DSC 214
Topological Data Analysis
Topic 2: Simplicial Complexes

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Overview

- Simplicial complex
  - a specific type of topological space commonly used in practice to model data

- Notations

- Commonly used simplicial complexes from point cloud data (PCD)

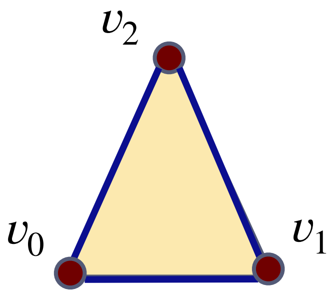
Introduction to Simplicial Complex

A (Geometric) Simplex

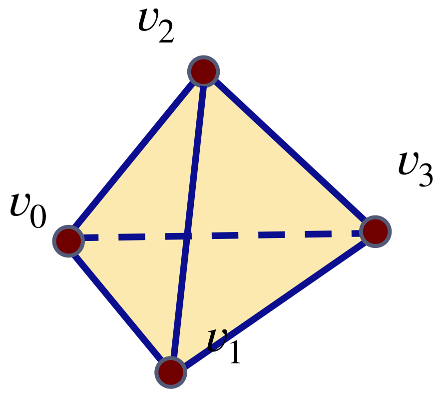
- Points $\{v_0, \ldots, v_p\} \subset \mathbb{R}^N$ are (affinely) independent
  - if vectors $v_i - v_0, i = 1, \ldots, p$ are linearly independent
- Geometric *p*-simplex $\sigma = \{\ v_0, v_1, \ldots,\ v_p\}$ $(\dim(\sigma) = p)$
  - Convex combination of $p + 1$ *(affinely) independent* points in R^N
    - $\sigma = \{\ \sum_{i=0}^{p} a_i v_i \mid a_i \geq 0, \sum a_i = 1\ \}$
- Examples

0-simplex

1-simplex



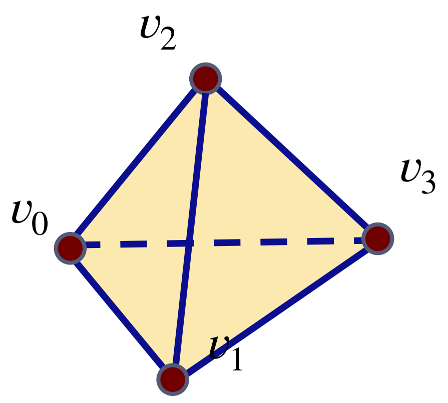
2-simplex



3-simplex

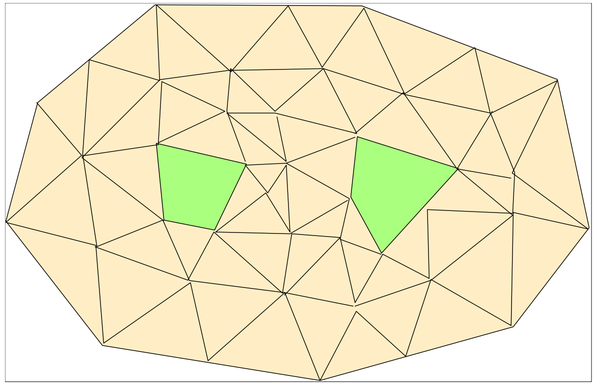
A (Geometric) Simplex

- Points $\{v_0, \ldots, v_p\} \subset \mathbb{R}^N$ are (affinely) independent
  - if vectors $v_i - v_0, i = 1, \ldots, p$ are linearly independent
- Geometric *p-simplex* $\sigma = \{ v_0, v_1, \ldots, v_p\}$
  - Convex combination of $p+1$ *affinely-independent* points in R^N
    - $\sigma = \{ \sum_{i=0}^{p} a_i v_i \mid a_i \geq 0, \sum a_i = 1 \}$
- Simplex τ formed by a subset of $\{ v_0, v_1, \ldots, v_p\}$ is called a face of σ, denoted by $\tau \subseteq \sigma$
  - τ is a proper face of σ if $\dim(\tau) = \dim(\sigma) - 1$
  - $\partial\sigma =$ collection of **all** proper faces of σ
- For a d-simplex σ
  - $\sigma \cong \mathbb{B}^d,\ \partial\sigma \cong \mathbb{S}^{d-1},\ \sigma^o \cong \mathbb{B}^d_0 \cong \mathbb{R}^d$



Simplicial complex

- A (geometric) simplicial complex K
  - A collection of simplices such that
    - If $\sigma \in K$, then any fact $\tau \subseteq \sigma$ is also in K
    - If $\sigma \cap \sigma' \neq \emptyset$, then $\sigma \cap \sigma'$ is a face of both simplices.
  - $\dim(K)$ = highest dim of any simplex in K



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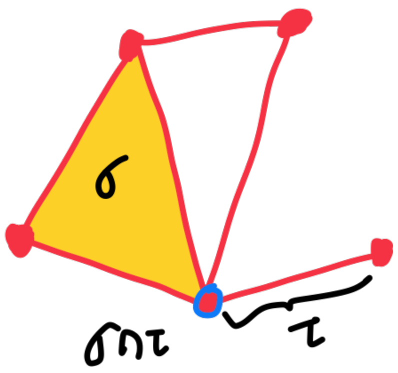

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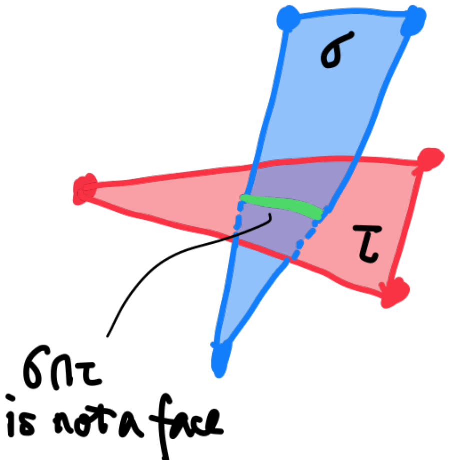
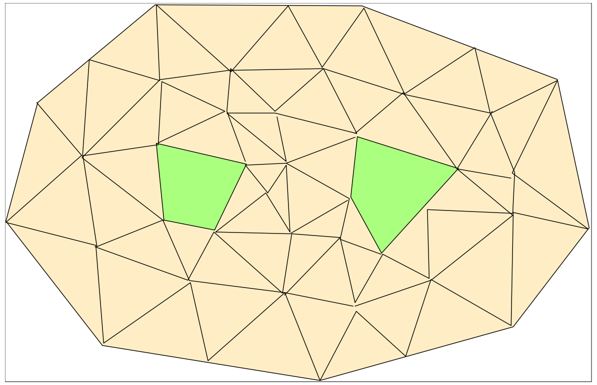


Simplicial complex

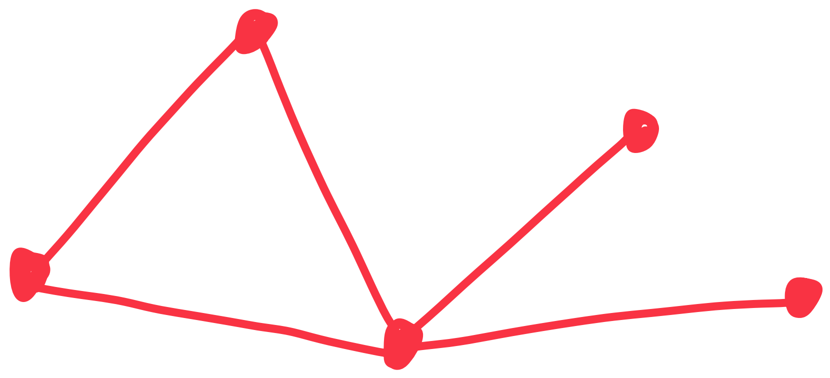


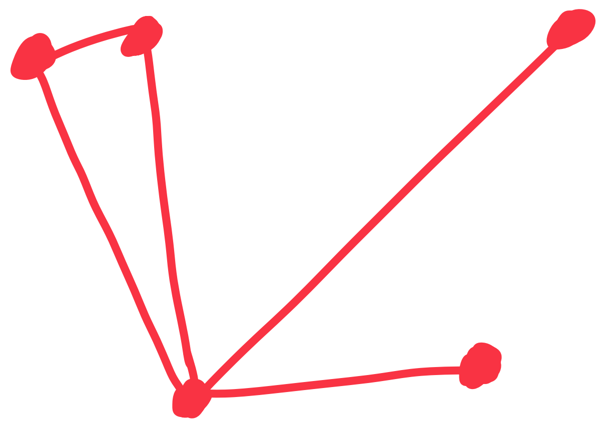
- A geometric simplicial complex K
  - A **collection** of simplices such that
    - If $\sigma \in K$, then any fact $\tau \subseteq \sigma$ is also in K
    - If $\sigma \cap \sigma' \neq \emptyset$, then $\sigma \cap \sigma'$ is a face of both simplices.
  - $\dim(K)$ = highest dim of any simplex in K

- Subcomplex: $L \subseteq K$ and L is a complex
- The p-**skeleton** of K consists of all simplices in K of dimension at most p

- Underlying space $|K|$ of K
  - is the union of all points in all simplices of K,
  - i.e, $|K| = \bigcup_{\sigma \in K} \{x \mid x \in \sigma\}$

- Geometric simplicial completes are nice for intuition / having a mental picture. But we are interested in topology





- Distinct geometrically but the same topologically (i.e., they are homeomorphic)
- A graph can be abstractly defined as $G = (V, E)$

Abstract simplicial complex

- An (abstract) p-simplex $\sigma = \{ v_0, v_1, \ldots, v_p \}$
  - a set of cardinality $p + 1$
  - A subset $\tau \subseteq \sigma$ is a face of σ
- An (abstract) simplicial complex $K = (V, \Sigma)$
  - A vertex set V
  - A collection Σ of simplices such that

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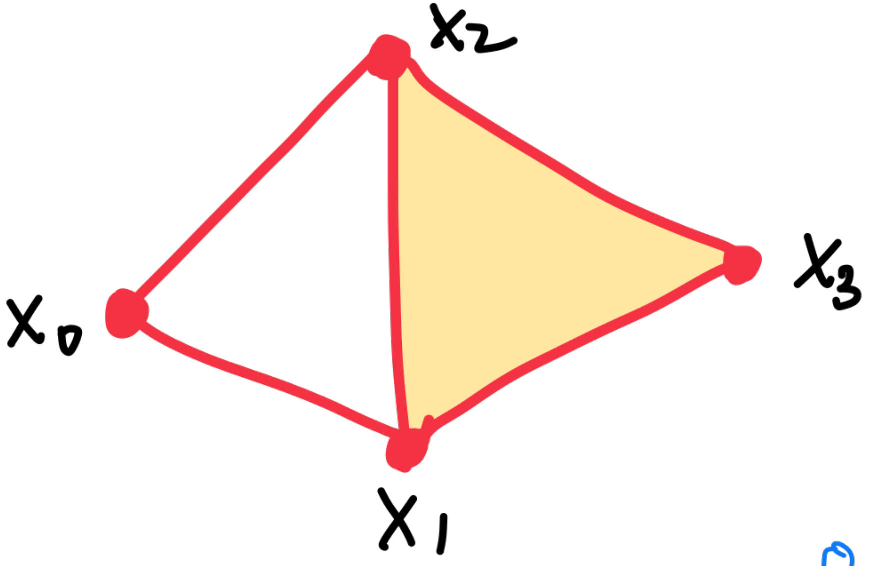


$$V = \{x_0, x_1, \dots, x_3\}$$

$$\Sigma = \left\{ \begin{array}{l} \{x_0\}, \{x_1\}, \{x_2\}, \{x_3\} \\ \{x_0, x_1\}, \{x_0, x_2\}, \{x_1, x_2\}, \\ \{x_1, x_3\}, \{x_2, x_3\} \\ \{x_1, x_2, x_3\} \end{array} \right\}$$

0-simplices

1-simplices

2-simplex

Abstract Simplicial Complex

- **Geometric realization** of an abstract simplicial complex K
  - Is a geometric simplicial complex S whose associated abstract simplicial complex $(V(S), \Sigma(S))$ is the "same" as $(V(K), \Sigma(K))$

Theorem
Any abstract simplicial complex K has a geometric realization. Any two geometric realizations of K have homeomorphic underlying spaces

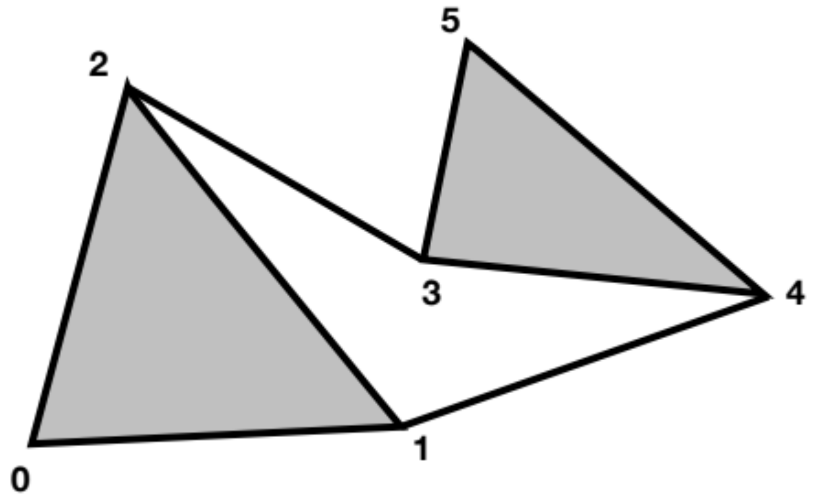
  - We use $|K|$ to denote the underlying space of a geometric realization of K and call $|K|$ the underlying space of K.

Theorem

Any abstract simplicial complex K has a geometric realization. Any two geometric realizations of K are homeomorphic to each other.

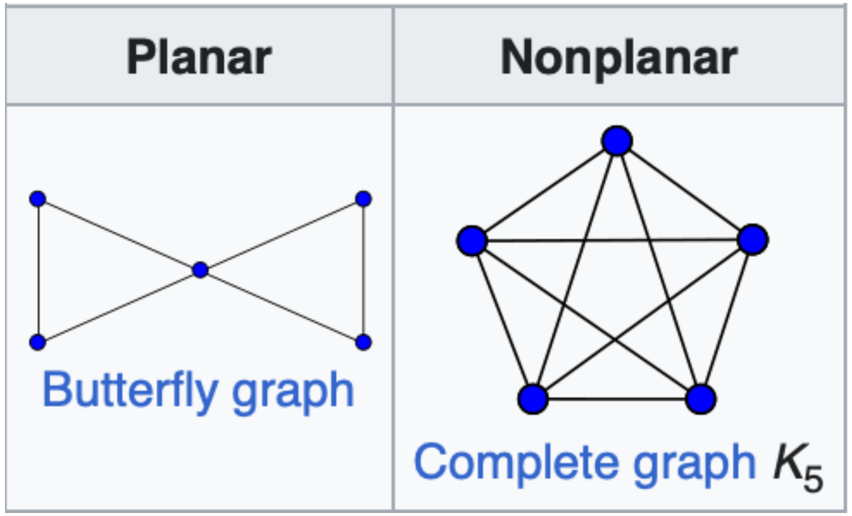
- If $V = \{v_0, \ldots, v_n\}$, embed V into $\mathbb{R}^{n+1}$ by $v_i \mapsto (0,\cdots,1\cdots,0) = e_i$
- For each simplex $\sigma = \{v_{i_0}, \ldots, v_{i_k}\}$, add geometric simplex $cnx\{e_{i_0}, \ldots, e_{i_k}\}$ to the realization

- The recipe in the proof is not efficient in terms of ambient dimension



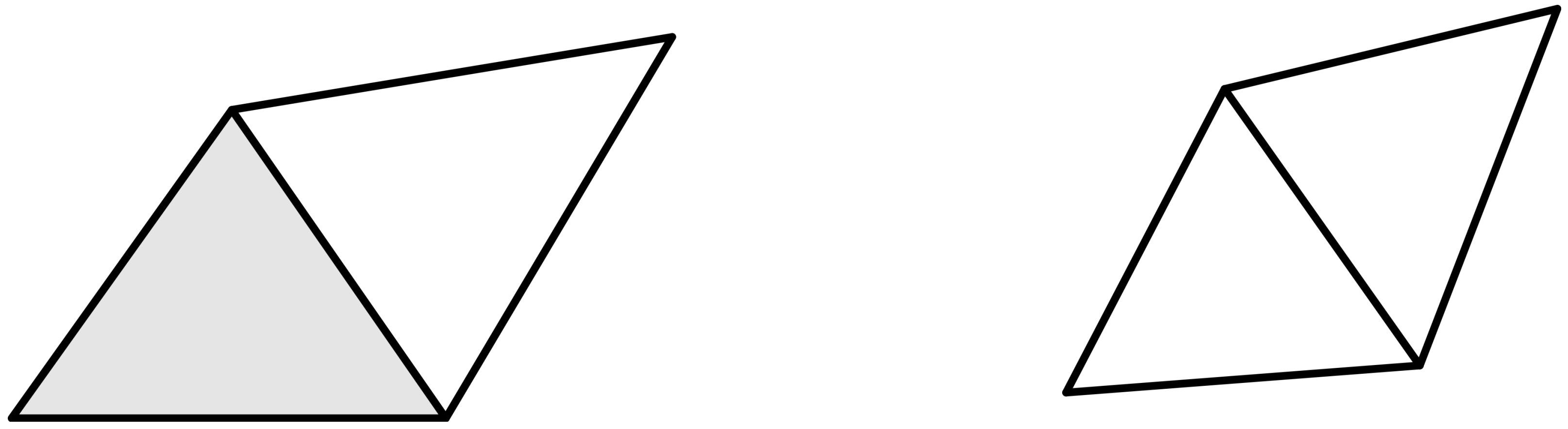
- Any finite d-dim abstract simplicial complex has a geometric realization in $\mathbb{R}^{2d+1}$

