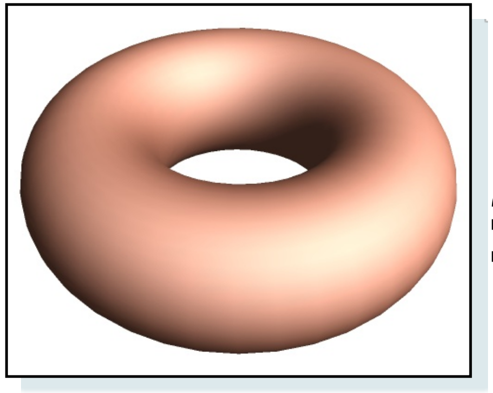
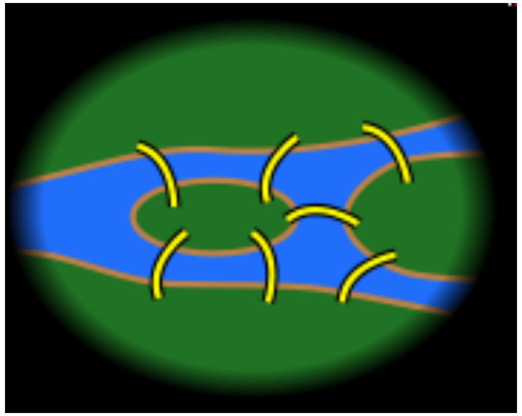
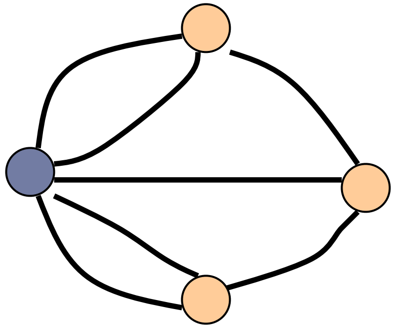
DSC 214
Topological Data Analysis
Topic 1: Basics

Instructor: Zhengchao Wan

Goal

 $\cong$ 

 $\simeq$ 

- Fundamental Questions
  - What is a topological space?
  - What is a "continuous" way of turning one space to another?
  - When can we say two spaces are the "same"?

Overview

- Fundamental concepts
  - Topological space
  - Continuous maps
  - Homeomorphisms and homotopies
  - Manifolds

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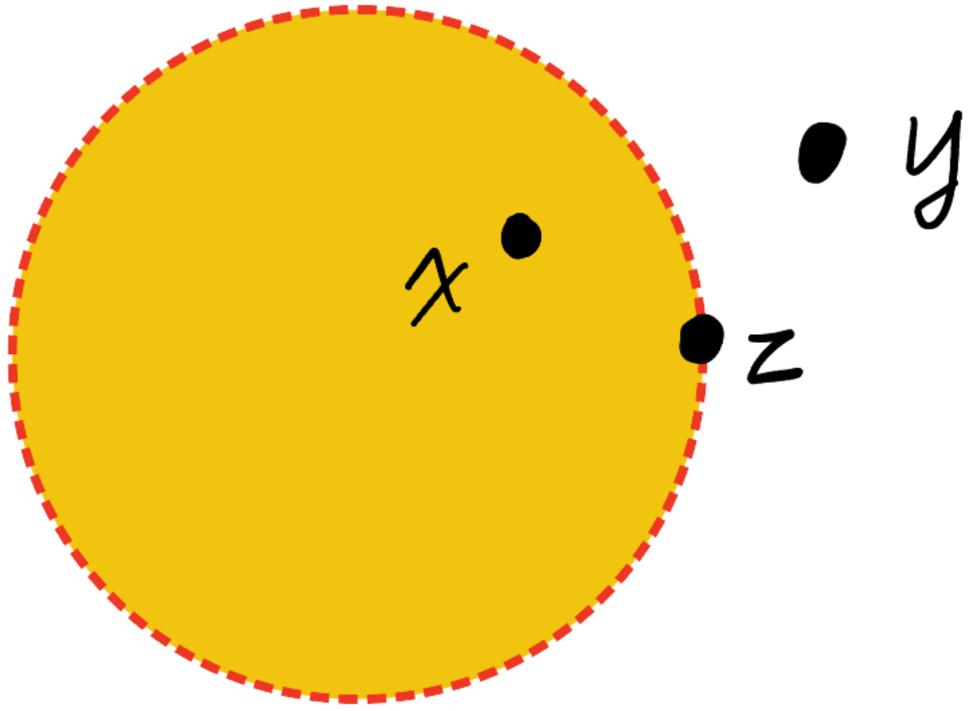
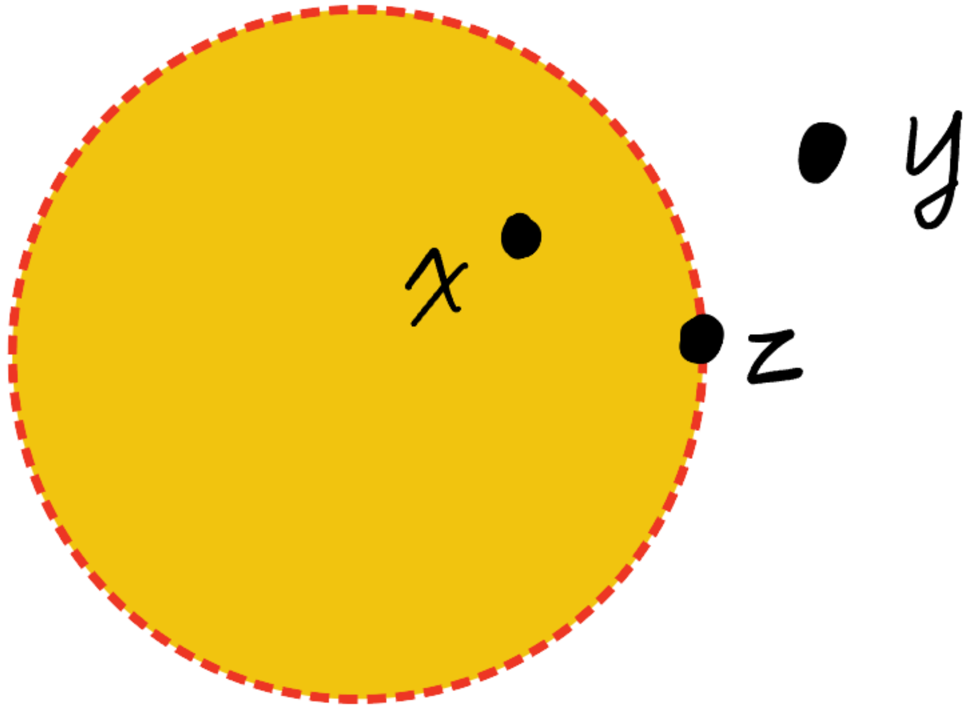
How we mathematically talk about space of interest

Set theory and beyond



Set theory and beyond

- Given a disk D (without boundary)



Set theory and beyond

- Given a disk D (without boundary)
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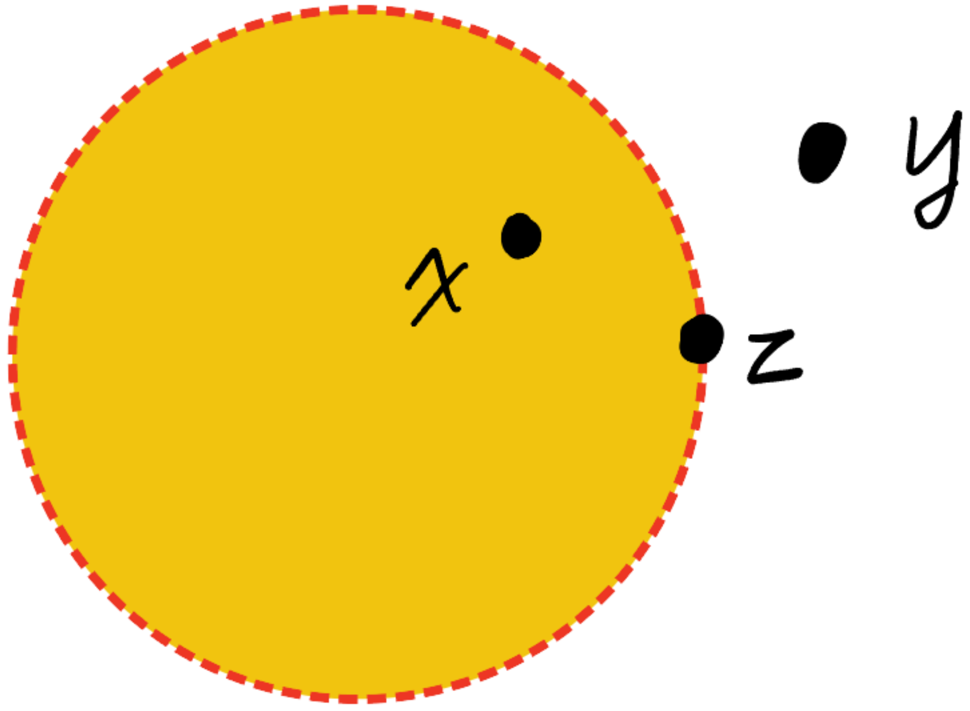

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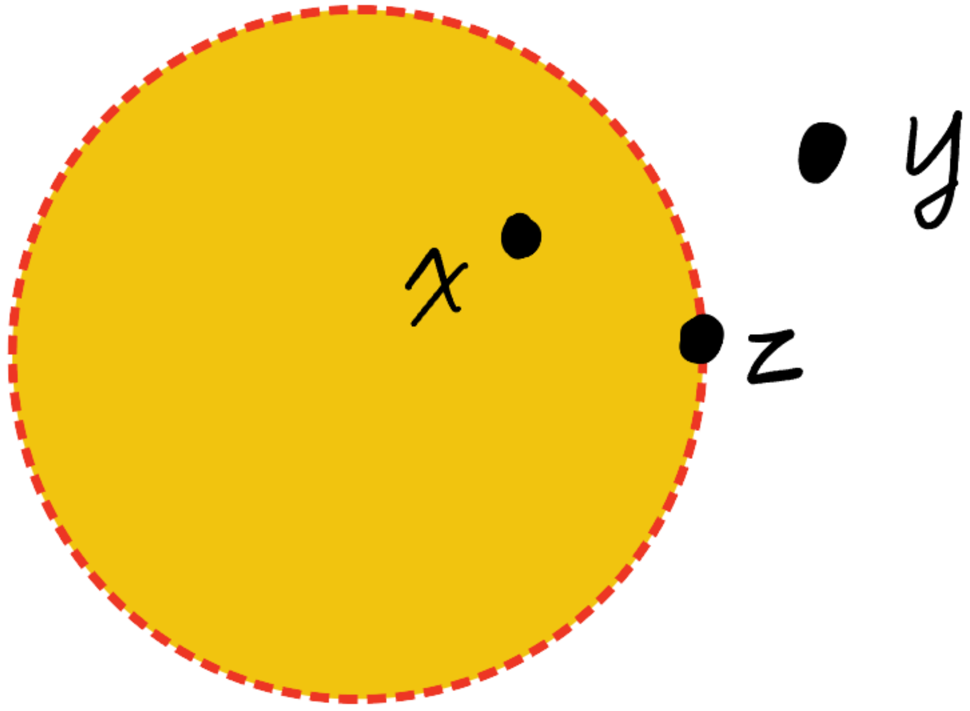

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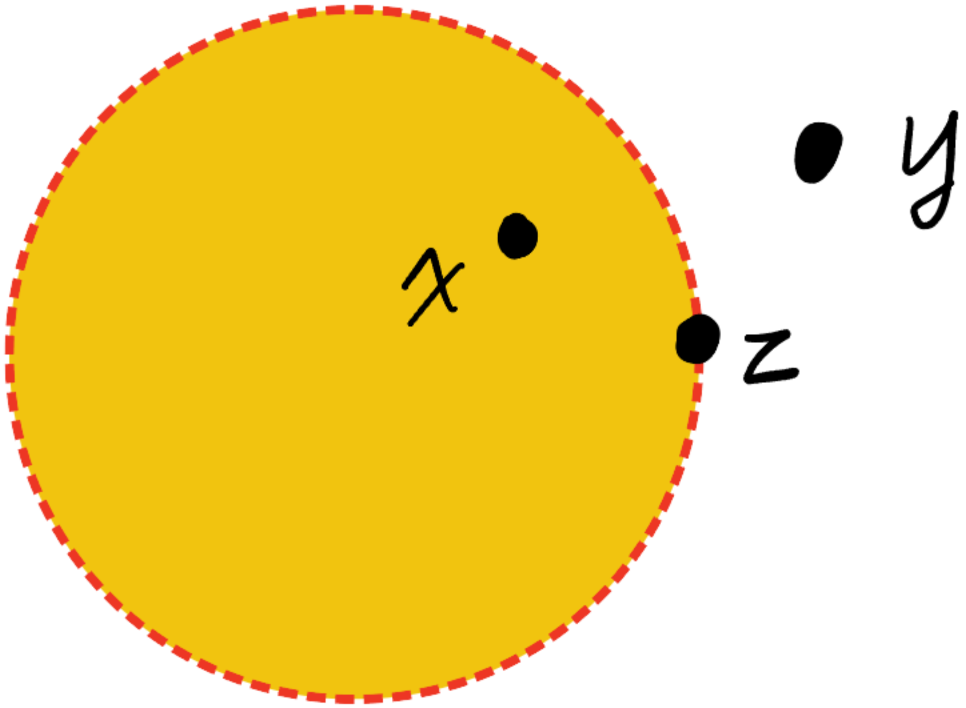

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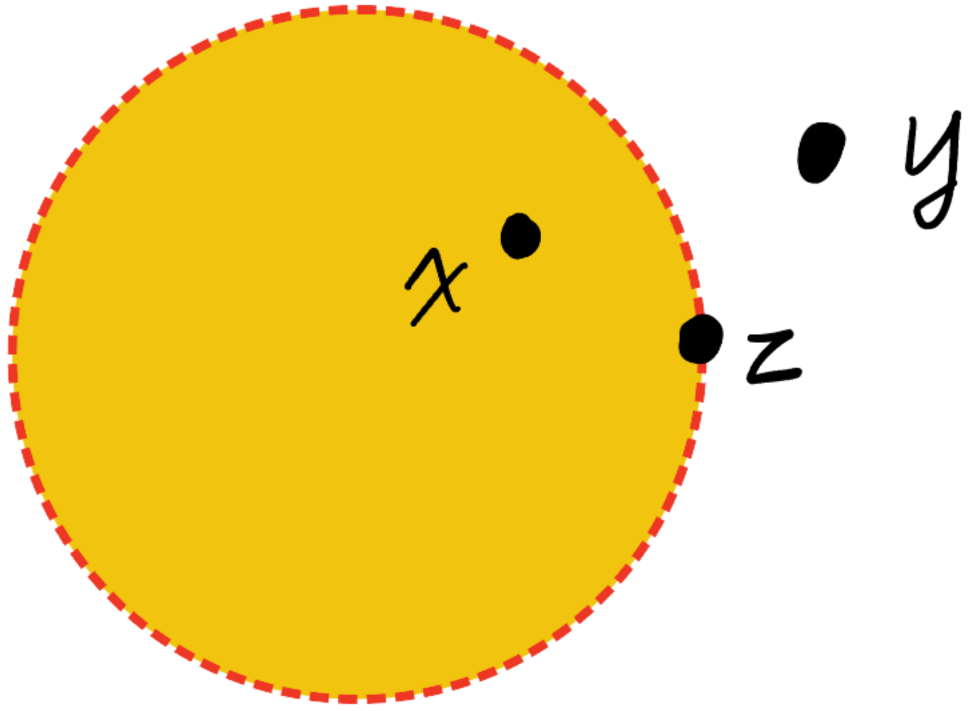

Set theory and beyond

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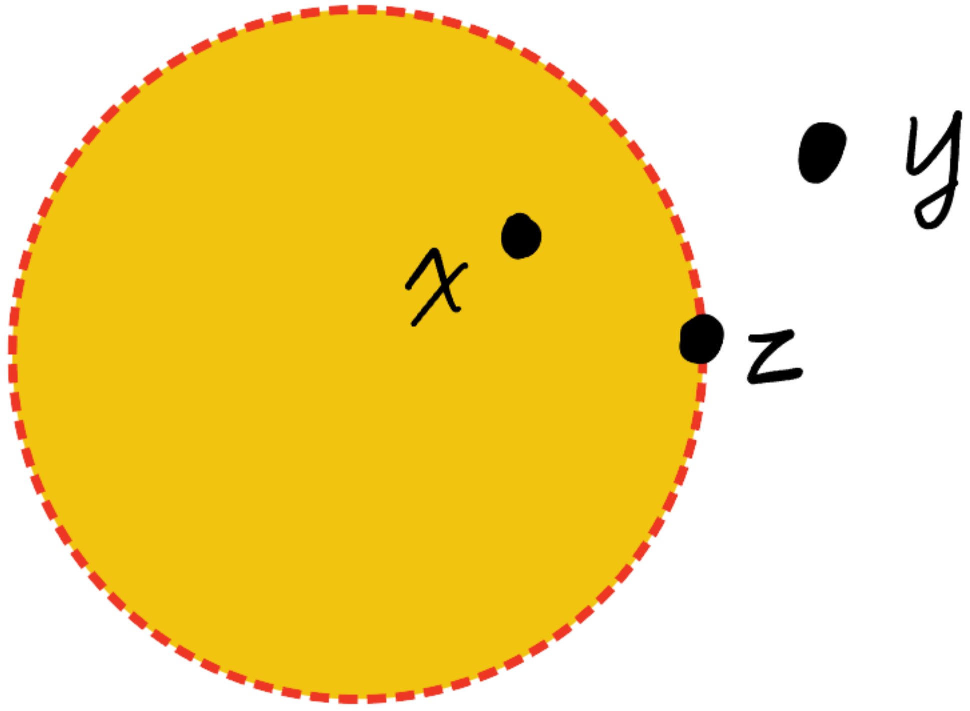
    - x and z are in the "**closure**" of D



Why do we care?

