DSC 214
Topological Data Analysis
Topic 4-A: Introduction to Persistent Homology

Instructor: Zhengchao Wan

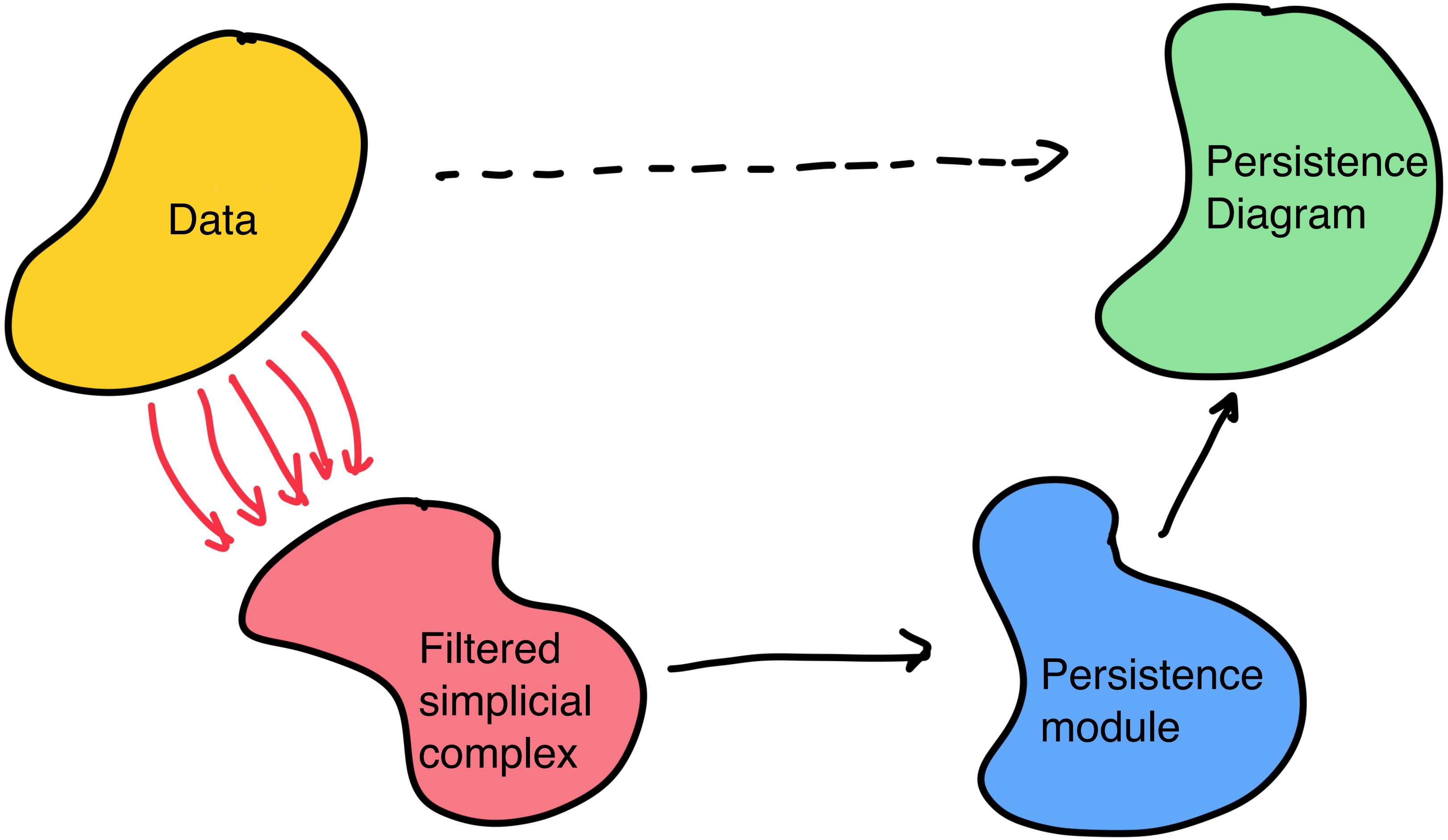
Persistent homology

- A modern extension of homology to ``sequence of spaces”
  - [Edelsbrunner, Letcher, and Zomorodian, FOCS 2000]
  - Significantly broaden its practical power

- What is persistent homology (PH)
  - Motivation
  - Persistent betti numbers and persistence diagrams

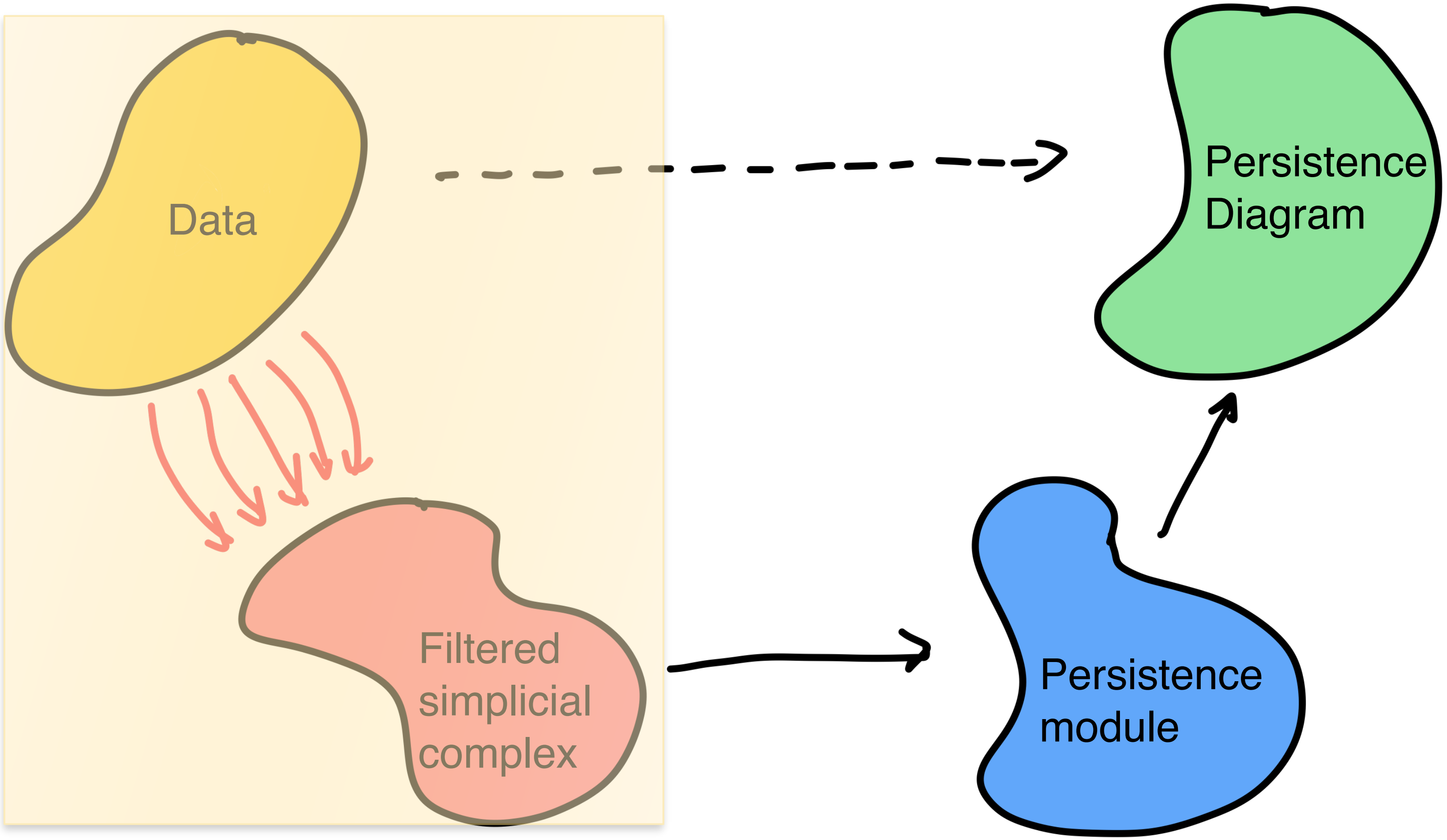
- Algorithm(s) for persistent homology

Mind picture

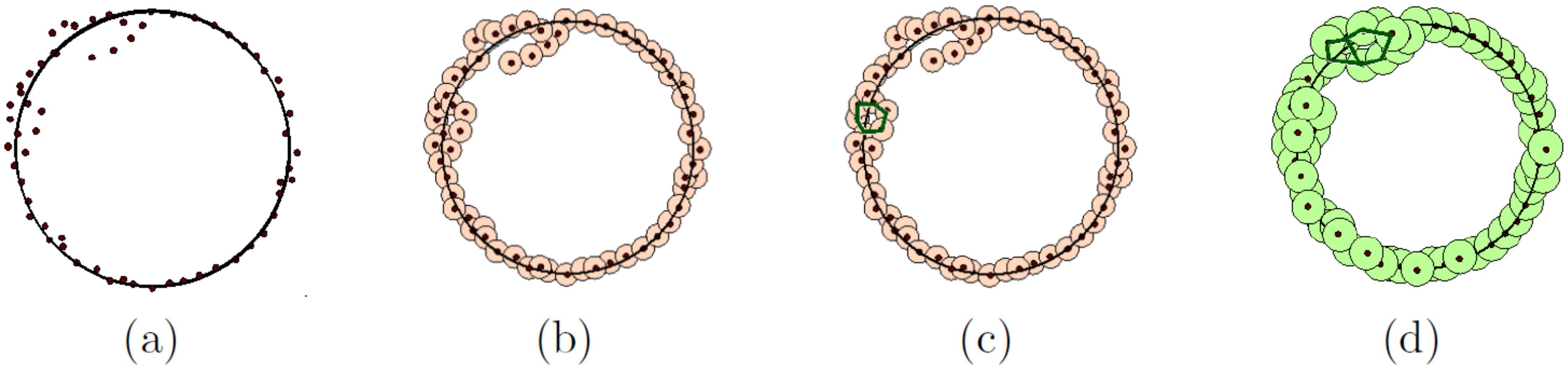


Section 1:
Persistent Homology

Filtered simplicial complex



Issue of Scale



(a) (b) (c) (d)

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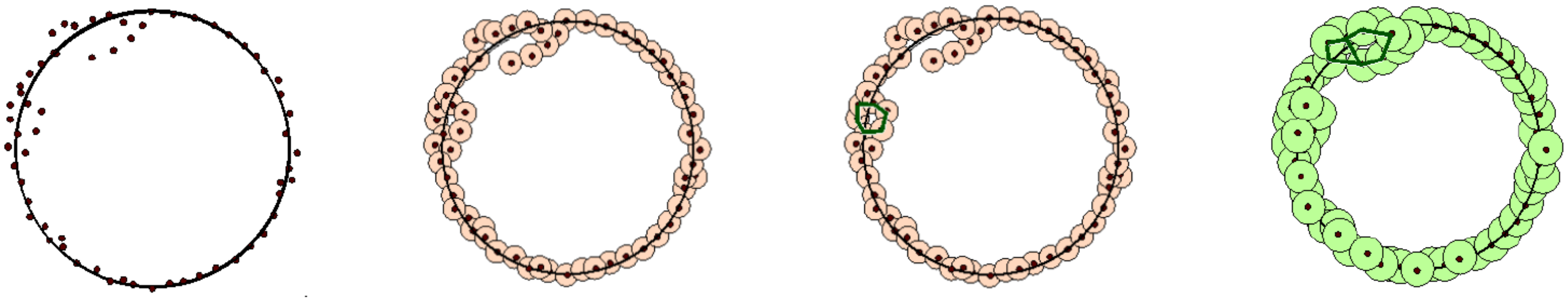

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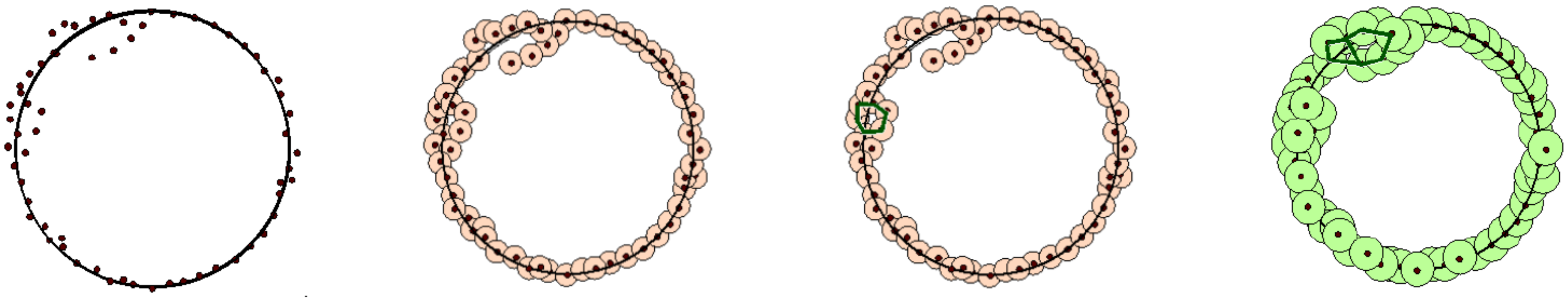

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- No single good scale!

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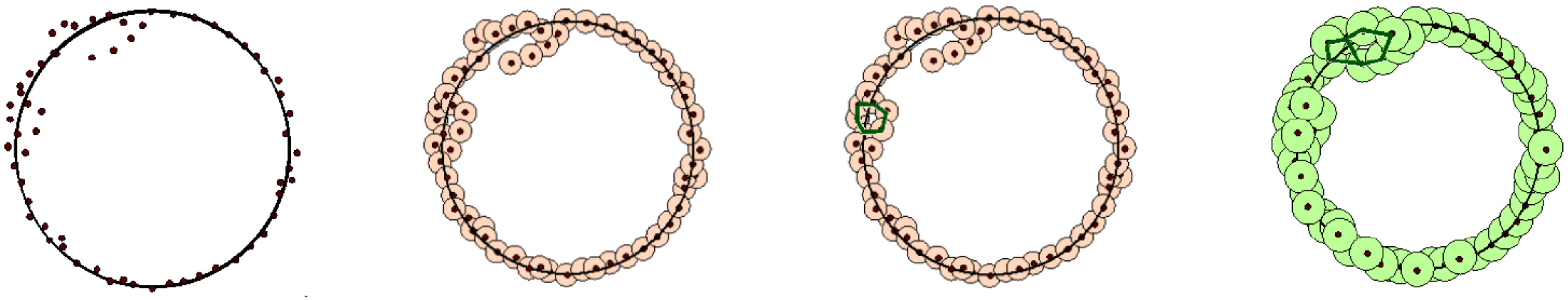

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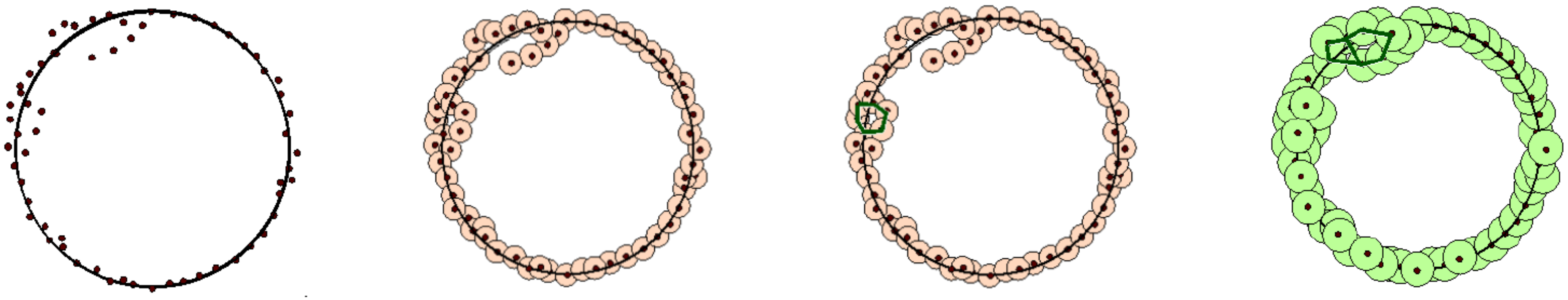
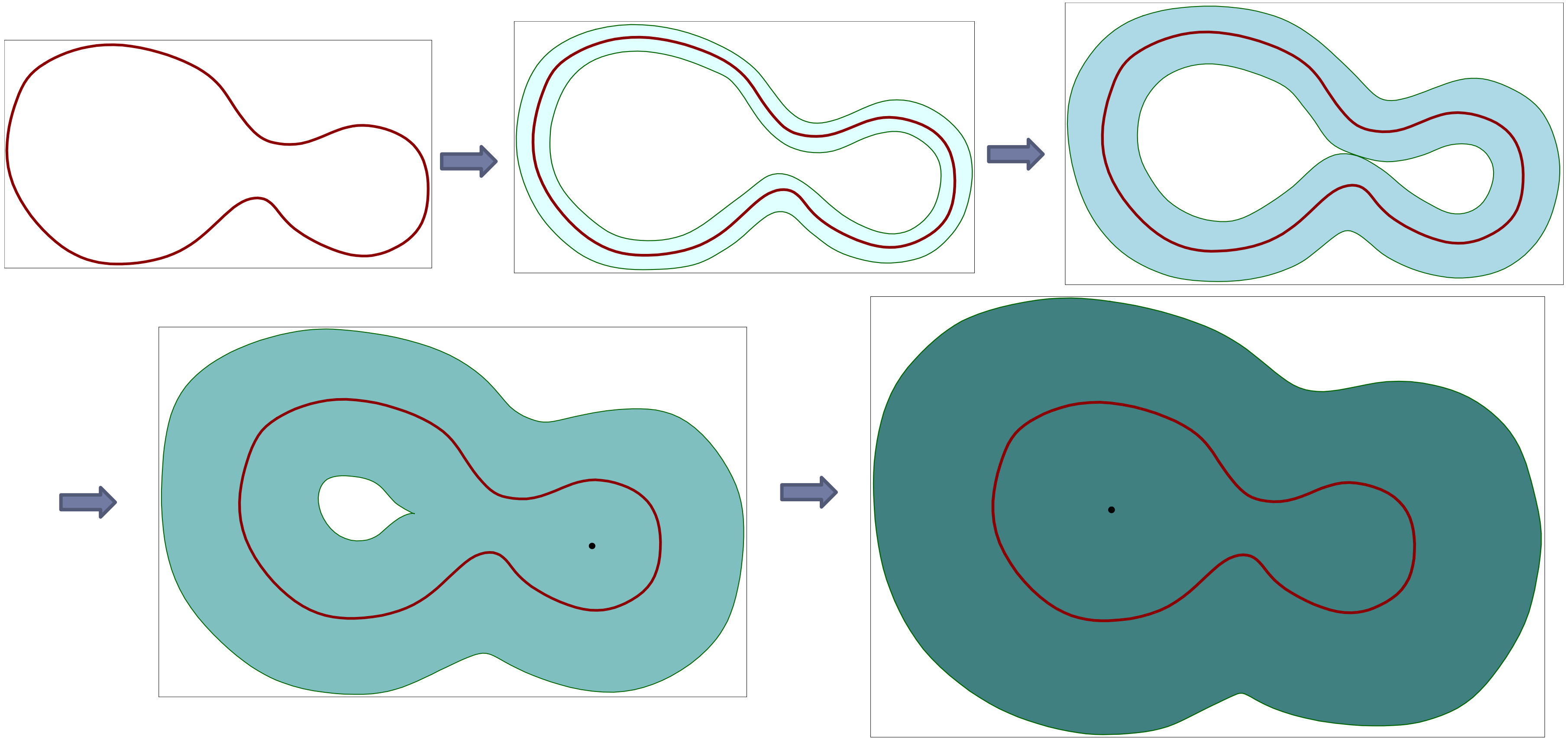


(a) (b) (c) (d)

- Which scale to take?
- No single good scale!
- All scales?
- Some ``features'' persists longer than others

Another Example



- Want to capture features of different ``sizes”

Čech Complex

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- Given a real value $r > 0$, the *Čech complex* $C^r(P)$ is the nerve of the set $\left\{ B\left(p_i, r\right) \right\}_{i \in [1,n]}$, where $B(p, r) = \{x \in \mathbb{R}^d \,|\, d(p, x) \leq r\}$

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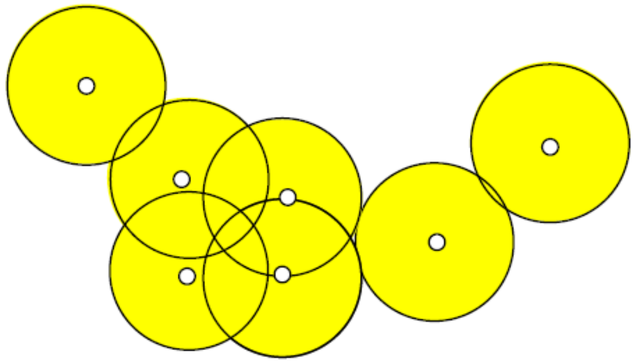

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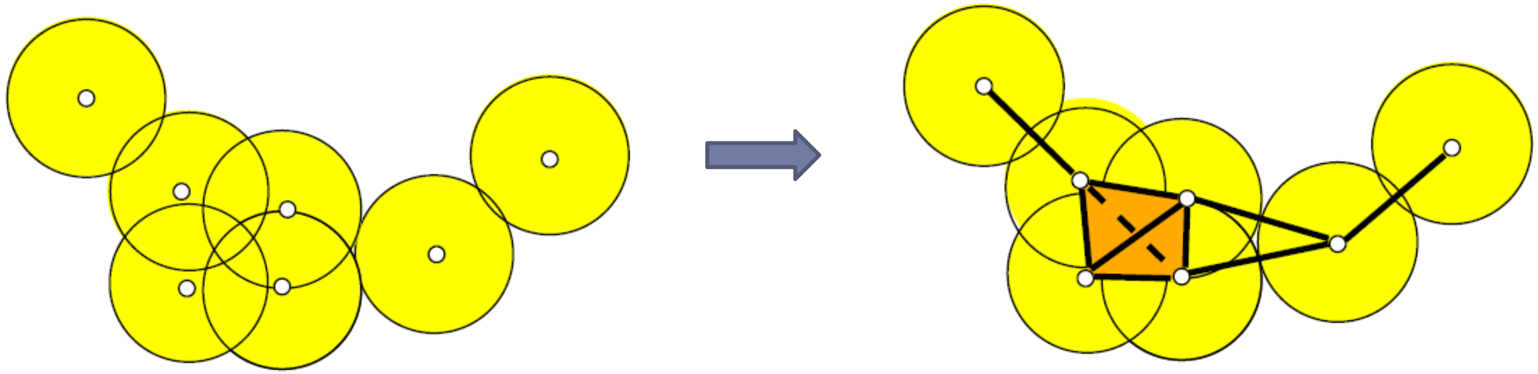

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- $(C^r(P))_{r \geq 0}$ is called the Čech filtration

Vietoris-Rips (Rips) Complex

- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$
- Given a real value $r > 0$, the *Vietoris-Rips (Rips) complex* $Rips^r(P)$ is:
  - $\left\{(p_{i_0}, \ldots, p_{i_k}) \mid B(p_{i_l}, r) \cap B(p_{i_j}, r) \neq \emptyset, \forall l, j = 0, \ldots, k\right\}$.
- More generally for P in a metric space (X, d):
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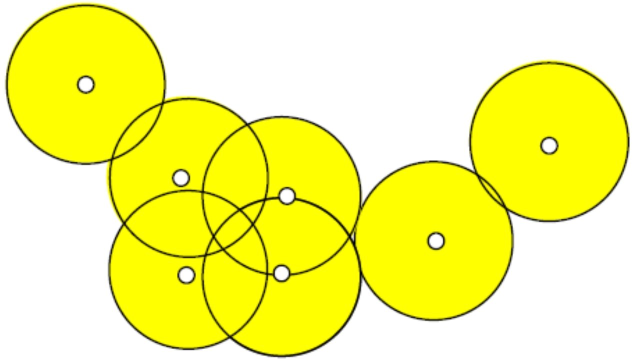

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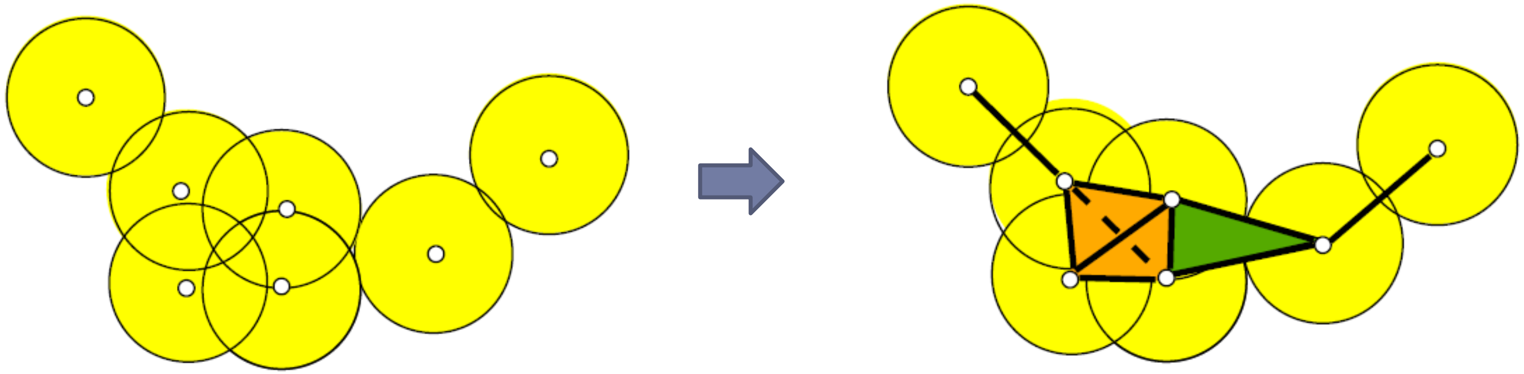

