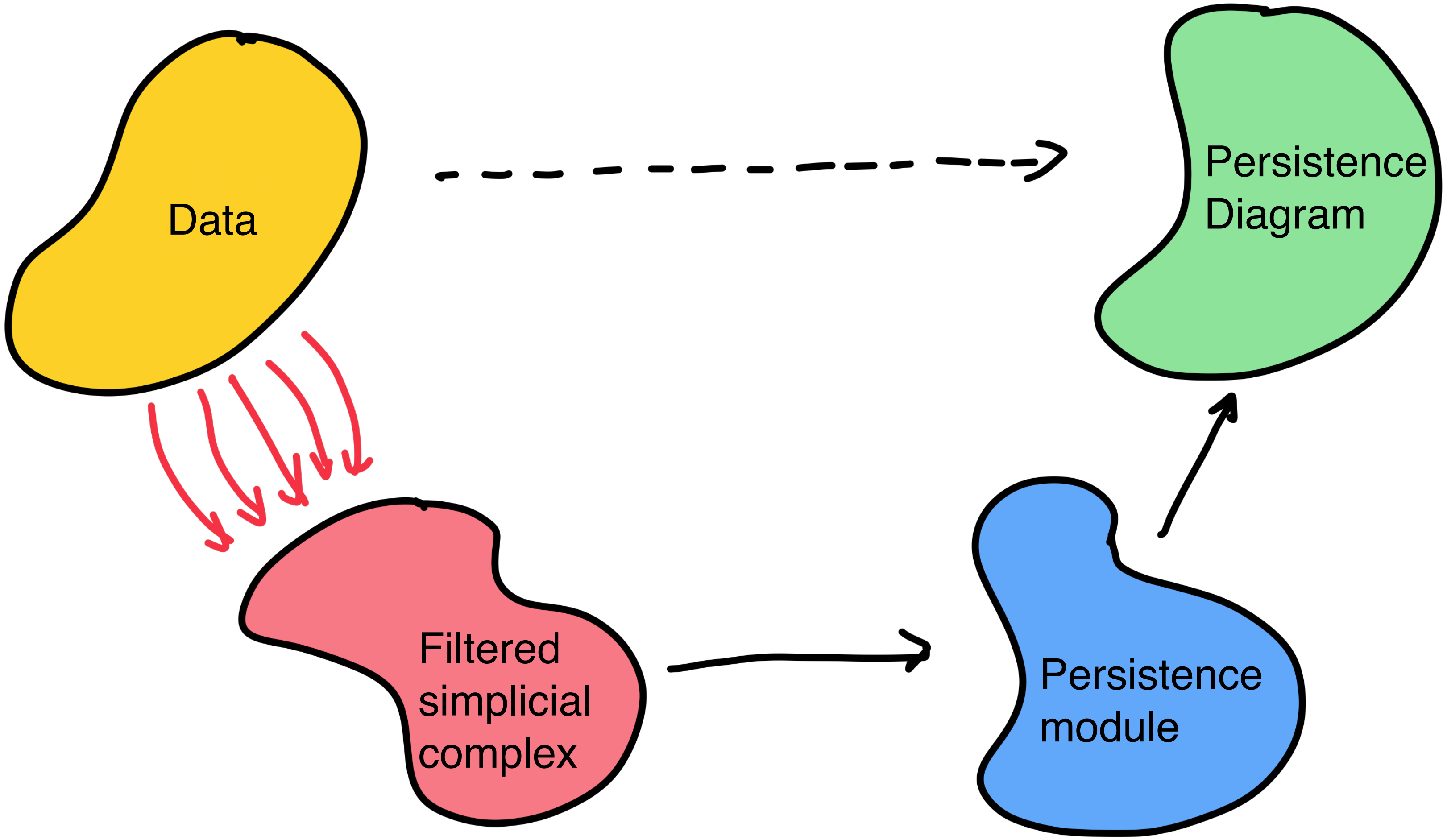
DSC 214
Topological Data Analysis
Topic 4-A: Introduction to Persistent Homology

Instructor: Zhengchao Wan

Persistent homology

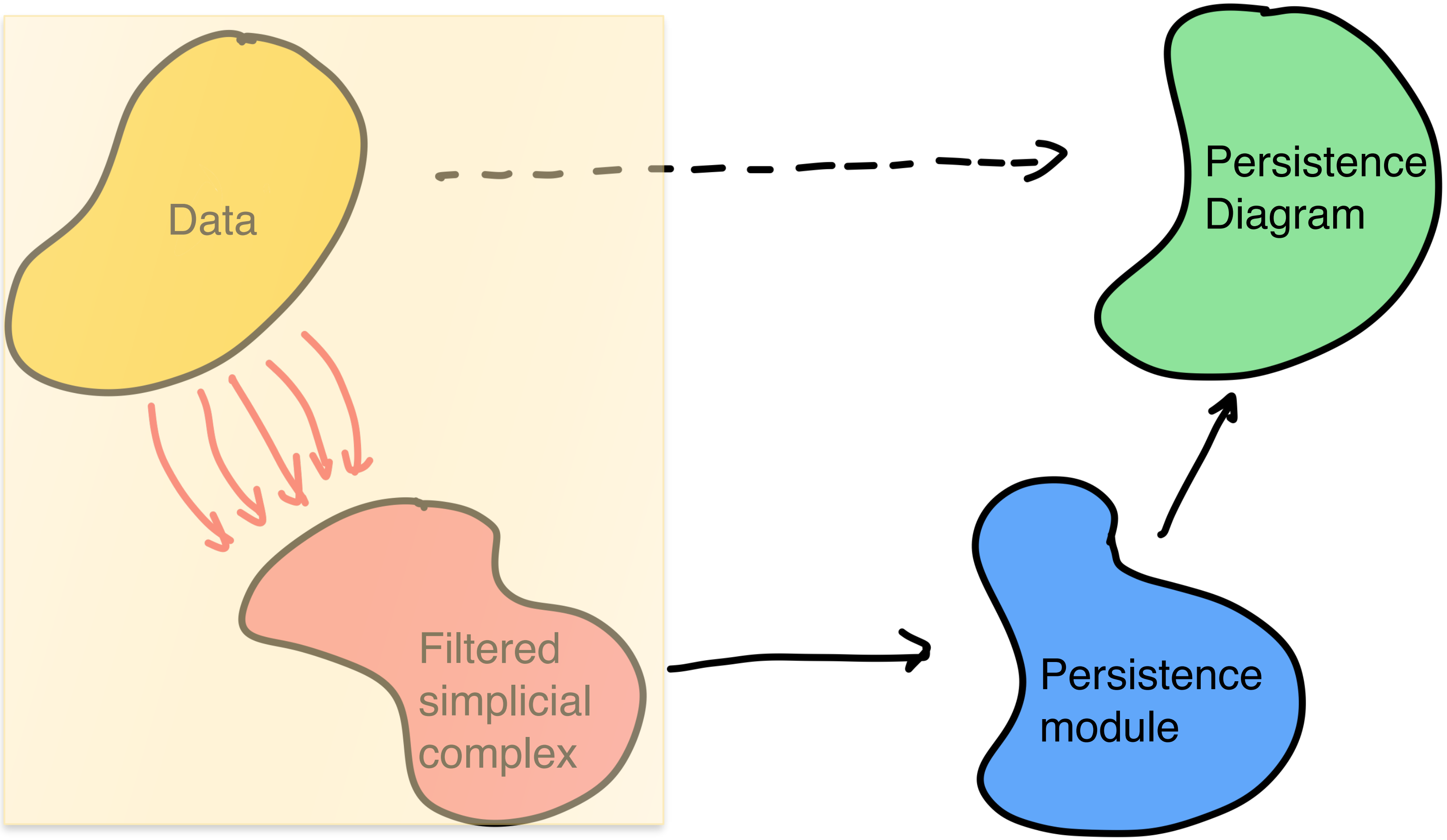
- A modern extension of homology to ``sequence of spaces”
  - [Edelsbrunner, Letcher, and Zomorodian, FOCS 2000]
  - Significantly broaden its practical power
- What is persistent homology (PH)
  - Motivation
  - Persistent betti numbers and persistence diagrams
- Algorithm(s) for persistent homology

Mind picture

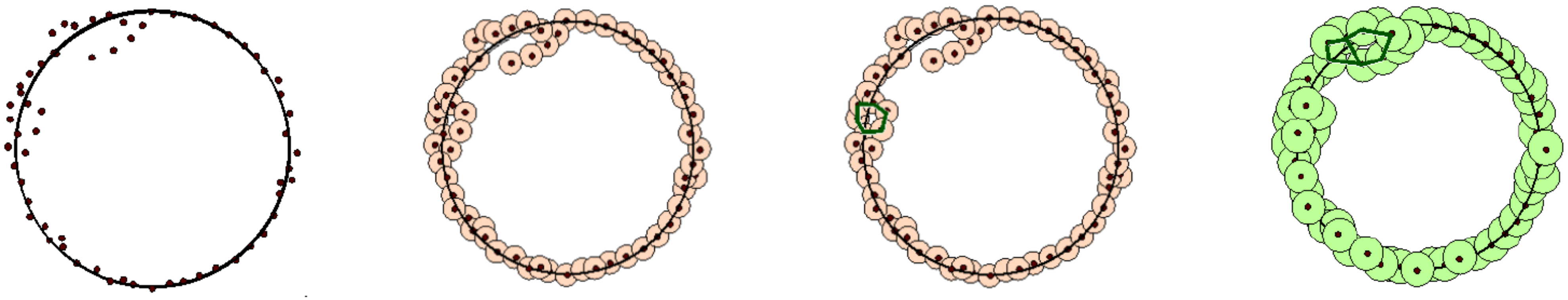


Section 1:
Persistent Homology

Filtered simplicial complex

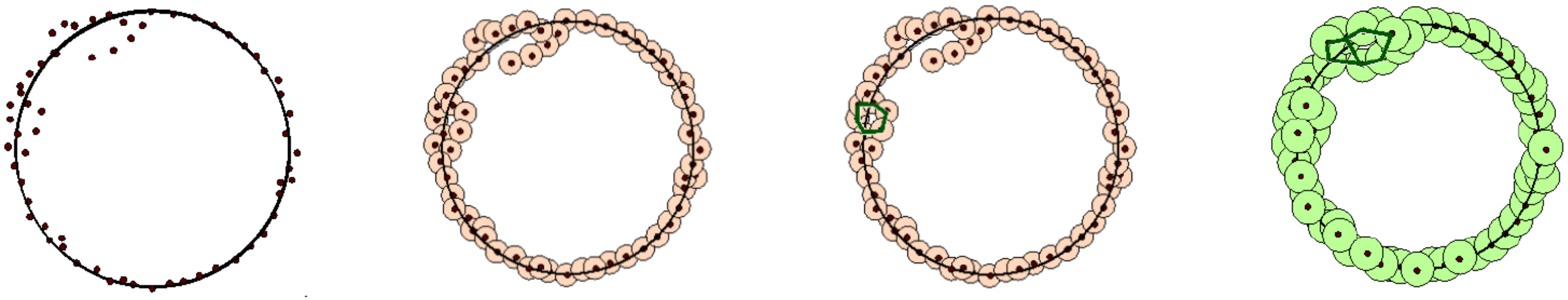


Issue of Scale



(a) (b) (c) (d)

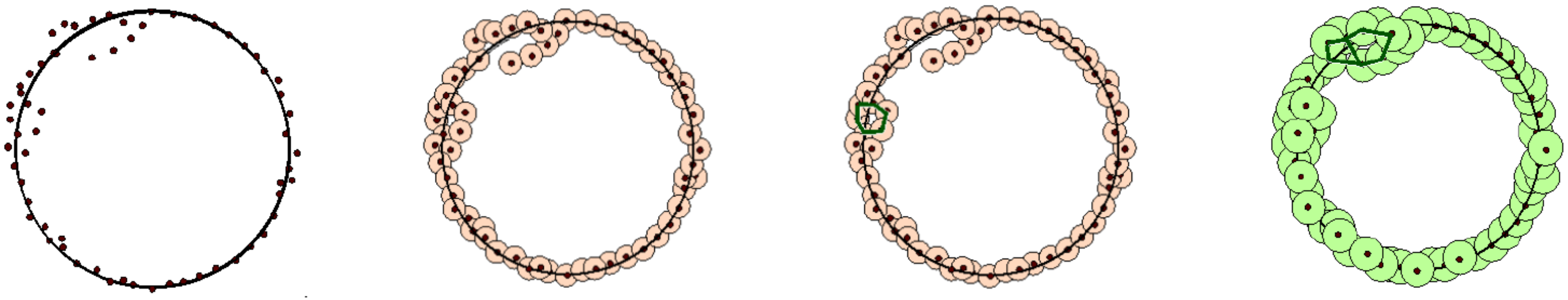
Issue of Scale



(a) (b) (c) (d)

- Which scale to take?

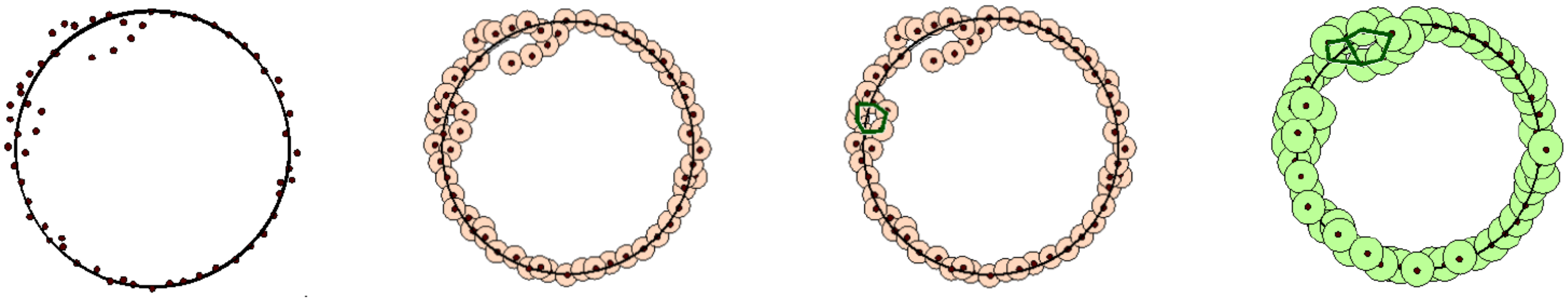
Issue of Scale



(a) (b) (c) (d)

- Which scale to take?
- No single good scale!

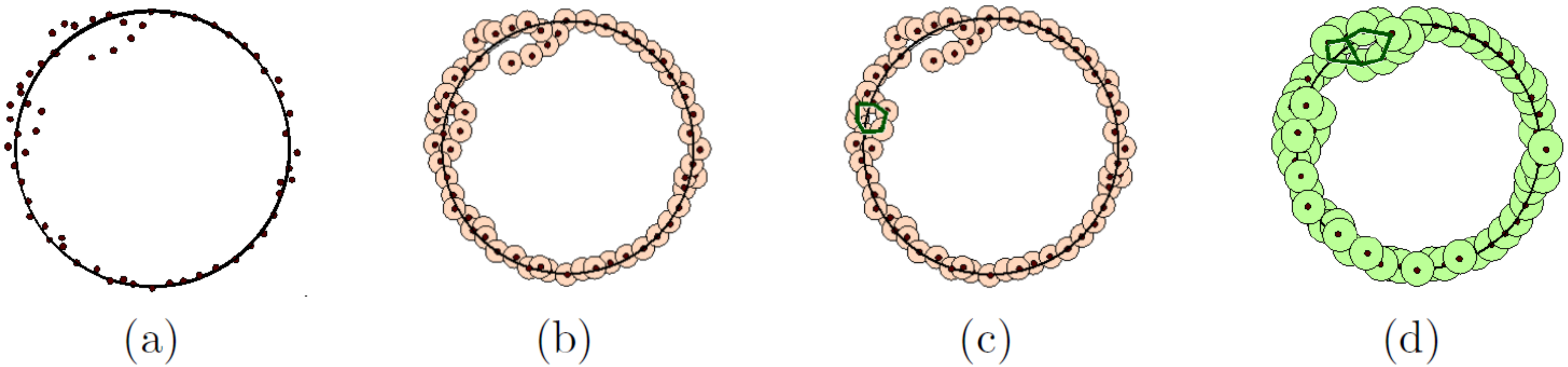
Issue of Scale



(a) (b) (c) (d)

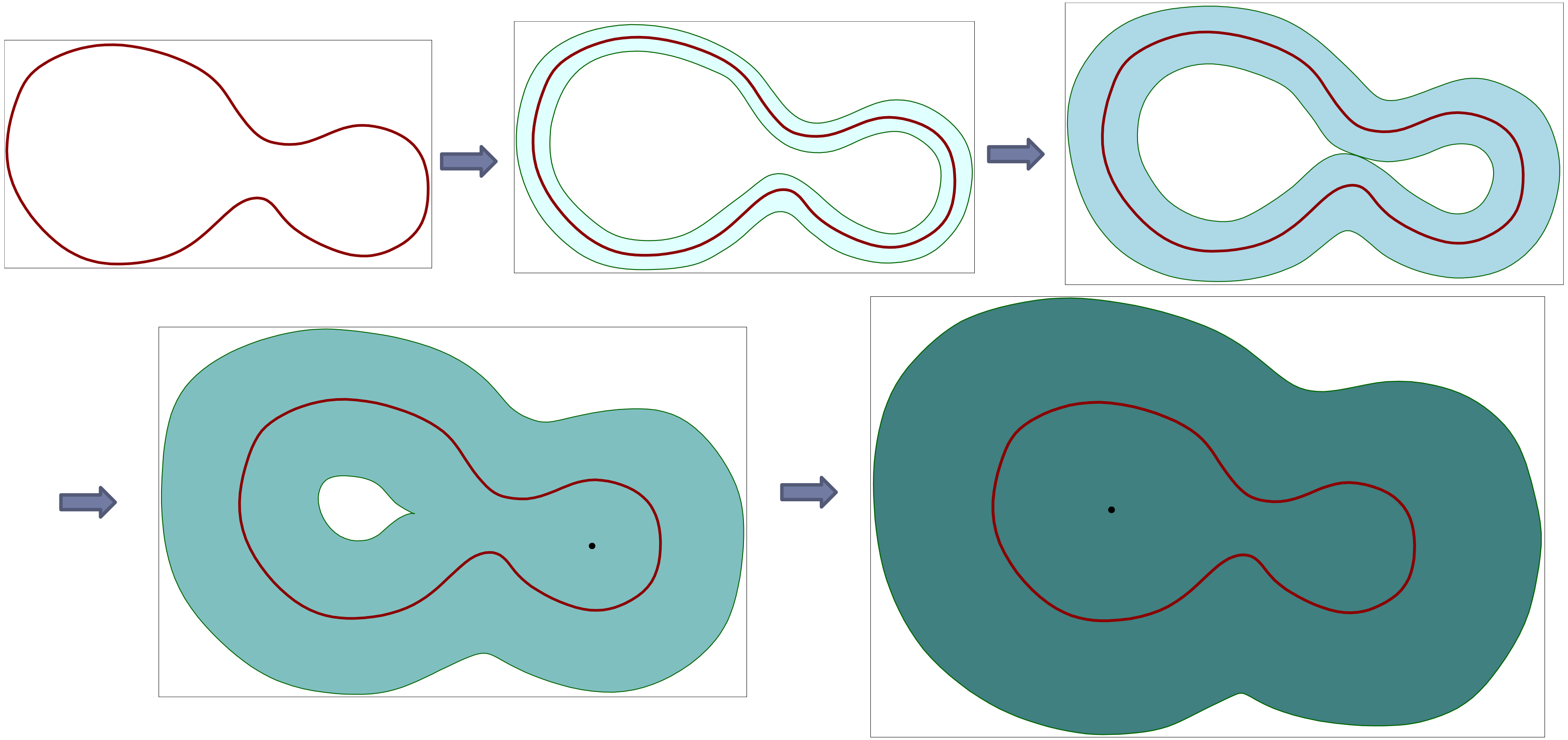
- Which scale to take?
- No single good scale!
- All scales?

Issue of Scale



- Which scale to take?
- No single good scale!
- All scales?
- Some ``features” persists longer than others

Another Example

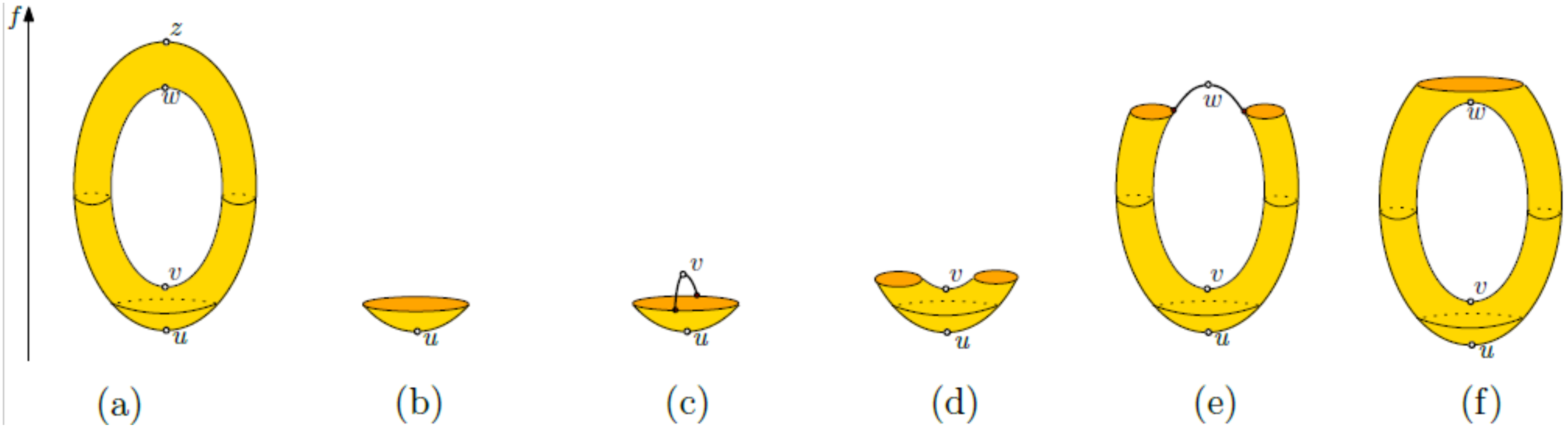


- Want to capture features of different ``sizes”

Another Example

- Want to capture homotopy type of the underlying space

Another Example



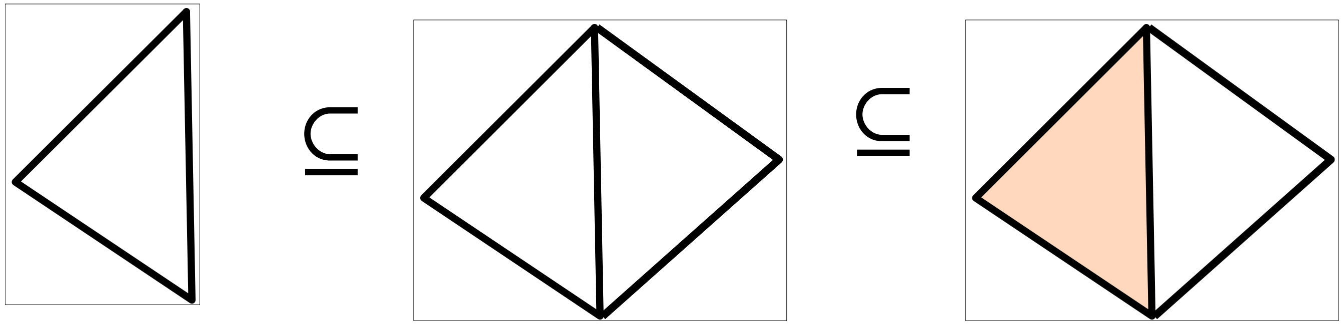
- Want to capture homotopy type of the underlying space

Filtration

- Inclusion map: $K \subseteq K'$ ($\iota\colon\ K \hookrightarrow\ K'$)
- A filtration
  - Given an index set $I \subseteq \mathbb{R}$
  - A sequence of simplicial complexes $(K_t)_{t \in I}$ is called a **filtered simplicial complex (or a filtration)** if $K_t \subseteq K_{t'}$ for $t < t'$

Filtration

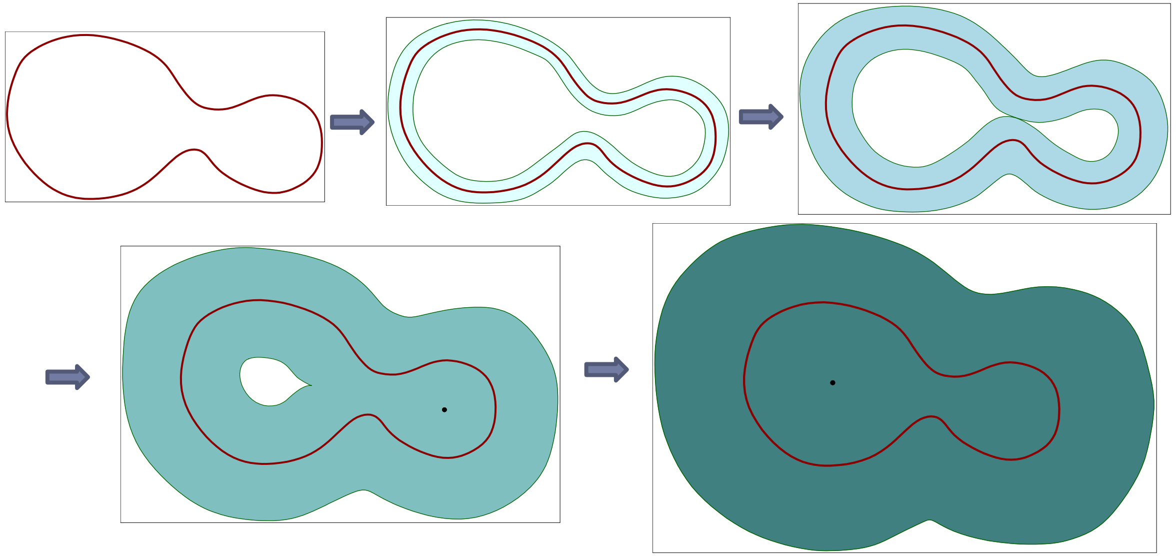
- Inclusion map: $K \subseteq K'$ ($\iota\colon\ K \hookrightarrow\ K'$)
- A filtration
  - Given an index set $I \subseteq \mathbb{R}$
  - A sequence of simplicial complexes $(K_t)_{t \in I}$ is called a **filtered simplicial complex (or a filtration)** if $K_t \subseteq K_{t'}$ for $t < t'$



