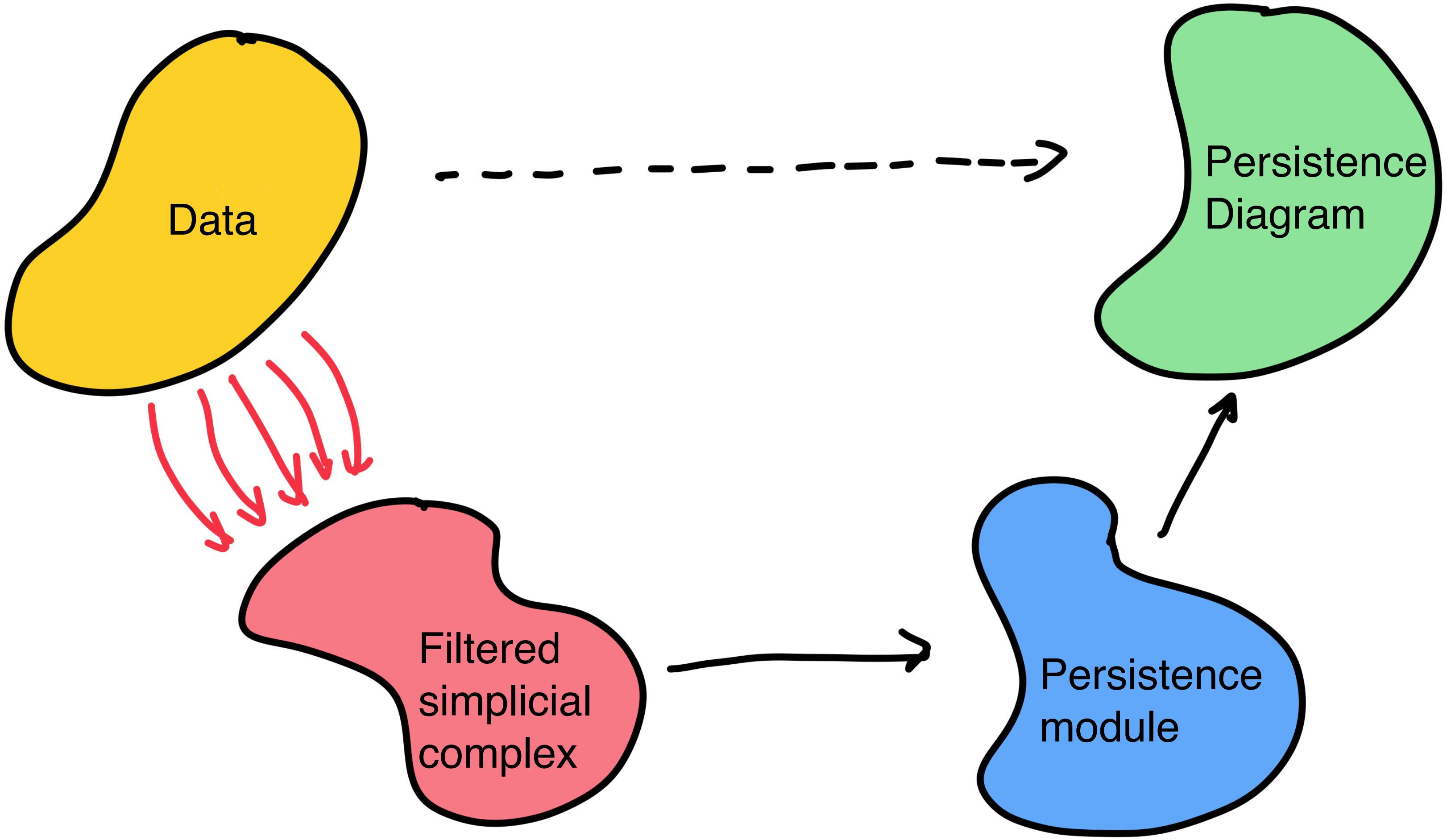
DSC 214
Topological Data Analysis
Topic 4-B: Persistent Homology for PCD and Functions

Instructor: Zhengchao Wan

Persistence-based Framework



Persistence-based Framework

- Persistence-based feature representation

Input data:
A collection of point sets, graphs, 2D/3D images, time-series , etc

→

Persistence feature space:
A collection of persistence-based feature representations (e.g, vectors)

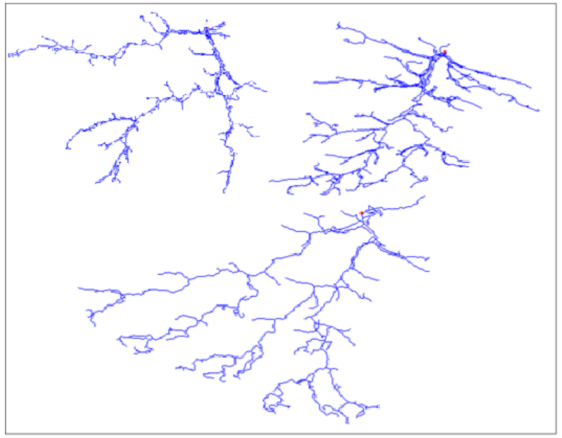
→

Further data analysis:
Feed to data analysis pipelines, say SVM, CNN, etc

Persistence-based Framework

- Persistence-based feature representation

Input data: A collection of point sets, graphs, 2D/3D images, time-series , etc	→	Persistence feature space: A collection of persistence-based feature representations (e.g, vectors)	→	Further data analysis: Feed to data analysis pipelines, say SVM, CNN, etc

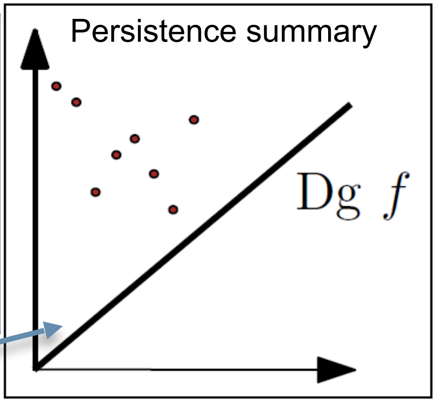


[Li, et al, PLOS One 2017]

Persistence-based Framework

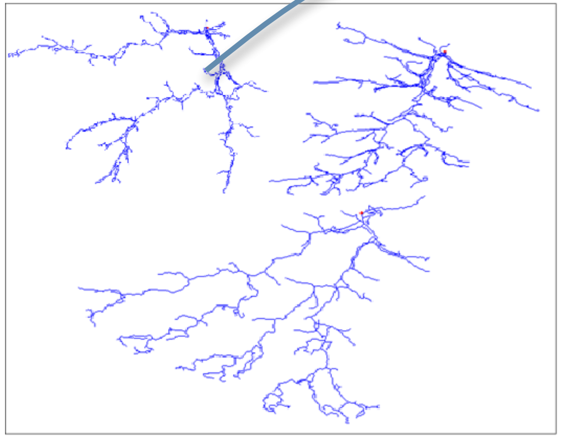
- Persistence-based feature representation

Input data:
A collection of point sets, graphs, 2D/3D images, time-series , etc



nce feature ace:
of persistence- l feature ns (e.g, vectors)

Further data analysis:
Feed to data analysis pipelines, say SVM, CNN, etc

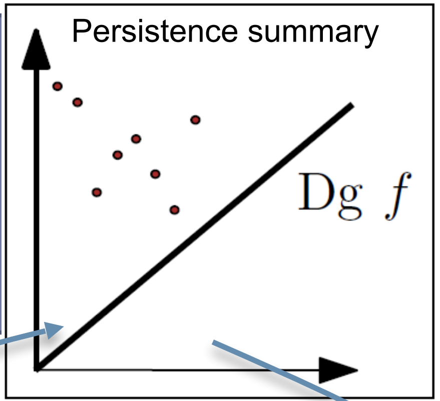


[Li, et al, PLOS One 2017]

Persistence-based Framework

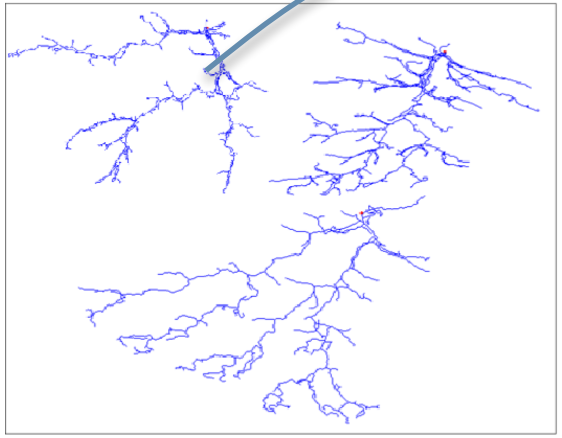
- Persistence-based feature representation