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  - A continuous map shouldn't tear things apart
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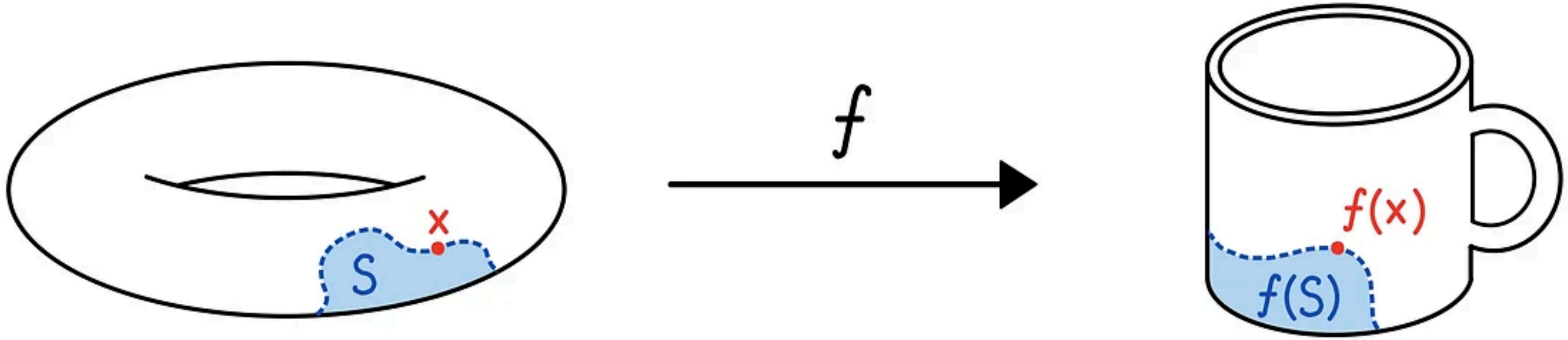

From https://wgyory.wixsite.com/toolatetopologize/post/post-1

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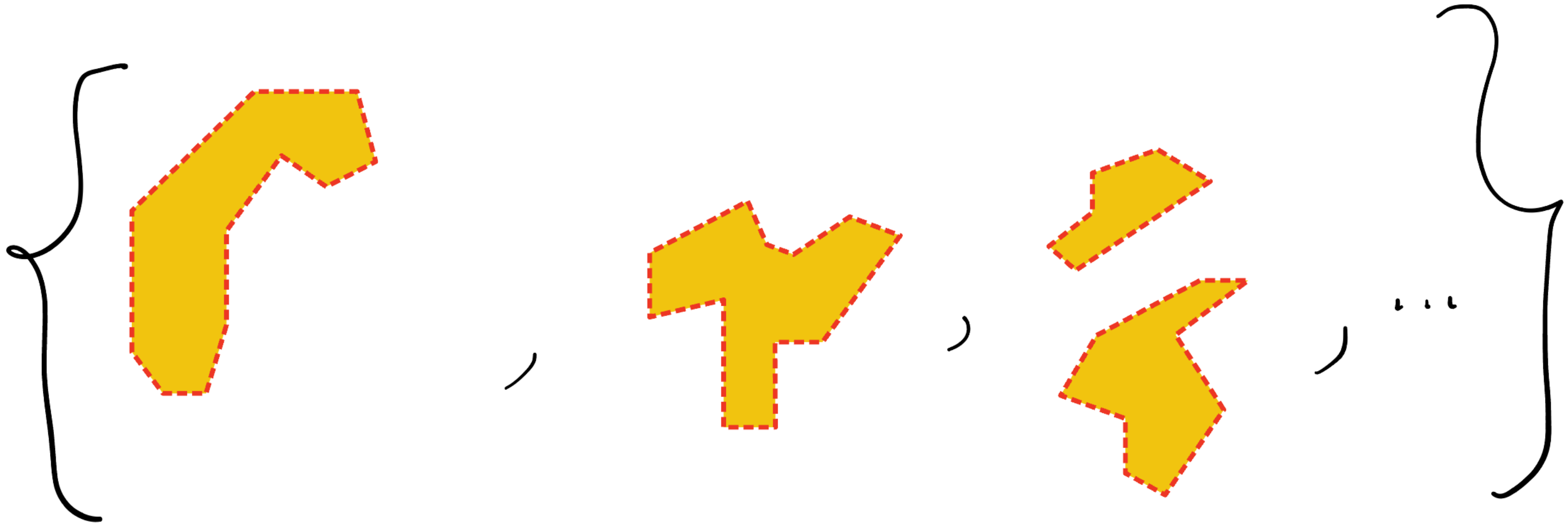
- We keep track of **ALL** the relations "S contacts x" to make the above intuition rigorous!

Topology

- It turns out that we just need to specify all “open” subsets to keep track of all relations “S contacts x” on a given set



Topological space

Definition 1.1 (Topological space) *A* topological space *is a set* X *endowed with a* topological structure *(a* topology*)* $\mathcal{T}$ *such that the following conditions are satisfied:*

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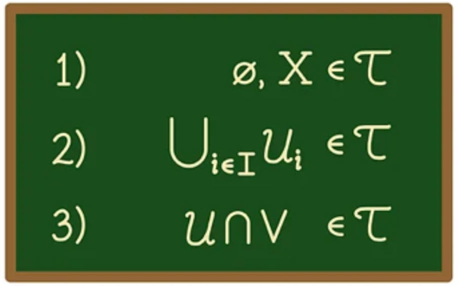

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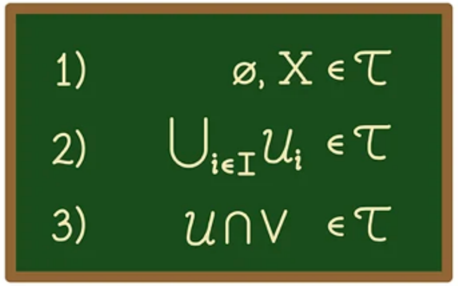

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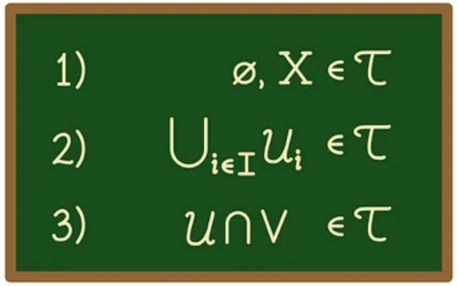

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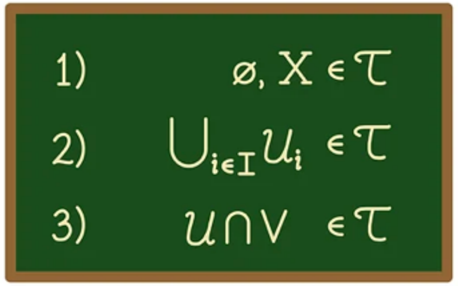

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  - **Metric space topology**

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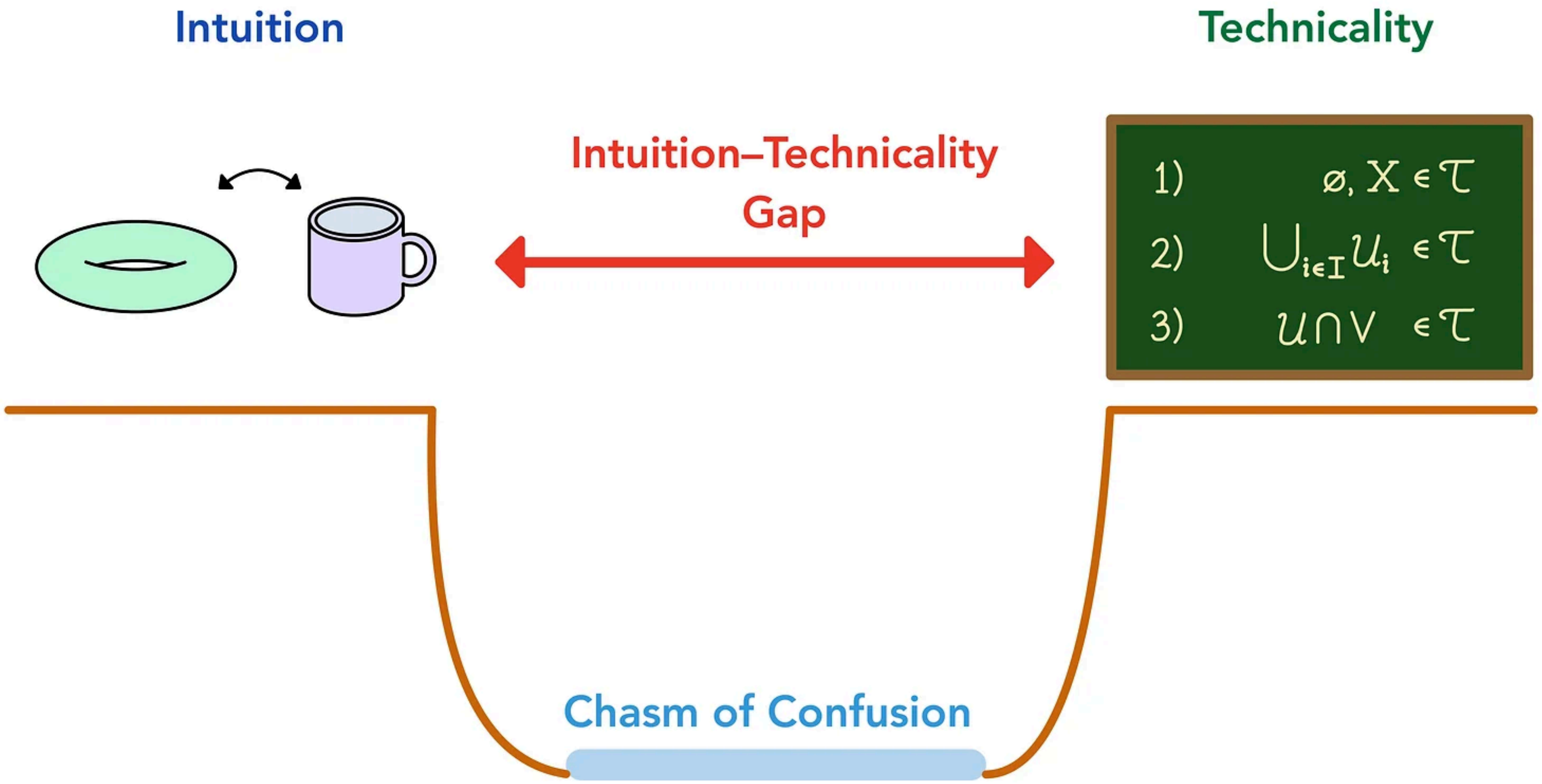

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Topology provides the formal language to keep track of all the Nearness relation: S contacts x. Hence, we can talk about limits, continuity, etc of functions as well as connectedness, compactness, etc of spaces.

Open / Closed sets

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- Each set $A \in T$ is called an *open set*
- A set B is *closed* if its complement is open
  - i.e., there exists A such that $B = X \setminus A$

Closure, interior, boundary

Closure, interior, boundary

- Recall a set is closed if its complement is open
- Given a topological space $(\mathsf{X}, \mathcal{T})$ and a subset $A \subseteq \mathsf{X}$:
  - the *closure* of A, denoted by $\bar{A}$, is the smallest closed set containing A.
    - $\bar{A} = \cap_{\text{closed } C \supset A} C$
  - its *interior* A^o is the union of all open subsets of A.
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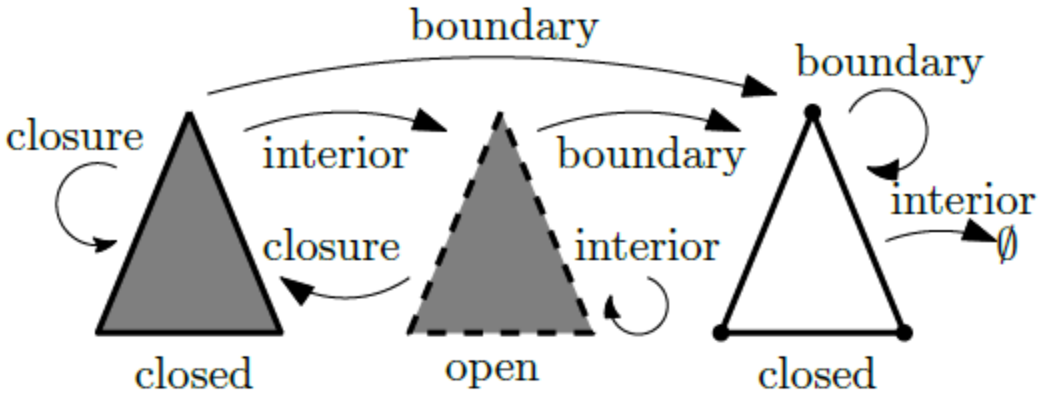

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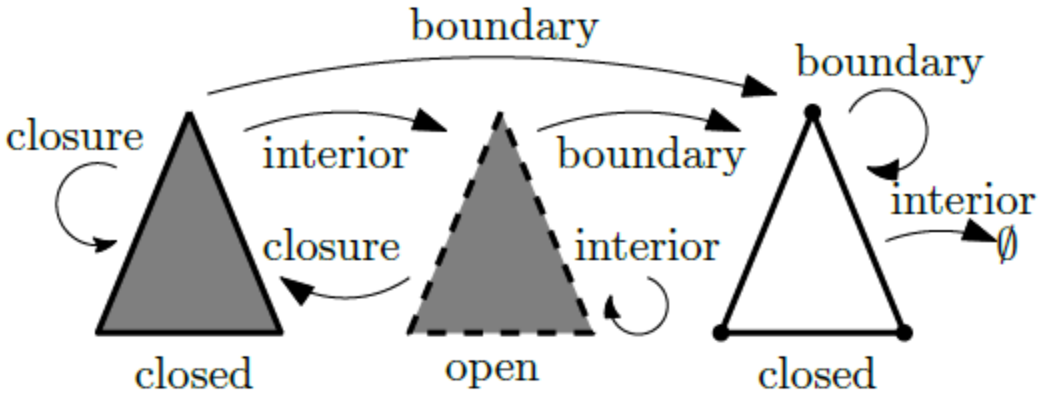
“S contacts x” can be formally defined as $x \in \bar{S}$



Examples in $\mathbb{R}$

- Let $A = [1,2)$
  - $\bar{A} = [1,2]$
  - $A^o = (1,2)$
  - $\partial A = \{1,2\}$

- For any given set X, one can define different topologies on top of that. Some of them can be bizarre, such as the trivial topology

- The most useful topology in this class is the **metric space topology**

Metric space

Definition 2 (Metric space). *A* metric space *is a pair* (X, d) *where* X *is a set and* d *is a distance function* $\mathsf{d} : \mathsf{X} \times \mathsf{X} \to \mathbb{R}$ *satisfying the following properties:*

- $\mathsf{d}(p, q) = 0$ *if and only if* $p = q$
- $\mathsf{d}(p, q) = \mathsf{d}(q, p), \forall p, q \in \mathsf{X}$*;*
- $\mathsf{d}(p, q) \leq \mathsf{d}(p, r) + \mathsf{d}(r, q), \forall p, q, r \in \mathsf{X}.$

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- Examples:
  - $(\mathbb{R}^k, \|\cdot\|_2)$ k-dimensional Euclidean space, equipped with the standard Euclidean distance $d(p, q) = \|p - q\|_2$
  - "Curved" space (manifolds), equipped with geodesic distance
    - e.g, the surface of earth.
  - Space can also be discrete, as very often in data analysis
    - $(P,\ d)$: a set of points with pairwise distance (or similarity) given.
    - or graphs, equipped with shortest path metric.

Metric space topology

- Open ball:
  - $B_o(c,\ r) = \{x \in X \mid d(c, x) < r\}$

Definition 3 (Metric space topology). *Given a metric space* X, *all metric balls* $\{B_o(c, r) \mid c \in \mathbb{T}$ *and* $0 < r \leq \infty\}$ *and their union constituting the open sets define a topology on* X.

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- The set of metric balls is called a *basis* for this topology on X
  - it generates all open sets in this topology
- In general, when we refer to a common metric space, say Euclidean space, we refer to this metric space topology induced by standard metric.

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Exercise: why?

Subspace topology

- A topological space $(\mathsf{X}, \mathcal{T})$, say the Euclidean space
- Given a subset $Y \subseteq \mathsf{X}$, the subspace topology $\left(Y, \mathcal{T}_Y\right)$, (inherited from $(\mathsf{X}, \mathcal{T})$), is such that $\mathcal{T}_Y$ consists of intersection between open sets in $\mathcal{T}$ and Y.

- Common subspaces of Euclidean space
  - Euclidean d-ball: $\mathbb{B}^d = \left\{x \in \mathbb{R}^d \mid \|x\| \leq 1\right\}$
  - Open Euclidean d-ball: $\mathbb{B}_o^d = \left\{x \in \mathbb{R}^d \mid \|x\| < 1\right\}$
  - Euclidean d-sphere: $\mathbb{S}^d = \left\{x \in \mathbb{R}^{d+1} \mid \|x\| = 1\right\}$
  - Euclidean half-space: $\mathbb{H}^d = \left\{x \in \mathbb{R}^d \mid x_d \geq 0\right\}$

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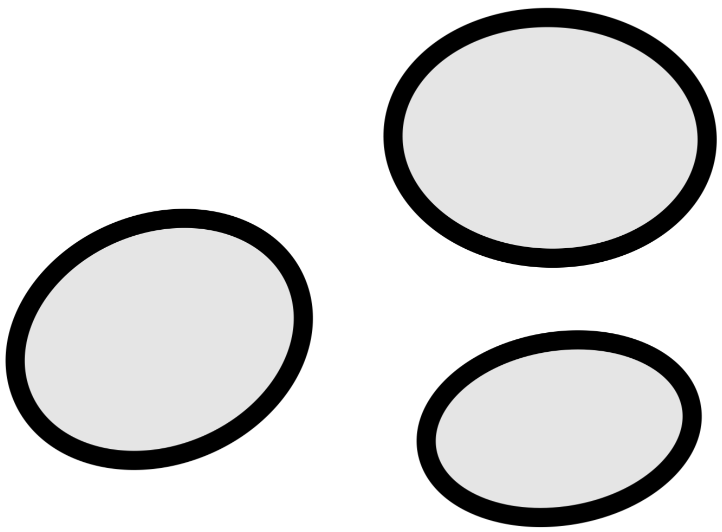

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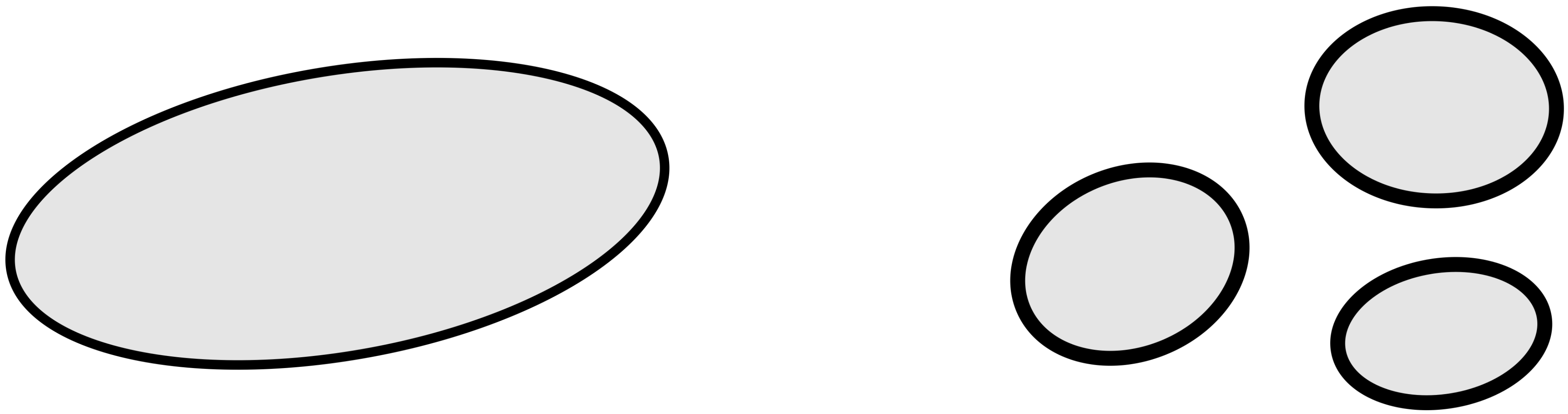

Compactness

- This generalizes the notion of **closed** and **bounded** sets in Euclidean space

- Open cover: $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ is an open cover for $(X, \mathscr{T})$ if $U_\alpha \in \mathscr{T}$ and $X = \cup_{\alpha \in A} U_\alpha$
- $(X, \mathscr{T})$ is called **compact** if for any open cover $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ there exists a finite subcover, i.e., a finite set $A' \subseteq A$ such that $X = \cup_{\alpha \in A'} U_\alpha$
- Example: $(0,1)$ is not compact but $[0,1]$ is compact

Check-in: Where are we?