- The recipe in the proof is not efficient in terms of ambient dimension
- Any finite d-dim abstract simplicial complex has a geometric realization in $\mathbb{R}^{2d+1}$ but may not have a geometric realization in $\mathbb{R}^{2d}$
    - A graph (1-d simplicial complex) can be plotted in $\mathbb{R}^3$ but not necessarily in $\mathbb{R}^2$



Graphs and Simplicial Complexes

- Any simple graph (without double edge and self-loop) is a simplicial complex
- The 1-skeleton of a simplicial complex is a graph



Graphs and Simplicial Complexes

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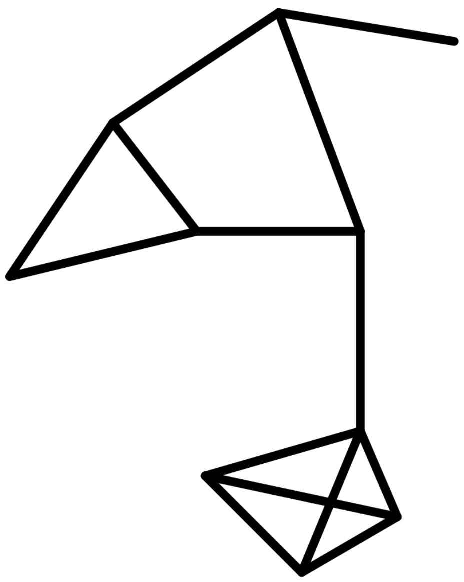
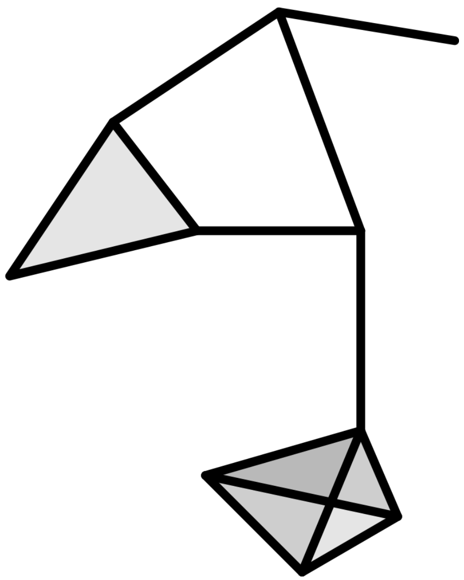
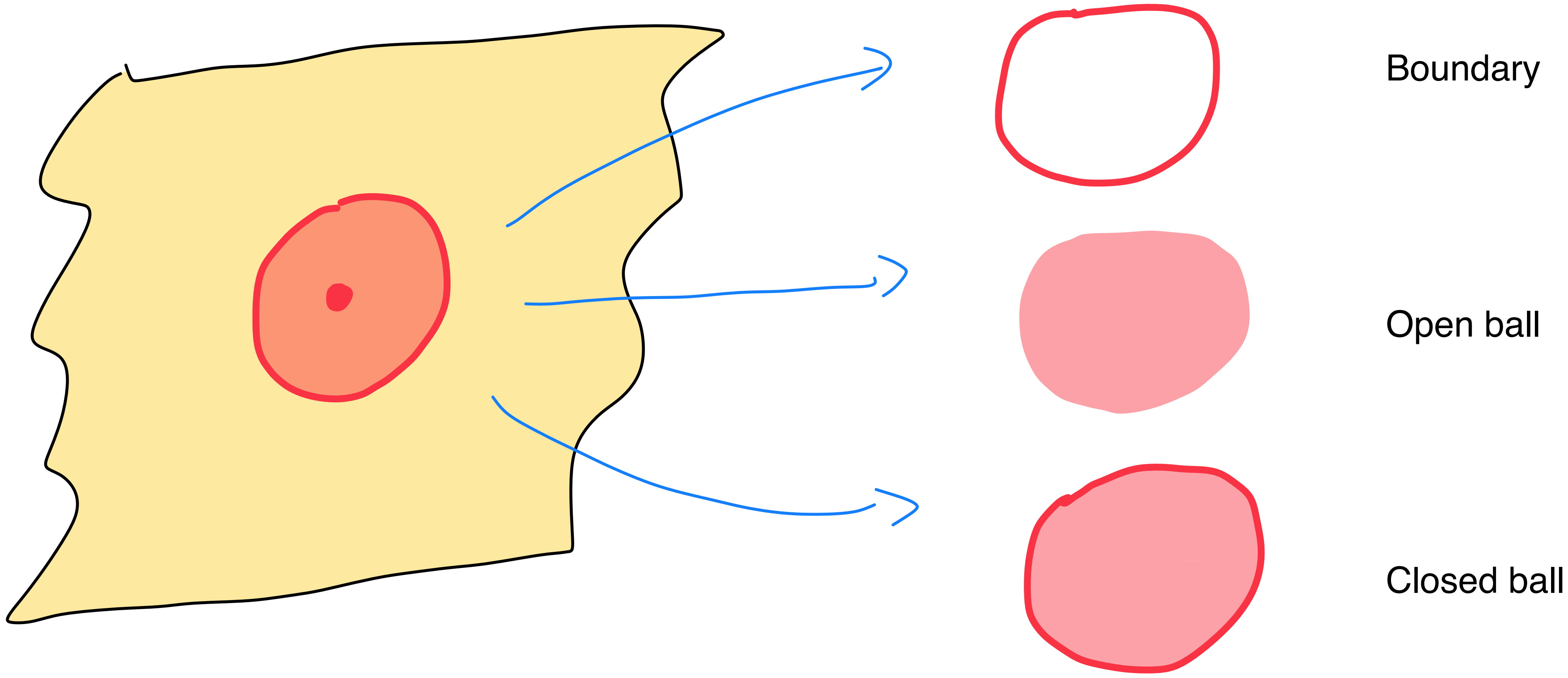
- The 1-skeleton of a simplicial complex is a graph
- **Clique complex** induced by a graph





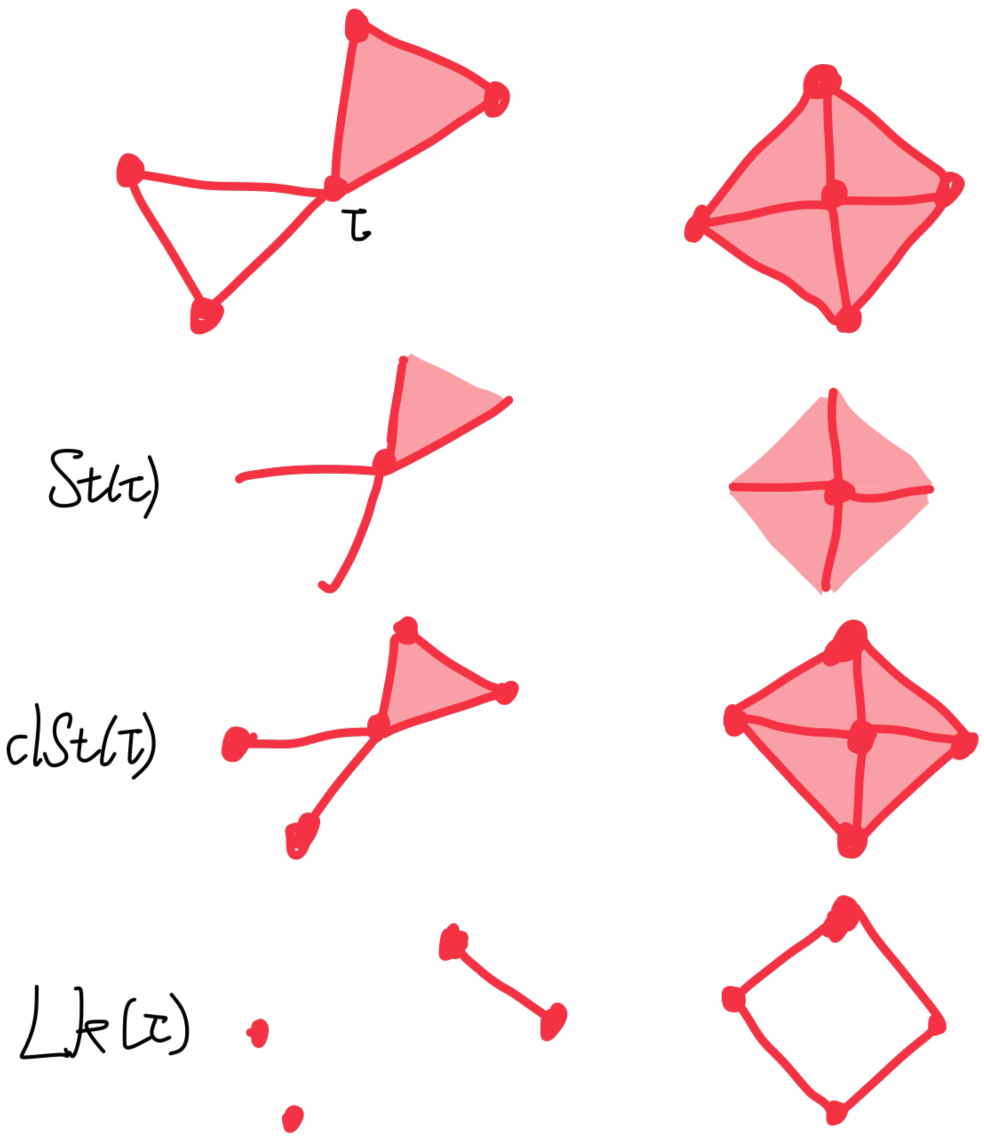
Some notions related to simplicial complexes

Star and links



Star and links

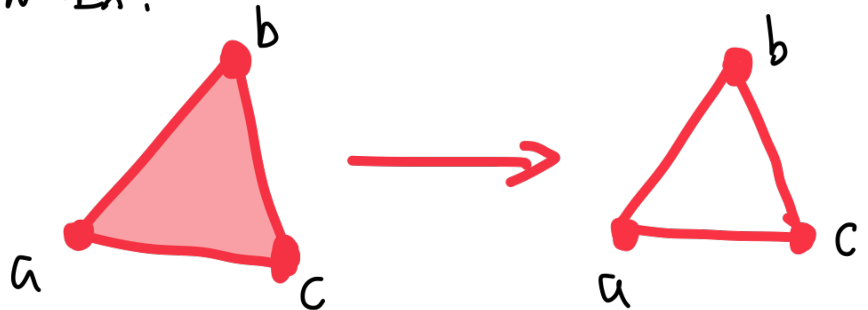
- Given a simplex $\tau \in K$
  - Star: $St(\tau) = \{\sigma \in K \mid \tau \subset \sigma\}$
    - A star may not be a simplicial complex
  - Closed star: $cl\,St(\tau) = \displaystyle\bigcup_{\sigma \in St(\tau)} \{\, \sigma' \mid \sigma' \subset \sigma \,\}$
  - Link: $Lk(\tau) = \{\, \sigma \in cl\,St(\tau) \mid \sigma \cap \tau = \varnothing \,\}$



Simplicial map

- Intuitively, analogous to continuous maps between topological spaces
- Given simplicial complexes K and L
  - a function $f : V(K) \to V(L)$ is called a simplicial map if
    - for any $\sigma = \{p_0, \ldots, p_d\} \in \Sigma(K)$, $f(\sigma) = \left\{ f\left(p_0\right), \ \ldots, \ f\left(p_d\right) \right\}$ spans a simplex in L, i.e., $f(\sigma) \in \Sigma(L)$.
  - A simplicial map is also denoted $f : K \to L$

NON-EX:



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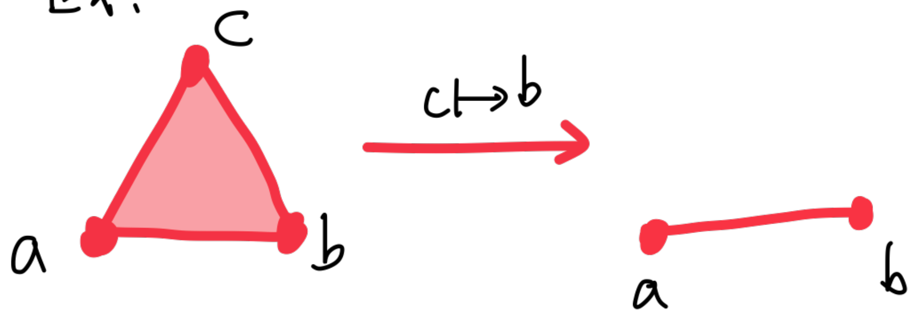

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  - A simplicial map is also denoted $f : K \to L$
- A simplicial map $f: K \to L$ is an **isomorphism**
  - if f is bijective between vertex sets and f^{-1} is a simplicial map

Simplicial map

- A simplicial map $f : K \to L$ induces a natural continuous function $f' : |K| \to |L|$
  - s.t $f'(x) = \sum_{i \in [0,d]} a_i f(p_i)$ for $x = \sum_{i \in [0,d]} a_i p_i \in \sigma = \{p_0, \ldots, p_d\}$

Simplicial map

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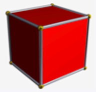
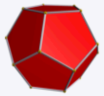
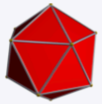
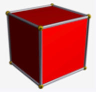
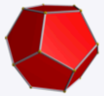
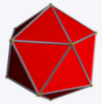
- Theorem:
  - An isomorphism $f: K \to L$ induces a **homeomorphism** $f': |K| \to |L|$

A topological invariant – Euler Characteristic

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V - E + F$
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

A topological invariant – Euler Characteristic

- For the surface of a polyhedron, the Euler Characteristic is defined as $\chi = V - E + F$.
- Euler's polyhedron formula:
  - $\chi = 2$ for surface of convex polyhedron

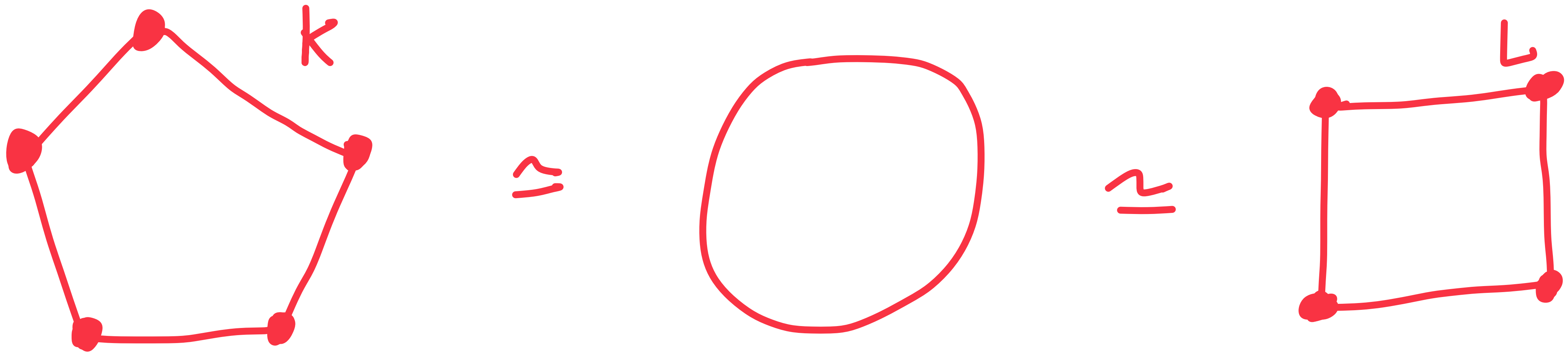
Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V - E + F$
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

A topological invariant – Euler Characteristic

A topological invariant – Euler Characteristic

- Given a d-dim simplicial complex K with n_i number of i-simplices
- the *Euler characteristic* of K is defined as:
  - $\chi(K) := \sum_{i=0} (-1)^i n_i$
- Euler characteristic is both a topological invariant and a homotopy invariant, meaning that it does not change under homeomorphism or homotopy equivalence.