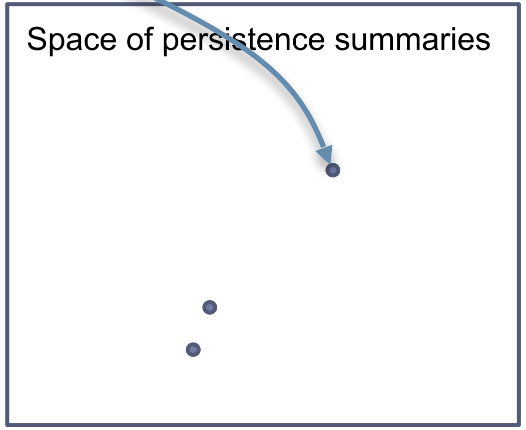
Input data:
A collection of point sets, graphs, 2D/3D images, time-series , etc



nce feature ace:
of persistence- l feature ns (e.g, vectors)

Further data analysis:
Feed to data analysis pipelines, say SVM, CNN, etc





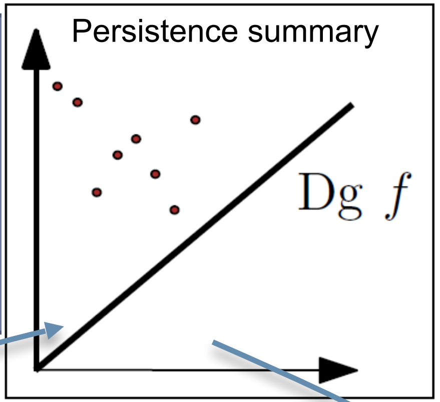
[Li, et al, PLOS One 2017]

Persistence-based Framework

- Persistence-based feature representation

Input data:

A collection of point sets, graphs, 2D/3D images, time-series , etc

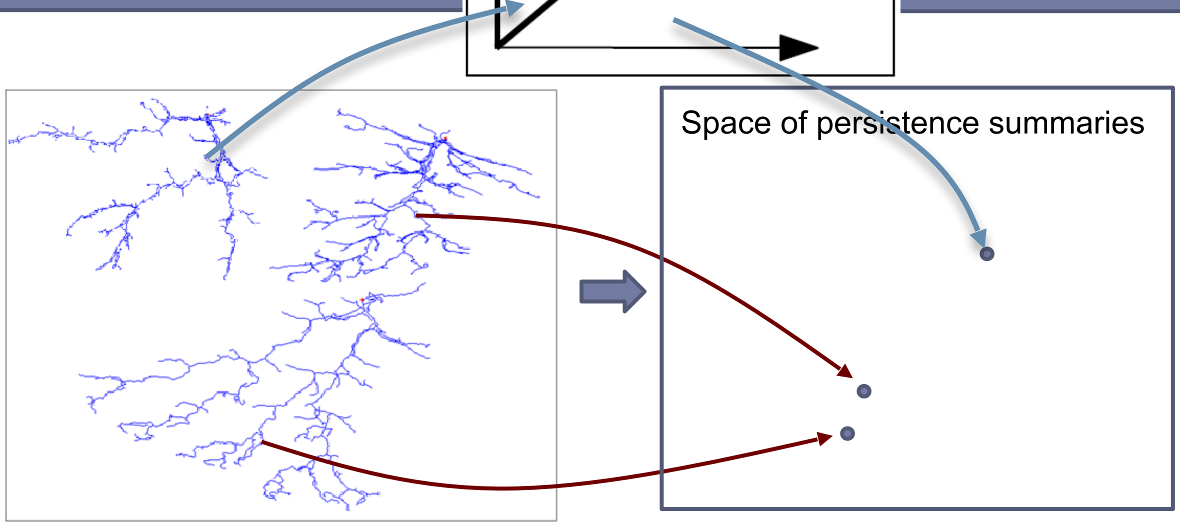


nce feature ace:

of persistence-
l feature
ns (e.g, vectors)

Further data analysis:

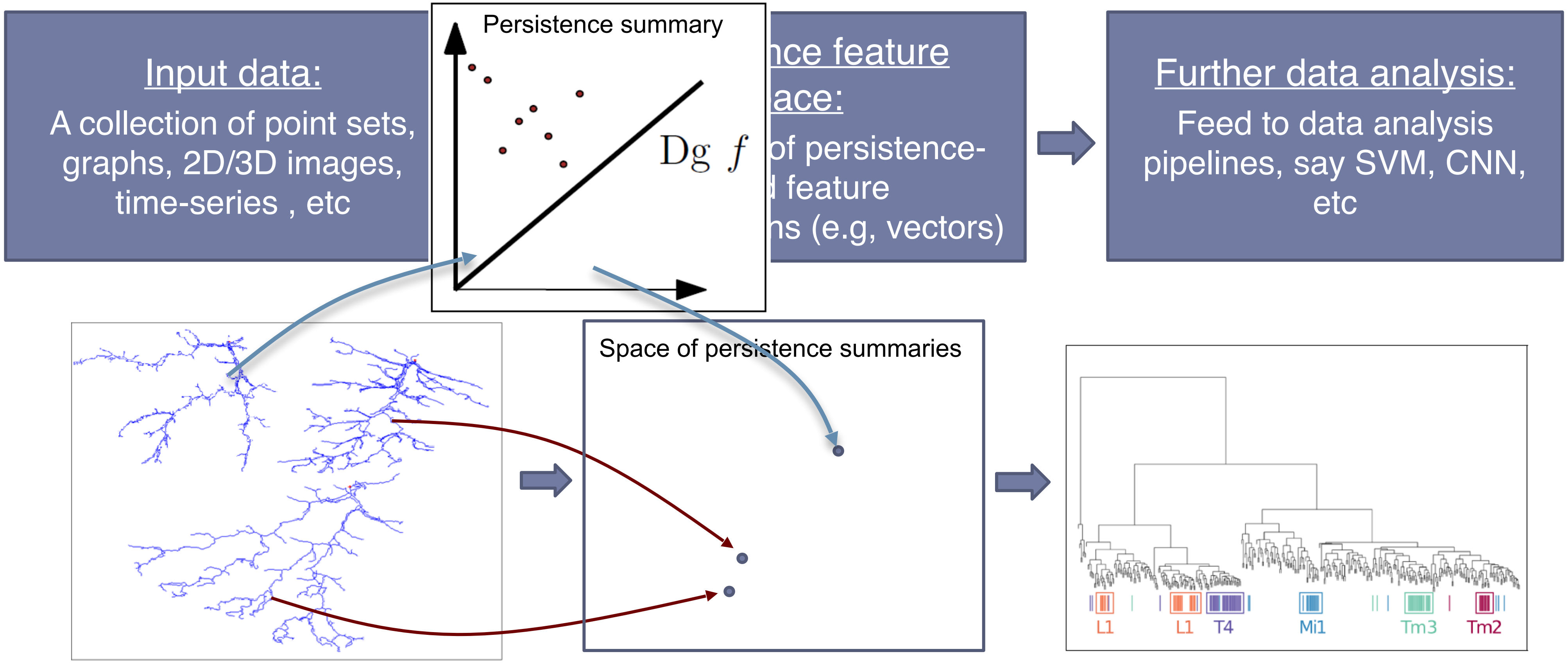
Feed to data analysis pipelines, say SVM, CNN, etc



[Li, et al, PLOS One 2017]

Persistence-based Framework

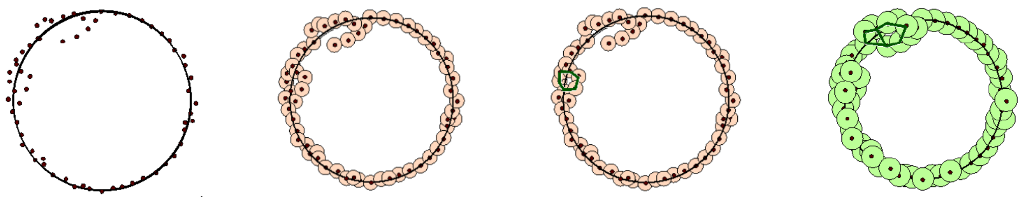
- Persistence-based feature representation



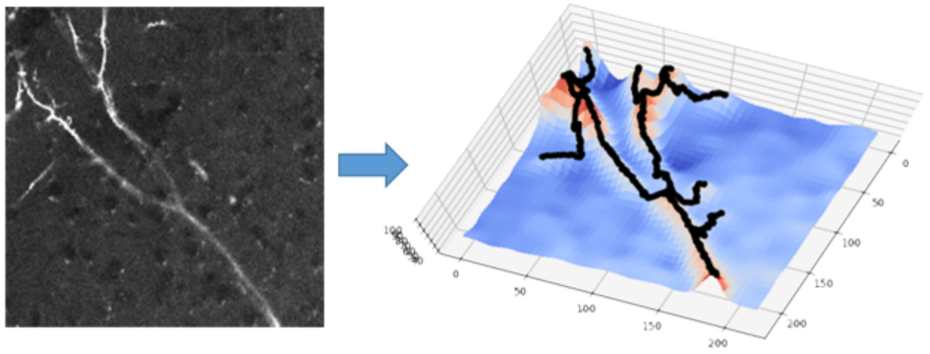
[Li, et al, PLOS One 2017]

- Recently, many methods for mapping persistence diagrams to a finite vector space or a Hilbert space
  - Persistence landscapes
    - [Bubenik 2012]
  - Persistence scale space kernel
    - [Reininghause et al., 2014]
  - Persistence images
    - [Adams et al., 2015, 2017]
  - Persistence weighted Gaussian kernel
    - [Kusano et al., 2017]
  - Sliced Wasserstein kernel
    - [Carriere et al., 2017]
  - Persistence Fisher kernel
    - [Le and Yamada 2018]
  - ....