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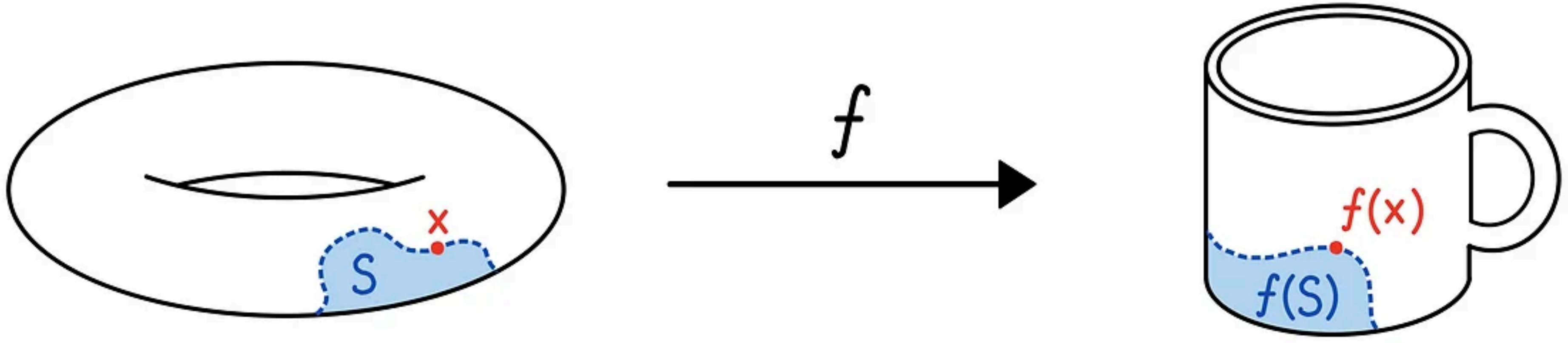
How we mathematically talk about space of interest

Now we need ways to connect different spaces!

Recall

- We want to rigorously define "continuous transformation"
  - A continuous map shouldn't tear things apart
  - If S "contacts" x, under a continuous transformation, we want that $f(S)$ "contacts" $f(x)$



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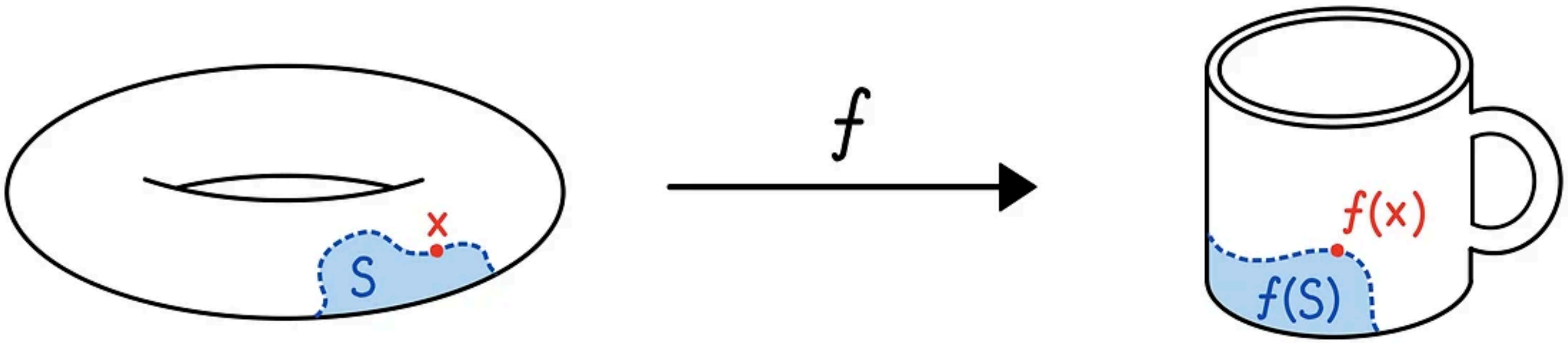
    - $f(\bar{S}) \subset \overline{f(S)}$
- A formal way describing "If S contacts x, then $f(S)$ contacts $f(x)$"



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1. Closure of image contains image of closure
2. Continuous map does not tear things apart

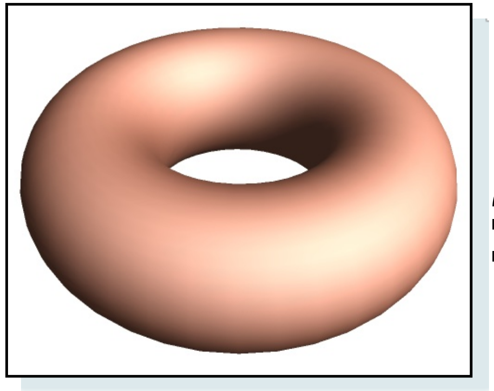
From https://wgyory.wixsite.com/toolatetopologize/post/post-1

Exercise

- Please try to prove the notion of continuity in calculus is compatible with the new definition of continuity
  - $\lim_{x \to x_0} f(x) = f(x_0)$ for all x_0

Check-in: Where are we?

- Fundamental concepts
  - Topological space → How we mathematically talk about space of interest
  - Continuous maps → Now we need ways to connect different spaces!
  - Homeomorphisms and homotopies → Describe relations of spaces
  - Manifolds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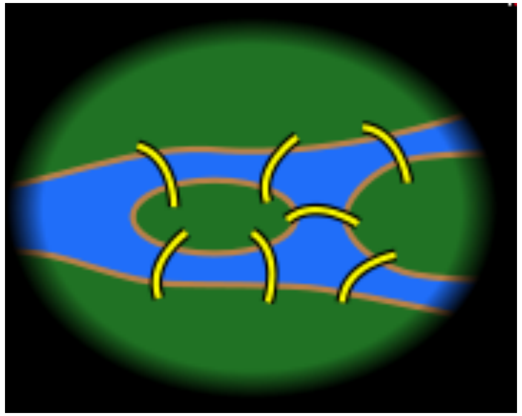
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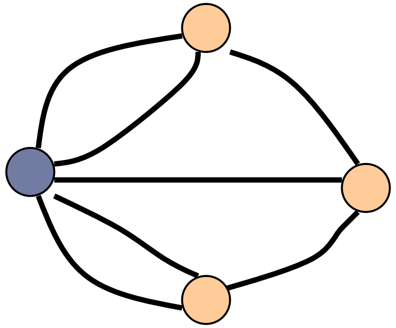

Homeomorphism = homoios + morphē = Similar shapes

Definition 5 (Homeomorphism) *Given two topological spaces* X *and* Y, *a* homeomorphism *between them is a map* $h : \mathsf{X} \to \mathsf{Y}$ *such that* h *is bijection and the inverse of* h *is also continous.*

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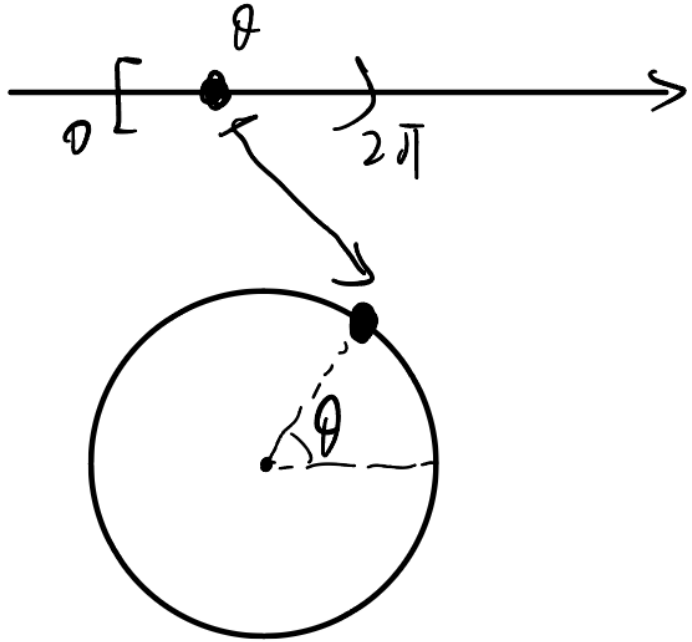
  - Caveat: not always true
- Homeomorphism preserves all topological quantities: number of connected components, number of holes, voids, etc.

Continuous bijective map may have discontinuous inverse

- $[0,2\pi) \to \mathbb{S}^1$ by $\theta \mapsto e^{i\theta}$



Examples

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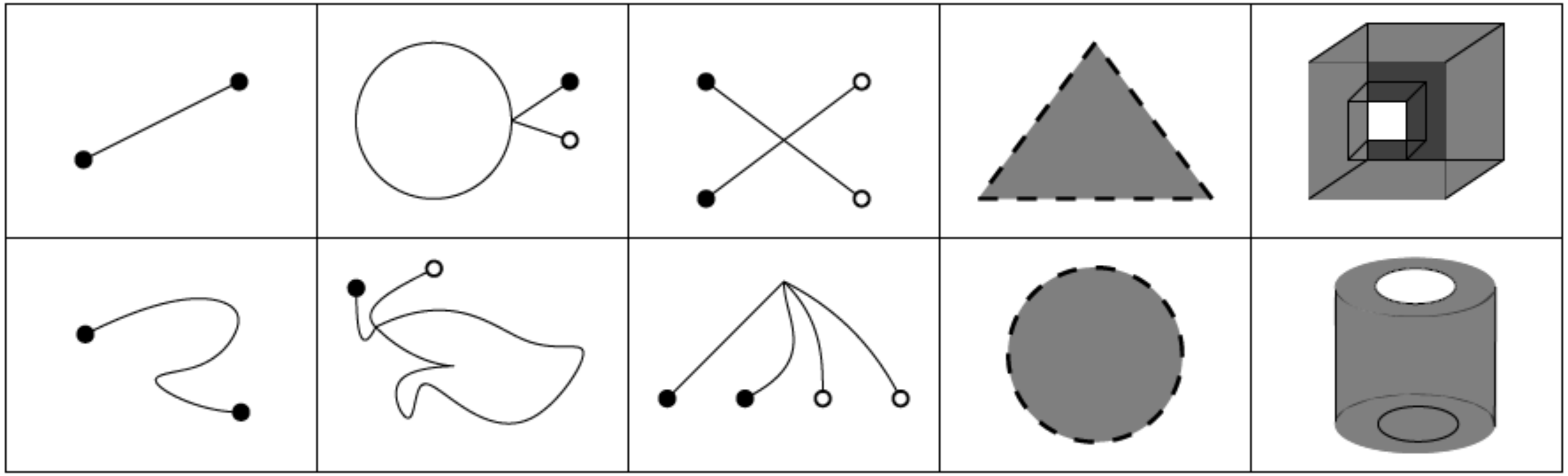

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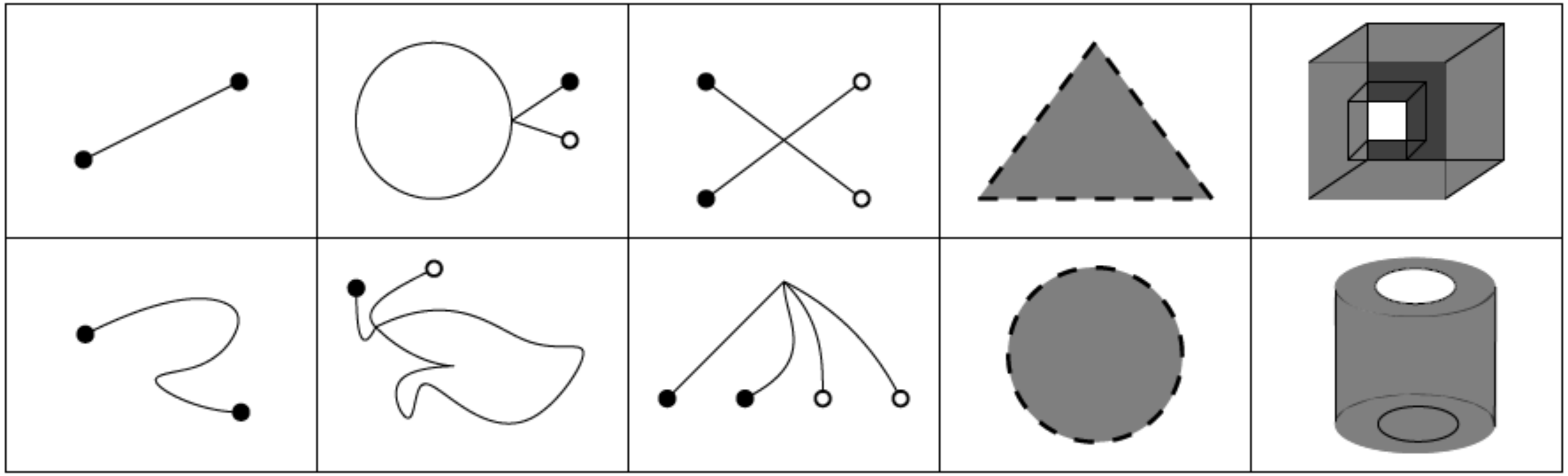

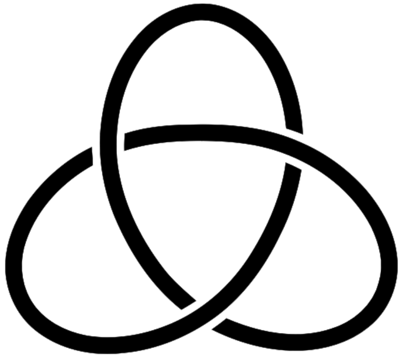

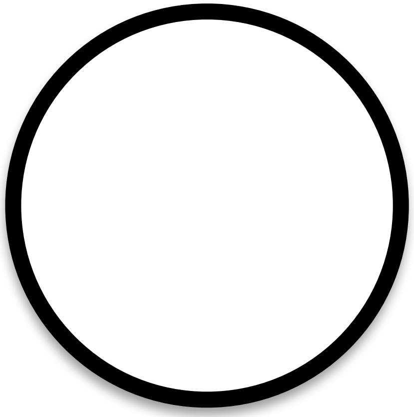

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- A trick: remove one point and check connected components
  - Y and I are not homeomorphic; X and Y are not homeomorphic
  - $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic
  - What about $\mathbb{R}^2$ and $\mathbb{R}^3$?

- In general, hard to decide whether two spaces are homeomorphic or not!

More Examples and Non Examples

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- The Euclidean space $\mathbb{R}^d$ is homeomorphic to any open ball $\mathbb{B}_o(c, r)$

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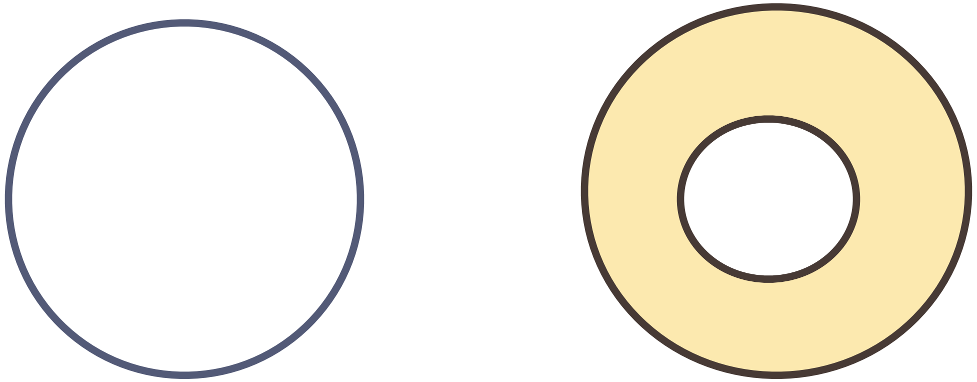
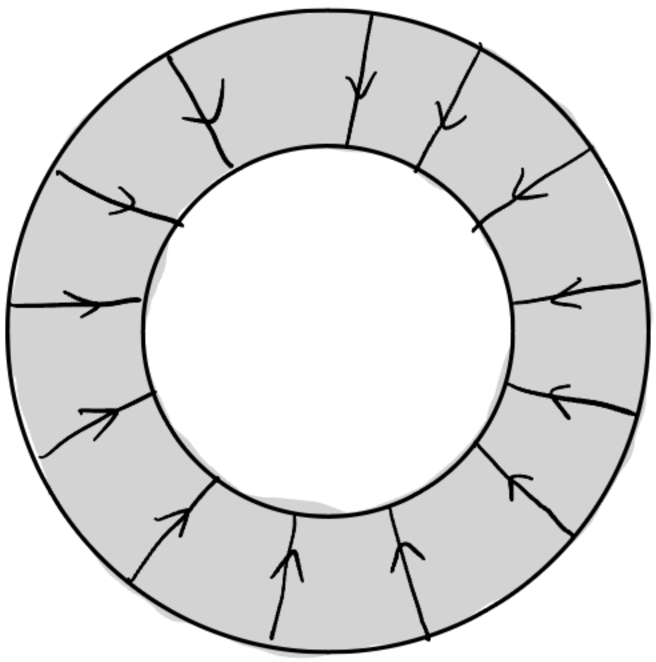
  - We will see why after knowing homotopy

Another level of similarity



- They are not homeomorphic (why?)
- But they look very similar

A closer look



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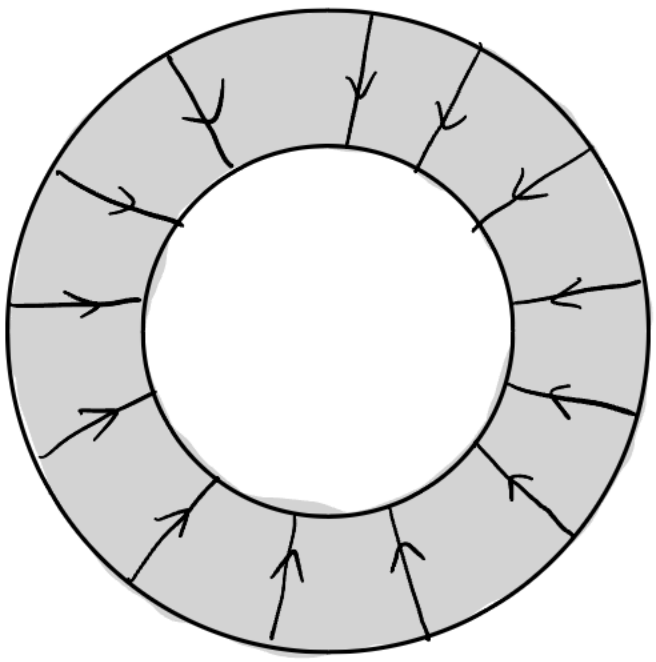
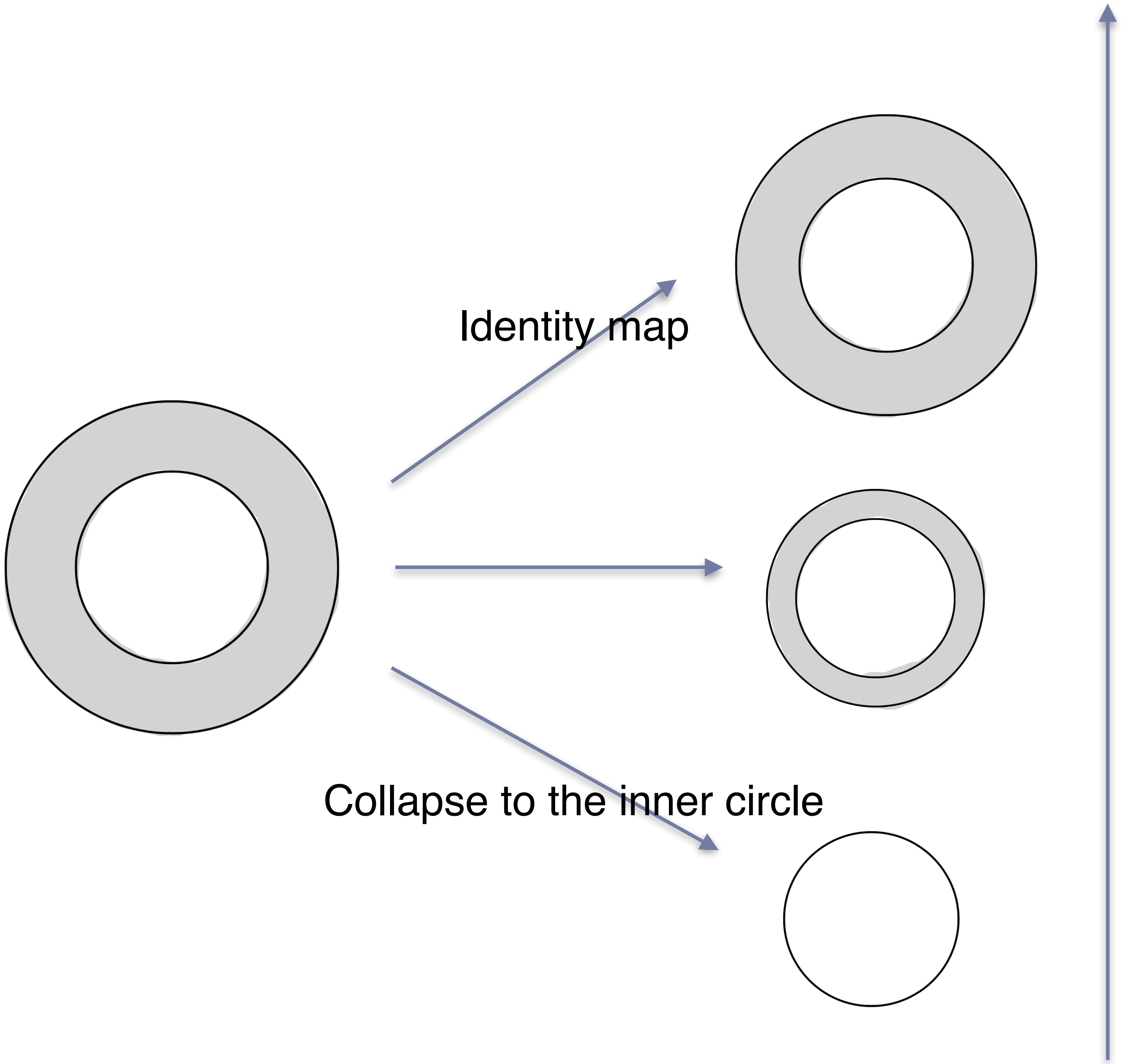
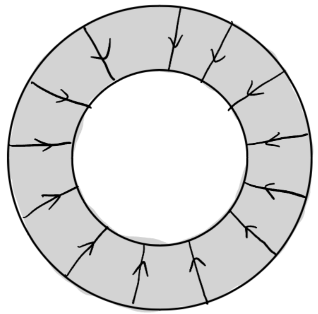
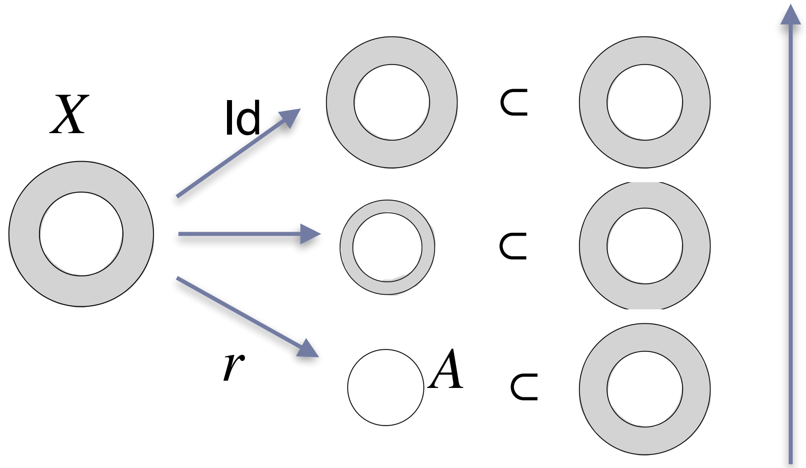