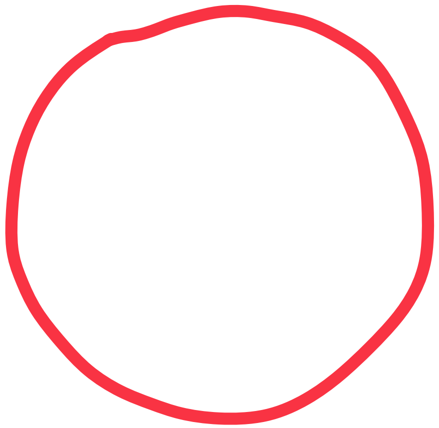
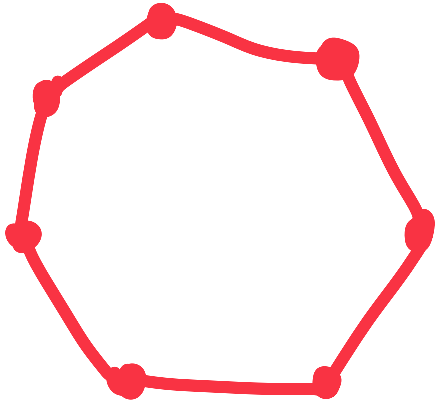
A topological invariant – Euler Characteristics

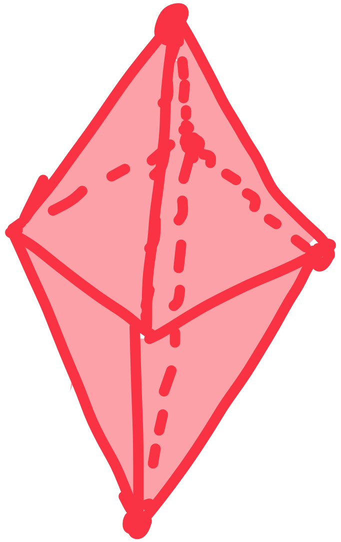


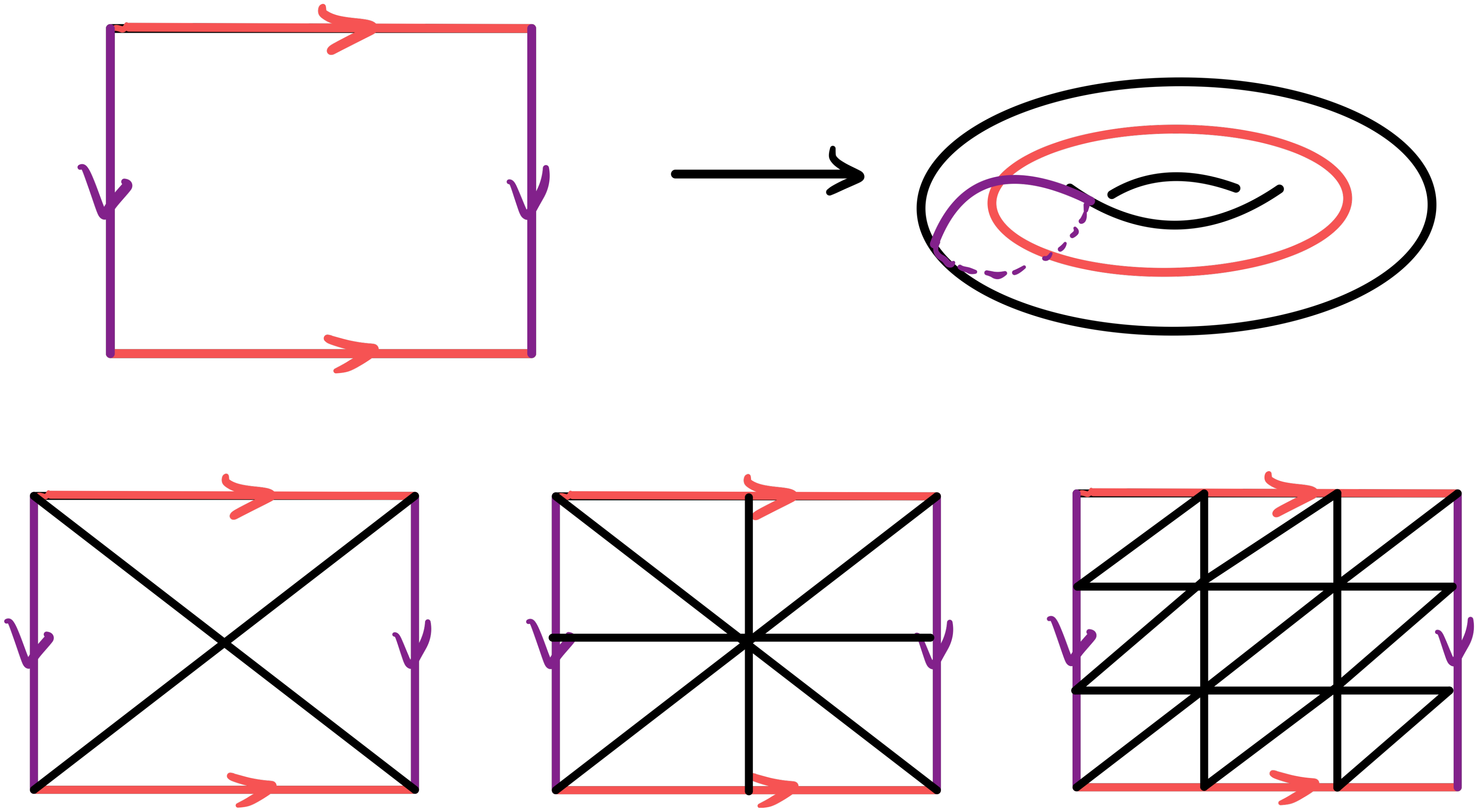
$\chi(K) = 5 - 5 = 0$ $\qquad$ $\chi(\mathbb{S}^1) = 0?$ $\qquad$ $\chi(L) = 4 - 4 = 0$

Triangulation of a manifold

- Given a manifold (with or without boundary) M, a simplicial complex K is a triangulation of M
  - if the underlying space $|K|$ of K is homeomorphic to M

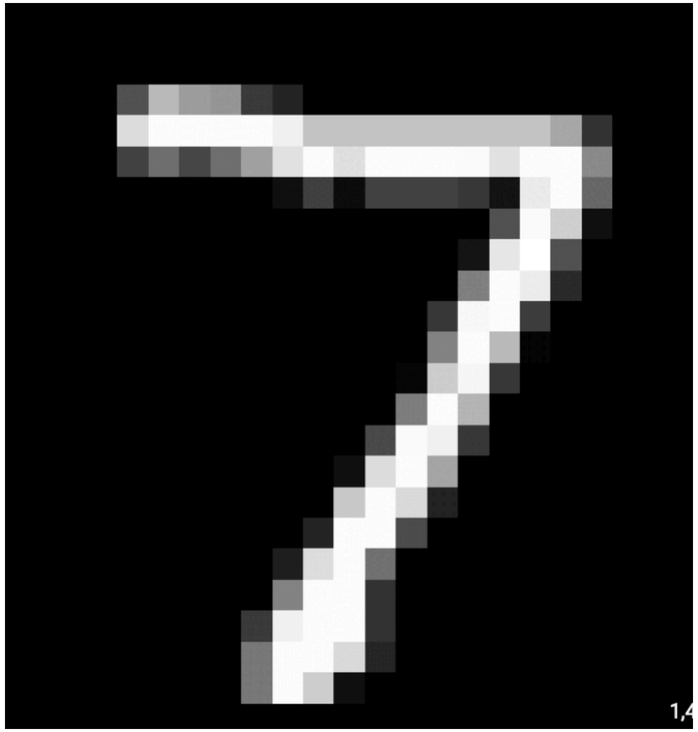





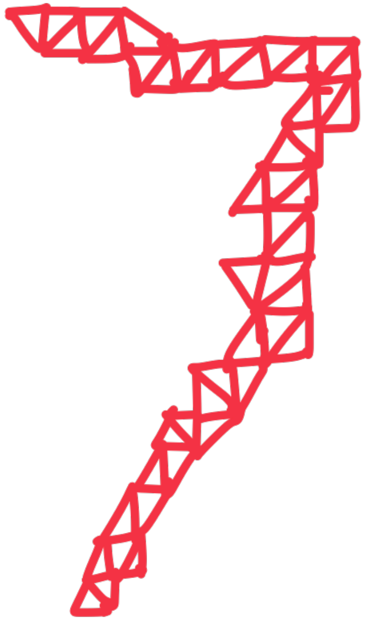


Other complexes

Image Data



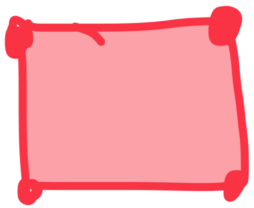
triangulation?

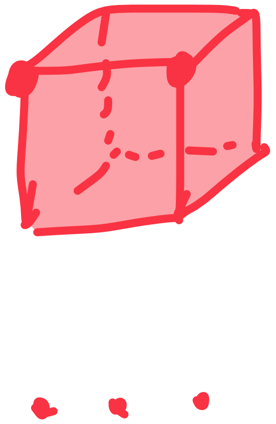


Cubical Complex

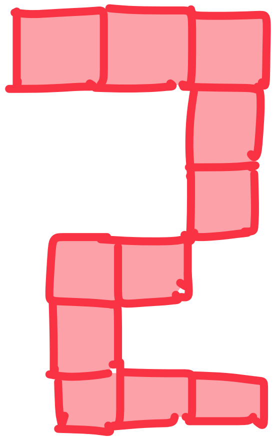
0-cube 

1-cube 

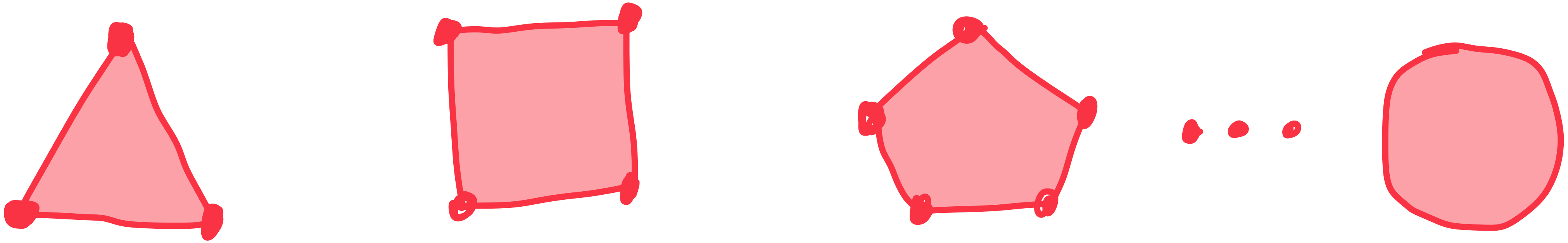
2-cube 

3-cube 

2-dim cubical complex



CW Complex



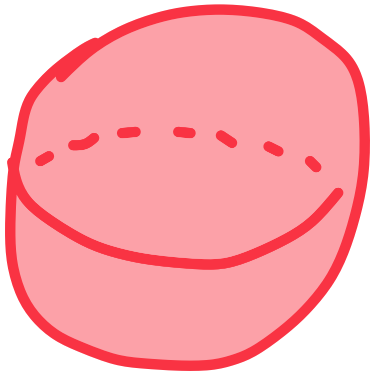
Triangle Rectangle Pentagon Disk

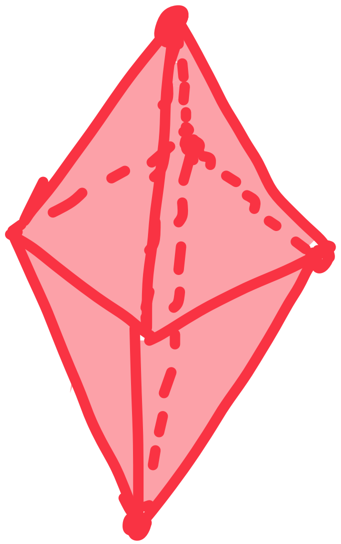
0-cell 1-cell 2-cell 3-cell K-cell

CW Complex

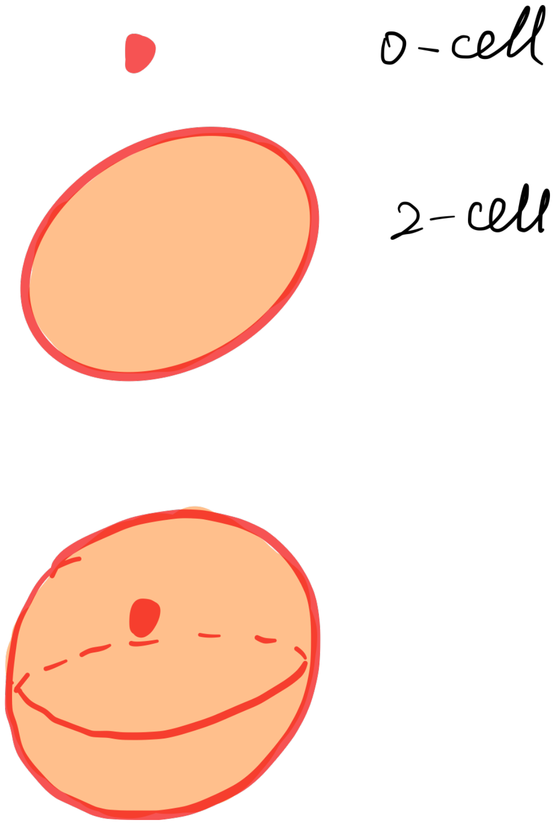
- A CW complex X is the union of a sequence of topological spaces
  - $\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \cdots$
  - Such that X_k is obtained from X_{k-1} by "gluing" k-cells $\{e_\alpha^k\}_\alpha$, each homeomorphic to $\mathbb{D}^k$, by continuous maps $\partial e_\alpha^k \to X_{k-1}$
  - Each X_k is called the k-skeleton of X