Vietoris-Rips (Rips) Filtration

- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$

- $(Rips^r(P))_{r \geq 0}$ is called the Vietoris-Rips (Rips) Filtration

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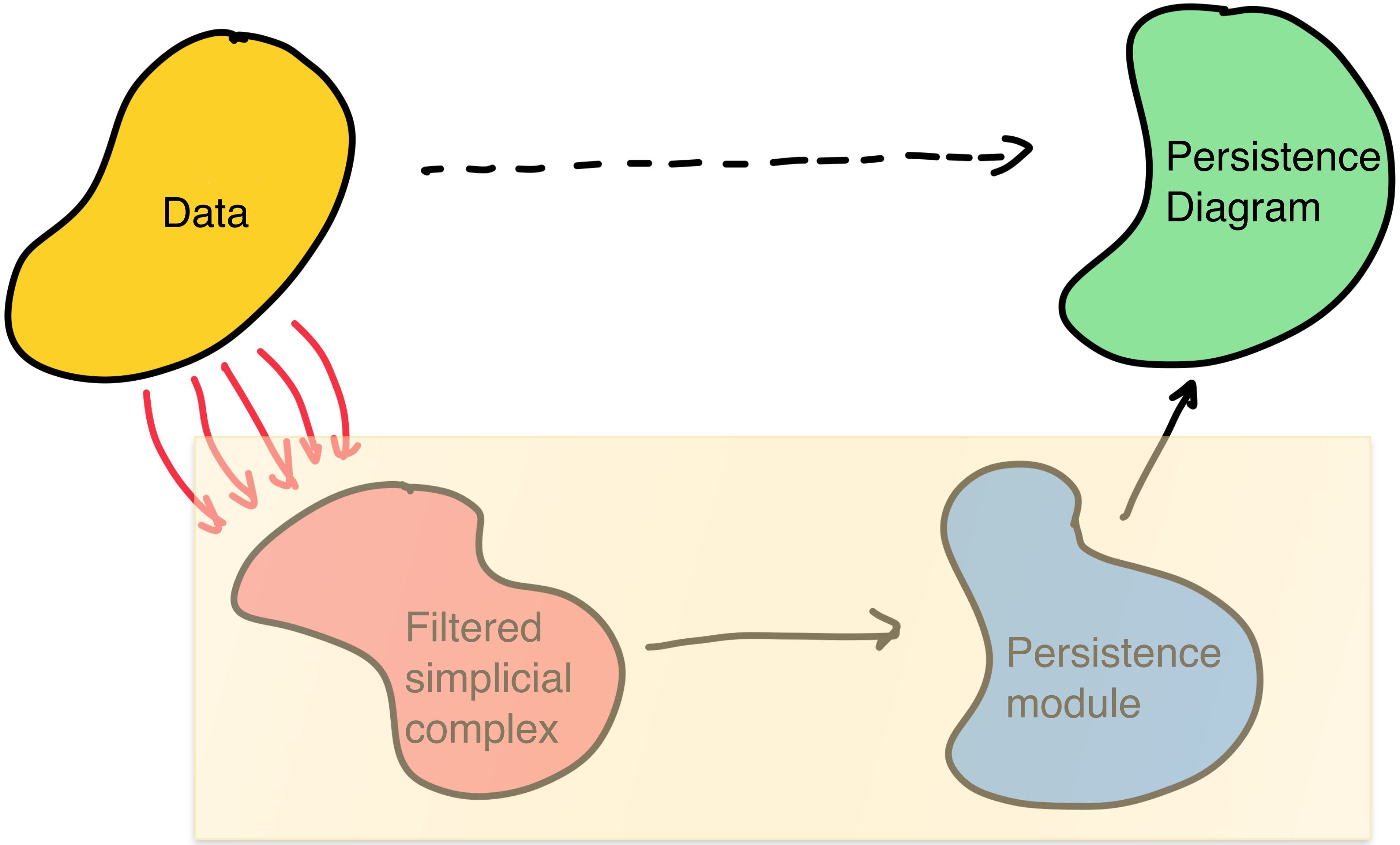
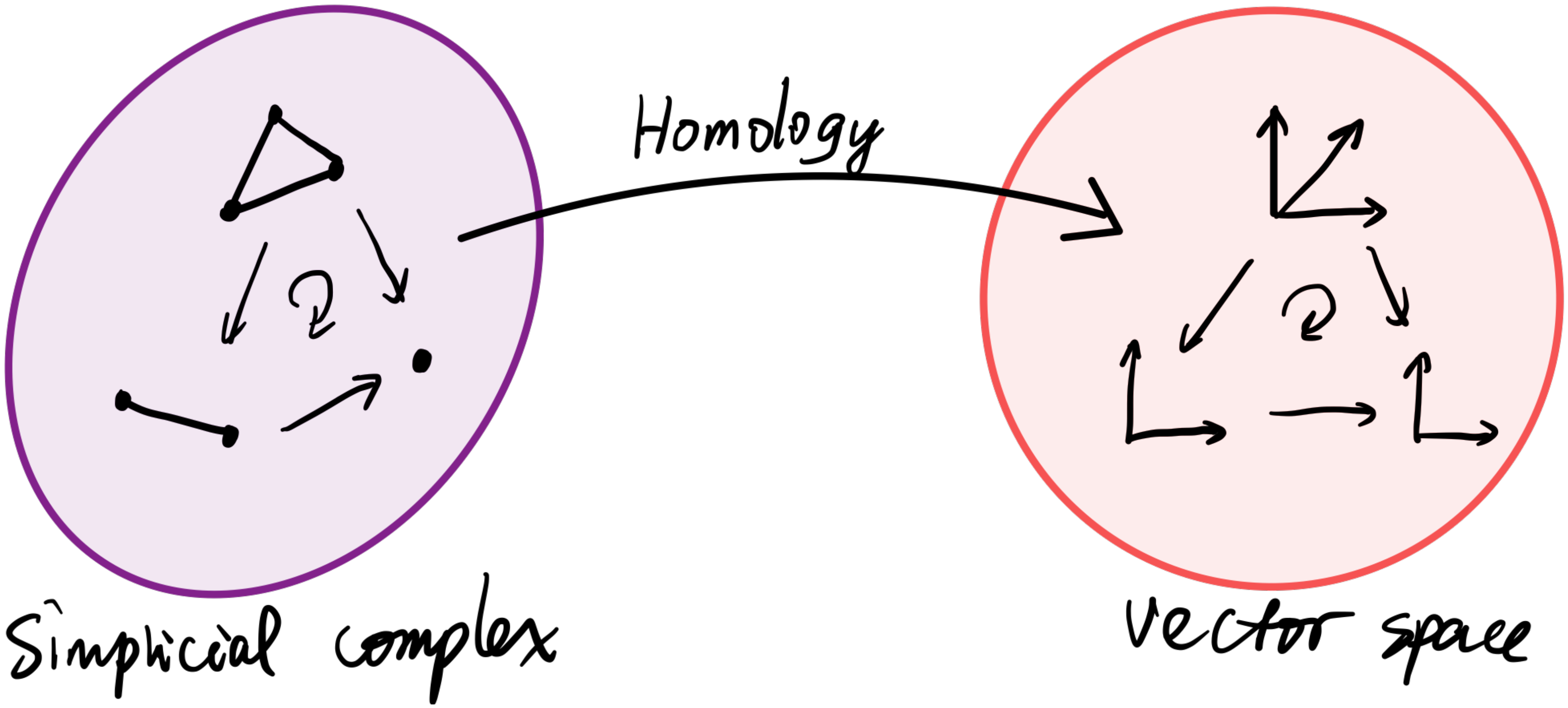
- So $(K_t)_{t\in[0,\infty)}$ is essentially the same as (or can be reconstructed from) $(K_{t_i})_{i=0,\ldots,n}$
- Both Čech and Rips filtrations are finitely represented

Persistence modules



Mind picture of functoriality



Persistence Modules

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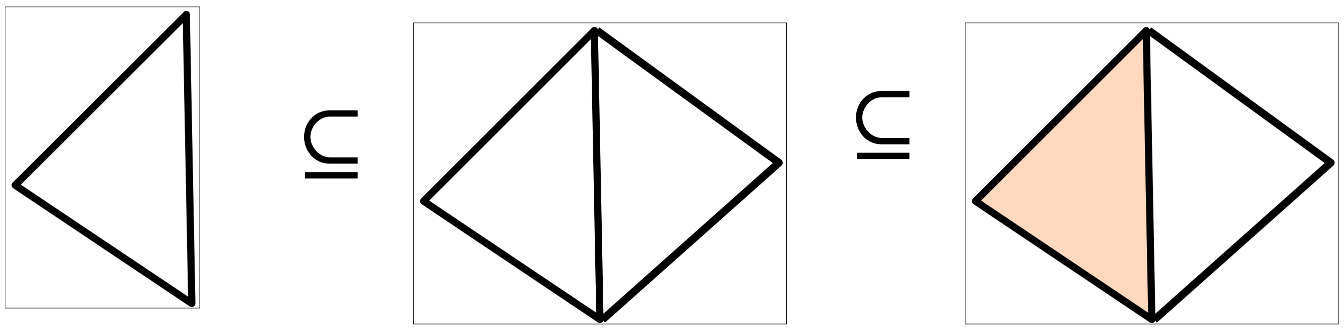

$$K_1 \subseteq K_2 \subseteq K_3$$

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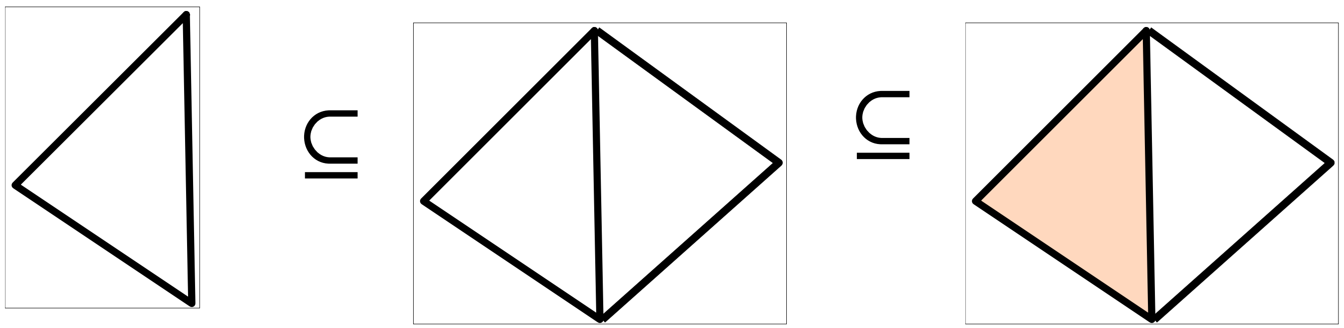
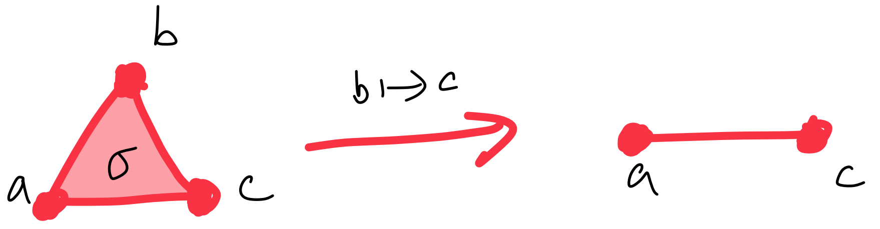


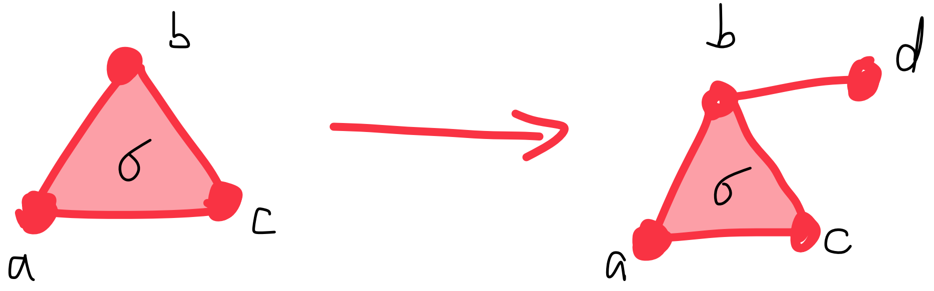
$$
\begin{array}{ccccc}
K_1 & \subseteq & K_2 & \subseteq & K_3 \\
H_1(K_1) & \to & H_1(K_2) & \to & H_1(K_3) \\
[c] & \mapsto & [c] & \mapsto & 0
\end{array}
$$

Construction of f_p

- Define $\bar{f}_p : C_p(K) \to C_p(K')$
  - $$\bar{f}_p(\sigma) = \begin{cases} f(\sigma) & \text{if } f(\sigma) \text{ is } p-\text{dimensional} \\ 0 & \text{otherwise} \end{cases}$$
  - Define $f_p : H_p(K) \to H_p(K')$
    - $f_p([c]) := [\bar{f}_p(c)]$



$\bar{f}_2(\sigma) = 0$ since $f(\sigma)=\{a,c\}$ is 1-dim



$\bar{f}_2(\sigma) = \sigma$

Understanding $H_p(K) \to H_p(K')$

- Define $\bar{f}_p : C_p(K) \to C_p(K')$
  - $\bar{f}_p(\sigma) = f(\sigma)$
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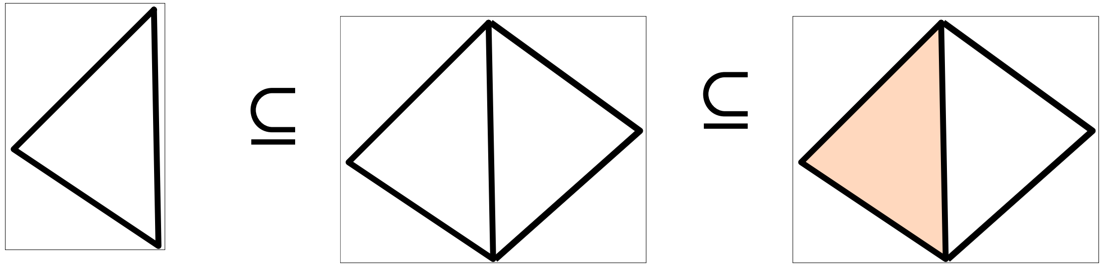
    - $f_p([c]) := [c]$



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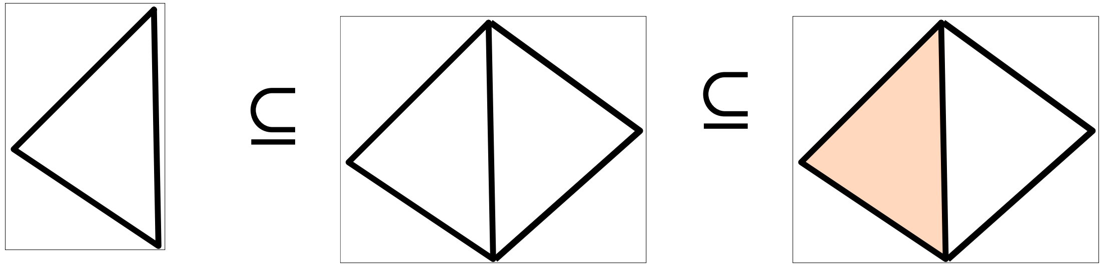

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$$[c] \mapsto [c] \mapsto 0$$

Persistence Modules

- $K \subseteq K' \Rightarrow \xi_p : H_p(K) \to H_p(K')$
  - Inclusion maps induce homomorphisms in homology groups (under Z_2-coefficients, linear maps in vector spaces)

$$\boxed{\begin{aligned} &K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K \\ &\Rightarrow \quad H_*(K_0) \to H_*(K_1) \to \cdots \to H_*(K_n) = H_*(K) \end{aligned}}$$

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$$\boxed{\begin{array}{l} K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K \\ \Rightarrow \quad H_*\left(K_0\right) \to H_*\left(K_1\right) \to \cdots \to H_*\left(K_n\right) = H_*(K) \end{array}}$$

- Define $\xi_*^{i,j} : H_*\left(K_i\right) \to H_*\left(K_j\right)$
  - $\xi_*^{i,j} = \xi_*^{j-1,\,j} \circ \cdots \circ \xi_*^{i,i+1}$
- Persistent module induced by the filtration
  - $\mathscr{P} = \left\{ H_*\left(K_i\right) \xrightarrow{\xi^{i,j}} H_*\left(K_j\right) \right\}_{0 \leq \mathrm{i} \leq \mathrm{j} \leq n}$

Persistence Vector Spaces

- A **persistence vector space** V over a field $\mathbb{F}$ is
  - a sequence of vector spaces $\{V_i\}_{i=0,\ldots,n}$
  - Together with maps $L_{i,j} : V_i \to V_j$ for $i \leq j$ such that
    - $L_{i,i} = Id_{V_i}$
    - For $i \leq j \leq k$, $L_{i,k} = L_{j,k} \circ L_{i,j}$
  - Write $V = \{L_{i,j} : V_i \to V_j\}$ or simply $V = \{V_i\}$

Persistence Vector Spaces

- Let $\{V_i\}$ and $\{W_i\}$ be two persistence vector spaces
  - a sequence of linear maps $\{\varphi_i : V_i \to W_i\}_{i=0,\ldots,n}$ is called a **linear transformation** from $\{V_i\}$ to $\{W_i\}$ if for any $i \leq j$

$$\begin{tikzcd} V_i \arrow[r, "L^V_{i,j}"] \arrow[d, "\varphi_i"'] & V_j \arrow[d, "\varphi_j"] \\ W_i \arrow[r, "L^W_{i,j}"'] & W_j \end{tikzcd}$$

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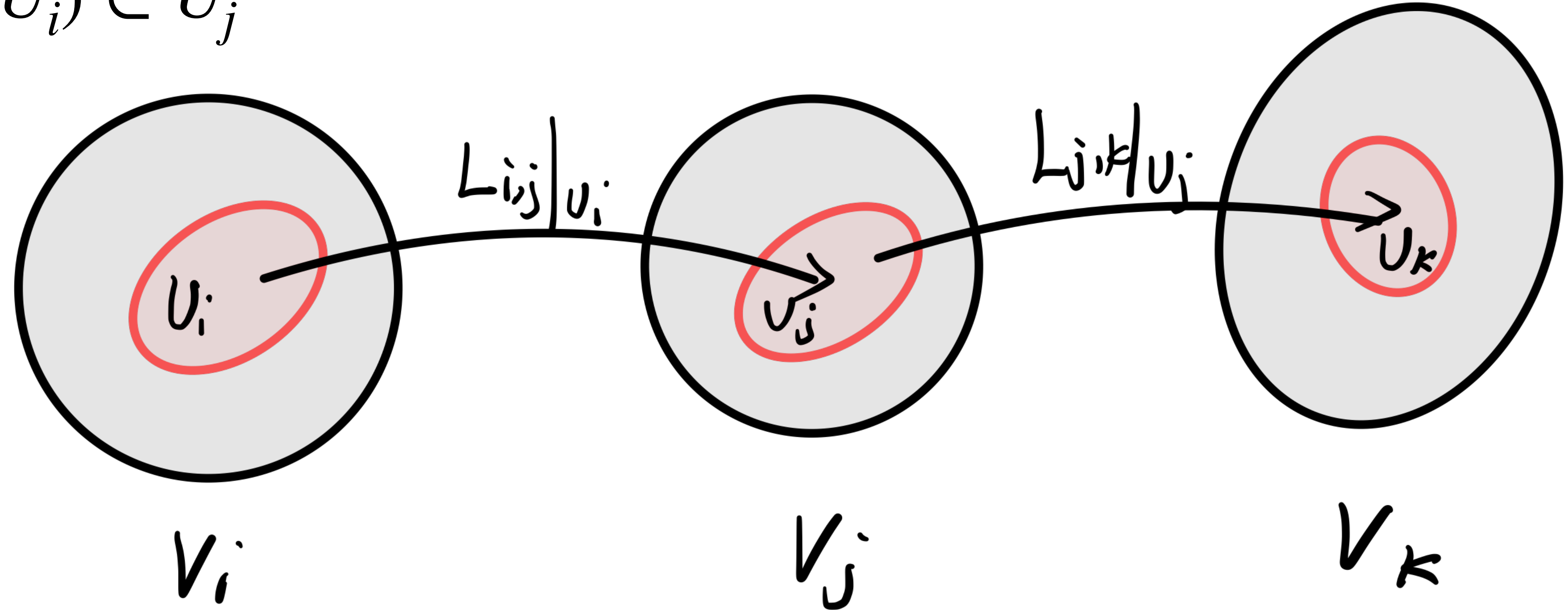
  - φ is called an isomorphism if each φ_i is an isomorphism

Persistence Vector Spaces

- A **sub-persistence vector space** is a collection $U = \{U_i \subset V_i\}$ such that
- $L_{i,j}(U_i) \subset U_j$



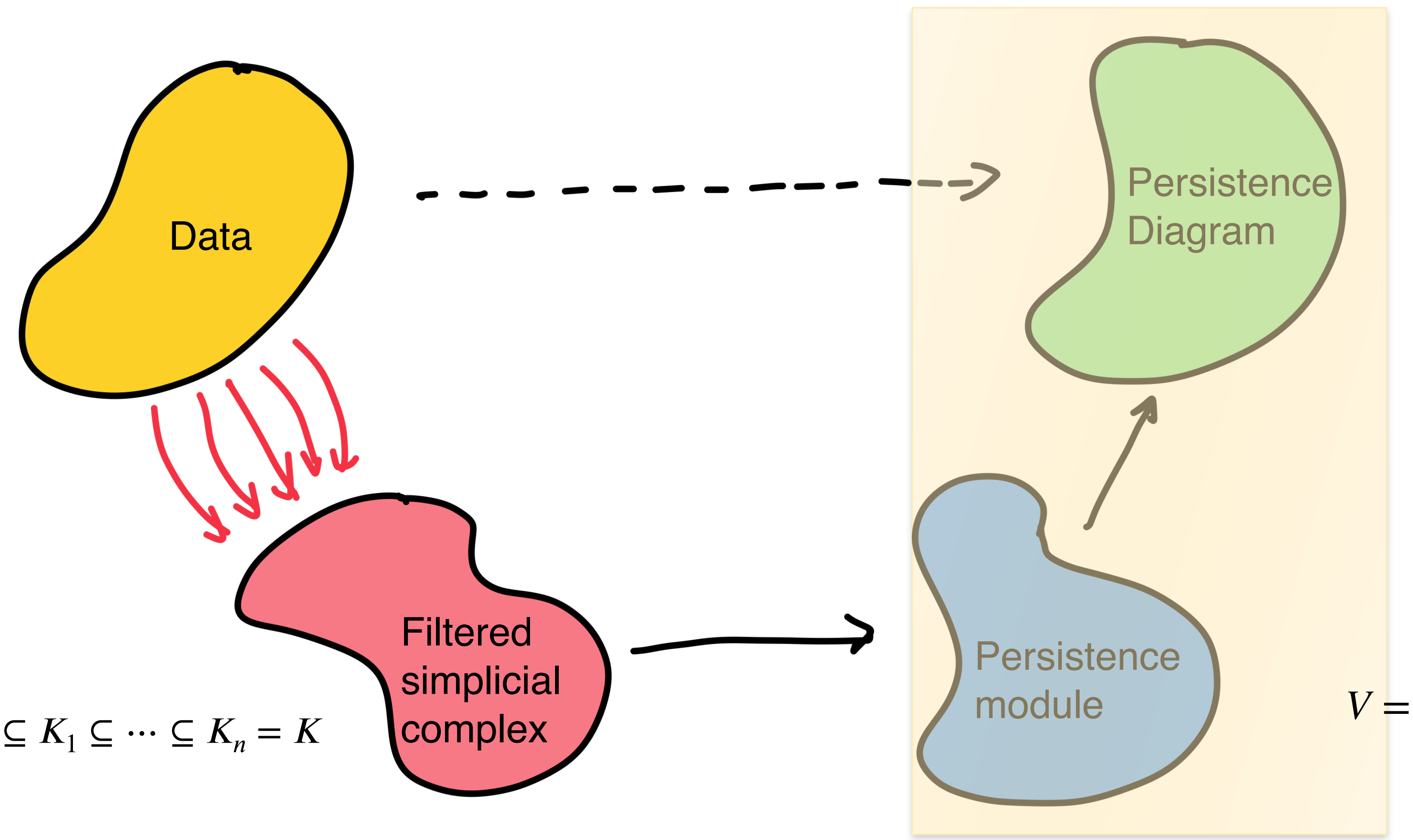
Persistence Vector Spaces

- Let $\{V_i\}$ and $\{W_i\}$ be two persistence vector spaces
- The **direct sum** $V \oplus W$ is the collection $\{V_i \oplus W_i\}$ with maps
- $L_{i,j}^{V\oplus W} = L_{i,j}^{V} \oplus L_{i,j}^{W}$ defined by $L_{i,j}^{V\oplus W}(v, w) = (L_{i,j}^{V}(v), L_{i,j}^{W}(w))$

- Dimension and basis are the most important objects of a vector space

- What are “dimension” and “basis” for a persistence vector space?

