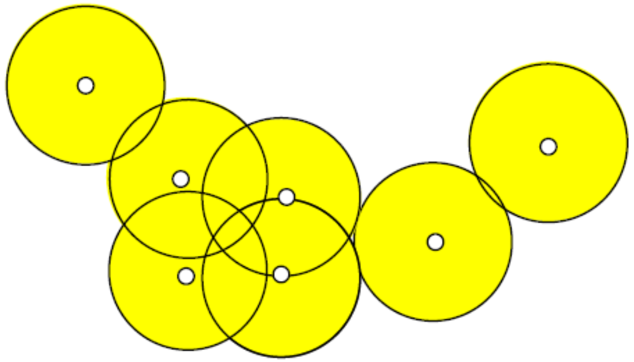

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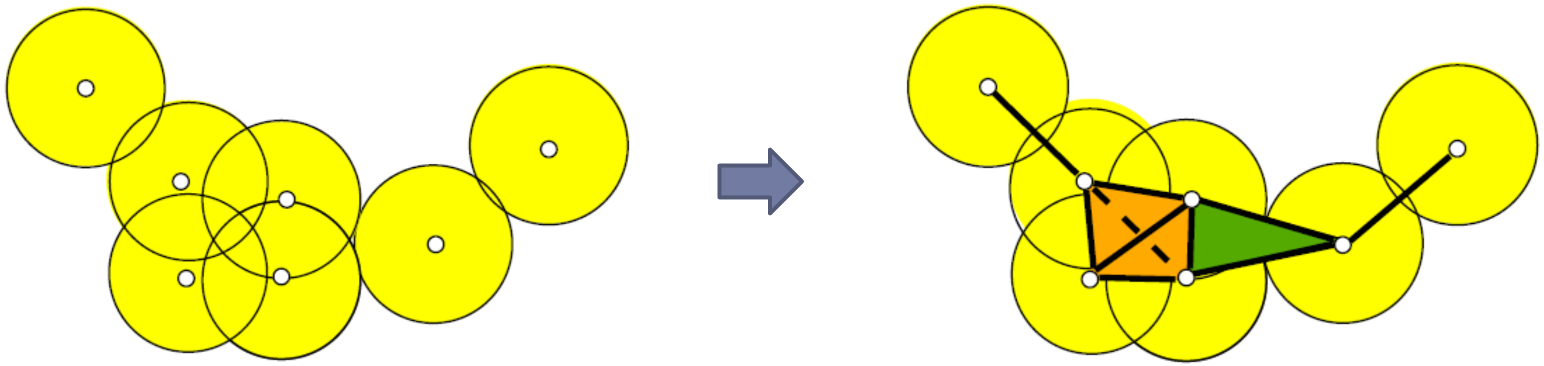

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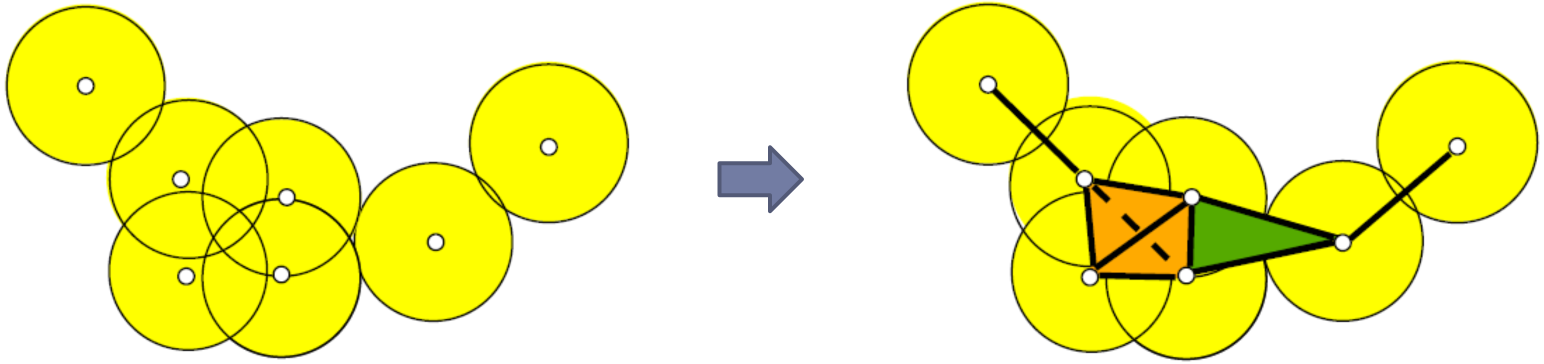


- Rips complex shares the same edge set as the Cech complex w.r.t same r.
- It is the clique complex induced by its edge set.

Rips and Čech Filtrations

- Relation in general metric spaces
  - $C^r(P) \subseteq R^r(P) \subseteq C^{2r}(P)$
  - Bounds better in Euclidean space

Target filtration (F_X): $X^{r_0} \subseteq X^r \subseteq \cdots X^r \subseteq \cdots$

Intermediate filtration: $P^{r_0} \subseteq P^{r_1} \subseteq \cdots P^r \subseteq \cdots$

$\cong$ By Nerve Lemma

Čech filtration ($\mathscr{C}_X$): $C^{r_0} \subseteq C^{r_1} \subseteq \cdots C^r \subseteq \cdots$

Two sequence interleave

Rips filtration ($\mathscr{R}_X$): $R^{r_0} \subseteq R^{r_1} \subseteq \cdots R^r \subseteq \cdots$

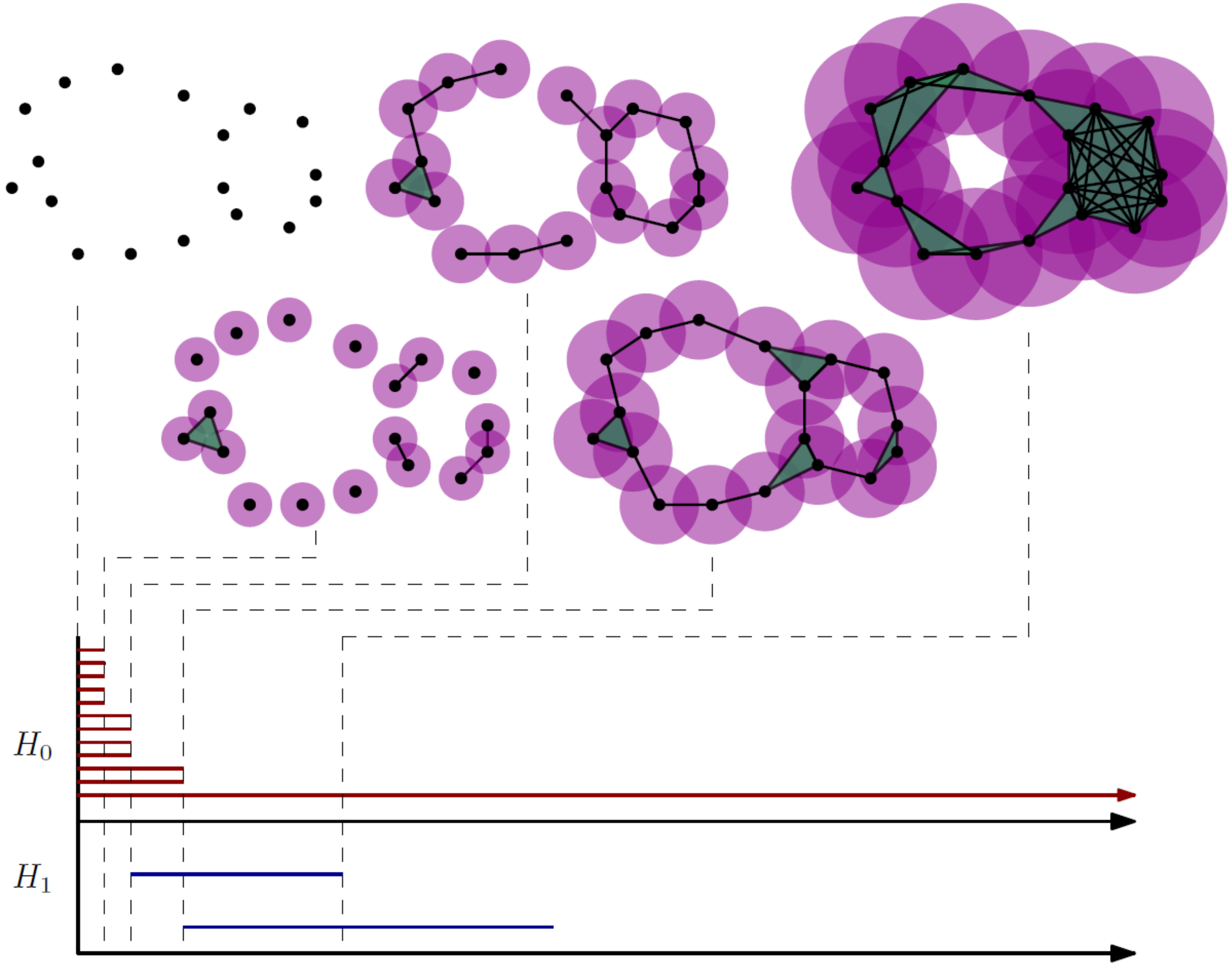
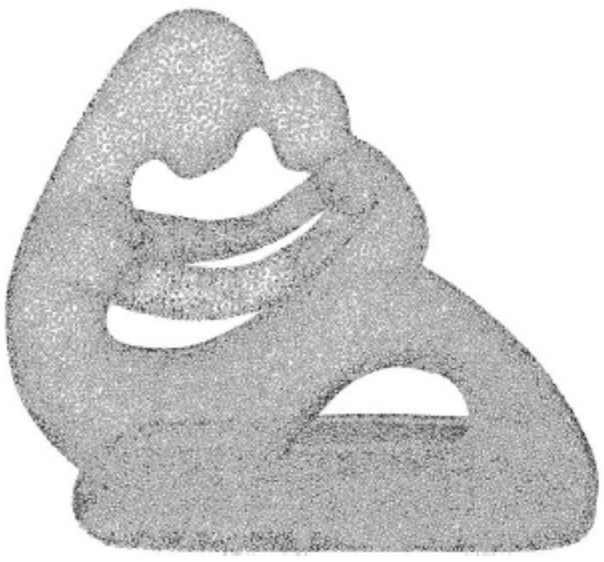
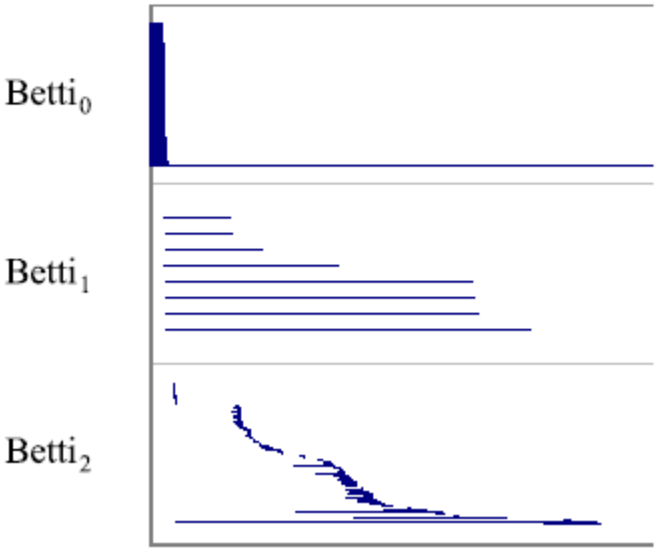


Image courtesy of T. K. Dey



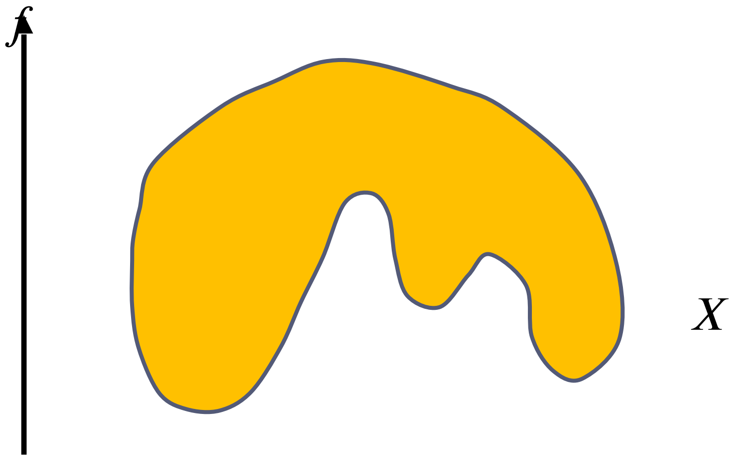
(a) MotherChild model

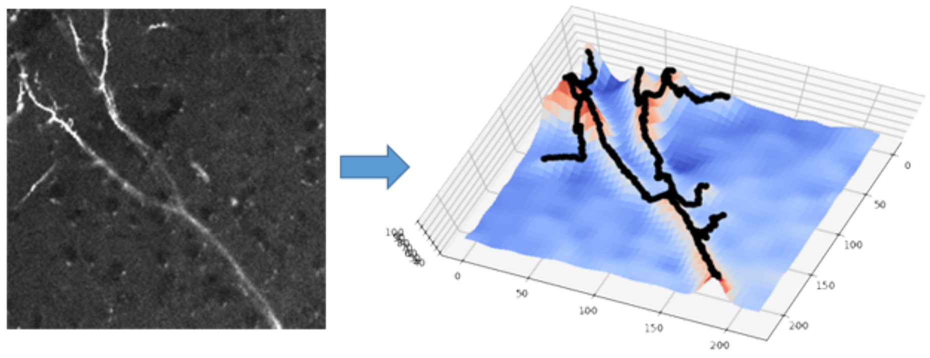


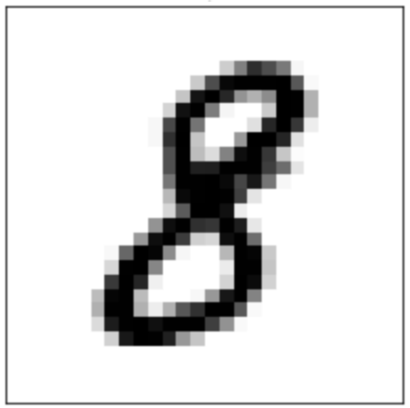
Section 2:
PH induced by Functions

Real-valued Functions

- Input data:
  - $f: X \to R$

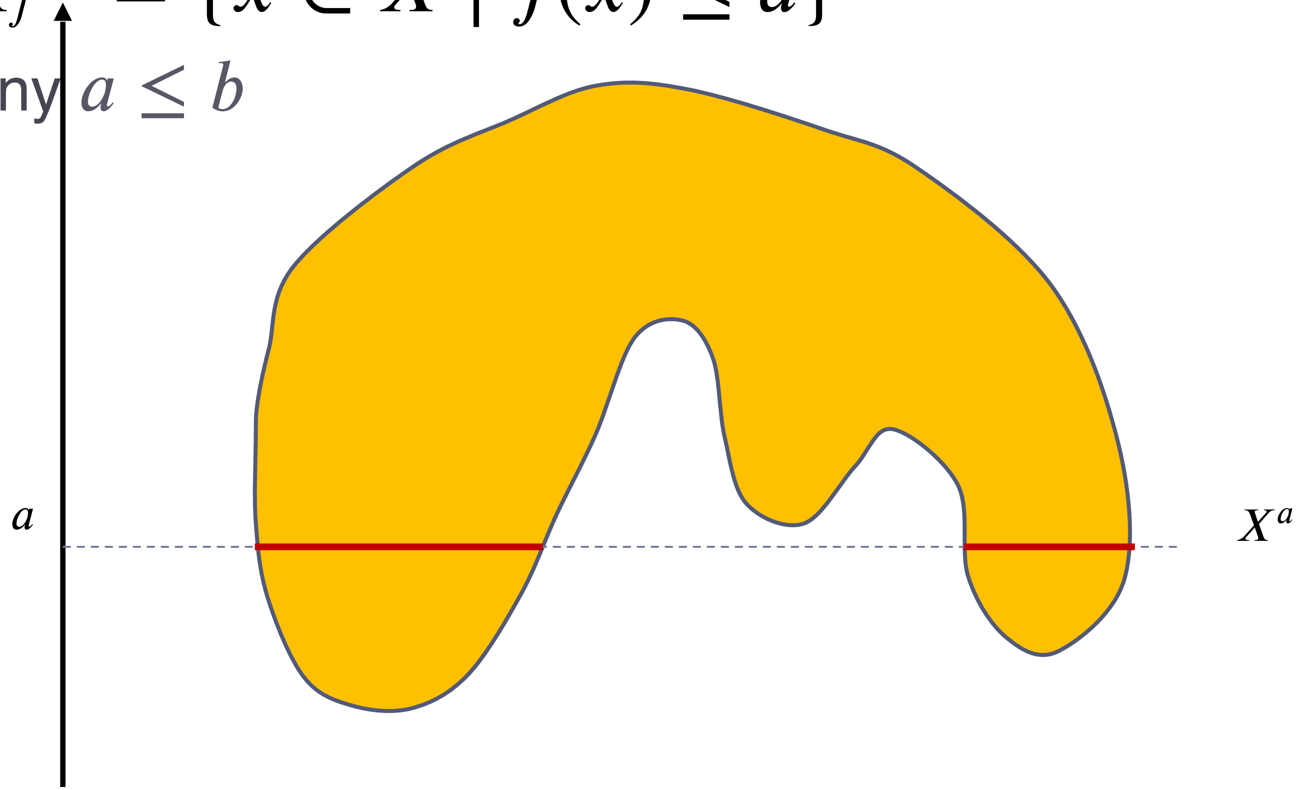






Notations

- Function: $f: X \to R$
- Level set: $X^a = \{x \in X \mid f(x) = a\}$,
- Sub-level set: $X_f^{\leq a} = \{x \in X \mid f(x) \leq a\}$
  - $X^{\leq a} \subseteq X^{\leq b}$ for any $a \leq b$



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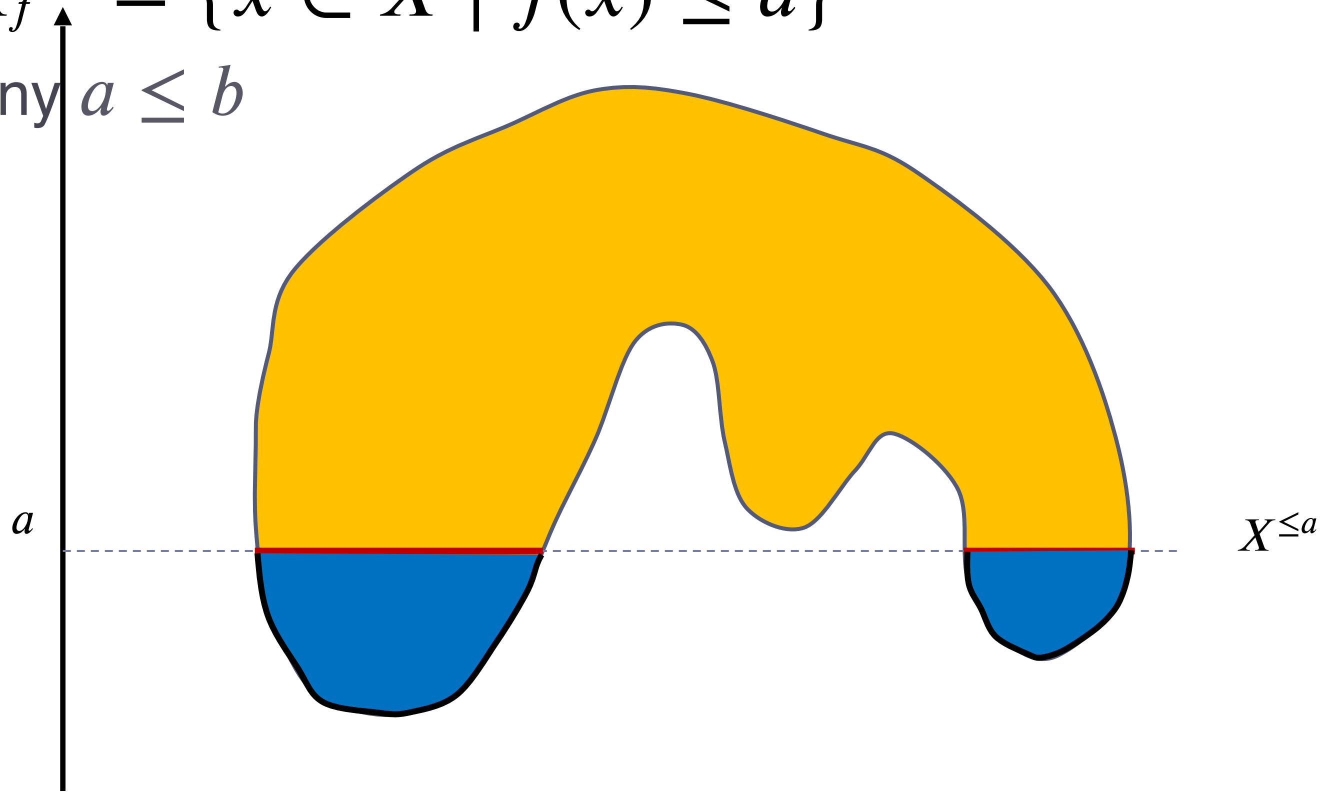