Homotopy and Deformation Retraction

Definition 6 (Homotopy) *First, two* $f, g : \mathsf{X} \to \mathsf{Y}$ *are* homotopic *if there is a continuous map* $H : \mathsf{X} \times [0,1] \to \mathsf{Y}$ *such that* $H_0 = f$ *and* $H_1 = g$. *This map* H *is called a* homotopy *connecting* f *and* g. If H exists, we say f and g are homotopic, denoted by $f \simeq g$





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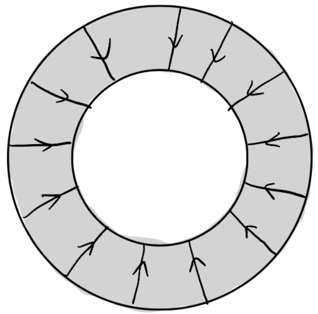
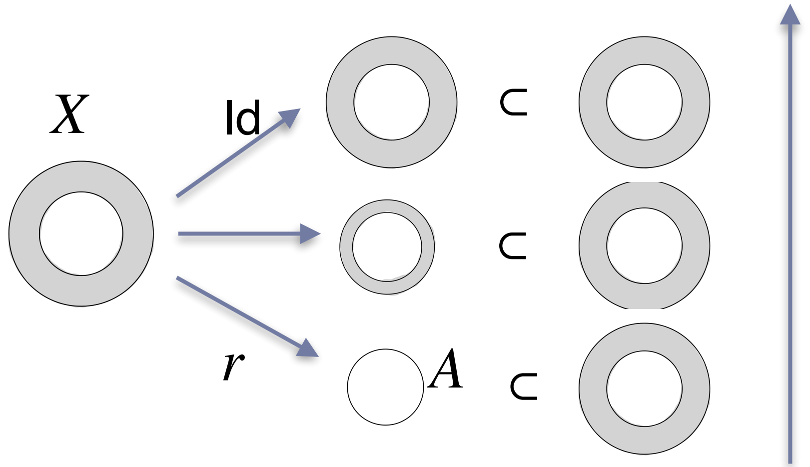


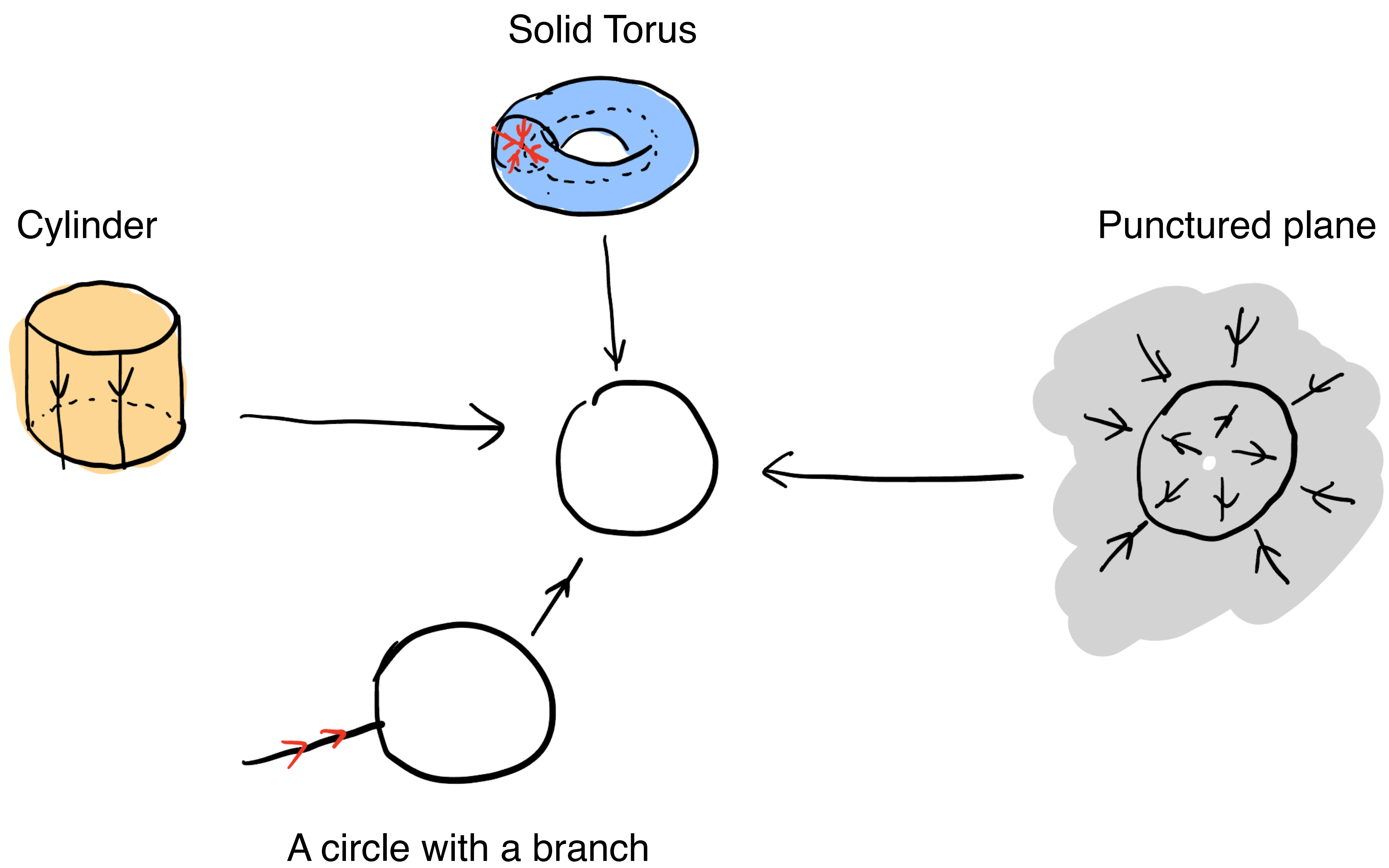


Definition 7 (Deformation retraction) *Let $A \subseteq X$ be a subspace of topological space X. A* retraction (map) *r is a continuous map $r : X \to A$ such that $r(x) = x$ for any $x \in A$.*

We say that $A \subseteq X$ is a deformation retract of X *if there is a retraction r that is homotopic to the identity map in X. This retraction map is called a* deformation retraction.

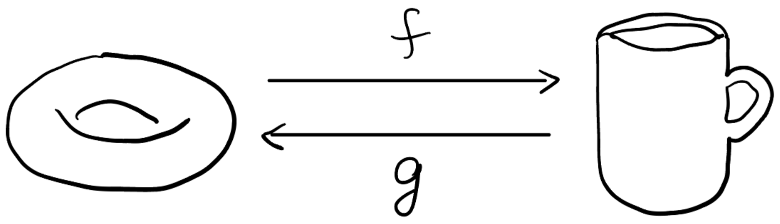
Equivalently, a continuous map $R : X \times [0,1] \to X$ is a deformation retraction of X onto A *if for every $x \in X$ and $a \in A$, $R(x,0) = x$; $R(x,1) \in A$ and $R(a,1) = a$.*

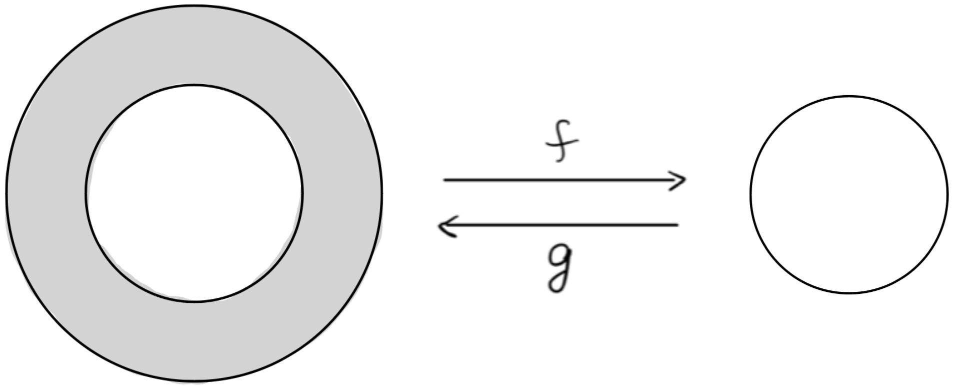
Examples of Deformation Retraction



A weaker notion of similarity: Homotopy equivalent

- When X, Y are homeomorphic, there exists continuous functions $f: X \to Y, g: Y \to X$ such that $f \circ g = Id_Y$ and $g \circ f = Id_X$

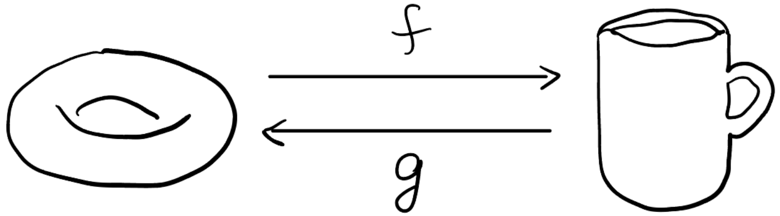


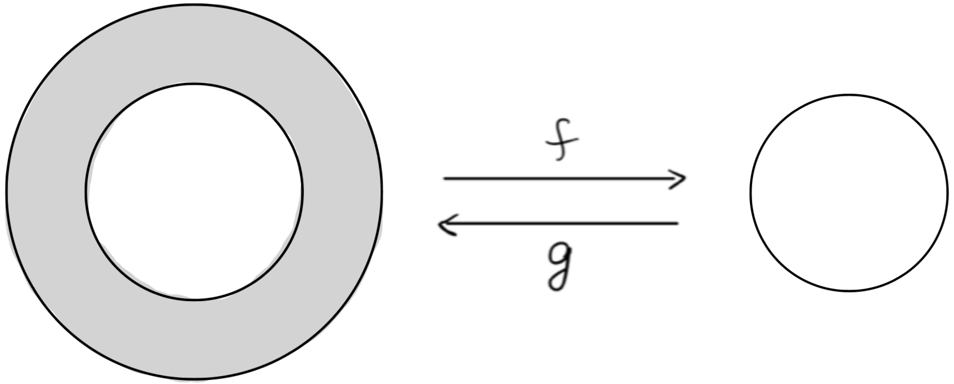


- We say X, Y are **homotopy equivalent**, if there exists continuous functions $f: X \to Y, g: Y \to X$ such that $f \circ g \simeq Id_Y$ and $g \circ f \simeq Id_X$

A weaker notion of similarity: Homotopy equivalent

- Homeomorphism allows stretching and shrinking





- Homotopy allows stretching, shrinking and **crushing/collapsing**

A weaker notion: Homotopy equivalent

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- Homotopy equivalent relation is transitive.

- In general, hard to establish homotopy equivalent relation as well

Definition 7 (Deformation retraction) *Let $A \subseteq X$ be a subspace of topological space X. A* retraction (map) *r is a continuous map $r : X \to A$ such that $r(x) = x$ for any $x \in A$.*

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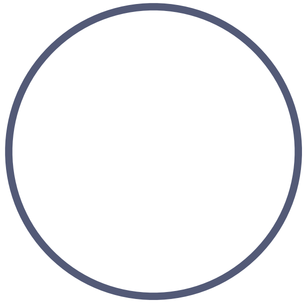
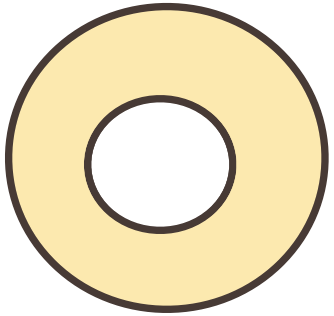
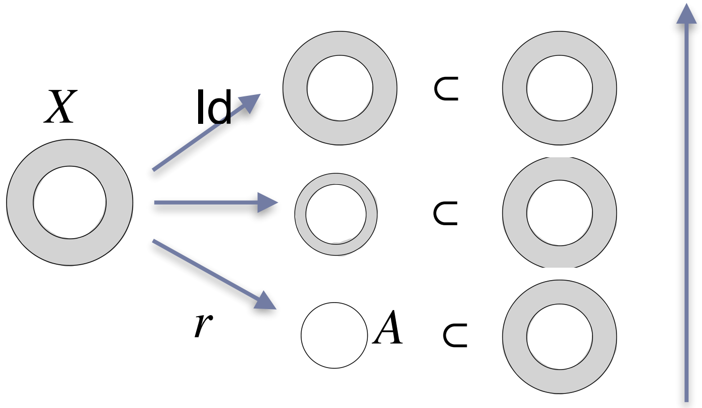
- Theorem:
  - If Y is a deformation retract of X, then X and Y are homotopy equivalent.

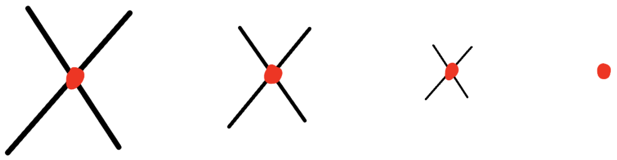




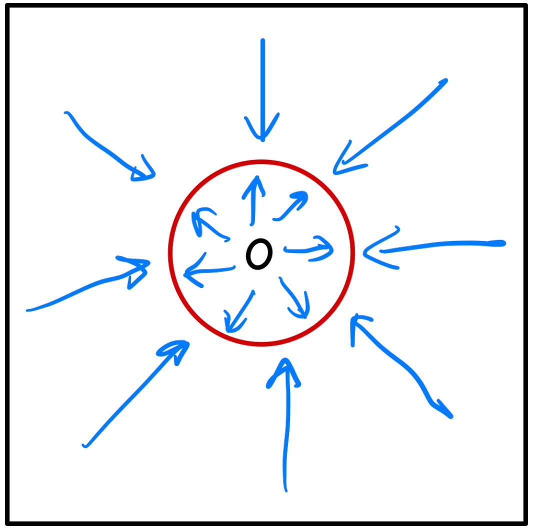


Examples of homotopy equivalence

- X and Y are homotopy equivalent but not homeomorphic



- A disk and a point
- A tree and a point
- A punctured plane and a circle



$$R : [0,1] \times \mathbb{R}^2 \backslash 0 \to \mathbb{R}^2 \backslash 0$$

$$R(t, x) = (1 - t)x + t\frac{x}{\|x\|}$$

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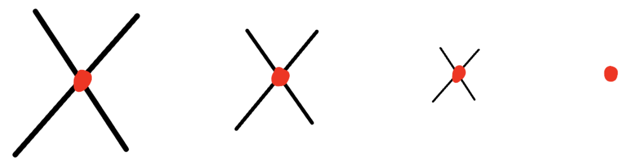

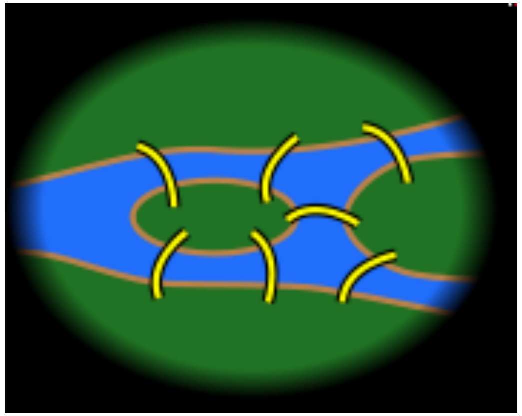

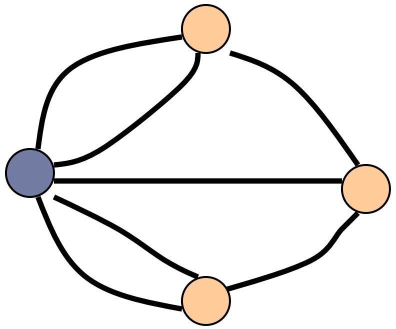

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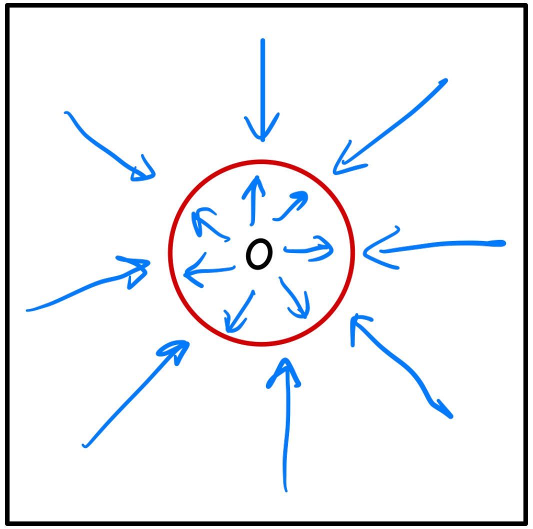

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Examples

- A punctured hollow torus ≈?

Examples

- A punctured hollow torus $\approx$?