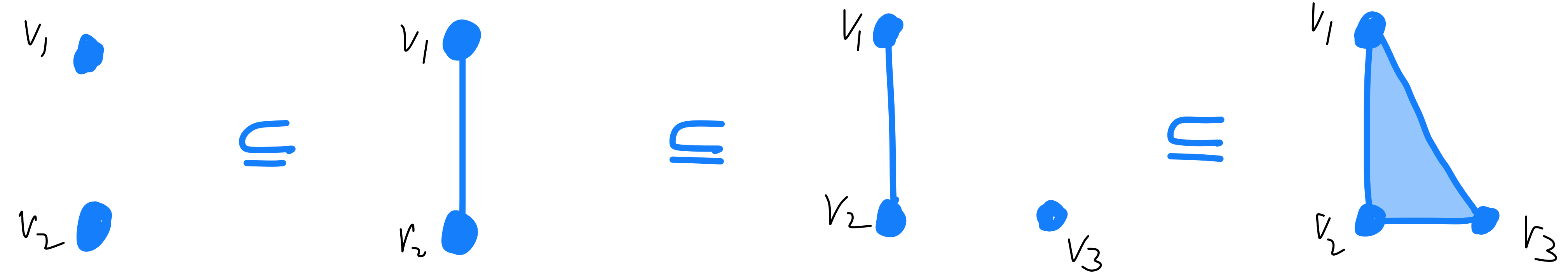
Persistence Diagram



$K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$

$V = \{L_{i,j} : V_i \to V_j\}$

Persistent Module Example



$$V_0 = \langle [v_1],[v_2]\rangle \longrightarrow V_1 = \langle [v_1]\rangle \longrightarrow V_2 = \langle [v_1],[v_3]\rangle \longrightarrow V_3 = \langle [v_1]\rangle$$

$$\xi^{0,1} = \begin{array}{cc} [v_1] & [v_2] \\ \end{array}$$
$$\begin{array}{c} \\ \end{array}\begin{pmatrix} 1, & 1 \end{pmatrix} [v_1]$$

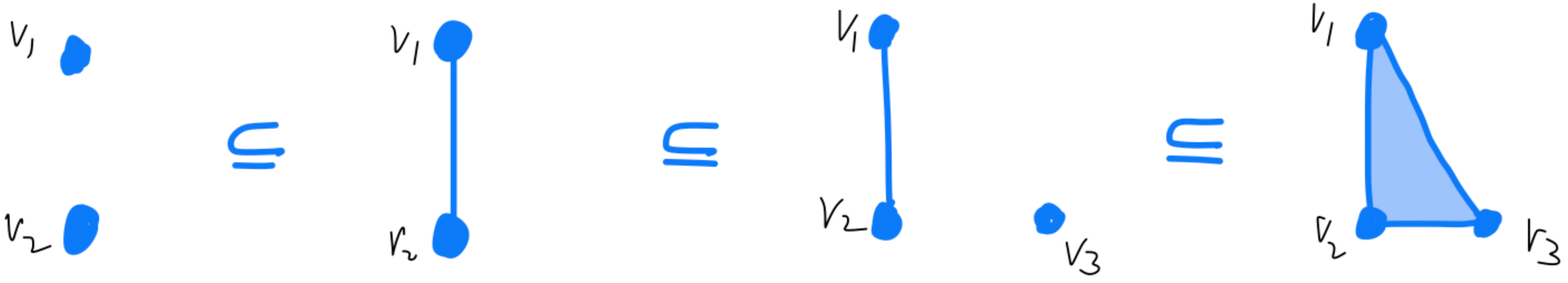
$$\xi^{1,2} = \begin{array}{cc} & [v_1] \\ \begin{array}{c}[v_1]\\ [v_3]\end{array} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array}$$

$$\xi^{2,3} = \begin{array}{cc} & \begin{array}{cc}[v_1] & [v_3]\end{array} \\ [v_1] & \begin{bmatrix} 1 & 1 \end{bmatrix} \end{array}$$

$$\langle [v_1]\rangle \longrightarrow \langle [v_1]\rangle \longrightarrow \langle [v_1]\rangle \longrightarrow \langle [v_1]\rangle$$

$$\langle [v_2]-[v_1]\rangle \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

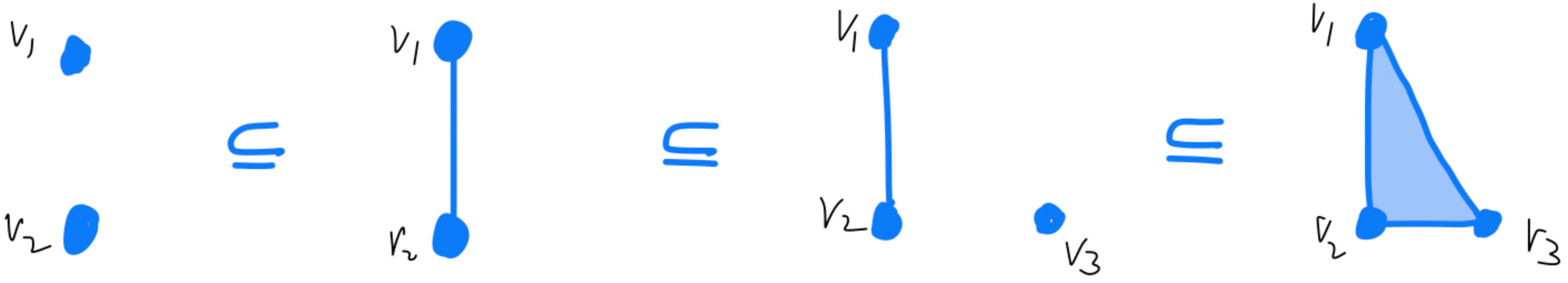
$$0 \longrightarrow 0 \xrightarrow{0} \langle [v_3]-[v_1]\rangle \longrightarrow 0$$



$$\langle [v_1] \rangle \to \langle [v_1] \rangle \to \langle [v_1] \rangle \to \langle [v_1] \rangle \qquad \langle [v_1] - [v_2] \rangle \to 0 \to 0 \to 0 \qquad 0 \to 0 \to \langle [v_3] - [v_1] \rangle \to 0$$

$$\cong \qquad\qquad \cong \qquad\qquad \cong$$

$$\mathbb{F} \to \mathbb{F} \to \mathbb{F} \to \mathbb{F} \qquad \mathbb{F} \to 0 \to 0 \to 0 \qquad 0 \to 0 \to \mathbb{F} \to 0$$



$$\langle [v_1] \rangle \to \langle [v_1] \rangle \to \langle [v_1] \rangle \to \langle [v_1] \rangle \qquad \langle [v_1] - [v_2] \rangle \to 0 \to 0 \to 0 \qquad 0 \to 0 \to \langle [v_3] - [v_1] \rangle \to 0$$

$$\cong \qquad\qquad \cong \qquad\qquad \cong$$

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$$V_0 \to V_1 \to V_2 \to V_3 \cong \begin{array}{l} \mathbb{F} \to \mathbb{F} \to \mathbb{F} \to \mathbb{F} \\ \oplus\ \mathbb{F} \to 0 \to 0 \to 0 \\ \oplus\ 0 \to 0 \to \mathbb{F} \to 0 \end{array}$$

Interval persistence vector spaces

- Given the index set $I = \{0, \ldots, n\}$
- Let $0 \leq b < d \leq n + 1$, the **interval persistence vector space**, denoted by $I[b, d)$ is defined as
- $I[b, d) = 0 \to \cdots \to 0 \to \mathbb{F} \to \mathbb{F} \to \cdots \to \mathbb{F} \to 0 \to \cdots \to 0$

  bth position $\qquad\qquad$ $d - 1$th position

- $I[b, n + 1) = 0 \to \cdots \to 0 \to \mathbb{F} \to \mathbb{F} \to \cdots \to \mathbb{F}$ is often written as $I[b, \infty)$

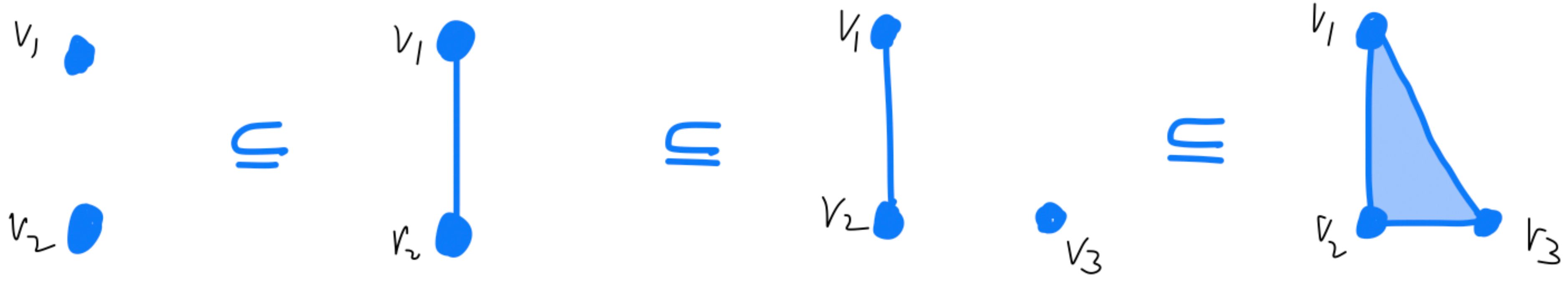
Decomposition Theorem

- Let $V = \{V_i\}_{i=0}^{n}$ be any persistence vector space. Then, there exist a collection of $0 \leq b_j < d_j \leq n+1, j = 1, \ldots, M$ such that
- $V \cong I[b_1, d_1) \oplus I[b_2, d_2) \oplus \cdots \oplus I[b_M, d_M)$
- The composition is unique up to reordering the summands.

Persistence Diagram and Barcodes

- $V \cong I[b_1, d_1) \oplus I[b_2, d_2) \oplus \cdots \oplus I[b_M, d_M)$
- Each (b_j, d_j) is called a **persistence pairing**
- The multiset $D = \{(b_j, d_j)\}_{j=1,\ldots,M} \subseteq \mathbb{R}^2$ is called the **persistence diagram** of V
- The collection of intervals $\{[b_j, d_j)\}_{j=1,\ldots,M}$ is called the **barcode** of V

Example



$$\langle[v_1]\rangle \to \langle[v_1]\rangle \to \langle[v_1]\rangle \to \langle[v_1]\rangle \qquad \langle[v_1 - v_2]\rangle \to 0 \to 0 \to 0 \qquad 0 \to 0 \to \langle[v_3] - [v_1]\rangle \to 0$$

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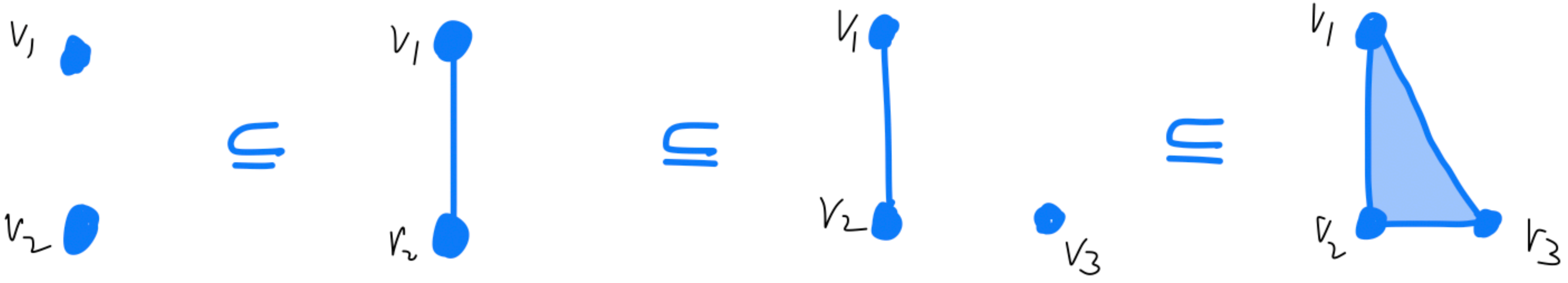

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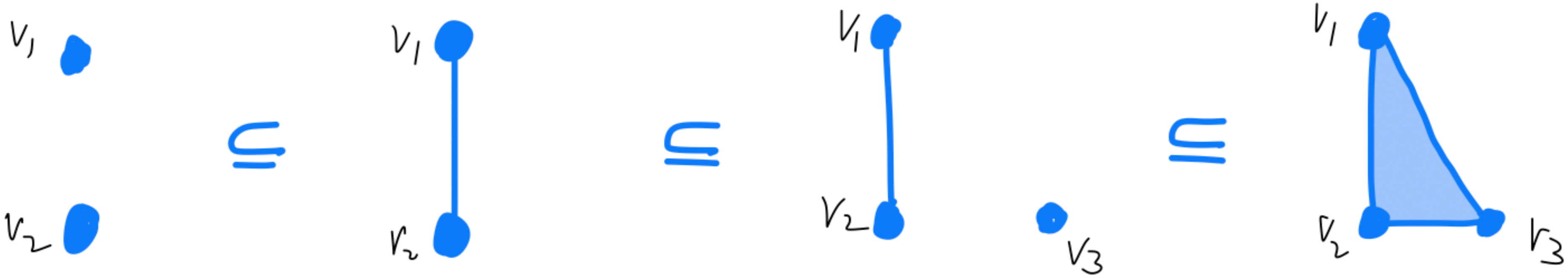
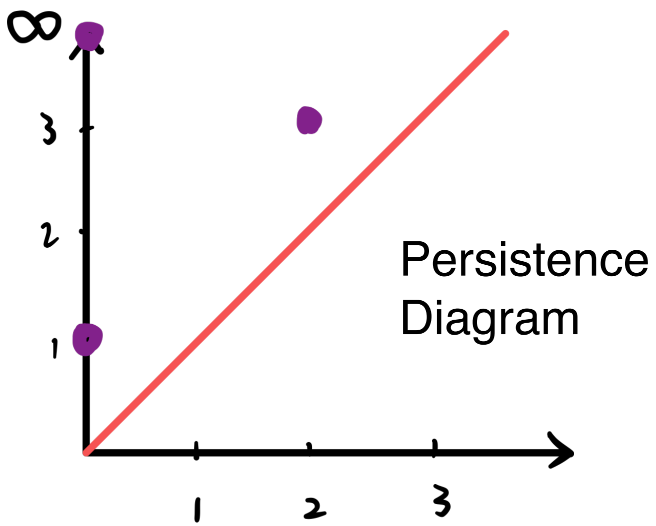
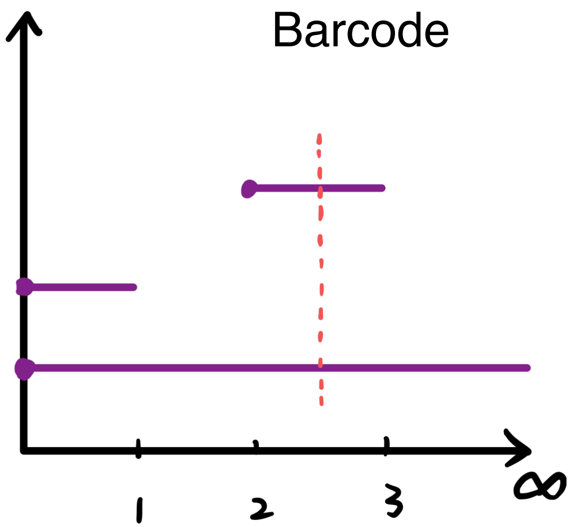
Example



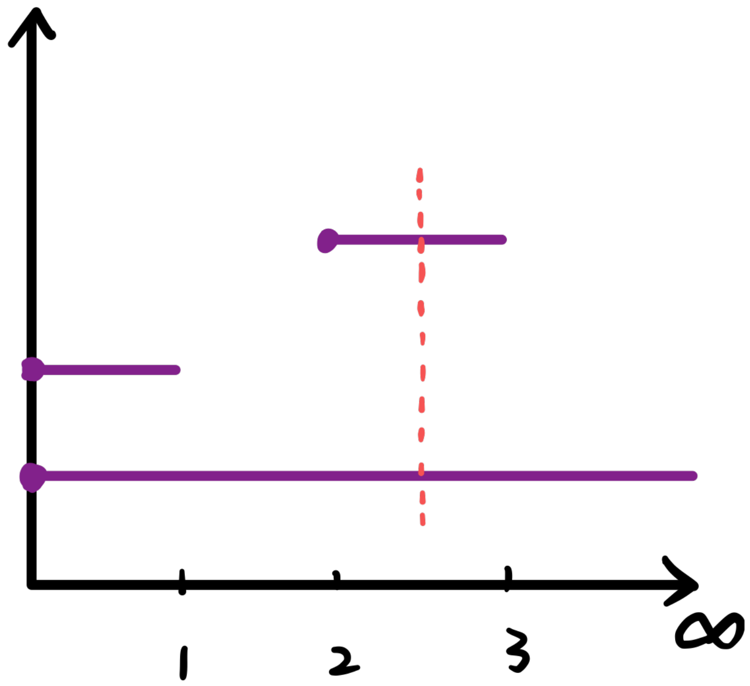
$$V_0 \to V_1 \to V_2 \to V_3 \cong I[0,\infty) \oplus I[0,1) \oplus I[2,3)$$



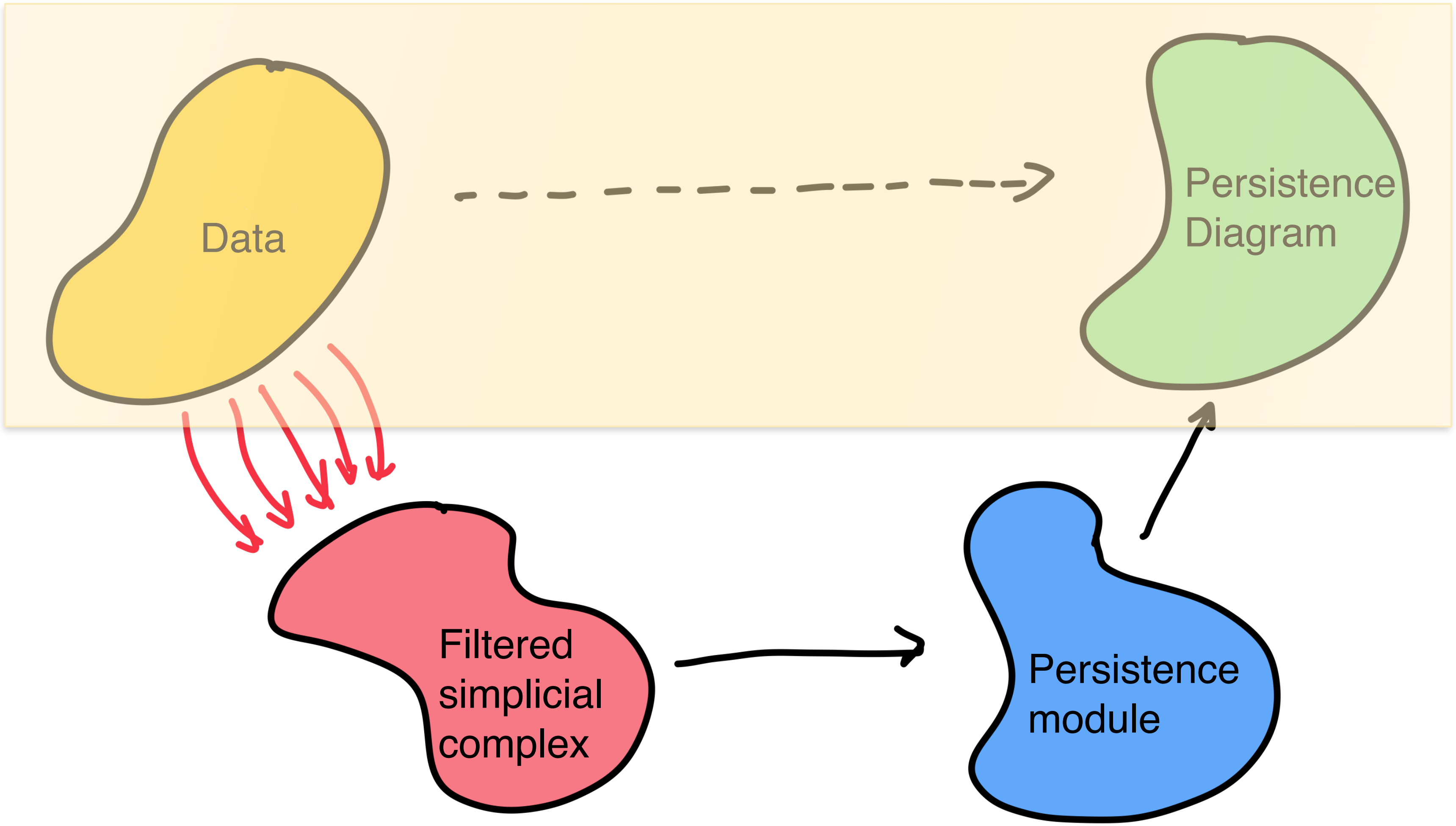


Remark

- Persistence diagrams (or barcodes) are serving the role of dimension of vector spaces



TDA in a nutshell



Persistence Diagram and Barcodes for filtrations

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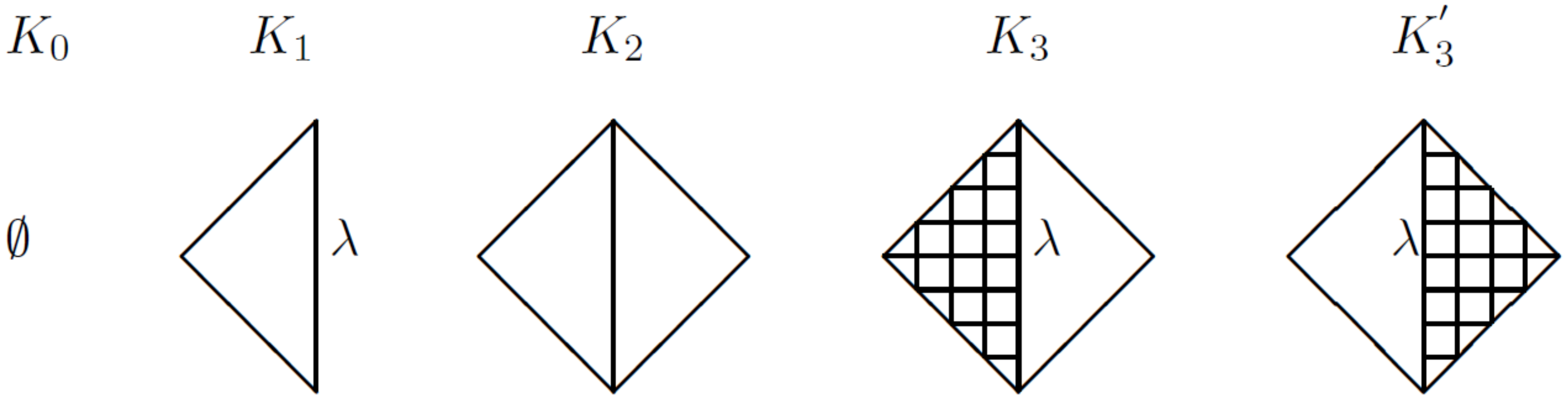
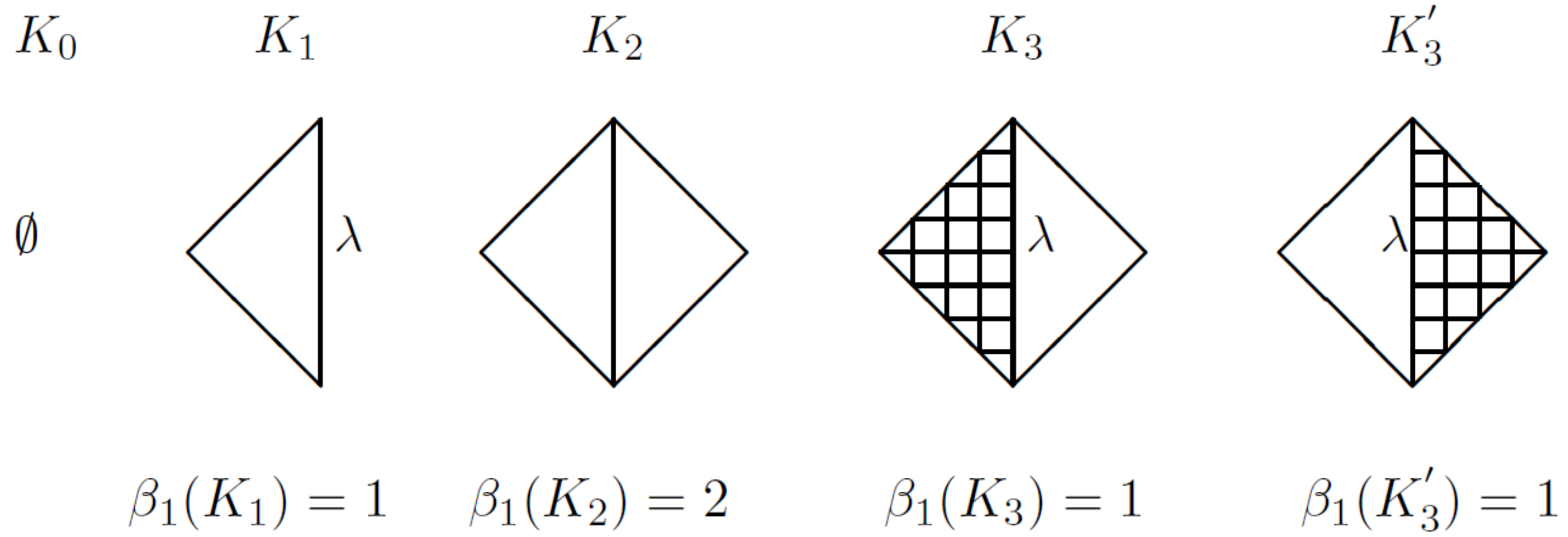