$K_1 \quad \subseteq \quad K_2 \quad \subseteq \quad K_3$

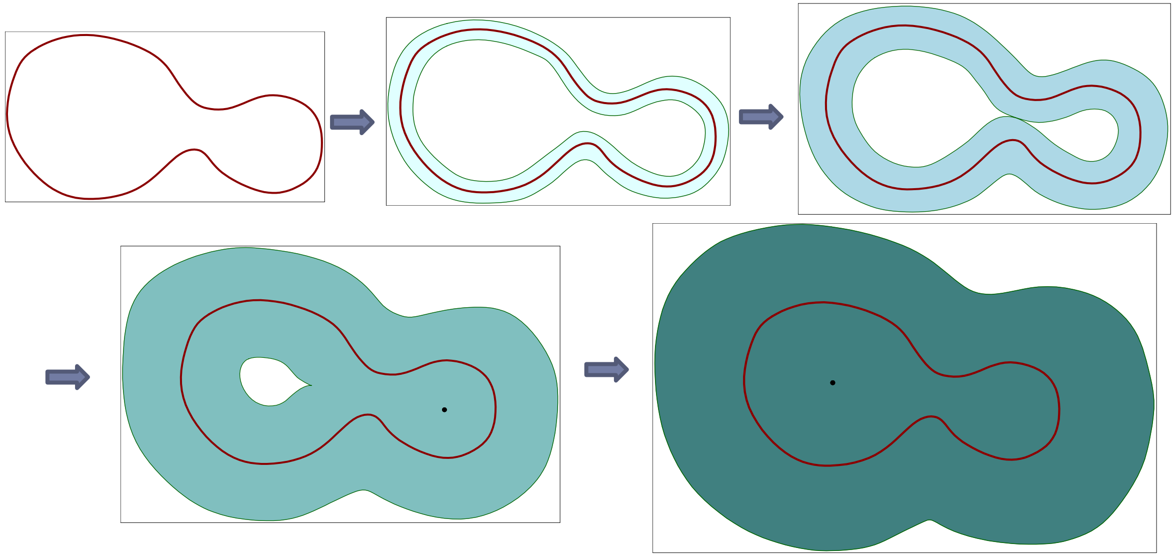
Filtration - variations

- Different index sets
  - $I = [0, \infty)$
  - $I = \{0 = t_0 < t_1 < \ldots < t_n\}$
  - $I = \{0, 1, \ldots, n\}$
    - $(K_{t_i})_i \to (K_i)_i$ by $K_i = K_{t_i}$
- Filtration of topological spaces



Filtration - variations

- Different index sets
  - $I = [0,\infty)$
  - $I = \{0 = t_0 < t_1 < \ldots < t_n\}$
  - $I = \{0,1,\ldots,n\}$
    - $(K_{t_i})_i \to (K_i)_i$ by $K_i = K_{t_i}$
- Filtration of topological spaces



Čech Complex

Čech Complex

- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$

Čech Complex

- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$
- Given a real value $r > 0$, the *Čech complex* $C^r(P)$ is the nerve of the set $\left\{ B\left(p_i, r\right) \right\}_{i \in [1,n]}$, where $B(p, r) = \{x \in \mathbb{R}^d \,|\, d(p, x) \leq r\}$

Čech Complex

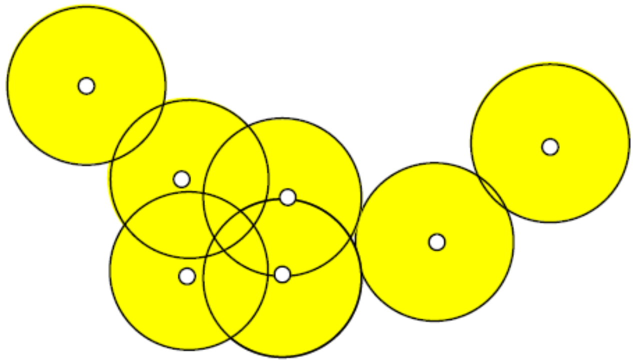
- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$
- Given a real value $r > 0$, the *Čech complex* $C^r(P)$ is the nerve of the set $\left\{ B\left(p_i, r\right) \right\}_{i \in [1,n]}$, where $B(p, r) = \{x \in \mathbb{R}^d \,|\, d(p, x) \leq r\}$
  - i.e, $\sigma = \left\{p_{i_0}, \ \ldots, p_{i_s}\right\} \in C^r(P)$ iff $\bigcap_{j \in [0,s]} B\left(p_{i_j}, \ r\right) \neq \emptyset$

Čech Complex

- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$
- Given a real value $r > 0$, the *Čech complex* $C^r(P)$ is the nerve of the set $\left\{ B\left(p_i, r\right) \right\}_{i \in [1,n]}$, where $B(p, r) = \{x \in \mathbb{R}^d \,|\, d(p, x) \leq r\}$
  - i.e, $\sigma = \left\{ p_{i_0}, \ \ldots, p_{i_s} \right\} \in C^r(P)$ iff $\bigcap_{j \in [0,s]} B\left(p_{i_j}, \ r\right) \neq \emptyset$
- The definition can be extended to a finite sample P of a metric space (X, d).

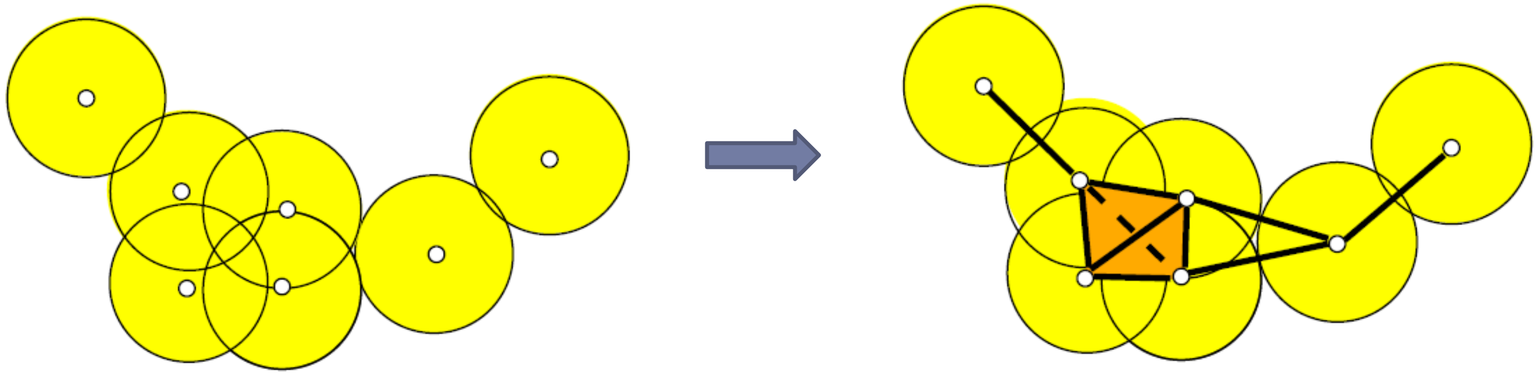
Čech Complex

- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$
- Given a real value $r > 0$, the *Čech complex* $C^r(P)$ is the nerve of the set $\left\{ B\left(p_i, r\right) \right\}_{i\in[1,n]}$, where $B(p, r) = \{x \in \mathbb{R}^d \,|\, d(p, x) \leq r\}$
  - i.e, $\sigma = \left\{p_{i_0}, \ \ldots, p_{i_s}\right\} \in C^r(P)$ iff $\bigcap_{j\in[0,s]} B\left(p_{i_j}, \ r\right) \neq \emptyset$
- The definition can be extended to a finite sample P of a metric space (X, d).



Čech Complex

- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$
- Given a real value $r > 0$, the *Čech complex* $C^r(P)$ is the nerve of the set $\left\{ B\left(p_i, r\right) \right\}_{i \in [1,n]}$, where $B(p, r) = \{x \in \mathbb{R}^d \,|\, d(p, x) \leq r\}$
  - i.e, $\sigma = \left\{ p_{i_0}, \ \ldots, p_{i_s} \right\} \in C^r(P)$ iff $\bigcap_{j \in [0,s]} B\left(p_{i_j}, \ r\right) \neq \emptyset$
- The definition can be extended to a finite sample P of a metric space (X, d).



Čech Filtration

Čech Filtration

- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$

Čech Filtration

- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$

Čech Filtration

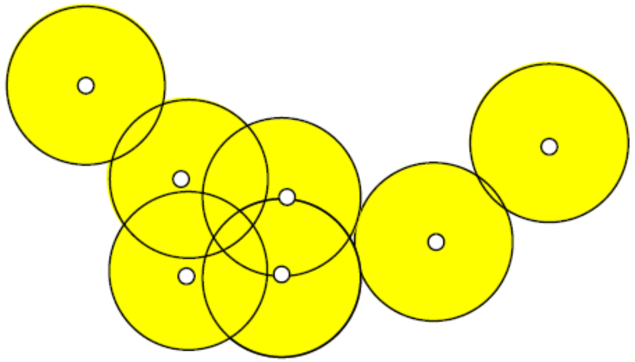
- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$
- $(C^r(P))_{r \geq 0}$ is called the Čech filtration

Vietoris-Rips (Rips) Complex

- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$
- Given a real value $r > 0$, the *Vietoris-Rips (Rips) complex* $Rips^r(P)$ is:
  - $\left\{(p_{i_0}, \ldots, p_{i_k}) \mid B(p_{i_l}, r) \cap B(p_{i_j}, r) \neq \emptyset, \forall l, j = 0, \ldots, k\right\}$.
- More generally for P in a metric space (X, d):
  - $Rips^r(P) = \left\{(p_{i_0}, \ldots, p_{i_k}) \mid d(p_{i_l}, p_{i_j}) \leq 2r, \forall l, j = 0, \ldots, k\right\}$.

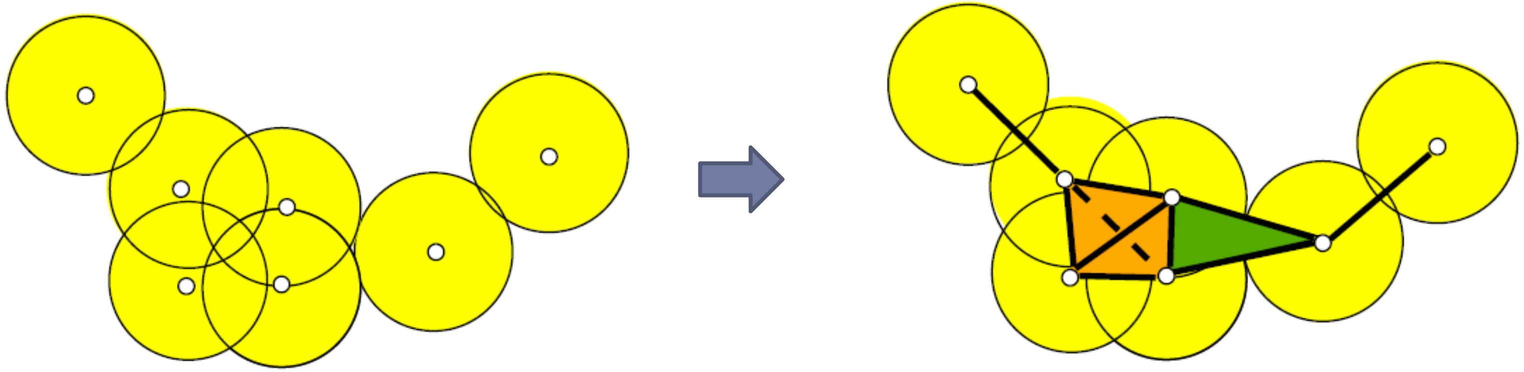
Vietoris-Rips (Rips) Complex

- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$
- Given a real value $r > 0$, the *Vietoris-Rips (Rips) complex* $Rips^r(P)$ is:
  - $\left\{ (p_{i_0}, \ldots, p_{i_k}) \,|\, B(p_{i_l}, r) \cap B(p_{i_j}, r) \neq \emptyset, \forall l, j = 0, \ldots, k \right\}$.
- More generally for P in a metric space (X, d):
  - $Rips^r(P) = \left\{ (p_{i_0}, \ldots, p_{i_k}) \,|\, d(p_{i_l}, p_{i_j}) \leq 2r, \forall l, j = 0, \ldots, k \right\}$.



Vietoris-Rips (Rips) Complex

- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$
- Given a real value $r > 0$, the *Vietoris-Rips (Rips) complex* $Rips^r(P)$ is:
  - $\left\{ (p_{i_0}, \ldots, p_{i_k}) \mid B(p_{i_l}, r) \cap B(p_{i_j}, r) \neq \varnothing, \forall l, j = 0, \ldots, k \right\}.$
- More generally for P in a metric space (X, d):
  - $Rips^r(P) = \left\{ (p_{i_0}, \ldots, p_{i_k}) \mid d(p_{i_l}, p_{i_j}) \leq 2r, \forall l, j = 0, \ldots, k \right\}.$



Vietoris-Rips (Rips) Filtration

- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$

- $(Rips^r(P))_{r \geq 0}$ is called the Vietoris-Rips (Rips) Filtration

Finitely presented filtration

Finitely presented filtration

- A filtration $(K_t)_{t\in[0,\infty)}$ is called **finitely represented** if

Finitely presented filtration

- A filtration $(K_t)_{t\in[0,\infty)}$ is called **finitely represented** if
  - There exist $0 = t_0 < t_1 < \cdots < t_n$ such that

Finitely presented filtration

- A filtration $(K_t)_{t\in[0,\infty)}$ is called **finitely represented** if
  - There exist $0 = t_0 < t_1 < \cdots < t_n$ such that
  - $K_t = K_{t'}, \quad \forall t_i \leq t < t' < t_{i+1}$ and $i = 0,\ldots,n$ $(t_{n+1} := \infty)$

Finitely presented filtration

- A filtration $(K_t)_{t\in[0,\infty)}$ is called **finitely represented** if
  - There exist $0 = t_0 < t_1 < \cdots < t_n$ such that
  - $K_t = K_{t'}, \quad \forall t_i \leq t < t' < t_{i+1}$ and $i = 0,\ldots,n$ $(t_{n+1} := \infty)$

Finitely presented filtration

- A filtration $(K_t)_{t\in[0,\infty)}$ is called **finitely represented** if
  - There exist $0 = t_0 < t_1 < \cdots < t_n$ such that
  - $K_t = K_{t'}, \quad \forall t_i \le t < t' < t_{i+1}$ and $i = 0,\ldots,n$ $(t_{n+1} := \infty)$

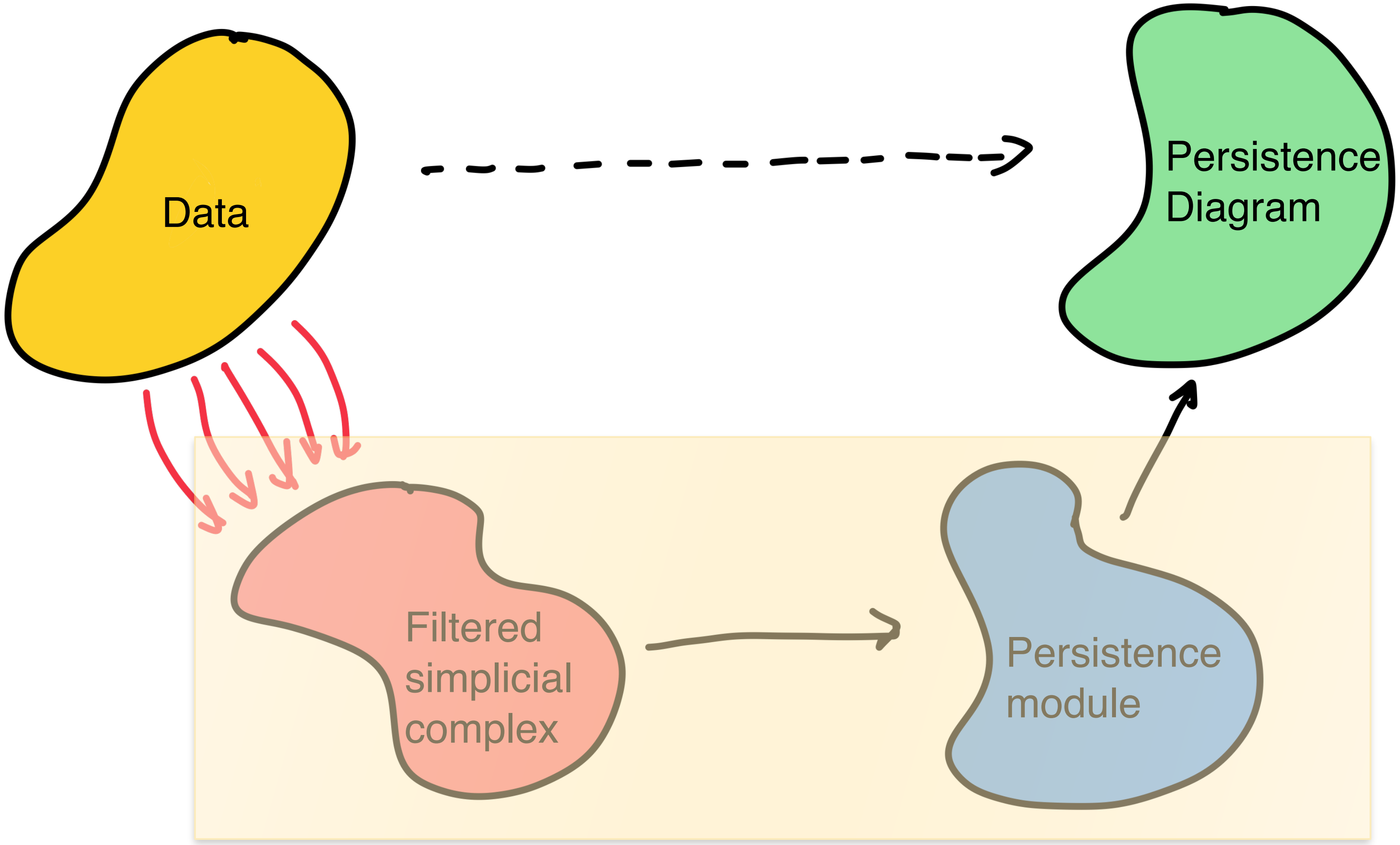
- So $(K_t)_{t\in[0,\infty)}$ is essentially the same as (or can be reconstructed from) $(K_{t_i})_{i=0,\ldots,n}$

Finitely presented filtration

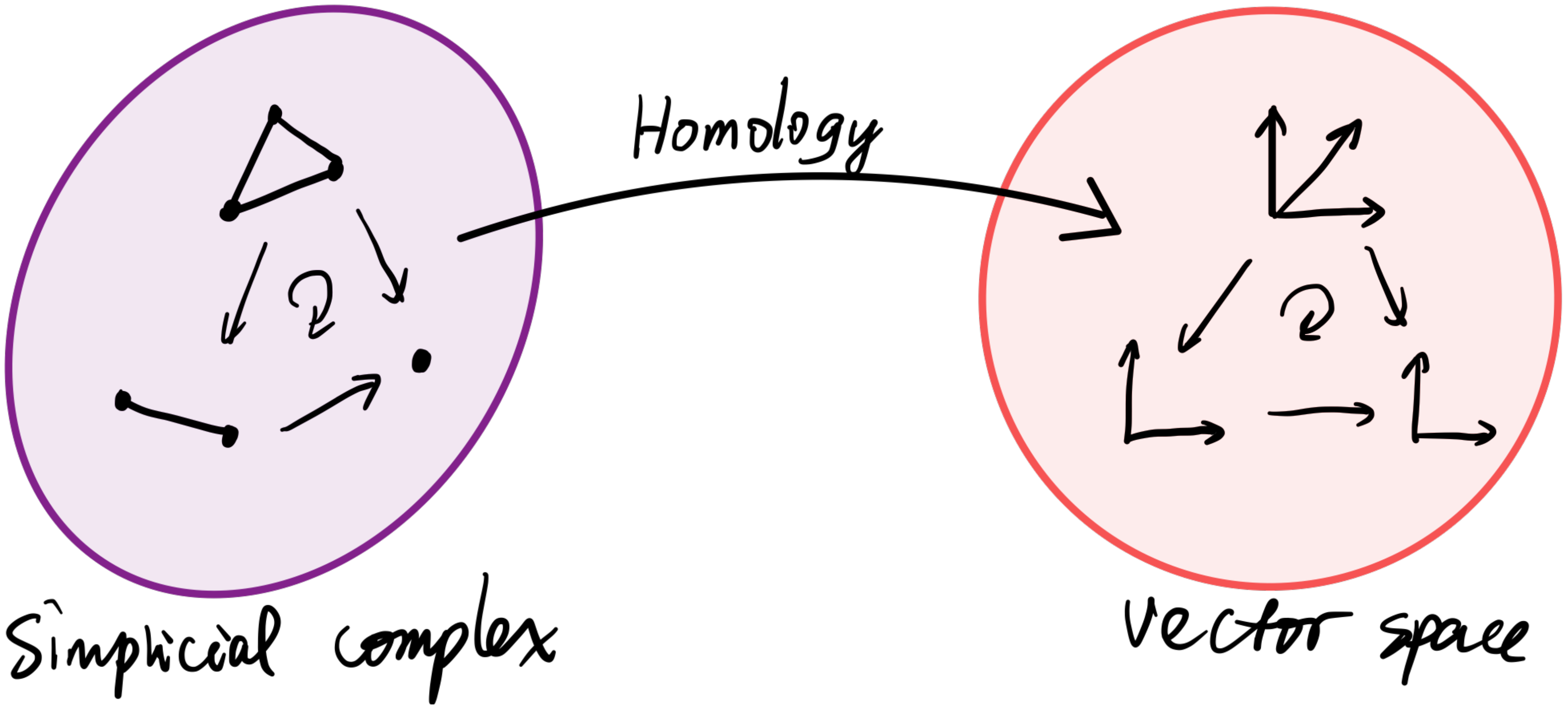
- A filtration $(K_t)_{t\in[0,\infty)}$ is called **finitely represented** if
  - There exist $0 = t_0 < t_1 < \cdots < t_n$ such that
  - $K_t = K_{t'}, \quad \forall t_i \leq t < t' < t_{i+1}$ and $i = 0,\ldots,n$ $(t_{n+1} := \infty)$

- So $(K_t)_{t\in[0,\infty)}$ is essentially the same as (or can be reconstructed from) $(K_{t_i})_{i=0,\ldots,n}$
- Both Čech and Rips filtrations are finitely represented

Persistence modules



Mind picture of functoriality

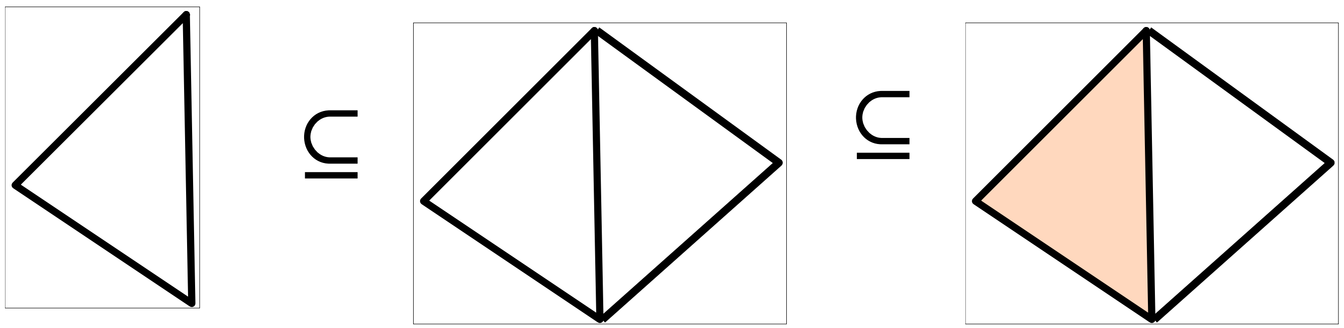


Persistence Modules

- $K \subseteq K' \;\Rightarrow \xi_p : H_p(K) \to H_p(K')$
  - Inclusion maps induce homomorphisms in homology groups (under Z_2-coefficients, linear maps in vector spaces)

Persistence Modules

- $K \subseteq K' \Rightarrow \xi_p : H_p(K) \to H_p(K')$
  - Inclusion maps induce homomorphisms in homology groups (under Z_2-coefficients, linear maps in vector spaces)

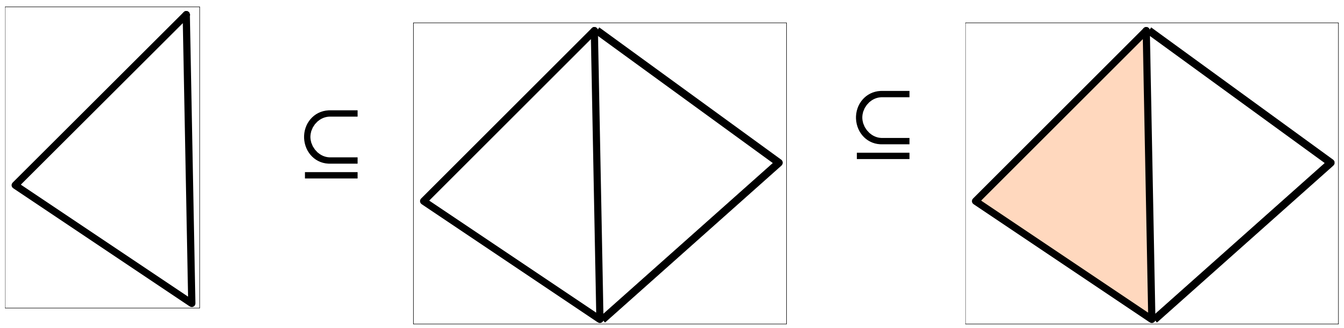


$$K_1 \subseteq K_2 \subseteq K_3$$

$$H_1(K_1) \to H_1(K_2) \to H_1(K_3)$$

Persistence Modules

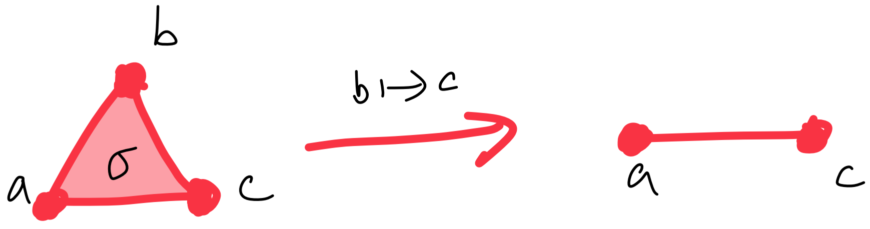
- $K \subseteq K' \Rightarrow \xi_p : H_p(K) \to H_p(K')$
  - Inclusion maps induce homomorphisms in homology groups (under Z_2-coefficients, linear maps in vector spaces)