Non Examples

Non Examples

- The Euclidean space $\mathbb{R}^n$ is homotopy equivalent to a point

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- Are $\mathbb{S}^n$ and $\mathbb{S}^m$ homotopy equivalent?
- Can we use computer to determine whether two topological spaces are homotopy equivalent or not?

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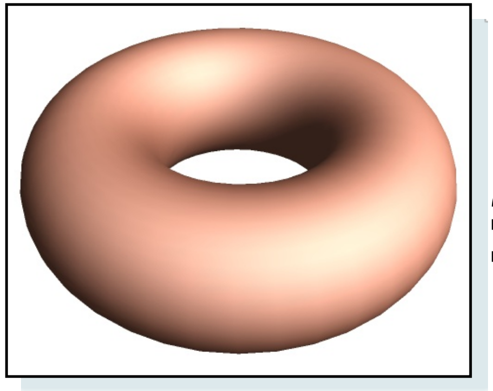
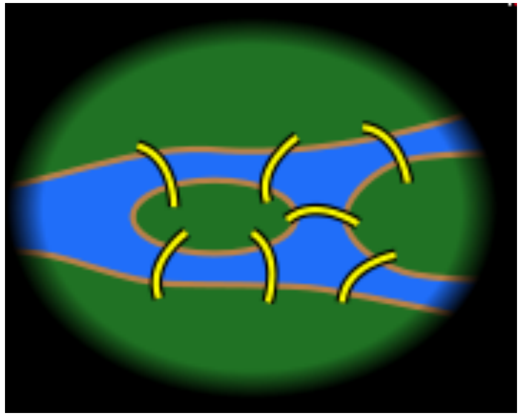
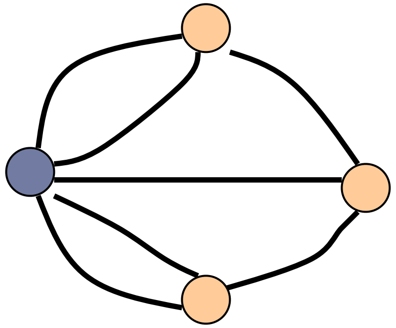
Both homeomorphism and homotopy equivalence can be hard to detect, and not computationally friendly in general. Later we will focus on homology, which is much easier to compute.

Here is what I promised

- $\mathbb{R}^d$ is not homeomorphic to the half space $\mathbb{H}^d = \{(x_1, \cdots, x_d) : x_1 \geq 0\}$

Summary

 $\cong$ 

 $\simeq$ 

- Fundamental Questions
  - What is a topological space?
  - What is a "continuous" way of turning one space to another?
  - When can we say two spaces are the "same"?

Check-in: Where are we?

- Fundamental concepts
  - Topological space
  - Continuous maps
  - Homeomorphisms and homotopies
  - Manifolds

How we mathematically talk about space of interest

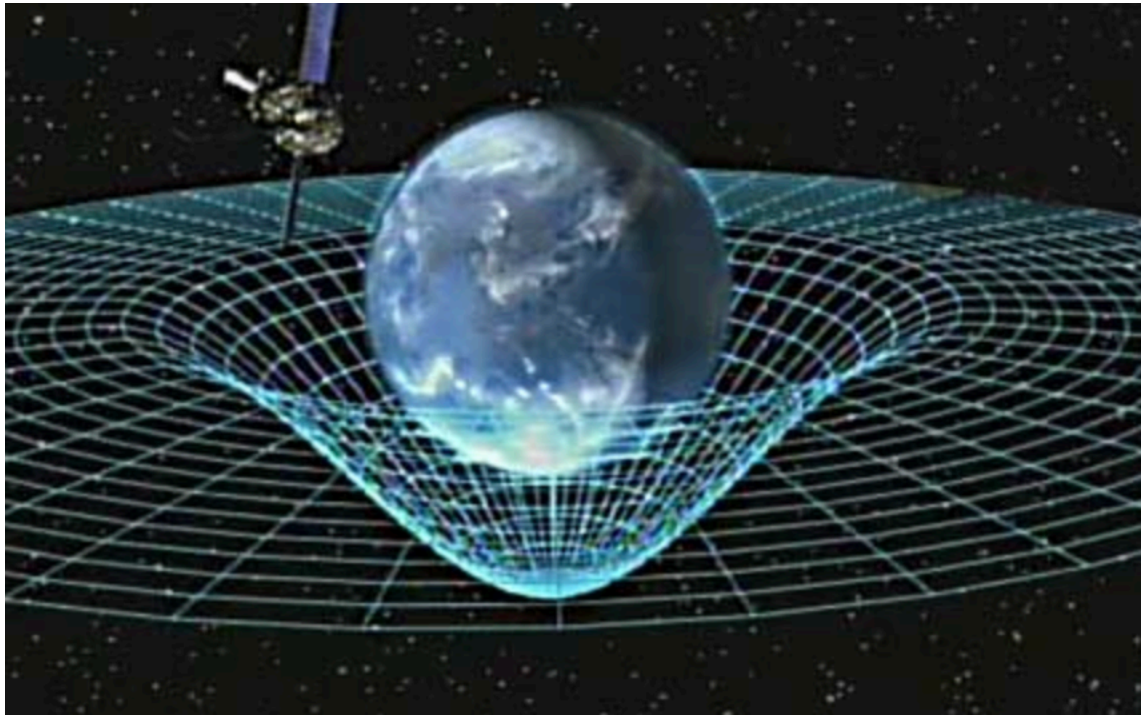
Now we need ways to connect different spaces!

Describe relations of spaces

A generalization of Euclidean space; a special type of topological space

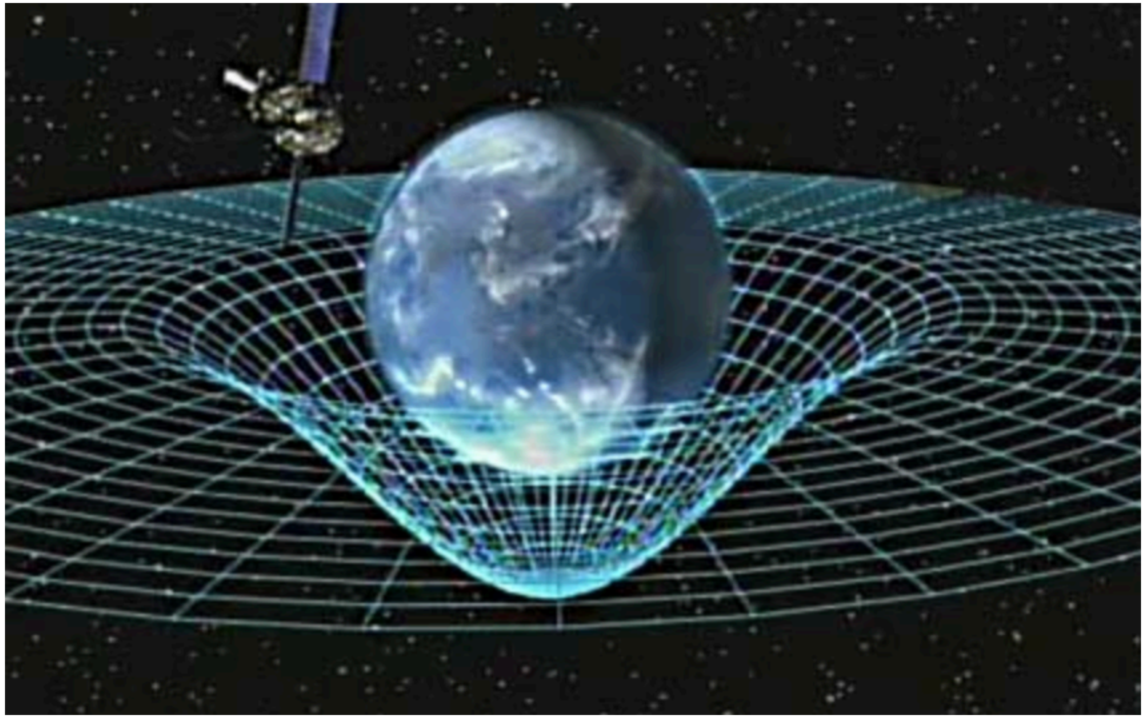
Why manifolds?

- We like Euclidean spaces. Why manifolds?



Why manifolds?

- We like Euclidean spaces. Why manifolds?
  - We live in non-Euclidean space



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  - Data and the space where data are sampled from often live in a subspace within the ambient space they are embedded in
    - Ambient space:
      - the space where data or the object/space of interest are embedded in.
    - Intrinsic space:
      - the space where data are sampled from **(manifold hypothesis)**

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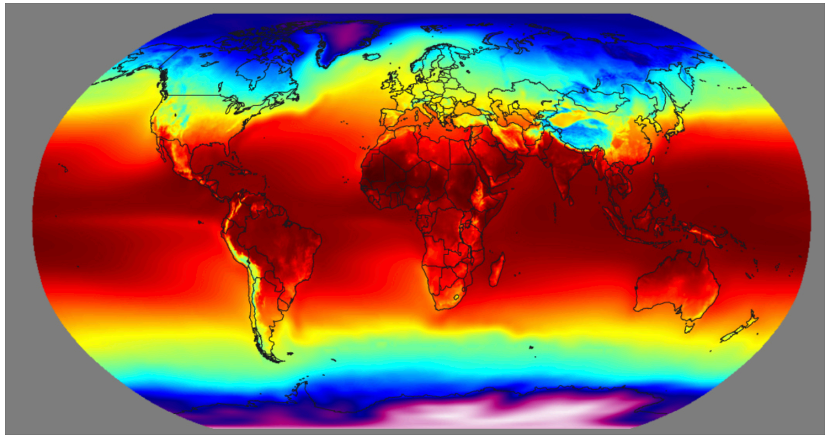

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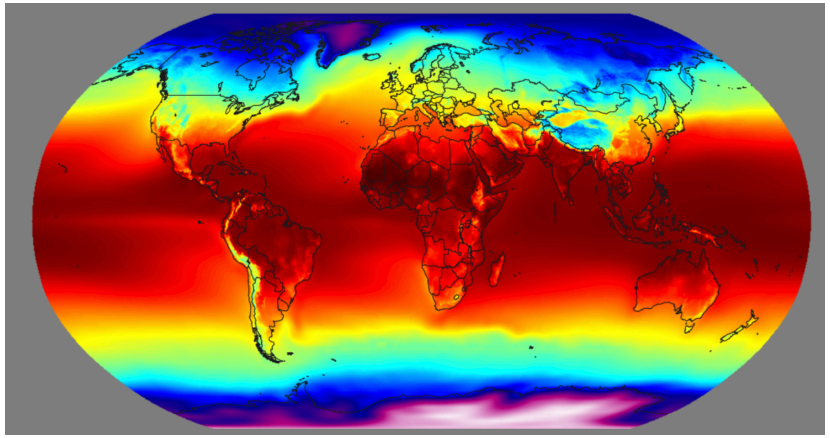
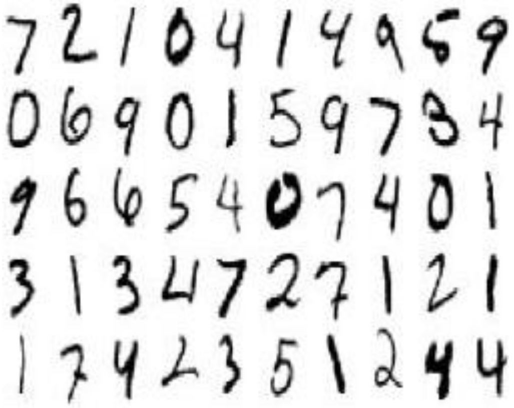
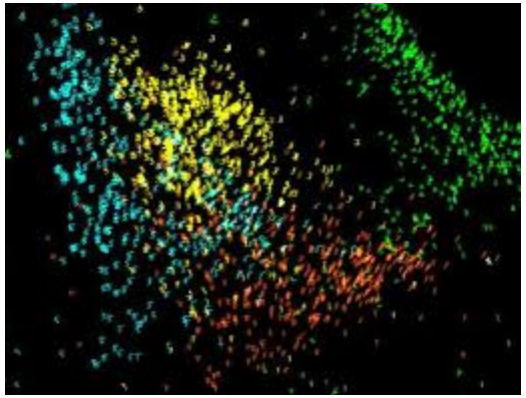
    - Intrinsic space:
      - the space where data are sampled from **(manifold hypothesis)**







Why manifolds?

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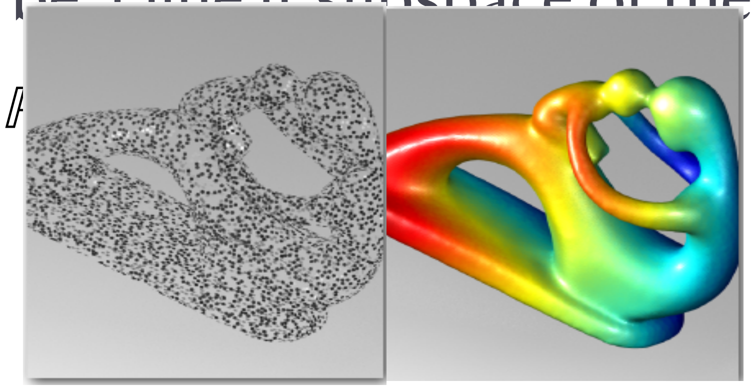

What are manifolds

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- In particular, locally it behaves like a linear space.

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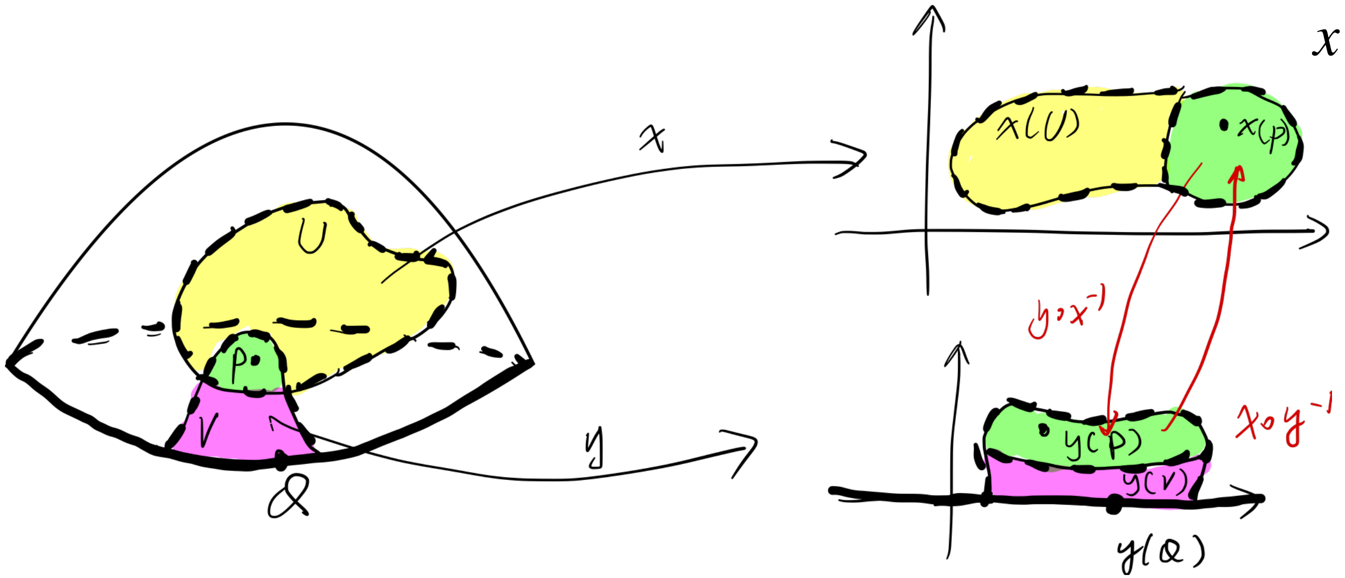

$x : U \to x(U) \subseteq \mathbb{R}^2$ is called a **coordinate chart**

$x \circ y^{-1}$ is a homeomorphism. If $x \circ y^{-1}$ is smooth, then M is called a **differentiable manifold**

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1-dim manifold

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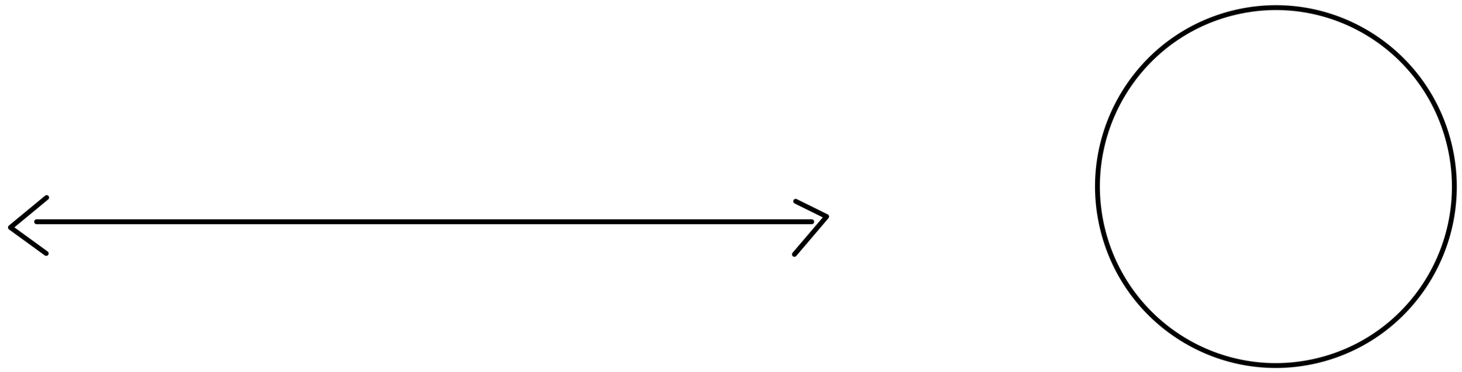
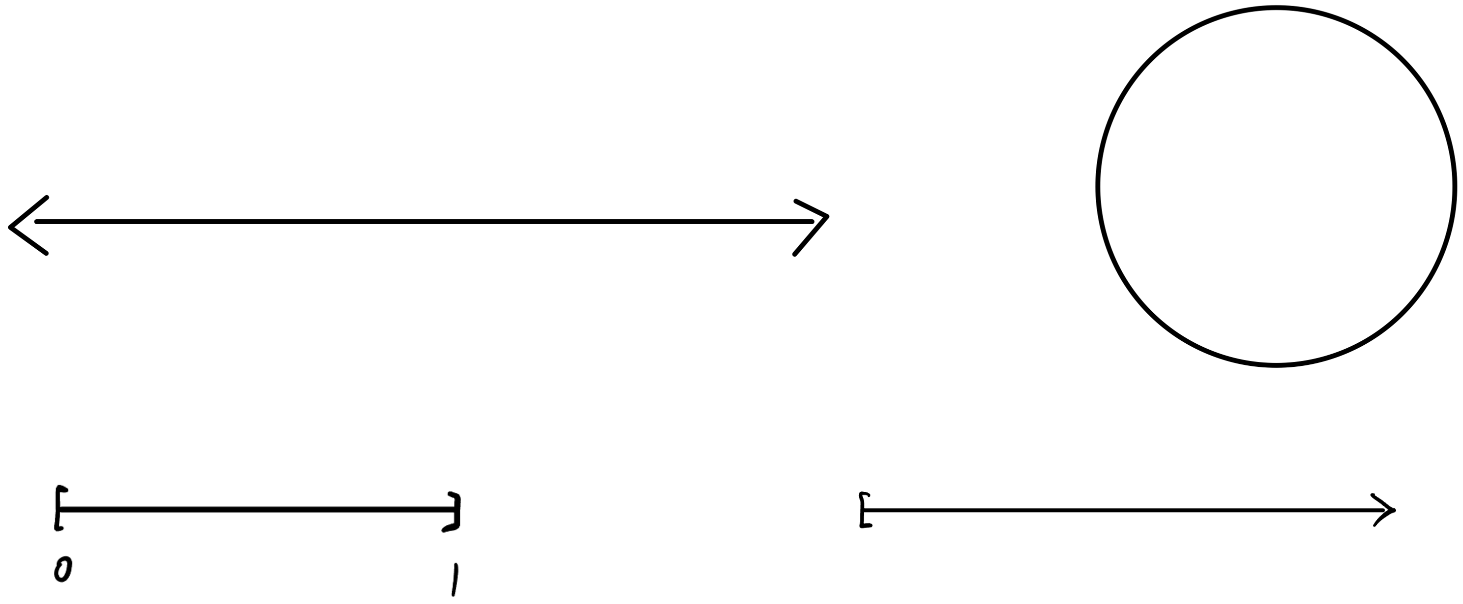