K_0
K_1
K_2
K_3
K_3'
$\emptyset$
λ
λ
λ



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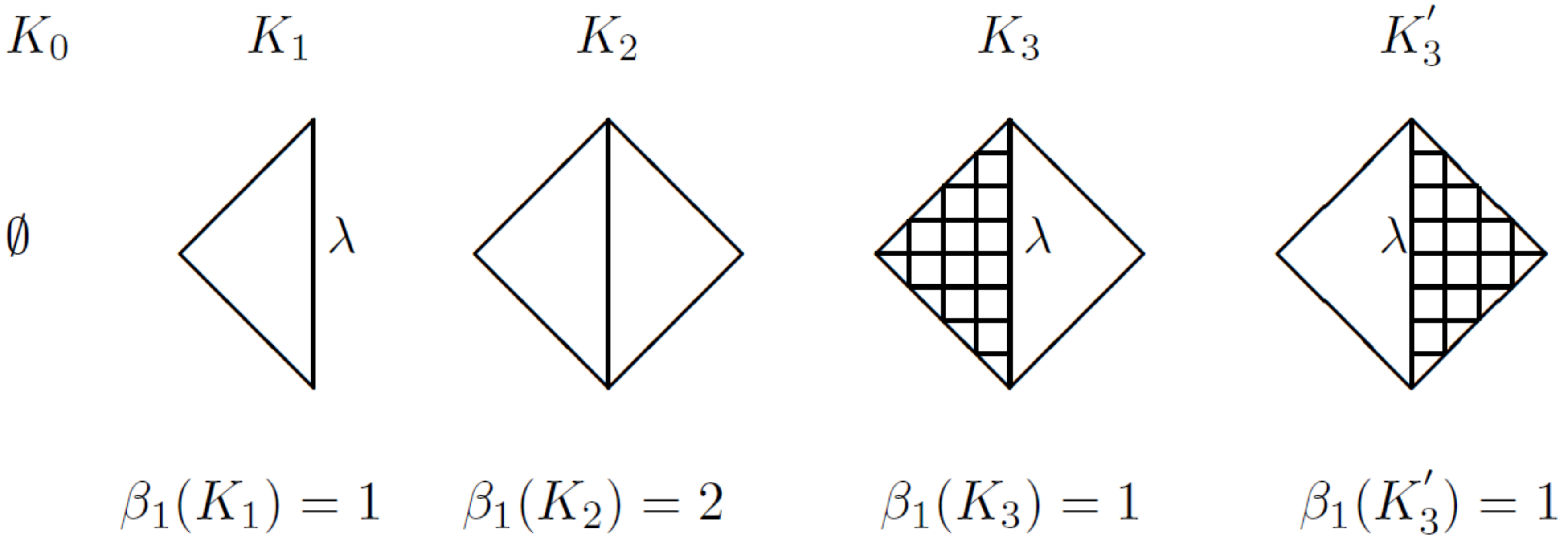
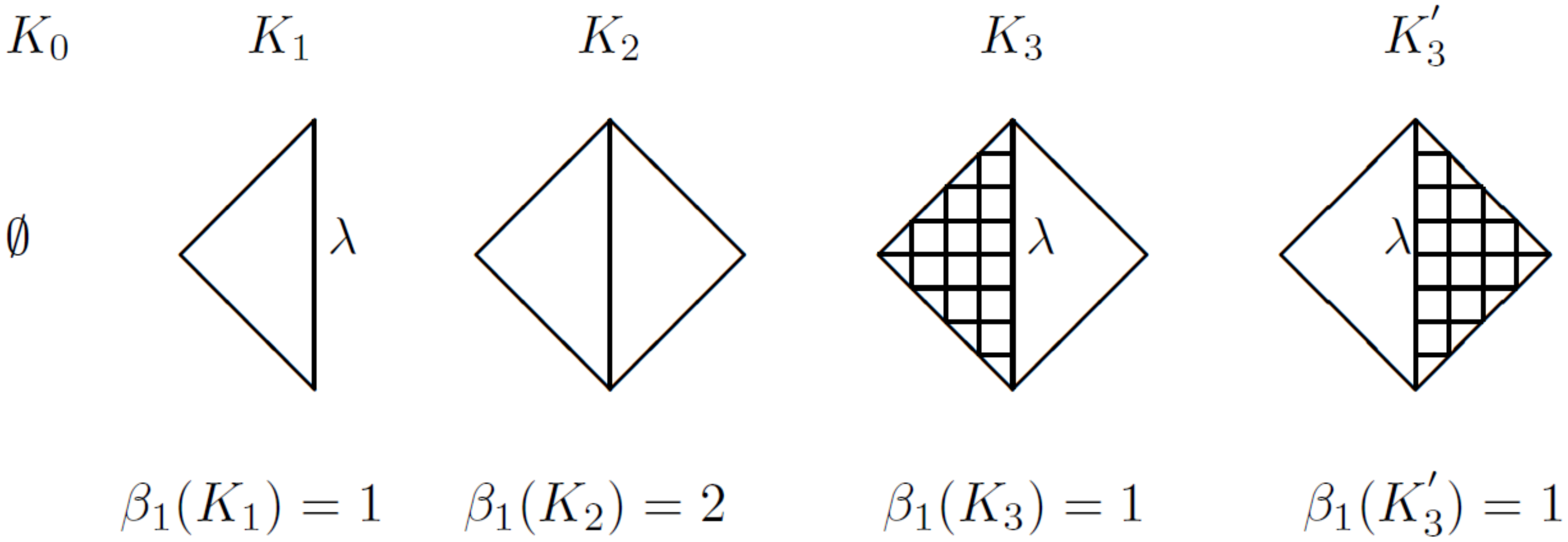


- For $K_0 \subset K_1 \subset K_2 \subset K_3$, what is $\mu_1^{1,3}$? $\mu_1^{1,2}$?



- For $K_0 \subset K_1 \subset K_2 \subset K_3$, what is $\mu_1^{1,3}$? $\mu_1^{1,2}$?
- How about for the filtration $K_0 \subset K_1 \subset K_2 \subset K_3'$?

How to compute the decomposition?

- Möbius inversion
- Simplex-wise filtration

Möbius Inversion - counting the number of persistence pairings

Persistence Modules

- $K \subseteq K' \Rightarrow \xi_p : H_p(K) \to H_p(K')$
  - Inclusion maps induce homomorphisms in homology groups (under Z_2-coefficients, linear maps in vector spaces)

$$\boxed{\begin{array}{l} K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K \\ \Rightarrow \quad H_*(K_0) \to H_*(K_1) \to \cdots \to H_*(K_n) = H_*(K) \end{array}}$$

- Define $\xi_*^{i,j} : H_*(K_i) \to H_*\left(K_j\right)$
  - $\xi_*^{i,j} = \xi_*^{j-1,\, j} \circ \cdots \circ \xi_*^{i,i+1}$
- Persistent module induced by the filtration
  - $\mathscr{P} = \left\{ H_*(K_i) \xrightarrow{\xi^{i,j}} H_*\left(K_j\right) \right\}_{0 \le i \le j \le n}$

Persistent Homology

- p-th persistent homology group from i to j:
  - $(H_p(K_j) \supset)\ H_p^{i,j} = \mathrm{Im}(\xi_p^{i,j})$
    - Subgroup of $H_p\left(K_j\right)$ that ``existed'' in $H_p\left(K_i\right)$

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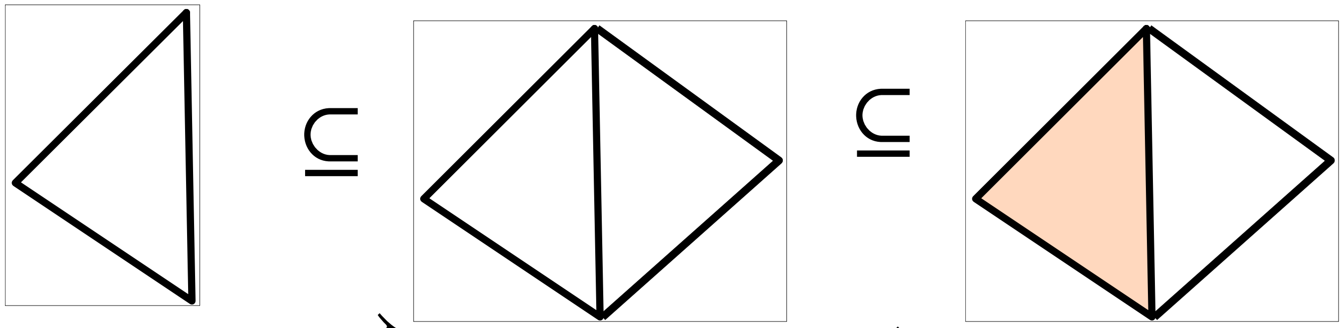
$\beta_1(K_1) = 1$ $\quad$ $\beta_1(K_2) = 2$ $\quad$ $\beta_1(K_3) = 1$



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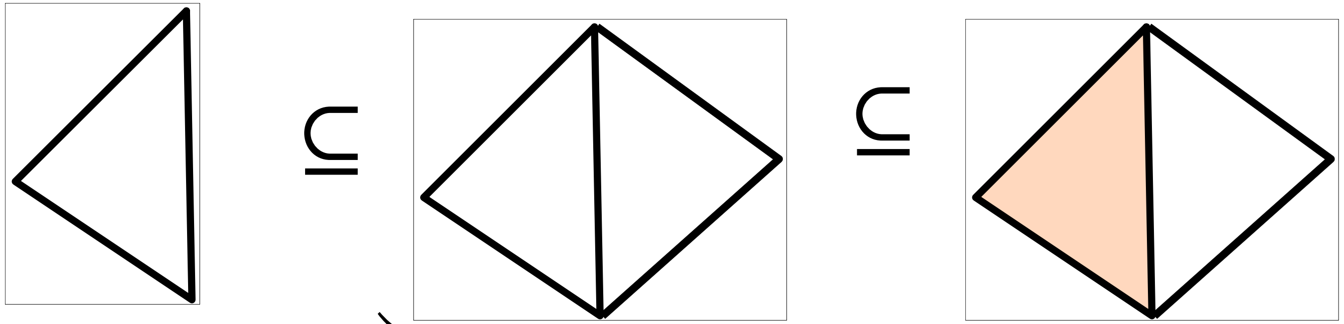

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$\beta_1^{1,3} = ?\ \beta_1^{1,2} = ?$

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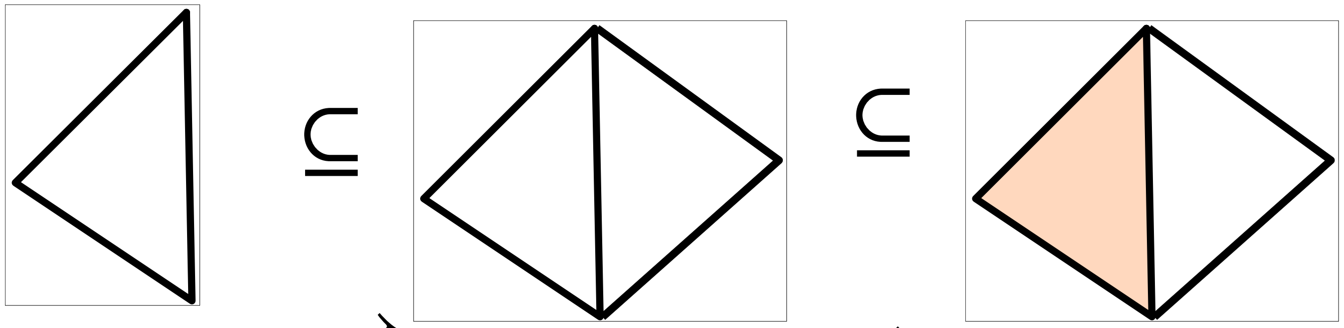

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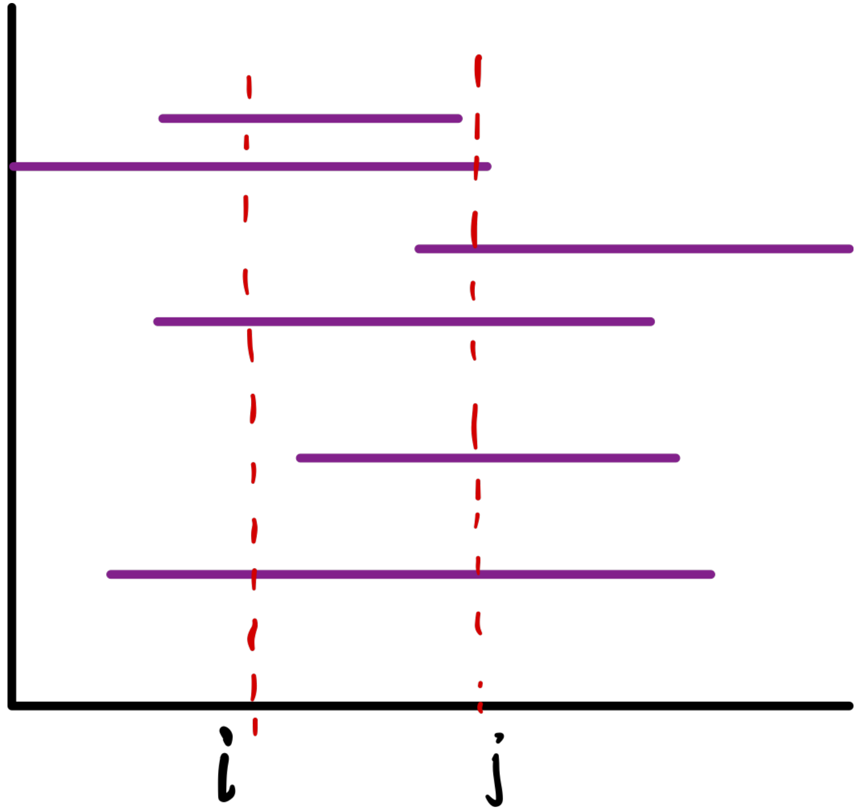
- p-th persistent betti number: $\beta_p^{i,j} = \dim H_p^{i,j}$
- $\beta_p^{i,j}$ denotes the number of homology classes existing at both K_i and K_j
- As long as one can write down a matrix representation Ξ of $\xi_p^{i,j} : H_p(K_i) \to H_p(K_j)$, one has that $\beta_p^{i,j} = \mathrm{rank}\Xi$

Connection to decomposition theorem

- Let $V = \{V_i = H_p(K_i)\}_{i=0}^{n}$ be the p-dim persistence module for the filtered simplicial complex $K = \{K_i\}$
- Assume that $V \cong I[b_1, d_1) \oplus I[b_2, d_2) \oplus \cdots \oplus I[b_M, d_M)$
- $\mu^{b,d}$:= number of intervals $I[b, d)$

Persistence Betti Number vs Barcode

$$\beta_p^{i,j} = \dim H_p^{i,j}$$



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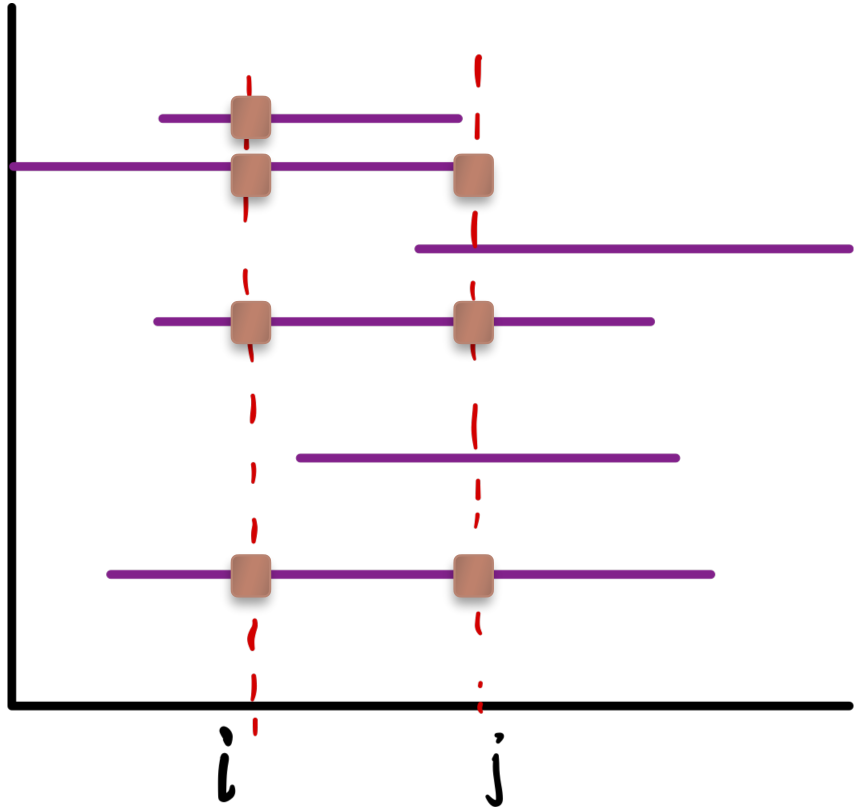
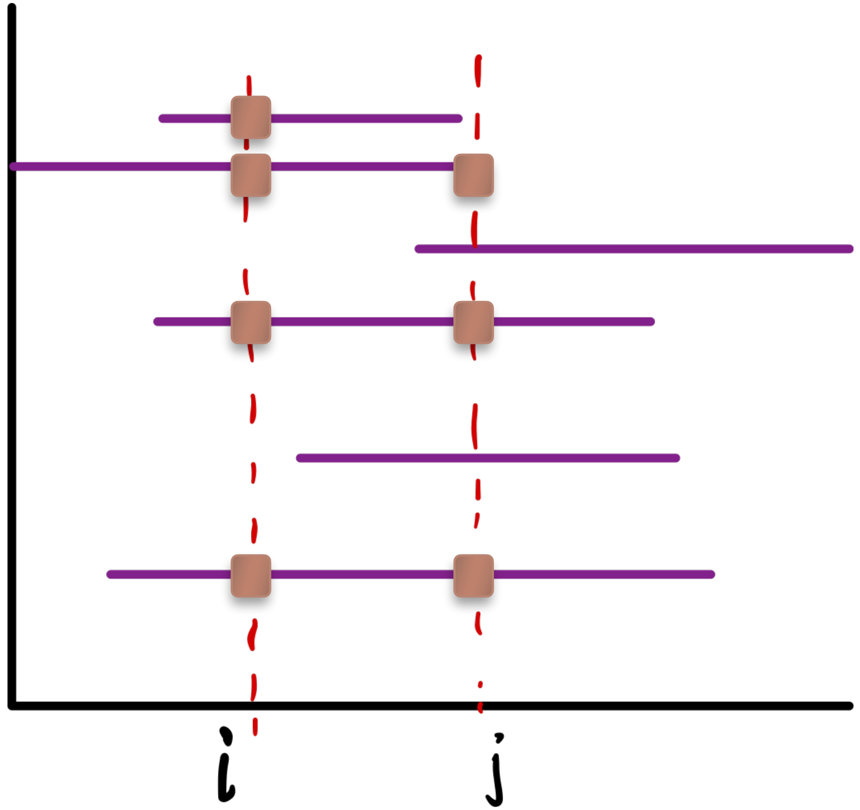