CW Complex



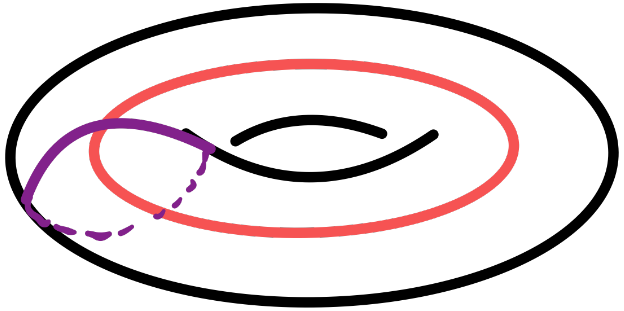


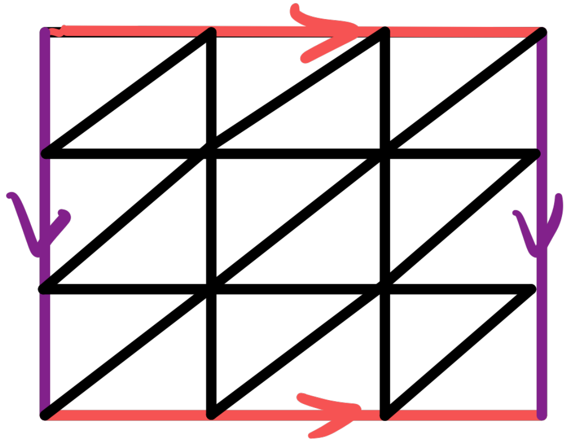
Triangulation of a sphere



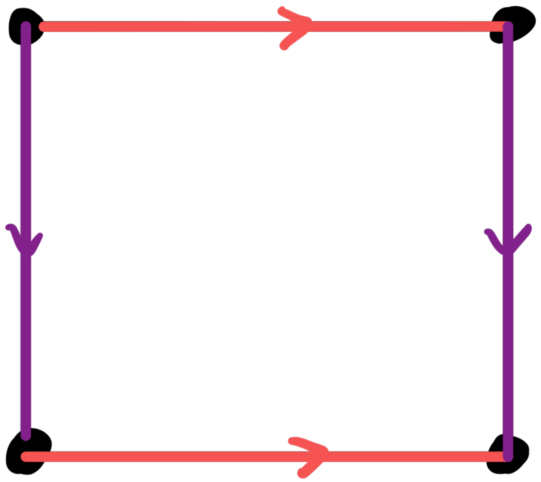
CW structure of a sphere

CW Complex





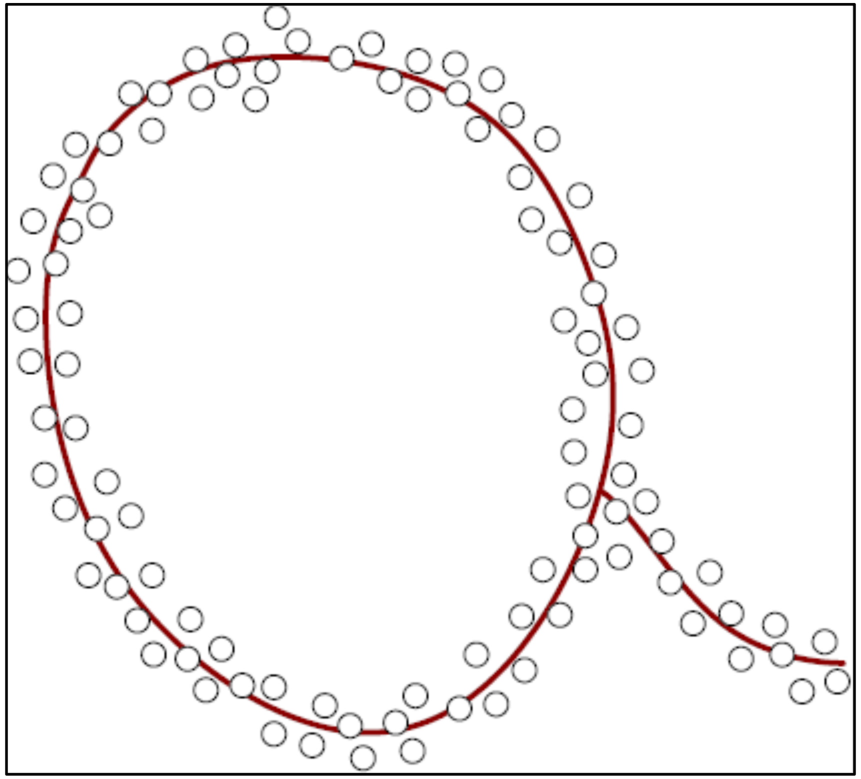
Triangulation of a torus



CW structure of a torus

Common Complexes

Goal: create simplicial complexes from dataset in order to use topological tools



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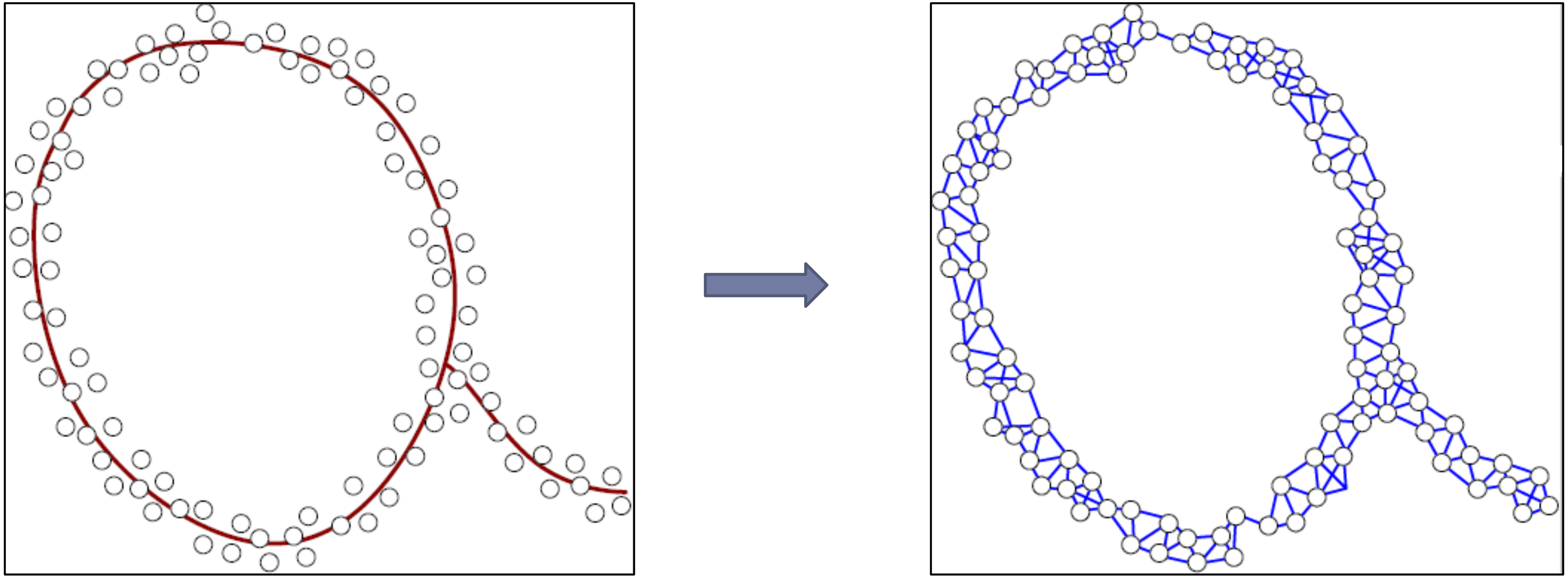
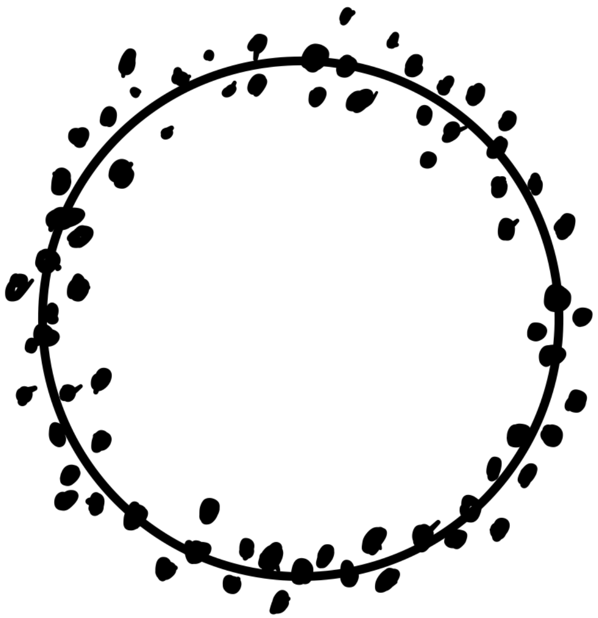
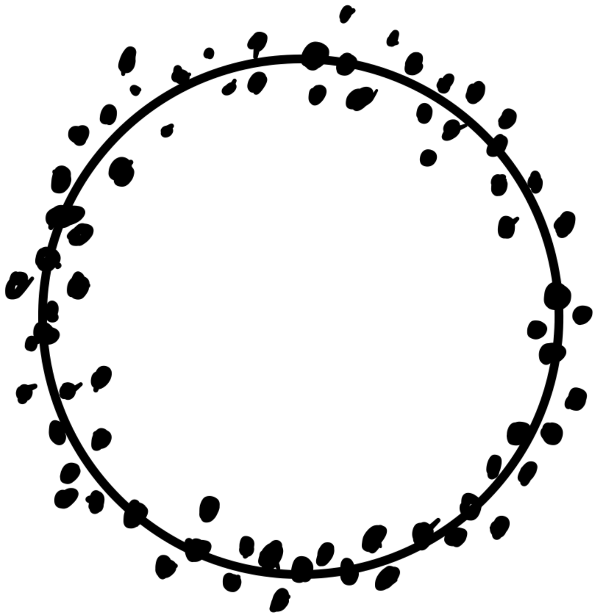


Thickening “recovers” the shape



Thickening “recovers” the shape



- The topology of the data set is trivial since we only have finitely many points

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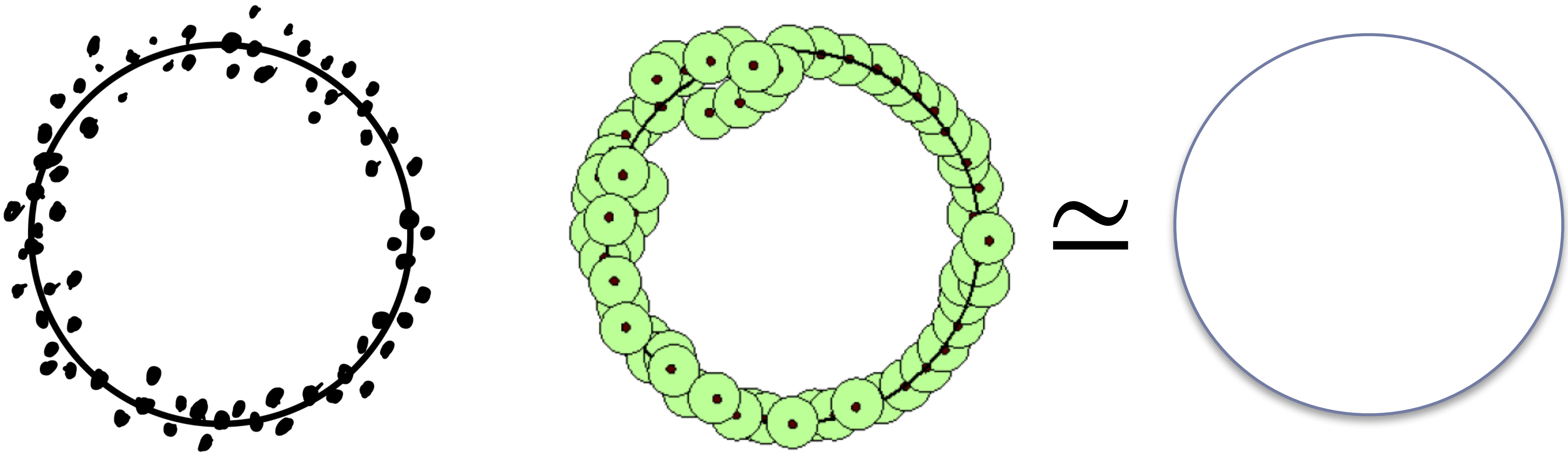

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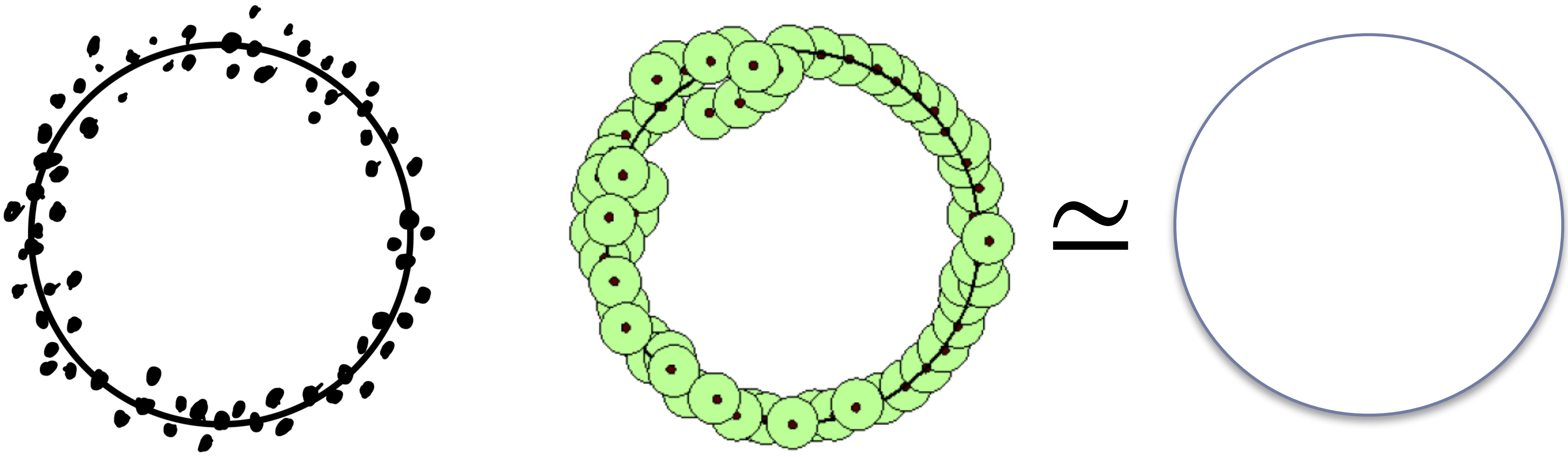
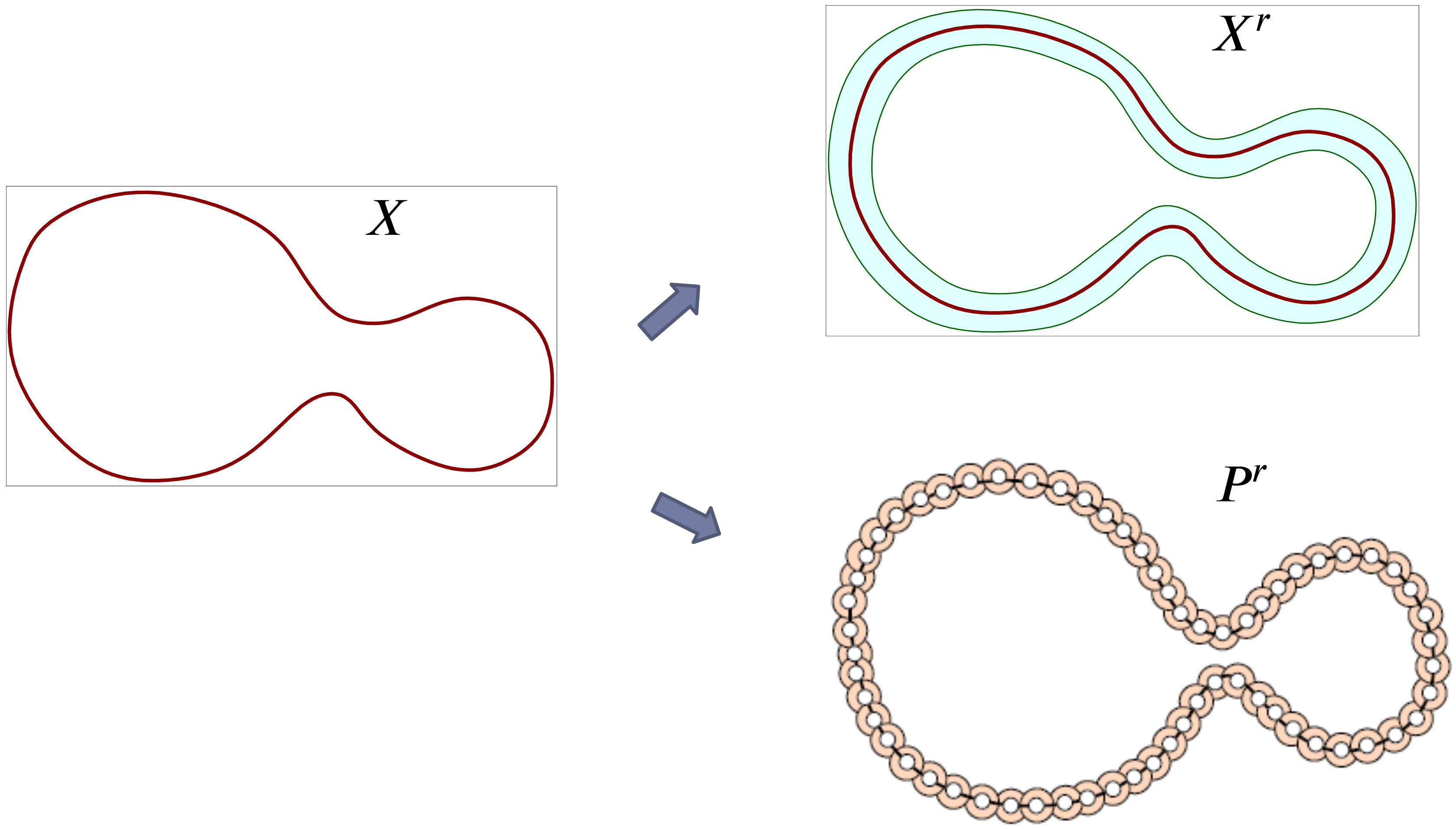
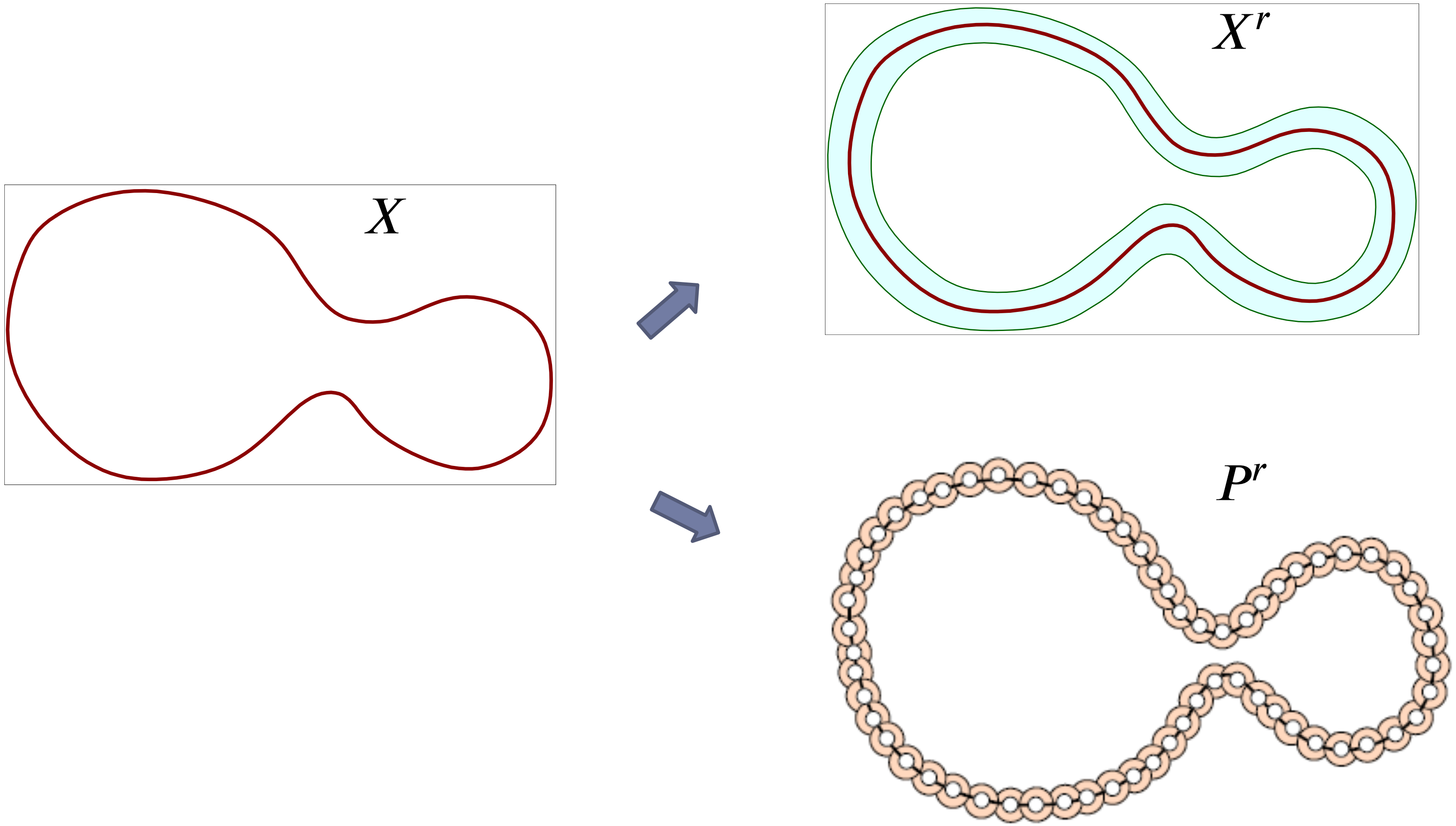


- The topology of the data set is trivial since we only have finitely many points
- Thickening can be used to recover the shape or the “homotopy type” of the underlying ground truth

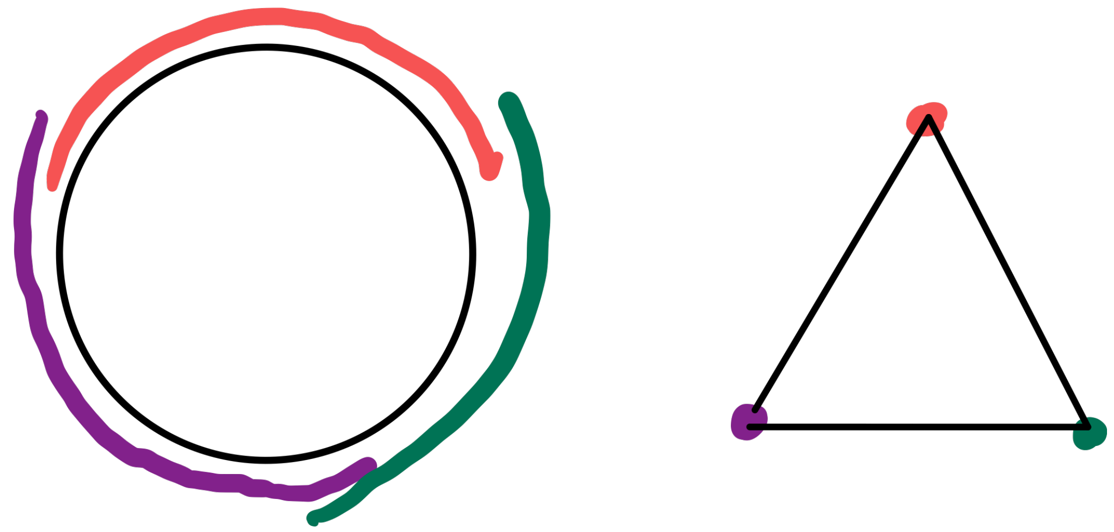

X
X^r
P^r



- Unions of balls around data points can recover the topological information of the underlying space
- How to create a simplicial complex from this "cover"?