1-dim manifold



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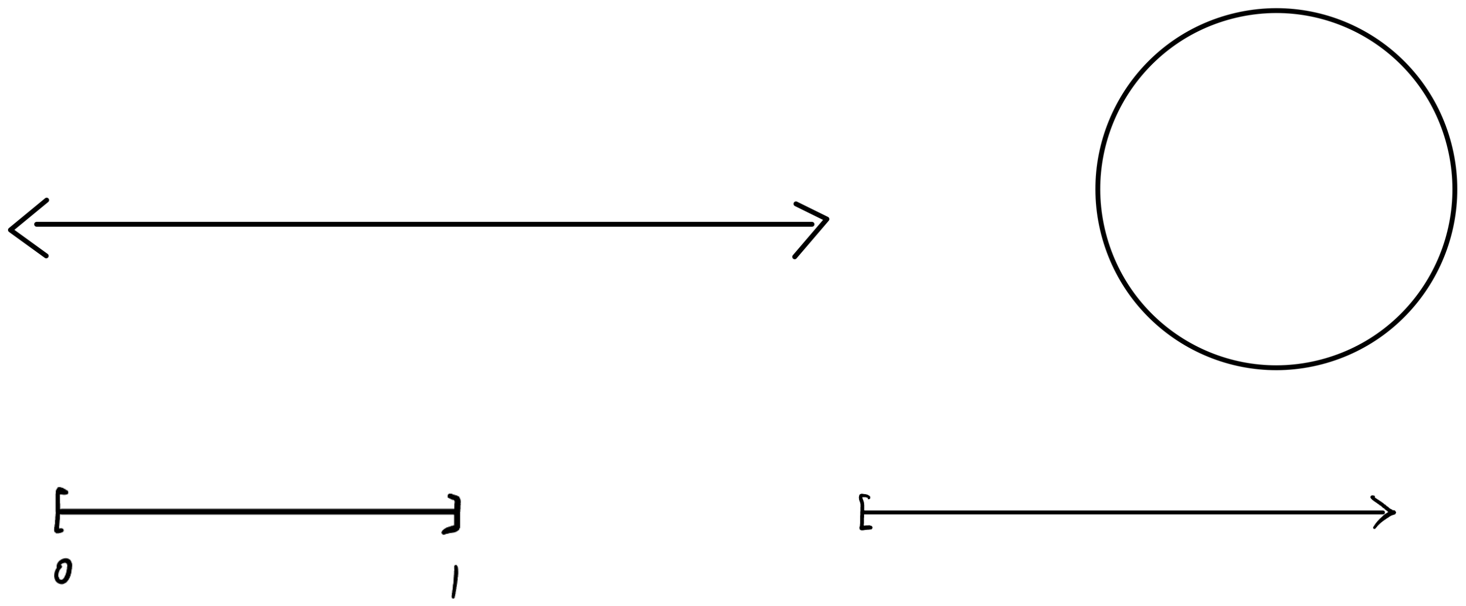
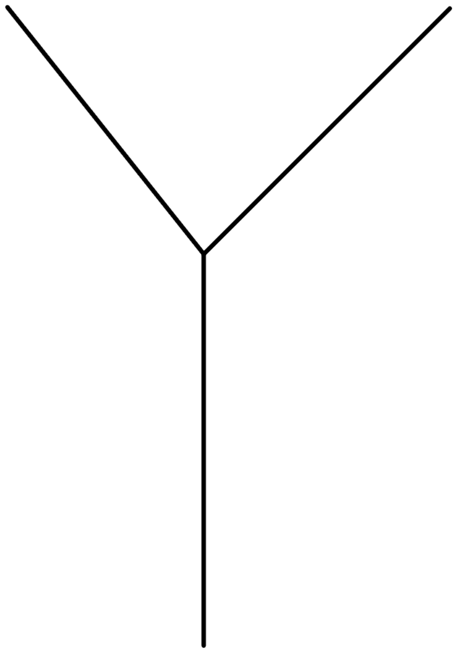


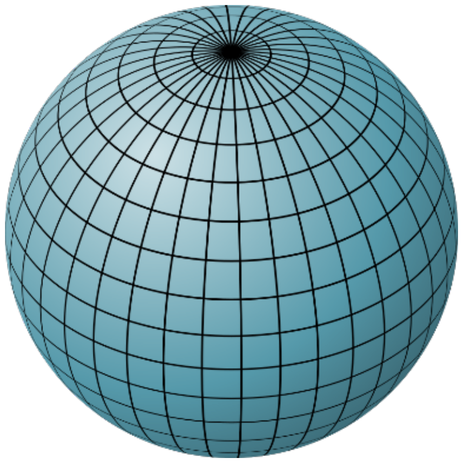
- These are everything!
- Any connected 1-dim manifold (with or without boundary) is homeomorphic to one of the above

A non-example

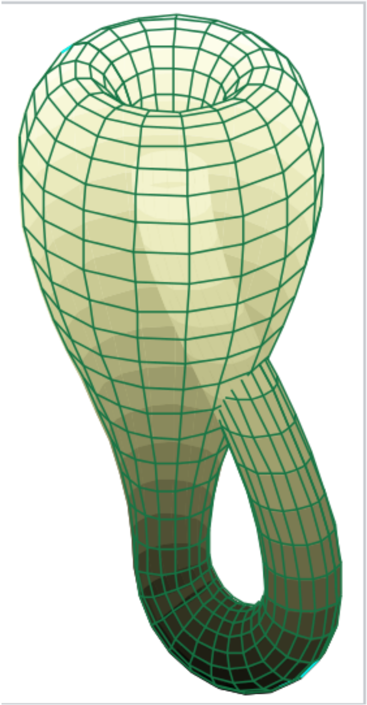


2-dim manifold

Sphere



Klein bottle



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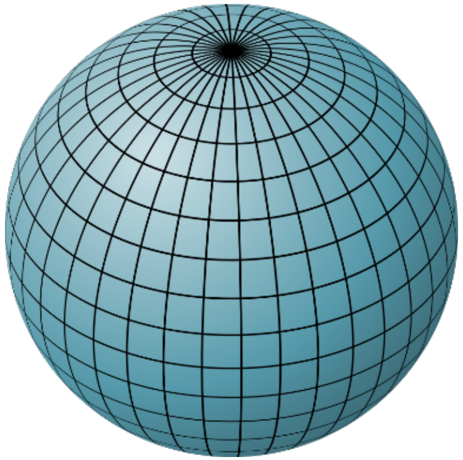

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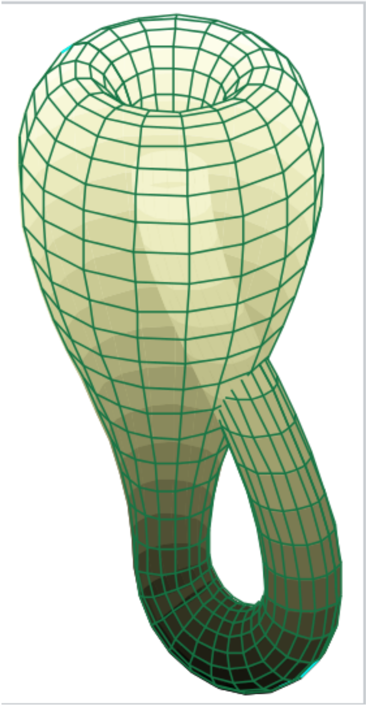
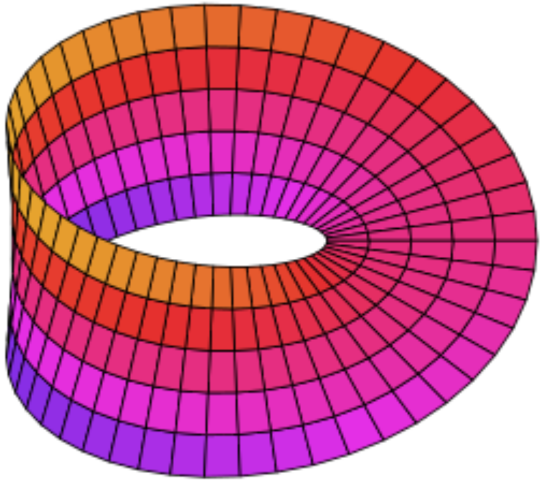
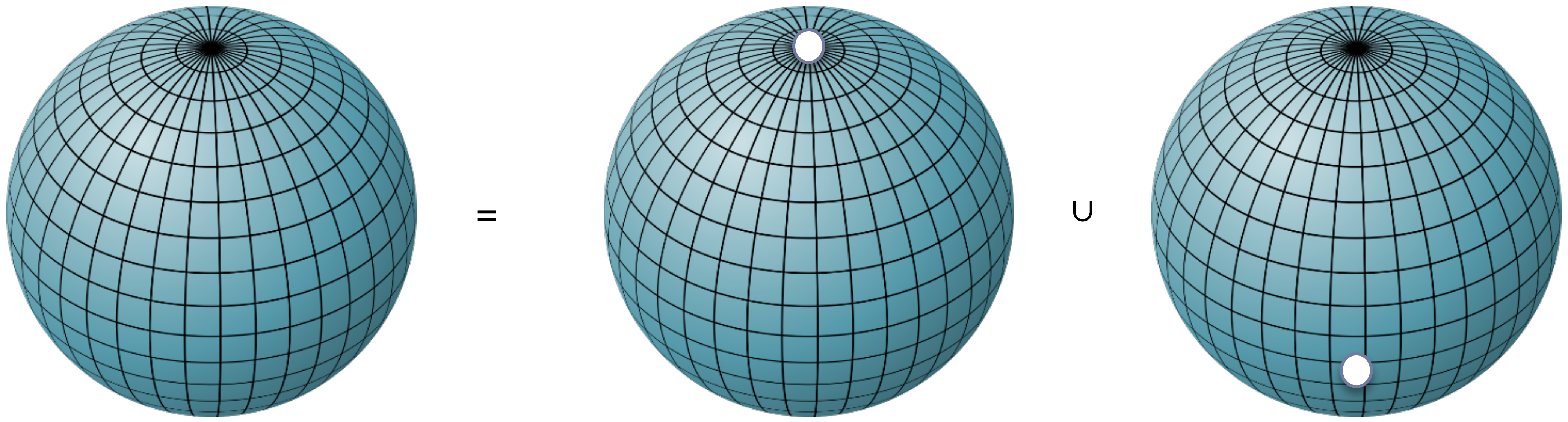


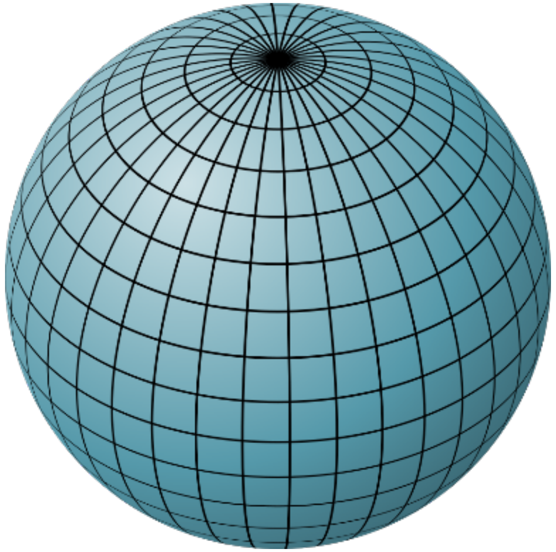
Möbius strip



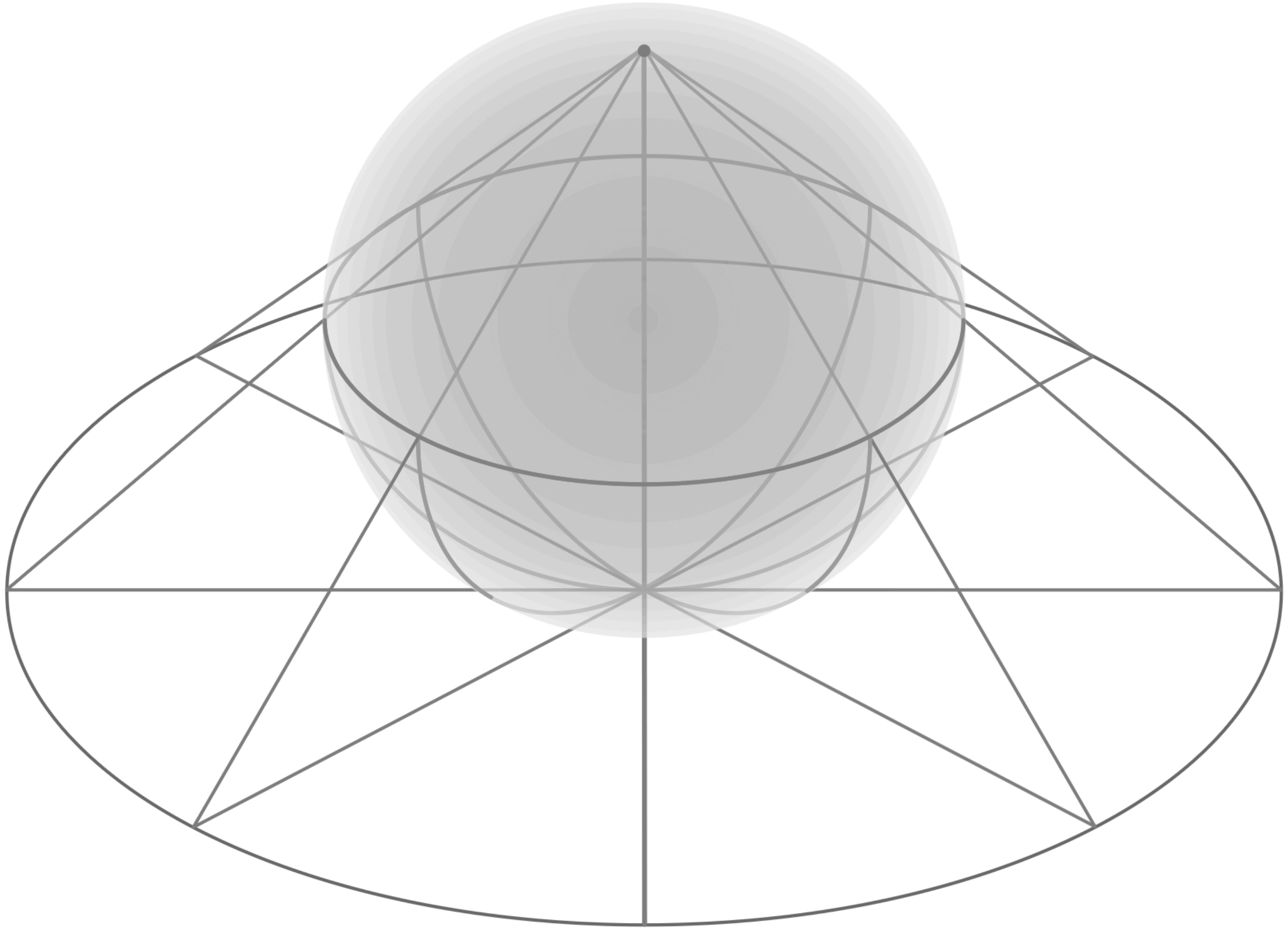
Charts of the sphere



Homeomorphism to $\mathbb{R}^d$



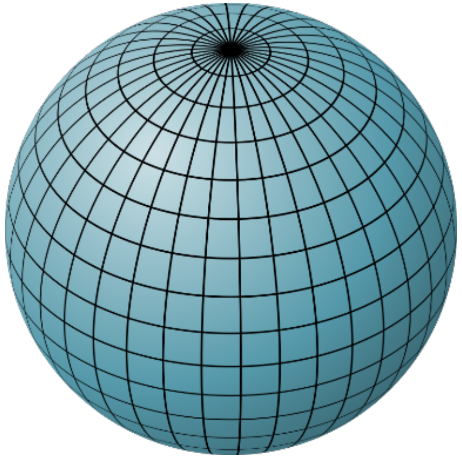
Stereographic projection (pic from wiki)

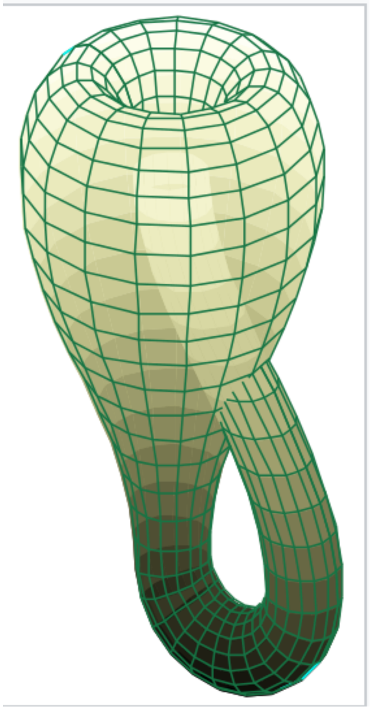


Is this a 2-d manifold?

Classification of 2d manifolds?