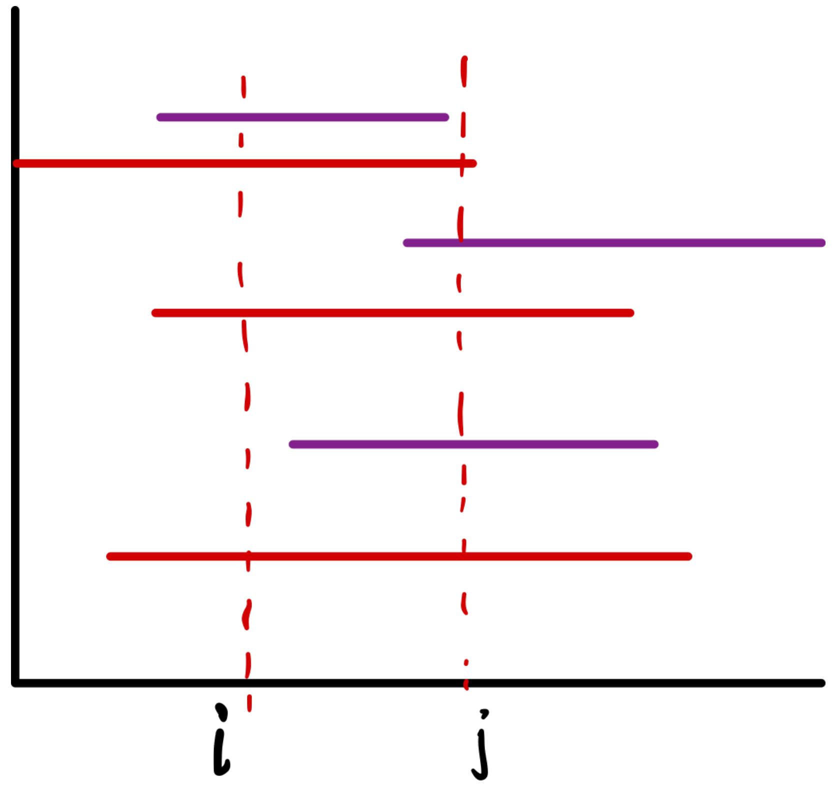
Persistence Betti Number vs Barcode

$$\beta_p^{i,j} = \dim H_p^{i,j}$$

$$\beta_p^{i,j} = 3$$

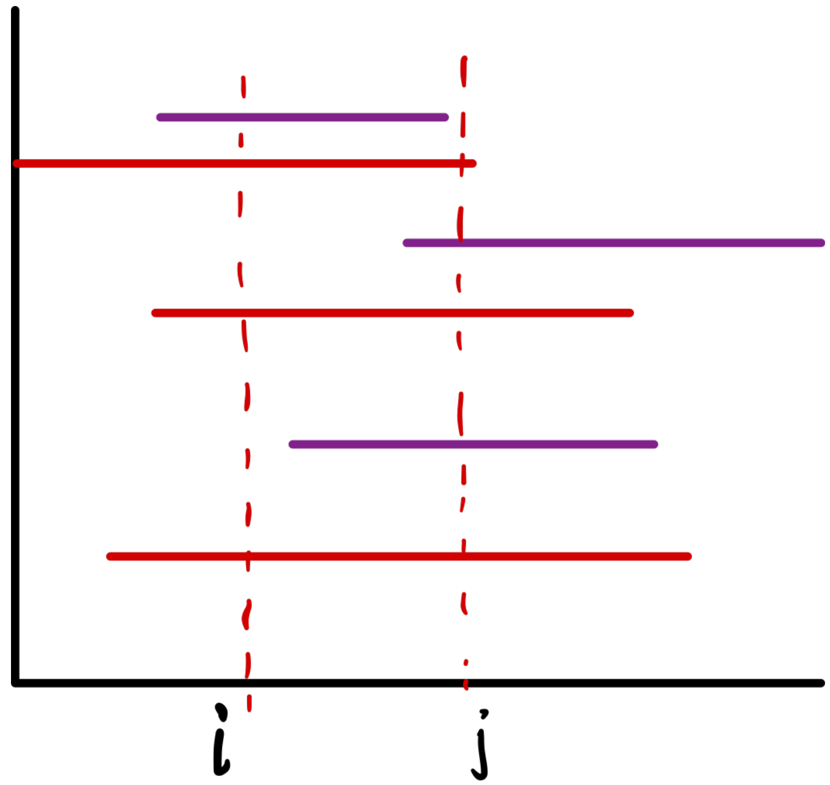


Persistence Betti Number vs Barcode



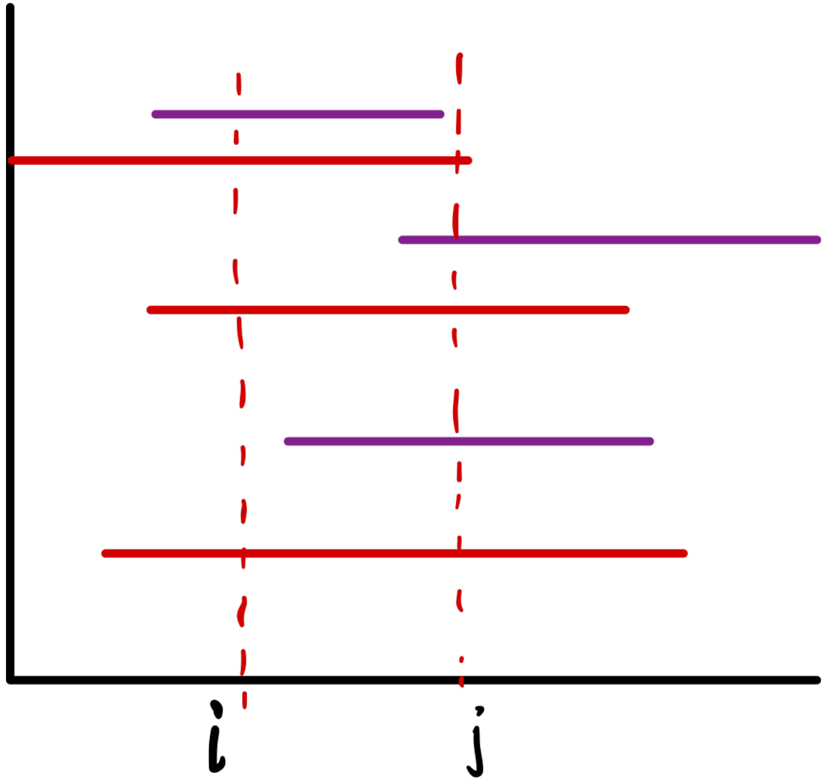
Persistence Betti Number vs Barcode

- $\beta_p^{i,j}$ = # of bars crossing both vertical lines

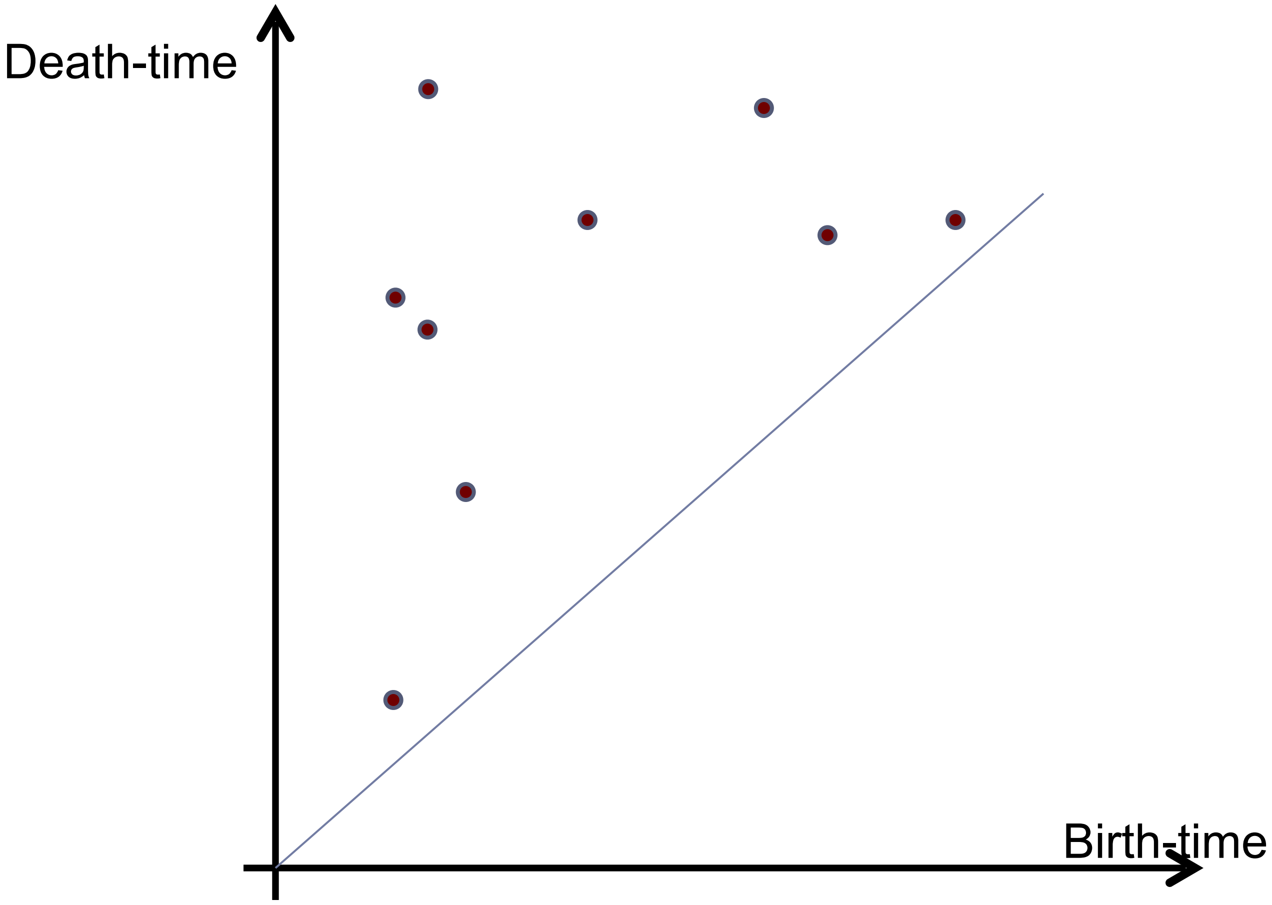


Persistence Betti Number vs Barcode

- $\beta_p^{i,j}$= # of bars crossing both vertical lines
- $$\beta_p^{i,j} = \sum_{k \le i, j < l} \mu_p^{k,l}$$



Persistence Betti Number vs Persistence Diagram



Persistence Betti Number vs Persistence Diagram

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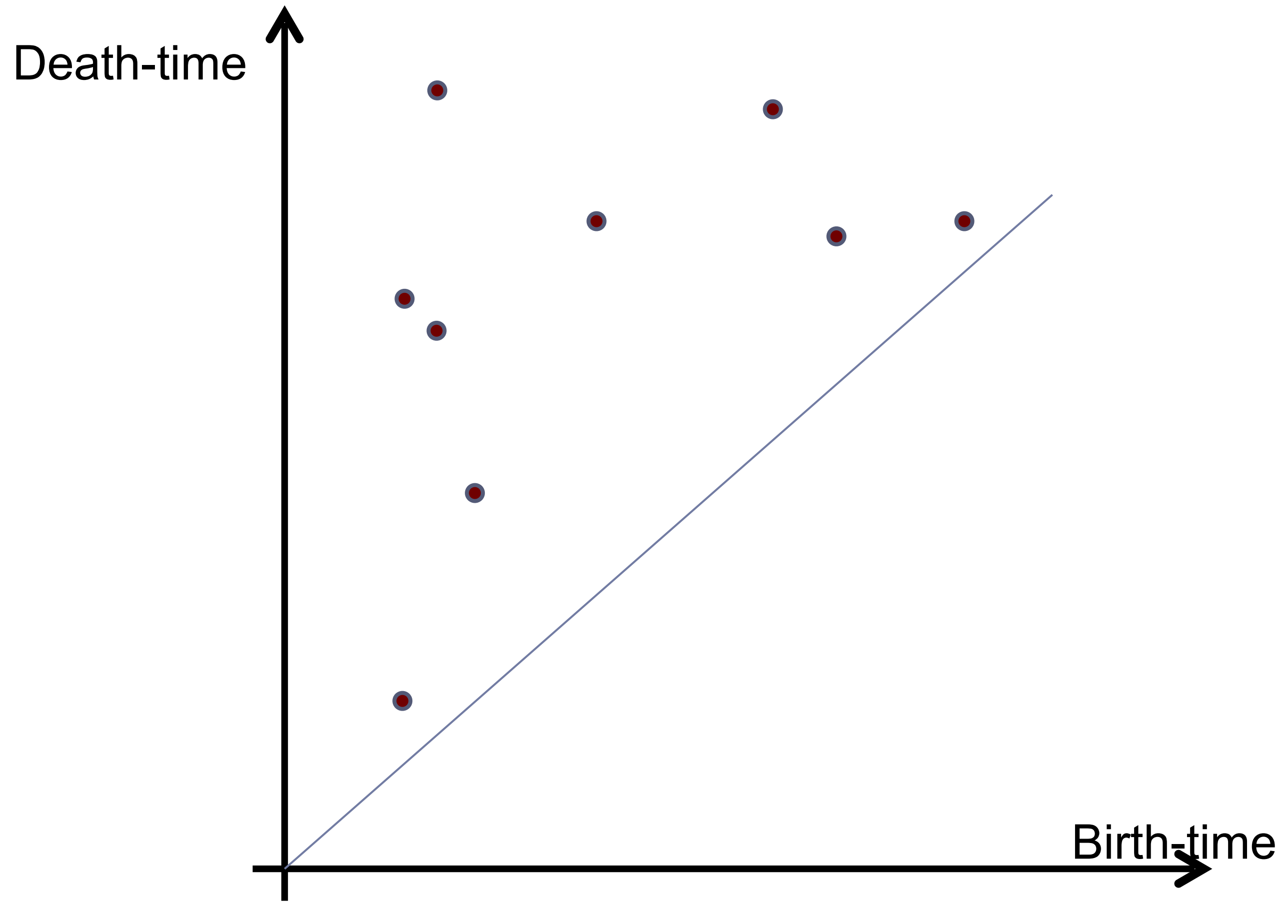

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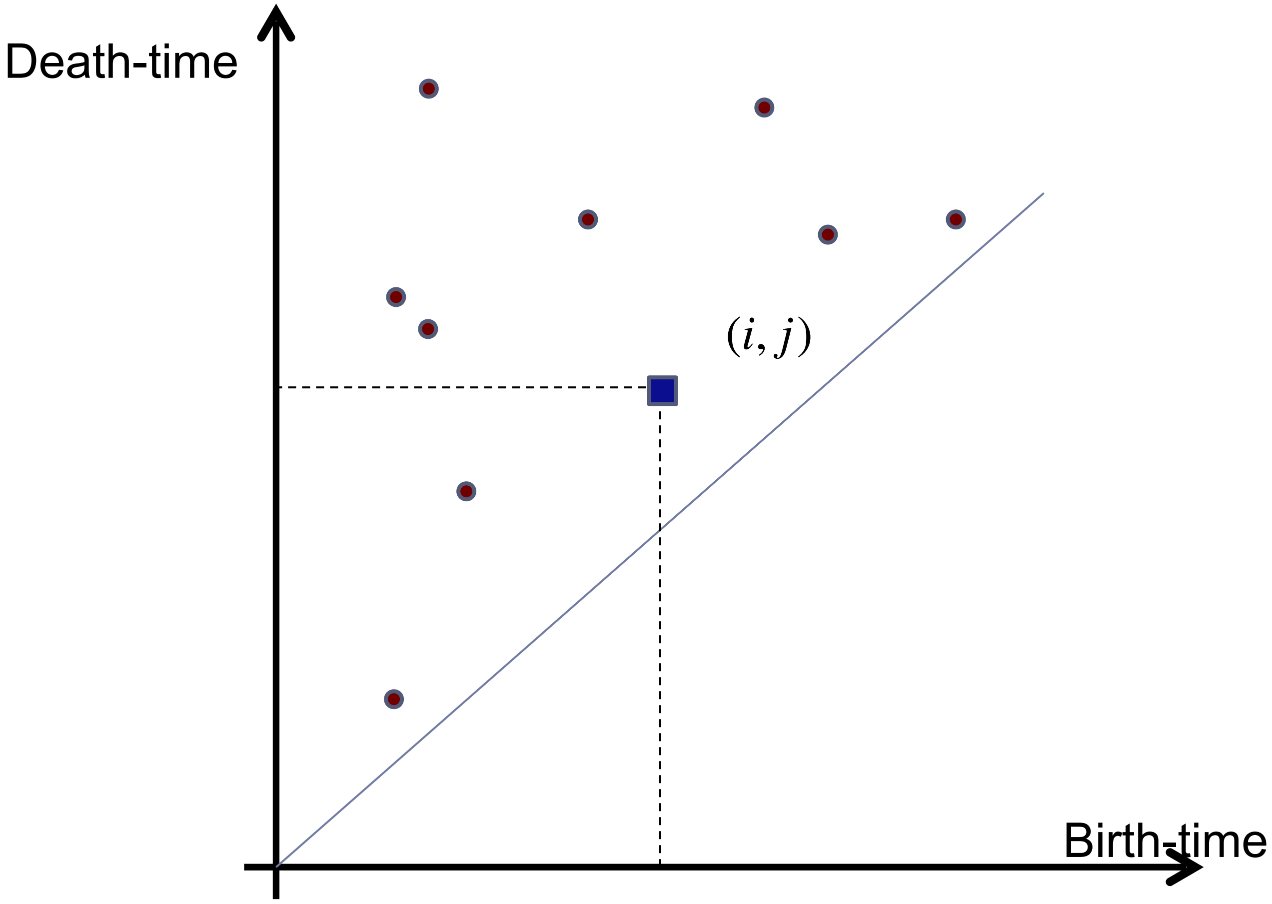
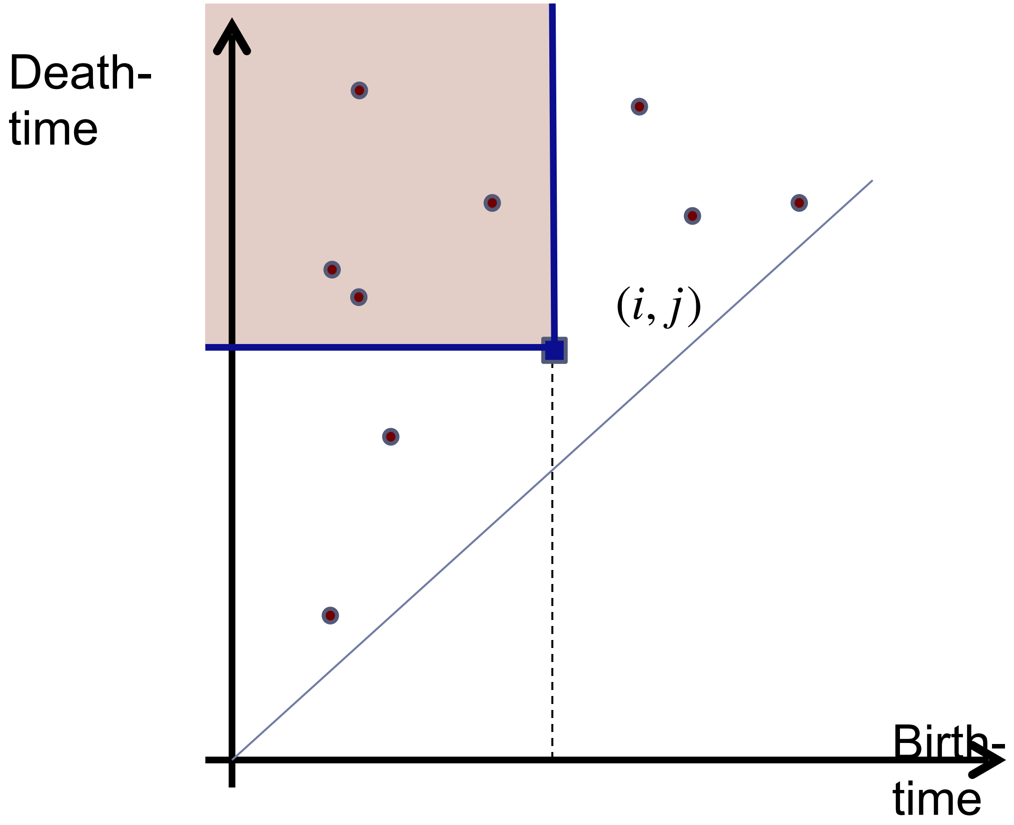


Persistence Betti Number vs Persistence Diagram

- Theorem:
  - $$\beta_p^{i,j} = \sum_{k \le i, j < l} \mu_p^{k,l}$$



Möbius inversion

- For $0 \leq i < j \leq n+1$, the multiplicity of (i, j) can be computed as follows
  - $\mu_p^{i,j} = (\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j})$
  - $\beta_p^{-1,j} = \beta^{i,n+1} = 0$

Möbius inversion

- Persistent pairing number:
  - $\mu_p^{i,j} = \underbrace{(\beta_p^{i,j-1} - \beta_p^{i,j})} - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j})$

$$K_i \to \cdots \to K_{j-1} \to K_j$$

$\beta_p^{i,j-1}$

Möbius inversion

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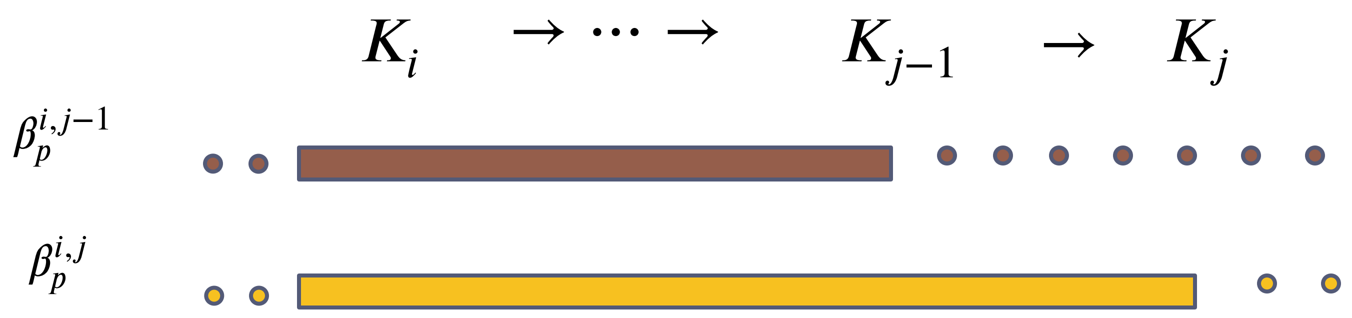

Möbius inversion

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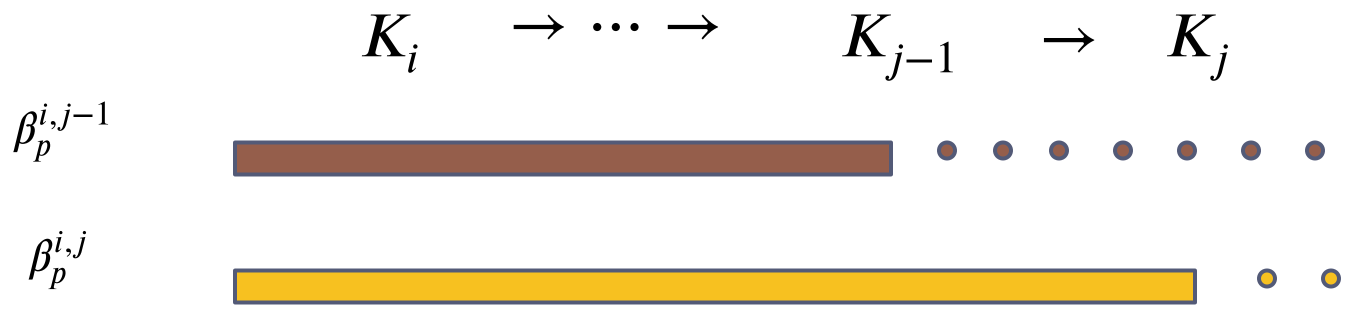

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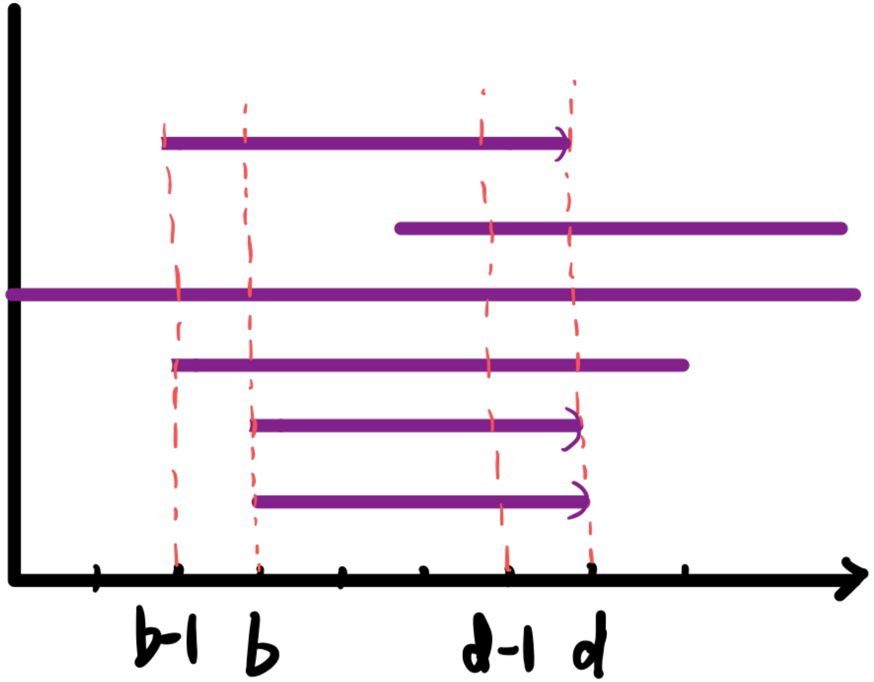
$$K_{i-1} \to K_i \to \cdots \to K_{j-1} \to K_j$$

Möbius inversion

- Persistent pairing number:
  - $$\mu_p^{i,j} = \underbrace{(\beta_p^{i,j-1} - \beta_p^{i,j})}_{\text{Number of independent homology classes from } K_i \text{ but died entering } K_j} - \underbrace{(\beta_p^{i-1,j-1} - \beta_p^{i-1,j})}_{\text{Number of independent homology classes from } K_{i-1} \text{ but died entering } K_j}$$
  - $\mu_p^{i,j}$ denotes the number of independent homology classes created at K_i and died entering K_j

Example

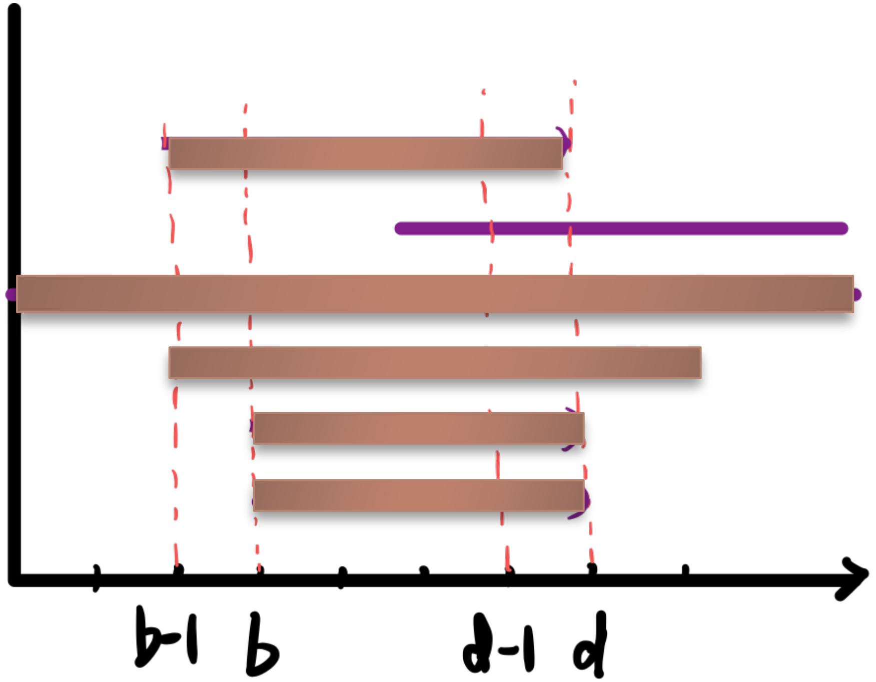
- $\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$



$\#I[b,d) = 2$

Example

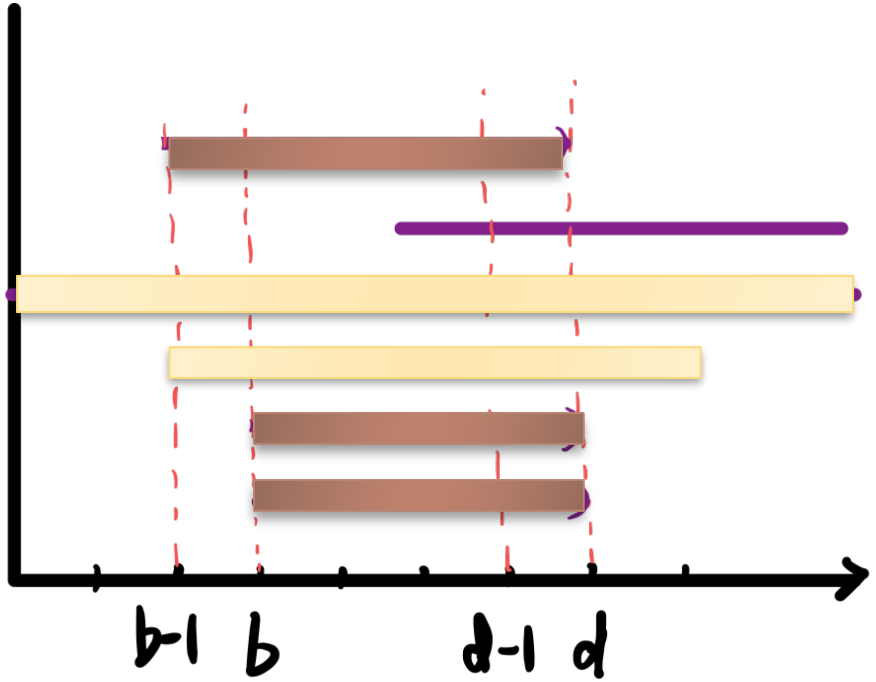
- $\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$



$\beta^{b,d-1} = 5$

Example

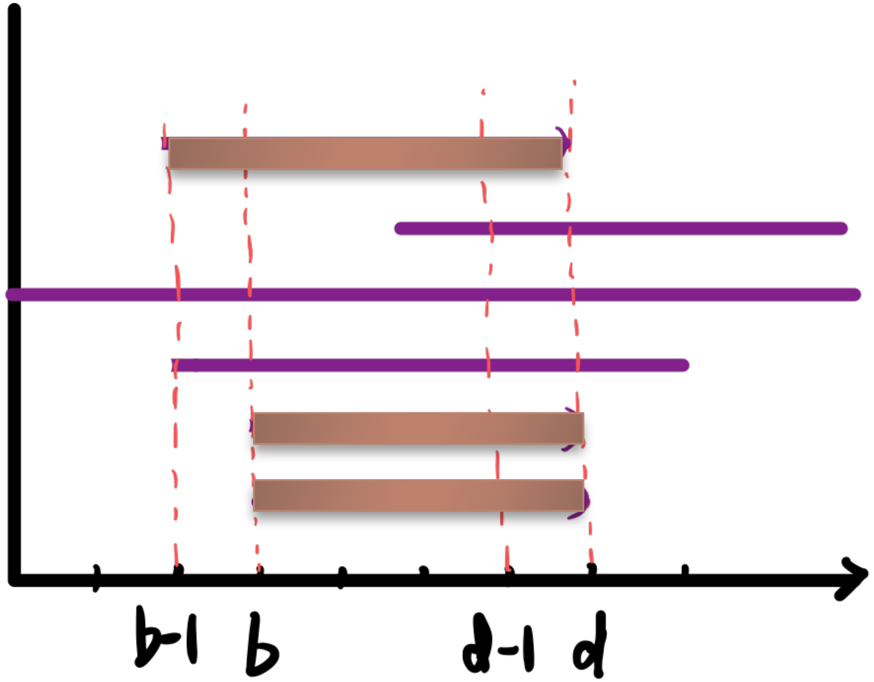
- $\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$