- How do we use persistent homology that introduced last time to different types of data? What information can we obtain?
- Two examples:
  - Point cloud data
  - Functions on triangulated spaces



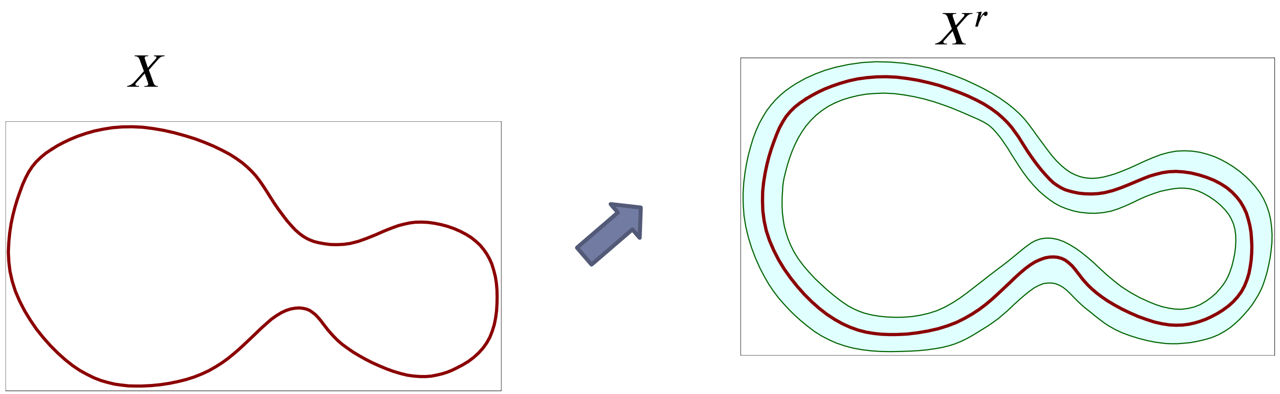
(a) (b) (c) (d)