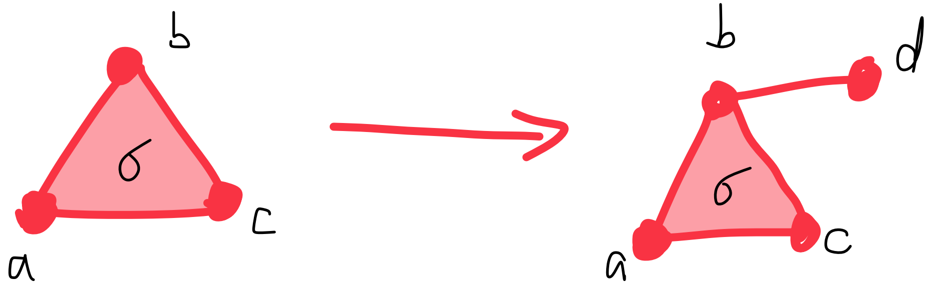
$$\begin{array}{ccccc}
K_1 & \subseteq & K_2 & \subseteq & K_3 \\
H_1(K_1) & \to & H_1(K_2) & \to & H_1(K_3) \\
[c] & \mapsto & [c] & \mapsto & 0
\end{array}$$

Construction of f_p

- Define $\bar{f}_p : C_p(K) \to C_p(K')$
  - $\bar{f}_p(\sigma) = \begin{cases} f(\sigma) & \text{if } f(\sigma) \text{ is } p-\text{dimensional} \\ 0 & \text{otherwise} \end{cases}$
  - Define $f_p : H_p(K) \to H_p(K')$
    - $f_p([c]) := [\bar{f}_p(c)]$



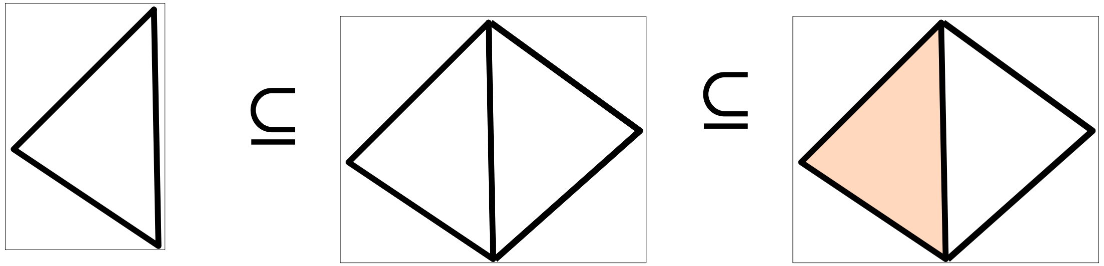
$\bar{f}_2(\sigma) = 0$ since $f(\sigma) = \{a, c\}$ is 1-dim



$\bar{f}_2(\sigma) = \sigma$

Understanding $H_p(K) \to H_p(K')$

- Define $\bar{f}_p : C_p(K) \to C_p(K')$
  - $\bar{f}_p(\sigma) = f(\sigma)$
  - Define $f_p : H_p(K) \to H_p(K')$
    - $f_p([c]) := [c]$

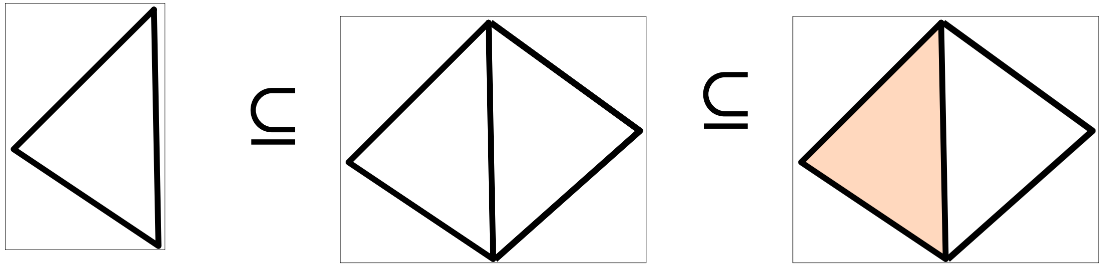


$$K_1 \subseteq K_2 \subseteq K_3$$

$$H_1(K_1) \to H_1(K_2) \to H_1(K_3)$$

Understanding $H_p(K) \to H_p(K')$

- Define $\bar{f}_p : C_p(K) \to C_p(K')$
  - $\bar{f}_p(\sigma) = f(\sigma)$
  - Define $f_p : H_p(K) \to H_p(K')$
    - $f_p([c]) := [c]$



$$
\begin{array}{ccccc}
K_1 & \subseteq & K_2 & \subseteq & K_3 \\
H_1(K_1) & \to & H_1(K_2) & \to & H_1(K_3) \\
[c] & \mapsto & [c] & \mapsto & 0
\end{array}
$$

Persistence Modules

- $K \subseteq K' \;\Rightarrow \xi_p : H_p(K) \to H_p(K')$
  - Inclusion maps induce homomorphisms in homology groups (under Z_2-coefficients, linear maps in vector spaces)

$$\boxed{\begin{aligned}&K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K \\ &\Rightarrow \quad H_*(K_0) \to H_*(K_1) \to \cdots \to H_*(K_n) = H_*(K)\end{aligned}}$$

Persistence Modules

- $K \subseteq K' \Rightarrow \xi_p : H_p(K) \to H_p(K')$
  - Inclusion maps induce homomorphisms in homology groups (under Z_2-coefficients, linear maps in vector spaces)

$$\boxed{\begin{array}{l} K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K \\ \Rightarrow \quad H_*(K_0) \to H_*(K_1) \to \cdots \to H_*(K_n) = H_*(K) \end{array}}$$

- Define $\xi_*^{i,j} : H_*(K_i) \to H_*(K_j)$
  - $\xi_*^{i,j} = \xi_*^{j-1,j} \circ \cdots \circ \xi_*^{i,i+1}$
- Persistent module induced by the filtration
  - $\mathscr{P} = \left\{ H_*(K_i) \xrightarrow{\xi^{i,j}} H_*(K_j) \right\}_{0 \le i \le j \le n}$

Persistence Vector Spaces

- A **persistence vector space** V over a field $\mathbb{F}$ is
  - a sequence of vector spaces $\{V_i\}_{i=0,\ldots,n}$
  - Together with maps $L_{i,j} : V_i \to V_j$ for $i \leq j$ such that
    - $L_{i,i} = Id_{V_i}$
    - For $i \leq j \leq k$, $L_{i,k} = L_{j,k} \circ L_{i,j}$
  - Write $V = \{L_{i,j} : V_i \to V_j\}$ or simply $V = \{V_i\}$

Persistence Vector Spaces

- Let $\{V_i\}$ and $\{W_i\}$ be two persistence vector spaces
  - a sequence of linear maps $\{\varphi_i : V_i \to W_i\}_{i=0,\ldots,n}$ is called a **linear transformation** from $\{V_i\}$ to $\{W_i\}$ if for any $i \leq j$

$$\begin{tikzcd}
V_i \arrow[r, "L^V_{i,j}"] \arrow[d, "\varphi_i"'] & V_j \arrow[d, "\varphi_j"] \\
W_i \arrow[r, "L^W_{i,j}"'] & W_j
\end{tikzcd}$$

Persistence Vector Spaces

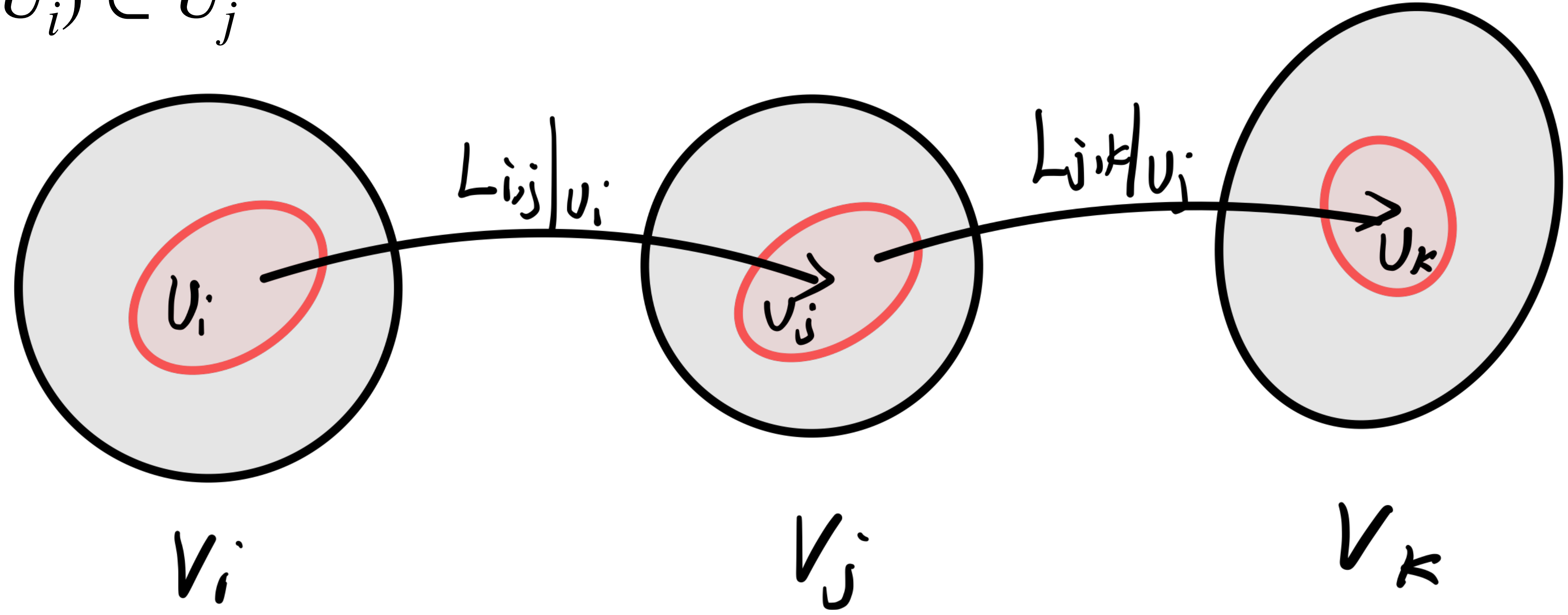
- Let $\{V_i\}$ and $\{W_i\}$ be two persistence vector spaces
  - a sequence of linear maps $\varphi = \{\varphi_i : V_i \to W_i\}_{i=0,\ldots,n}$ is called a **linear transformation** from $\{V_i\}$ to $\{W_i\}$ if for any $i \leq j$

$$\begin{tikzcd} V_i \arrow[r, "L^V_{i,j}"] \arrow[d, "\varphi_i"'] & V_j \arrow[d, "\varphi_j"] \\ W_i \arrow[r, "L^W_{i,j}"'] & W_j \end{tikzcd}$$

  - φ is called an isomorphism if each φ_i is an isomorphism

Persistence Vector Spaces

- A **sub-persistence vector space** is a collection $U = \{U_i \subset V_i\}$ such that
- $L_{i,j}(U_i) \subset U_j$



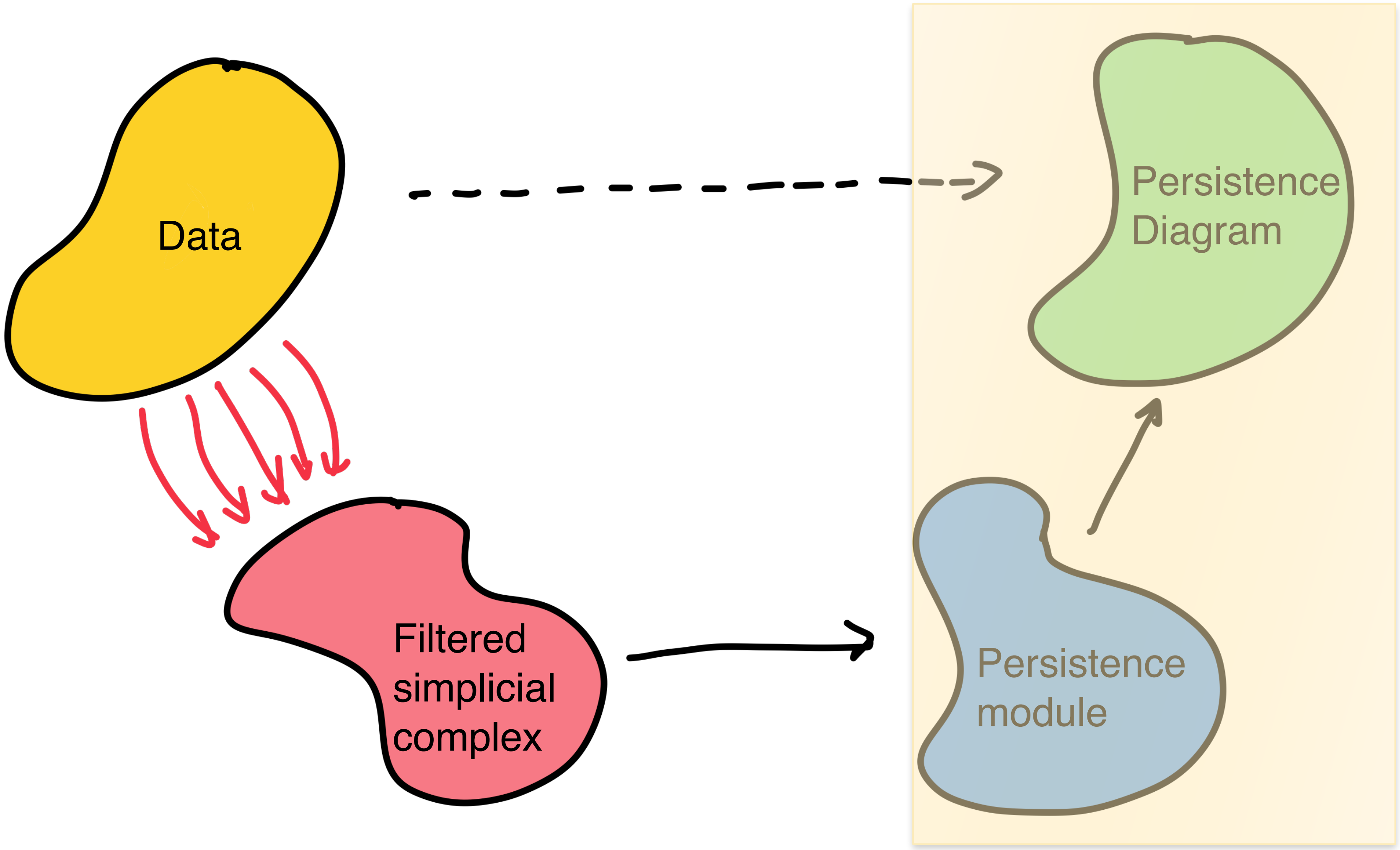
Persistence Vector Spaces

- Let $\{V_i\}$ and $\{W_i\}$ be two persistence vector spaces
- The **direct sum** $V \oplus W$ is the collection $\{V_i \oplus W_i\}$ with maps
- $L_{i,j}^{V \oplus W} = L_{i,j}^{V} \oplus L_{i,j}^{W}$ defined by $L_{i,j}^{V \oplus W}(v, w) = (L_{i,j}^{V}(v), L_{i,j}^{W}(w))$

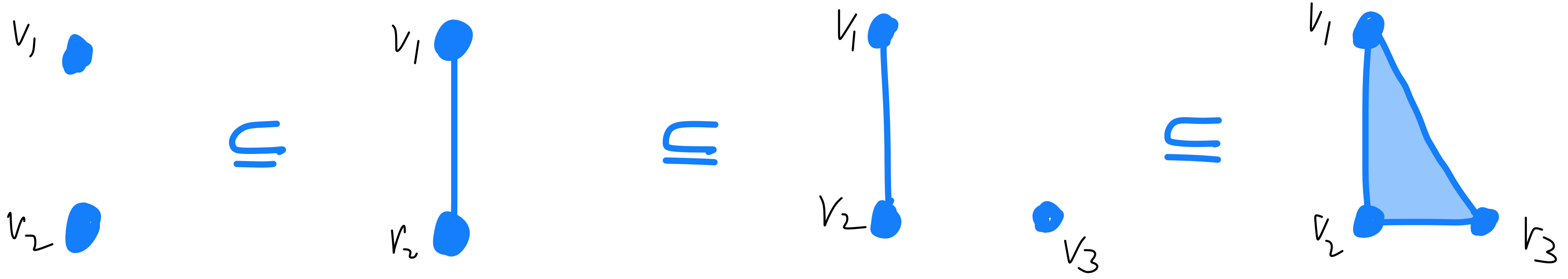
- Dimension and basis are the most important objects of a vector space

- What are “dimension” and “basis” for a persistence vector space?

Persistence Diagram



Persistent Module Example



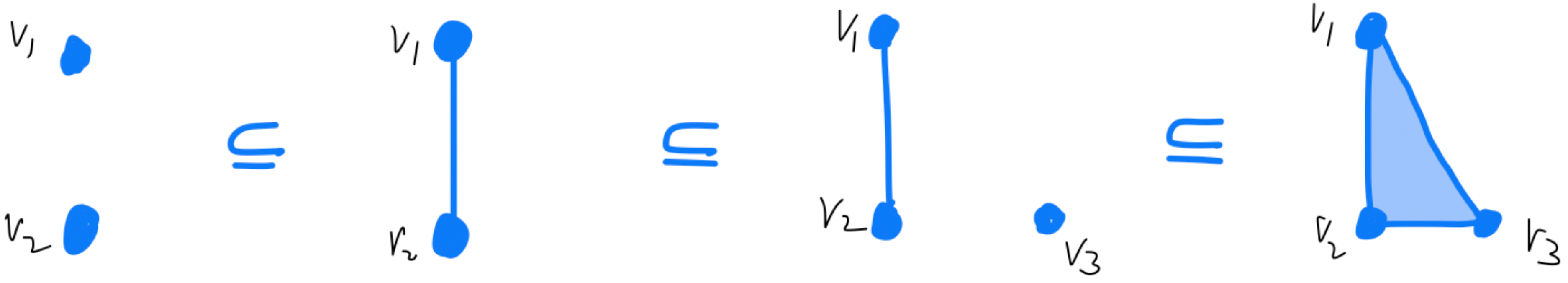
$$V_0 = \langle [v_1], [v_2] \rangle \longrightarrow V_1 = \langle [v_1] \rangle \longrightarrow V_2 = \langle [v_1], [v_3] \rangle \longrightarrow V_3 = \langle [v_1] \rangle$$

$$\xi^{0,1} = \begin{array}{cc} {\scriptstyle [v_1]} & {\scriptstyle [v_2]} \\ \left(1 \right., & \left. 1 \right) \end{array} {\scriptstyle [v_1]} \qquad \xi^{1,2} = \begin{array}{c} {\scriptstyle [v_1]} \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} {\scriptstyle [v_1]} \\ {\scriptstyle [v_3]} \end{array} \qquad \xi^{2,3} = \begin{array}{cc} {\scriptstyle [v_1]} & {\scriptstyle [v_3]} \\ \left[1 \right. & \left. 1 \right] \end{array} {\scriptstyle [v_1]}$$

$$\langle [v_1] \rangle \longrightarrow \langle [v_1] \rangle \longrightarrow \langle [v_1] \rangle \longrightarrow \langle [v_1] \rangle$$

$$\langle [v_2] - [v_1] \rangle \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

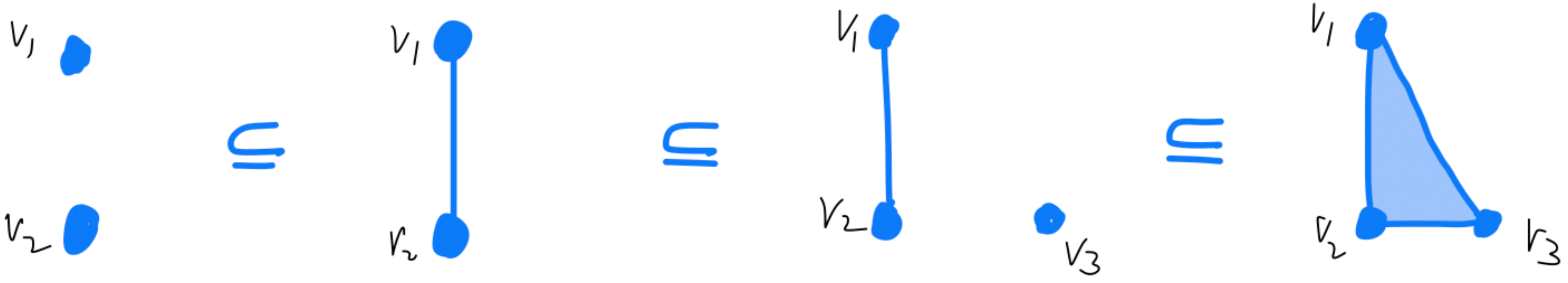
$$0 \longrightarrow 0 \xrightarrow{0} \langle [v_3] - [v_1] \rangle \longrightarrow 0$$



$$\langle [v_1] \rangle \to \langle [v_1] \rangle \to \langle [v_1] \rangle \to \langle [v_1] \rangle \quad \langle [v_1] - [v_2] \rangle \to 0 \to 0 \to 0 \quad 0 \to 0 \to \langle [v_3] - [v_1] \rangle \to 0$$

$$\cong \qquad \cong \qquad \cong$$

$$\mathbb{F} \to \mathbb{F} \to \mathbb{F} \to \mathbb{F} \quad \mathbb{F} \to 0 \to 0 \to 0 \quad 0 \to 0 \to \mathbb{F} \to 0$$



$$\langle [v_1] \rangle \to \langle [v_1] \rangle \to \langle [v_1] \rangle \to \langle [v_1] \rangle \qquad \langle [v_1] - [v_2] \rangle \to 0 \to 0 \to 0 \qquad 0 \to 0 \to \langle [v_3] - [v_1] \rangle \to 0$$

$$\cong \qquad\qquad \cong \qquad\qquad \cong$$

$$\mathbb{F} \to \mathbb{F} \to \mathbb{F} \to \mathbb{F} \qquad \mathbb{F} \to 0 \to 0 \to 0 \qquad 0 \to 0 \to \mathbb{F} \to 0$$

$$V_0 \to V_1 \to V_2 \to V_3 \cong \begin{array}{l} \mathbb{F} \to \mathbb{F} \to \mathbb{F} \to \mathbb{F} \\ \oplus\ \mathbb{F} \to 0 \to 0 \to 0 \\ \oplus\ 0 \to 0 \to \mathbb{F} \to 0 \end{array}$$

Interval persistence vector spaces

- Given the index set $I = \{0, \ldots, n\}$
- Let $0 \leq b < d \leq n + 1$, the **interval persistence vector space**, denoted by $I[b, d)$ is defined as
- $I[b, d) = 0 \to \cdots \to 0 \to \mathbb{F} \to \mathbb{F} \to \cdots \to \mathbb{F} \to 0 \to \cdots \to 0$

bth position $\quad$ $d - 1$th position

- $I[b, n + 1) = 0 \to \cdots \to 0 \to \mathbb{F} \to \mathbb{F} \to \cdots \to \mathbb{F}$ is often written as $I[b, \infty)$

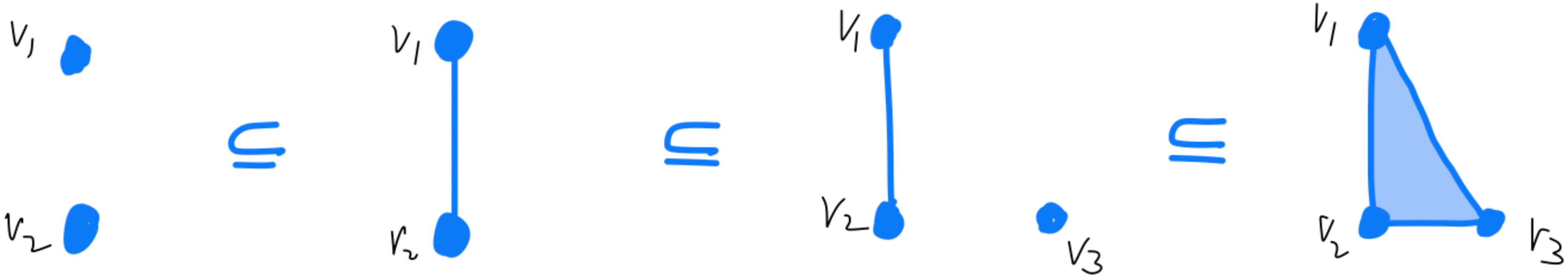
Decomposition Theorem

- Let $V = \{V_i\}_{i=0}^{n}$ be any persistence vector space. Then, there exist a collection of $0 \leq b_j < d_j \leq n+1, j = 1,\ldots, M$ such that
- $V \cong I[b_1, d_1) \oplus I[b_2, d_2) \oplus \cdots \oplus I[b_M, d_M)$
- The composition is unique up to reordering the summands.