$\beta^{b,d} = 2$

Example

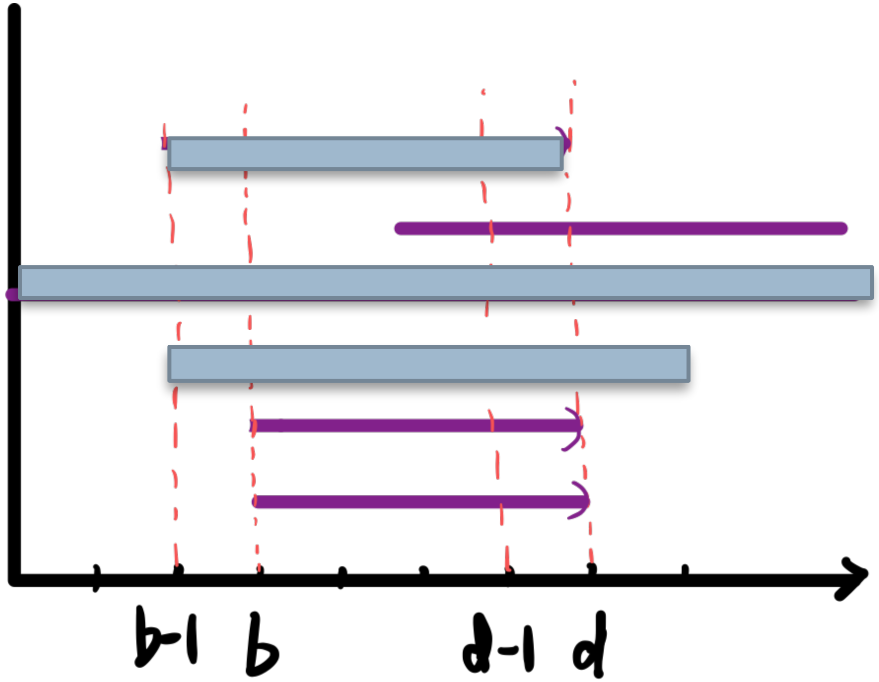
- $\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$



$\beta^{b,d-1} - \beta^{b,d} = 3$

Example

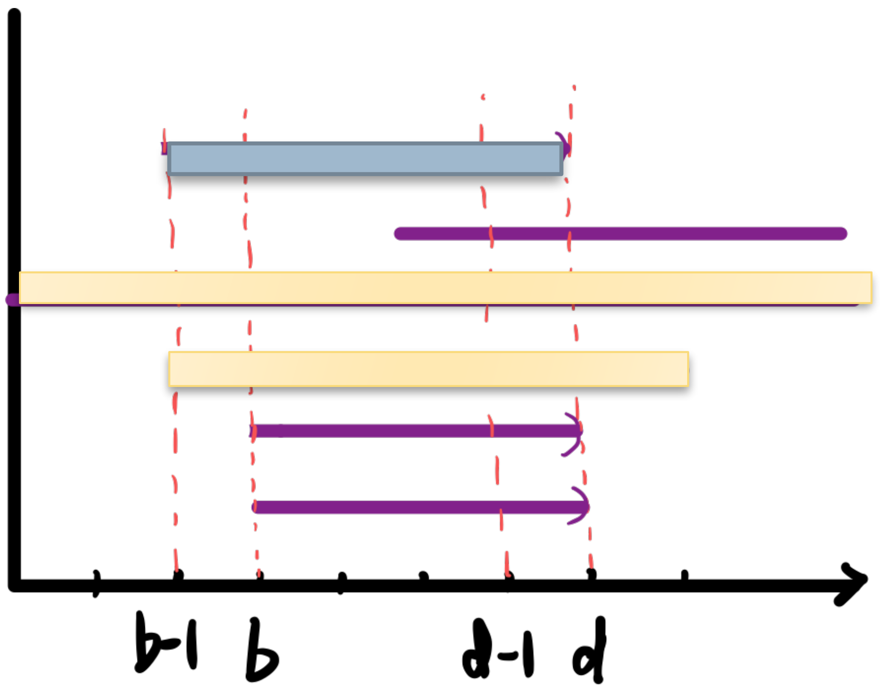
- $\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$



$\beta^{b-1,d-1} = 3$

Example

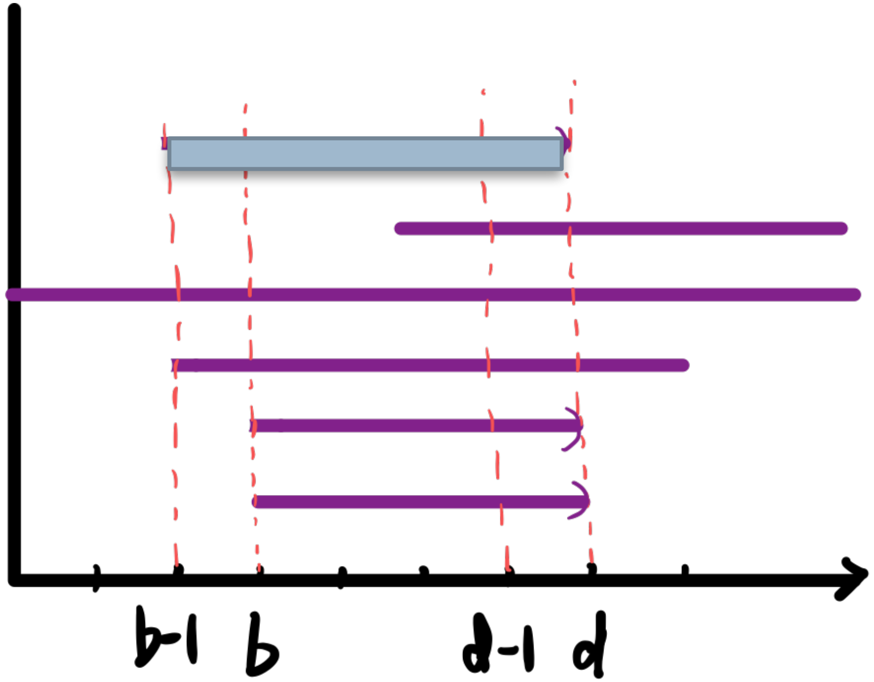
- $\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$



$\beta^{b-1,d} = 2$

Example

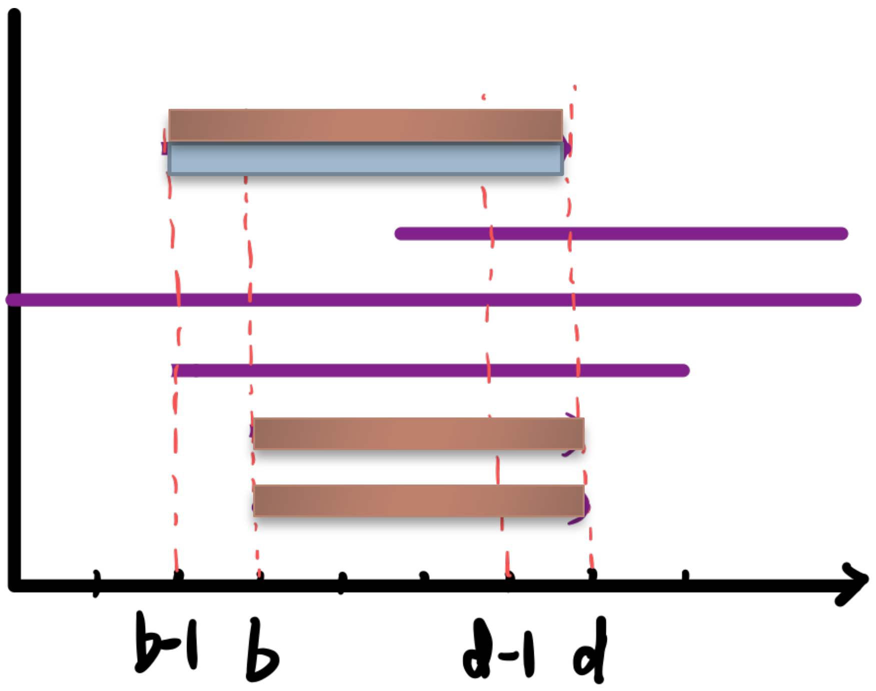
- $\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$



$\beta^{b-1,d-1} - \beta^{b-1,d} = 1$

Example

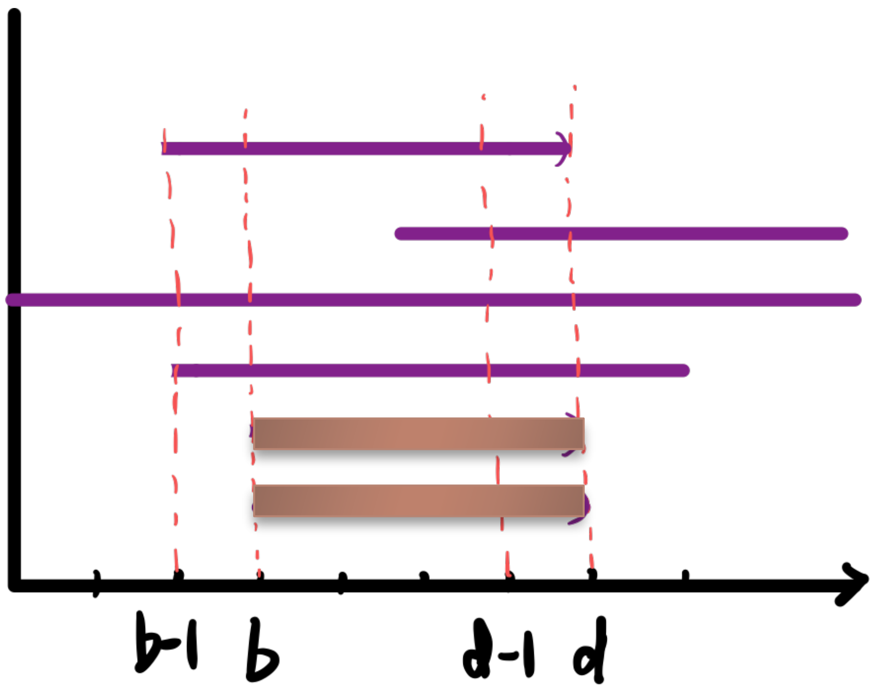
- $\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$



$\mu^{b,d} = 3 - 1 = 2$

Example

- $\mu^{b,d} = (\beta^{b,d-1} - \beta^{b,d}) - (\beta^{b-1,d-1} - \beta^{b-1,d})$



$\mu^{b,d} = 3 - 1 = 2$

A more refined topological view - write down barcodes explicitly

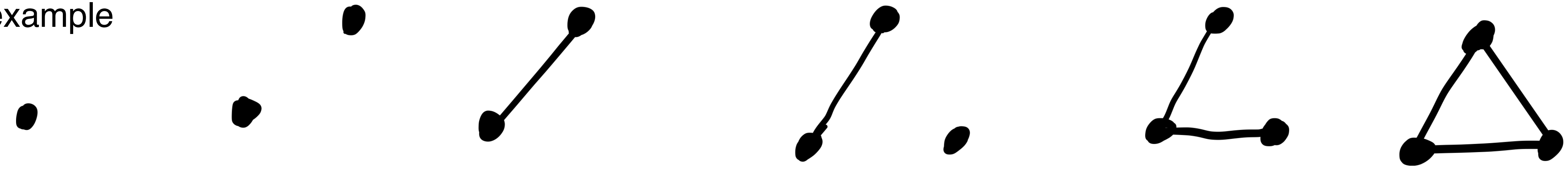
An alternative view

- **Simplex-wise filtration**

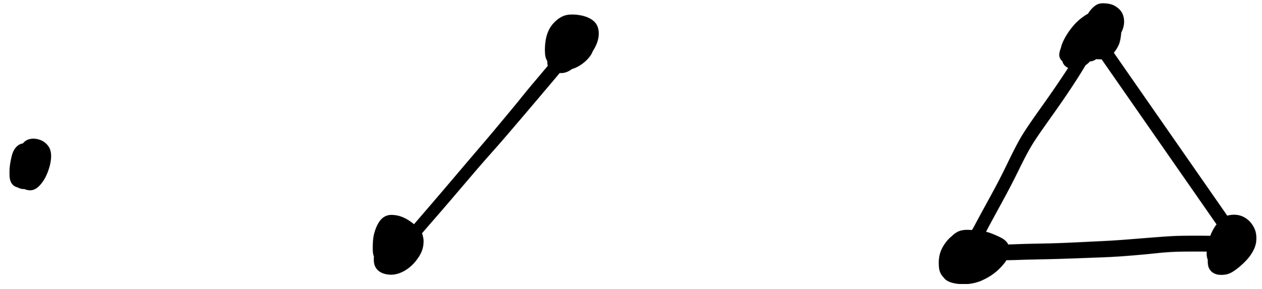
$$\varnothing = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K$$

  - s.t , $\sigma_i = K_i \setminus K_{i-1}$

example



non-example



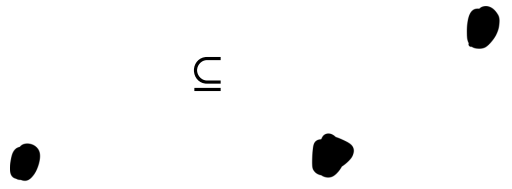
An alternative view

- **Simplex-wise filtration**

$\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K$

  - s.t , $\sigma_i = K_i \setminus K_{i-1}$

- Suppose we are at K_i, and consider p-simplex $\sigma = \sigma_{i+1}$
  - creator: adding σ creates a p-cycle
    - this cycle then must be ``new'', creates a homology class which is not in the image of $H_p(K_i) \to H_p(K_{i+1})$
    - hence β_p++



An alternative view

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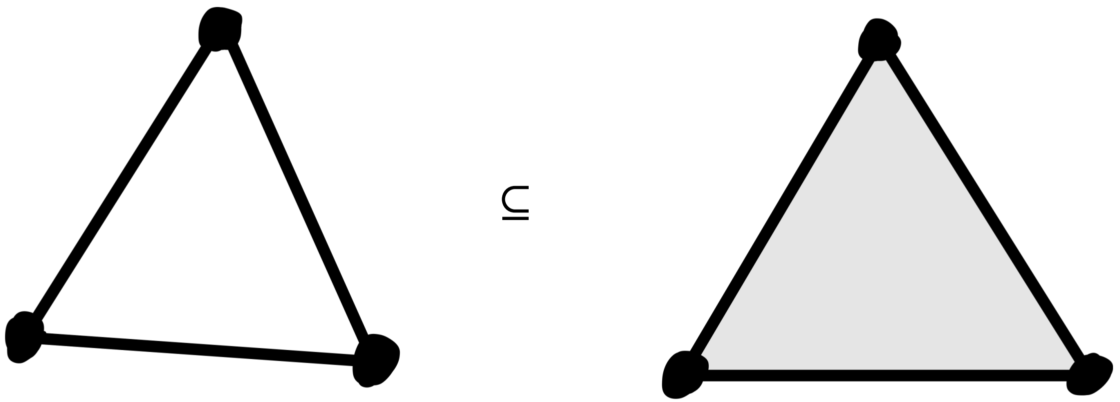

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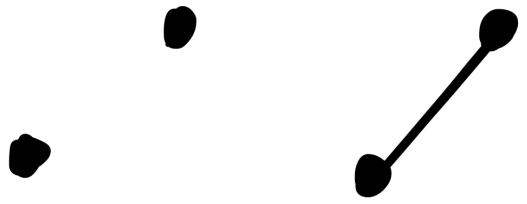
  - creator: adding σ creates a p-cycle
    - this cycle then must be ``new'', creates a homology class which is not in the image of $H_p(K_i) \to H_p(K_{i+1})$
    - hence $\beta_p + +$
  - destroyer: killing a $(p-1)$-cycle
    - this $(p-1)$-cycle is $\partial\sigma$, and $[\partial\sigma] \neq 0$ in $H_{p-1}(K_i)$, but trivial in $H_{p-1}(K_{i+1})$
    - hence $\beta_{p-1} + +$

- Let $V = \{V_i = H_p(K_i)\}_{i=0}^n$ and consider the persistence diagram $V \cong I[b_1, d_1) \oplus I[b_2, d_2) \oplus \cdots \oplus I[b_M, d_M)$ for a simplex-wise filtration
- Each $I[b, d)$ corresponds to
  - adding a p simplex σ_b at time b to create a p-cycle
  - adding a $p + 1$ simplex σ_d at time d to kill the above p-cycle

Subtlety of non-uniqueness

- Which cycle is killed?



  - The younger one will be killed

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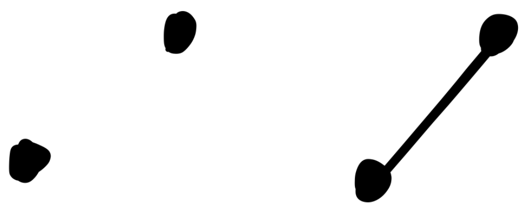
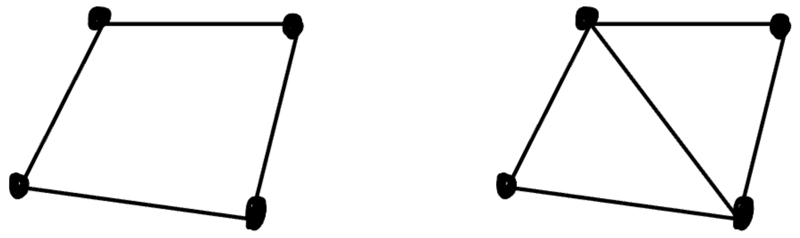


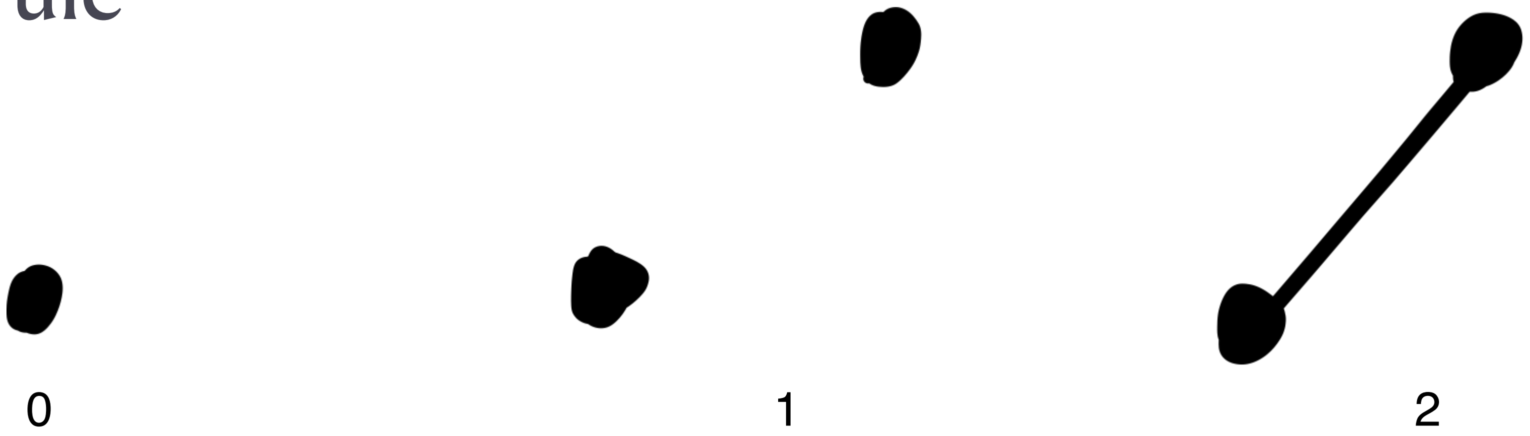
  - The younger one will be killed

- Which cycle is created?

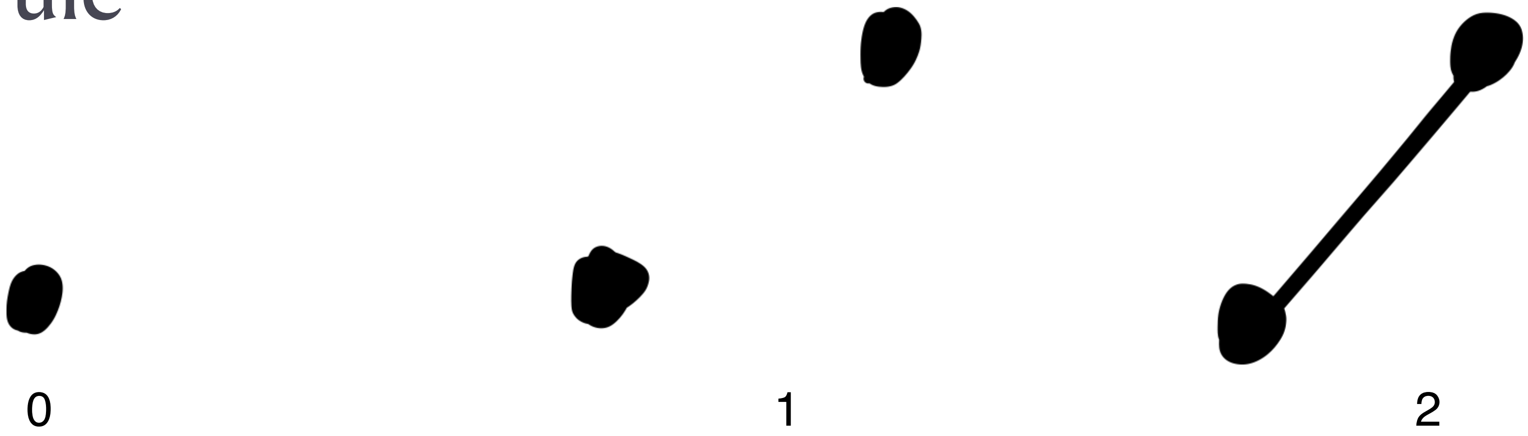


  - Several cycle classes are born
  - But the dimension only increases by 1

Elder rule

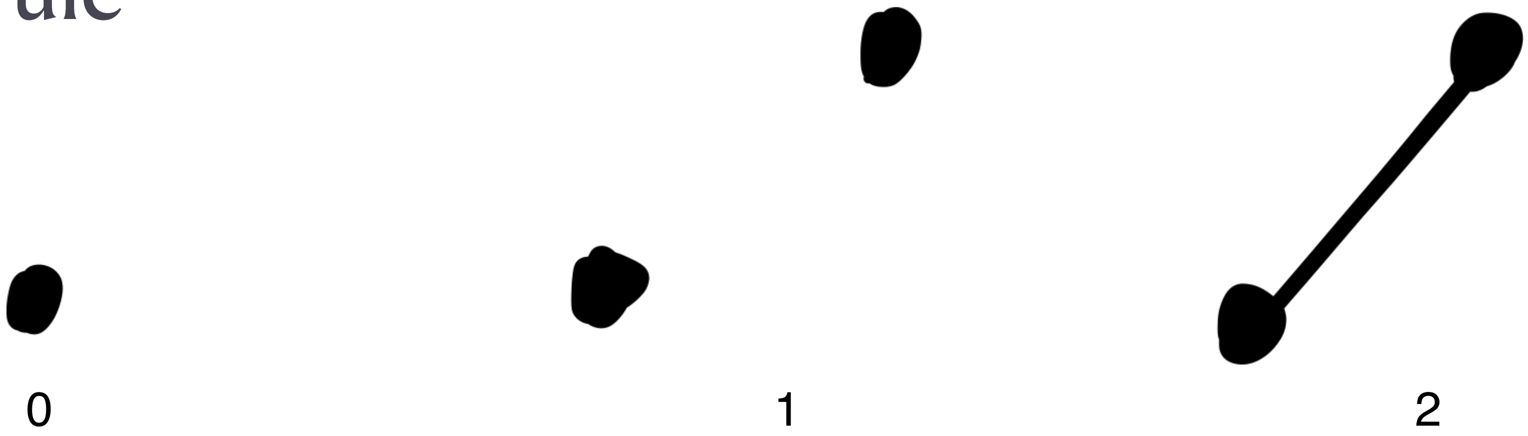


Elder rule



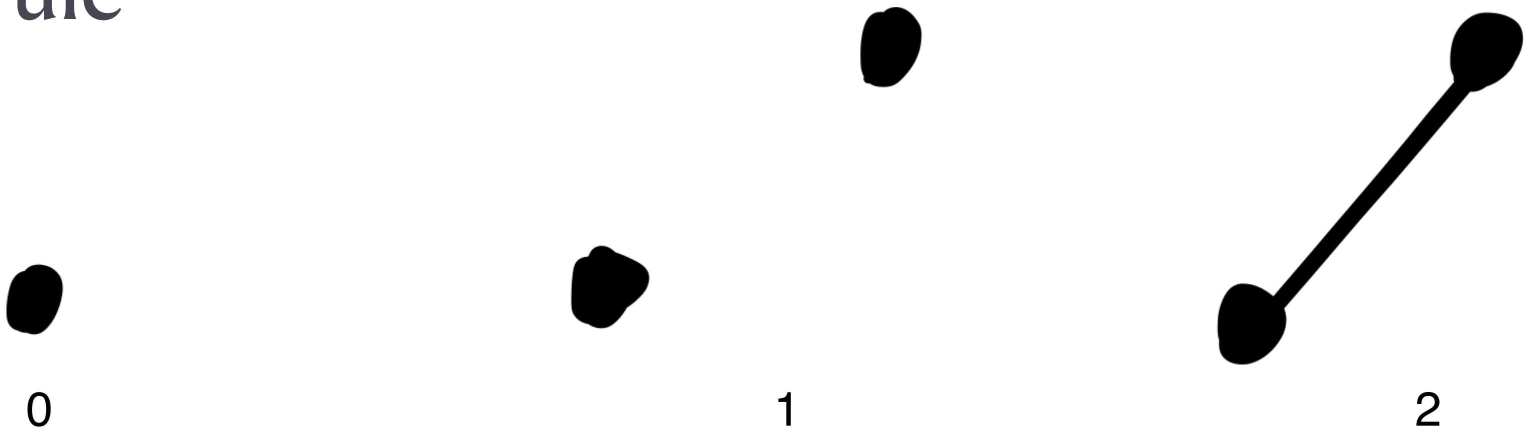
- [0,2) or [1,2)?