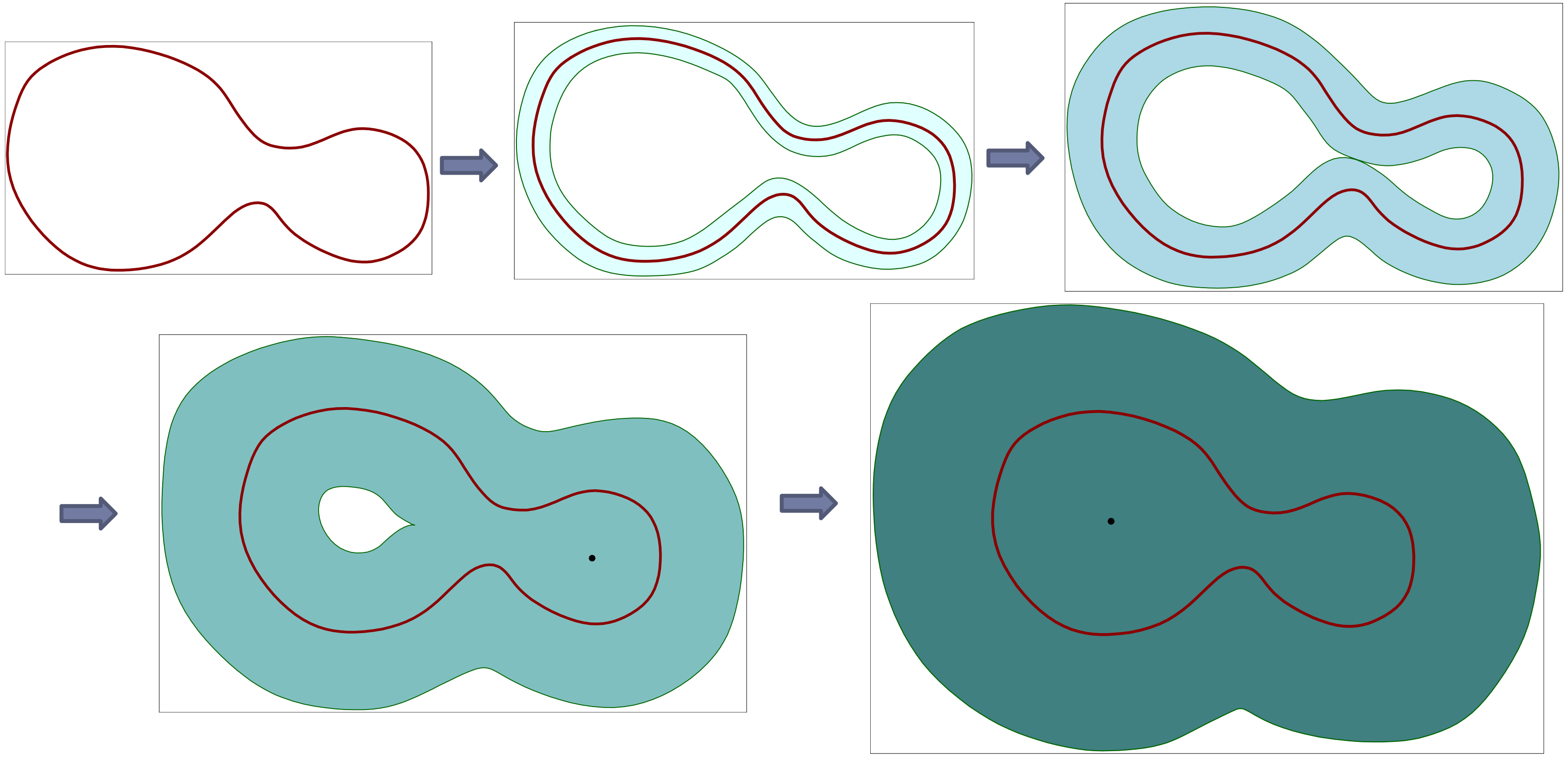
Section 1:
PH for Point Cloud Data

Offset – Union of Balls

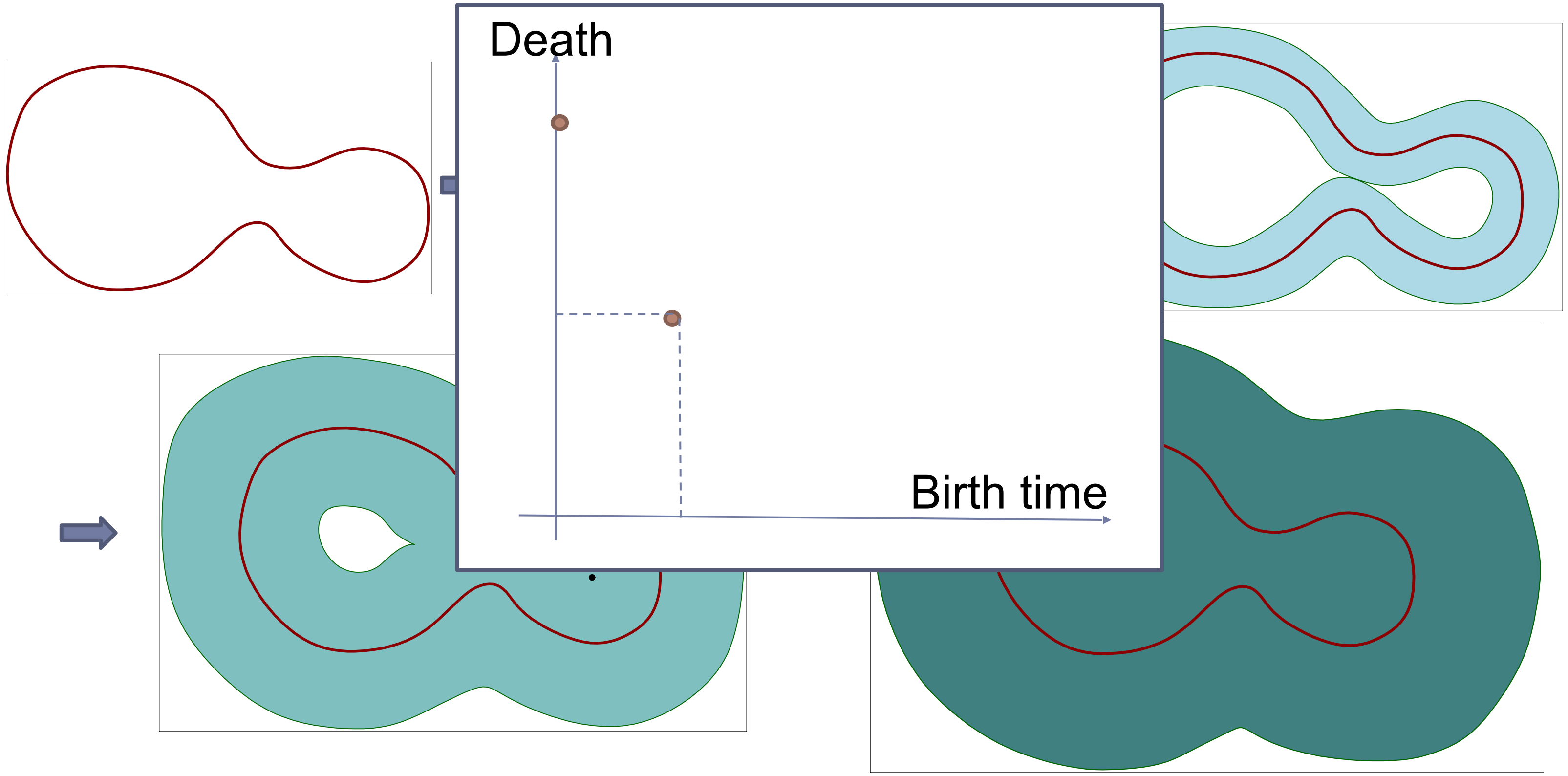
- What can we do with the underlying ground truth?
- $X \subset R^d$: a compact subset of R^d
  - a hidden space of interests
- X^r: r-offset of X
  - $X^r = \left\{ y \in R^d \mid d\left(y,\ X\right) \leq r \right\} = \bigcup_{x \in X} B(x,\ r)$



- Target filtration: $X^{\alpha_0} \subseteq X^{\alpha_1} \subseteq \cdots X^{\alpha} \subseteq \cdots$
- PH induced by this filtration provides a topological summary of X



- Target filtration: $X^{\alpha_0} \subseteq X^{\alpha_1} \subseteq \cdots X^{\alpha} \subseteq \cdots$
- PH induced by this filtration provides a topological summary of X



Remark

Remark

- One can theoretically extend Persistence diagrams to general topological spaces

Remark

- One can theoretically extend Persistence diagrams to general topological spaces
- The persistence diagrams capture essential topological information of the underlying topological spaces

Remark

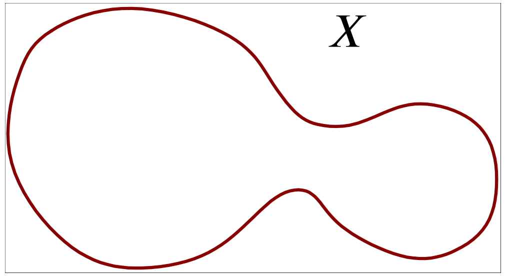
- One can theoretically extend Persistence diagrams to general topological spaces
- The persistence diagrams capture essential topological information of the underlying topological spaces
- We don’t have access to ground truth topological spaces in general

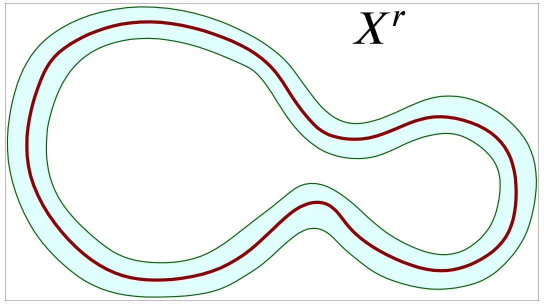
Remark

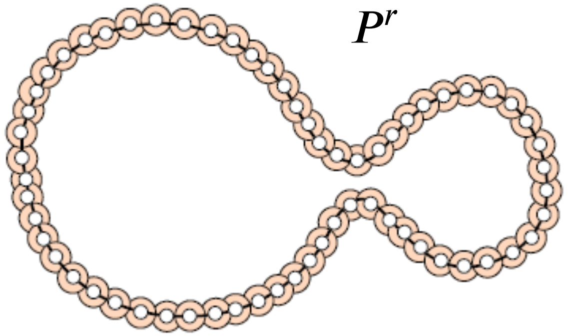
- One can theoretically extend Persistence diagrams to general topological spaces
- The persistence diagrams capture essential topological information of the underlying topological spaces
- We don't have access to ground truth topological spaces in general
- We hope that we can approximate the persistence diagram of the ground truth using samples

Union of Balls

- Target filtration: $X^{r_0} \subseteq X^{r_1} \subseteq \cdots X^r \subseteq \cdots$
- Instead of X, we are only given PCD P
  - $P^r = \bigcup_{p \in P} B(p, r)$
  - Intuitively, P^r approximates X^r





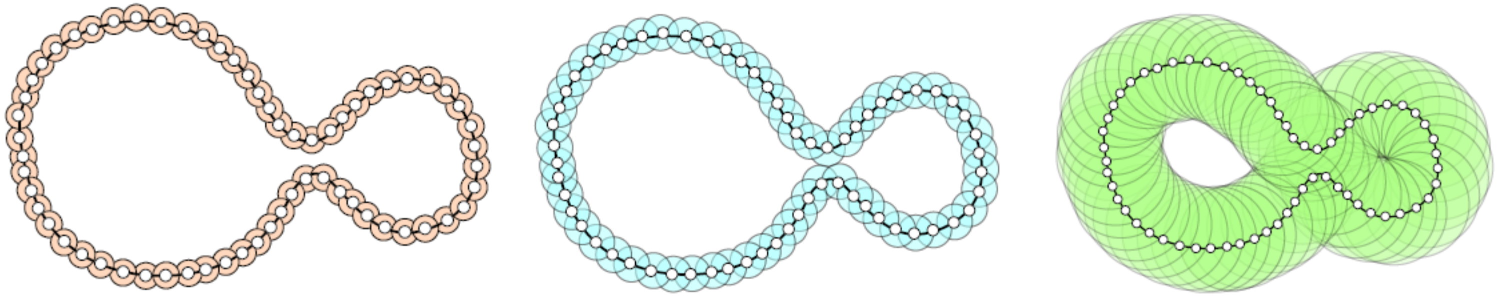


Union of Balls

- Target filtration: $X^{r_0} \subseteq X^{r_1} \subseteq \cdots X^{r} \subseteq \cdots$
- Instead of X, we are only given PCD P
  - $P^r = \bigcup_{p \in \mathrm{P}} B(p,\ r)$
  - Intuitively, P^r approximates X^r
  - As we increase r:
  - Intermediate filtration: $P^{r_0} \subseteq P^{r_1} \subseteq \cdots P^{r} \subseteq \cdots$

Union of Balls

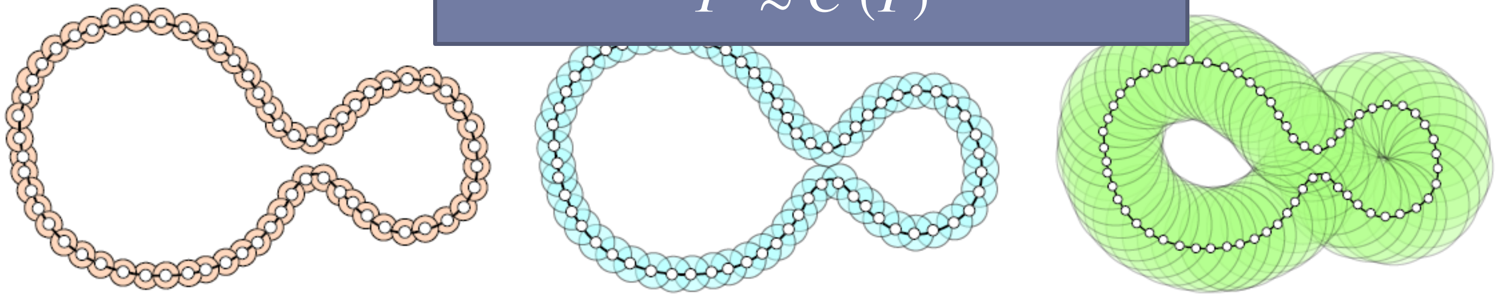
- Target filtration: $X^{r_0} \subseteq X^{r_1} \subseteq \cdots X^{r} \subseteq \cdots$
- Instead of X, we are only given PCD P
  - $P^r = \bigcup_{p \in \mathrm{P}} B(p,\ r)$
  - Intuitively, P^r approximates X^r
  - As we increase r:
  - Intermediate filtration: $P^{r_0} \subseteq P^{r_1} \subseteq \cdots P^{r} \subseteq \cdots$