Persistence Diagram and Barcodes

- $V \cong I[b_1, d_1) \oplus I[b_2, d_2) \oplus \cdots \oplus I[b_M, d_M)$
- Each (b_j, d_j) is called a **persistence pairing**
- The multiset $D = \{(b_j, d_j)\}_{j=1,\ldots,M} \subseteq \mathbb{R}^2$ is called the **persistence diagram** of V
- The collection of intervals $\{[b_j, d_j)\}_{j=1,\ldots,M}$ is called the **barcode** of V

Example



$$\langle [v_1] \rangle \to \langle [v_1] \rangle \to \langle [v_1] \rangle \to \langle [v_1] \rangle$$

$$\cong$$

$$\mathbb{F} \to \mathbb{F} \to \mathbb{F} \to \mathbb{F}$$

$$\langle [v_1 - v_2] \rangle \to 0 \to 0 \to 0$$

$$\cong$$

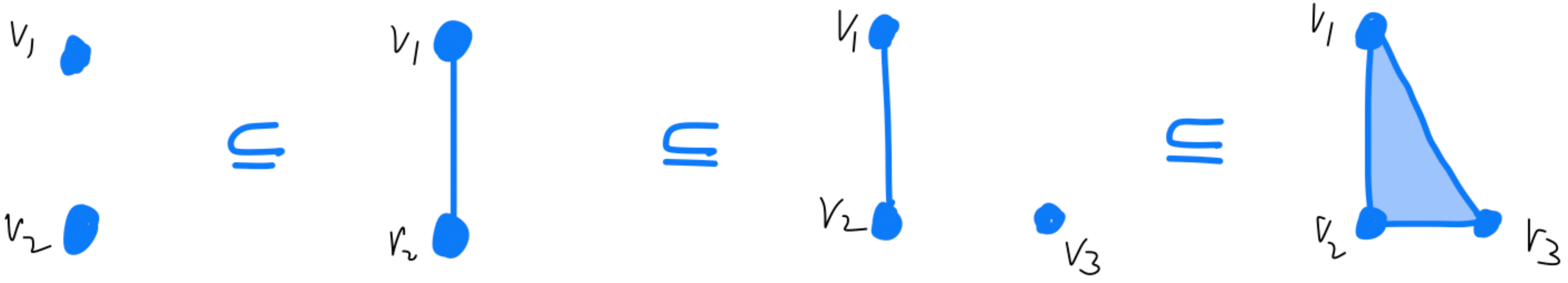
$$\mathbb{F} \to 0 \to 0 \to 0$$

$$0 \to 0 \to \langle [v_3] - [v_1] \rangle \to 0$$

$$\cong$$

$$0 \to 0 \to \mathbb{F} \to 0$$

Example



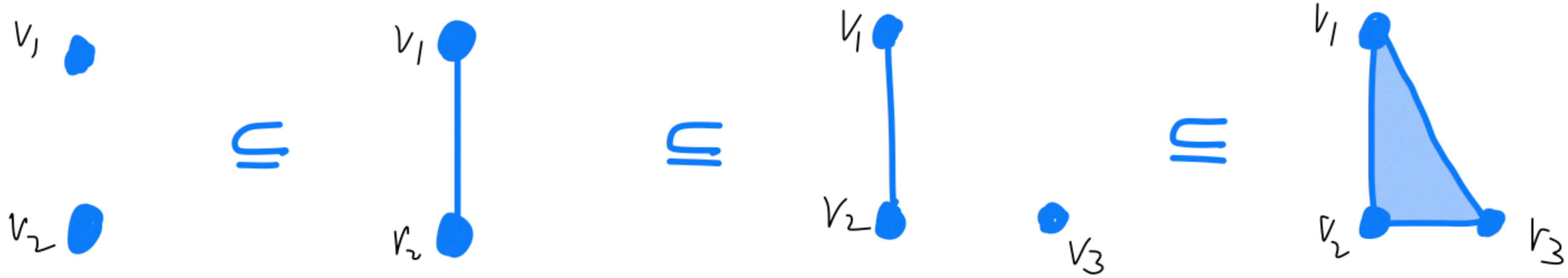
$$\langle [v_1] \rangle \to \langle [v_1] \rangle \to \langle [v_1] \rangle \to \langle [v_1] \rangle \qquad \langle [v_1 - v_2] \rangle \to 0 \to 0 \to 0 \qquad 0 \to 0 \to \langle [v_3] - [v_1] \rangle \to 0$$

$$\cong \qquad \cong \qquad \cong$$

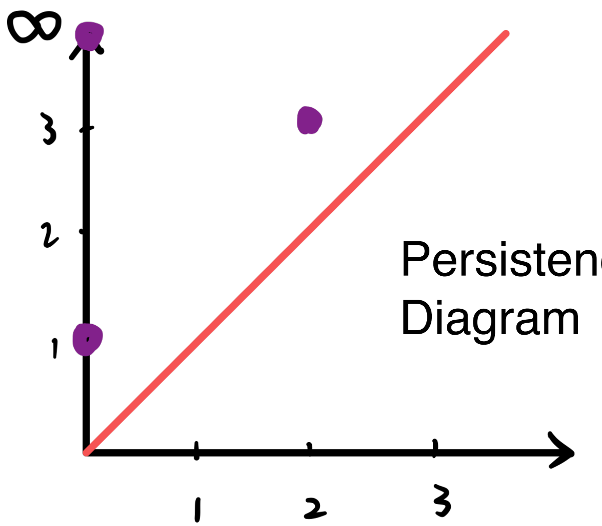
$$\mathbb{F} \to \mathbb{F} \to \mathbb{F} \to \mathbb{F} \qquad \mathbb{F} \to 0 \to 0 \to 0 \qquad 0 \to 0 \to \mathbb{F} \to 0$$

$$\boxed{V_0 \to V_1 \to V_2 \to V_3 \cong \begin{array}{l} \quad\ \mathbb{F} \to \mathbb{F} \to \mathbb{F} \to \mathbb{F} \\ \oplus\ \mathbb{F} \to 0 \to 0 \to 0 \\ \oplus\ 0 \to 0 \to \mathbb{F} \to 0 \end{array}}$$

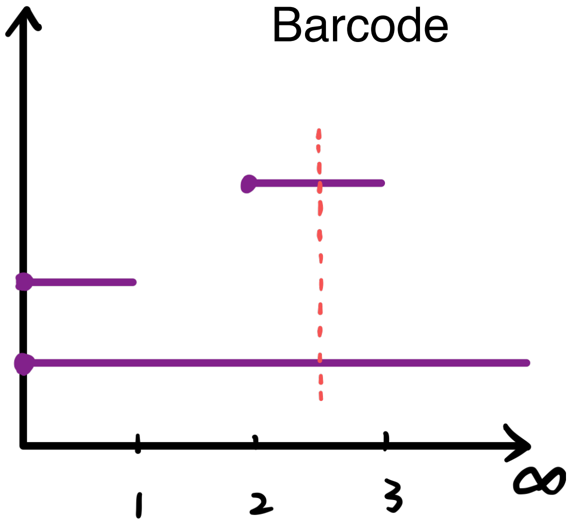
Example



$$V_0 \to V_1 \to V_2 \to V_3 \cong I[0,\infty) \oplus I[0,1) \oplus I[2,3)$$



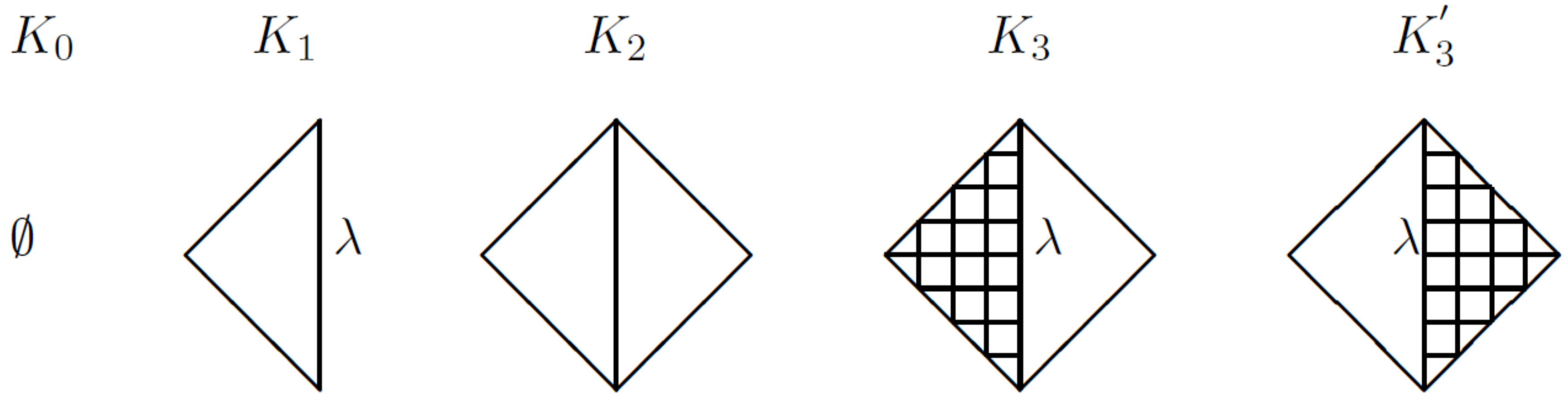
Persistence Diagram

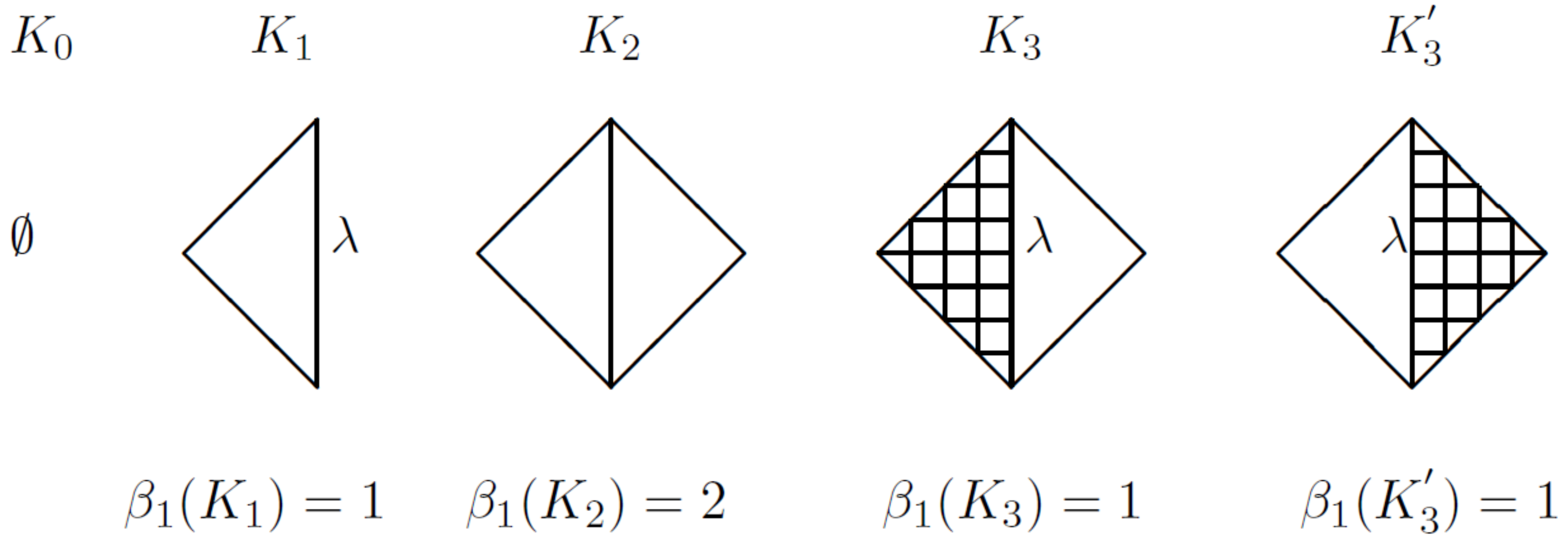


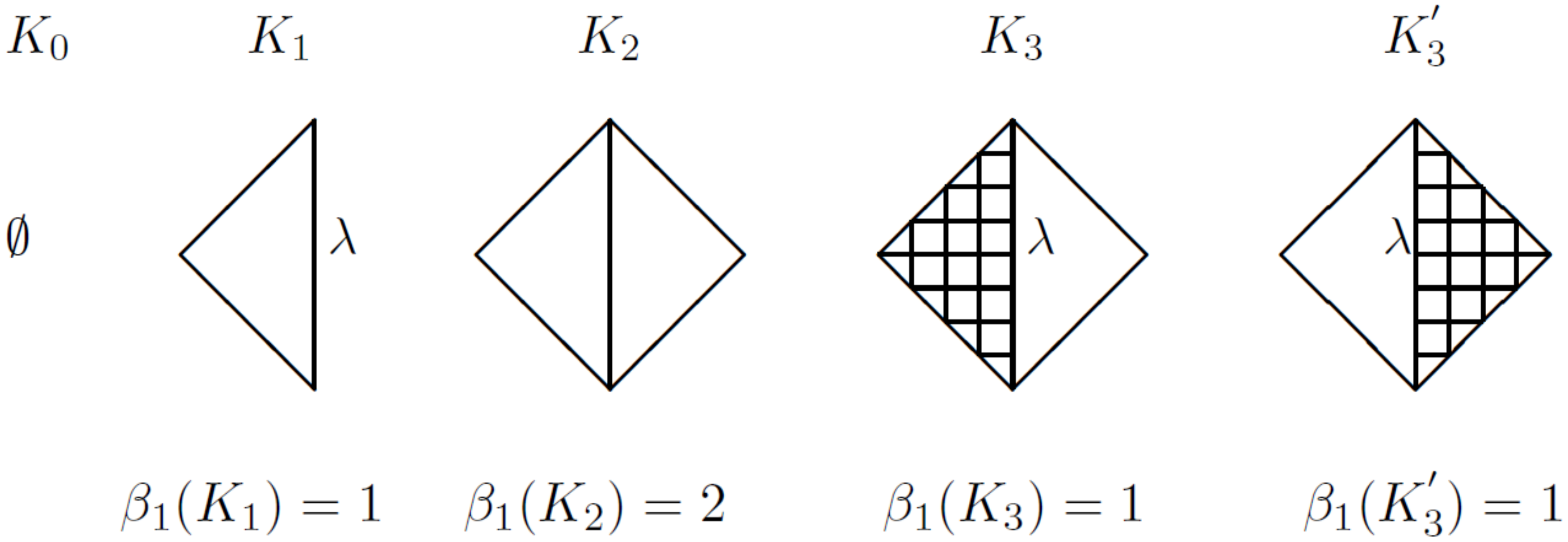
Barcode

Persistence Diagram and Barcodes for filtrations

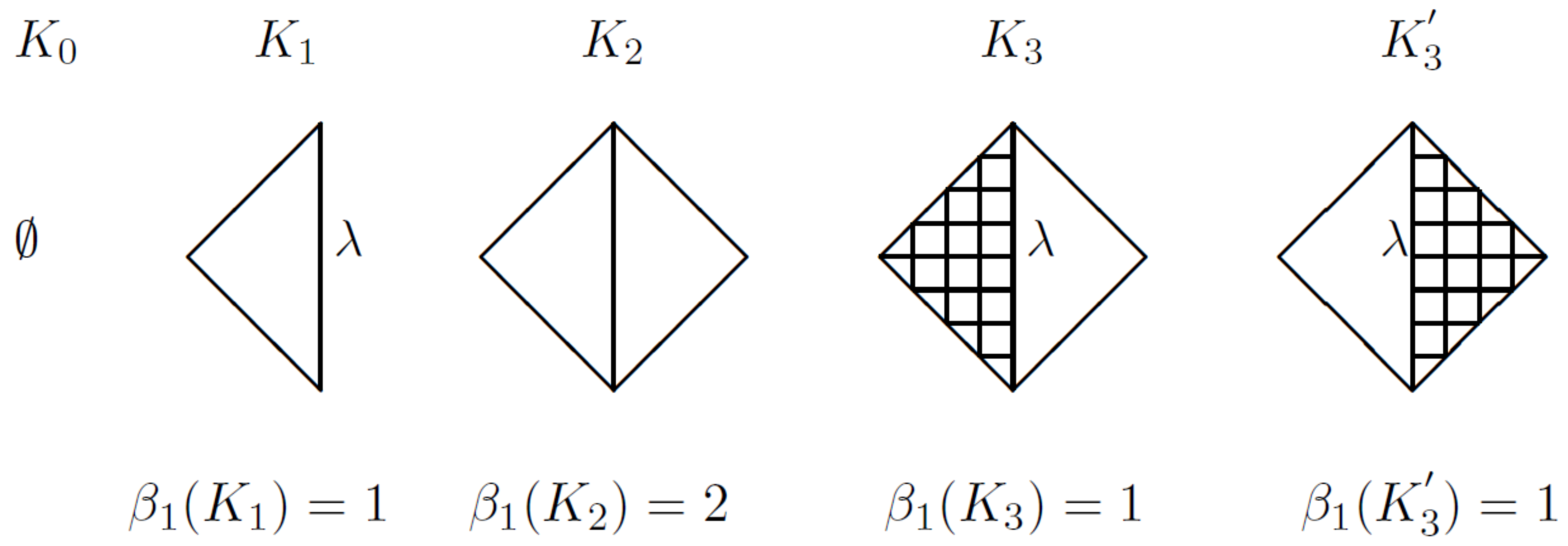
- Let $V = \{V_i = H_p(K_i)\}_{i=0}^{n}$ be the p-dim persistence module for the filtered simplicial complex $K = \{K_i\}$
- Assume that $V \cong I[b_1, d_1) \oplus I[b_2, d_2) \oplus \cdots \oplus I[b_M, d_M)$
- The multiset $Dgm_p(K) = \{(b_j, d_j)\}_{j=1,\ldots,M} \subseteq \mathbb{R}^2$ is called the degree p **persistence diagram** of K
  - $\mu_p^{b,d}$ denotes the multiplicity of (b, d): it denotes the number of independent homology classes created at K_b and died entering K_d


K_0
K_1
K_2
K_3
K_3'
$\emptyset$
λ
λ
λ


K_0
K_1
K_2
K_3
K_3'
$\emptyset$
λ
λ
λ
$\beta_1(K_1) = 1$
$\beta_1(K_2) = 2$
$\beta_1(K_3) = 1$
$\beta_1(K_3') = 1$

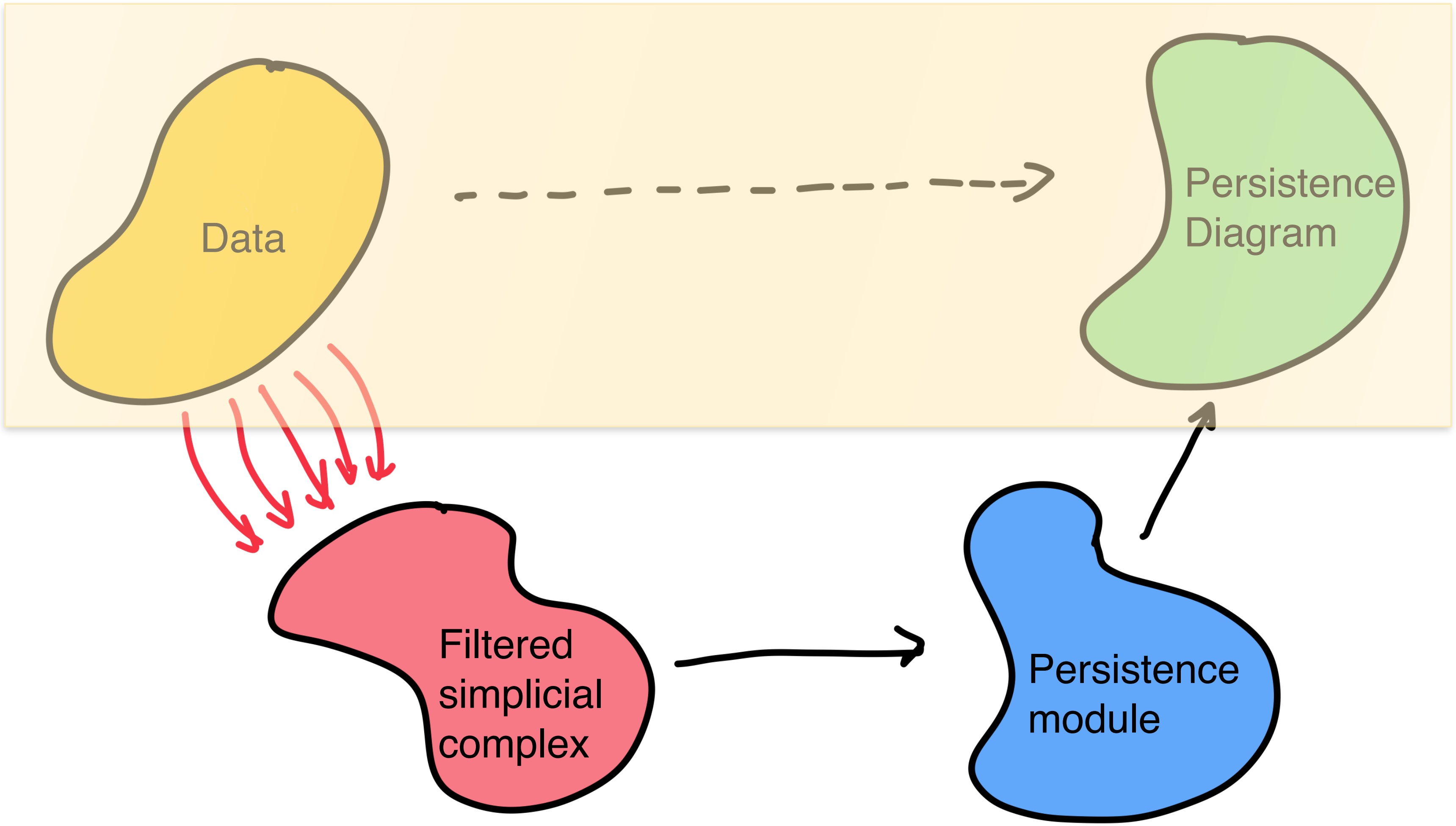


- For $K_0 \subset K_1 \subset K_2 \subset K_3$, what is $\mu_1^{1,3}$? $\mu_1^{1,2}$?



- For $K_0 \subset K_1 \subset K_2 \subset K_3$, what is $\mu_1^{1,3}$? $\mu_1^{1,2}$?
- How about for the filtration $K_0 \subset K_1 \subset K_2 \subset K_3'$?

TDA in a nutshell



Möbius Inversion - counting the number of persistence pairings

Persistence Modules

- $K \subseteq K' \;\Rightarrow \xi_p : H_p(K) \to H_p(K')$
  - Inclusion maps induce homomorphisms in homology groups (under Z_2-coefficients, linear maps in vector spaces)

$$\boxed{\begin{array}{l} K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K \\ \Rightarrow \quad H_*\left(K_0\right) \to H_*\left(K_1\right) \to \cdots \to H_*\left(K_n\right) = H_*(K) \end{array}}$$

- Define $\xi_*^{i,j} : H_*\left(K_i\right) \to H_*\left(K_j\right)$
  - $\xi_*^{i,j} = \xi_*^{j-1,\,j} \circ \cdots \circ \xi_*^{i,i+1}$
- Persistent module induced by the filtration
  - $\mathscr{P} = \left\{ H_*\left(K_i\right) \xrightarrow{\xi^{i,j}} H_*\left(K_j\right) \right\}_{0 \le \mathrm{i} \le \mathrm{j} \le n}$

Persistent Homology

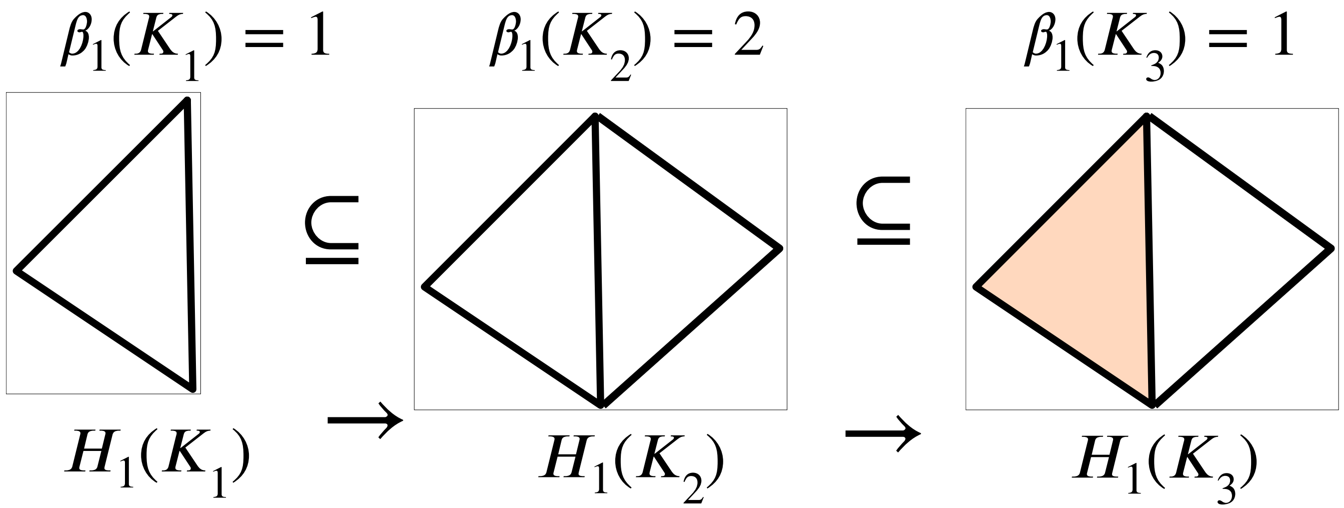
- p-th persistent homology group from i to j:
  - $(H_p(K_j) \supset)\, H_p^{i,j} = \mathrm{Im}(\xi_p^{i,j})$
    - Subgroup of $H_p\left(K_j\right)$ that ``existed'' in $H_p\left(K_i\right)$

Persistent Homology

- p-th persistent homology group from i to j:
  - $(H_p(K_j) \supset)\, H_p^{i,j} = \mathrm{Im}(\xi_p^{i,j} : H_p(K_i) \to H_p(K_j))$
    - Subgroup of $H_p\left(K_j\right)$ that ``existed'' in $H_p\left(K_i\right)$
- p-th persistent betti number: $\beta_p^{i,j} = \dim H_p^{i,j}$
- $\beta_p^{i,j}$ denotes the number of homology classes existing at both K_i and K_j

Persistent Homology

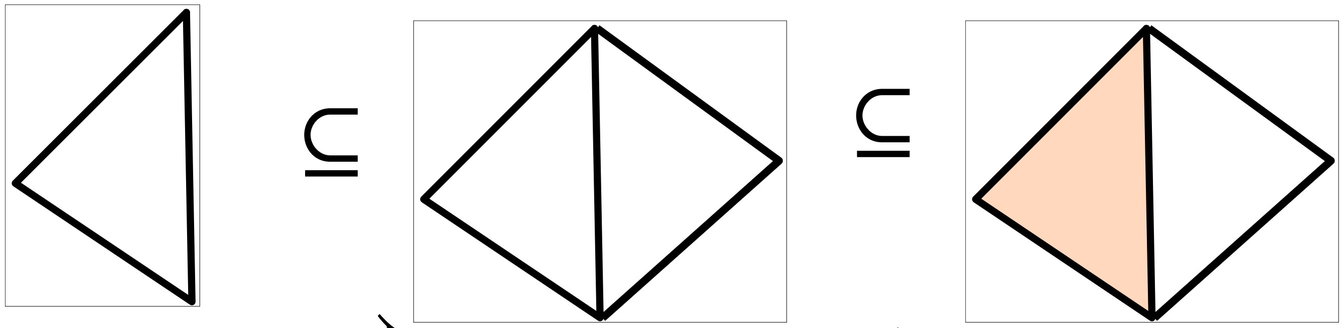
- p-th persistent homology group from i to j, where $0 \leq i \leq j \leq n$:
  - $(H_p(K_j) \supset)\ H_p^{i,j} = \mathrm{Im}(\xi_p^{i,j})$
    - Subgroup of $H_p\left(K_j\right)$ that ``existed'' in $H_p\left(K_i\right)$
- p-th persistent betti number: $\beta_p^{i,j} = \dim H_p^{i,j}$
- $\beta_p^{i,j}$ denotes the number of homology classes existing at both K_i and K_j



Persistent Homology

- p-th persistent homology group from i to j:
  - $(H_p(K_j) \supset)\ H_p^{i,j} = \mathrm{Im}(\xi_p^{i,j})$
    - Subgroup of $H_p\left(K_j\right)$ that ``existed'' in $H_p\left(K_i\right)$
- p-th persistent betti number: $\beta_p^{i,j} = \dim H_p^{i,j}$
- $\beta_p^{i,j}$ denotes the number of homology classes existing at both K_i and K_j