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- Interval-level set: $X^I = \left\{x \in X \mid f(x) \in I\right\}$

Function-induced Filtration

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- For any sequence $a_1 \leq a_2 \leq \cdots \leq a_n$ (with $a_n \geq f_{max}$)
  - Sublevel set filtration of X w.r.t f:
  - $X^{\leq a_1} \subseteq X^{\leq a_2} \subseteq \cdots \subseteq X^{\leq a_n}$

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- Persistence module
  - $H_*(X^{\leq a_1}) \to H_*\left(X^{\leq a_2}\right) \to \cdots \to H_*(X^{\leq a_n})$

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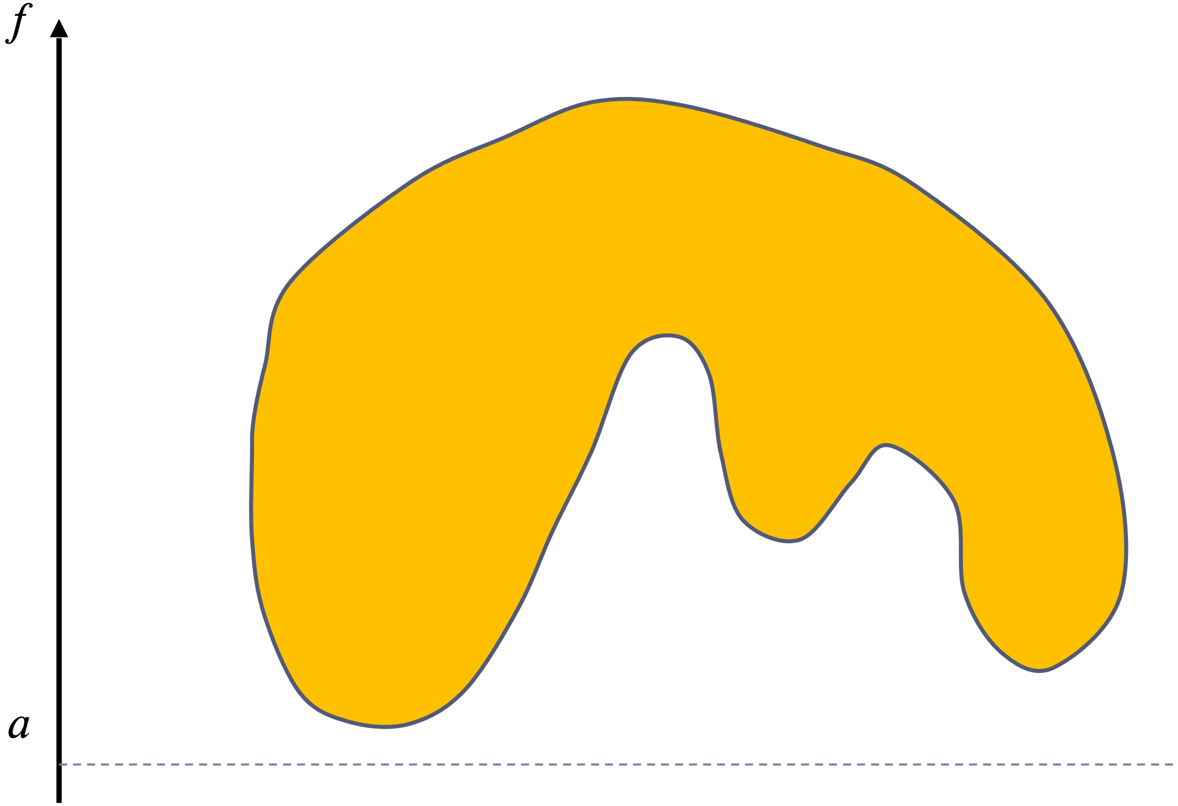
  - $X^{\leq a_1} \subseteq X^{\leq a_2} \subseteq \cdots \subseteq X^{\leq a_n}$
- Persistence module
  - $H_*(X^{\leq a_1}) \to H_*\left(X^{\leq a_2}\right) \to \cdots \to H_*(X^{\leq a_n})$
- General persistence module indexed by real
  - $\mathscr{P}_f = \left\{ H_*\left(X^{\leq a}\right) \to H_*\left(X^{\leq b}\right) \right\}_{a \leq b}$

Off-set filtration is a sub-level set filtration

- Given a compact set $X \subset \mathbb{R}^d$
- Function: $f : \mathbb{R}^d \to \mathbb{R}$ defined by $f(x) := d(x, X)$
- Sub-level set:
  $(\mathbb{R}^d)^{\leq a} = \{x \in \mathbb{R}^d : f(x) \leq a\} = \{x \in \mathbb{R}^d : d(x, X) \leq a\} = X^a$