- A 2d manifold without boundary is also called a **surface**
- All connected compact surfaces can be classified into a specific class of surfaces constructed from polygons





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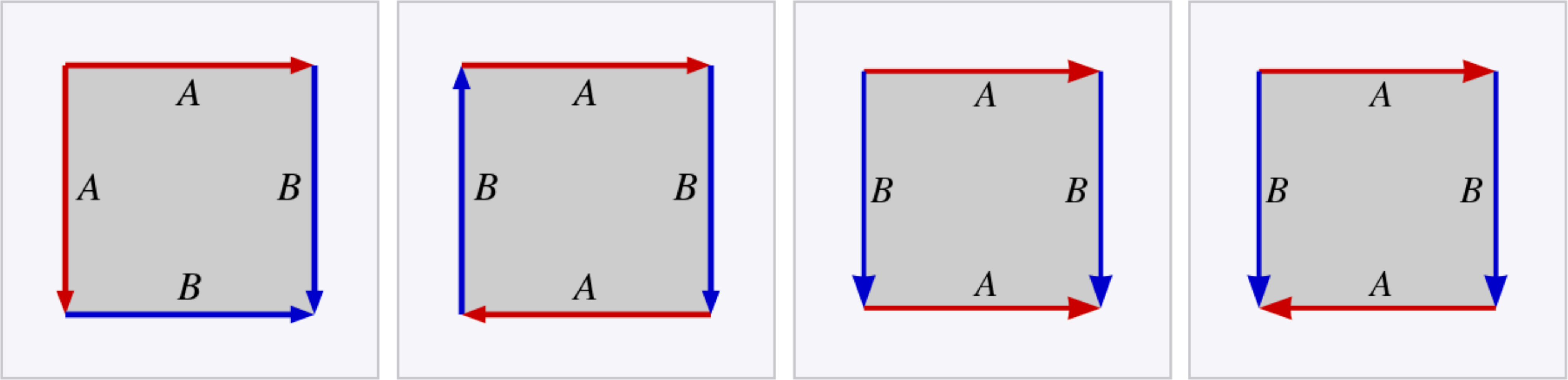
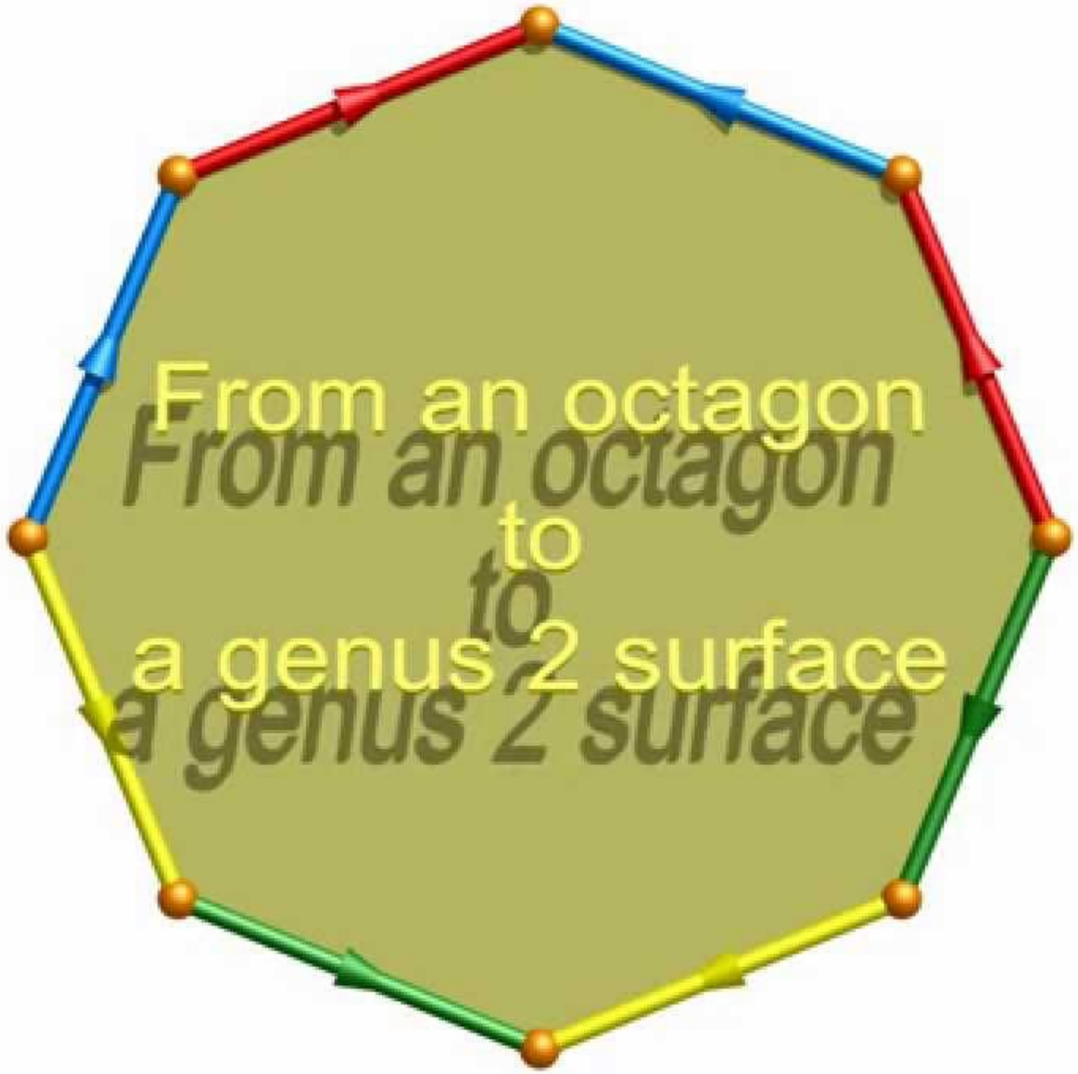


sphere   real projective plane   torus   Klein bottle

Visualization of double Torus



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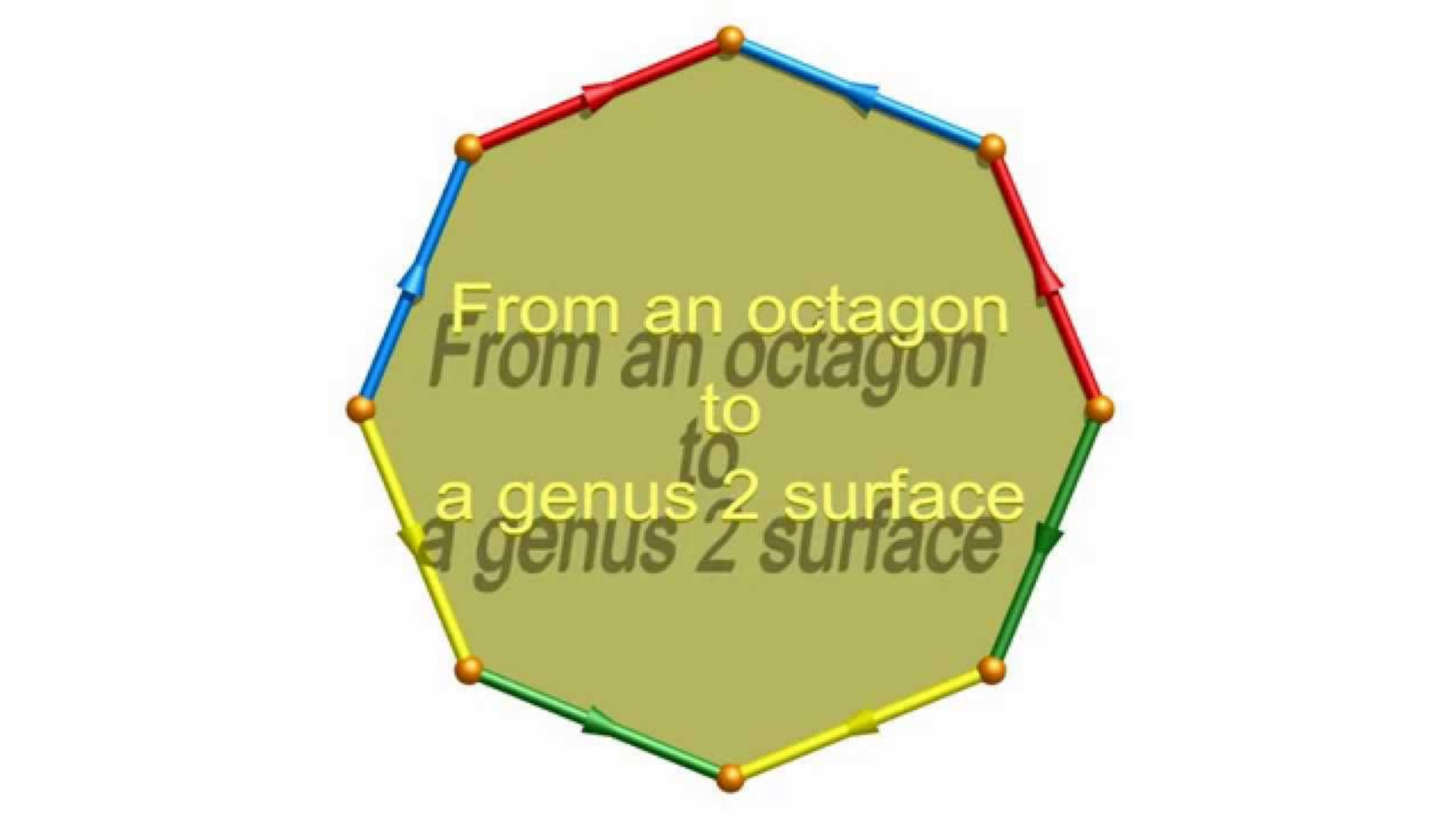
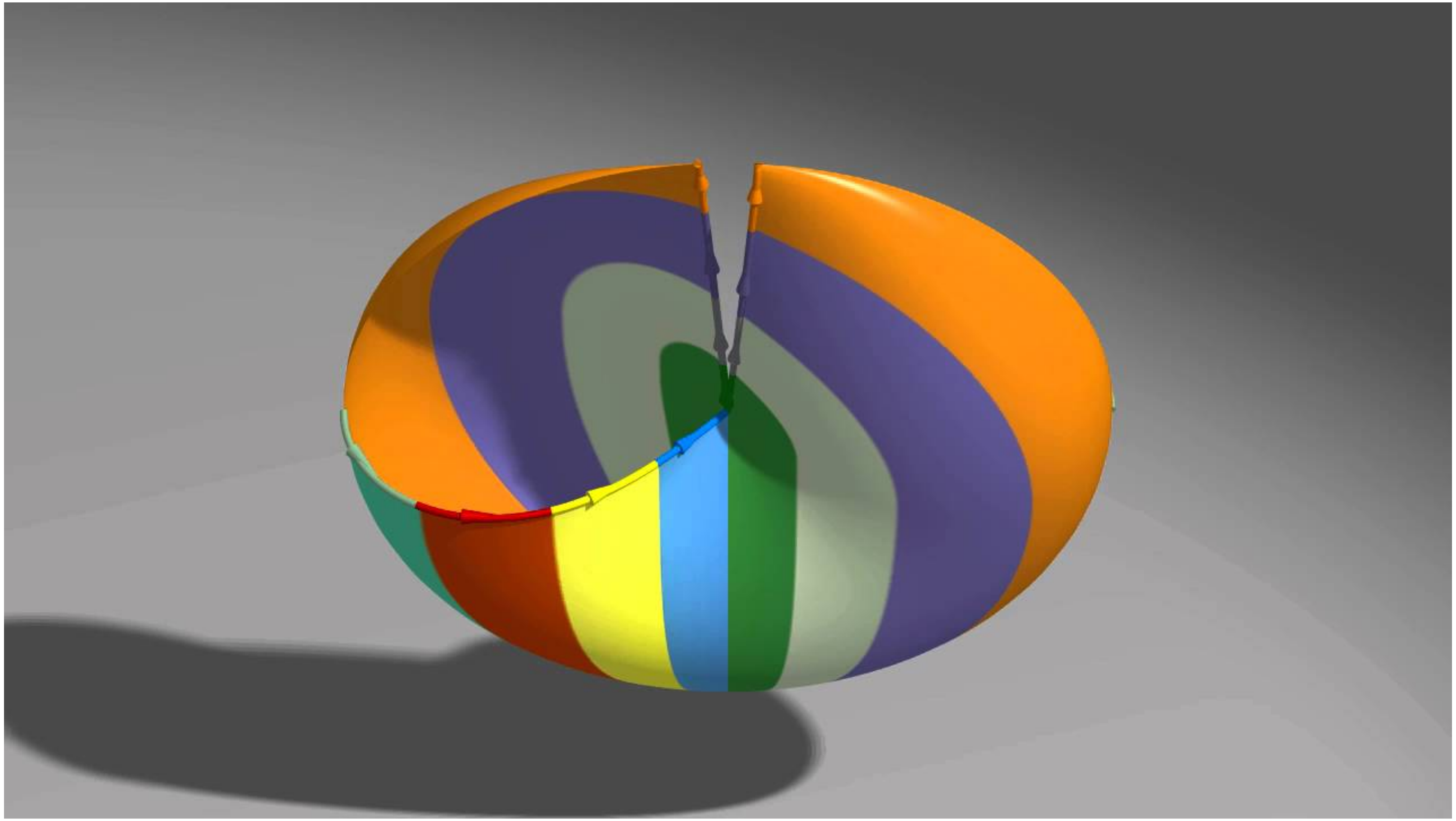


Visualization of projective plane



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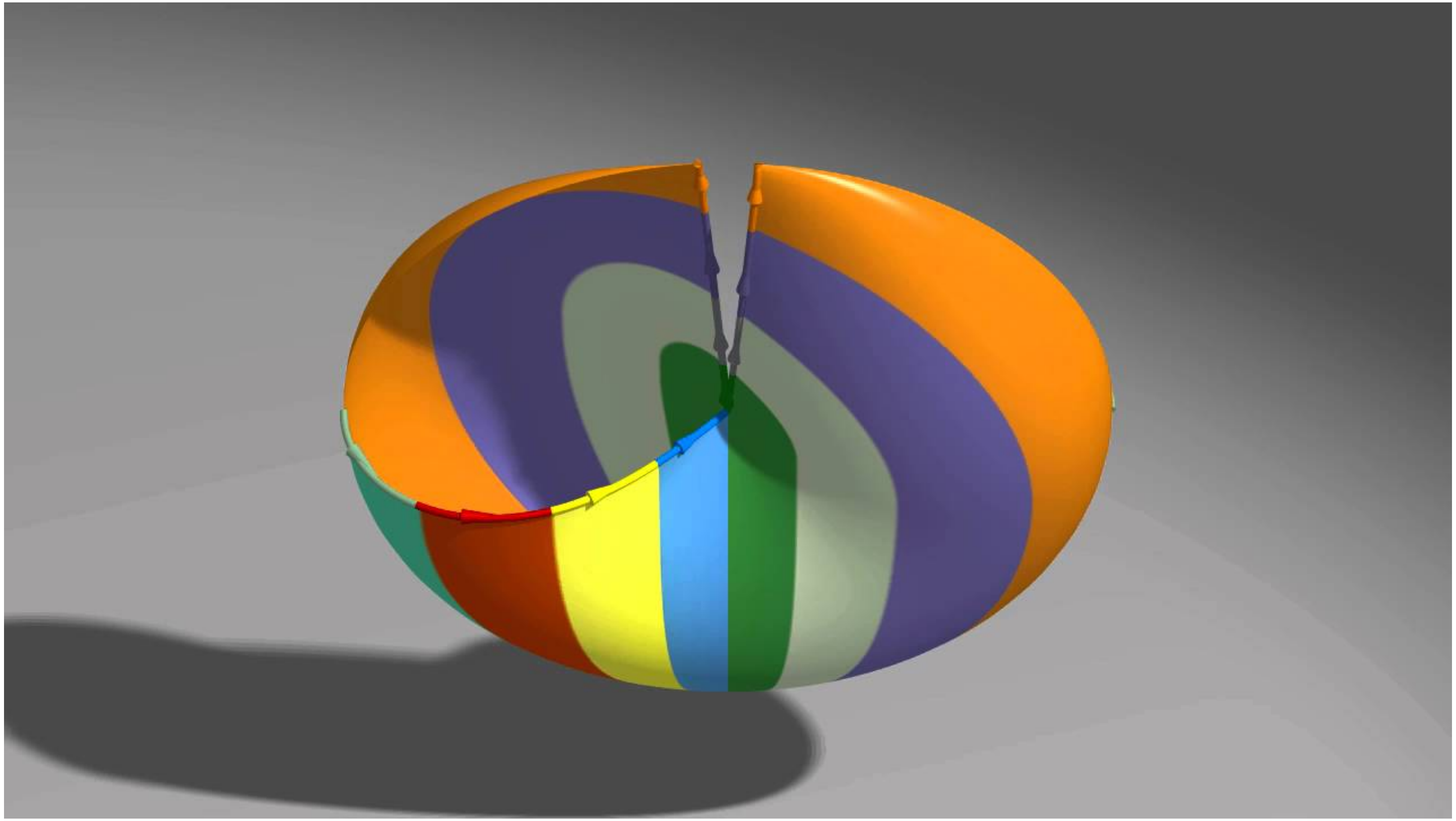
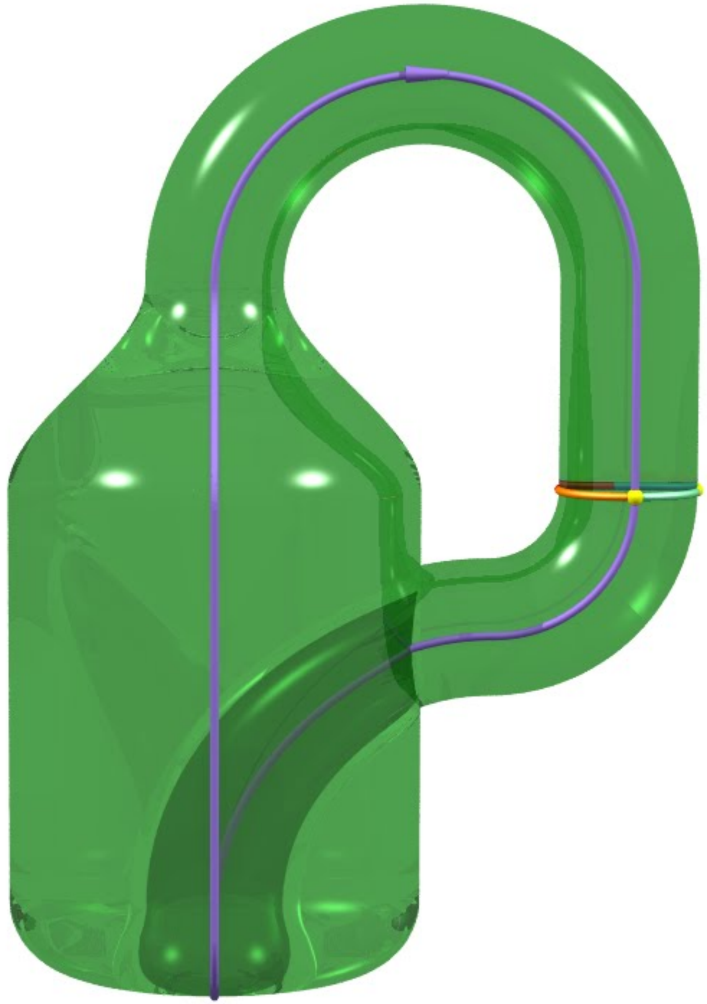