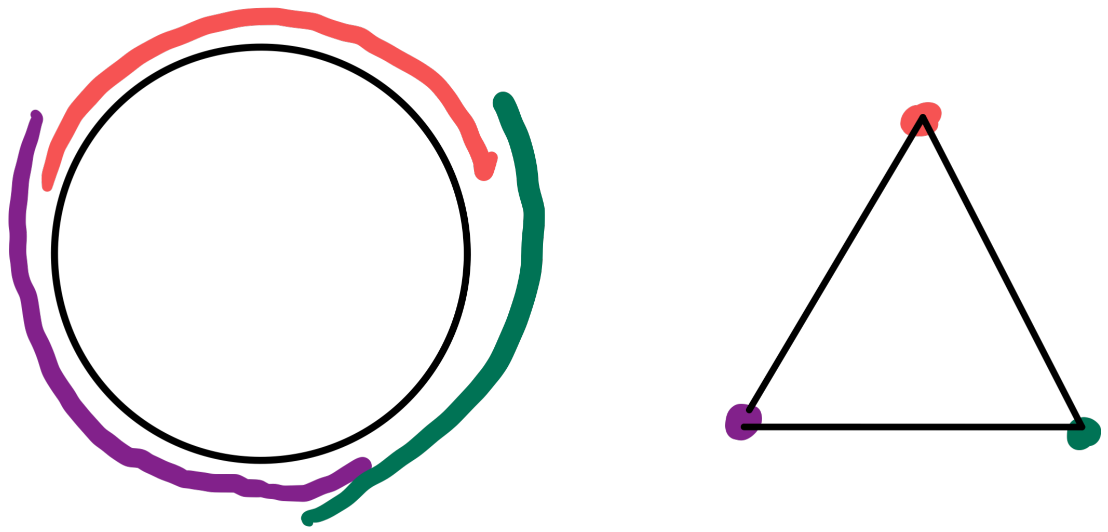
Nerve complex

- Given a finite collection of sets $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$, its nerve complex $Nrv(\mathscr{U})$ is a simplicial complex

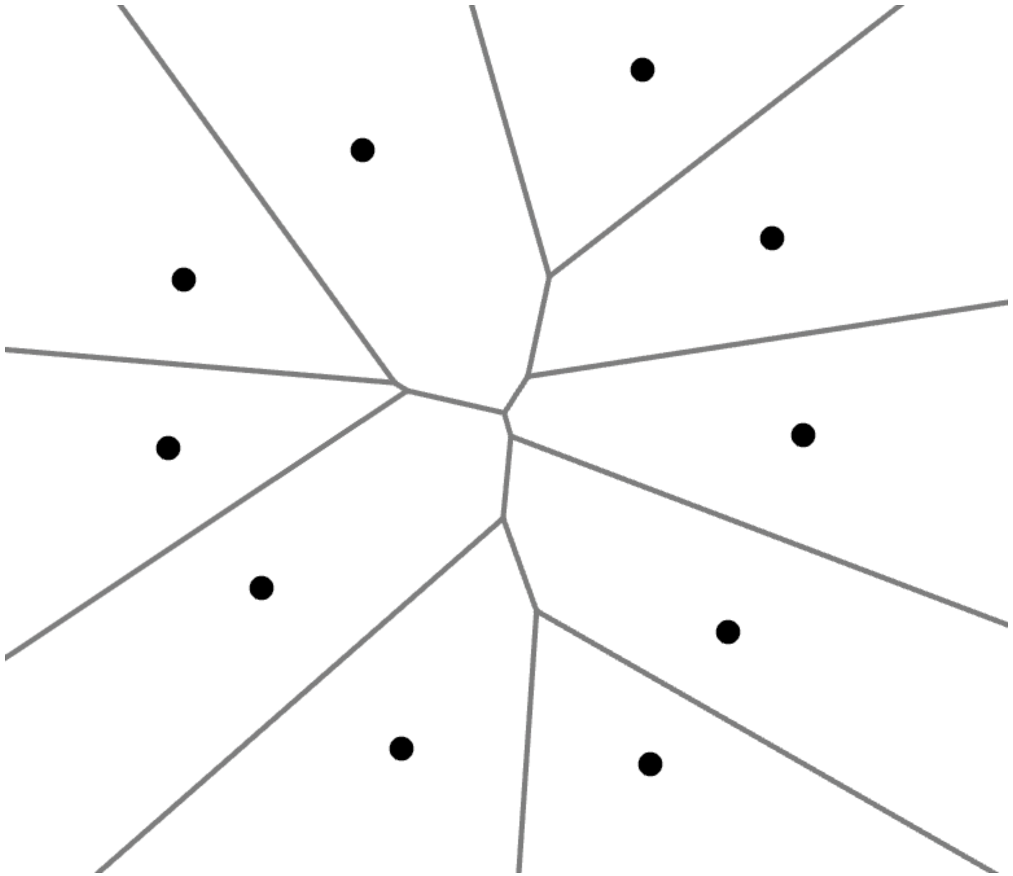


Nerve complex

- Given a finite collection of sets $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$, its nerve complex $Nrv(\mathscr{U})$ is a simplicial complex
  - The vertex set $V = A$
  - $\{\alpha_0, \ldots, \alpha_k\} \in \Sigma$ iff $\cap_{i=0}^{k} U_{\alpha_i} \neq \emptyset$



Example



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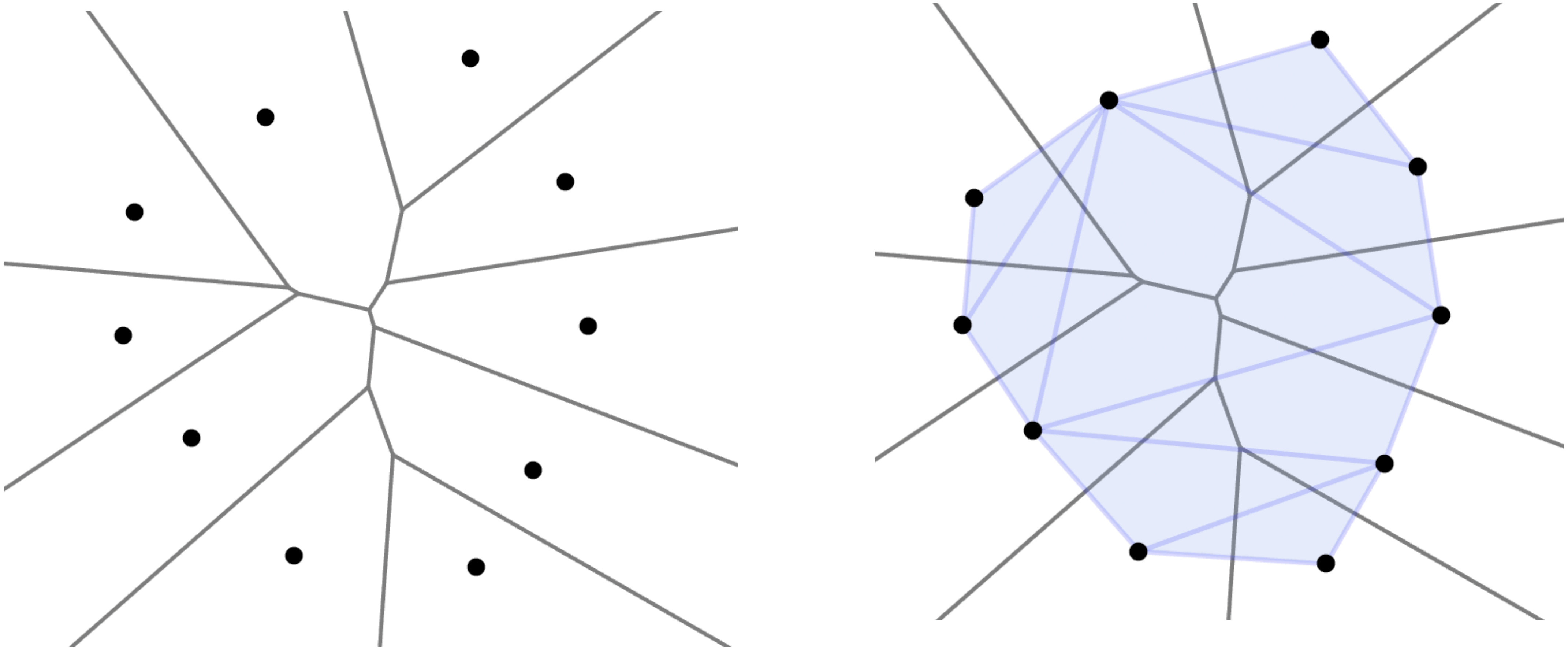

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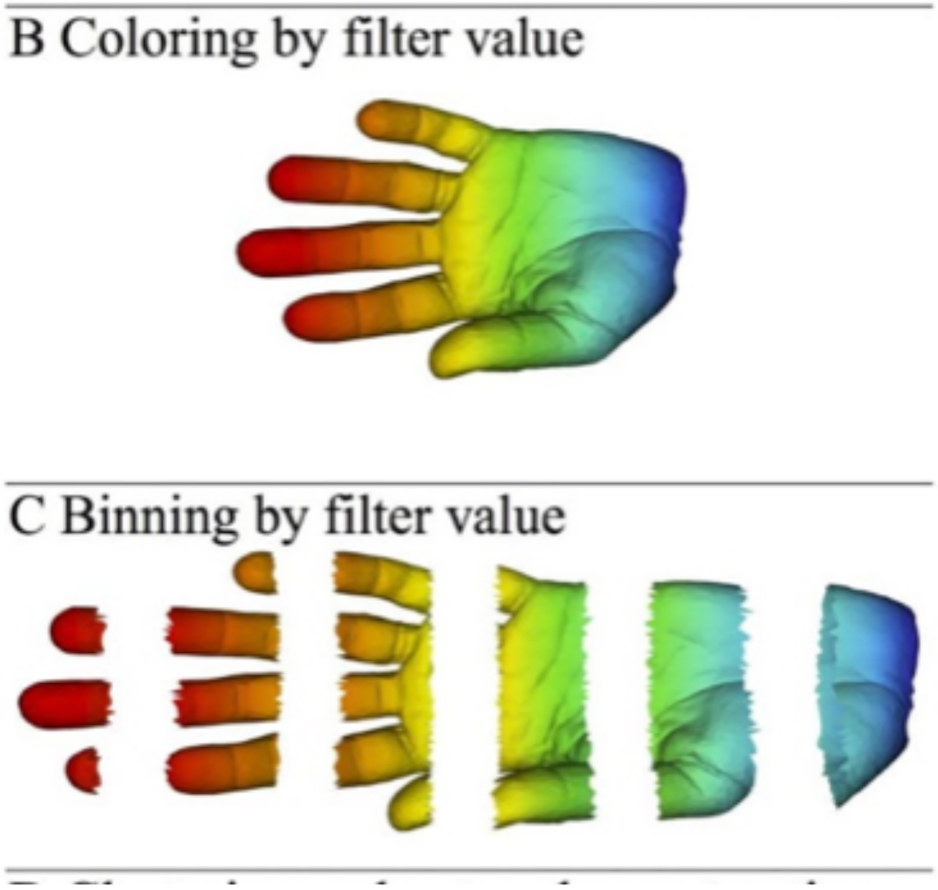
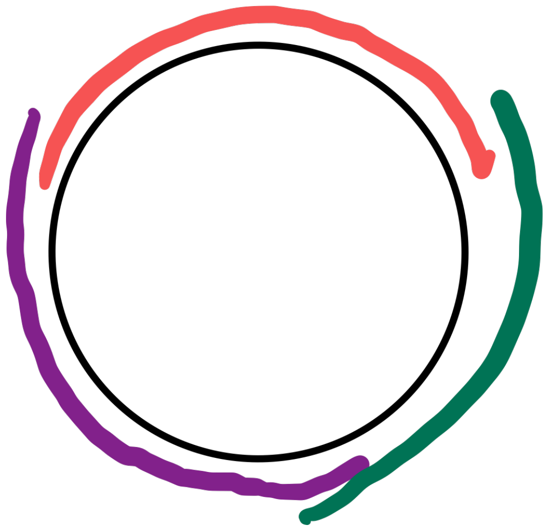
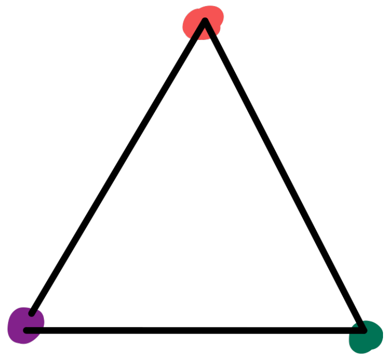
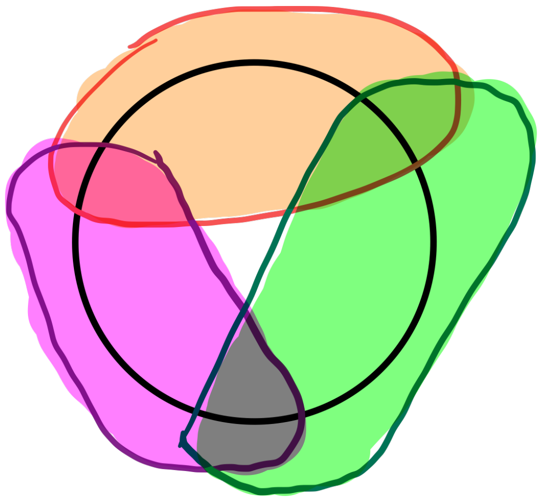
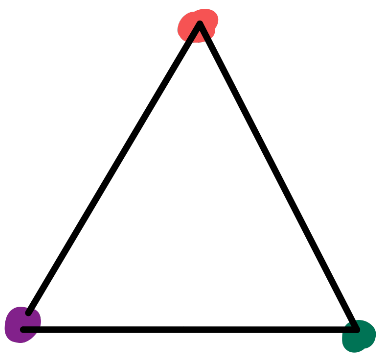


Nerve complex recovers homotopy type of the cover

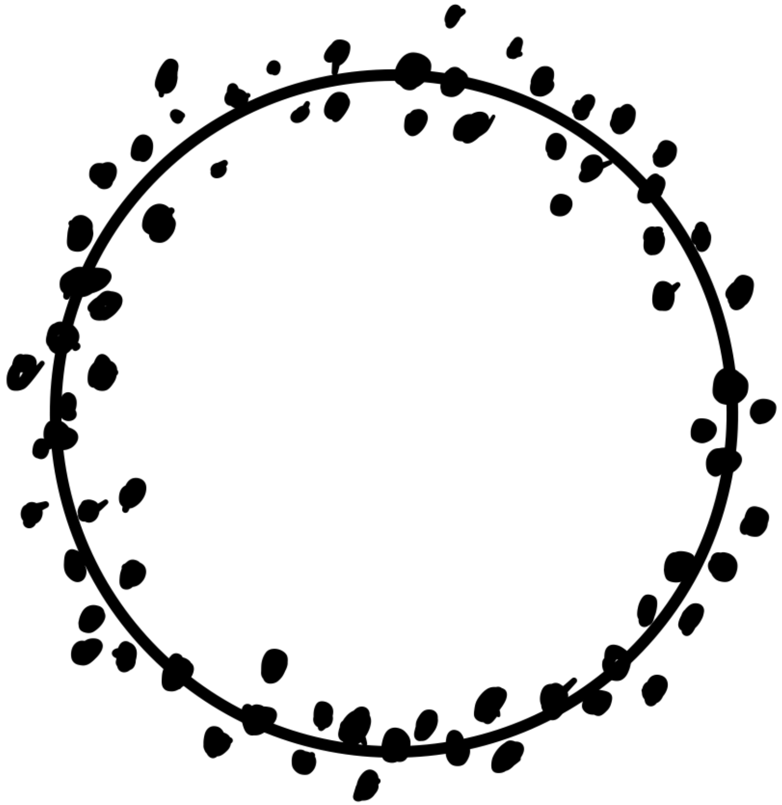
- Nerve Lemma (intrinsic):
  - Let $\mathcal{U}$ be an **open** cover of a metric space X such that $\cap_{i=1}^{k} U_{\alpha_i}$ is contractible for any finite elements in $\mathcal{U}$.
  - Then $|Nrv(\mathcal{U})| \simeq X$.