$\beta_1(K_1) = 1$ $\beta_1(K_2) = 2$ $\beta_1(K_3) = 1$



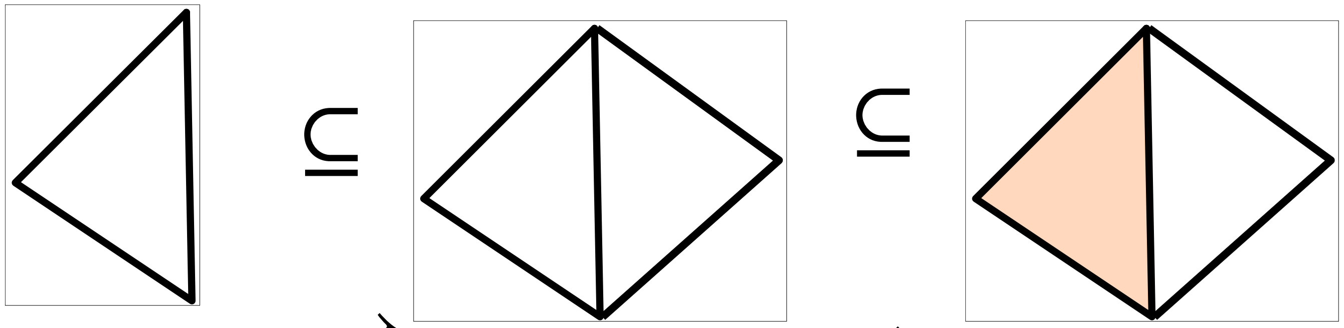
$H_1(K_1) \rightarrow H_1(K_2) \rightarrow H_1(K_3)$

$\beta_1^{1,3} = ?\ \beta_1^{1,2} = ?$

Persistent Homology

- p-th persistent homology group from i to j:
  - $(H_p(K_j) \supset)\ H_p^{i,j} = \mathrm{Im}(\xi_p^{i,j})$
    - Subgroup of $H_p\left(K_j\right)$ that ``existed'' in $H_p\left(K_i\right)$
- p-th persistent betti number: $\beta_p^{i,j} = \dim H_p^{i,j}$
- $\beta_p^{i,j}$ denotes the number of homology classes existing at both K_i and K_j

$\beta_1(K_1) = 1$ $\quad\beta_1(K_2) = 2$ $\quad\beta_1(K_3) = 1$



$H_1(K_1) \rightarrow H_1(K_2) \rightarrow H_1(K_3)$

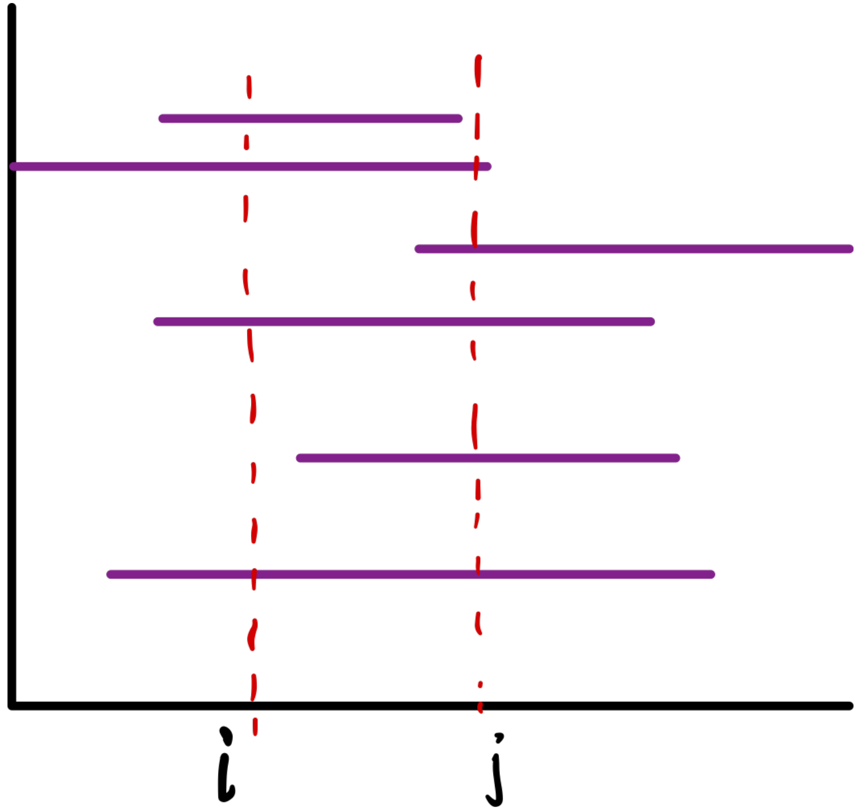
$\beta_1^{1,3} = 0 \quad \beta_1^{1,2} = 1$

Connection to decomposition theorem

- Let $V = \{V_i = H_p(K_i)\}_{i=0}^{n}$ be the p-dim persistence module for the filtered simplicial complex $K = \{K_i\}$
- Assume that $V \cong I[b_1, d_1) \oplus I[b_2, d_2) \oplus \cdots \oplus I[b_M, d_M)$
- $\mu^{b,d}$:= number of intervals $I[b, d)$

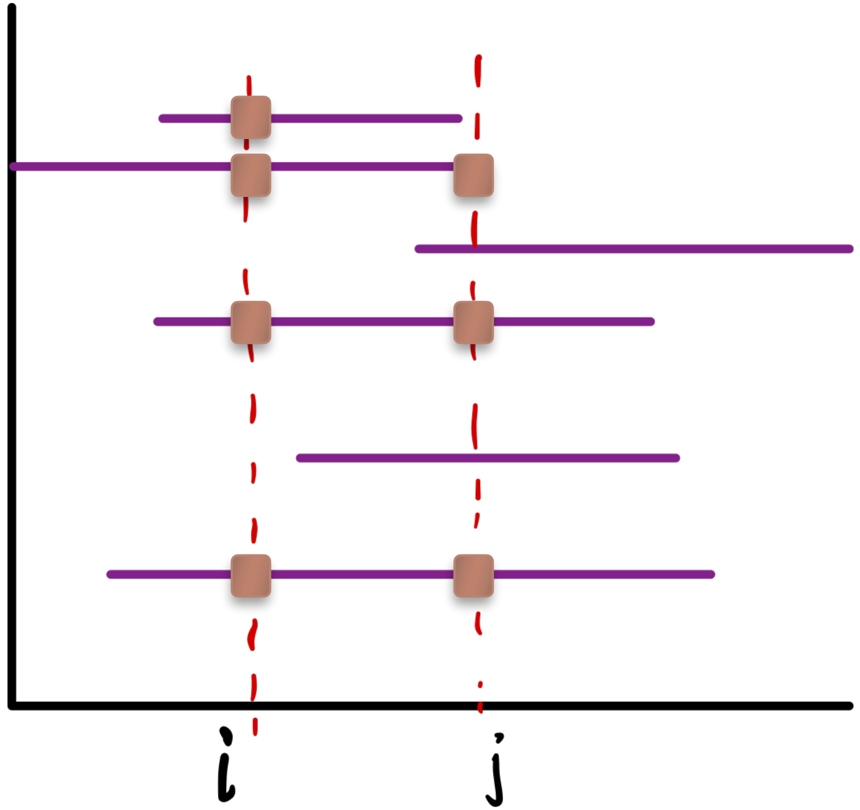
Persistence Betti Number vs Barcode

$$\beta_p^{i,j} = \dim H_p^{i,j}$$



Persistence Betti Number vs Barcode

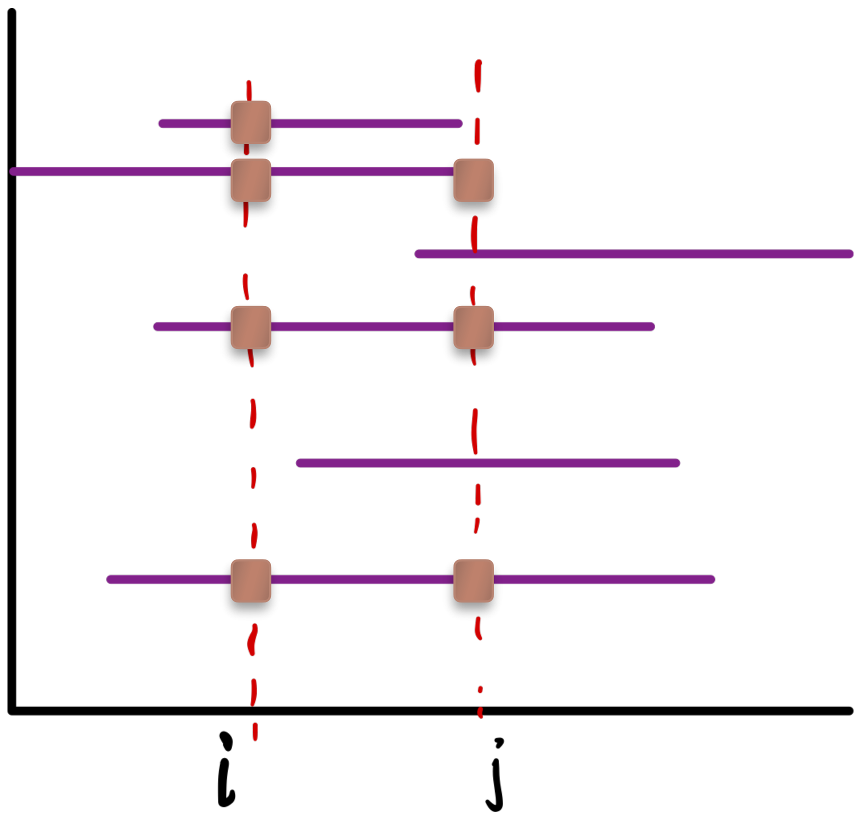
$$\beta_p^{i,j} = \dim H_p^{i,j}$$



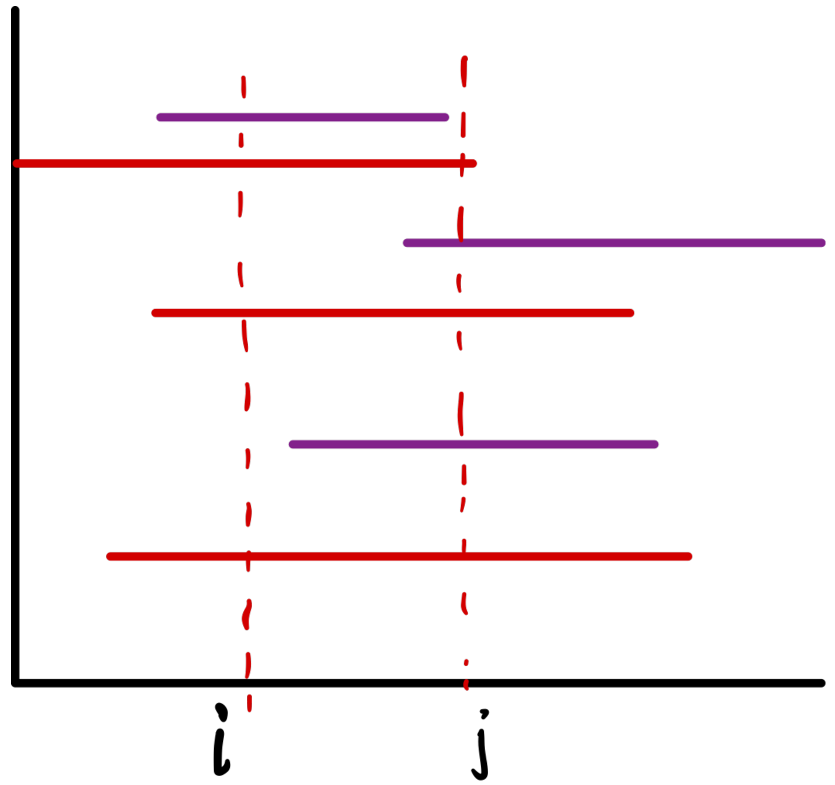
Persistence Betti Number vs Barcode

$$\beta_p^{i,j} = \dim H_p^{i,j}$$

$$\beta_p^{i,j} = 3$$

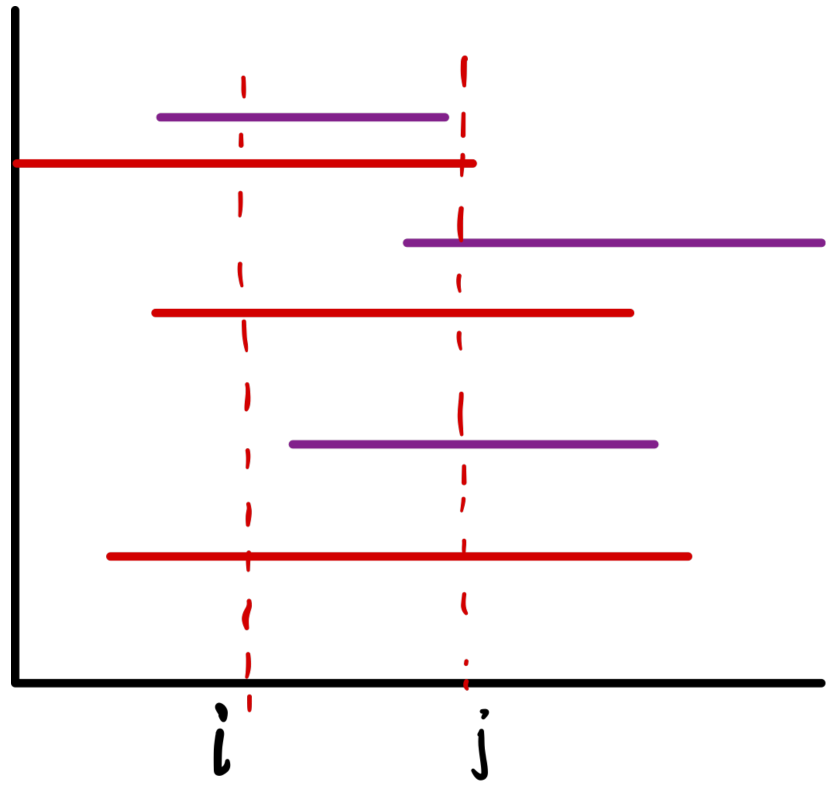


Persistence Betti Number vs Barcode



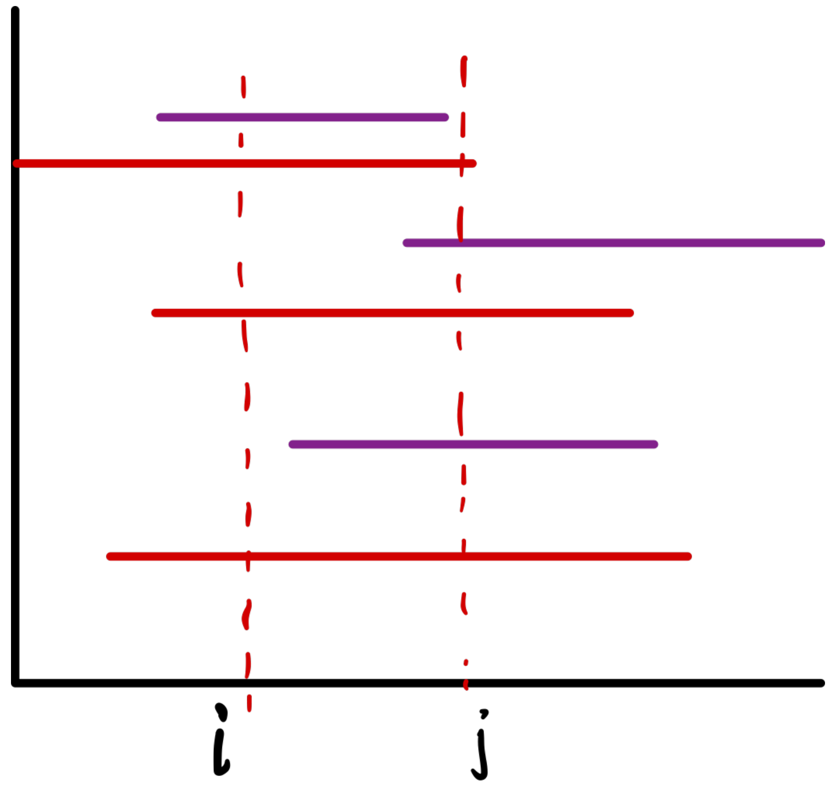
Persistence Betti Number vs Barcode

- $\beta_p^{i,j}$ = # of bars crossing both vertical lines

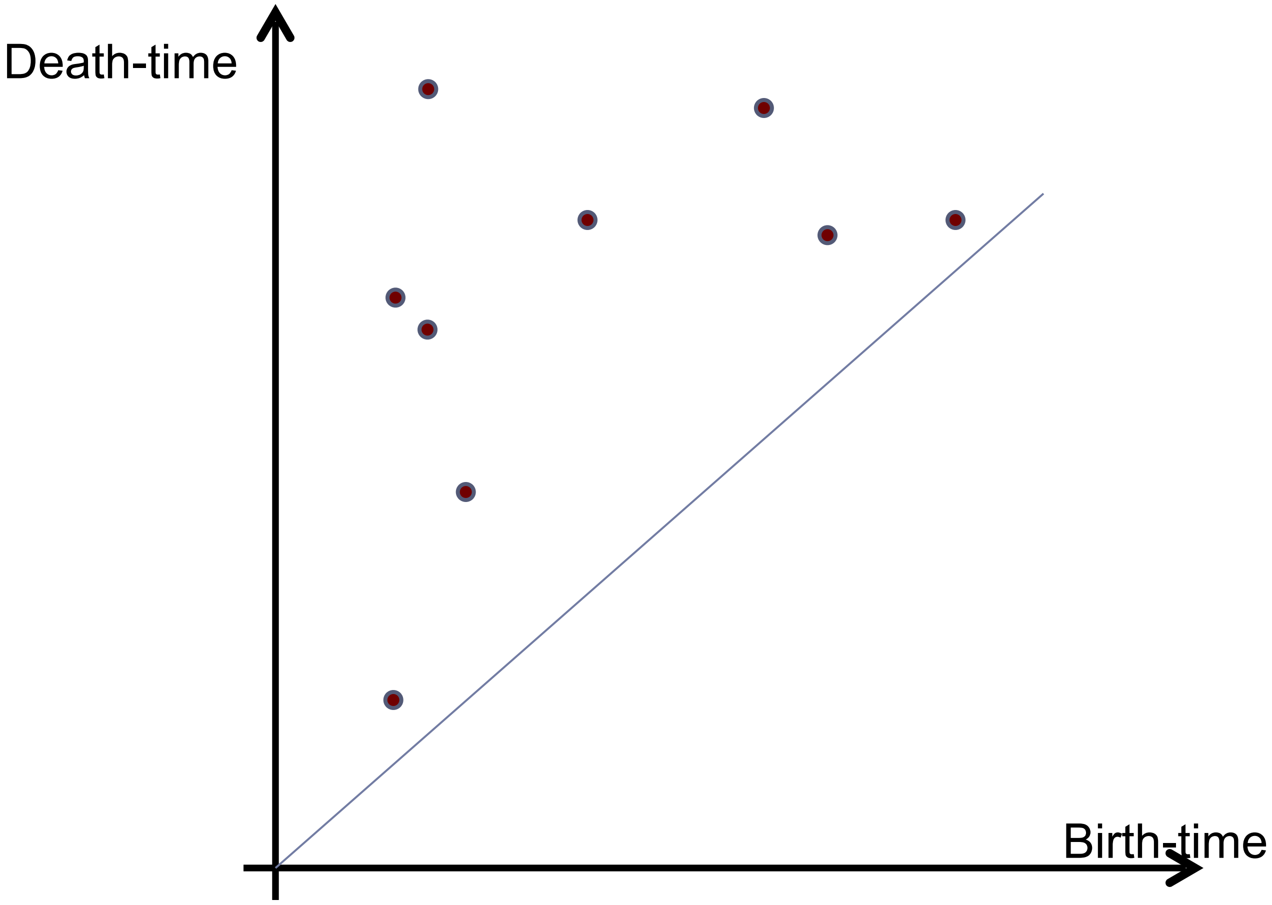


Persistence Betti Number vs Barcode

- $\beta_p^{i,j}$ = # of bars crossing both vertical lines
- $\beta_p^{i,j} = \sum_{k \le i, j < l} \mu_p^{k,l}$

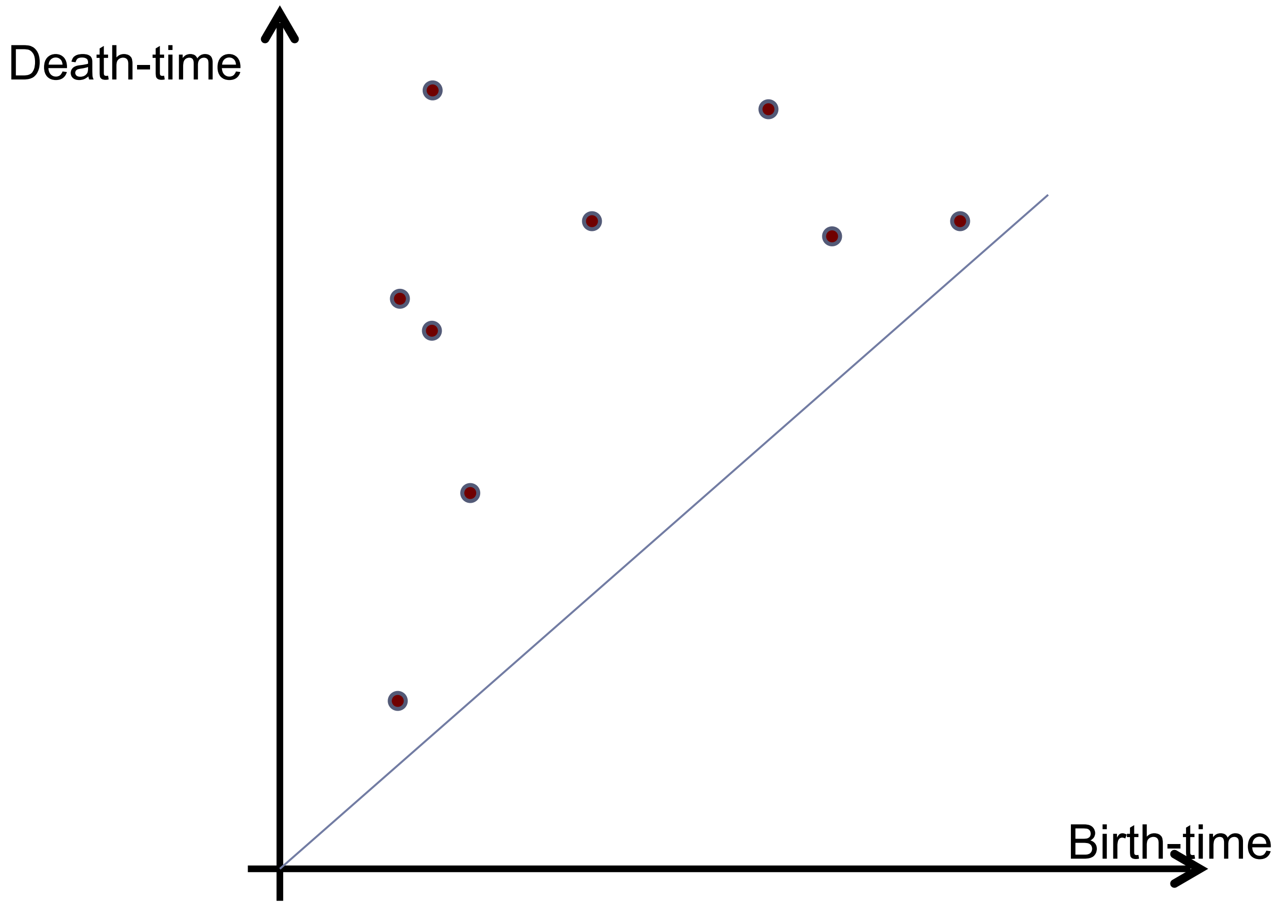


Persistence Betti Number vs Persistence Diagram



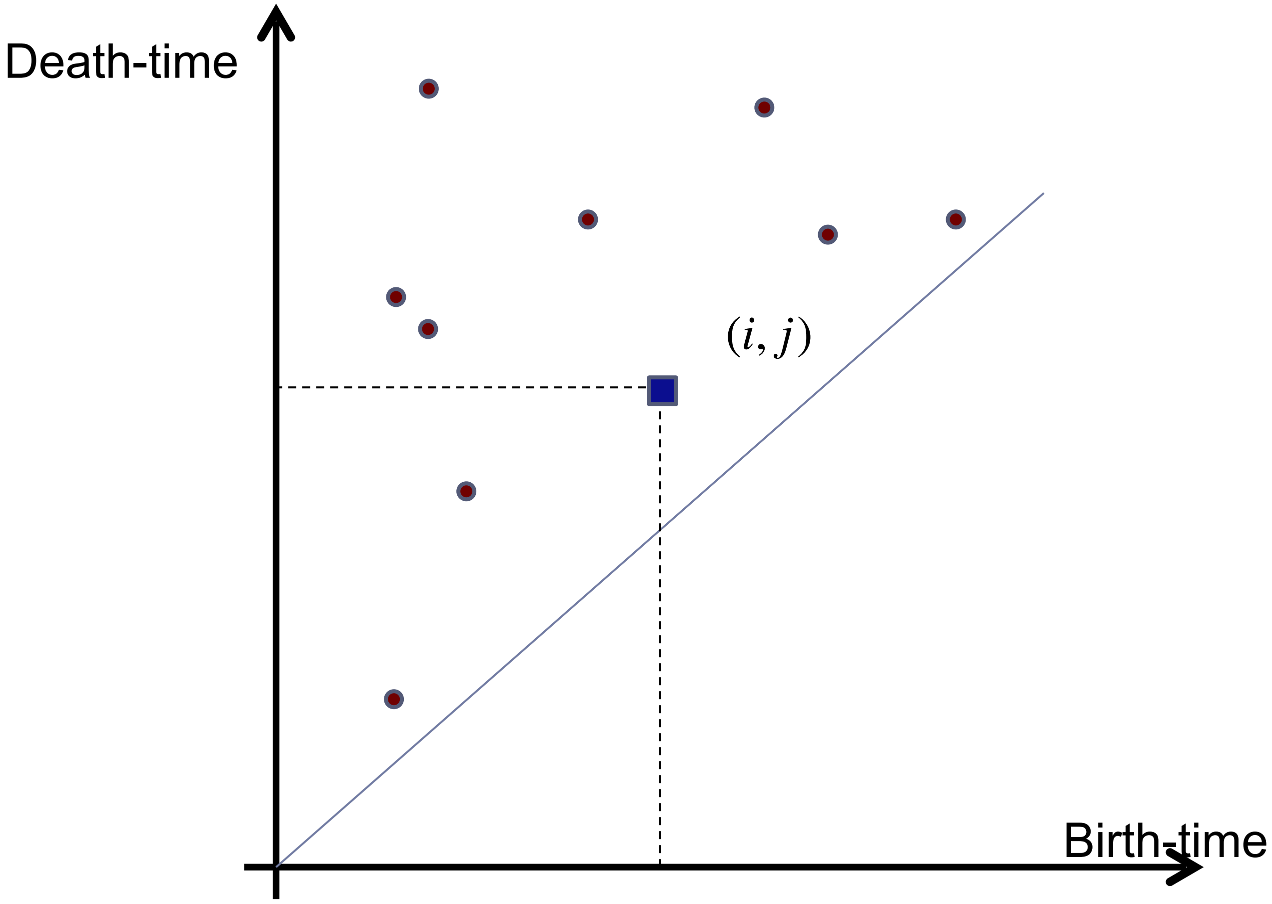
Persistence Betti Number vs Persistence Diagram