Visualization of Klein bottle



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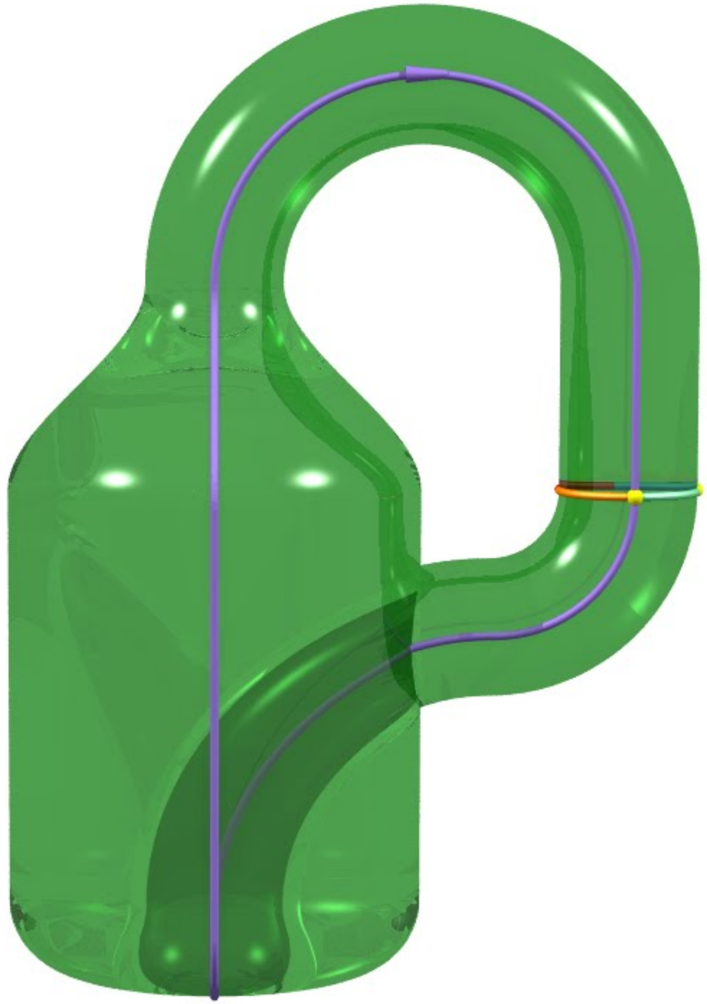
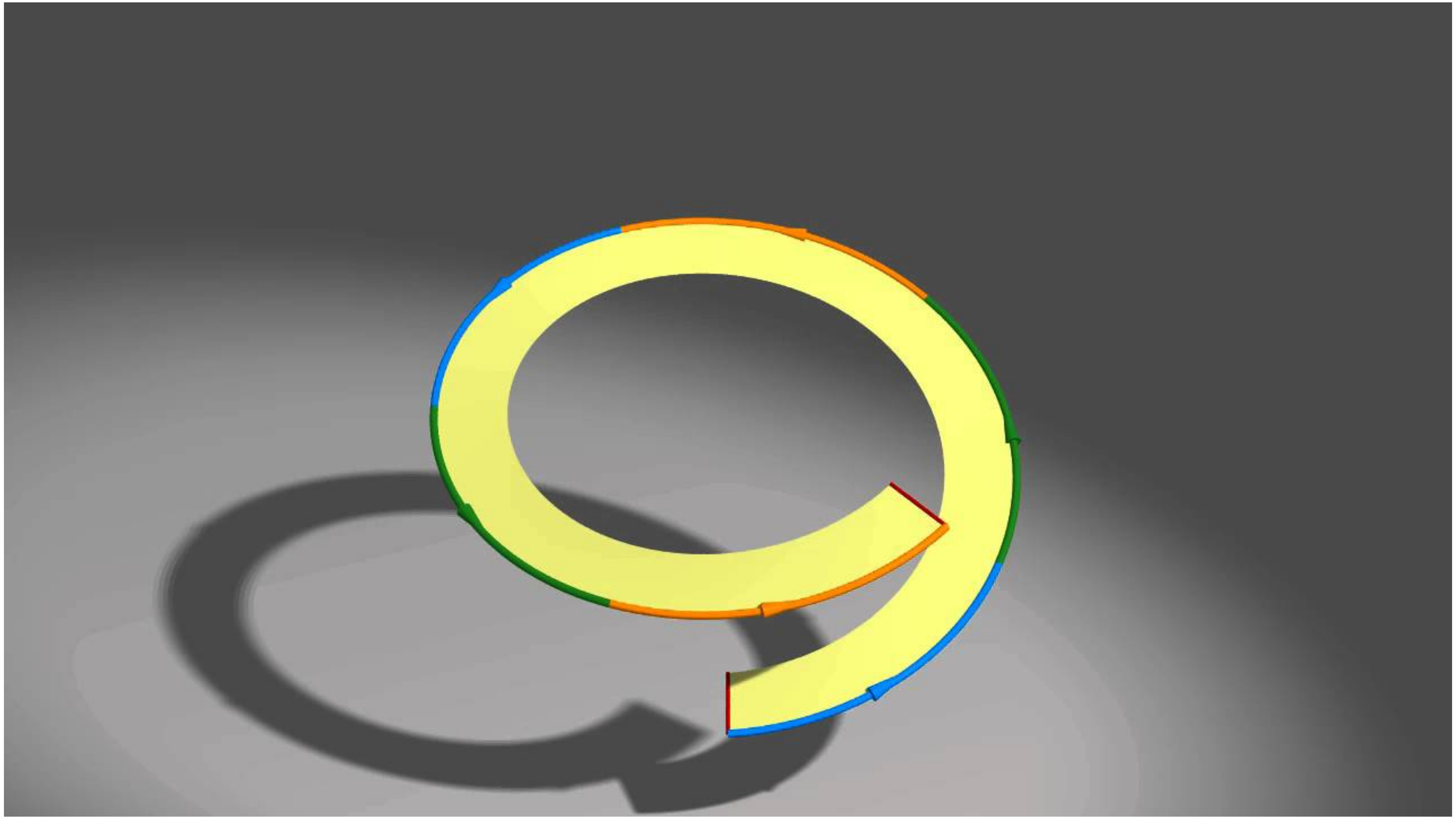


Projective plane = Mobius strip + a disk



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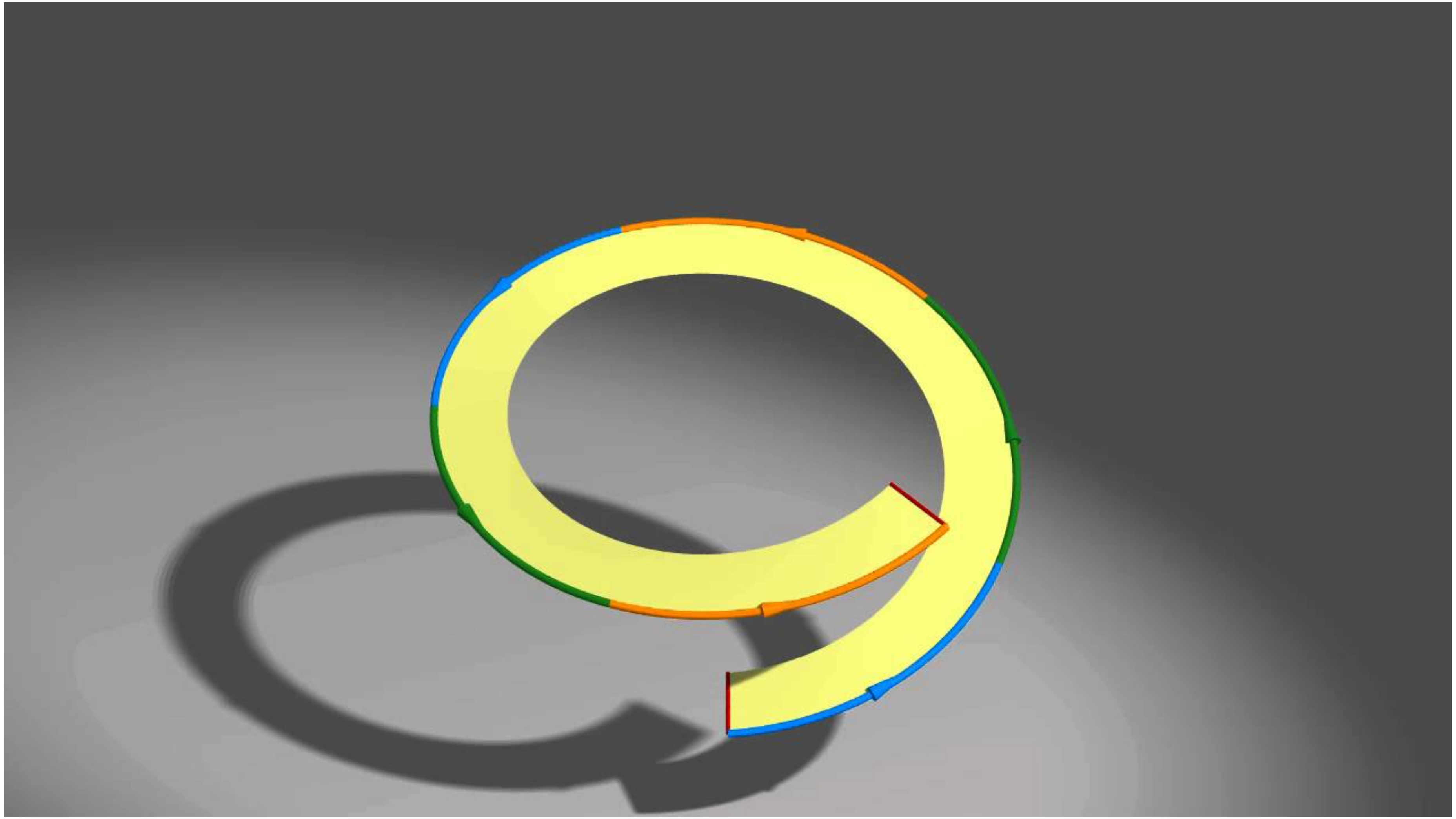
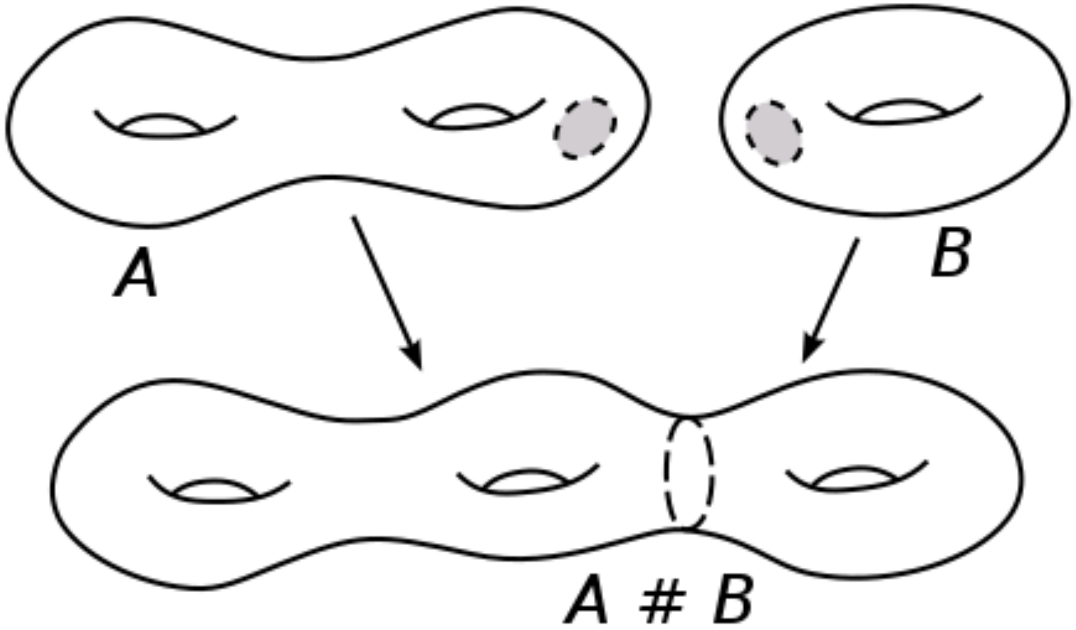


Connected sum operation

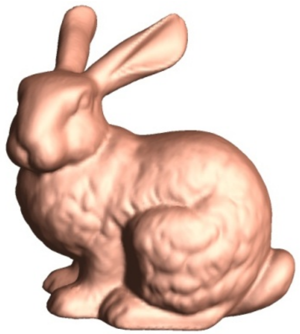
- Given two compact surface M and S, the connected sum $M\#S$ intuitively "merge" the two by cutting off a small disk (cap) from each surface, and the glue the remaining of the two surfaces along the boundary after the cutting.

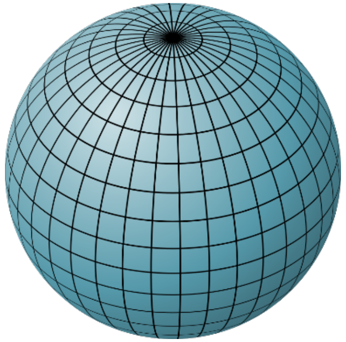


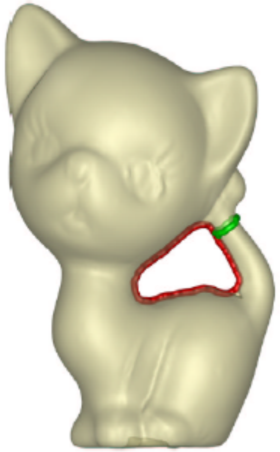
Wikipedia

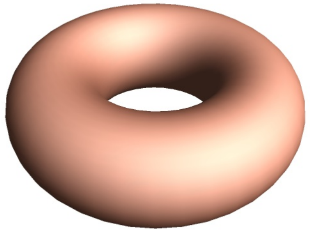
Classification of compact surfaces

Theorem 2 (Classification Theorem) *The two infinite families* $\mathbb{S}$, $\mathbb{T}$, $\mathbb{T}\#\mathbb{T}$, $\ldots$, *and* $\mathbb{P}$, $\mathbb{P}\#\mathbb{P}$, $\ldots$, *exhaust the family of compact 2-manifold without boundary (upto homeomorphism). The first family of surfaces are all orientable; while the second family are all non-orientable. Furthermore, no two surfaces in these sequences are homeomorphic.*

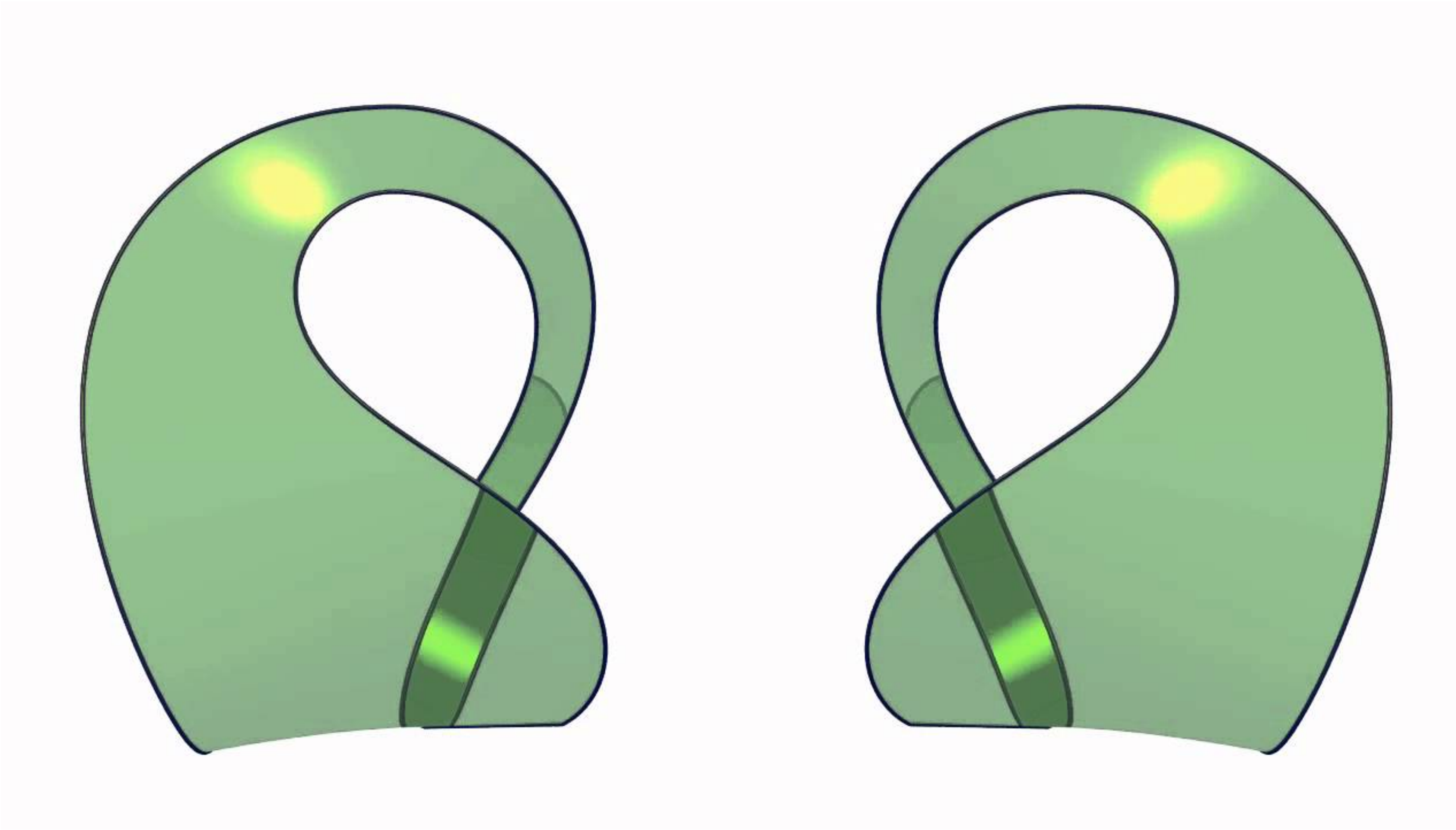








How does $\mathbb{P}\#\mathbb{P}$ become the Klein bottle?



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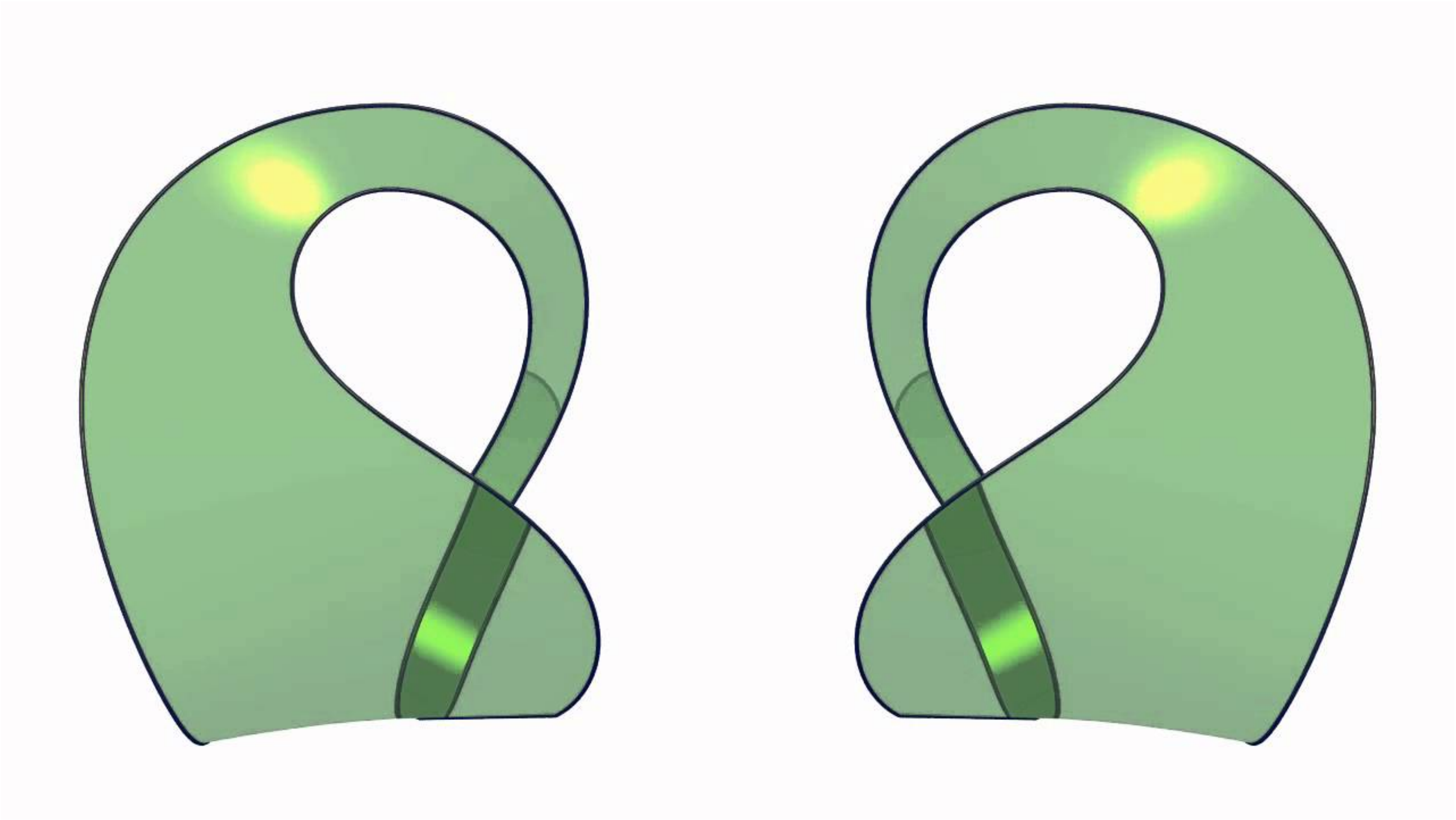

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- Intuitively
  - all orientable surfaces without boundaries can be generated by gluing handles to a sphere
- The number of $\mathbb{T}$ or $\mathbb{P}$ needed is called the genus g of the surface M
  - Sphere has genus 0, torus has genus 1, double-torus has genus 2.
- Hence the genus of a surface completely decides its topology upto homeomorphism
  - Any two compact surfaces with the same genus are homeomorphic

3d and beyond?

- Super complicated
- Way way way beyond the scope of this class
- Fields Medal level of mathematics

FIN