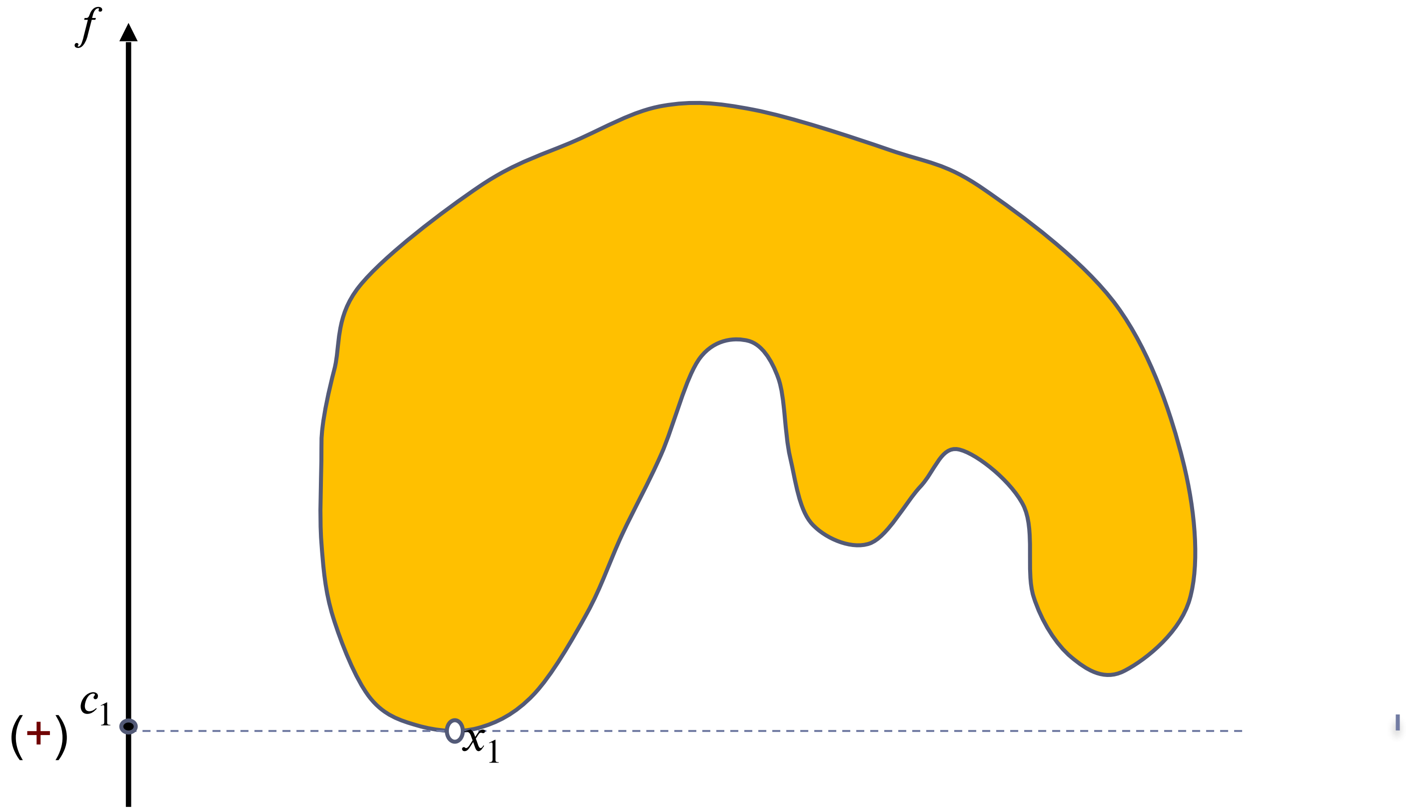
A Simple Example

- 0-th homology



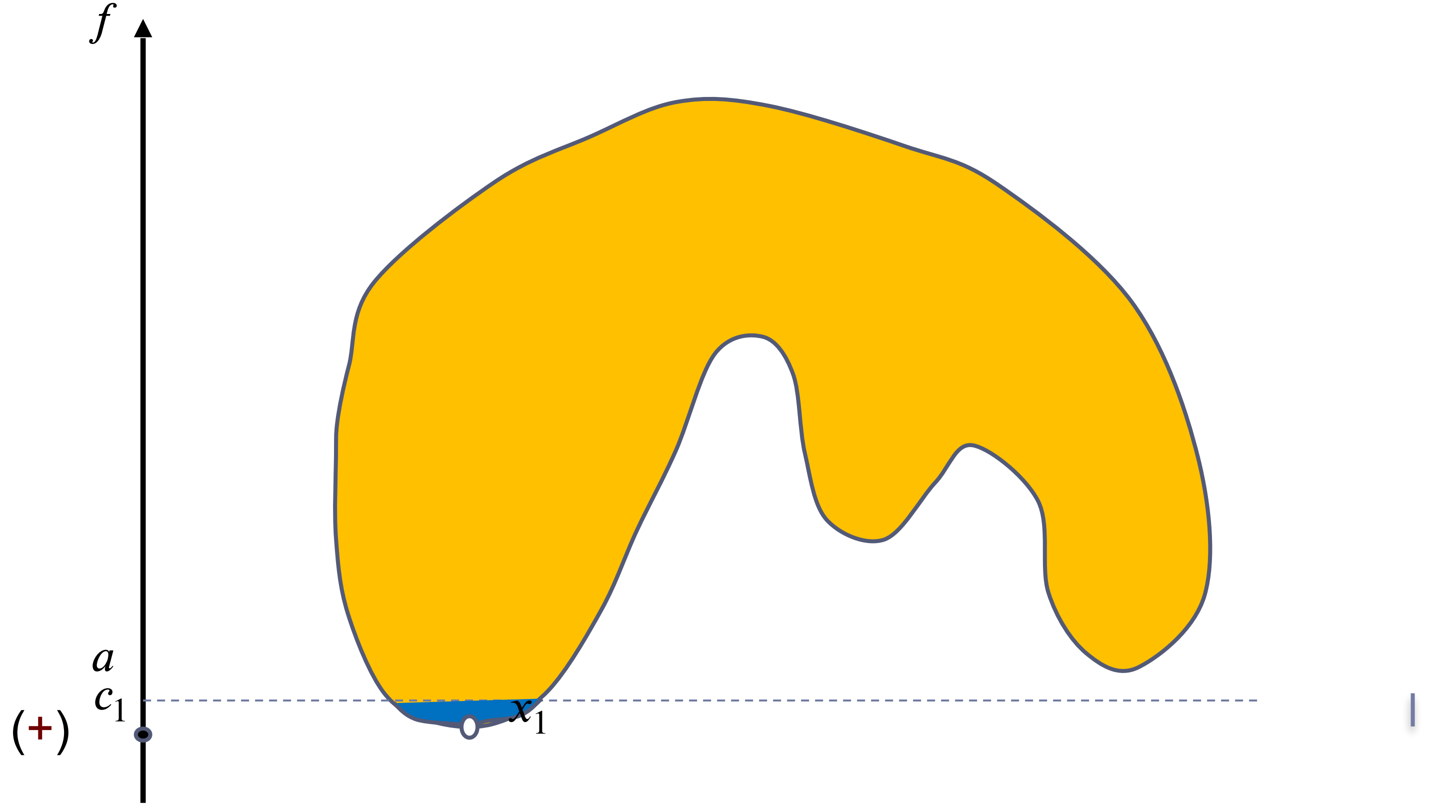
A Simple Example

- 0-th homology



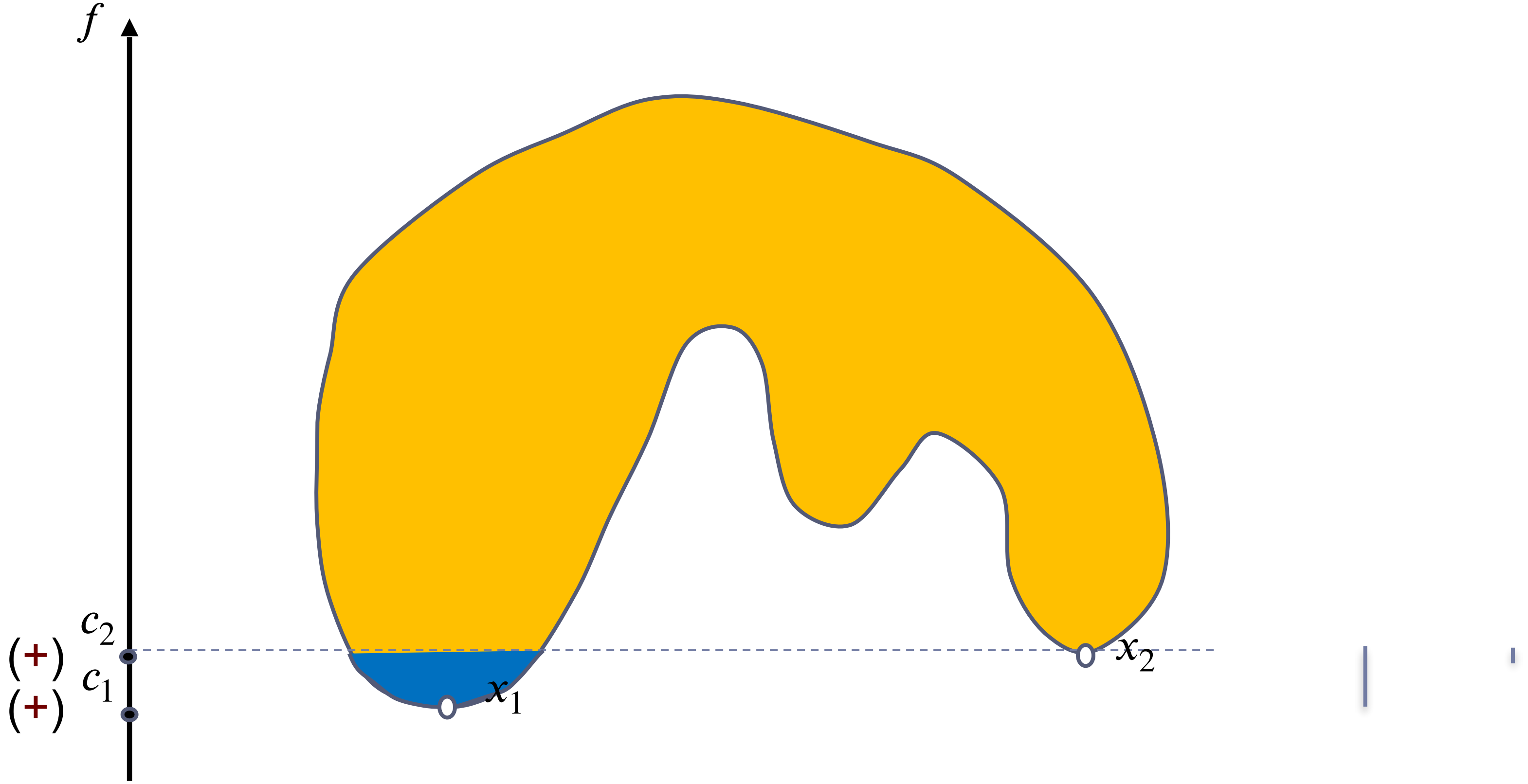
A Simple Example

- 0-th homology



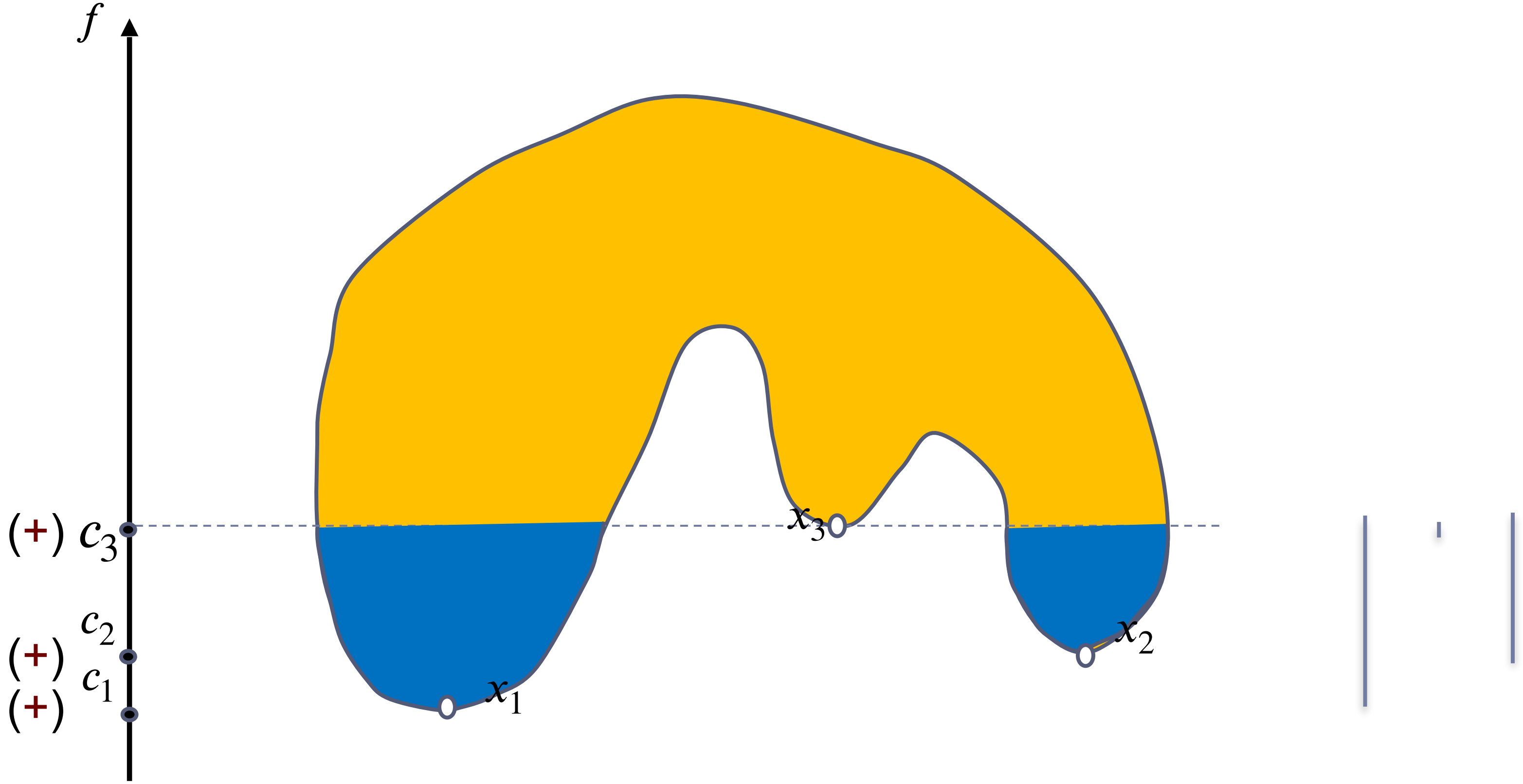
A Simple Example

- 0-th homology



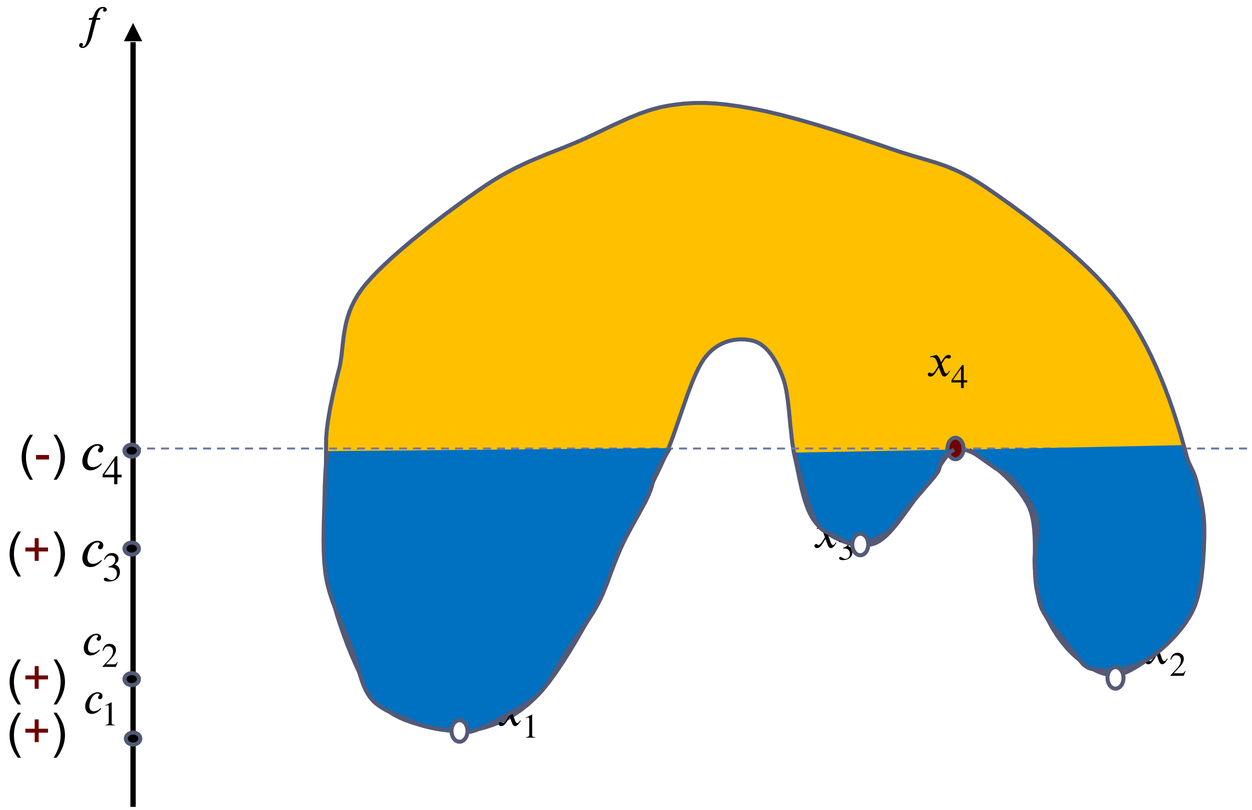
A Simple Example

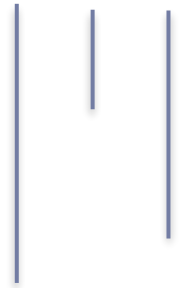
- 0-th homology



A Simple Example

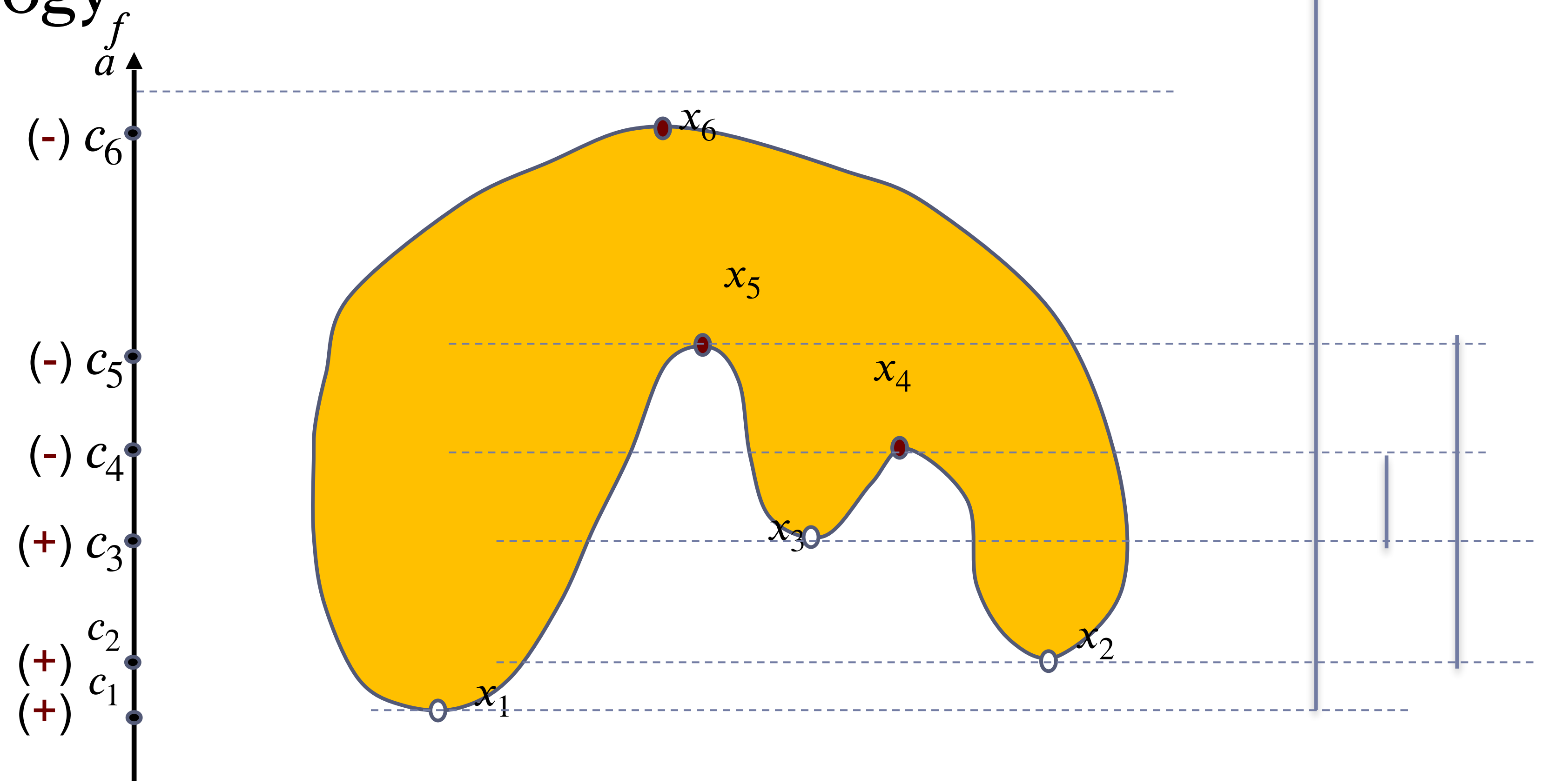
- 0-th homology





A Simple Example

- 0-th homology



Observations

- $H_*(a)$ only potentially changes at *critical values*
  - which, in the case of f is a smooth function defined on a smooth manifold, are exactly the function values of critical points of f

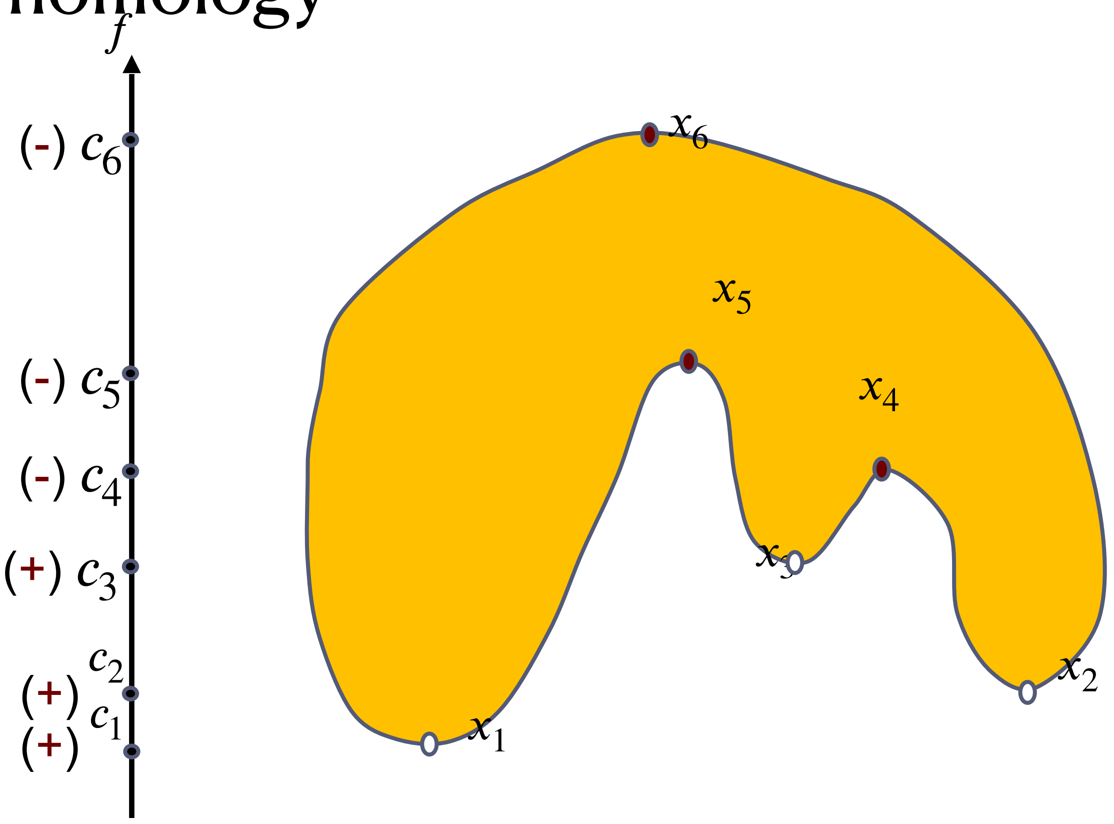
Observations

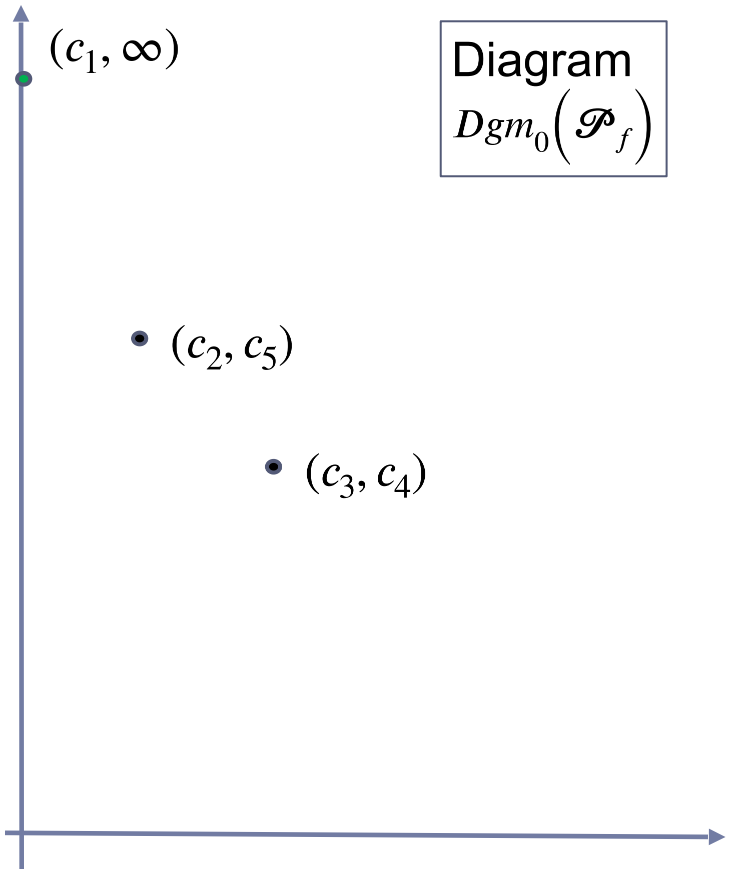
- $H_*(a)$ only potentially changes at *critical values*
    - which, in the case of f is a smooth function defined on a smooth manifold, are exactly the function values of critical points of f
    - $H_*(a) \to H_*(b)$ is isomorphism for $a < b$ within two consecutive critical values

- Persistence pairings for the sub-level set persistence module $\mathscr{P}_f = \left\{ H_*\left(X^{\leq a}\right) \to H_*\left(X^{\leq b}\right) \right\}_{a \leq b}$
    - are of the form $(b,\ d)$, with both b and d being critical values
    - We sometimes also refer the pairing to between corresponding critical points.

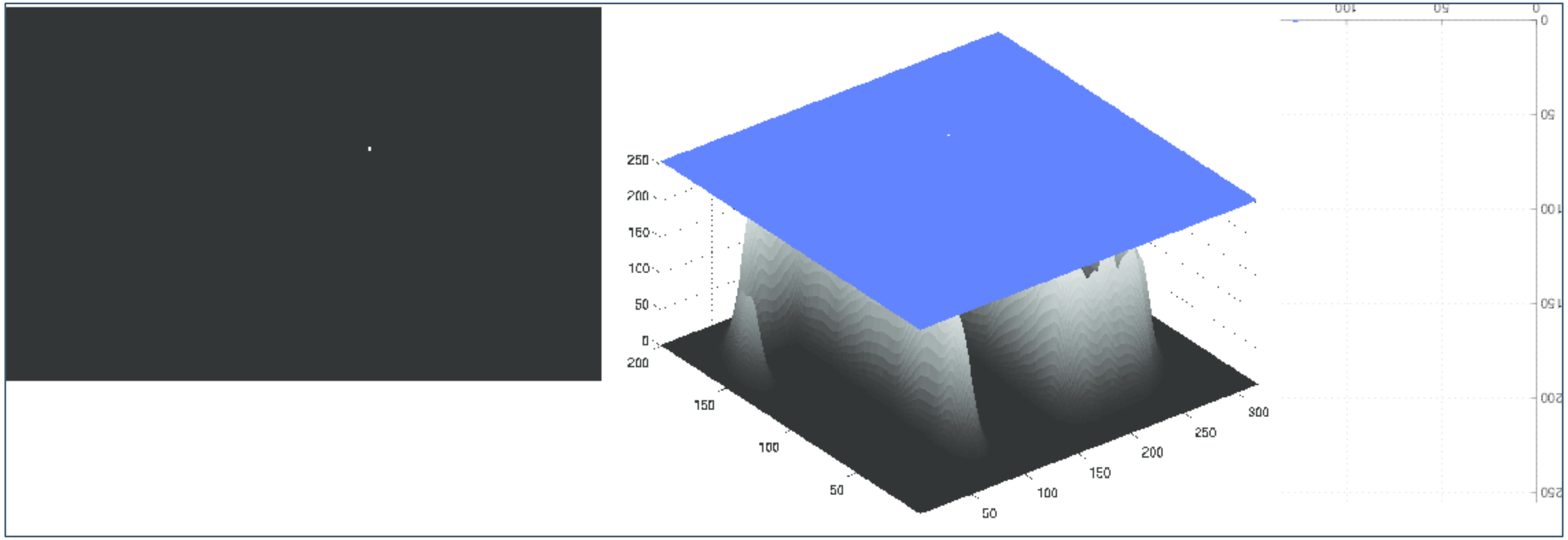
A Simple Example

- 0-th homology





- An example on images
  - By Chao Chen (Stony Brook Univ)



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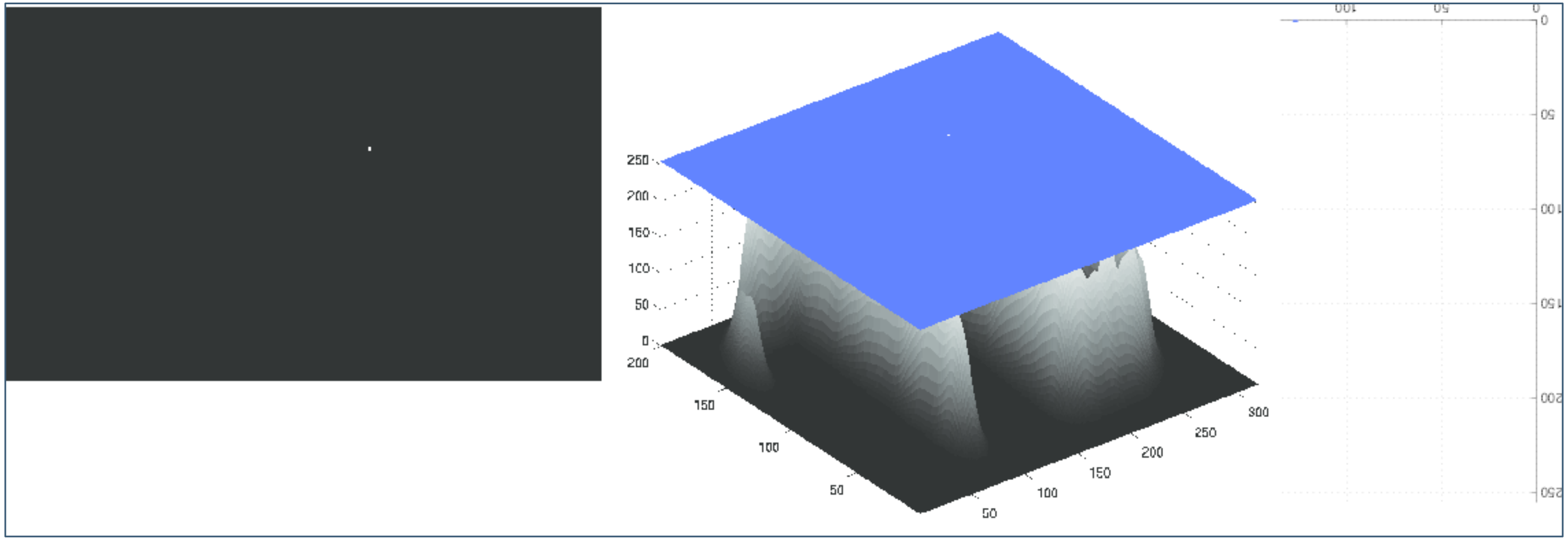


Remarks 1: Sublevel set filtration

- In general, for a smooth function $f: X \to R$ defined on a d-manifold X, under sublevel-set filtration,
  - The persistence pairings are between critical values, induced by pairings between critical points whose indices differ by 1.
    - E.g, for d = 2,
      - (min, saddle) or (saddle, max)
    - E.g, for d = 3,
      - (min, index-1 saddle), (index-1 saddle, index-2 saddle), or (index-2 saddle, max)
  - Pairs of the form $(a, \ \infty)$
    - Correspond to homology of X, i.e, $H_*(X)$
    - This is because $X^{\leq \infty} = X$

Remark 2: Super-level set filtration

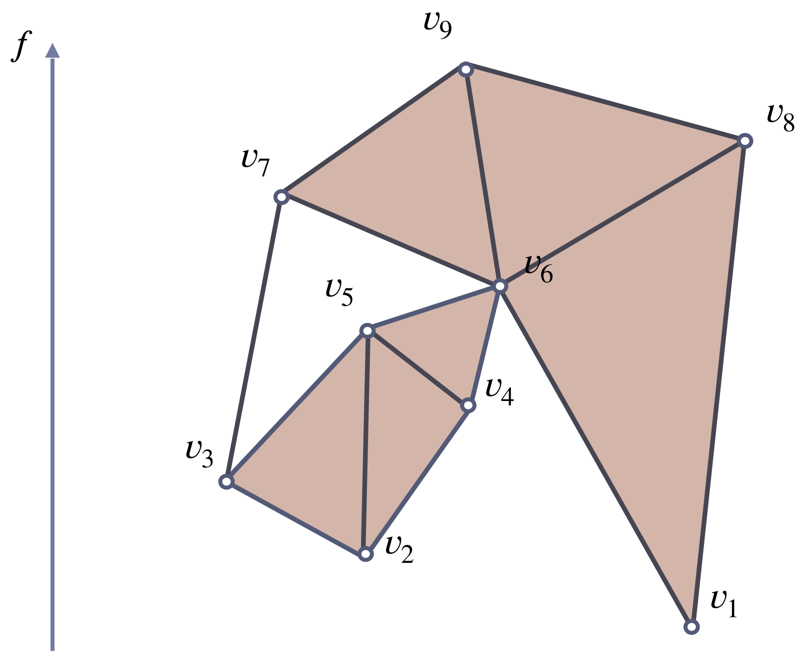
- Symmetrically, one can also consider super-level set filtration
- Super-level set: $X^{\geq a} = \{x \in X \mid f(x) \geq a\}$
  - $X^{\geq a} \subseteq X^{\geq b}$ for any $a \geq b$
- For any sequence $a_1 \geq a_2 \cdots \geq a_n$ (s.t $a_1 \geq f_{max},\ a_n \leq f_{min}$)
  - Sublevel set filtration of X w.r.t f:
  - $X^{\geq a_1} \subseteq X^{\geq a_2} \subseteq \cdots \subseteq X^{\geq a_n}$
- Persistence module
  - $H_*(X^{\geq a_1}) \to H_*\left(X^{\geq a_2}\right) \to \cdots \to H_*(X^{\geq a_n})$
- General persistence module indexed by real
  - $\mathcal{P}_f = \left\{H_*\left(X^{\geq a}\right) \to H_*\left(X^{\geq b}\right)\right\}_{a \geq b}$

Section 3 :
Computation in the PL-case

- We cannot deal with smooth manifolds or smooth functions directly
  - Triangulate smooth manifolds to be simplicial complexes
  - Approximate smooth functions by piecewise linear functions

Computation – PL Function

- K: a simplicial complex, $|K|$ its underlying space (e.g. a triangulation of a manifold)
- Piecewise linear (PL) function $f: |K| \to R$
  - f defined at vertices (0-simplices) V of K and linearly interpolated within each simplex $\sigma \in K$



Computation – PL Function

- Given PL-function $f : \left| K \right| \to R,$ consider the persistence module induced by its sub-level set filtration
  - $\mathscr{P}_f = \left\{ H_*\left(|K|^{\leq a}\right) \to H_*\left(|K|^{\leq b}\right) \right\}_{a \leq b}$
- $\{\, |K|^{\leq a} \subset |K|^{\leq b} \}$ is still a filtration of topological spaces
- To compute persistence pairings for $\mathscr{P}_f,$ we want to simulate sub-level set filtration by a **filtered simplicial complex**