Elder rule



- [0,2) or [1,2)?
  - Older class will continue: [1,2)

Elder rule



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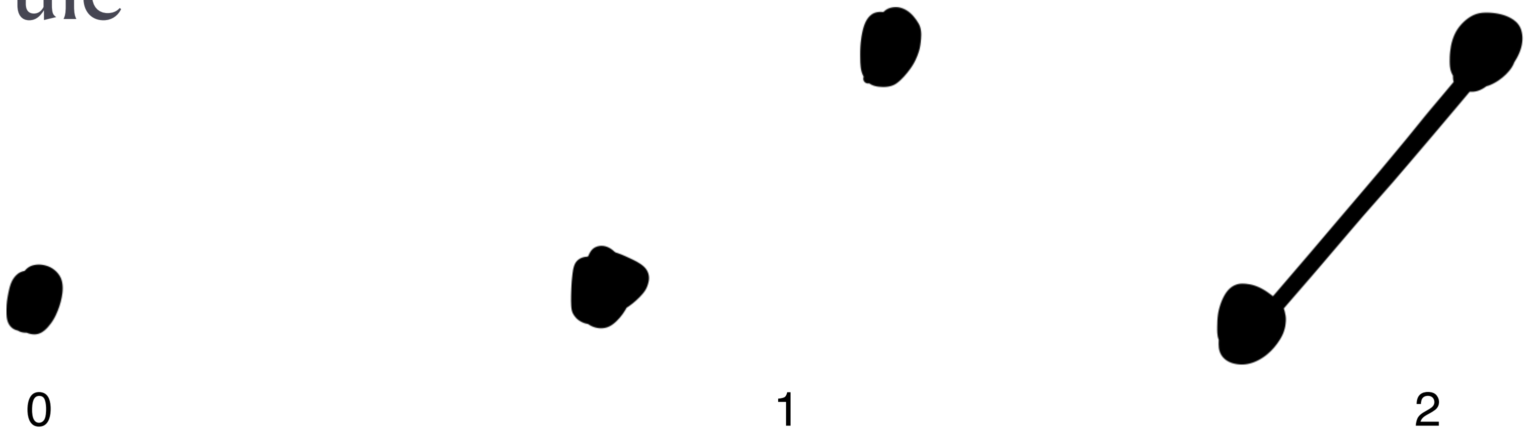

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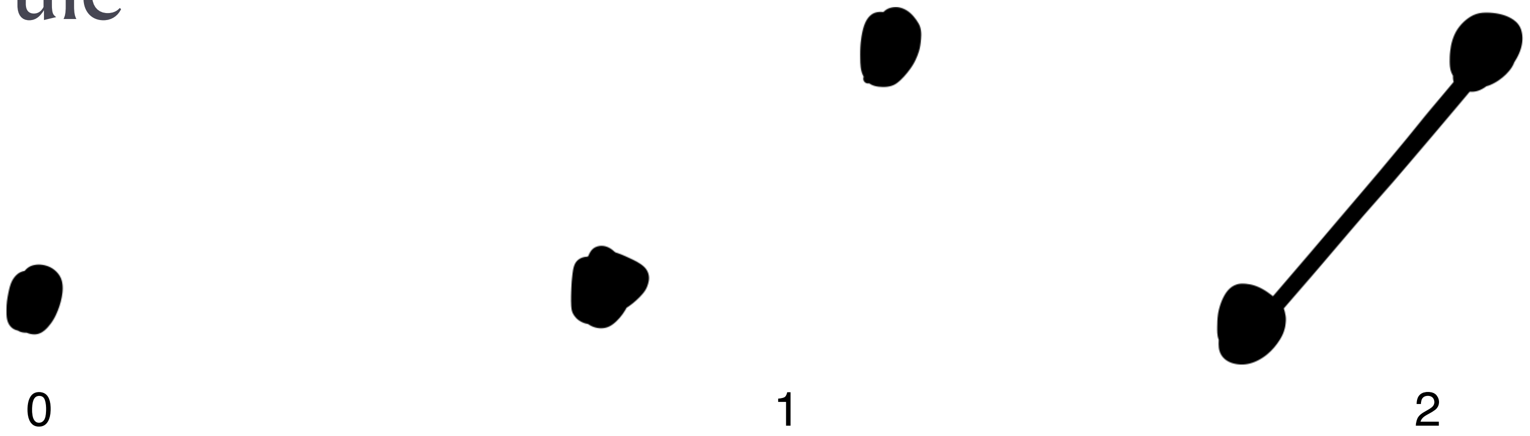
- One can then write down the barcodes directly

Elder rule



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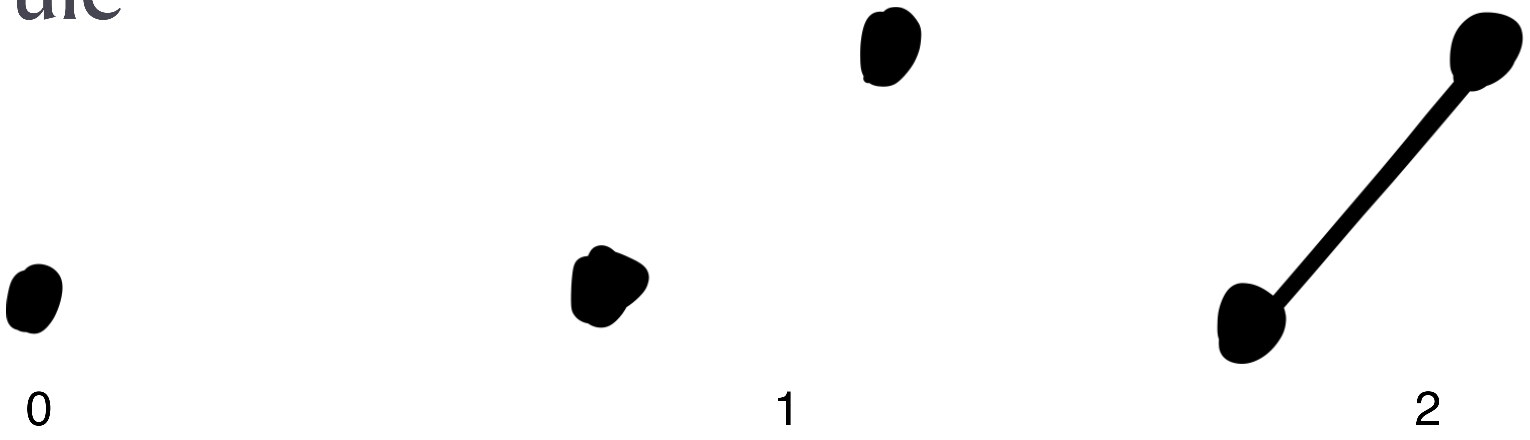
Elder rule



- [0,2) or [1,2)?
  - Older class will continue: [1,2)

- One can then write down the barcodes directly
- A persistence pairing (i, j) can be specified as (σ_i, σ_j)
- All (i, ∞) or (σ_i, ∞) correspond to homology classes of the final simplicial complex

- See board for an example

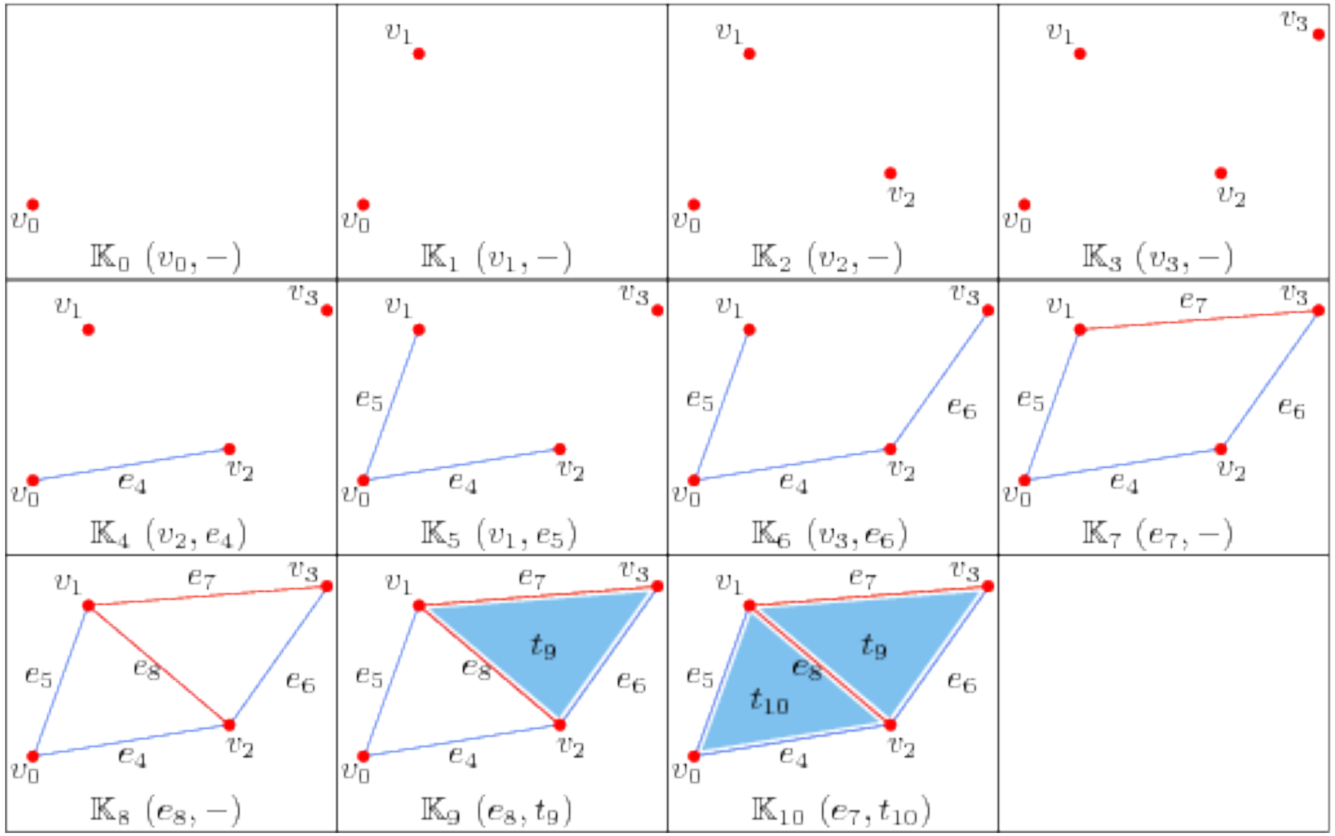
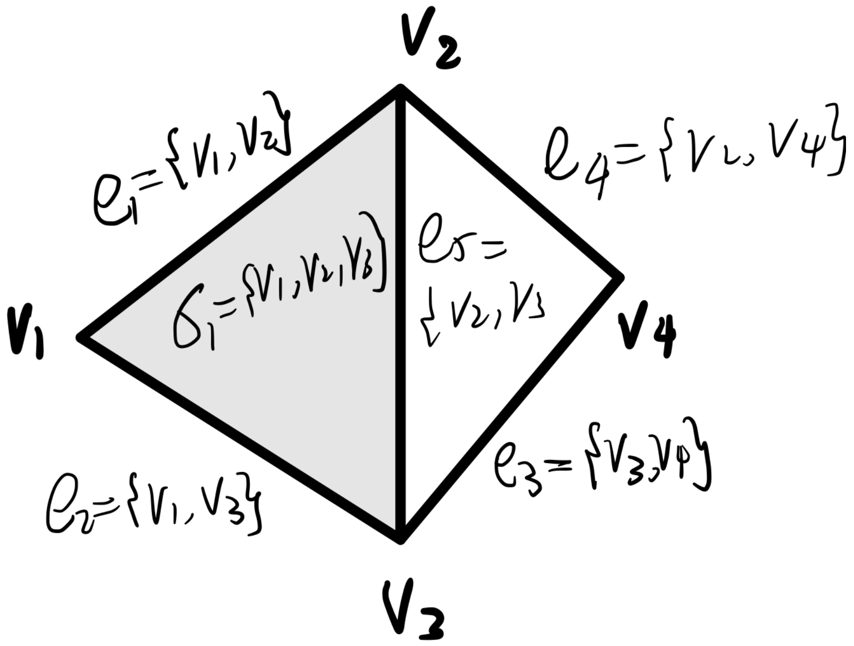


Image courtesy of T.K.Dey

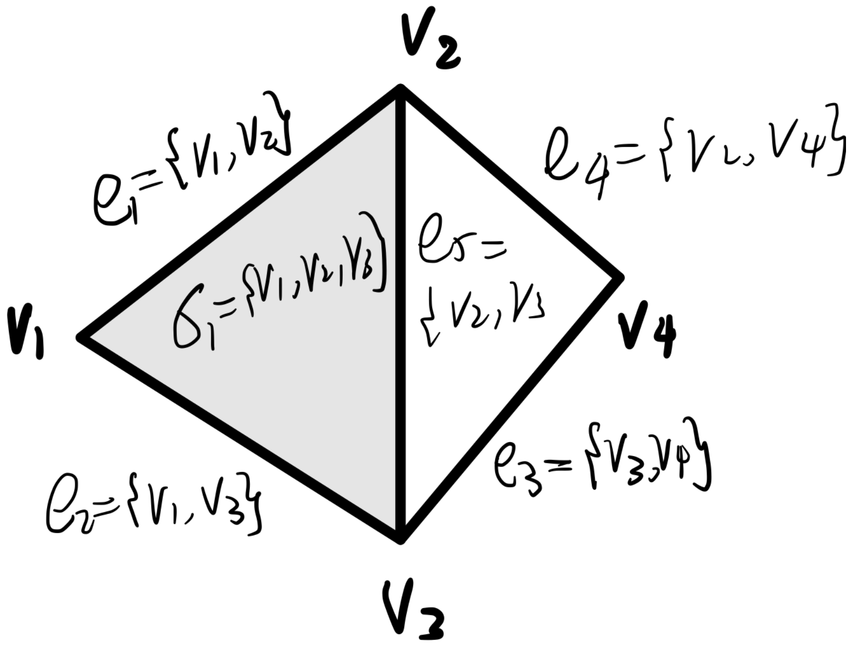
Section 2:
Persistence Algorithm

Recall homology algorithm



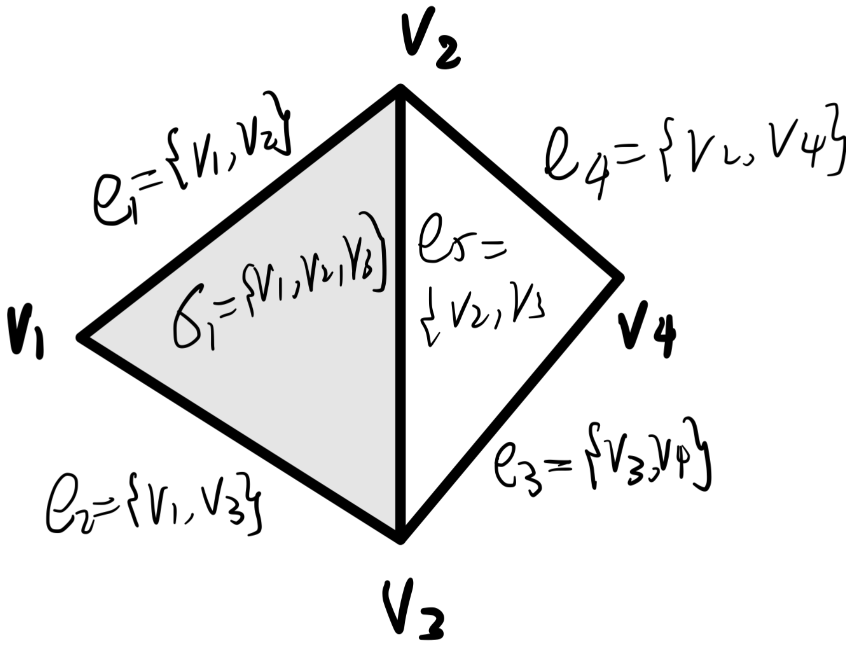
	e1	e2	e3	e4	e5
v1	1	1	0	0	0
v2	1	0	0	1	1
v3	0	1	1	0	1
v4	0	0	1	1	0

Recall homology algorithm



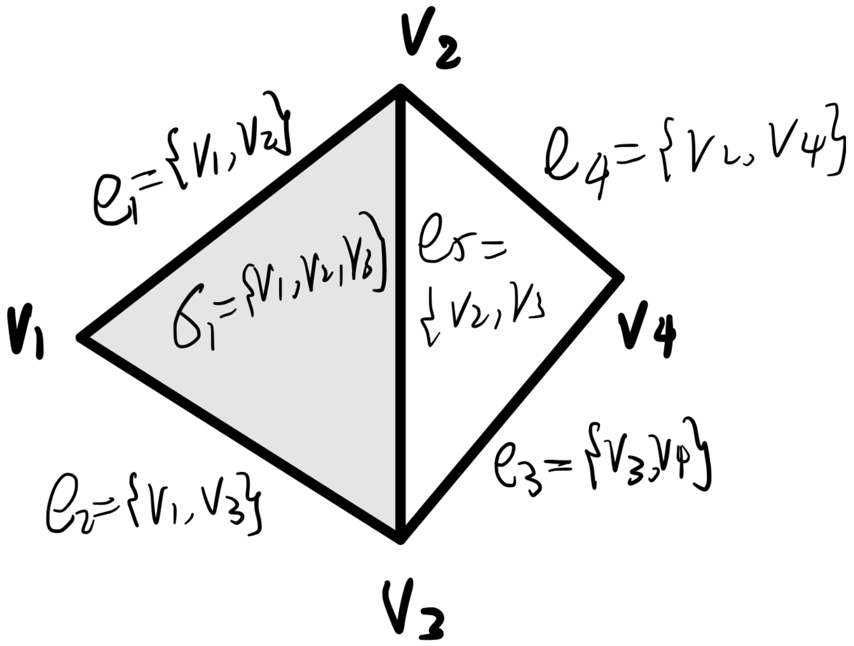
	e1	e2	e3	e4+e3	e5
v1	1	1	0	0	0
v2	1	0	0	1	1
v3	0	1	1	1	1
v4	0	0	1	0	0

Recall homology algorithm



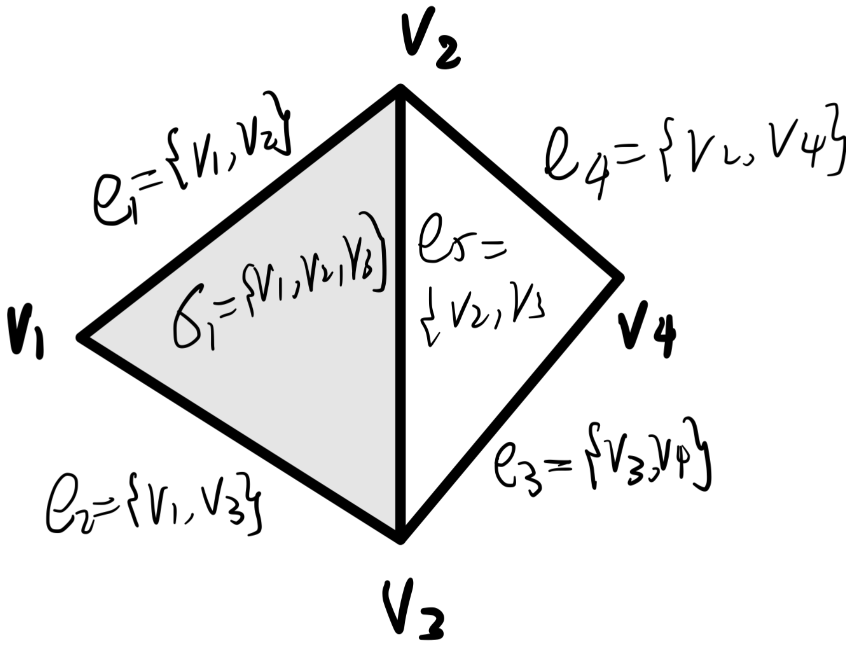
	e1	e2	e3	e4+e3+e2	e5
v1	1	1	0	1	0
v2	1	0	0	1	1
v3	0	1	1	0	1
v4	0	0	1	0	0

Recall homology algorithm



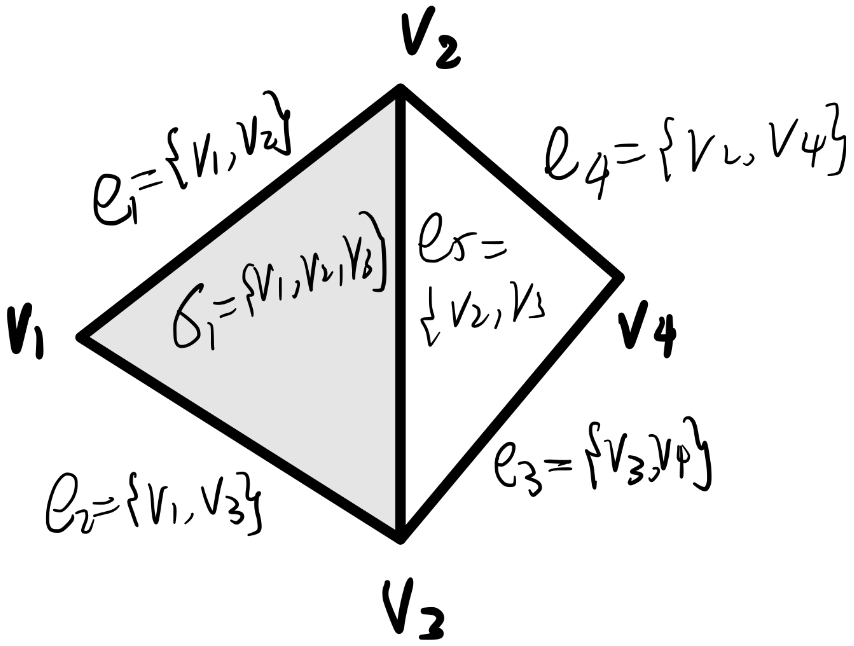
	e1	e2	e3	e4+e3+e2 +e1	e5
v1	1	1	0	0	0
v2	1	0	0	0	1
v3	0	1	1	0	1
v4	0	0	1	0	0

Recall homology algorithm



	e1	e2	e3	e4+e3+e2 +e1	e5+e2
v1	1	1	0	0	1
v2	1	0	0	0	1
v3	0	1	1	0	0
v4	0	0	1	0	0

Recall homology algorithm



	e1	e2	e3	e4+e3+e2+e1	e5+e2+e1
v1	1	1	0	0	0
v2	1	0	0	0	0
v3	0	1	1	0	0
v4	0	0	1	0	0

Persistent Algorithm

- Simplex-wise filtration $\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n = K$
  - s.t., $\sigma_i = K_i \setminus K_{i-1}$
  - i.e, filtration induced by an ordered sequence of simplices $\sigma_1, \sigma_2, \ldots, \sigma_n$ s.t. $K_i = \{\sigma_1, \cdots, \sigma_i\}$
- Let A be boundary matrix for K with $Col_A[i] = \partial\sigma_i$
- $lowId_M(j)$: index of lowest 1-entry in $Col_M[j]$