Union of Balls

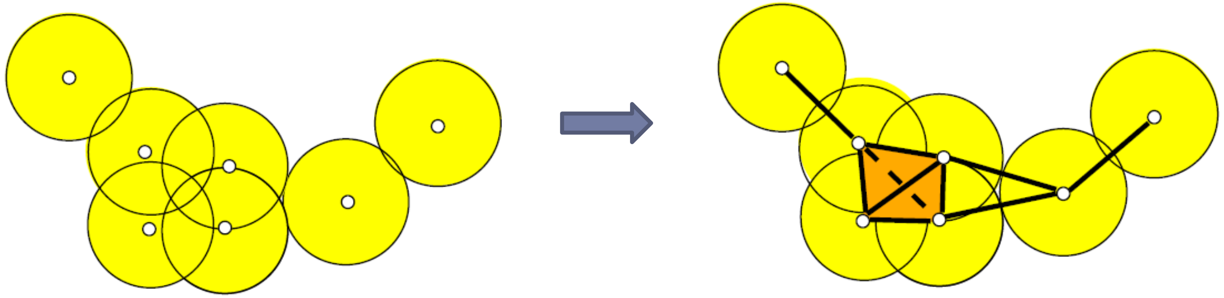
- Target filtration: $X^{r_0} \subseteq X^{r_1} \subseteq \cdots X^{r} \subseteq \cdots$
- Instead of X, we are only given PCD P
  - $P^r = \bigcup_{p \in P} B(p, r)$
  - Intuitively, P^r approximates X^r
  - As we increase r:
  - Intermediate filtration: $P^{r_0} \subseteq P^{r_1} \subseteq \cdots P^{r} \subseteq \cdots$

By Nerve Lemma,
$P^r \approx C^r(P)$



Recall: Čech Complex

- Given a set of points $P = \{ p_1, p_2, \ldots, p_n \} \subset R^d$
- Given a real value $r > 0$, the *Čech complex* $C^r(P)$ is the nerve of the set $\left\{ B(p_i, r) \right\}_{i\in[1,n]}$
  - i.e, $\sigma = \left\{ p_{i_0}, \ldots, p_{i_s} \right\} \in C^r(P)$ iff $\bigcap_{j\in[0,s]} B\left(p_{i_j}, r\right) \neq \emptyset$
- The definition can be extended to a finite sample P of a metric space.



Nerves

- Given a finite set F, its nerve complex $Nrv(F)$ is
  - defined as all non-empty subset of F with non-empty common intersection
  - i.e, $Nrv(F) = \left\{ X \subseteq F \mid \bigcap_{\sigma \in X} \sigma \neq \emptyset \right\}$

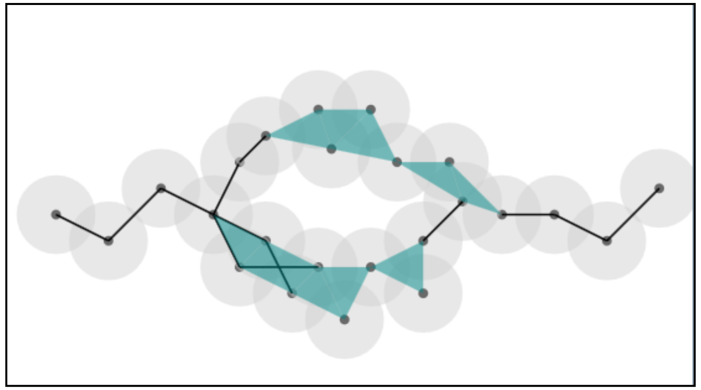
Nerves

- Given a finite set F, its nerve complex $Nrv(F)$ is
  - defined as all non-empty subset of F with non-empty common intersection
  - i.e, $Nrv(F) = \left\{ X \subseteq F \mid \bigcap_{\sigma \in X} \sigma \neq \emptyset \right\}$

- Hence Čech complex $C^r(P)$
  - is the nerve of $F = \left\{ B\left(p,\ r\right) \mid p \in P \right\}$
  - i.e, $C^r(P) = Nrv(F)$

Nerve Lemma

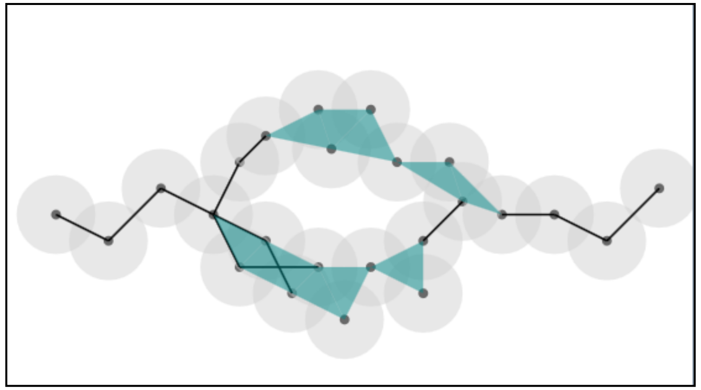
- Nerve Lemma (a simplified version):
  - Let $\mathscr{U}$ be a finite collection of closed, convex subsets in R^d. Then $|Nrv(\mathscr{U})| \simeq \cup_{\alpha \in A} U_\alpha \subset \mathbb{R}^d$.



Nerve Lemma

- Nerve Lemma (a simplified version):
  - Let $\mathscr{U}$ be a finite collection of closed, convex subsets in R^d. Then $|Nrv(\mathscr{U})| \simeq \cup_{\alpha \in A} U_\alpha \subset \mathbb{R}^d$.

- Corollary:
  - $|C^r(P)| \simeq \cup_{p \in P} B(p, r)$, i.e, $|C^r(P)|$ is homotopy equivalent to the union of r-balls around points in P



Persistent Homology Inference from PCD

- Input:
  - A set of points $P \subseteq R^d$ sampled on/around X
- Question:
  - How to approximate the persistence module induced by F_X?

Target filtration (F_X):	$X^{r_0} \subseteq X^{r_1} \subseteq \cdots X^r \subseteq \cdots$
Intermediate filtration:	$P^{r_0} \subseteq P^{r_1} \subseteq \cdots P^r \subseteq \cdots$

$\cong$ By Nerve Lemma

Čech filtration ($\mathscr{C}_X$):	$C^{r_0} \subseteq C^{r_1} \subseteq \cdots C^r \subseteq \cdots$

Persistent Homology Inference from PCD

- Input:
  - A set of points $P \subseteq R^d$ sampled on/around X
- Question:
  - How to approximate the persistence module induced by F_X?

Target filtration (F_X): $X^{r_0} \subset X^{r_1} \subset \cdots X^{r} \subset \cdots$

Intermedi

The approximation of their persistence diagrams can be made precise by using the so-called interleaving distance.

$\cong$ By Nerve Lemma

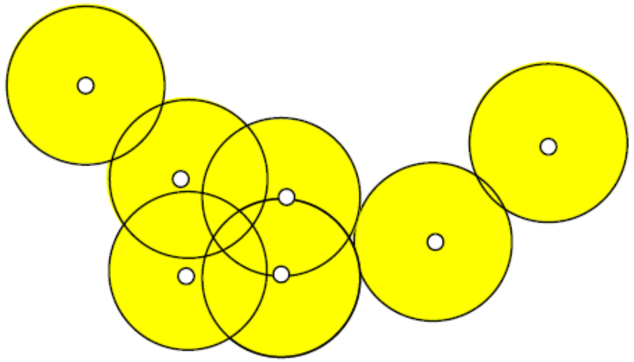
Čech filtration ($\mathscr{C}_X$): $C^{r_0} \subseteq C^{r_1} \subseteq \cdots C^{r} \subseteq \cdots$

Recall: Rips Complex

- Given a set of points $P = \{ p_1, p_2, \ldots, p_n \} \subset R^d$
- Given a real value $r > 0$, the *Vietoris-Rips (Rips) complex* $R^r(P)$ is:
  - $\{ \left(p_{i_0}, p_{i_1}, \ldots, p_{i_k} \right) \mid B_r\left(p_{i_l}\right) \cap B_r\left(p_{i_j}\right) \neq \varnothing, \ \forall\, l, \ j \in [0, k\,] \}.$