Lower Star filtration

Lower Star filtration

Lower Star filtration

- Assume vertices $\{v_1, \ldots, v_n\}$ sorted in non-decreasing order by function value f

Lower Star filtration

- Assume vertices $\{v_1, \ldots, v_n\}$ sorted in non-decreasing order by function value f
- Consider discrete values $a_1 \leq \cdots \leq a_n$ with $a_i = f\left(v_i\right)$
  - $K_i := \{\sigma \in K \mid f(v) \leq a_i, \ \forall v \in \sigma\}$
  - Consider filtration $\varnothing = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$

Lower Star filtration

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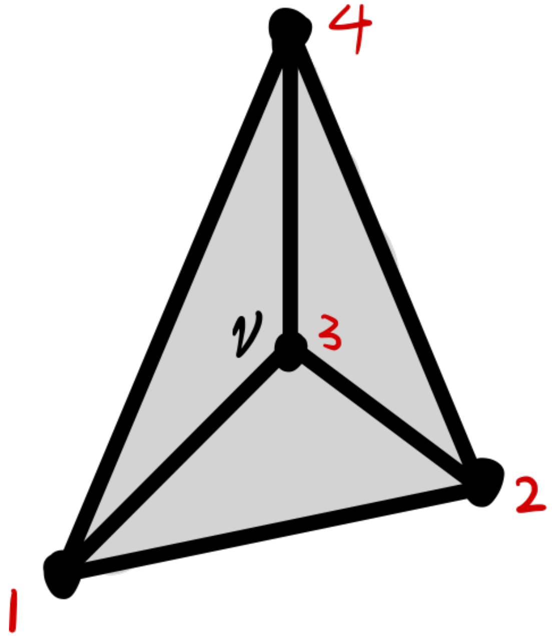
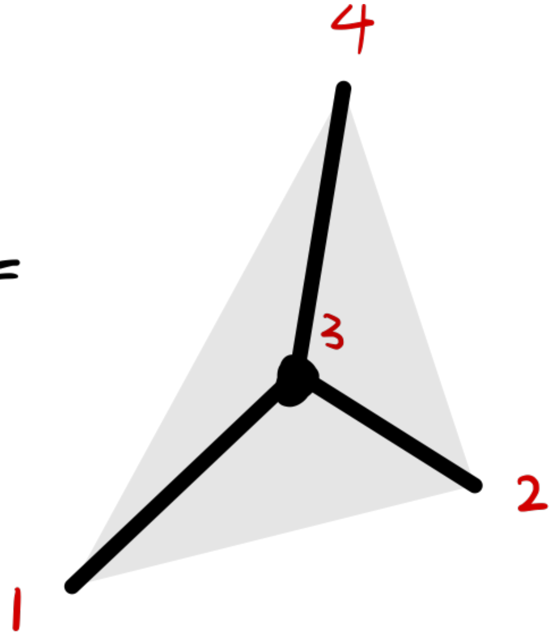
- Consider discrete values $a_1 \leq \cdots \leq a_n$ with $a_i = f\left(v_i\right)$
  - $K_i := \{\sigma \in K \mid f(v) \leq a_i, \ \forall v \in \sigma\}$
  - Consider filtration $\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$
- Called lower star filtration as
  - $LowSt\left(v_i\right) = K_i \setminus K_{i-1}$
    - where $LowSt\,(v) := \{\sigma \in K \mid v \in \sigma \ and \ f(u) \leq f(v) \ for \ any \ u \in \sigma\}$
  - $K_i = \bigcup_{j \leq i} LowSt(v_j)$

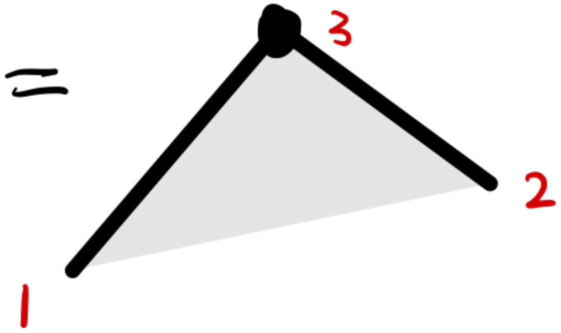
Lower Star vs Star



$St(v) =$

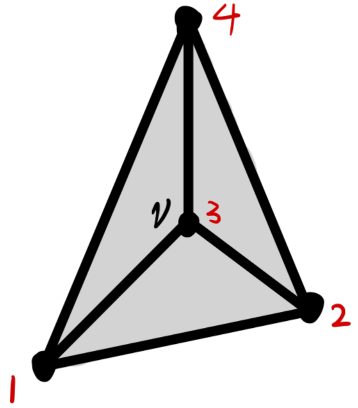


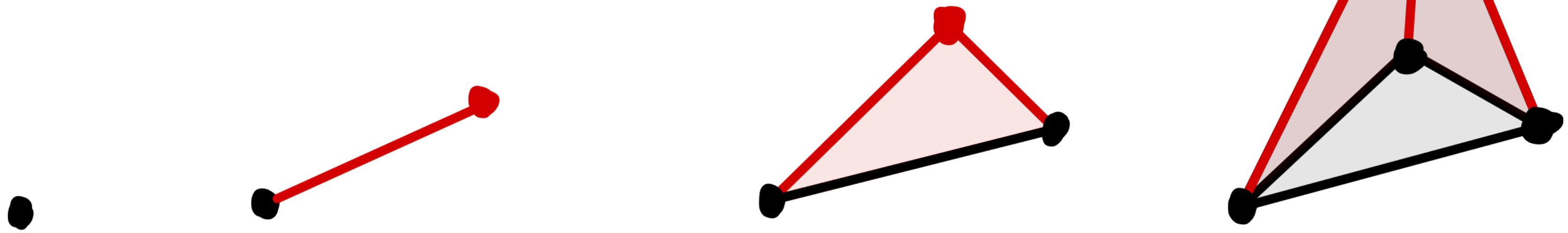
$LowSt(v) =$



Lower Star filtration

- Consider discrete values $a_1 \leq \cdots \leq a_n$ with $a_i = f\left(v_i\right)$
  - $K_i \coloneqq \{\sigma \in K \mid f(v) \leq a_i,\ \forall v \in \sigma\}$
  - Consider filtration $\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$





Computation – PL Function

- Goal: persistence pairings for
  - $\mathscr{P}_f = \left\{ H_*\left(|K|^{\leq a}\right) \to H_*\left(|K|^{\leq b}\right) \right\}_{a \leq b}$
- Simulate sub-level set filtration by ***lower star filtration***
  - Assume vertices $\{v_1, \ldots, v_n\}$ sorted in non-decreasing order by function value f
  - Consider discrete values $a_1 \leq \cdots \leq a_n$ with $a_i = f\left(v_i\right)$
  - $K_i := \{\sigma \in K \mid f(v) \leq a_i, \ \forall v \in \sigma\}$
  - Consider filtration $\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$

Computation – PL Function

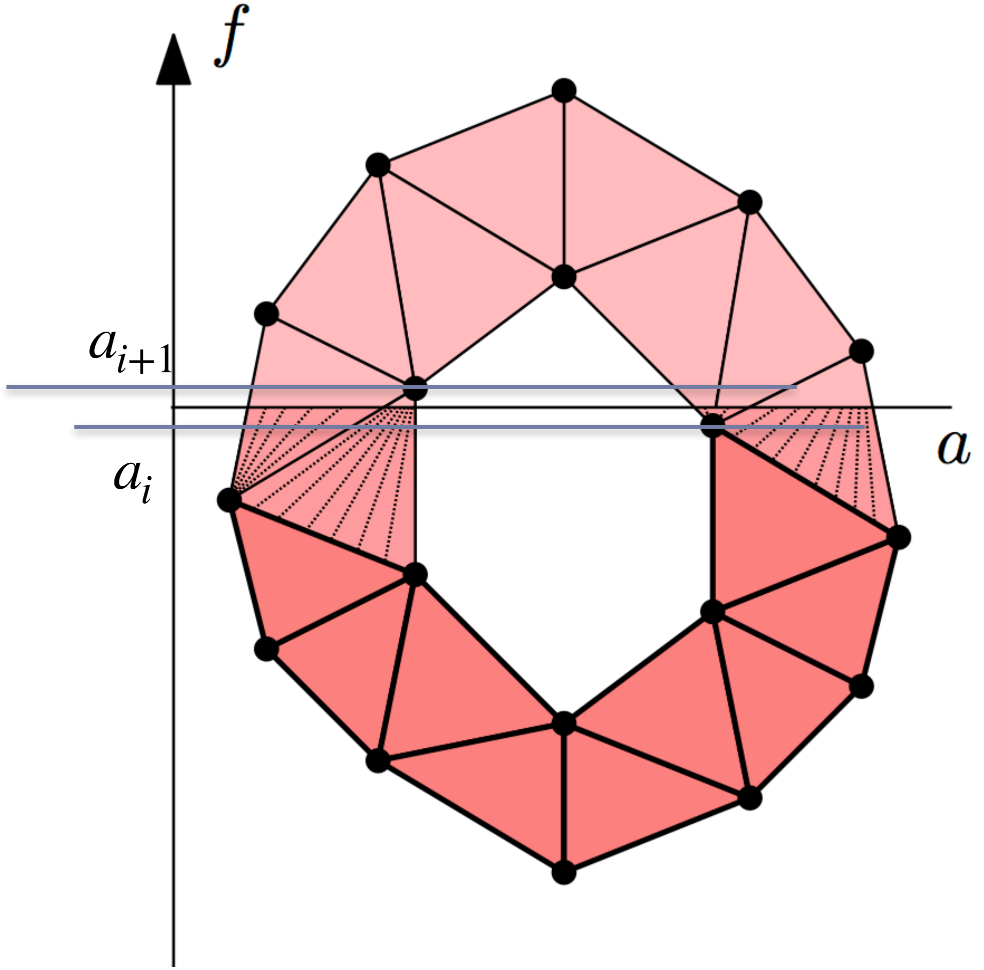
- Goal: persistence pairings for
  - $\mathcal{P}_f = \left\{ H_*\left(|K|^{\leq a}\right) \to H_*\left(|K|^{\leq b}\right) \right\}_{a \leq b}$

- Simulate sub-level set filtration by *lower star filtration*
  - $\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$
    - where $K_i := \left\{\sigma \in K \mid f(v) \leq a_i,\ \forall v \in \sigma\right\}$
  - $\Rightarrow H_*\left(K_0\right) \to H_*\left(K_1\right) \to H_*\left(K_2\right) \to \cdots \to H_*\left(K_n\right)$

Sub-level set vs Lower Star filtrations

- For any a, if $a_i \leq a < a_{i+1}$, then

- $|K|^{\leq a} \simeq K_i$

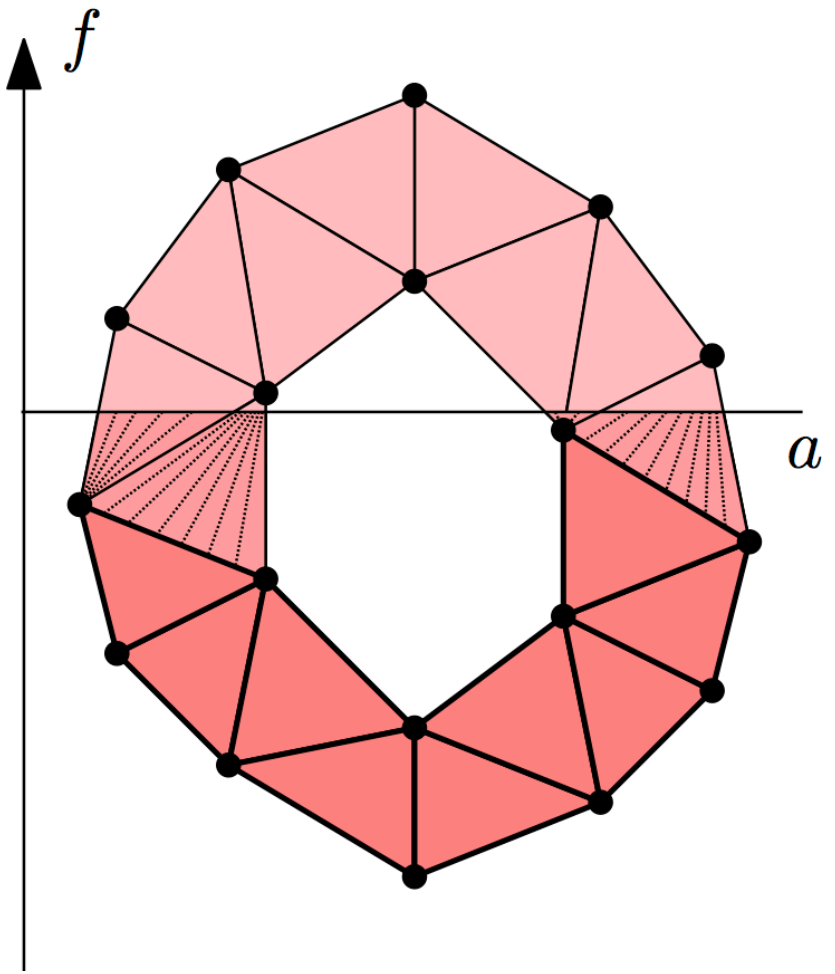


Sub-level set vs Lower Star filtrations

- Lower star filtration
  - $\varnothing \subset K_1 \subset \cdots \subset K_n$
- Sub-level set filtration

  $\varnothing \subset |K|^{\leq a_1} \subset \cdots \subset |K|^{\leq a_n}$

- They induce **isomorphic** persistence modules



Computation of PD for lower star filtration

- A simplex-wise filtration *realizes* the lower star filtration if

$$\underbrace{\sigma_1, \dots, \sigma_{I_1}}_{Lst(v_1)}, \underbrace{\sigma_{I_1+1}, \dots, \sigma_{I_2}}_{\{v_2\} \sqcup Lst(v_2)}, \dots, \underbrace{\sigma_{I_{j-1}+1}, \dots, \sigma_{I_{j+1}}}_{\{v_j\} \sqcup Lst(v_j)}, \dots, \underbrace{\sigma_{I_{n-1}+1}, \dots, \sigma_{I_n}}_{\{v_n\} \sqcup Lst(v_n)}$$

K_1

K_2

...

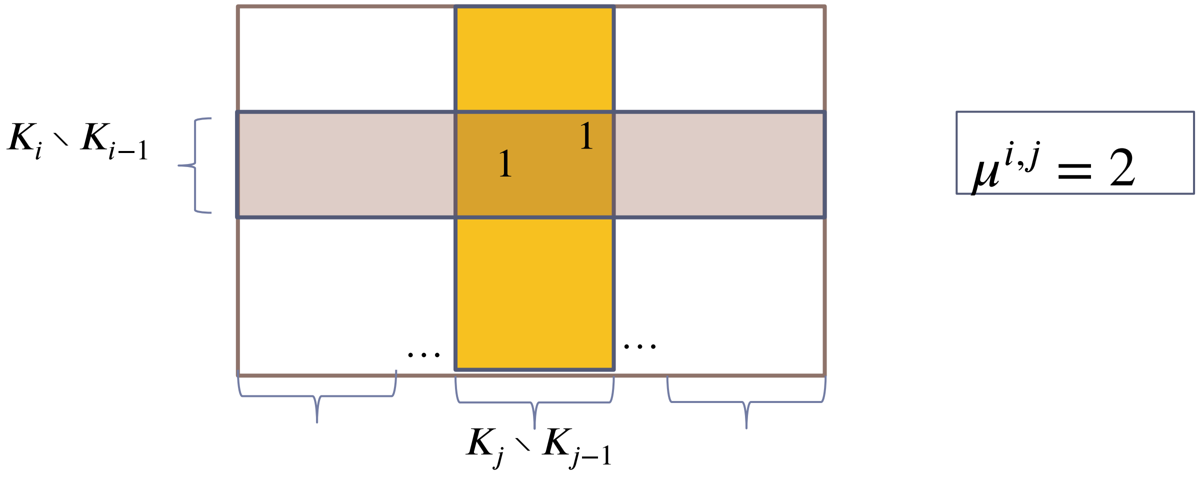
K_i

...

K_n

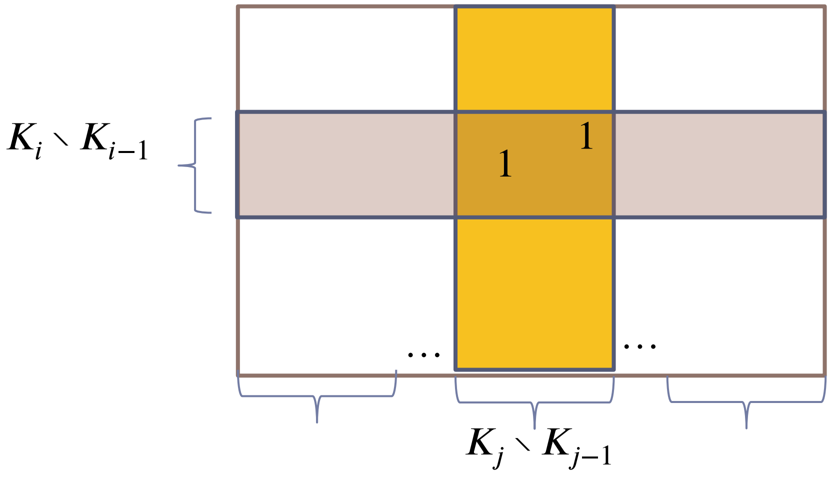
PL-implementation

- Given a PL function $f: K \to R$, perform the persistence algorithm for any simplex-wise lower star filtration.
  - Let P denote the output set of paired simplices
  - Then, $\mu^{i,j} > 0$ if and only if there exists $(\sigma, \tau) \in P$ such that $\sigma \in K_i \setminus K_{i-1}$, while $\tau \in K_j \setminus K_{j-1}$
    - $\mu^{i,j}$ equals the cardinality of the set of such pairs.



PL-implementation

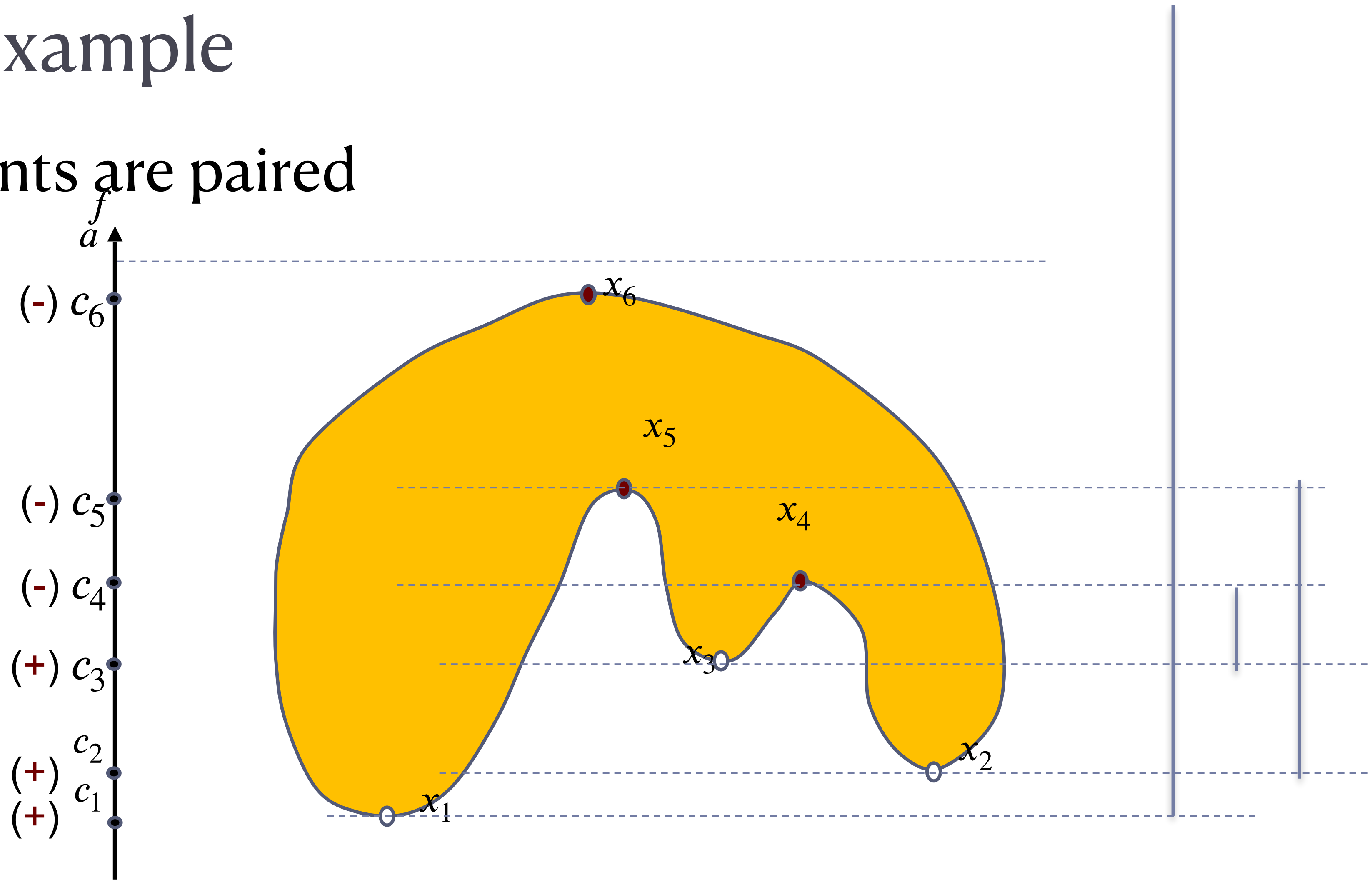
- Given a PL function $f: K \to R$, perform the persistence algorithm for any simplex-wise lower star filtration.
  - Let P denote the output set of paired simplices
  - Then, $\mu^{i,j} > 0$ if a What can we say about (σ, τ)? uch that $\sigma \in K_i \setminus K_{i-1}$, while $\tau \in K_j \setminus K_{j-1}$
    - $\mu^{i,j}$ equals the cardinality of the set of such pairs.