Persistent Algorithm

- Assume input filtration $\emptyset = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_m = K$
  - $\sigma_i = K_i \setminus K_{i-1}$
  - i.e, filtration induced by an ordered sequence of simplices $\sigma_1, \sigma_2, \ldots, \sigma_n$ s.t. $K_i = \{\sigma_1, \cdots, \sigma_i\}$
  - Let A be boundary matrix for K with $Col_A[i] = \partial\sigma_i$
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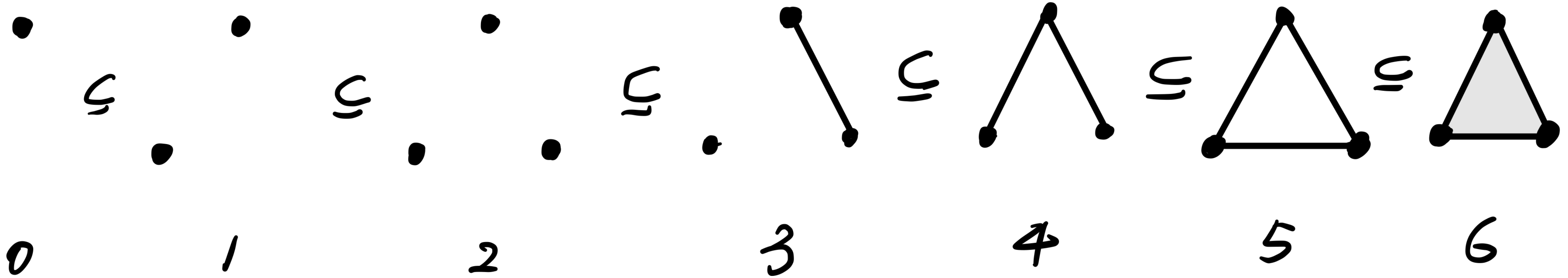
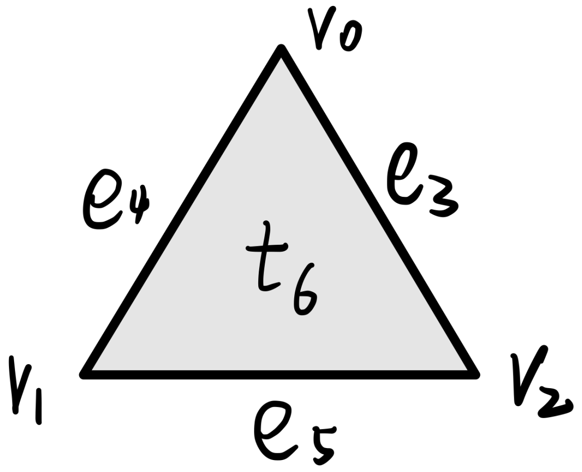
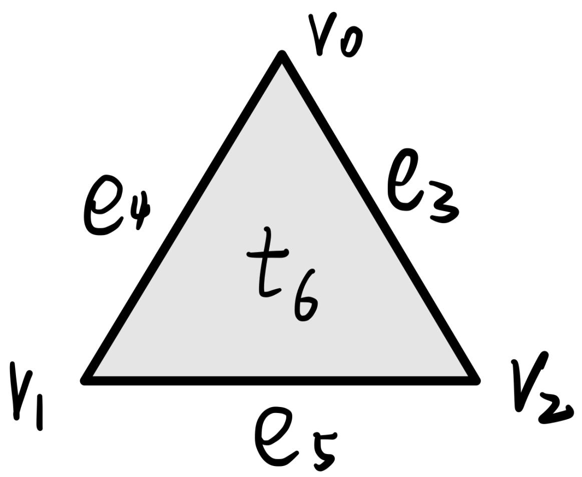
Algorithm 1 Right-Reduction(A)

```
R = A;
for j = 1 → m do
   while there exists j0 < j with lowId(j0) = lowId(j)  do
      add column j0 of R to column j of R
   end while
end for
```

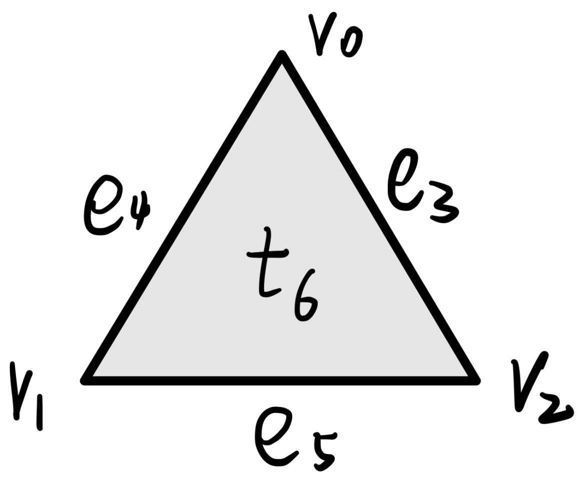
Example



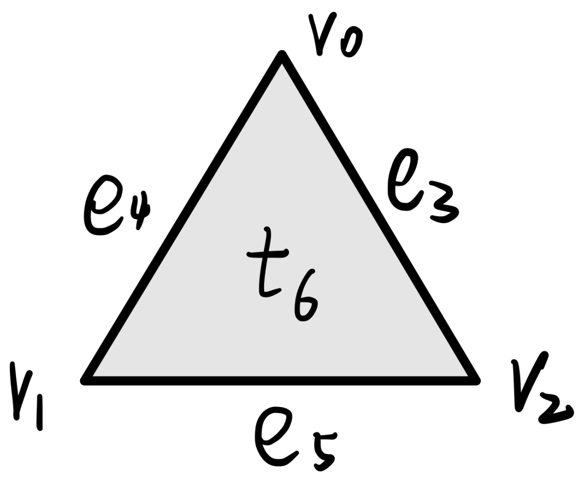




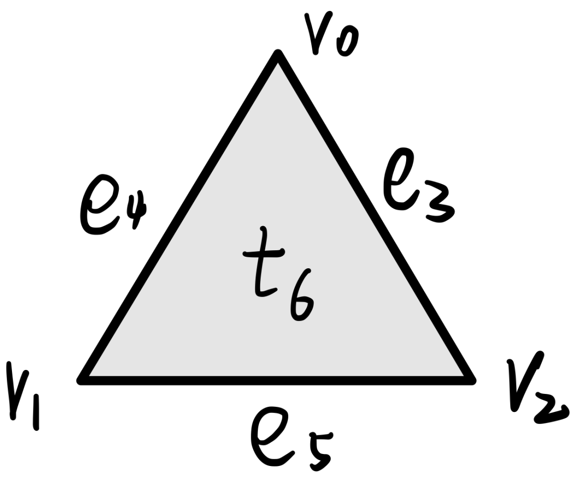
	v0	v1	v2	e3	e4	e5	t6
v0	0	0	0	1	1	0	0
v1	0	0	0	0	1	1	0
v2	0	0	0	1	0	1	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0

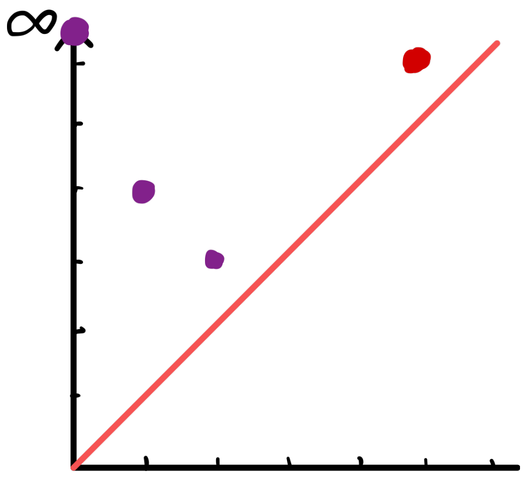


	v0	v1	v2	e3	e4	e5+e3	t6
v0	0	0	0	1	1	1	0
v1	0	0	0	0	1	1	0
v2	0	0	0	1	0	0	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0



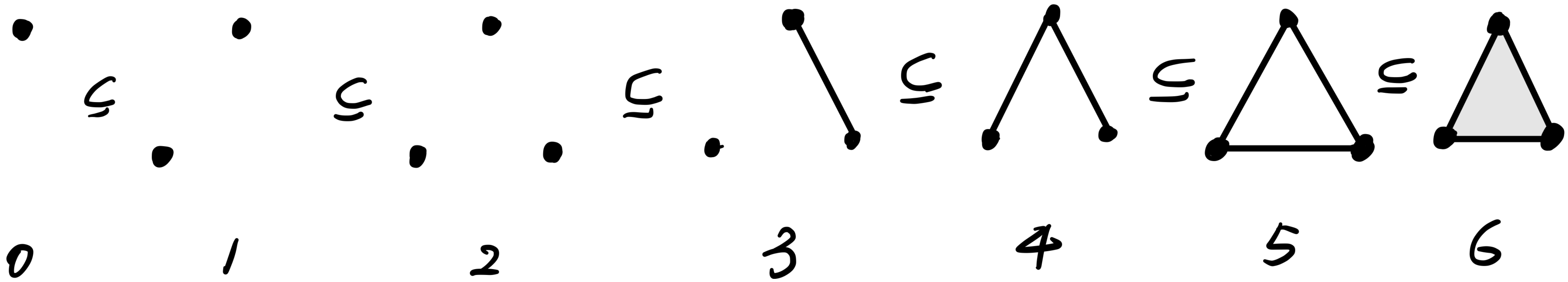
	v0	v1	v2	e3	e4	e5+e3+e4	t6
v0	0	0	0	1	1	0	0
v1	0	0	0	0	1	0	0
v2	0	0	0	1	0	0	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0

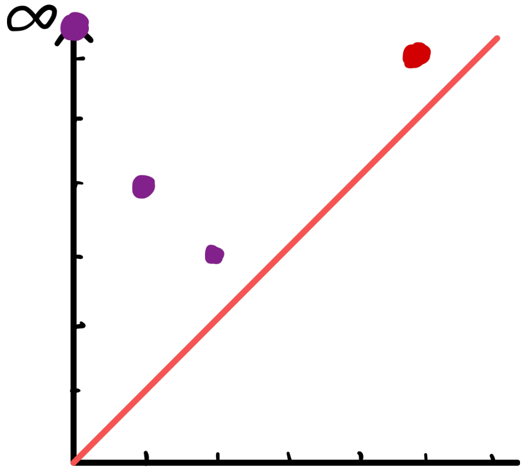




	v0	v1	v2	e3	e4	e5+e3+ e4	t6
v0	0	0	0	1	1	0	0
v1	0	0	0	0	1	0	0
v2	0	0	0	1	0	0	0
e3	0	0	0	0	0	0	1
e4	0	0	0	0	0	0	1
e5	0	0	0	0	0	0	1
t6	0	0	0	0	0	0	0

- Homology classes born at 0,1,2,5
- $(v_0, \infty), (v_1, e_4), (v_2, e_3), (e_5, t_6)$
- $Dgm_0 = \{(0,\infty), (1,4), (2,3)\}$
- $Dgm_1 = \{(5,6)\}$





- Homology classes born at 0,1,2,5
- $(v_0, \infty), (v_1, e_4), (v_2, e_3), (e_5, t_6)$
- $Dgm_0 = \{(0,\infty), (1,4), (2,3)\}$
- $Dgm_1 = \{(5,6)\}$

Persistent Pairings

- **Theorem A:**
  - Consider the output matrix R of algorithm Right-Reduction(A).

    Then $\mu^{i,j} = 1$ iff $lowId_R(j) = i$

Persistent Pairings

- Theorem A:
  - Consider the output matrix R of algorithm Right-Reduction(A). Then $\mu^{i,j} = 1$ iff $lowId_R(j) = i$

- Theorem B:
  - Given boundary matrix A, perform **any** sequence of right-column-addition operations only to convert it into the reduced form R. Then

$$\mu^{i,j} = 1 \text{ iff } lowId_R(j) = i$$

Generating cycles

- For any intermediate matrix M
  - Each column i is associated with a p-chain Γ^i
  - The column $Col_M[i]$ corresponds to the boundary of Γ^i
  - If $Col_M[i] = \left[0\ 0\ \ ..\ \ 0\right]^T,$ it is a *boundary cycle*
    - Death event
  - Otherwise, it is a cycle generating a new homology class
    - Birth event

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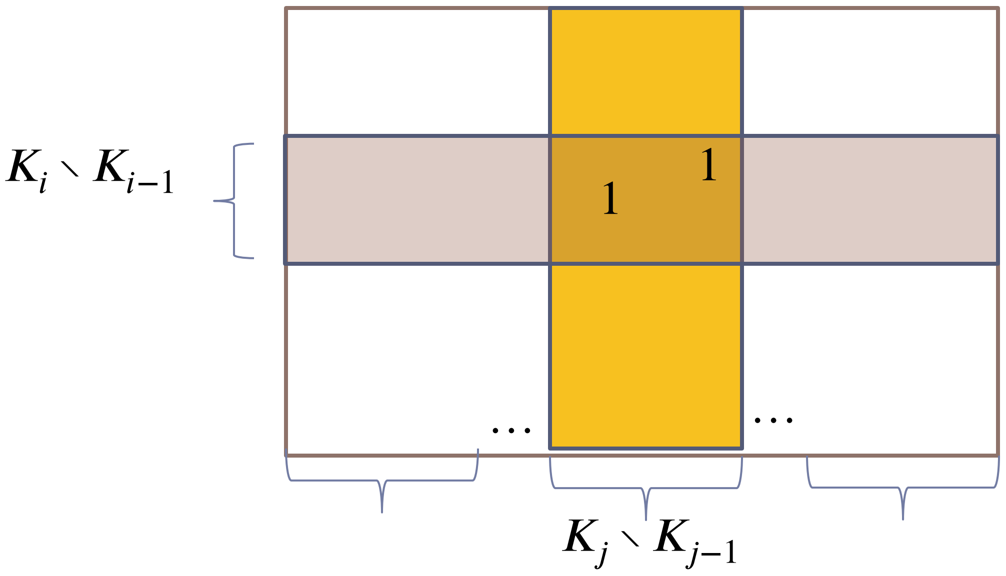
Generating cycle if this column is all-zero!

Computation

- Right-Reduction(A) runs in $O\left(N^3\right)$ time
  - where N is total number of simplices
- Can be improved to matrix multiplication time

General Filtration

- Given $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n$, let $\sigma_1, \sigma_2, \ldots, \sigma_N$ be an ordering of simplices consistent with the filtration
- i.e, $\underbrace{\sigma_1, \ldots, \sigma_{I_1}}_{K_1}, \underbrace{\sigma_{I_1+1}, \ldots, \sigma_{I_2}}_{K_2 \setminus K_1}, \ldots, \underbrace{\sigma_{I_{n-1}+1}, \ldots, \sigma_{I_n}}_{K_n \setminus K_{n-1}}$



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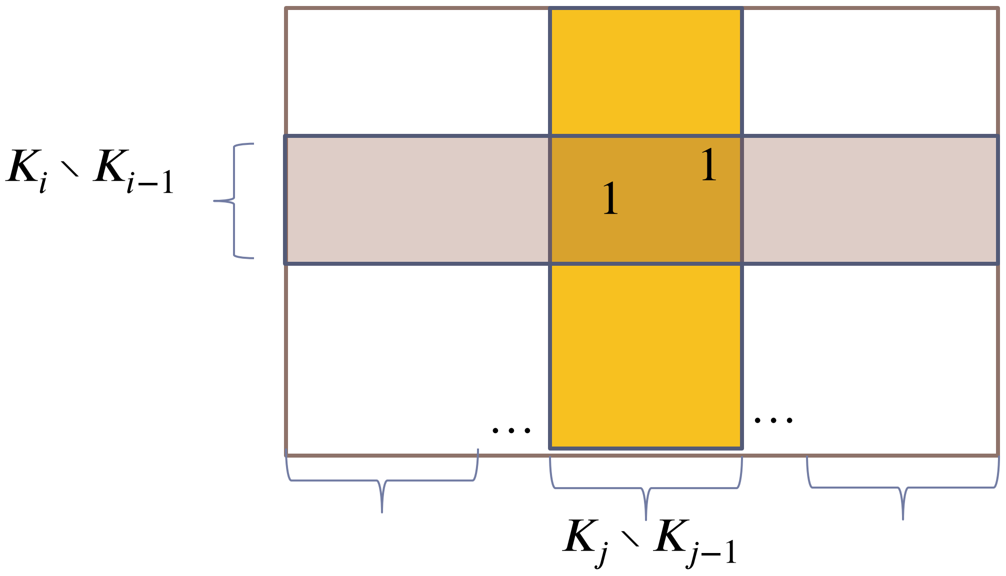
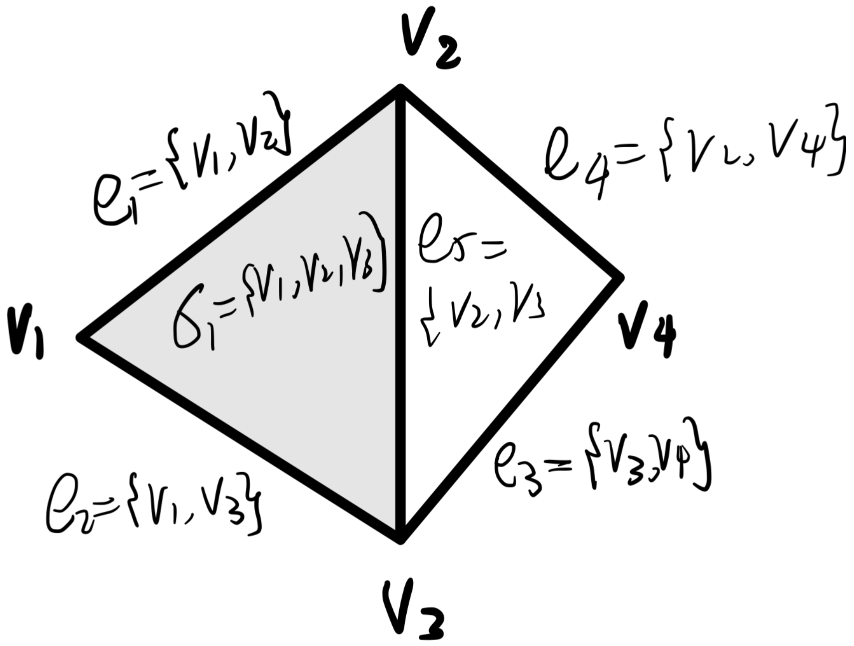
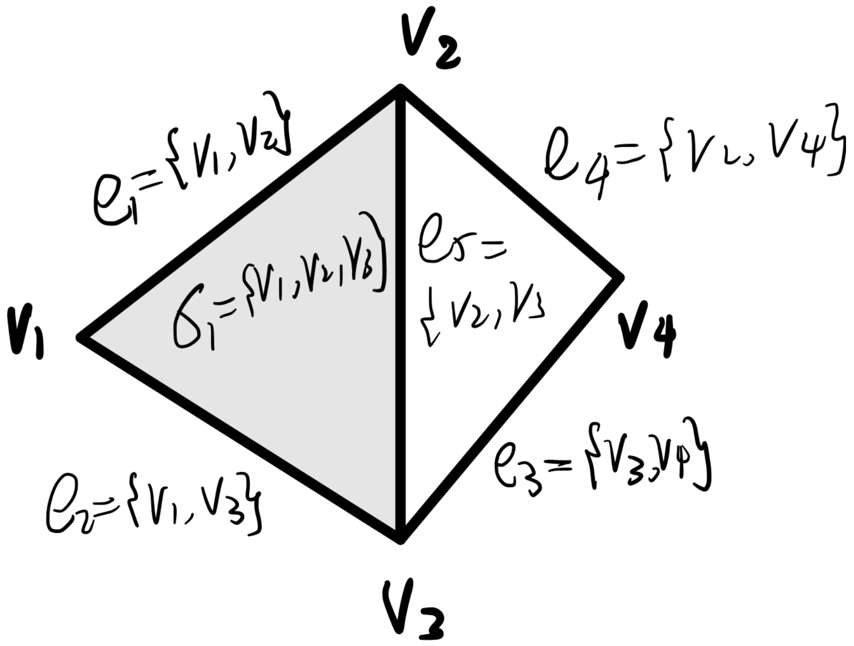


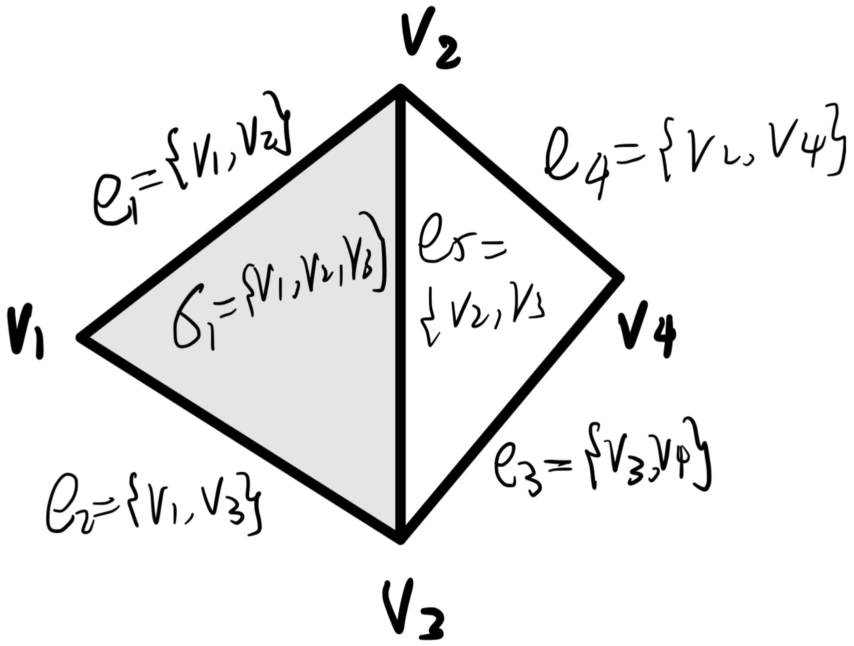
$\boxed{\mu^{i,j} = 2}$



	v1	v2	v3	v4	e1	e2	e3	e4	e5	S1
v1					1	1				
v2					1			1	1	
v3						1	1		1	
v4							1	1		
e1										1
e2										1
e3										
e4										
e5										1
S1										



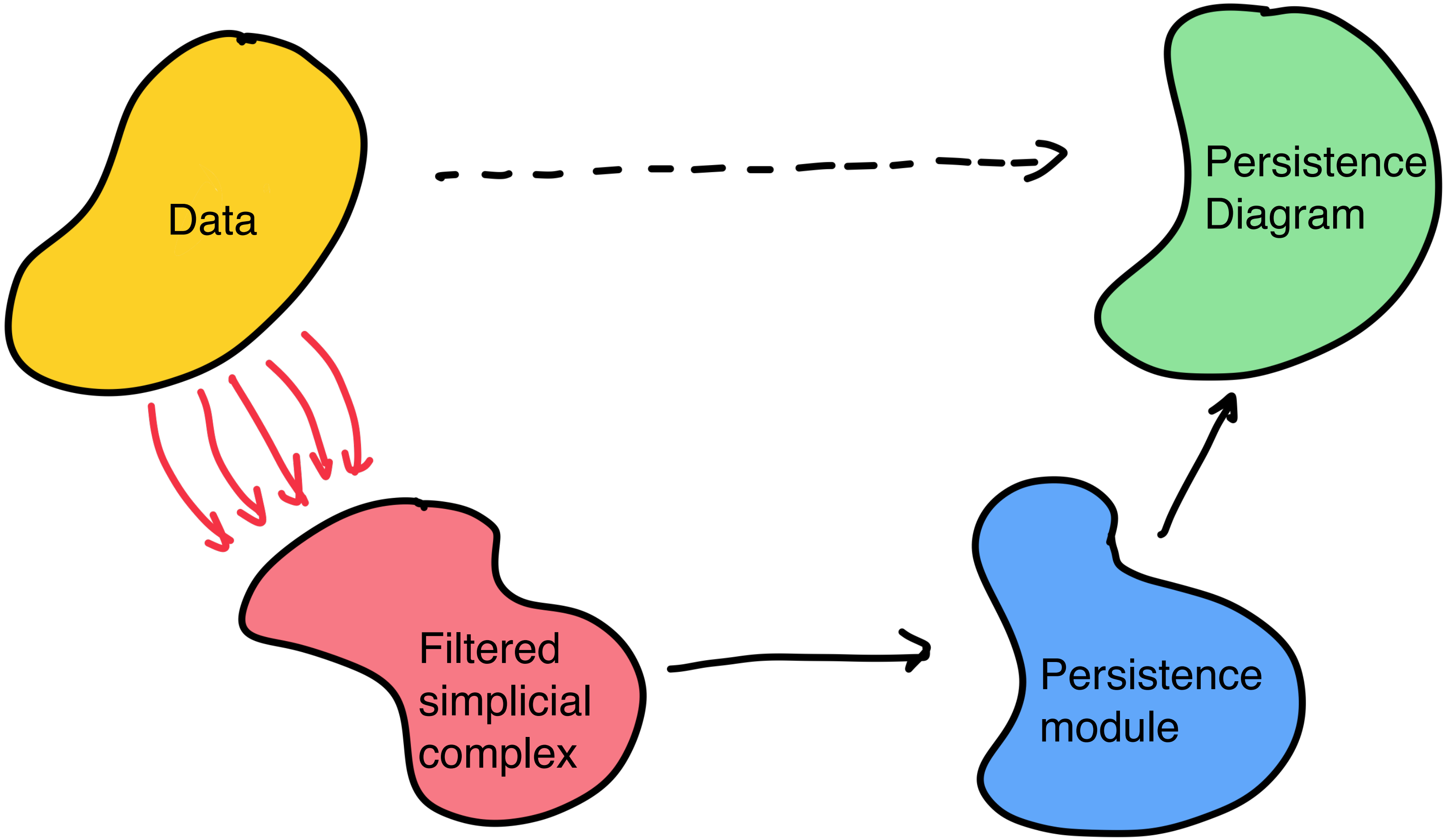
	v1	v2	v3	v4	e1	e2	e3	e4+e3+e2+e1	e5+e2+e1	S1
v1					1	1				
v2					1					
v3						1	1			
v4							1			
e1										1
e2										1
e3										
e4										
e5										1
S1										



	v1	v2	v3	v4	e1	e2	e3	e4+e3+ e2+e1	e5+e2 +e1	S1
v1					1	1				
v2					1					
v3						1	1			
v4							1			
e1										1
e2										1
e3										
e4										
e5										1
S1										

- $(v_1, \infty), (v_2, e_1), (v_3, e_2), (v_4, e_3), (e_4, \infty),(e_5, s_1)$
- $Dgm_0 = \{(0,\infty)\}$
- $Dgm_1 = \{(0,\infty)\}$
- $(v_1, \infty), (e_4, \infty)$ correspond to two homology classes

Mind picture



FIN