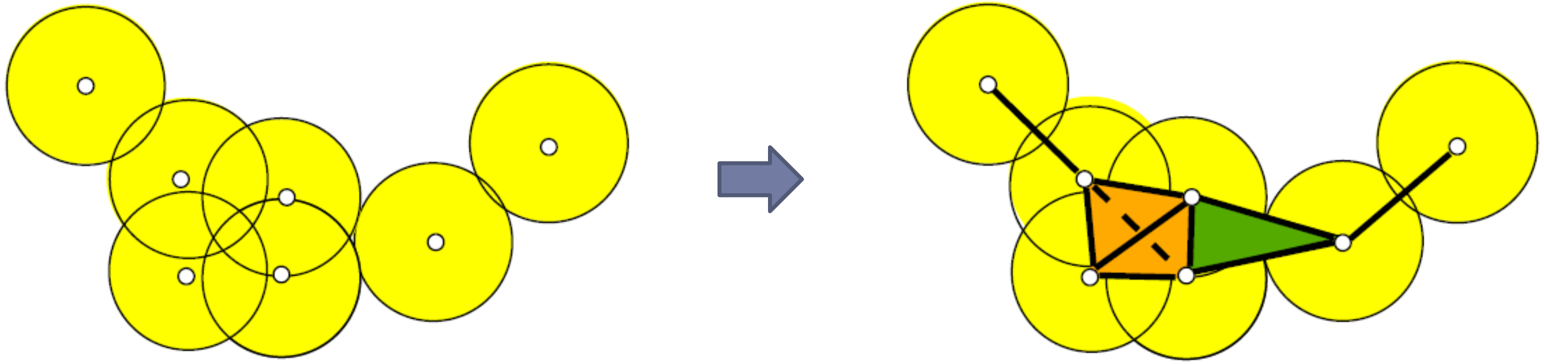
Recall: Rips Complex

- Given a set of points $P = \left\{ p_1, p_2, \ldots, \ p_n \right\} \subset R^d$
- Given a real value $r > 0$, the *Vietoris-Rips (Rips) complex* $R^r(P)$ is:
  - $\{ \left(p_{i_0}, p_{i_1}, \ldots, \ p_{i_k} \right) \mid B_r\left(p_{i_l} \right) \cap B_r\left(p_{i_j} \right) \neq \emptyset, \ \forall \ l, \ j \in [0, k \ ] \}.$



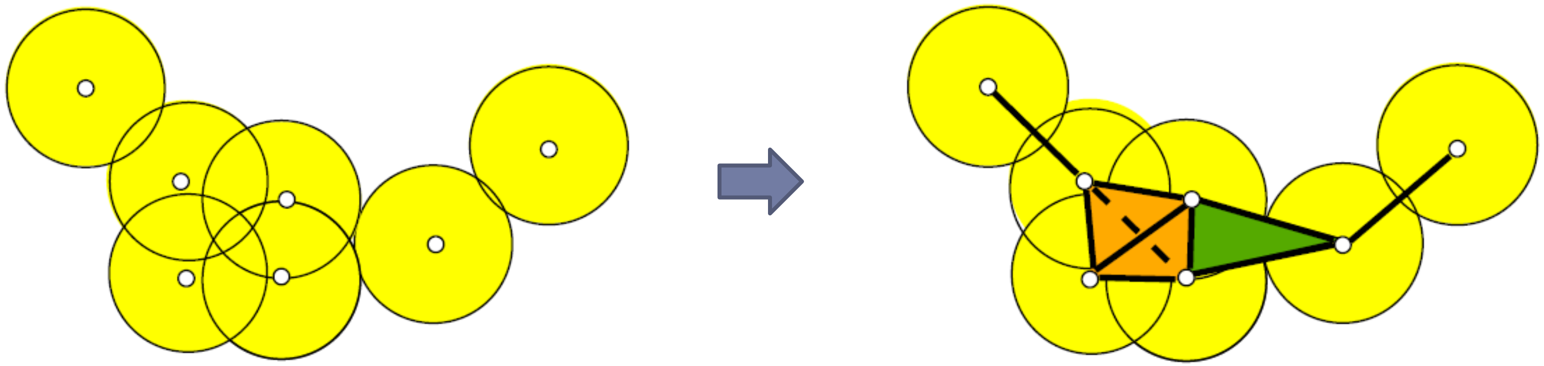
Recall: Rips Complex

- Given a set of points $P = \{ p_1, p_2, \ldots, p_n \} \subset R^d$
- Given a real value $r > 0$, the *Vietoris-Rips (Rips) complex* $R^r(P)$ is:
  - $\{ \left(p_{i_0}, p_{i_1}, \ldots, p_{i_k}\right) \mid B_r\left(p_{i_l}\right) \cap B_r\left(p_{i_j}\right) \neq \varnothing, \ \forall\, l, \ j \in [0, k] \}$.



Recall: Rips Complex

- Given a set of points $P = \{ p_1, p_2, \ldots, p_n \} \subset R^d$
- Given a real value $r > 0$, the *Vietoris-Rips (Rips) complex* $R^r(P)$ is:
  - $\{ \left(p_{i_0}, p_{i_1}, \ldots, p_{i_k}\right) \mid B_r\left(p_{i_l}\right) \cap B_r\left(p_{i_j}\right) \neq \emptyset, \ \forall\, l, \ j \in [0, k\,]\}.$



- Rips complex shares the same edge set as the Cech complex w.r.t same r.
- It is the clique complex induced by its edge set.

Rips and Čech Filtrations

- Relation in general metric spaces
  - $C^{r}(P) \subseteq R^{r}(P) \subseteq C^{2r}(P)$
  - Bounds better in Euclidean space

Target filtration (F_X): $X^{r_0} \subseteq X^{r} \subseteq \cdots X^{r} \subseteq \cdots$

Intermediate filtration: $P^{r_0} \subseteq P^{r_1} \subseteq \cdots P^{r} \subseteq \cdots$

$\cong$ By Nerve Lemma

Čech filtration ($\mathscr{C}_X$): $C^{r_0} \subseteq C^{r_1} \subseteq \cdots C^{r} \subseteq \cdots$

Two sequence interleave

Rips filtration ($\mathscr{R}_X$): $R^{r_0} \subseteq R^{r_1} \subseteq \cdots R^{r} \subseteq \cdots$