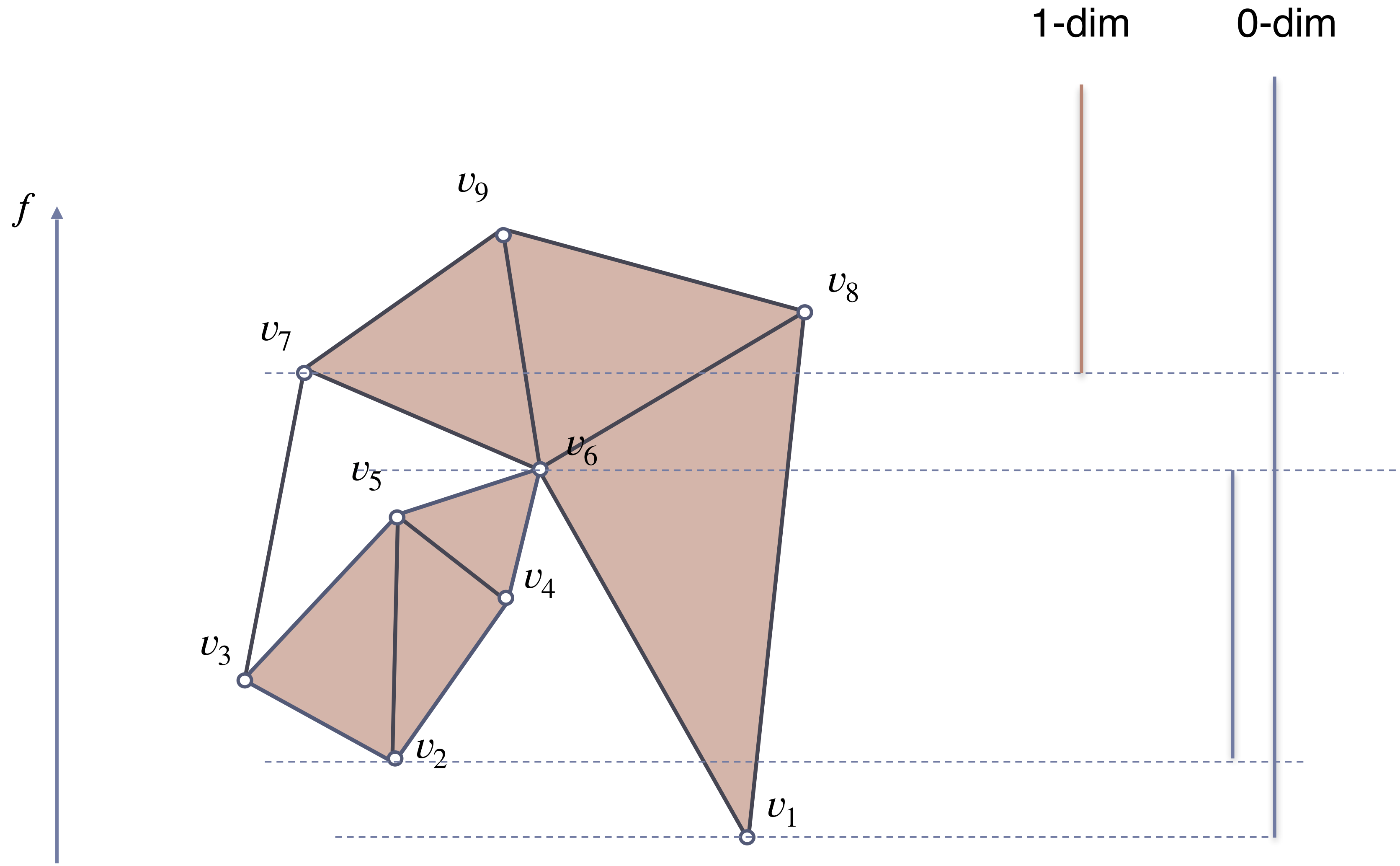
$\mu^{i,j} = 2$

A Simple Example

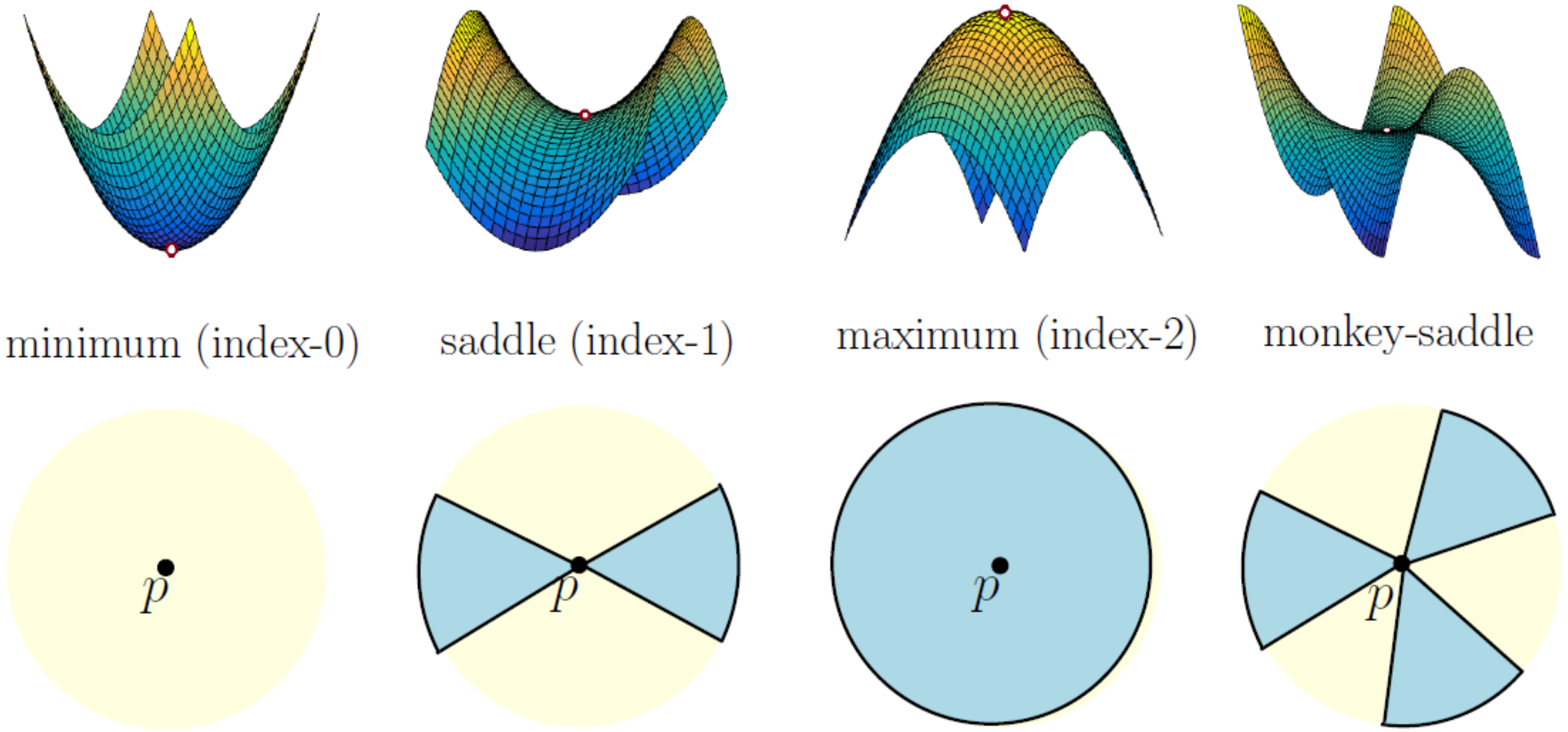
- Critical points are paired



Example



Recall: Critical points and local view



Local View

- For d-dimensional manifold M
- An open neighborhood $B_r(p)$ of p
  - $B_r(p)$ is an m-dimensional open ball
  - Consider the boundary of the closure of $B_r(p)$

Local View

- For d-dimensional manifold M
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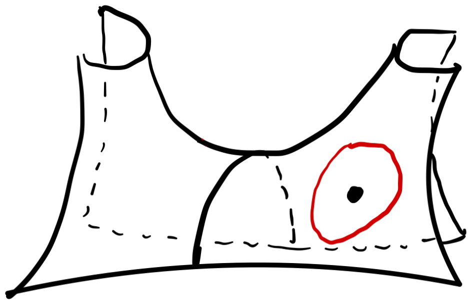
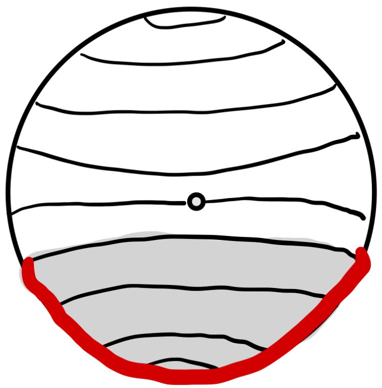


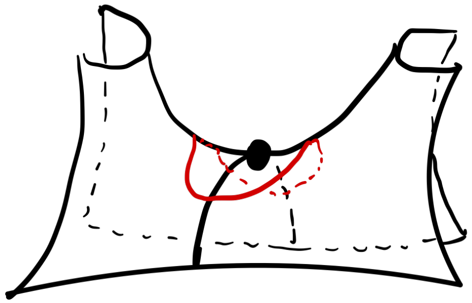


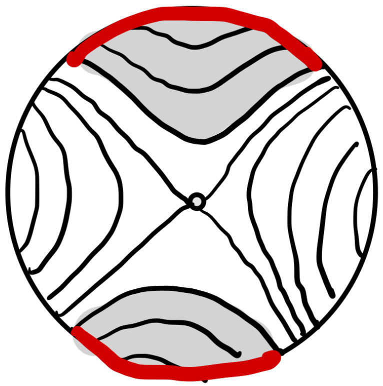
$\tilde{\beta}_0(X) = \beta_0(X) - 1 = 0$

$\beta_p(X) = 0$ for $p \geq 1$

Local View

- For *d*-dimensional manifold *M*
- An open neighborhood $B_r(p)$ of p
  - $B_r(p)$ is an d-dimensional open ball
  - Consider the boundary of the closure of $B_r(p)$ intersecting the sub-level set $M^{\leq f(p)-\epsilon}$ for some function $f : M \to \mathbb{R}$





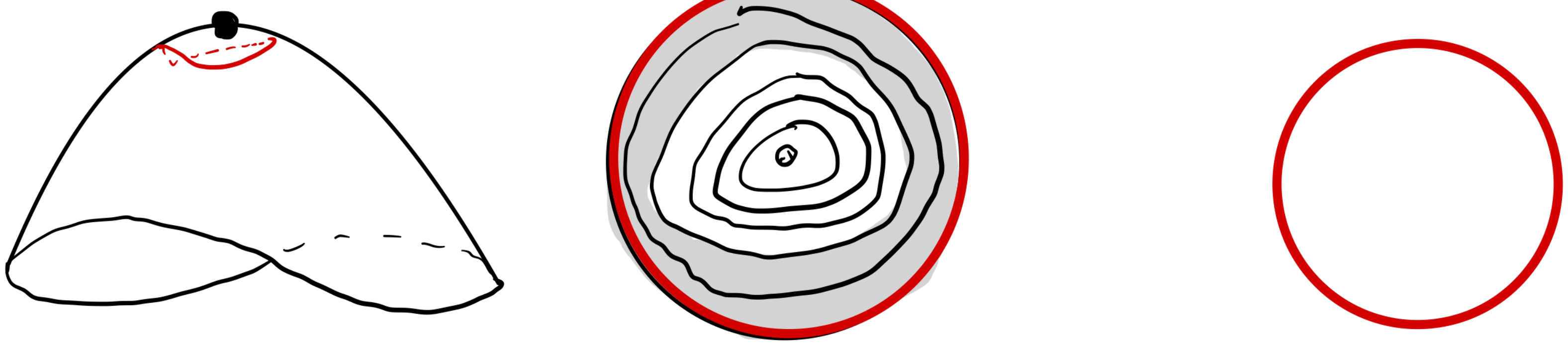
$\tilde{\beta}_0(X) > 0$

$\beta_p(X) = 0$ for $p \geq 1$

Index=1

Local View

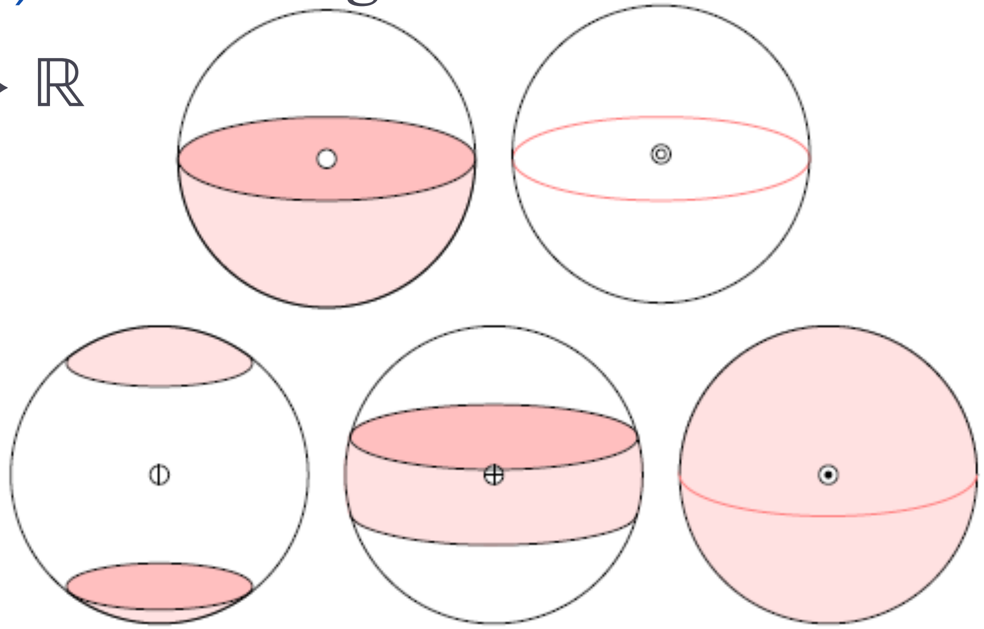
- For d-dimensional manifold M
- An open neighborhood $B_r(p)$ of p
  - $B_r(p)$ is an d-dimensional open ball
  - Consider the boundary of the closure of $B_r(p)$ intersecting the sub-level set $M^{\leq f(p)-\epsilon}$ for some function $f : M \to \mathbb{R}$



$\tilde{\beta}_0(X) = 0$
$\beta_1(X) = 1$
$\beta_p(X) = 0$ for $p \geq 2$
Index=2

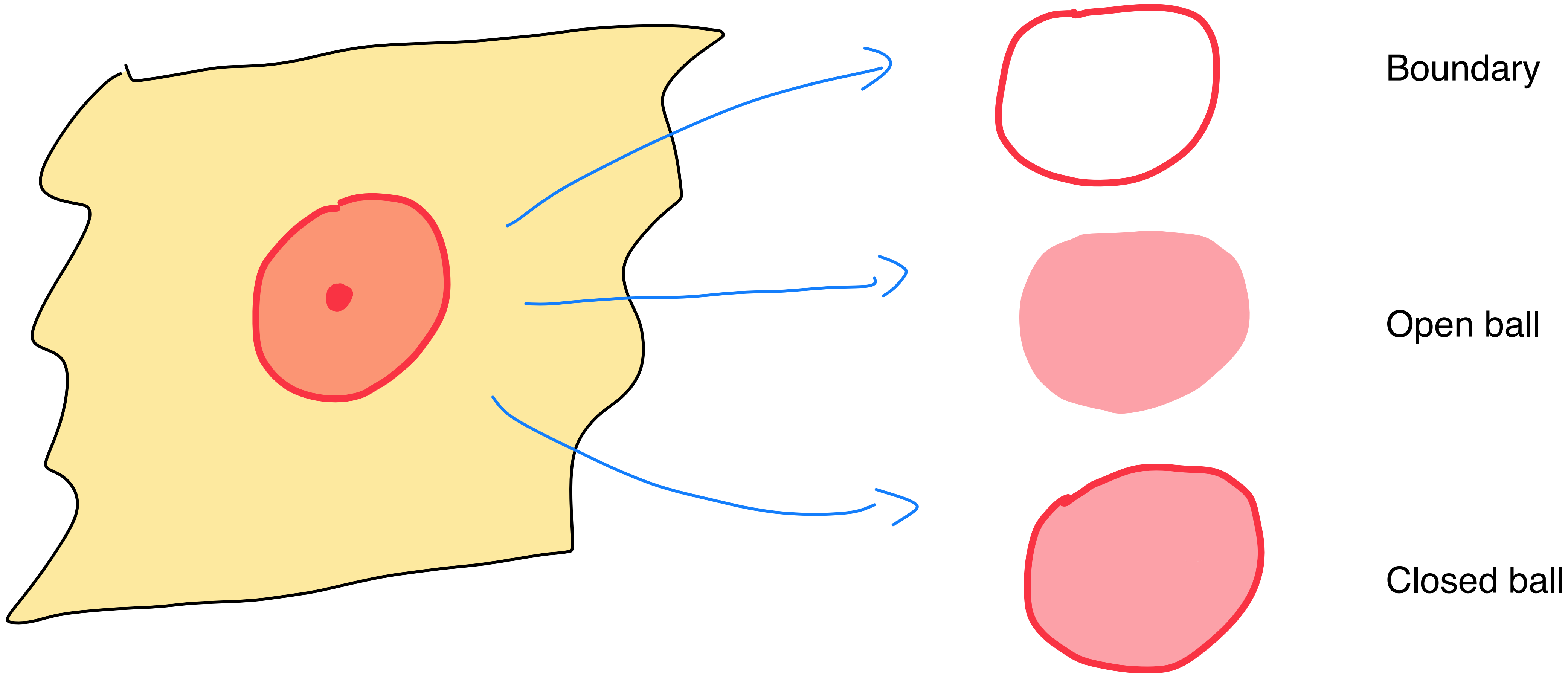
Local View

- For *d*-dimensional manifold *M*
- An open neighborhood $B_r(p)$ of *p*
  - $B_r(p)$ is an d-dimensional open ball
  - Consider the boundary of the closure of $B_r(p)$ intersecting the sub-level set $M^{\leq f(p)-\epsilon}$ for some function $f : M \rightarrow \mathbb{R}$