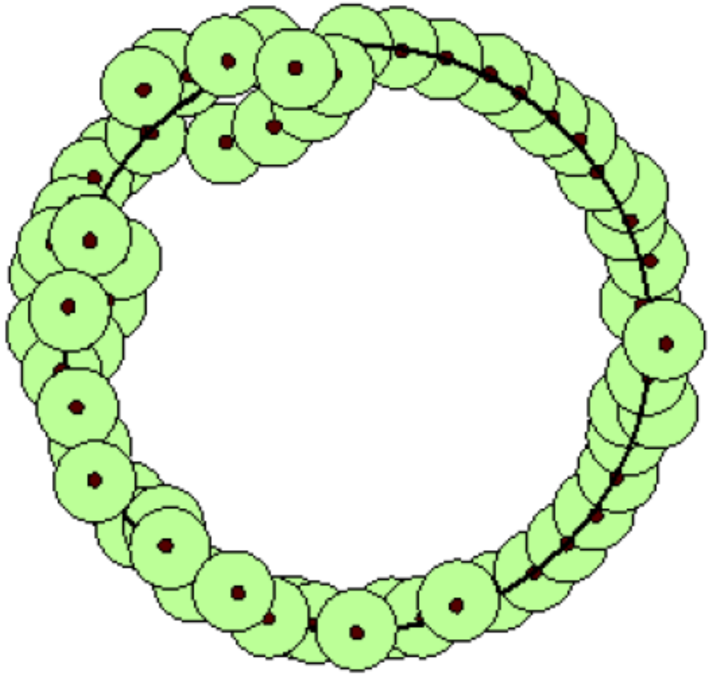
- Nerve Lemma (an alternative version on Euclidean space):
  - Let $\mathcal{U}$ be a finite collection of **closed**, **convex** subsets in R^d. Then $|Nrv(\mathcal{U})| \simeq \cup_{\alpha \in A} U_\alpha \subset \mathbb{R}^d$.

Čech complex

- Create a collection of balls with given radius
- Build the nerve of the given collection





Čech Complex

Čech Complex

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- Given a real value $r > 0$, the *Čech complex* $C^r(P)$ is the nerve of the set of closed balls $\{B(p_i, r)\}_{i=1,\ldots,n}$, where $B(p, r) = \{x \in \mathbb{R}^d \,|\, d(p, x) \leq r\}$

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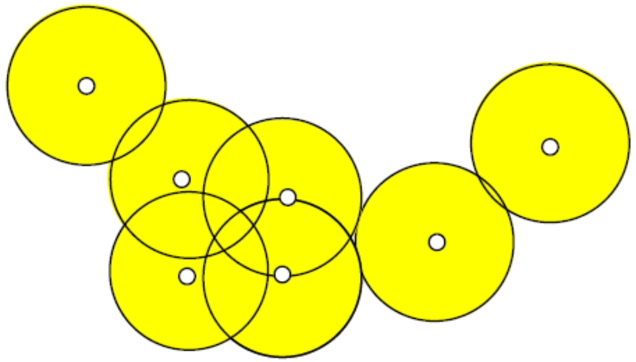

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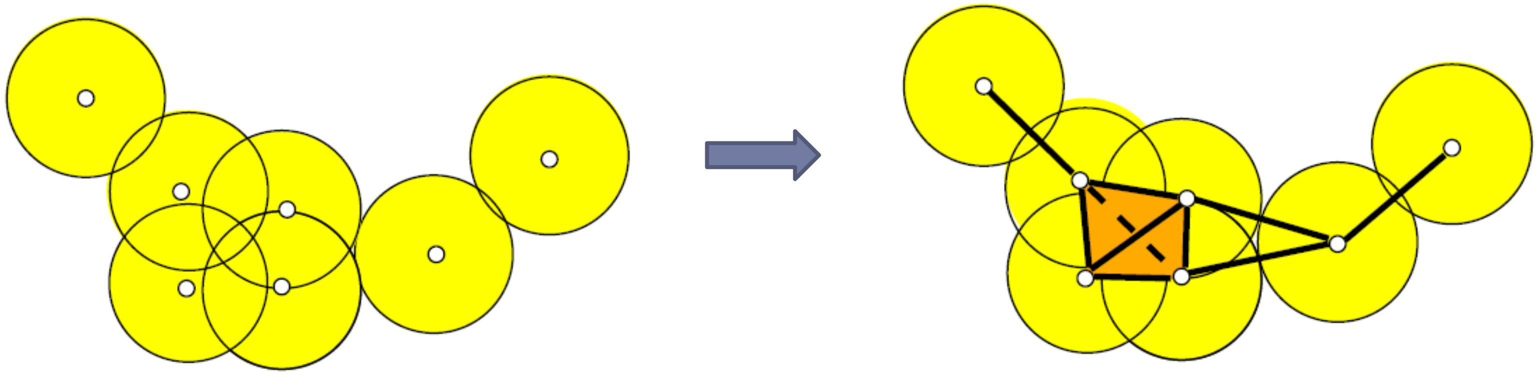

Nerve Lemma

- Nerve Lemma (a simplified version):
  - Let $\mathscr{U}$ be a finite collection of closed, convex subsets in R^d. Then $|Nrv(\mathscr{U})| \simeq \cup_{\alpha \in A} U_\alpha \subset \mathbb{R}^d$.

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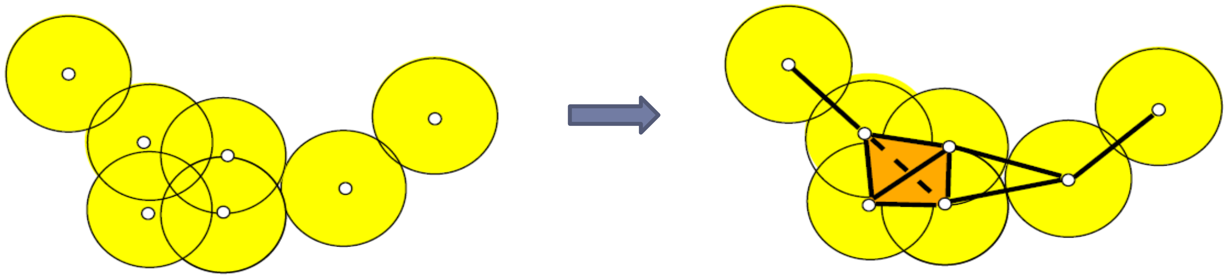

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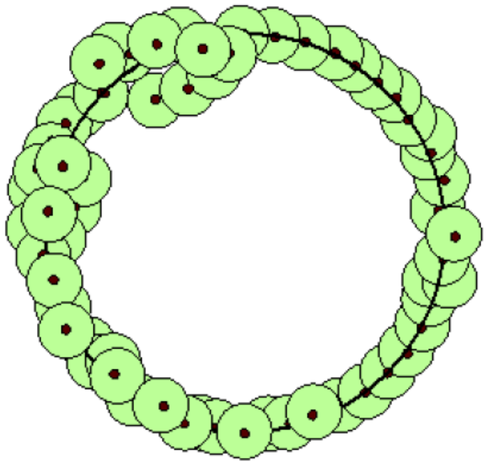
- Given a set of points P
  - approximating a hidden domain M
  - $U^r(P) = \bigcup_{p \in P} B(p, r)$ approximates M
  - $C^r(P)$ approximates $U^r(P)$



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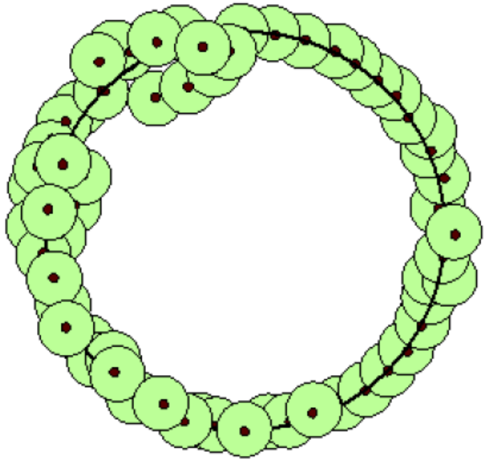


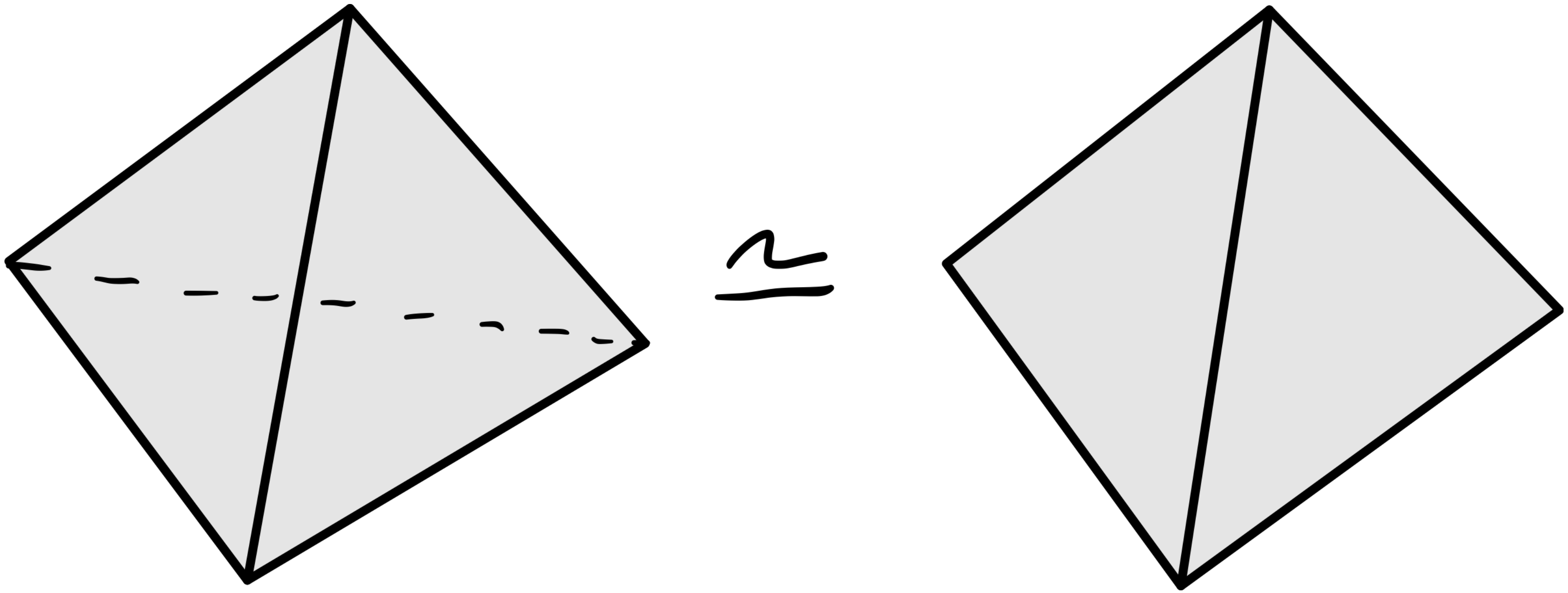
More on Čech

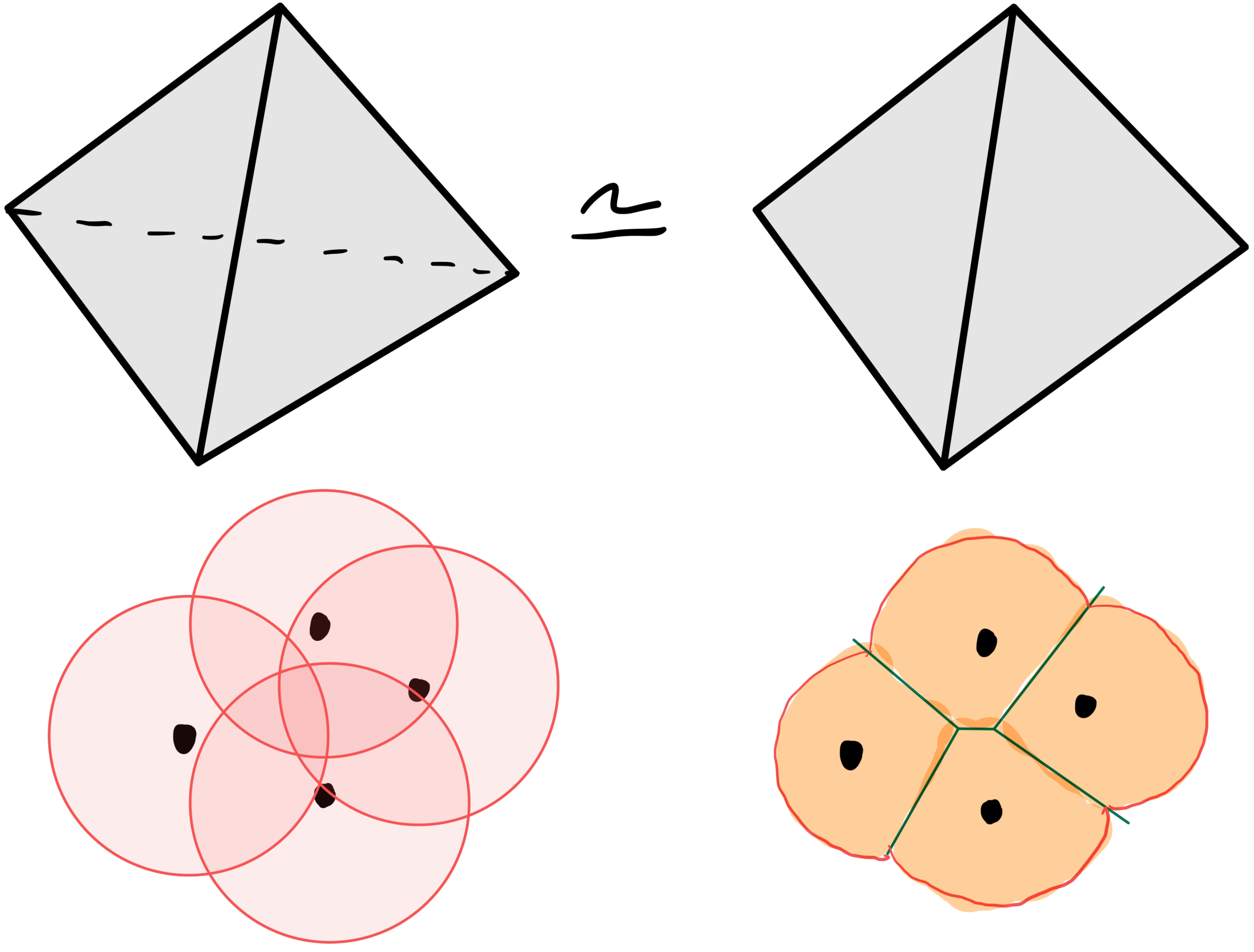
See Demo by Henry Adams

- Given a set of points $P \subset R^d$
  - $C^r(P)$ could have simplex of dimension larger than d
    - In particular, $C^\infty(P)$ is the same as n-simplex.
  - often only d-skeleton of $C^r(P)$ is needed
    - as $U^r(P)$ has trivial topology beyond dimension d
- $C^r(P)$ can be huge!! When r is large enough, there exists $O(2^n)$ many simplices!