- $$\beta_p^{i,j} = \sum_{k \leq i, j < l} \mu_p^{k,l}$$



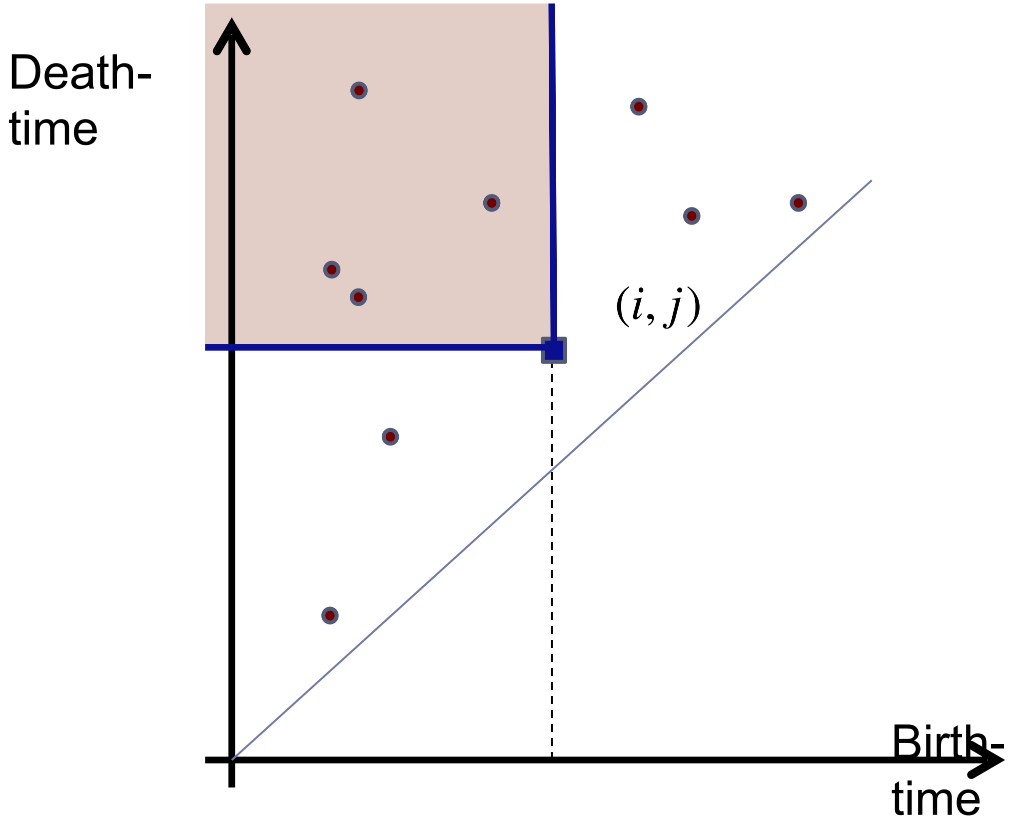
Persistence Betti Number vs Persistence Diagram

- $$\beta_p^{i,j} = \sum_{k \leq i, j < l} \mu_p^{k,l}$$



Persistence Betti Number vs Persistence Diagram

- Theorem:
  - $$\beta_p^{i,j} = \sum_{k \le i, j < l} \mu_p^{k,l}$$



Möbius inversion

- For $0 \leq i < j \leq n+1$, the multiplicity of (i, j) can be computed as follows
  - $\mu_p^{i,j} = (\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j})$
  - $\beta_p^{-1,j} = \beta^{i,n+1} = 0$

Möbius inversion

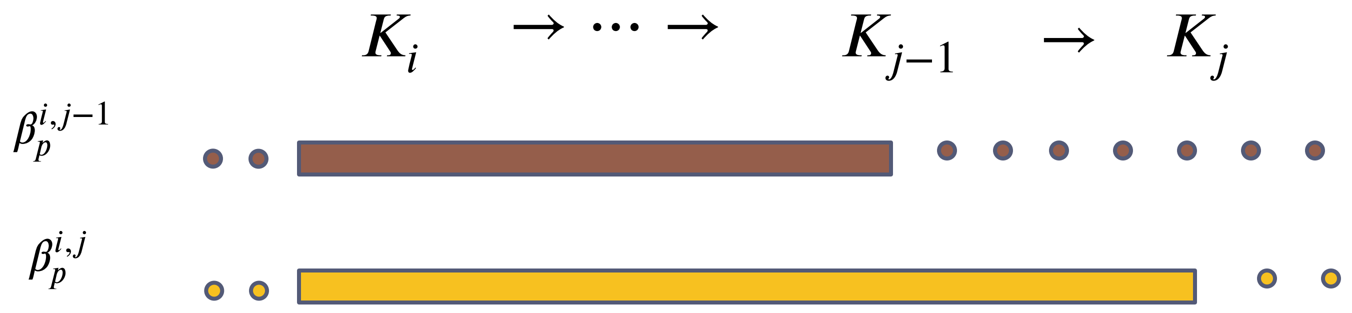
- Persistent pairing number:
  - $\mu_p^{i,j} = \underbrace{(\beta_p^{i,j-1} - \beta_p^{i,j})} - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j})$

$$K_i \to \cdots \to K_{j-1} \to K_j$$

$\beta_p^{i,j-1}$

Möbius inversion

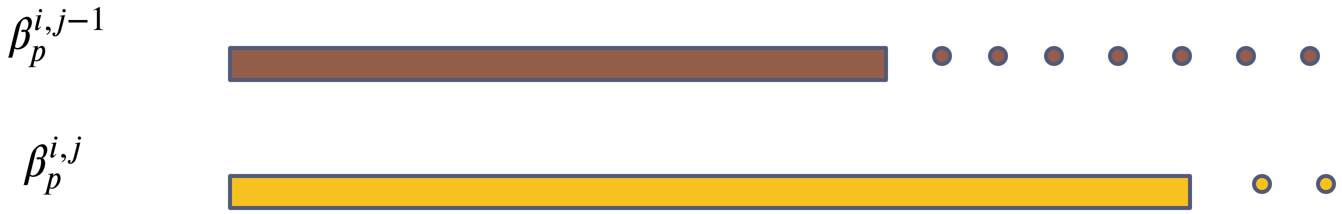
- Persistent pairing number:
  - $\mu_p^{i,j} = \underbrace{(\beta_p^{i,j-1} - \beta_p^{i,j})}_{} - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j})$



Möbius inversion

- Persistent pairing number:
  - $\mu_p^{i,j} = \underbrace{(\beta_p^{i,j-1} - \beta_p^{i,j})}_{\text{Number of independent homology classes from } K_i \text{ but died entering } K_j} - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j})$

$$K_i \quad \to \cdots \to \quad K_{j-1} \quad \to \quad K_j$$



Möbius inversion

- Persistent pairing number:
  - $\mu_p^{i,j} = (\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j})$

Number of independent homology classes from K_i but died entering K_j

Number of independent homology classes from K_{i-1} but died entering K_j

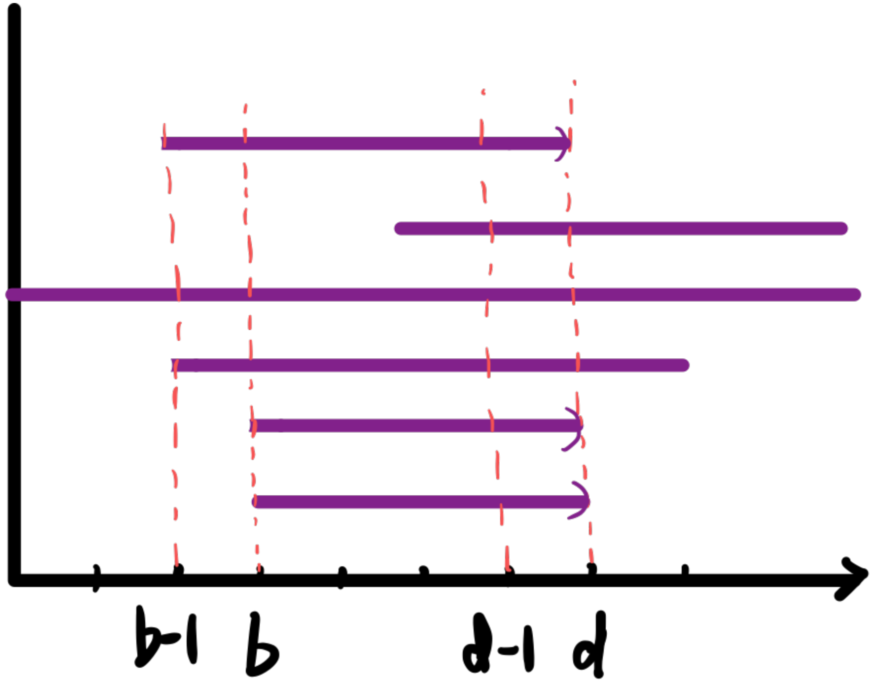
$$K_{i-1} \to K_i \to \cdots \to K_{j-1} \to K_j$$

Möbius inversion

- Persistent pairing number:
  - $$\mu_p^{i,j} = \underbrace{(\beta_p^{i,j-1} - \beta_p^{i,j})}_{\boxed{\begin{array}{c}\text{Number of independent}\\ \text{homology classes from}\\ K_i \text{ but died entering } K_j\end{array}}} - \underbrace{(\beta_p^{i-1,j-1} - \beta_p^{i-1,j})}_{\boxed{\begin{array}{c}\text{Number of independent}\\ \text{homology classes from}\\ K_{i-1} \text{ but died entering } K_j\end{array}}}$$
  - $\mu_p^{i,j}$ denotes the number of independent homology classes created at K_i and died entering K_j

Example

- $\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$



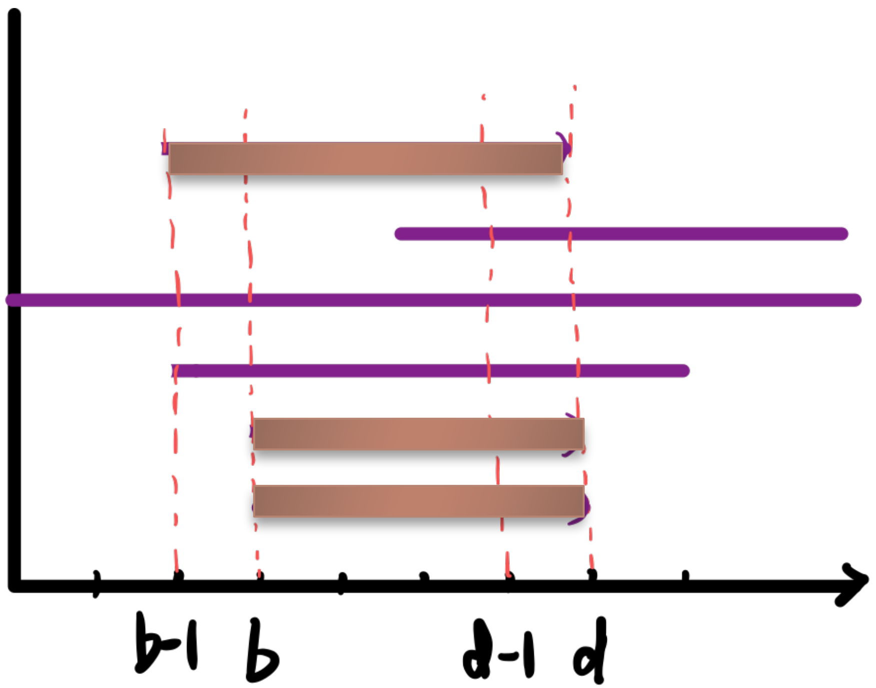
$\# I[b,d) = 2$

$\beta^{b,d-1} = 5, \ \beta^{b,d} = 2, \ \beta^{b-1,d-1} = 2, \ \beta^{b-1,d} = 1$

$\mu^{b,d} = (5-2) - (2-1) = 2$

Example

- $\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$



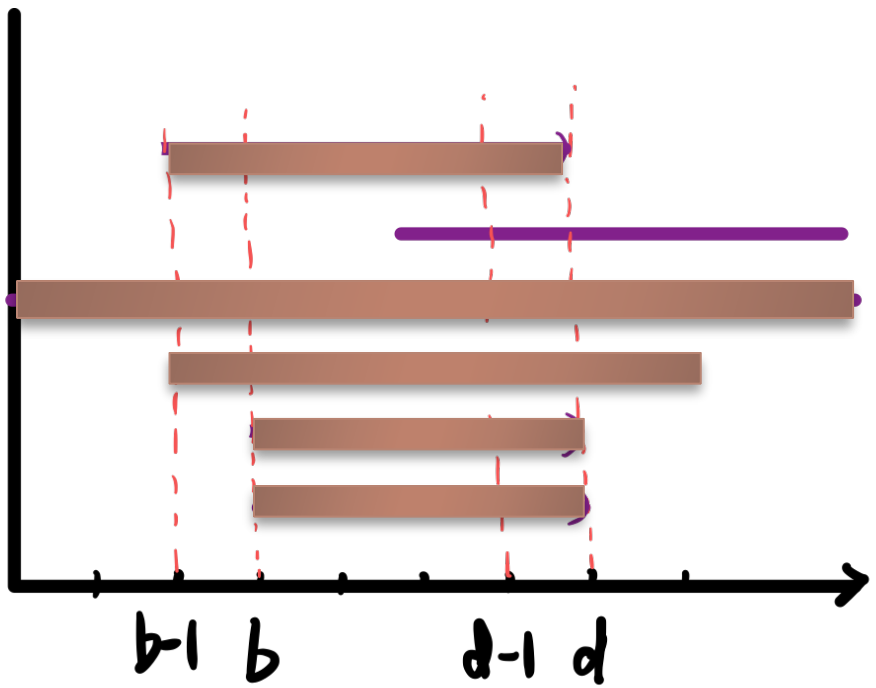
$\# I[b,d) = 2$

$\beta^{b,d-1} = 5, \ \beta^{b,d} = 2, \ \beta^{b-1,d-1} = 2, \ \beta^{b-1,d} = 1$

$\mu^{b,d} = (5-2) - (2-1) = 2$

Example

- $\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$



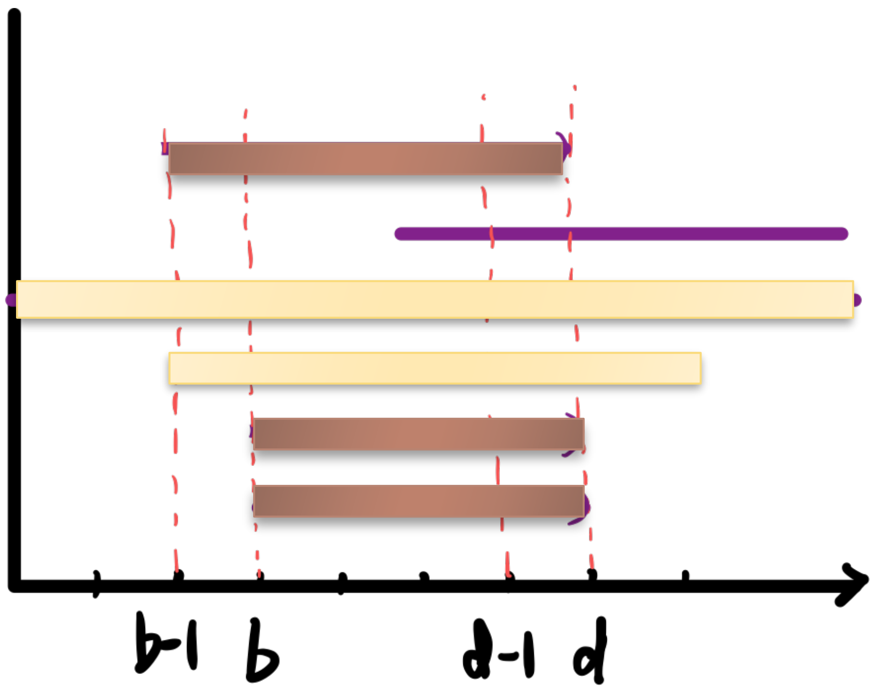
$\# I[b,d) = 2$

$\beta^{b,d-1} = 5, \ \beta^{b,d} = 2, \ \beta^{b-1,d-1} = 2, \ \beta^{b-1,d} = 1$

$\mu^{b,d} = (5-2) - (2-1) = 2$

Example

- $\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$



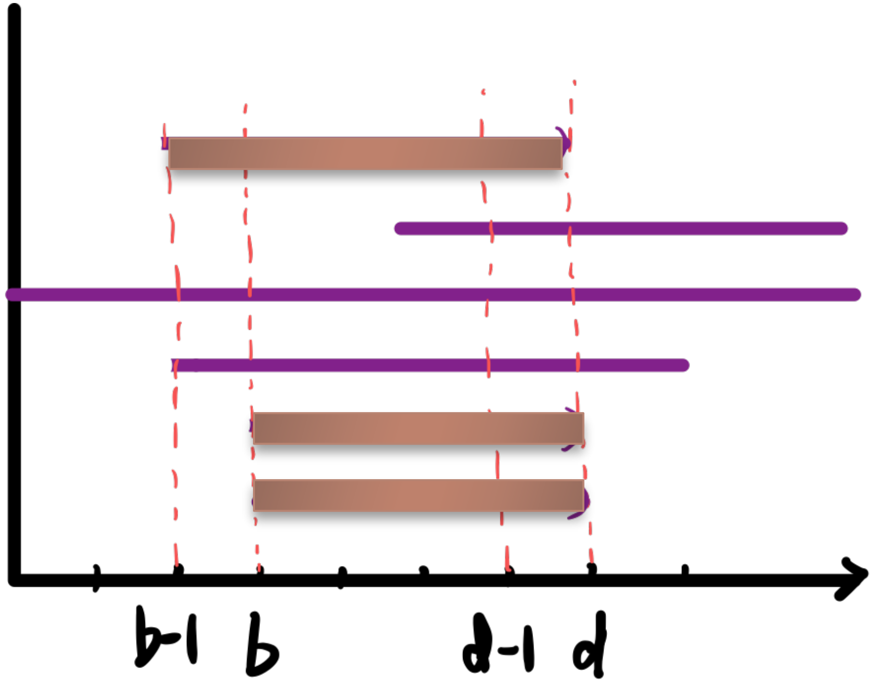
$\# I[b,d) = 2$

$\beta^{b,d-1} = 5, \ \beta^{b,d} = 2, \ \beta^{b-1,d-1} = 2, \ \beta^{b-1,d} = 1$

$\mu^{b,d} = (5-2) - (2-1) = 2$

Example

- $\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$



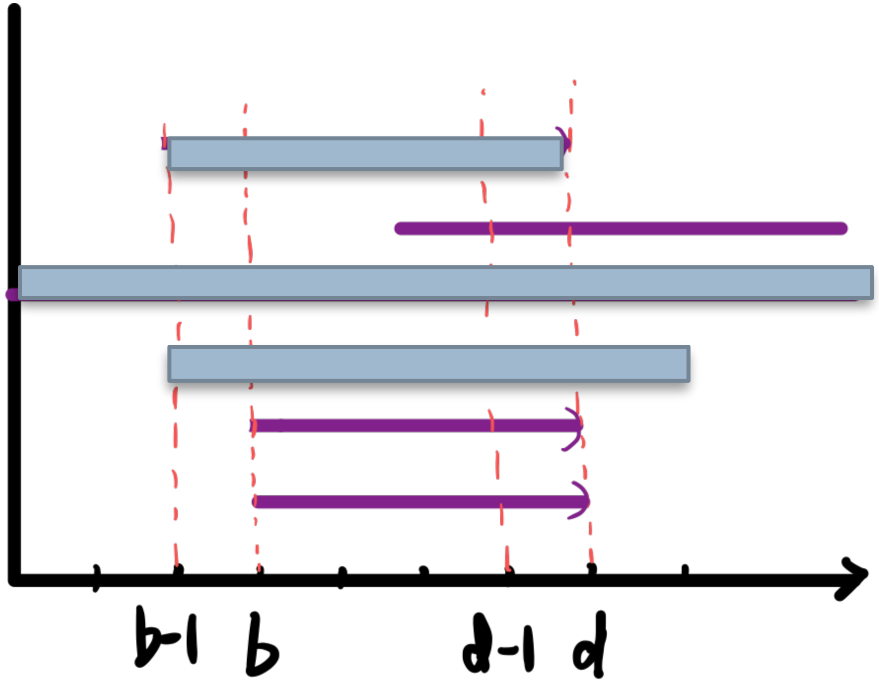
$\# I[b,d) = 2$

$\beta^{b,d-1} = 5, \ \beta^{b,d} = 2, \ \beta^{b-1,d-1} = 2, \ \beta^{b-1,d} = 1$

$\mu^{b,d} = (5-2) - (2-1) = 2$

Example

- $\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$



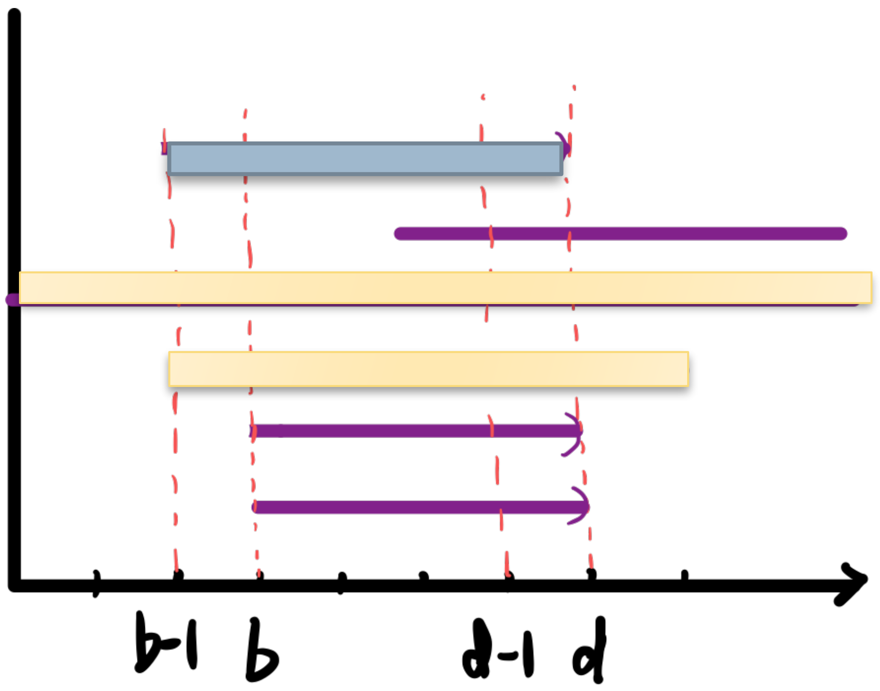
I[b, d) = 2

$\beta^{b,d-1} = 5, \ \beta^{b,d} = 2, \ \beta^{b-1,d-1} = 2, \ \beta^{b-1,d} = 1$

$\mu^{b,d} = (5-2) - (2-1) = 2$

Example

- $\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$



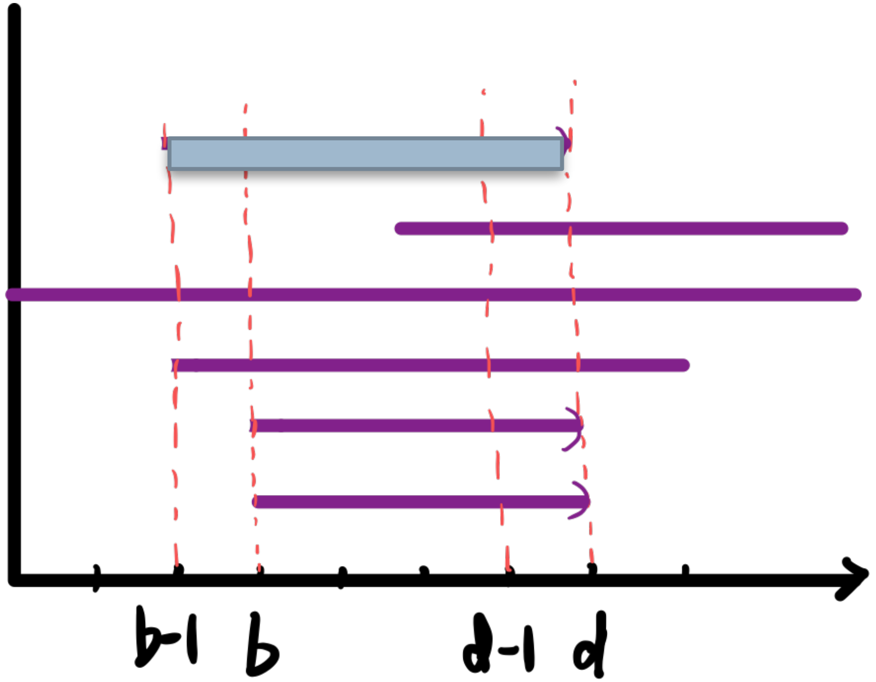
$\# I[b,d) = 2$

$\beta^{b,d-1} = 5, \ \beta^{b,d} = 2, \ \beta^{b-1,d-1} = 2, \ \beta^{b-1,d} = 1$

$\mu^{b,d} = (5-2) - (2-1) = 2$

Example

- $\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$



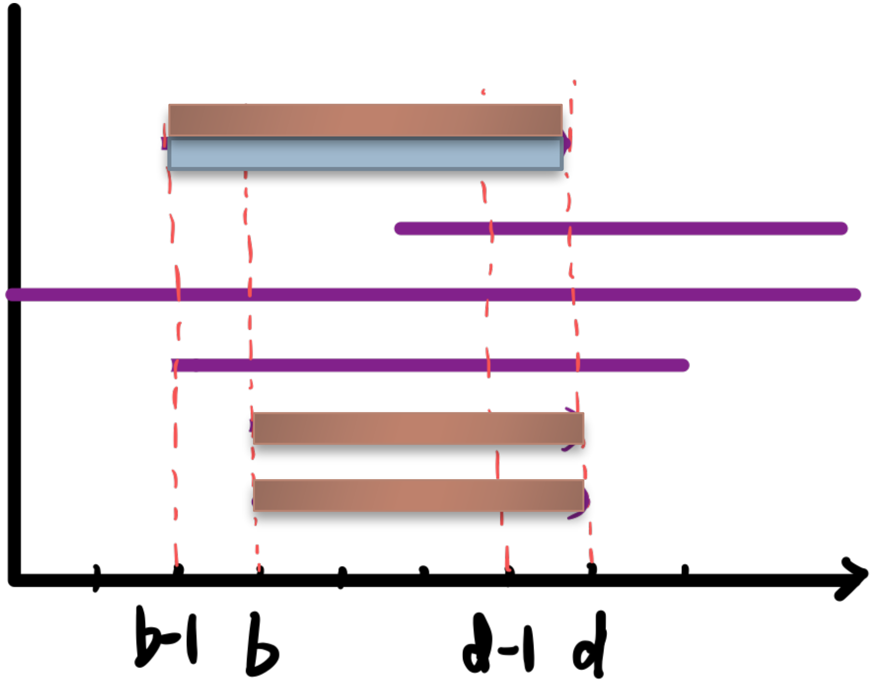
$\# I[b,d) = 2$

$\beta^{b,d-1} = 5, \ \beta^{b,d} = 2, \ \beta^{b-1,d-1} = 2, \ \beta^{b-1,d} = 1$