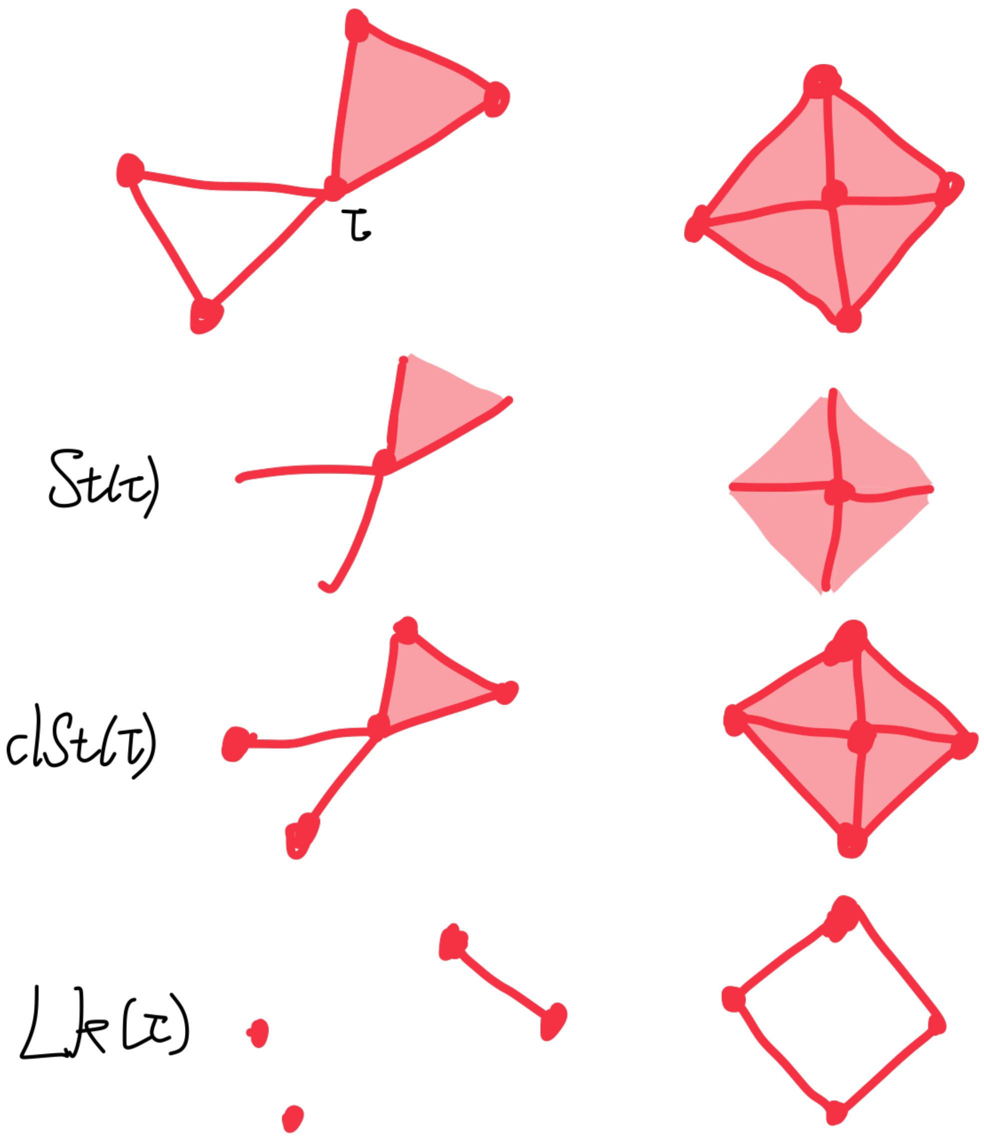
Courtesy of Edelsbrunner 2006

Star and links



Star and links

- Given a simplex $\tau \in K$
  - Star: $St(\tau) = \{\sigma \in K \mid \tau \subset \sigma\}$
  - Closed star: $clSt(\tau) = \bigcup_{\sigma \in St(\tau)} \{\, \sigma' \mid \sigma' \subset \sigma \,\}$
  - Link: $Lk(\tau) = \{\, \sigma \in clSt(\tau) \mid \sigma \cap \tau = \emptyset \,\}$

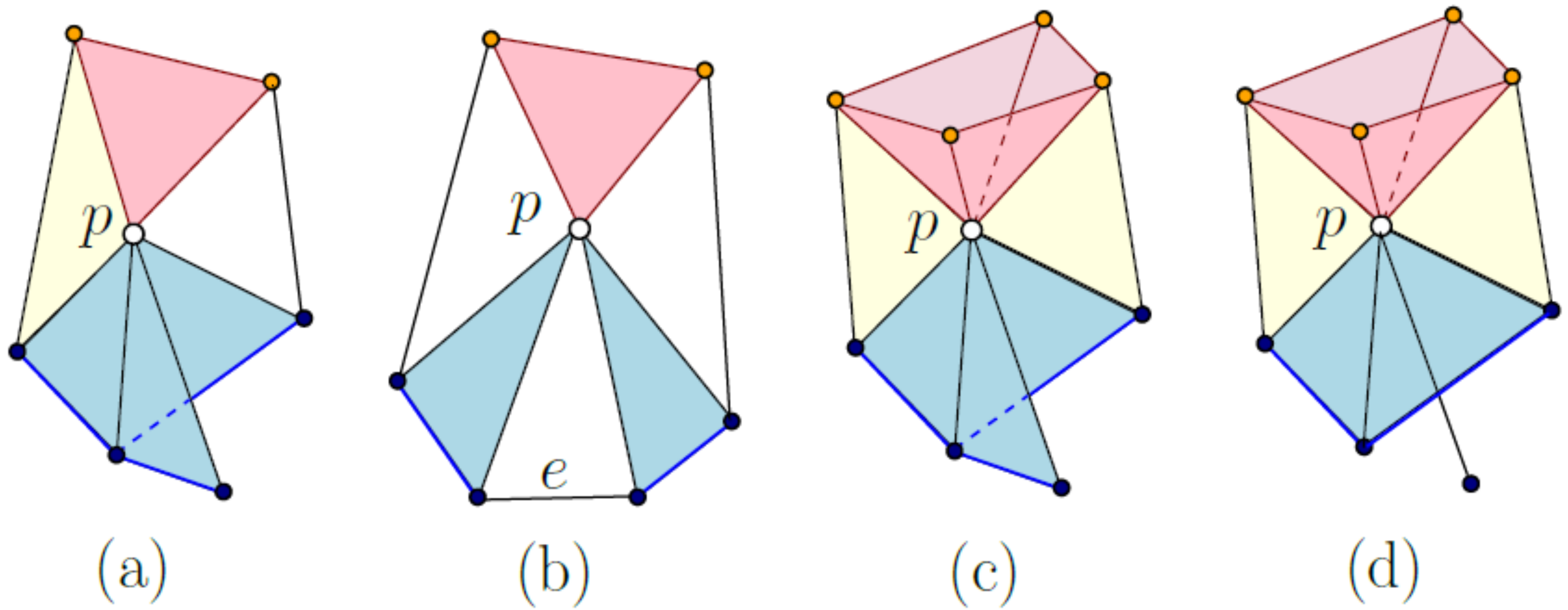


Upper and lower link

- $LowSt\,(v) := \{\sigma \in K \mid v \in \sigma \; and \; f(u) \leq f(v) \; for \; any \; u \in \sigma\}$
- $clLowSt(v) := \{\sigma \in K | \sigma \subset \tau \in LowSt(v)\}$
- $Llk(v) = clLowSt(v) \backslash LowSt(v)$

- $UpSt(v) := \{\sigma \in K | v \in \sigma \text{ and } f(u) \geq f(v), \forall u \in \sigma\}$
- $clUpSt(v) := \{\sigma \in K | \sigma \subset \tau \in UpSt(v)\}$
- $Ulk(v) = clUpSt(v) \backslash UpSt(v)$

Examples



PL-critical points

- Analogous to critical points in the smooth case.

Definition 1 (Reduced Betti number). $\tilde{\beta}_p(X) = \beta_p(X)$ for $p > 0$. For $p = 0$, $\tilde{\beta}_0(X) = \beta_0(X) - 1$ and $\tilde{\beta}_{-1}(X) = 0$ if X is not empty; otherwise, $\tilde{\beta}_0(X) = 0$ and $\tilde{\beta}_{-1}(X) = 1$.

- $\tilde{\beta}_0(X) =$ number of connected components - 1

PL-critical points

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Definition 2 (PL-critical points). Given a PL-function $f : |K| \to \mathbb{R}$, we say that a vertex $v \in K$ is a *regular* vertex or point if $\tilde{\beta}_p(\mathrm{Llk}(v)) = 0$ and $\tilde{\beta}_p(\mathrm{Ulk}(v)) = 0$ for any $p \geq -1$. It is called *PL-critical* (or simply *critical*) vertex or point otherwise.

Furthermore, we say that v has lower-link-index p if $\tilde{\beta}_{p-1}(\mathrm{Llk}(v)) > 0$. Similarly v has upper-link-index p if $\tilde{\beta}_{p-1}(\mathrm{Ulk}(v)) > 0$. The function value of a critical point is a *critical value for* f.

PL-critical points

Definition 2 (PL-critical points)**.** Given a PL-function $f : |K| \to \mathbb{R}$, we say that a vertex $v \in K$ is a *regular* vertex or point if $\tilde{\beta}_p(\mathrm{Llk}(v)) = 0$ and $\tilde{\beta}_p(\mathrm{Ulk}(v)) = 0$ for any $p \geq -1$. It is called *PL-critical* (or simply *critical*) vertex or point otherwise.

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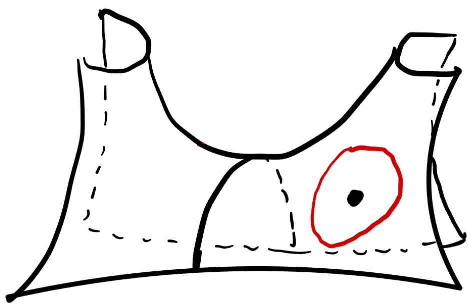
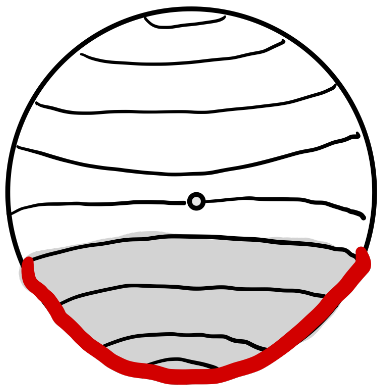
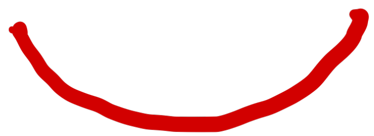




$\tilde{\beta}_0(X) = \beta_0(X) - 1 = 0$

$\beta_p(X) = 0$ for $p \geq 1$



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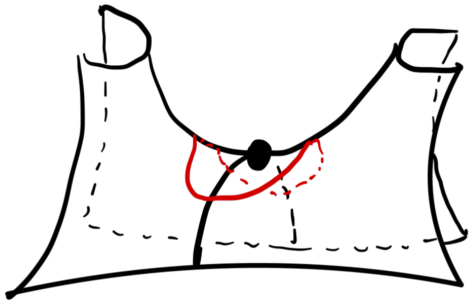
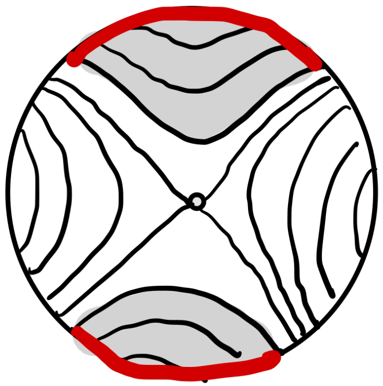
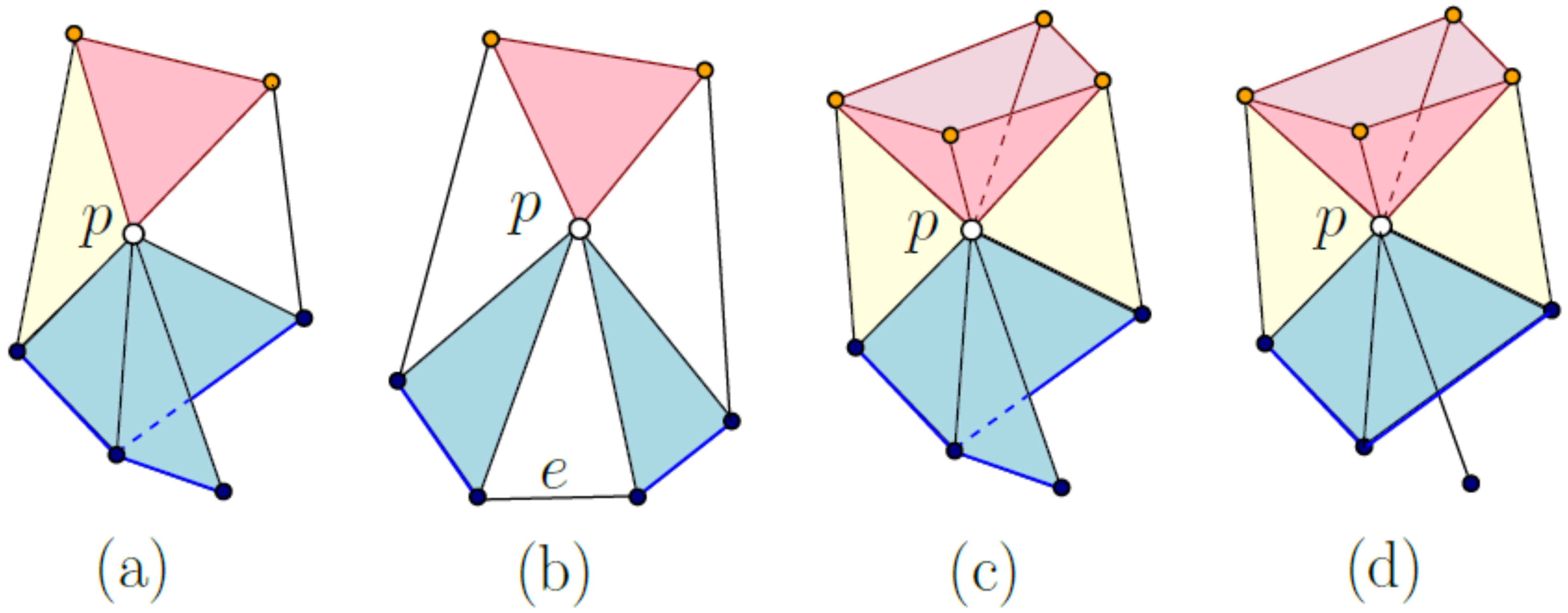