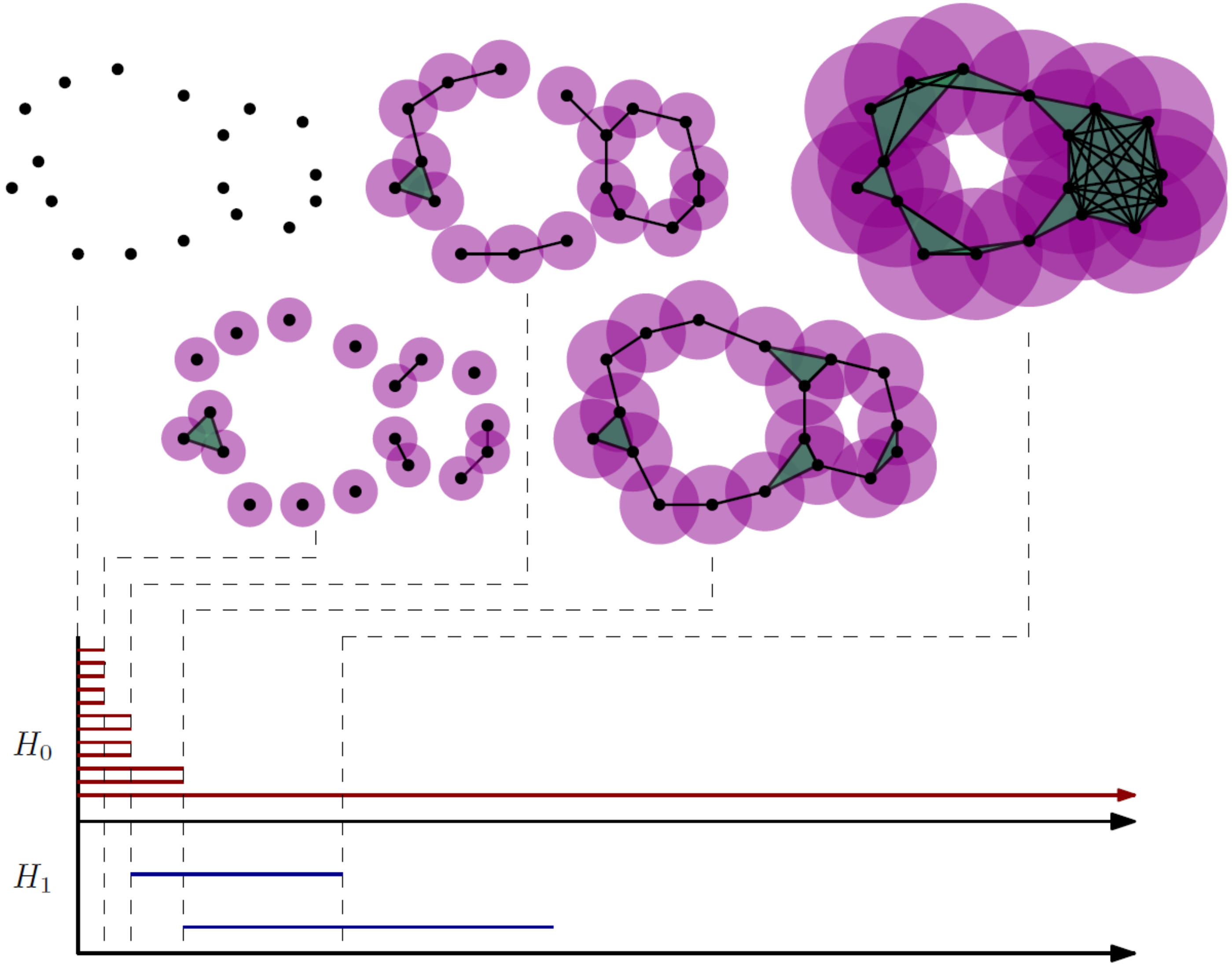
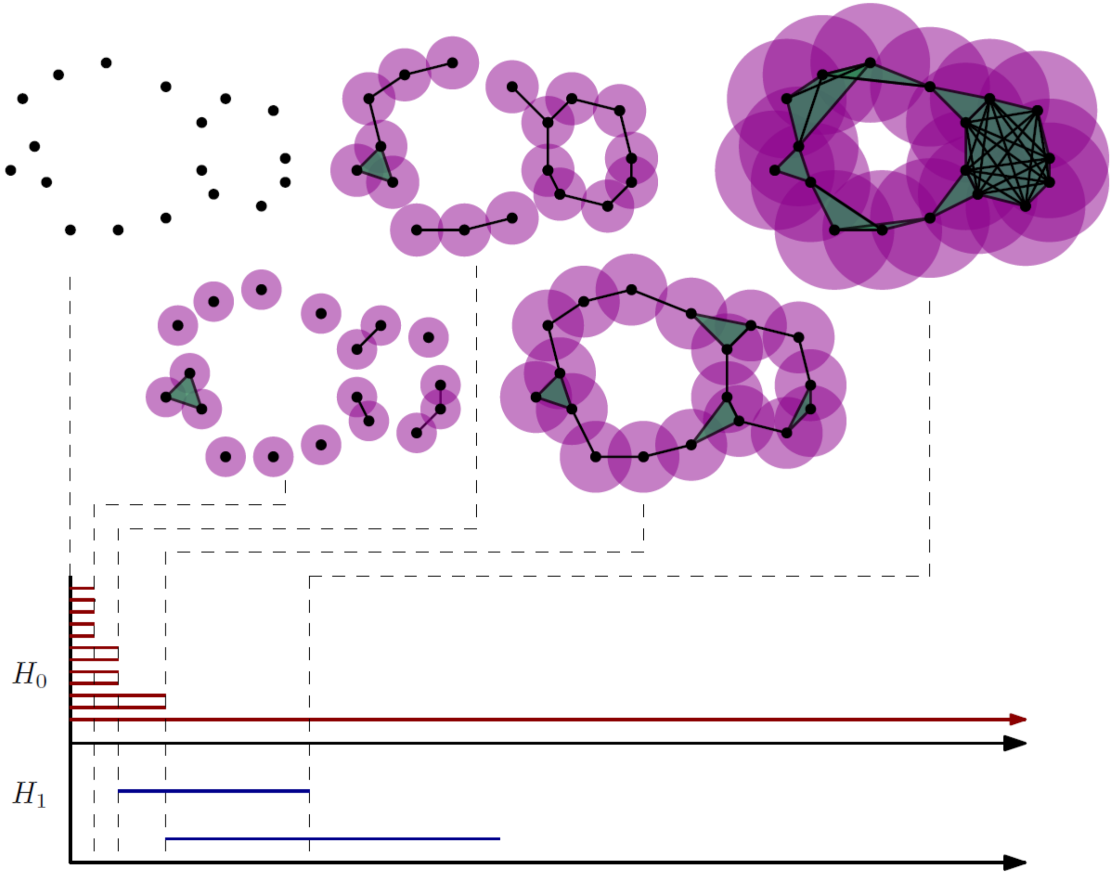