$\mu^{b,d} = (5-2) - (2-1) = 2$

Example

- $\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$



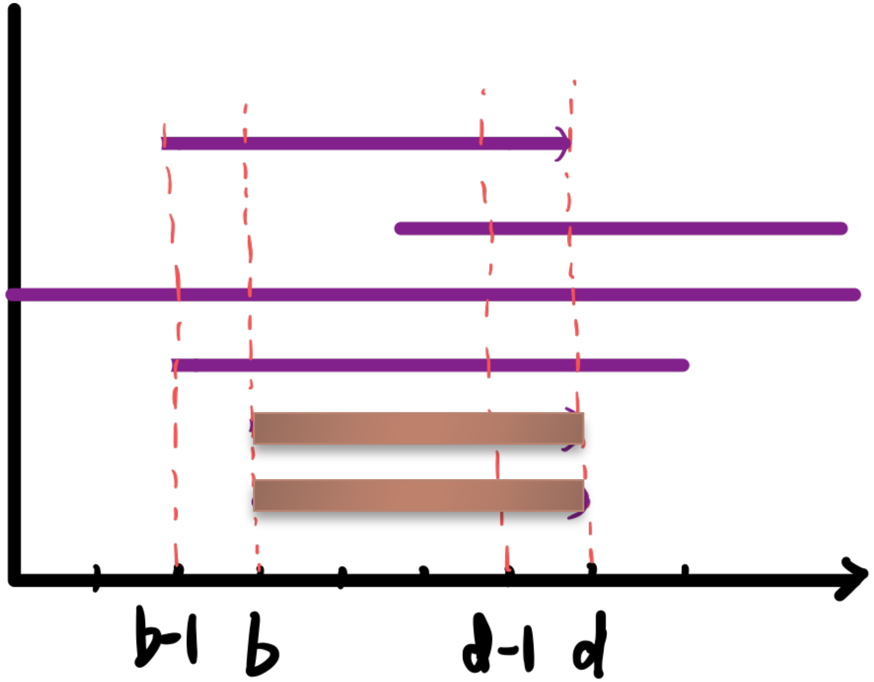
$I[b,d) = 2$

$\beta^{b,d-1} = 5, \ \beta^{b,d} = 2, \ \beta^{b-1,d-1} = 2, \ \beta^{b-1,d} = 1$

$\mu^{b,d} = (5-2) - (2-1) = 2$

Example

- $\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$



$\# I[b,d) = 2$

$\beta^{b,d-1} = 5,\ \beta^{b,d} = 2,\ \beta^{b-1,d-1} = 2,\ \beta^{b-1,d} = 1$

$\mu^{b,d} = (5-2) - (2-1) = 2$

A more refined topological view

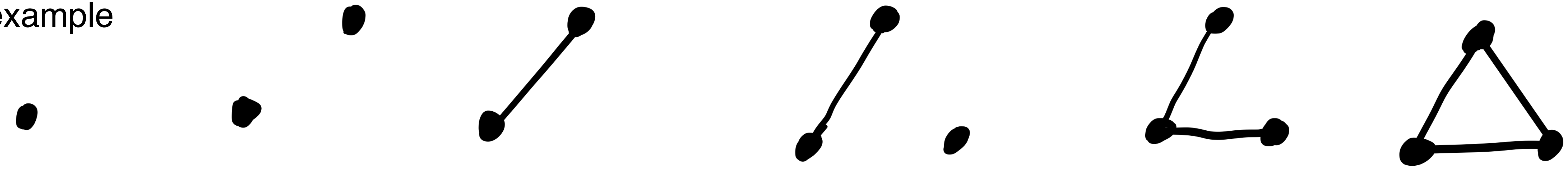
An alternative view

- **Simplex-wise filtration**

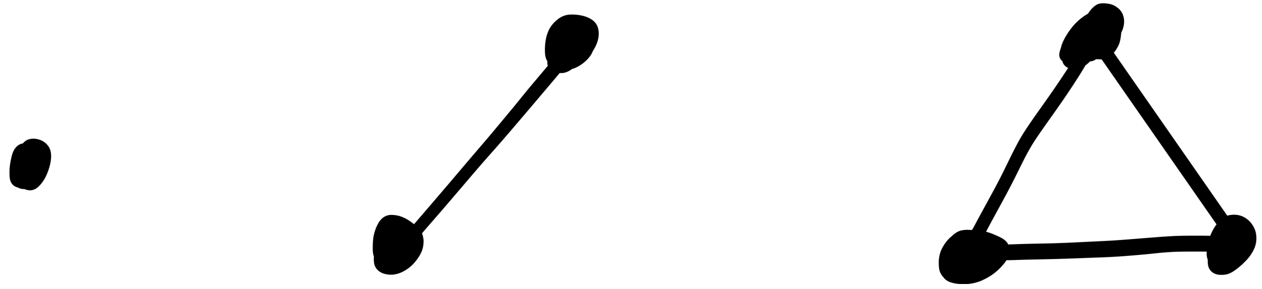
$$\varnothing = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K$$

  - s.t , $\sigma_i = K_i \smallsetminus K_{i-1}$

example



non-example



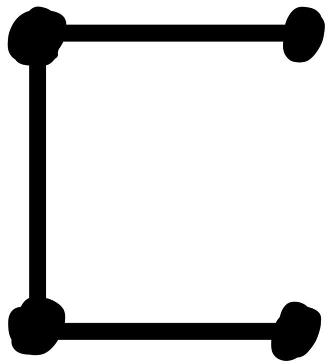
An alternative view

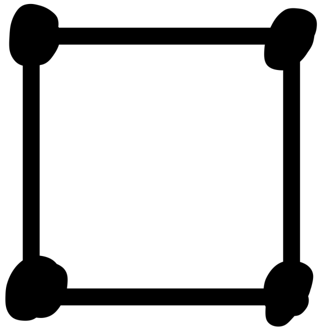
- **Simplex-wise filtration**

$\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K$

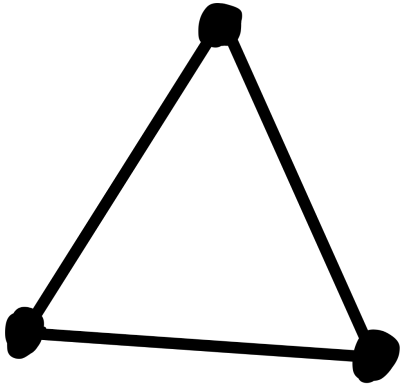
  - s.t , $\sigma_i = K_i \setminus K_{i-1}$

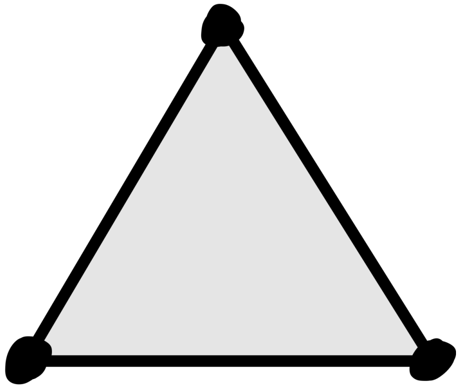
- Suppose we are at K_i, and consider p-simplex $\sigma = \sigma_{i+1}$
  - creator: adding σ creates a p-cycle
    - this cycle then must be ``new'', creates a homology class which is not in the image of $H_p(K_i) \to H_p(K_{i+1})$
    - hence $\beta_p + +$





An alternative view





- **Simplex-wise filtration**

$\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K$

  - s.t , $\sigma_i = K_i \setminus K_{i-1}$

- Suppose we are at K_i, and consider p-simplex $\sigma = \sigma_{i+1}$
  - creator: adding σ creates a p-cycle
    - this cycle then must be ``new'', creates a homology class which is not in the image of $H_p\left(K_i\right) \to H_p(K_{i+1})$
    - hence $\beta_p + +$
  - destroyer: killing a $(p-1)$-cycle
    - this $(p-1)$-cycle is $\partial\sigma$, and $\left[\partial\sigma\right] \neq 0$ in $H_{p-1}\left(K_i\right)$, but trivial in $H_{p-1}\left(K_{i+1}\right)$
    - hence $\beta_{p-1} + +$

An alternative view

- **Simplex-wise filtration**

$\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K$

  - s.t , $\sigma_i = K_i \setminus K_{i-1}$

- Suppose we are at K_i, and consider p-simplex $\sigma = \sigma_{i+1}$
  - creator: adding σ creates a p-cycle
    - this cycle then must be ``new'', creates a homology class which is not in the image of $H_p(K_i) \to H_p(K_{i+1})$
    - hence $\beta_p + +$
  - destroyer: killing a $(p-1)$-cycle
    - this $(p-1)$-cycle is $\partial\sigma$, and $[\partial\sigma] \neq 0$ in $H_{p-1}(K_i)$, but trivial in $H_{p-1}(K_{i+1})$
    - hence $\beta_{p-1} + +$

An alternative view

- **Simplex-wise filtration**

$\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K$

  - s.t , $\sigma_i = K_i \setminus K_{i-1}$

- Suppose we are at K_i, and consider p-simplex $\sigma = \sigma_{i+1}$
  - creator: adding σ creates a p-cycle
    - this cycle then must be ``new'', creates a homology class which is not in the image of $H_p\left(K_i\right) \rightarrow H_p(K_{i+1})$
    - hence $\beta_p + +$
  - destroyer: killing a $(p-1)$-cycle
    - this $(p-1)$-cycle is $\partial\sigma$, and $\left[\partial\sigma\right] \neq 0$ in $H_{p-1}\left(K_i\right)$, but trivial in $H_{p-1}\left(K_{i+1}\right)$
    - hence $\beta_{p-1} + +$

An alternative view

- **Simplex-wise filtration**

$\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K$

  - s.t , $\sigma_i = K_i \setminus K_{i-1}$

- Suppose we are at K_i, and consider p-simplex $\sigma = \sigma_{i+1}$
  - creator: adding σ creates a p-cycle
    - this cycle then must be ``new'', creates a homology c… …nage of $H_p(K_i) \to H_p(K_{i+1})$
    - hence $\beta_p{+}{+}$

Not unique

  - destroyer: killing a $(p-1)$-cycle
    - this $(p-1)$-cycle is $\partial\sigma$, and $[\partial\sigma] \neq 0$ in $H_{p-1}($… h… …$_1)$
    - hence $\beta_{p-1}{+}{+}$

Not unique

An alternative view

- Simplex-wise filtration

$\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K$

  - s.t , $\sigma_i = K_i \setminus K_{i-1}$

- Suppose we are at K_i, and consider p-simplex $\sigma = \sigma_{i+1}$
  - cr
    - $_{+1})$
  - de
    - this $(p-1)$ cycle is $\partial\sigma$, and $[\partial\sigma] \neq 0$ in $H_{p-1}(K_i)$, but trivial in $H_{p-1}(K_{i+1})$
    - hence $\beta_{p-1}++$

- Intuitively, the persistence pairing $(i,\ j)$ means that adding σ_j destroys a homology class created when adding σ_i.
- Hence, $\dim\left(\sigma_j\right) = \dim\left(\sigma_i\right) + 1$

- See board for an example

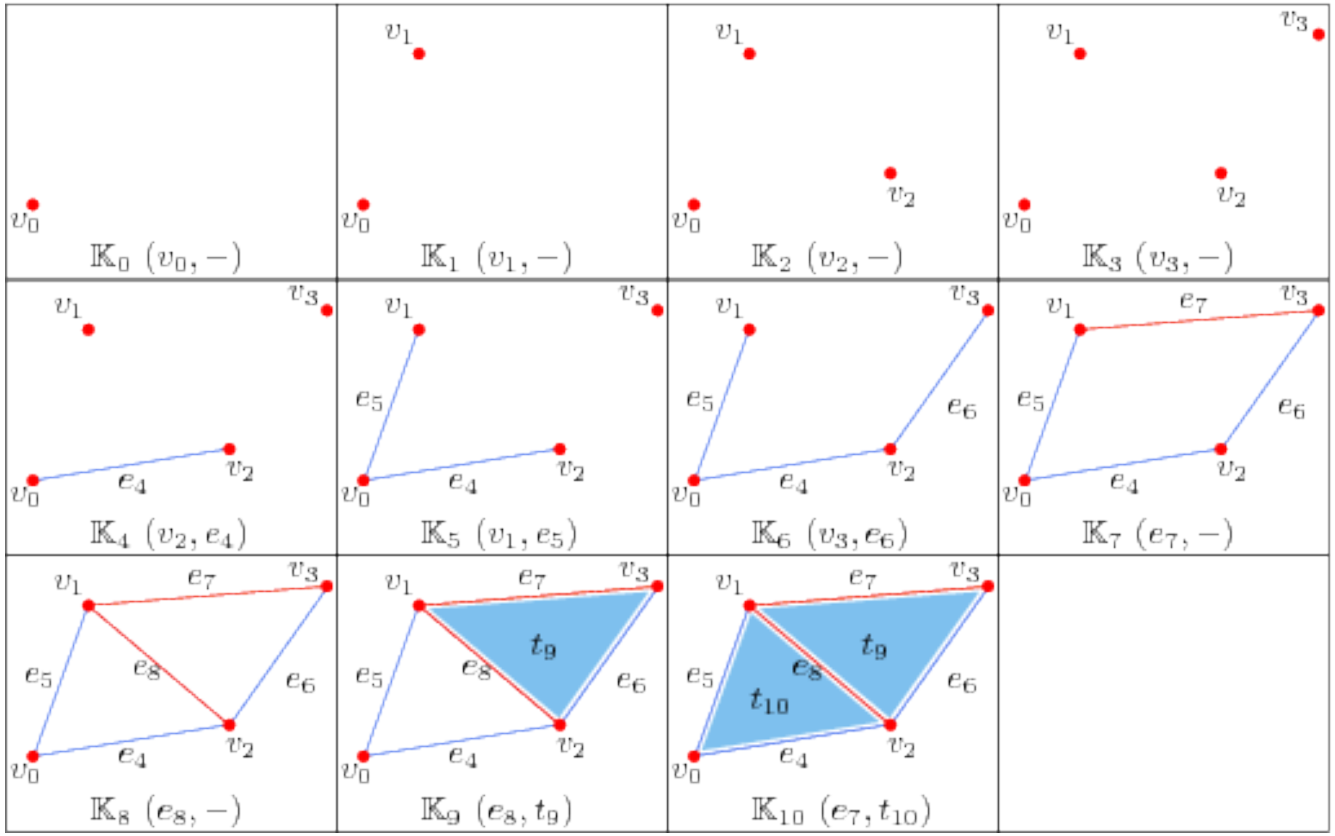
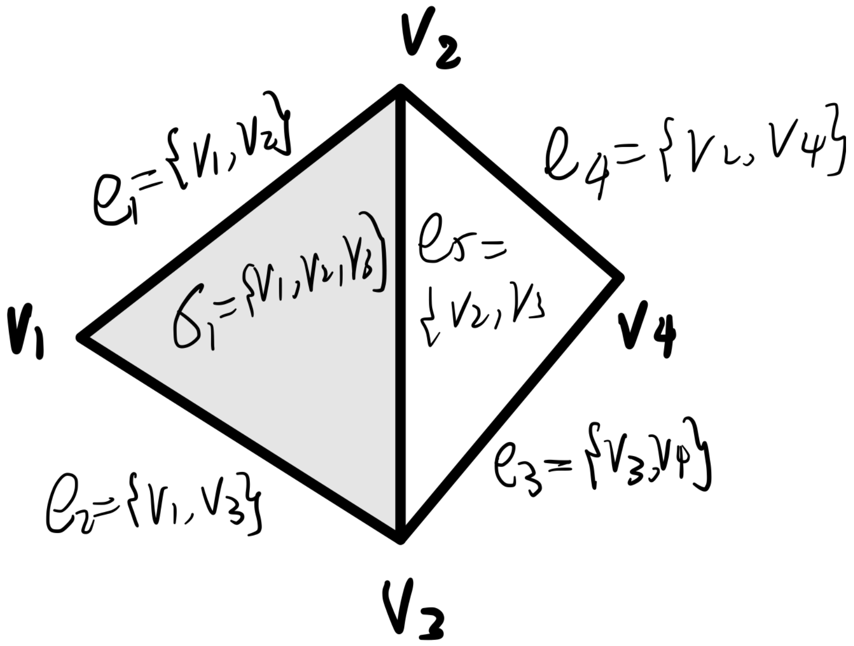


Image courtesy of T.K.Dey

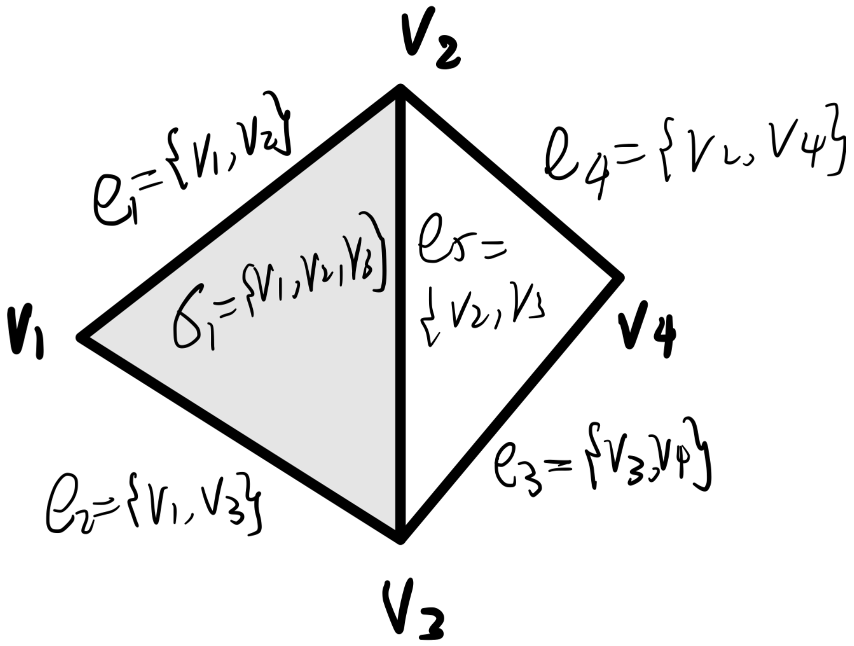
Section 2:
Persistence Algorithm

Recall homology algorithm



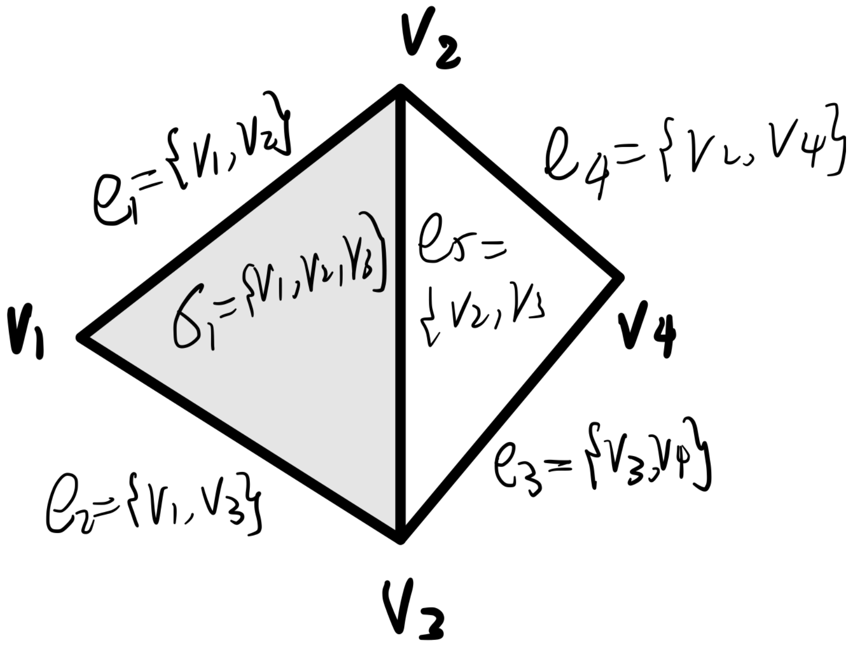
	e1	e2	e3	e4	e5
v1	1	1	0	0	0
v2	1	0	0	1	1
v3	0	1	1	0	1
v4	0	0	1	1	0

Recall homology algorithm



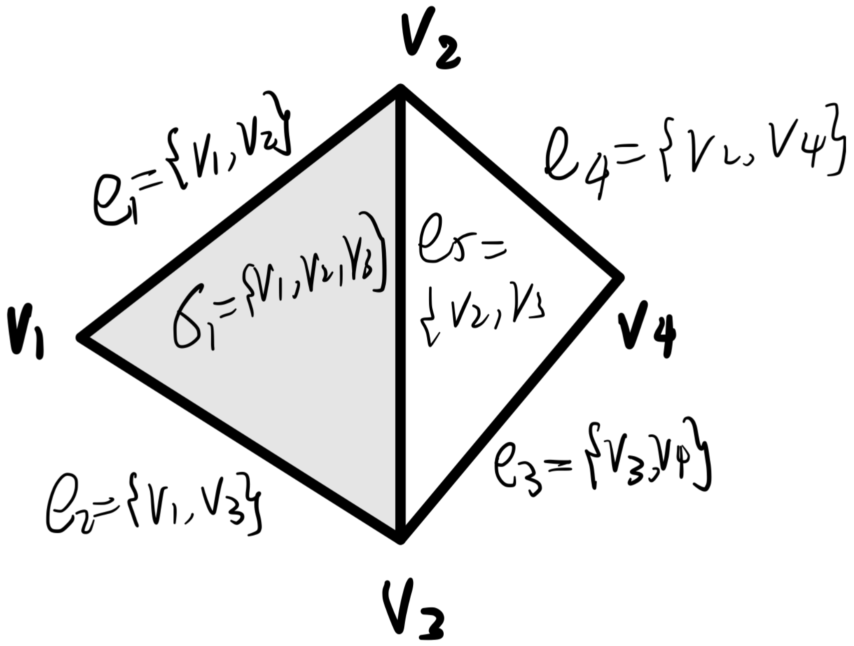
	e1	e2	e3	e4+e3	e5
v1	1	1	0	0	0
v2	1	0	0	1	1
v3	0	1	1	1	1
v4	0	0	1	0	0

Recall homology algorithm



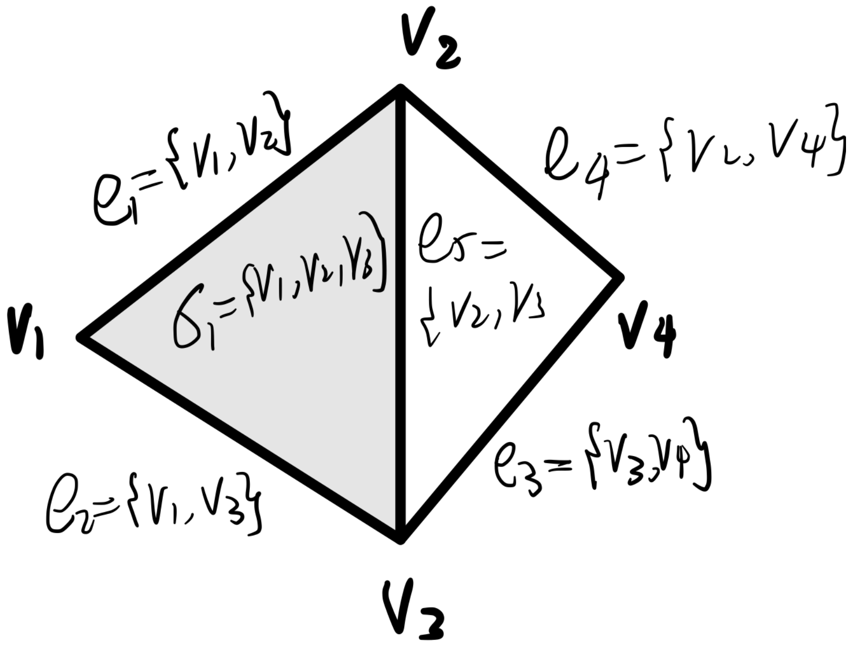
	e1	e2	e3	e4+e3+e2	e5
v1	1	1	0	1	0
v2	1	0	0	1	1
v3	0	1	1	0	1
v4	0	0	1	0	0

Recall homology algorithm



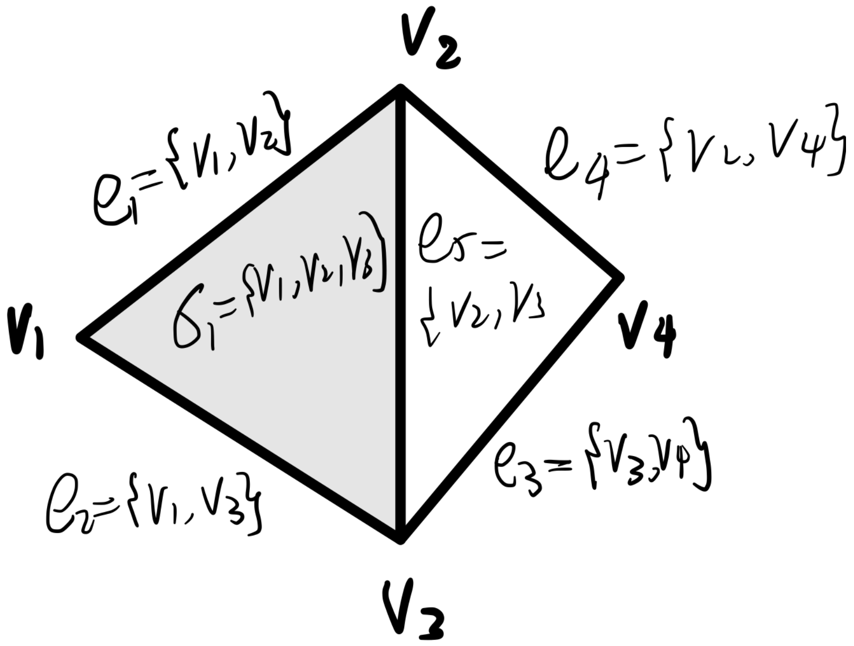
	e1	e2	e3	e4+e3+e2 +e1	e5
v1	1	1	0	0	0
v2	1	0	0	0	1
v3	0	1	1	0	1
v4	0	0	1	0	0