Alpha complex

- Some high dimension simplicies are redundant

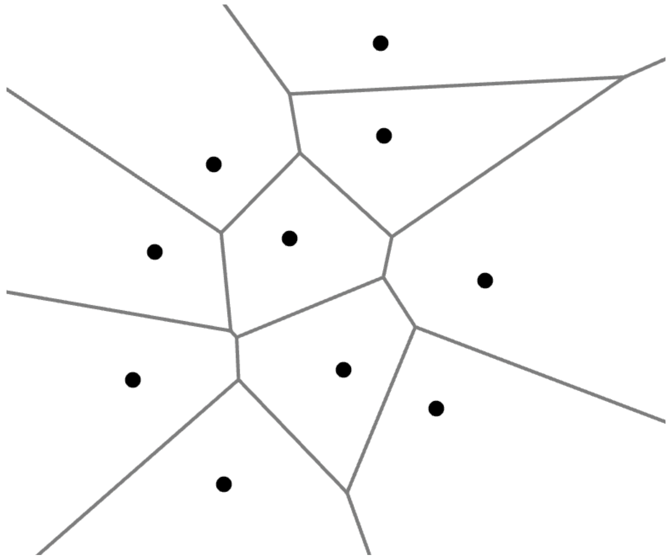




- Change the cover to create a simplified simplicial complex

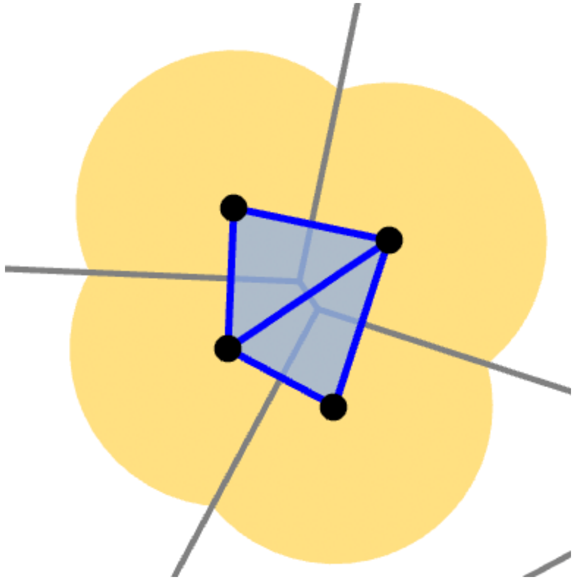
Voronoi Diagram

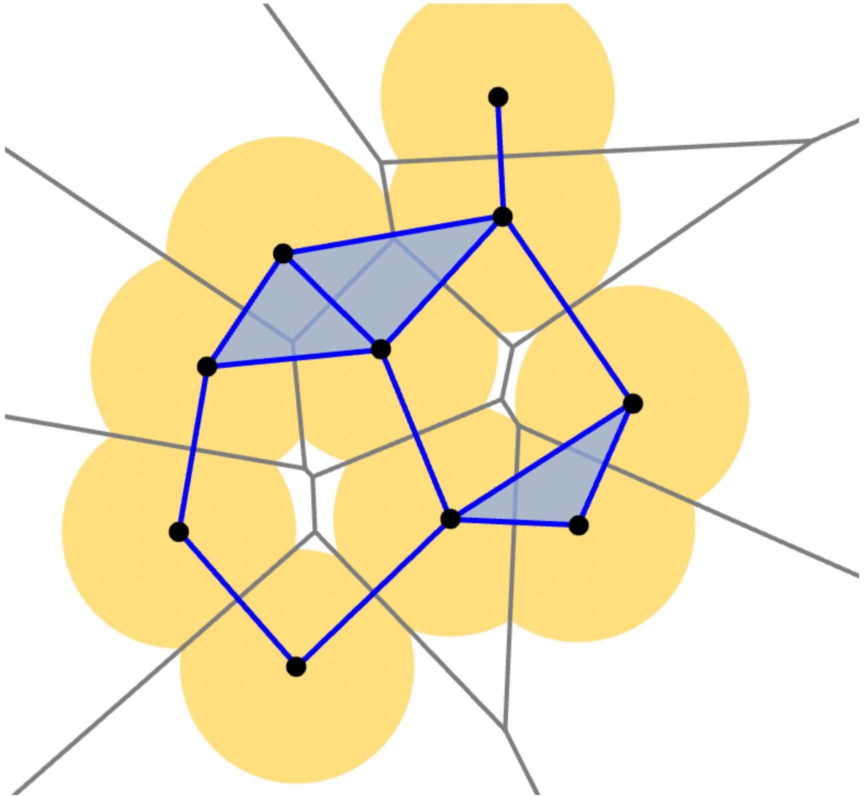
- Given a finite set $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$, the **Voronoi cell** of p_i is
  - $Vor(p_i) = \{x \in \mathbb{R}^d \mid \|x - p_i\| \leq \|x - p_j\|, \forall j \neq i\}$
- The **Voronoi Diagram** of P is the collection of all Voronoi cells.



Alpha complex

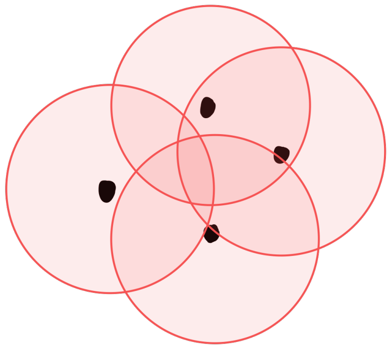
- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$
- Given a real value $r > 0$, the *Alpha complex* $Del^r(P)$ is the nerve of the set $\{B(p_i, r) \cap Vor(p_i)\}_{i=1}^{n}$

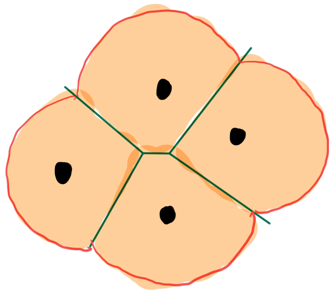




Alpha complex vs Čech complex

- $Del^r(P) \subset C^r(P)$
- $|Del^\infty(P)| = O(n^{\frac{d}{2}})$ whereas $|C^\infty(P)| = O(2^n)$
- $\dim Del^r(P) \leq d$ for generic P





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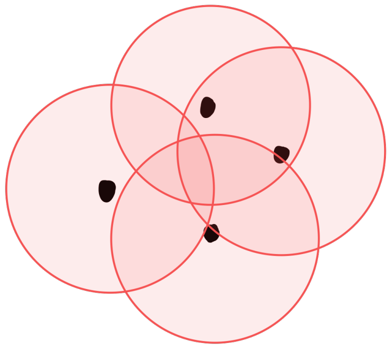
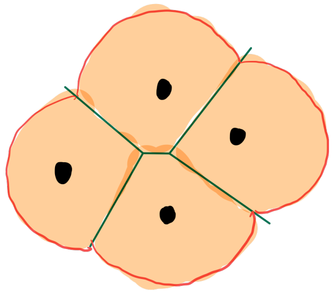
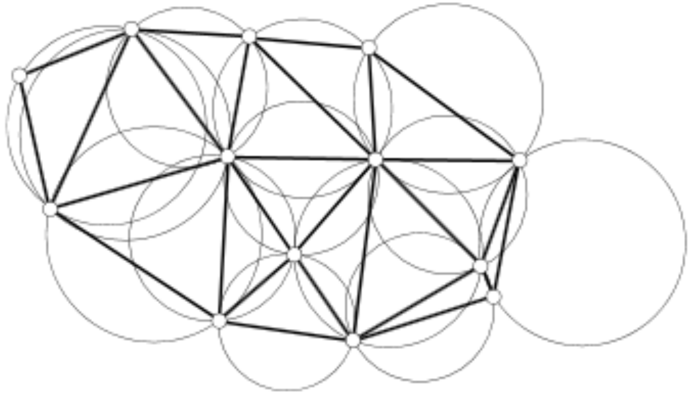




- Proposition:
  - $Del^r(P) \simeq C^r(P) \simeq \cup_p B(p, r)$, i.e, $C^r(P)$ and $Del^r(P)$ are homotopy equivalent.

Delaunay Complex

- $Del^{\infty}(P)$ is called the **Delaunay complex** of P, denote by $Del(P)$
  - $Del(P) = Nrv(\left\{Vor\left(p\right) \mid p \in P\right\})$
- Delaunay complex $Del(P)$
  - A simplex $\sigma = \left[p_{i_0},\ p_{i_1},\ \ldots,\ p_{i_k}\right]$ is in $Del(P)$ if and only if
    - There exists a ball B whose boundary contains vertices of σ, and that the interior of B contains no other point from P.



Delaunay Complex

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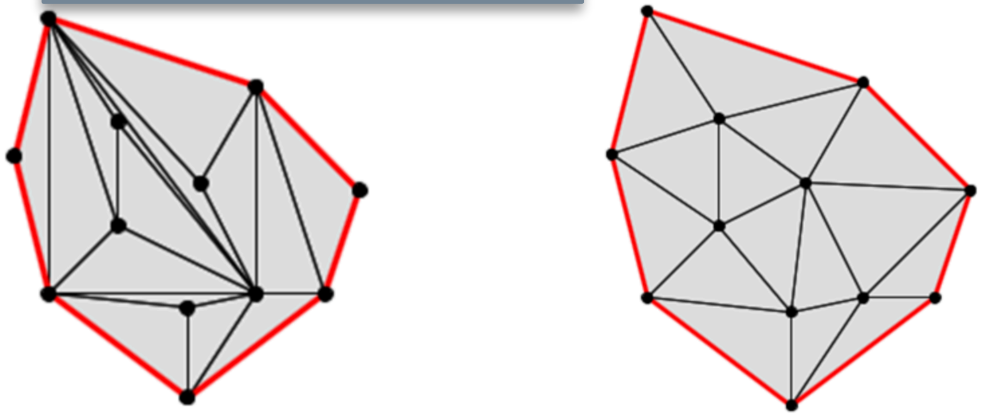
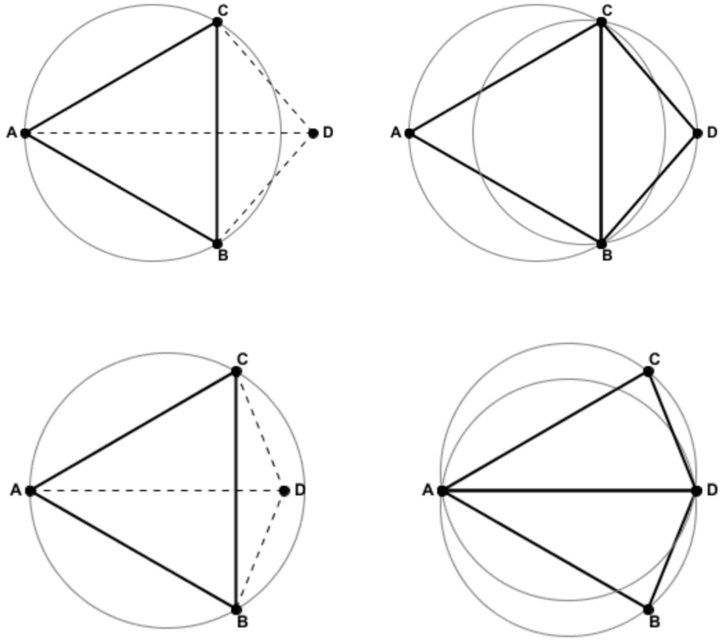
Images from online blog



See Demo by Ondrej Draganov



Delaunay Complex

- Foundation for surface reconstruction and meshing in 3D
  - *[Dey, Curve and Surface Reconstruction, 2006],*
  - *[Cheng, Dey and Shewchuk, Delaunay Mesh Generation, 2012]*

- However,
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in **high dimensions**

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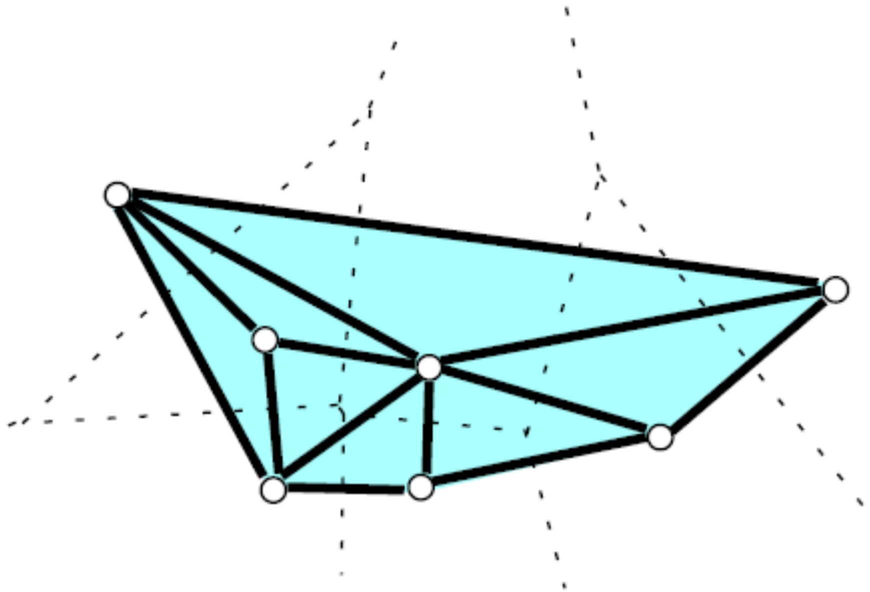

Vietoris Rips complex

Vietoris-Rips (Rips) Complex

- Given a set of points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$
- Given a real value $r > 0$, the *Vietoris-Rips (Rips) complex* $Rips^r(P)$ is:
  - $\left\{(p_{i_0}, \ldots, p_{i_k}) \mid B(p_{i_l}, r) \cap B(p_{i_j}, r) \neq \emptyset, \forall l, j = 0, \ldots, k\right\}$.
- More generally for P in a metric space (X, d):
  - $Rips^r(P) = \left\{(p_{i_0}, \ldots, p_{i_k}) \mid d(p_{i_l}, p_{i_j}) \leq 2r, \forall l, j = 0, \ldots, k\right\}$.

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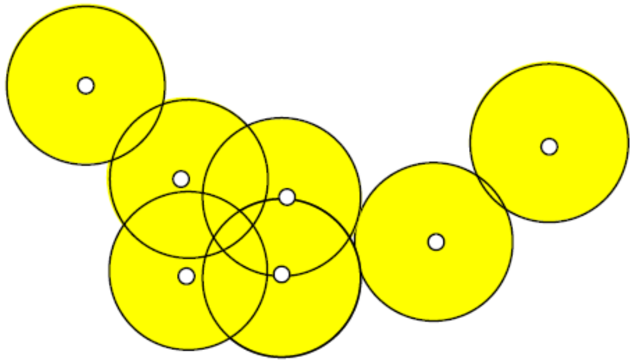

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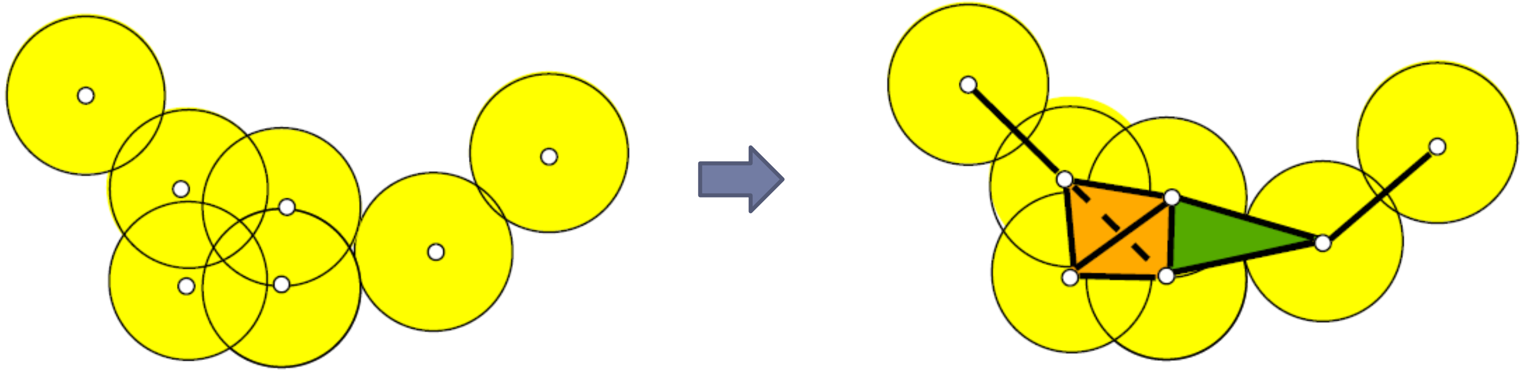

Vietoris-Rips (Rips) Complex

- The 1-skeleton of $Rips^r(P)$ is the $2r$ neighborhood graph of P, i.e., $\{p_i, p_j\} \in E$ if $d(p_i, p_j) \leq 2r$
  - Same for Čech
- $Rips^r(P)$ is the **clique complex** of its 1-skeleton
  - If $\{\{p_{i_k}, p_{i_l}\}\}_{k \neq l \in 0, \ldots, m}$ are edges,
  - then $d(p_{i_k}, p_{i_l}) \leq 2r$ for $k \neq l \in 0, \ldots, m$
  - Hence $\{p_{i_0}, \ldots, p_{i_m}\} \in Rips^r(P)$

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- $Rips^r(P)$ is the clique complex of its 1-skeleton

- Computing $Rips^r(P)$ reduces to computing the $2r$ neighborhood graph and finding its clique complex

See Demo by [Henry Adams]

Rips vs Čech

Rips vs Čech

- $C^r(P) \subset Rips^r(P)$

Rips vs Čech

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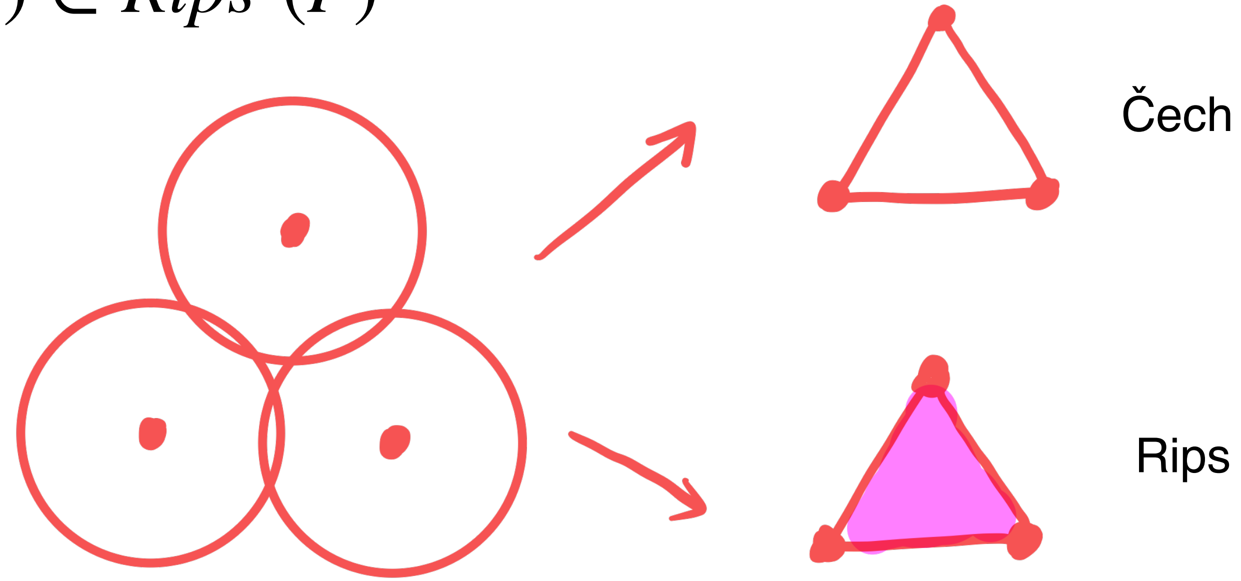