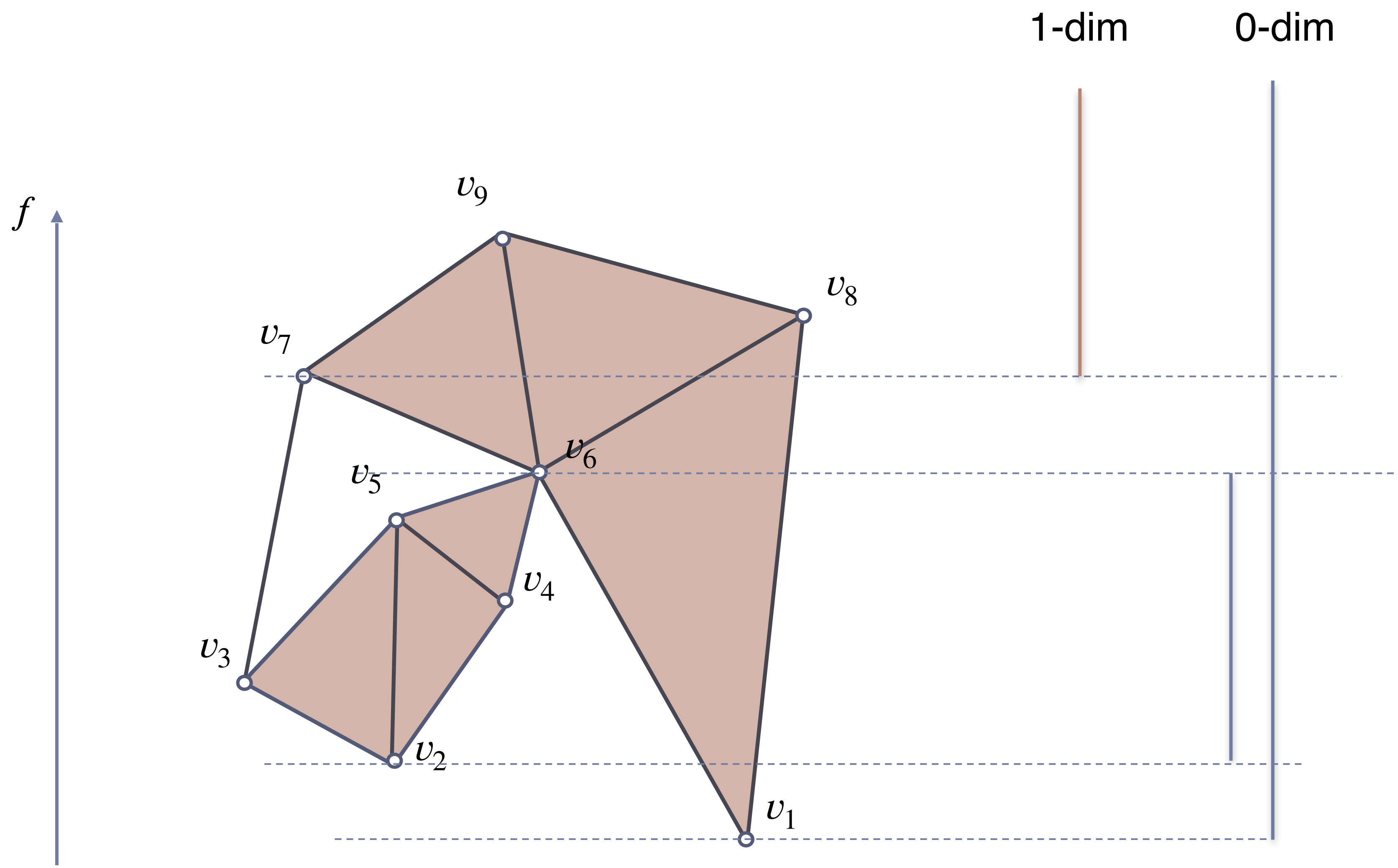


$\tilde{\beta}_0(X) > 0$
$\beta_p(X) = 0$ for $p \geq 1$
Index=1

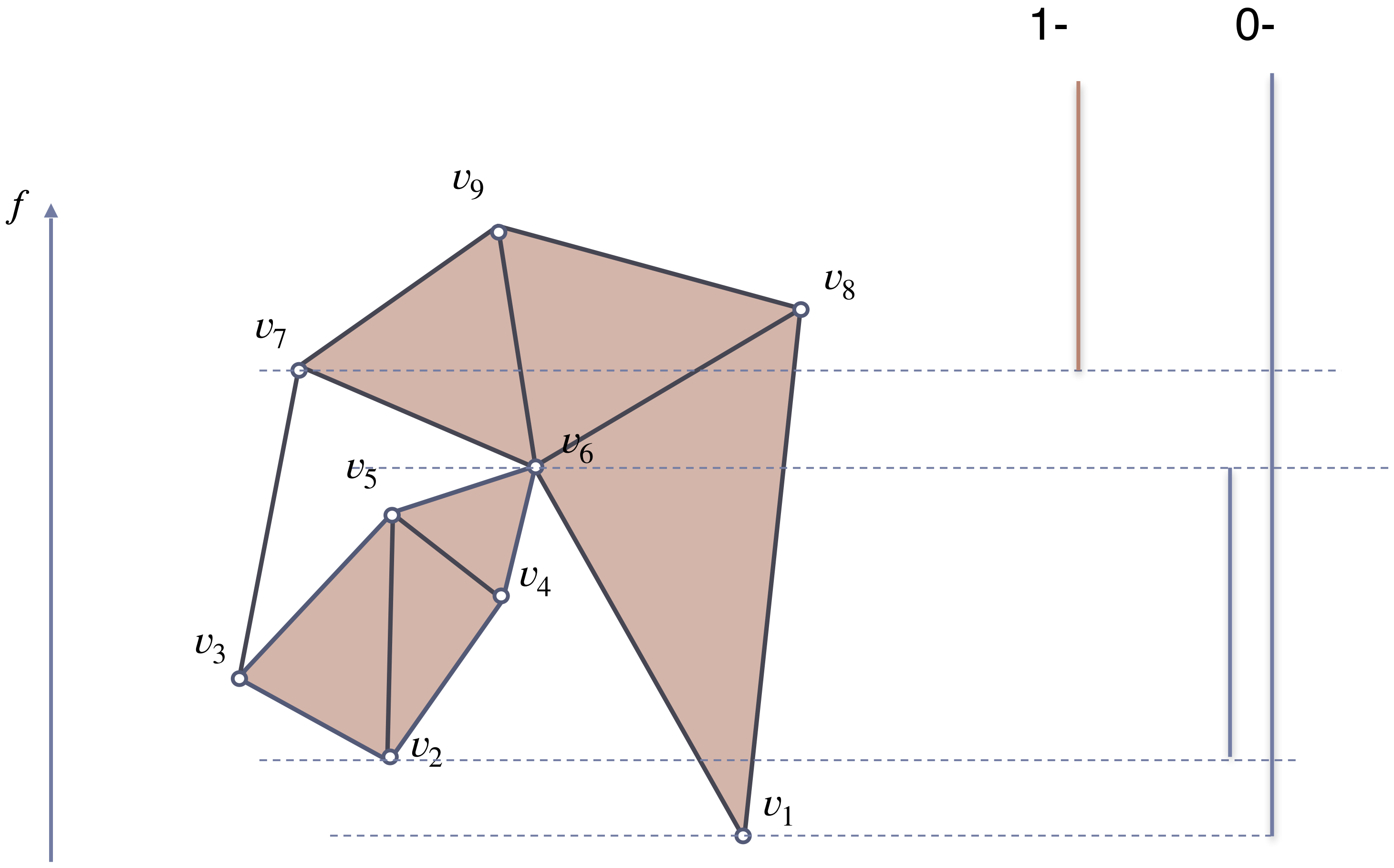
Examples



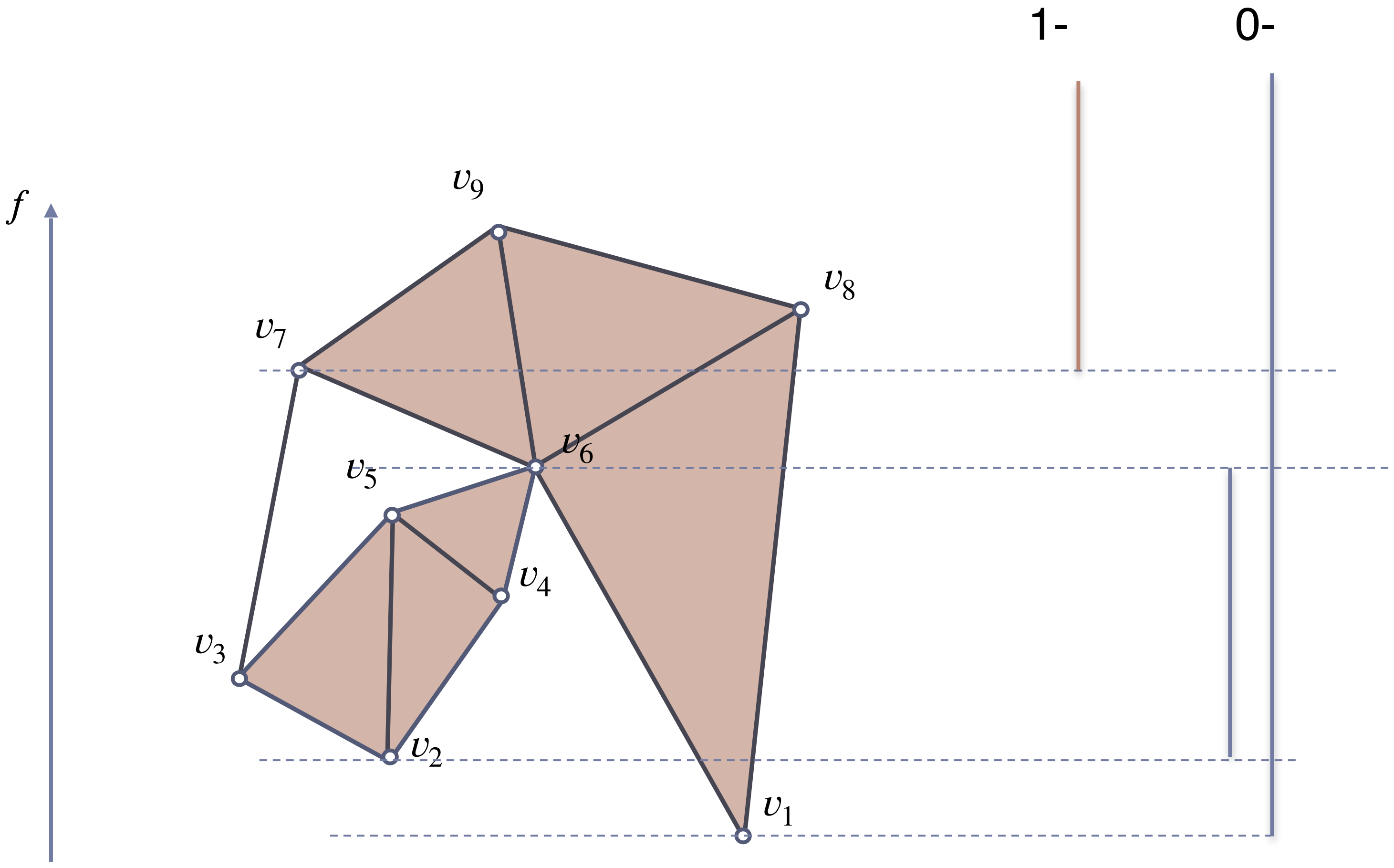
Example



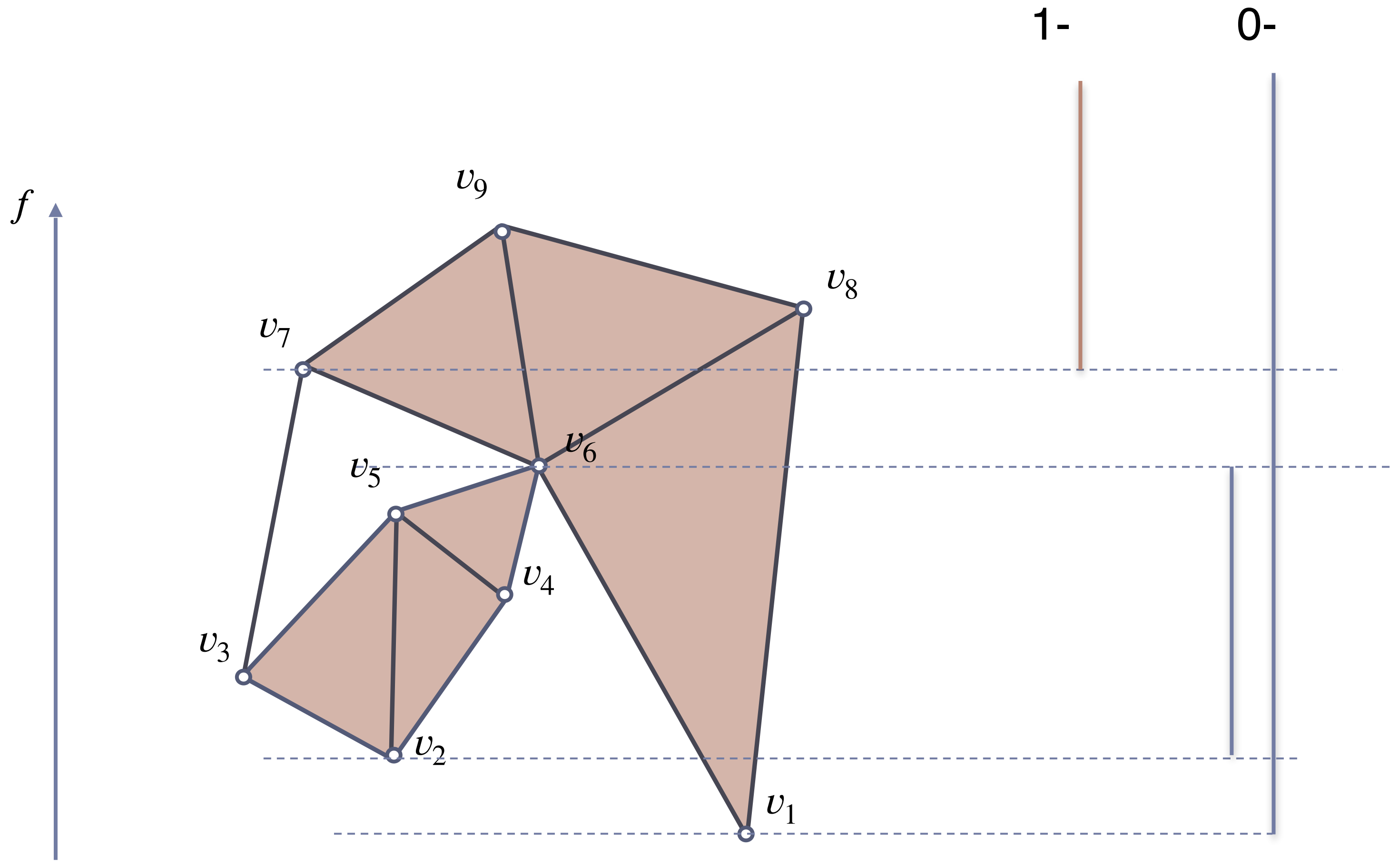
Example



Example

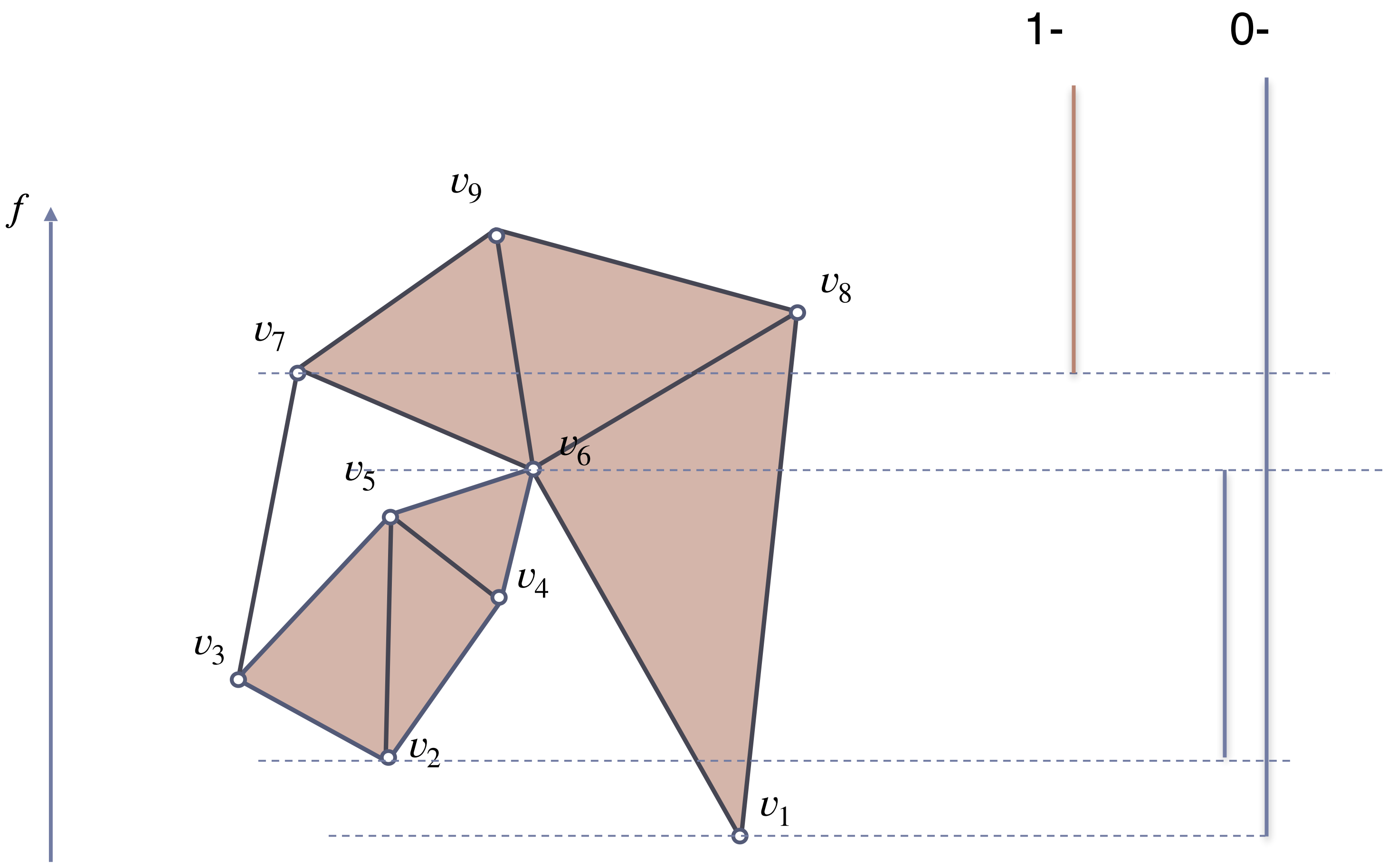


Example



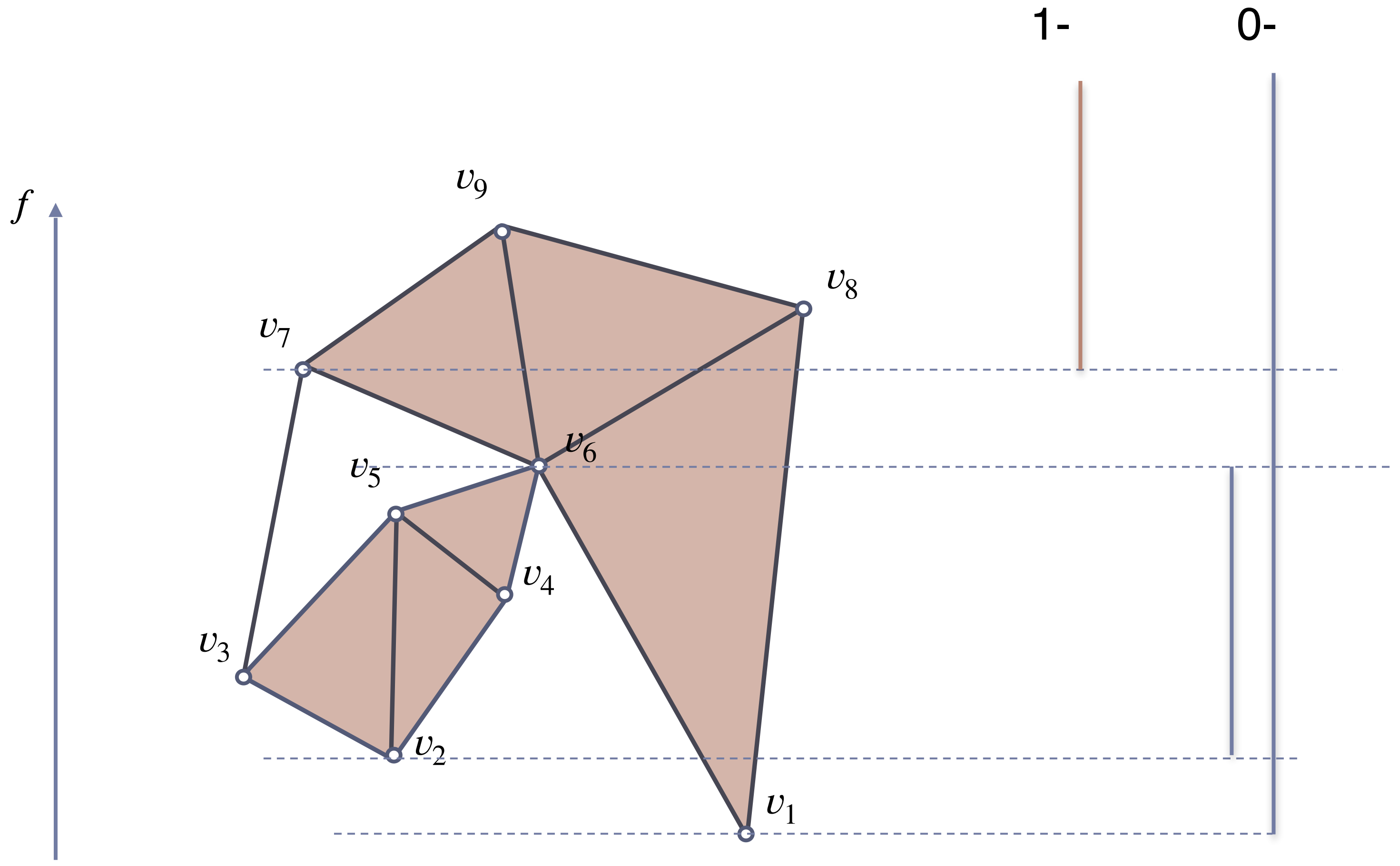
lower-link-index

Example



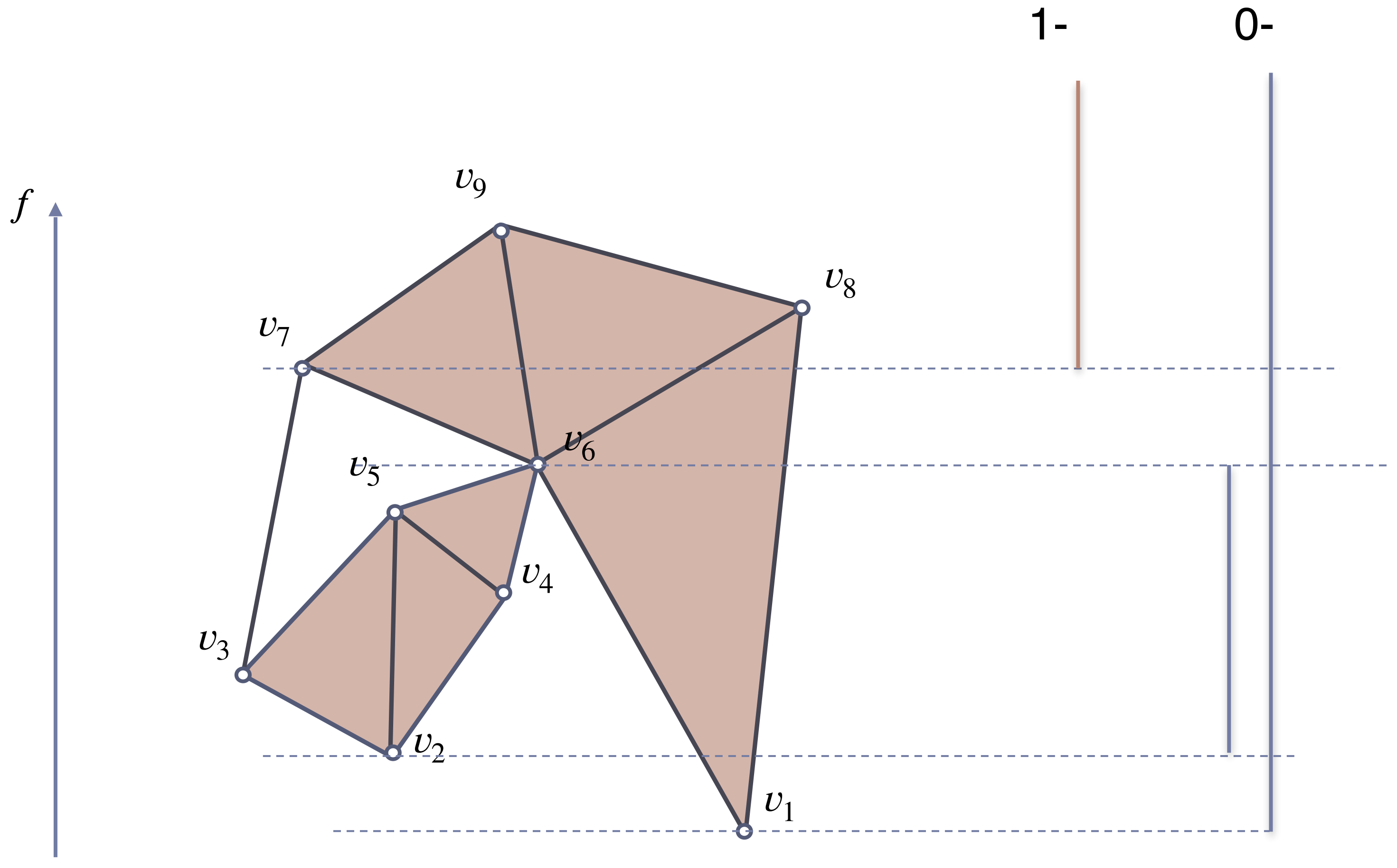
	v1
lower-link-index	0

Example



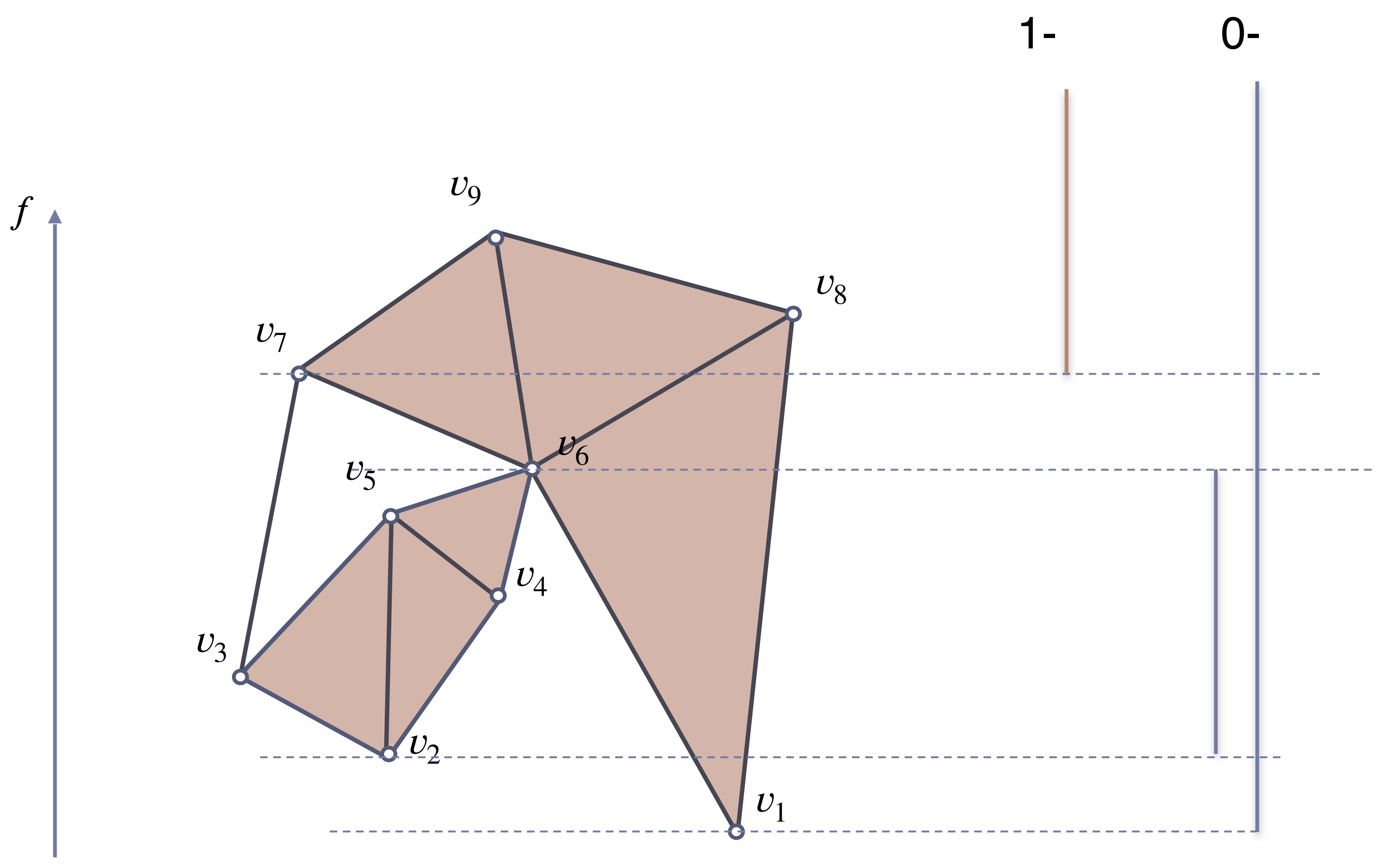
	v1	v2
lower-link-index	0	0

Example



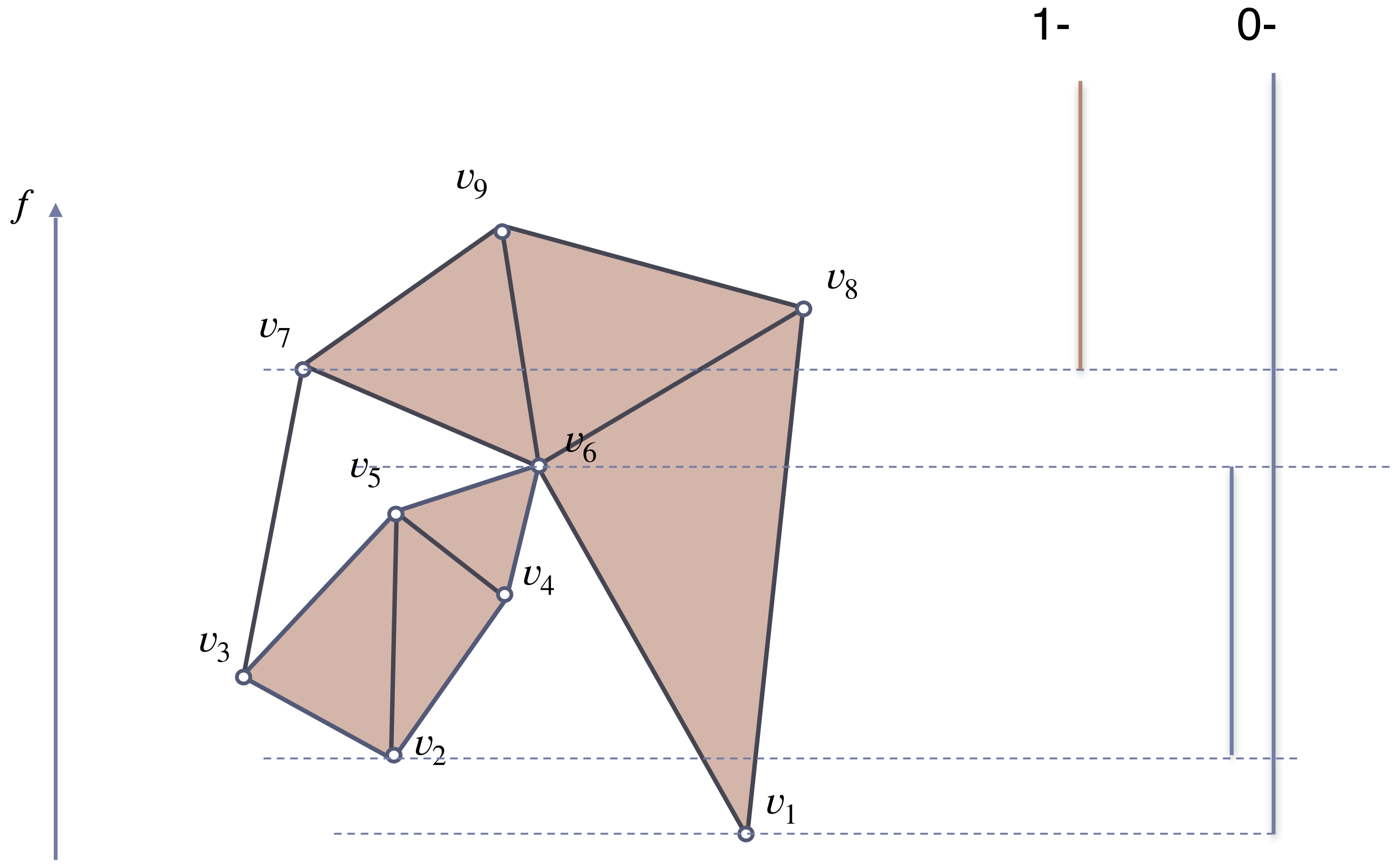
	v1	v2	v3
lower-link-index	0	0	NA

Example



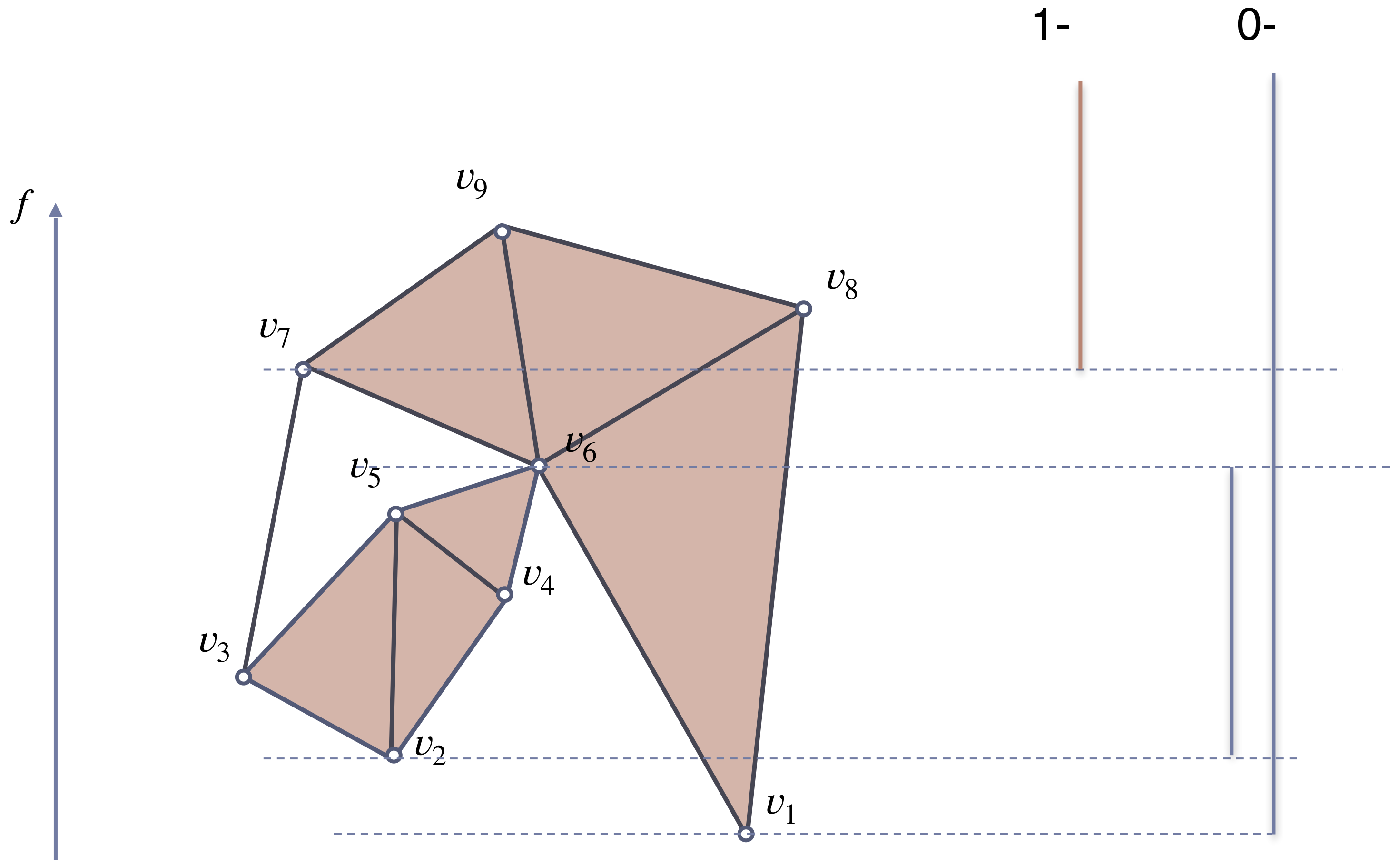
	v1	v2	v3	v4
lower-link-index	0	0	NA	NA

Example



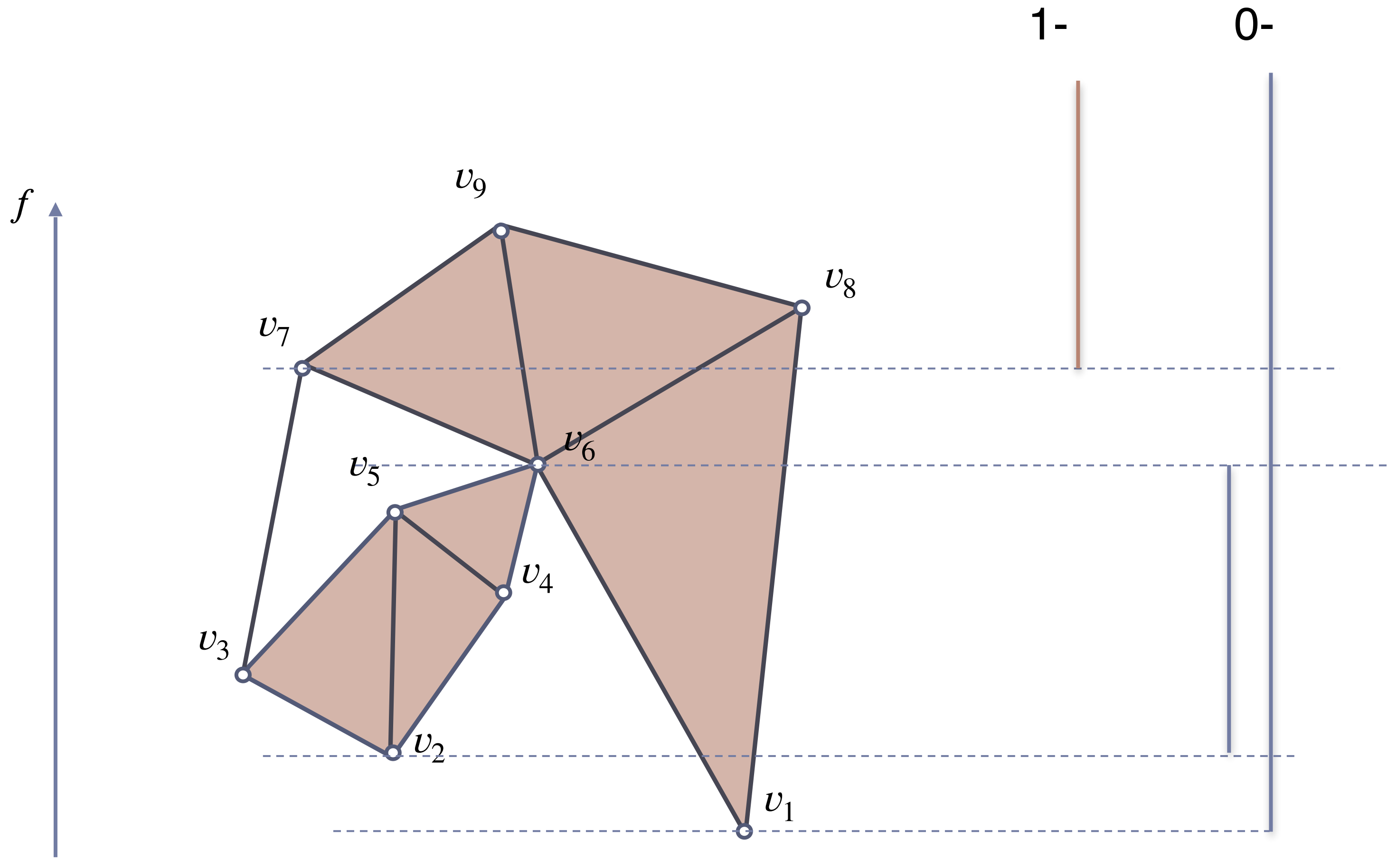
	v1	v2	v3	v4	v5
lower-link-index	0	0	NA	NA	NA

Example