Recall homology algorithm



	e1	e2	e3	e4+e3+e2 +e1	e5+e2
v1	1	1	0	0	1
v2	1	0	0	0	1
v3	0	1	1	0	0
v4	0	0	1	0	0

Recall homology algorithm



	e1	e2	e3	e4+e3+e2+e1	e5+e2+e1
v1	1	1	0	0	0
v2	1	0	0	0	0
v3	0	1	1	0	0
v4	0	0	1	0	0

Persistent Algorithm

- Simplex-wise filtration $\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K$
  - s.t., $\sigma_i = K_i \setminus K_{i-1}$
  - i.e, filtration induced by an ordered sequence of simplices $\sigma_1, \sigma_2, \ldots, \sigma_n$ s.t. $K_i = \{\sigma_1, \cdots, \sigma_i\}$
- Let A be boundary matrix for K with $Col_A[i] = \partial\sigma_i$
- $lowId_M(j)$: index of lowest 1-entry in $Col_M[j]$

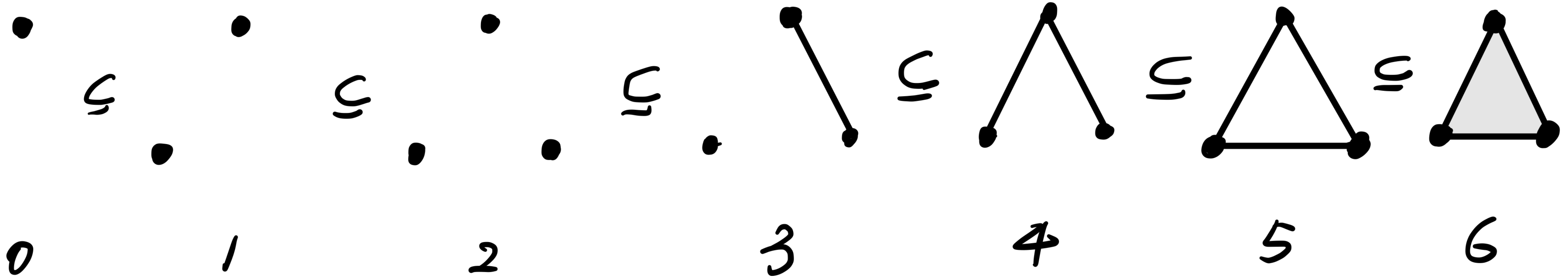
Persistent Algorithm

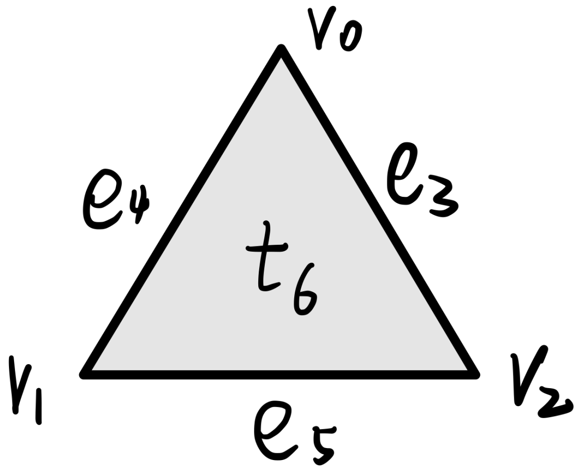
- Assume input filtration $\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_m = K$
  - $\sigma_i = K_i \setminus K_{i-1}$
  - i.e, filtration induced by an ordered sequence of simplices $\sigma_1, \sigma_2, \ldots, \sigma_n$ s.t. $K_i = \{\sigma_1, \cdots, \sigma_i\}$
  - Let A be boundary matrix for K with $Col_A[i] = \partial\sigma_i$
- $lowId_M(j)$: index of lowest 1-entry in $Col_M[j]$

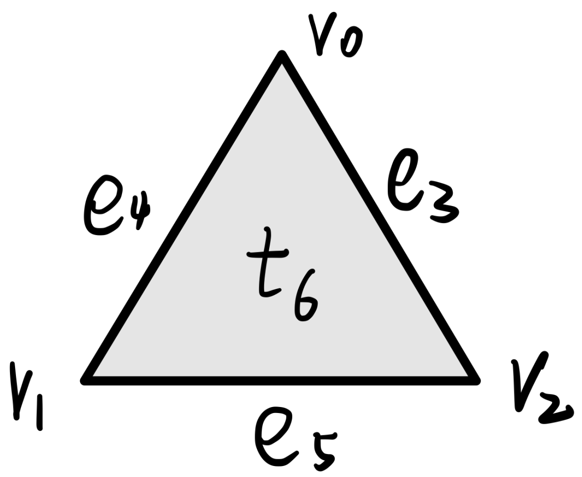
Algorithm 1 Right-Reduction(A)

```
R = A;
for j = 1 → m do
   while there exists j0 < j with lowId(j0) = lowId(j) do
      add column j0 of R to column j of R
   end while
end for
```

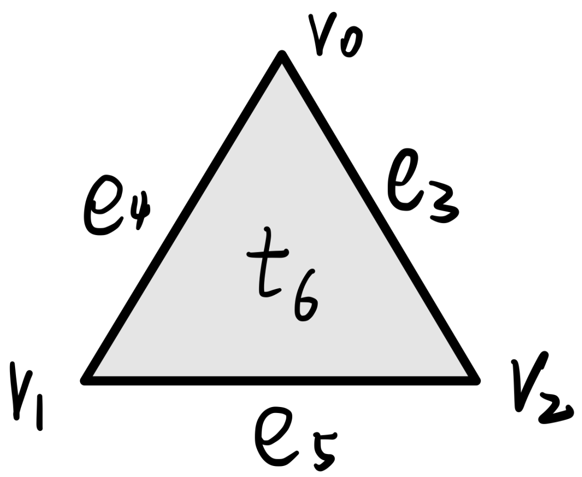
Example



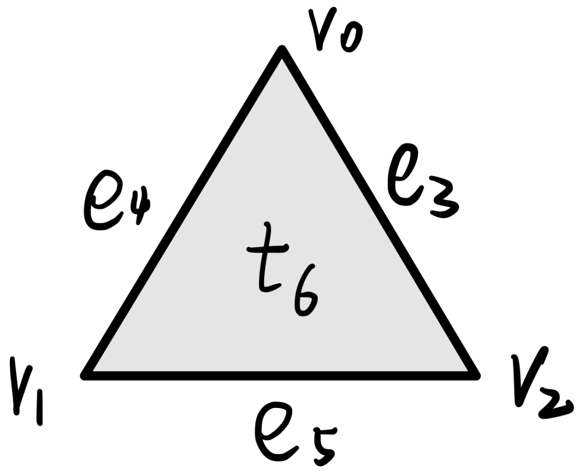




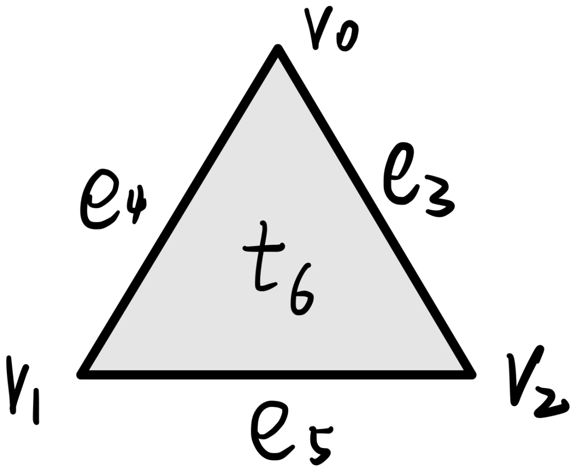
	v0	v1	v2	e3	e4	e5	t6
v0	0	0	0	1	1	0	0
v1	0	0	0	0	1	1	0
v2	0	0	0	1	0	1	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0

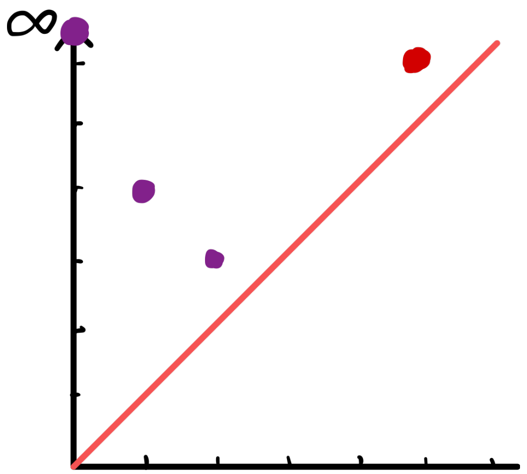


	v0	v1	v2	e3	e4	e5+e3	t6
v0	0	0	0	1	1	1	0
v1	0	0	0	0	1	1	0
v2	0	0	0	1	0	0	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0



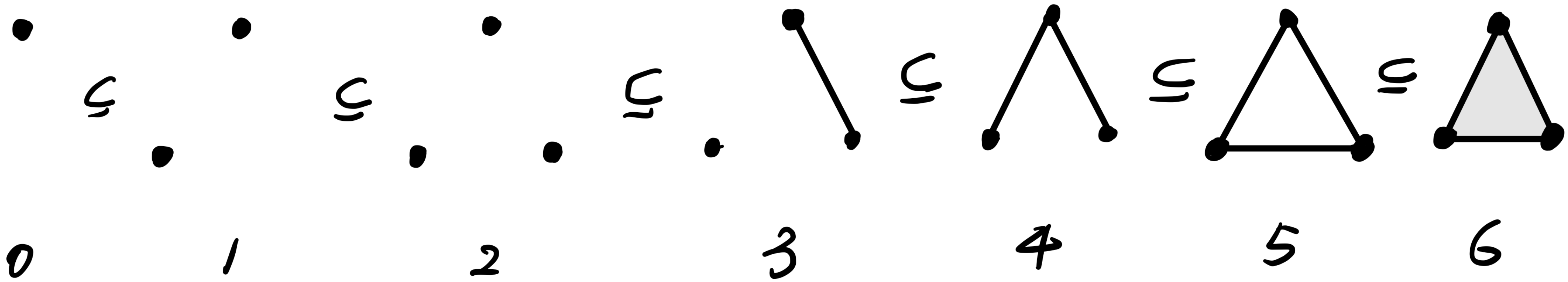
	v0	v1	v2	e3	e4	e5+e3+e4	t6
v0	0	0	0	1	1	0	0
v1	0	0	0	0	1	0	0
v2	0	0	0	1	0	0	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0

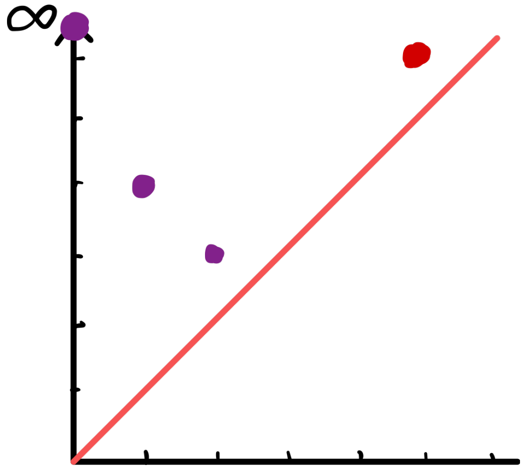




	v0	v1	v2	e3	e4	e5+e3+ e4	t6
v0	0	0	0	1	1	0	0
v1	0	0	0	0	1	0	0
v2	0	0	0	1	0	0	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0

- Homology classes born at 0,1,2,5
- $(v_0, \infty), (v_1, e_4), (v_2, e_3), (e_5, t_6)$
- $Dgm_0 = \{(0,\infty), (1,4), (2,3)\}$
- $Dgm_1 = \{(5,6)\}$





- Homology classes born at 0,1,2,5
- $(v_0, \infty), (v_1, e_4), (v_2, e_3), (e_5, t_6)$
- $Dgm_0 = \{(0,\infty), (1,4), (2,3)\}$
- $Dgm_1 = \{(5,6)\}$

Invariance

- Column addition ~ change of basis for C_p
  - Rank does not change

- For any intermediate matrix M
  - Each column i is associated with a p-chain Γ^i
  - The column $Col_M[i]$ corresponds to the boundary of Γ^i
  - If $Col_M[i] = \begin{bmatrix} 0 & 0 & .. & 0 \end{bmatrix}^T$, it is a cycle generating a new homology class
    - Birth event
  - Otherwise, it is a *boundary cycle*
    - Death event

Persistent Pairings

- **Theorem A:**
  - Consider the output matrix R of algorithm Right-Reduction(A).

    Then $\mu^{i,j} = 1$ iff $lowId_R\left(j\right) = i$

Persistent Pairings

- Theorem A:
  - Consider the output matrix R of algorithm Right-Reduction(A). Then $\mu^{i,j} = 1$ iff $lowId_R(j) = i$

- Theorem B:
  - Given boundary matrix A, perform **any** sequence of right-column-addition operations only to convert it into the reduced form R. Then

$$\mu^{i,j} = 1 \text{ iff } lowId_R(j) = i$$

Generating cycles

- For any intermediate matrix M
  - Each column i is associated with a p-chain Γ^i
  - The column $Col_M[i]$ corresponds to the boundary of Γ^i
  - If $Col_M[i] = \begin{bmatrix} 0 \ 0 \ .. \ 0 \end{bmatrix}^T$, it is a *boundary cycle*
    - Death event
  - Otherwise, it is a cycle generating a new homology class
    - Birth event

Generating cycles

- For any intermediate matrix M
  - Each column i is associated with a p-chain Γ^i
  - The column $Col_M[i]$ corresponds to the boundary of Γ^i
  - If $Col_M[i] = \left[0\ 0\ \ ..\ \ 0\right]^T$, it is a *boundary cycle*
    - Death event
  - Otherwise, it is a cycle generating a new homology class
    - Birth event

Generating cycles

- For any intermediate matrix M
  - Each column i is associated with a $\boxed{p\text{-chain } \Gamma^i}$
  - The column $Col_M[i]$ corresponds to the boundary of Γ^i
  - If $Col_M[i] = [0\ 0\ ..\ 0]^T$, it is a *boundary cycle*
    - Death event
  - Otherwise, it is a cycle generating a new homolo
    - Birth event

Generating cycle if this column is all-zero!

Computation

- Right-Reduction(A) runs in $O\left(N^3\right)$ time
  - where N is total number of simplices
- Can be improved to matrix multiplication time

Efficient implementation

Efficient implementation

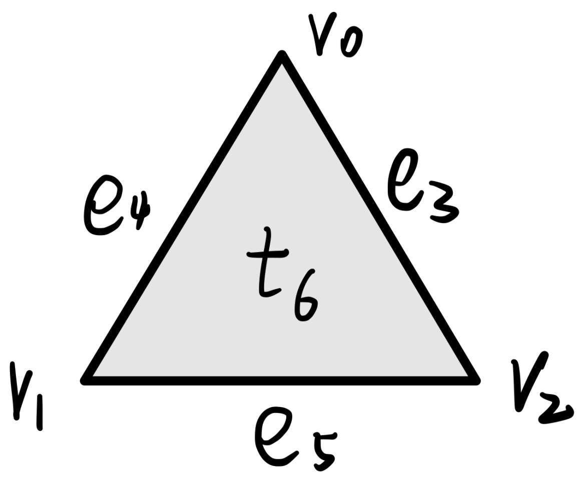
- Given a persistence pair (σ_i, σ_j), the column of σ_i will become zero eventually

Efficient implementation

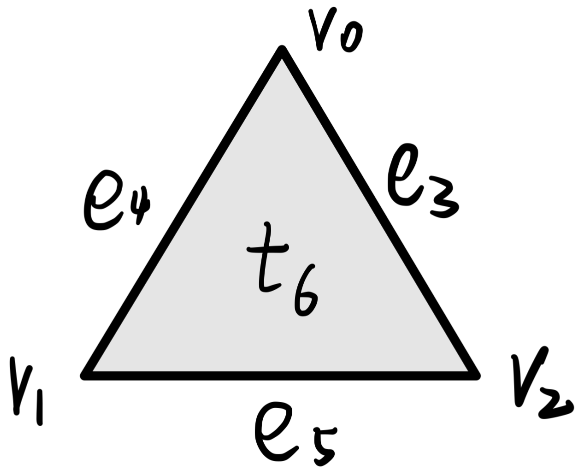
- Given a persistence pair (σ_i, σ_j), the column of σ_i will become zero eventually
- It is then more efficient to set the column to be 0 instead of adding columns iteratively

Efficient implementation

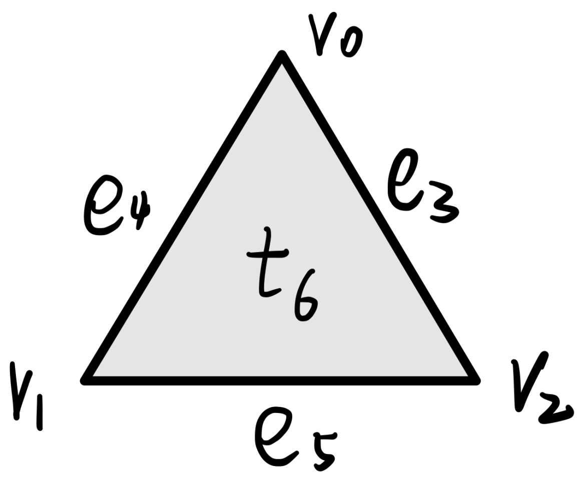
- Given a persistence pair (σ_i, σ_j), the column of σ_i will become zero eventually
- It is then more efficient to set the column to be 0 instead of adding columns iteratively
- Clearing:
  - Break the total boundary matrix to $A_1, A_2, \ldots, A_d$
  - Apply right reduction to $A_d, \ldots, A_1$
  - When (σ_i, σ_j) appears in reduced A_d, assign 0 to the column corresponding to σ_i in A_{d-1}
    - Record Γ_i by the column of σ_j



	v0	v1	v2	e3	e4	e5	t6
v0	0	0	0	1	1	0	0
v1	0	0	0	0	1	1	0
v2	0	0	0	1	0	1	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0



	v0	v1	v2	e3	e4	e5+e4 +e3	t6
v0	0	0	0	1	1	0	0
v1	0	0	0	0	1	0	0
v2	0	0	0	1	0	0	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0

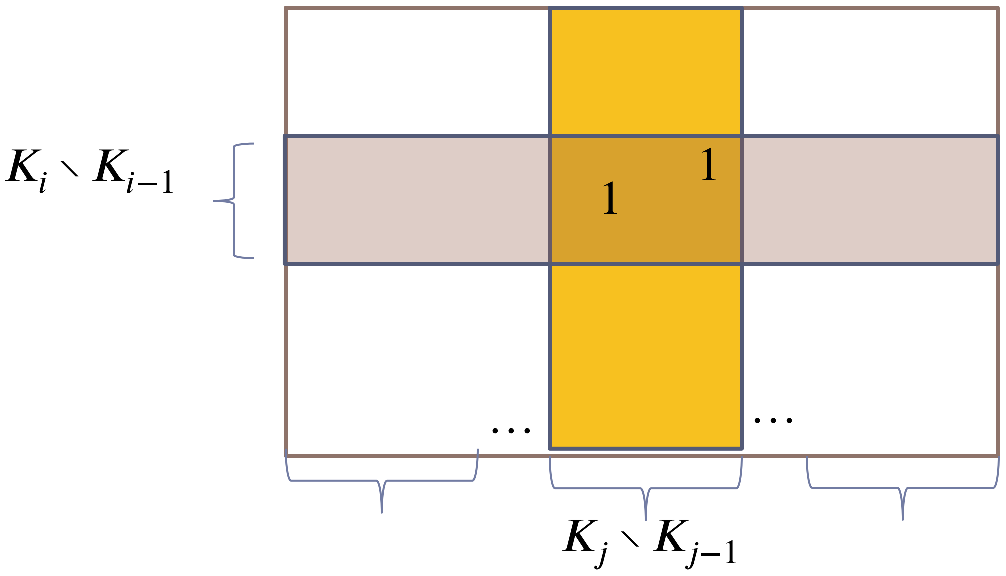


	v0	v1	v2	e3	e4	e5+e4 +e3	t6
v0	0	0	0	1	1	0	0
v1	0	0	0	0	1	0	0
v2	0	0	0	1	0	0	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0

- See more acceleration tricks in this video

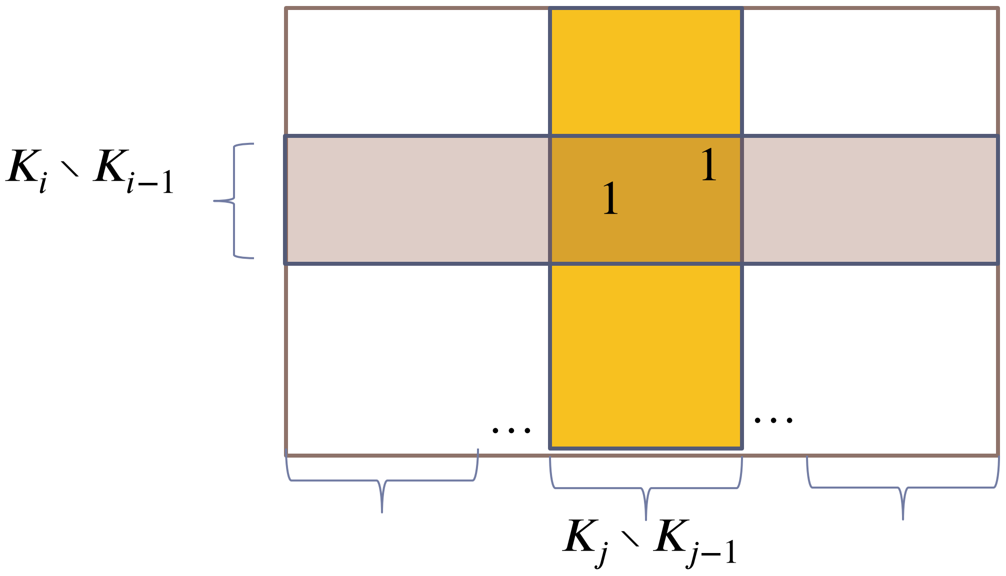
General Filtration

- Given $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n$, let $\sigma_1, \sigma_2, \ldots, \sigma_N$ be an ordering of simplices consistent with the filtration
- i.e, $\underbrace{\sigma_1, \ldots, \sigma_{I_1}}_{K_1}, \underbrace{\sigma_{I_1+1}, \ldots, \sigma_{I_2}}_{K_2 \setminus K_1}, \ldots, \underbrace{\sigma_{I_{n-1}+1}, \ldots, \sigma_{I_n}}_{K_n \setminus K_{n-1}}$



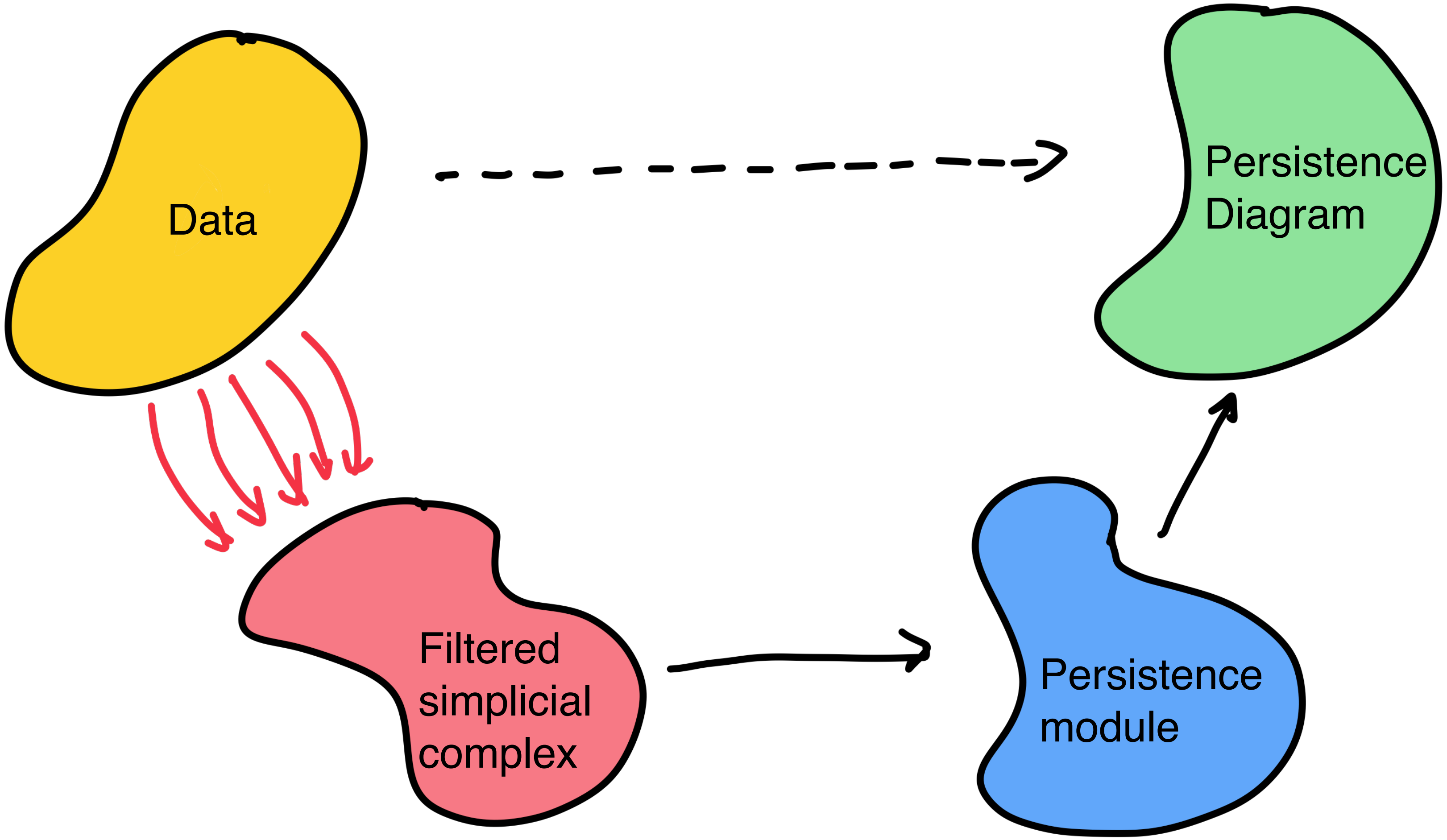
General Filtration

- Given $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n$, let $\sigma_1, \sigma_2, \ldots, \sigma_N$ be an ordering of simplices consistent with the filtration
- i.e, $\underbrace{\sigma_1, \ldots, \sigma_{I_1}}_{K_1}, \underbrace{\sigma_{I_1+1}, \ldots, \sigma_{I_2}}_{K_2 \setminus K_1}, \ldots, \underbrace{\sigma_{I_{n-1}+1}, \ldots, \sigma_{I_n}}_{K_n \setminus K_{n-1}}$



$\boxed{\mu^{i,j} = 2}$

Mind picture



FIN