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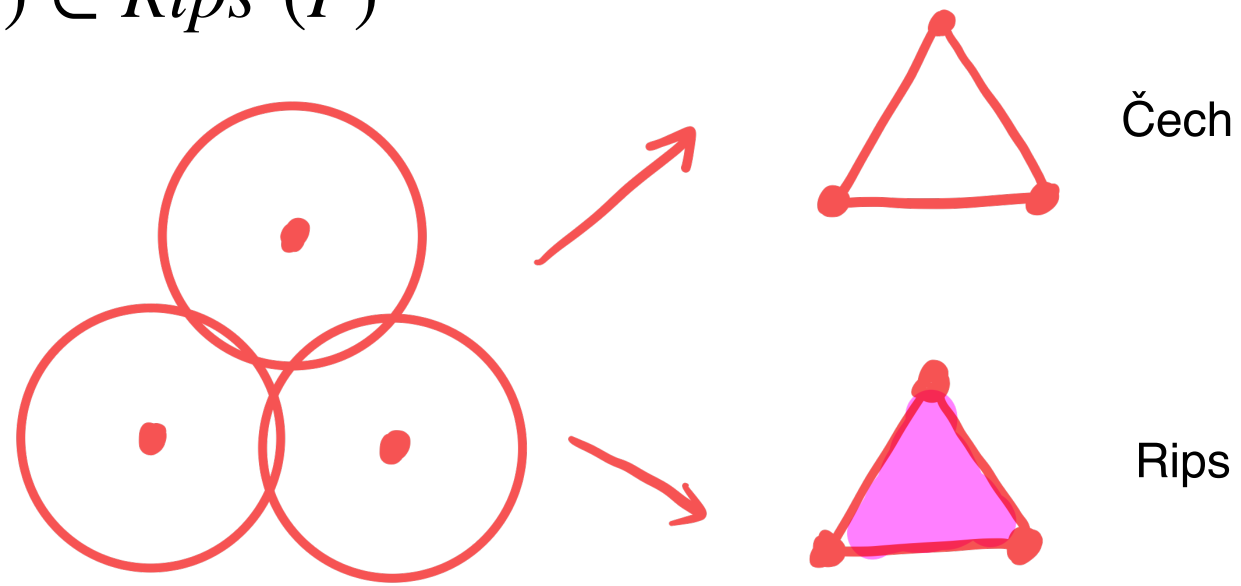

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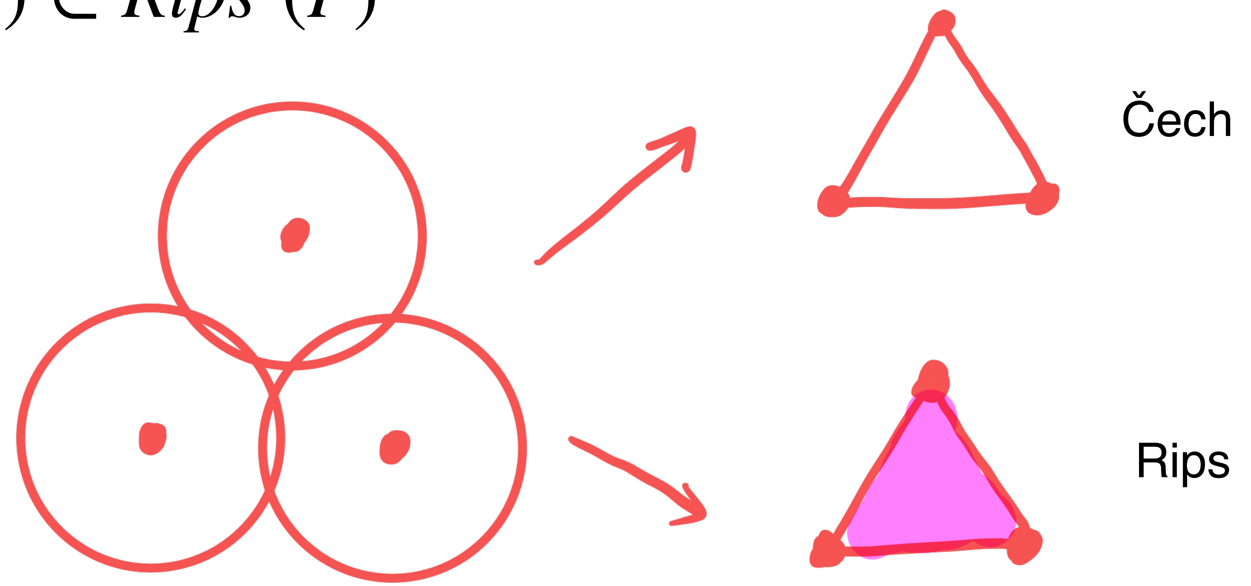

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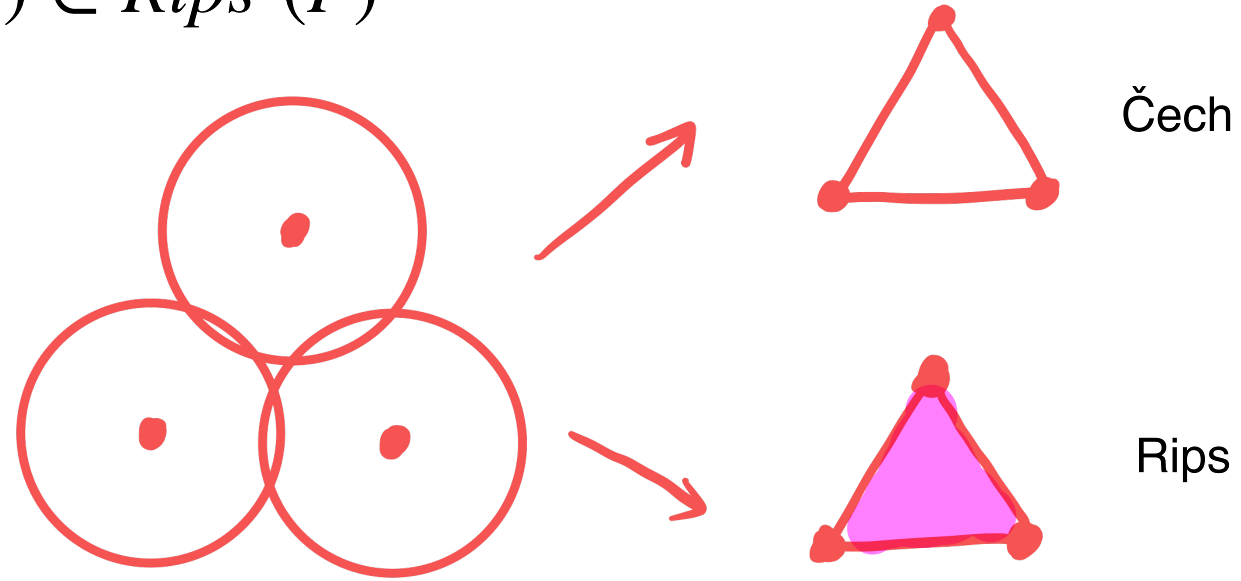

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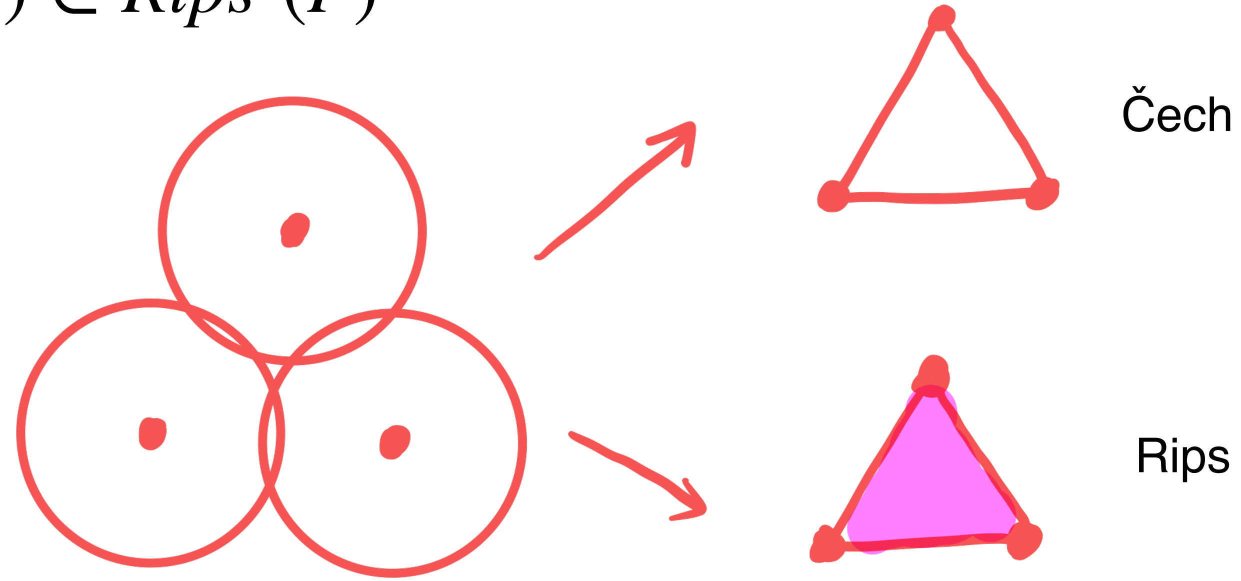

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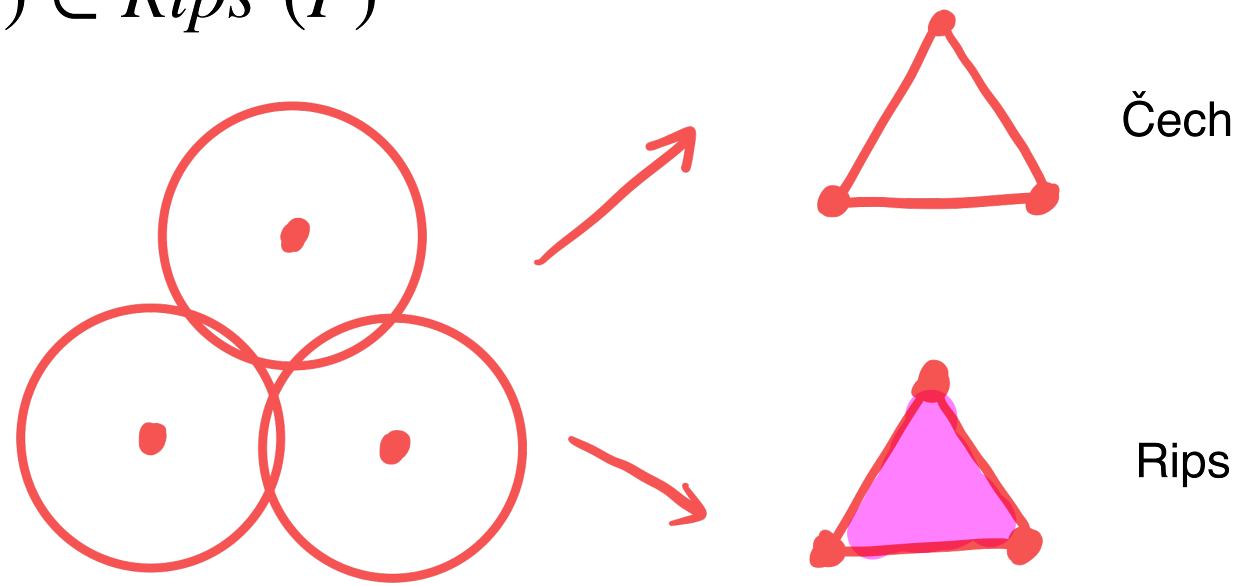

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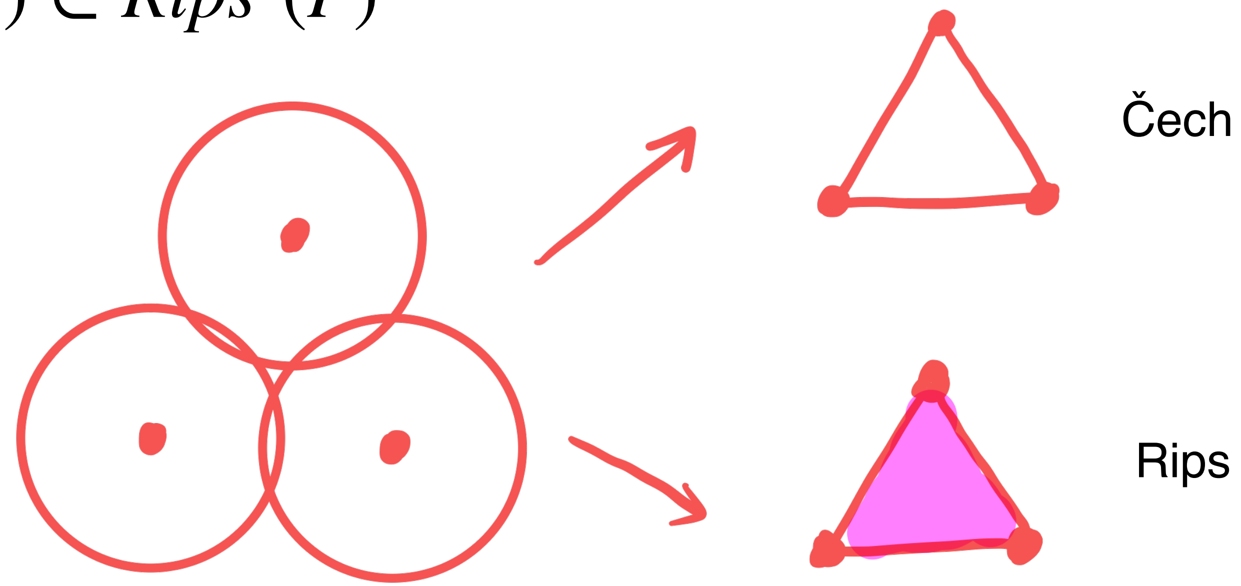

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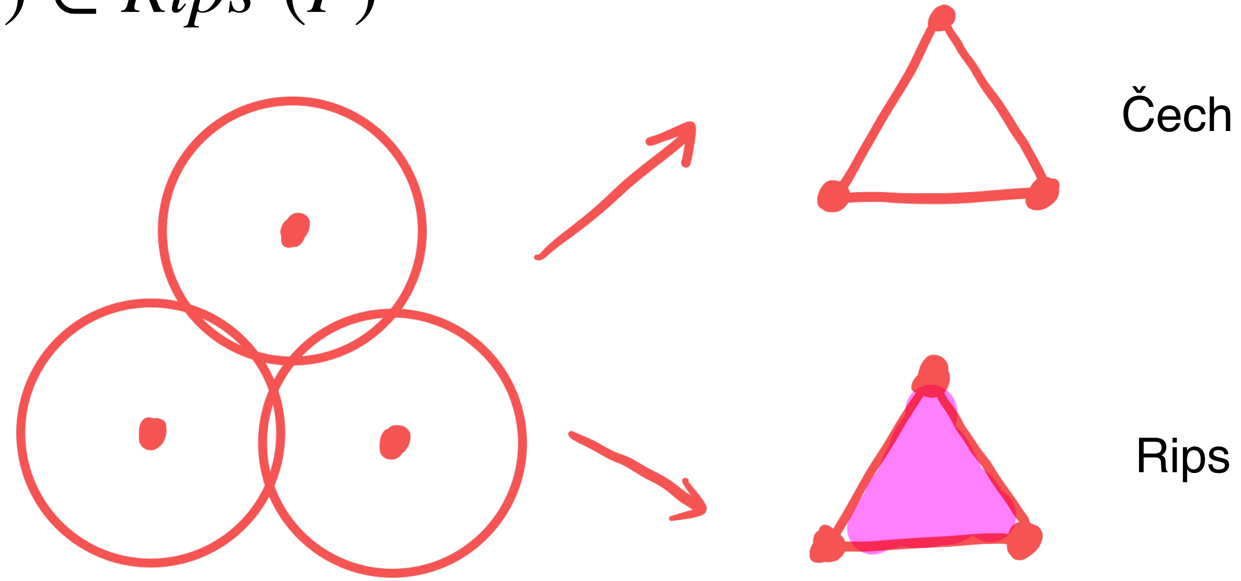

- $C^r(P) \subset Rips^r(P) \subset C^{2r}(P)$
  - $C^r(P) \subseteq Rips^r(P) \subseteq C^{\sqrt{2}r}(P)$ when $P \subseteq \mathbb{R}^d$

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  - Many optimized packages for Rips complex

	Recover the desired topology of data	Number of simplicies	Computation
Čech complex	Yes	Large	Hard in high dimension
Alpha complex	Yes	Small	Hard in high dimension
Rips complex	somewhat	Large	Doesn’t care dimension

FIN