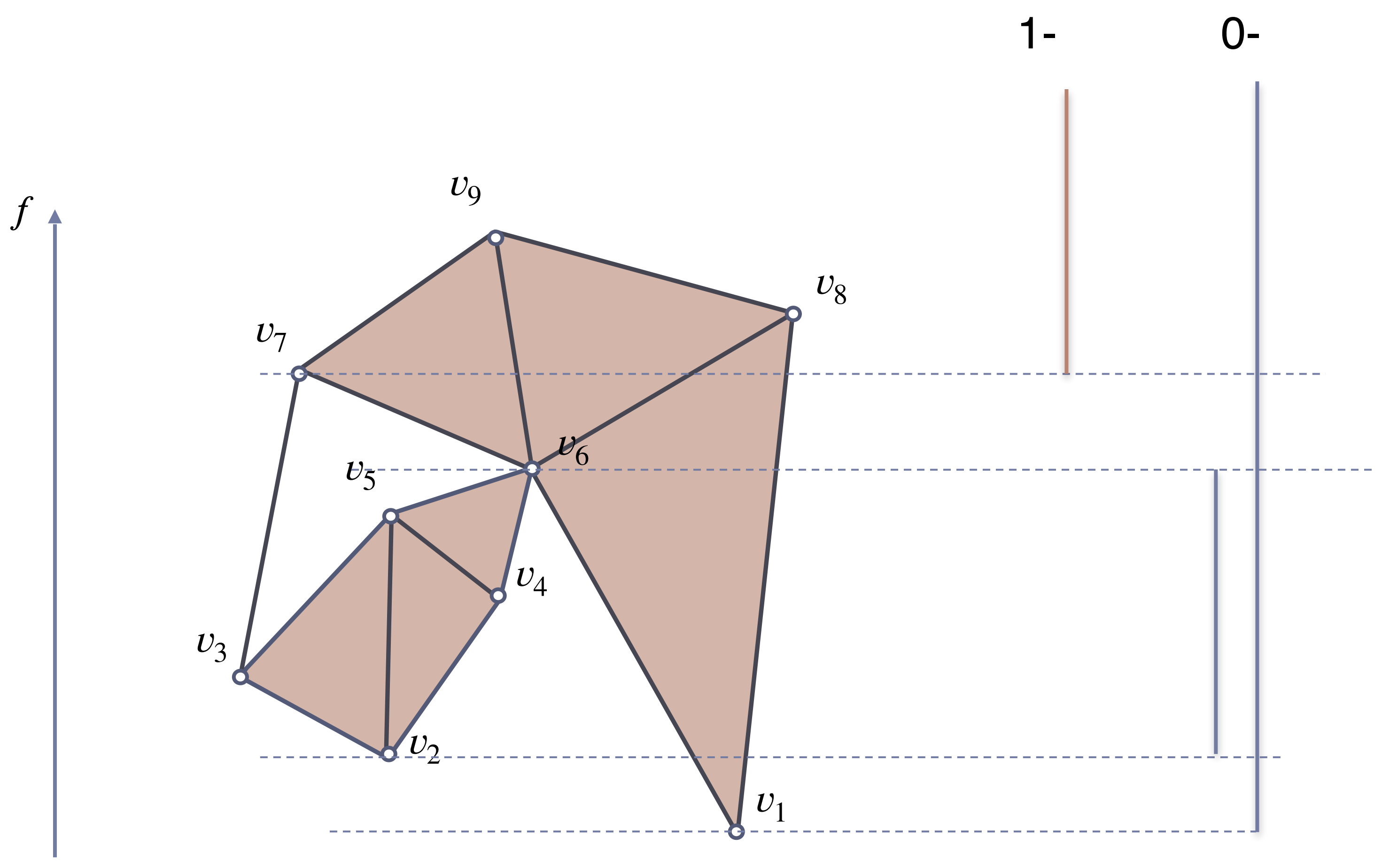
	v1	v2	v3	v4	v5	v6
lower-link-index	0	0	NA	NA	NA	1

Example



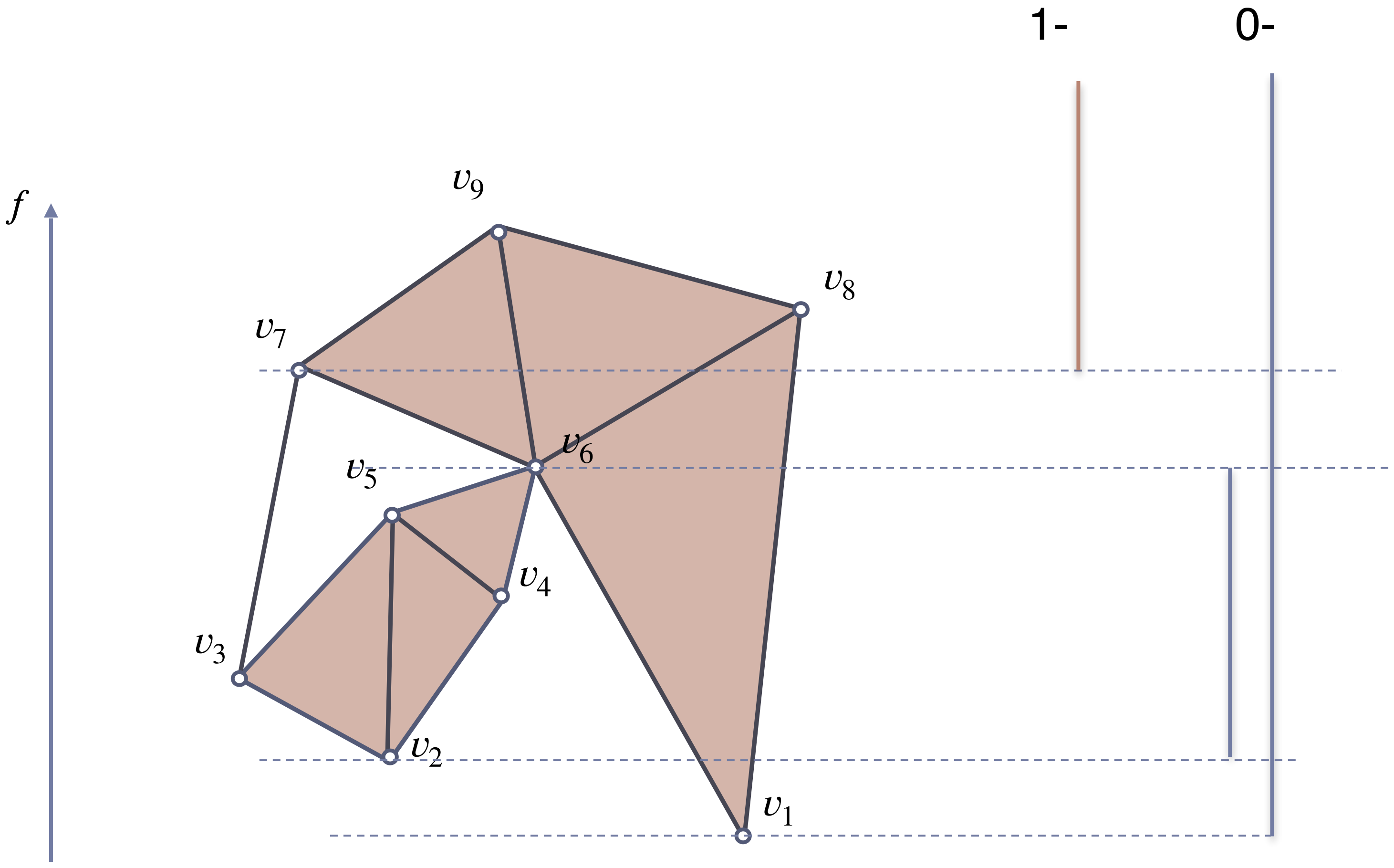
	v1	v2	v3	v4	v5	v6	v7
lower-link-index	0	0	NA	NA	NA	1	1

Example



	v1	v2	v3	v4	v5	v6	v7	v8
lower-link-index	0	0	NA	NA	NA	1	1	NA

Example



	v1	v2	v3	v4	v5	v6	v7	v8	V9
lower-link-index	0	0	NA	NA	NA	1	1	NA	NA

- Theorem
  - Given a PL function $f : |K| \to \mathbb{R}$, for any $2 \leq r \leq n$ and dimension p, $K_{r-1} \subset K_r$ induces an isomorphism $H_p(K_{r-1}) \cong H_p(K_r)$ unless v_r has lower-link-index p or $p+1$

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- **Theorem**
  - Given a PL function $f : |K| \to \mathbb{R}$ and compute its persistent homology as described. Then for each persistence pair $(i,\ j)$, both v_i and v_j must be PL-critical.

- Dionysus
- A GitHub repo

FIN