Image courtesy of T. K. Dey



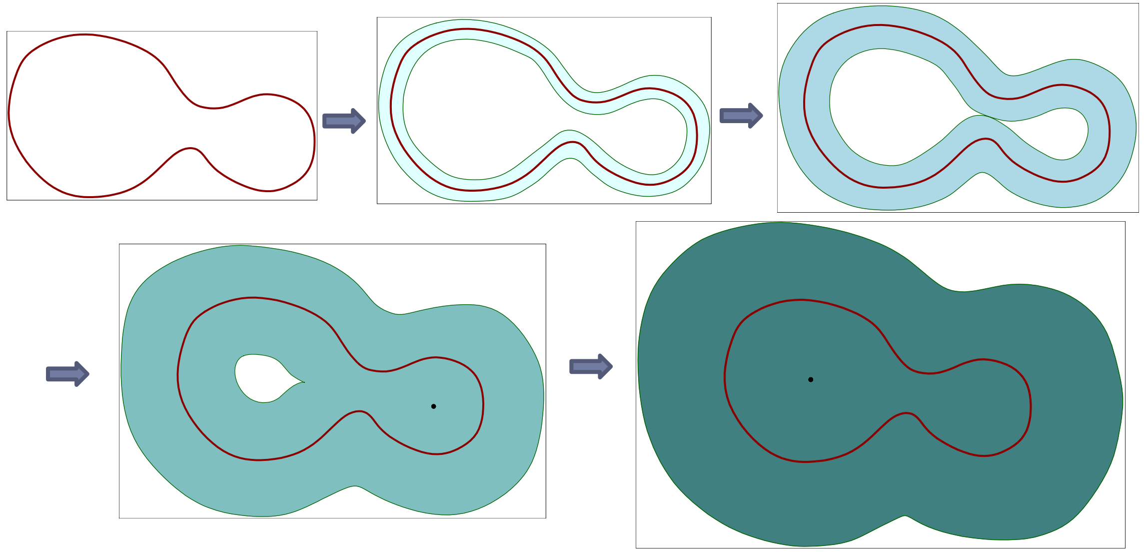


Image courtesy of T. K. Dey

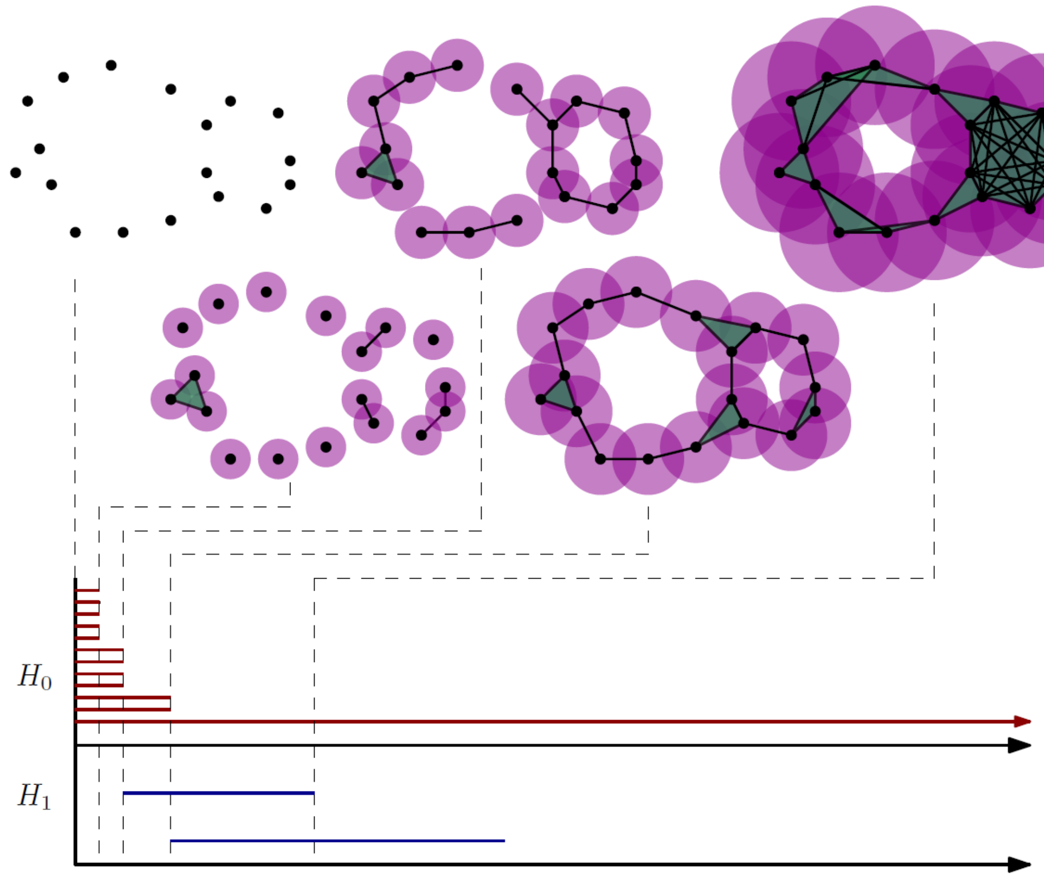
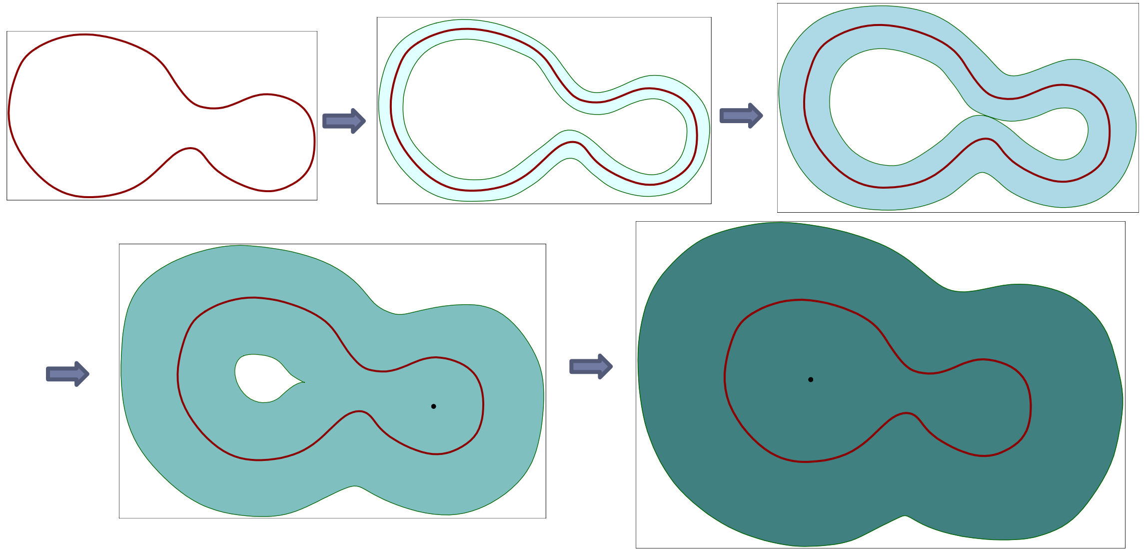
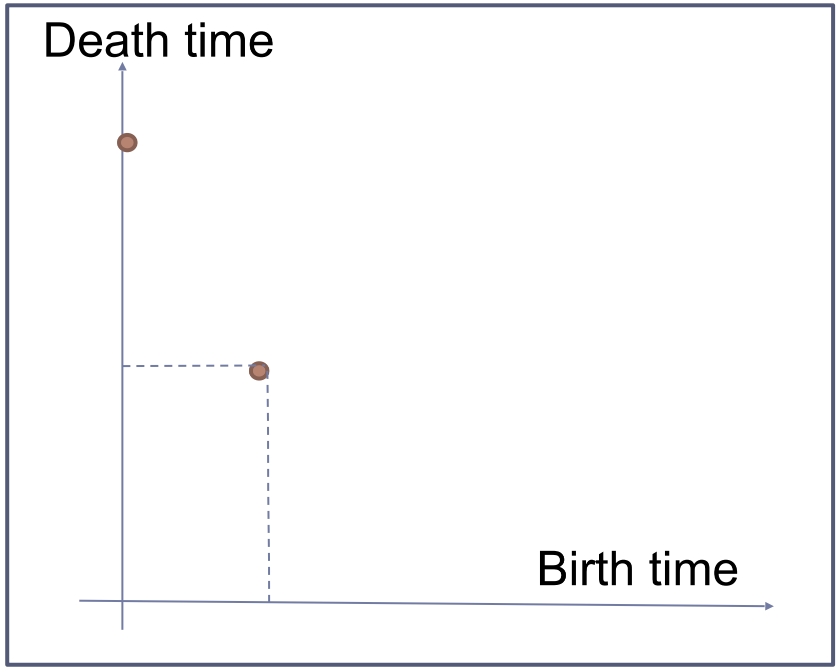
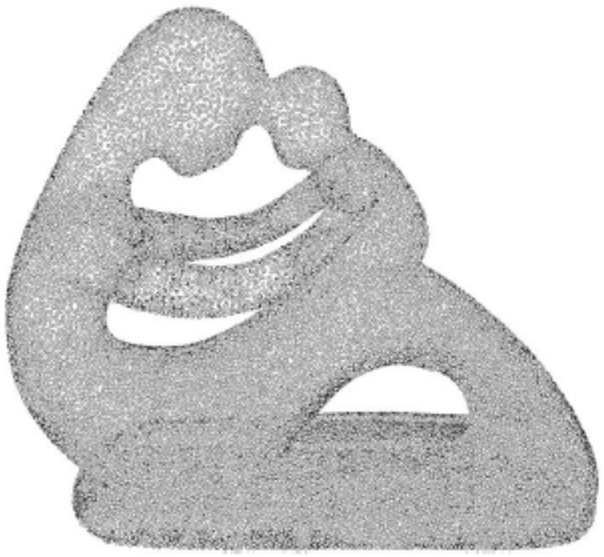


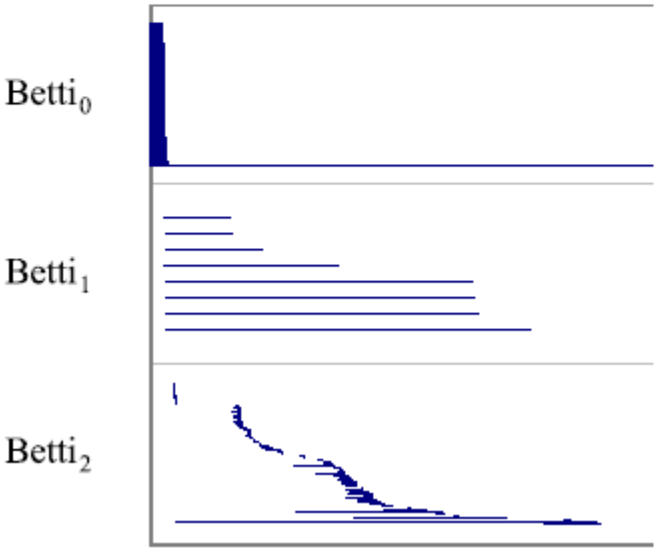
Image courtesy of T. K. Dey







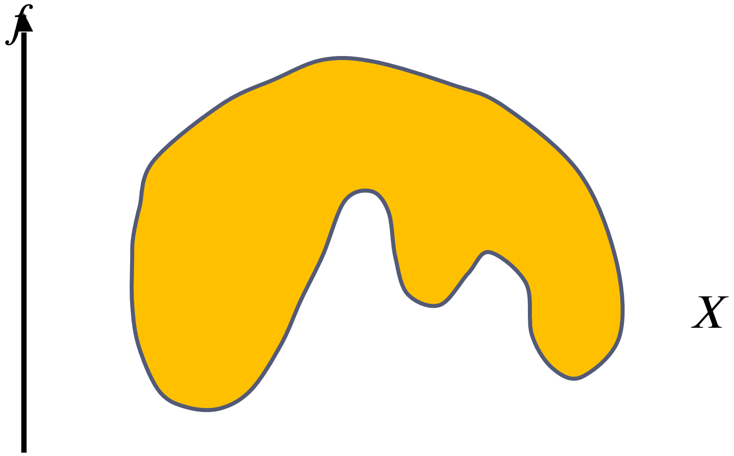
(a) MotherChild model

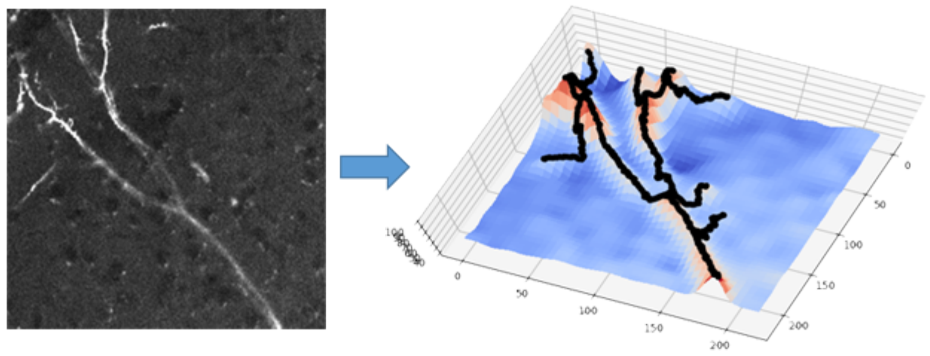


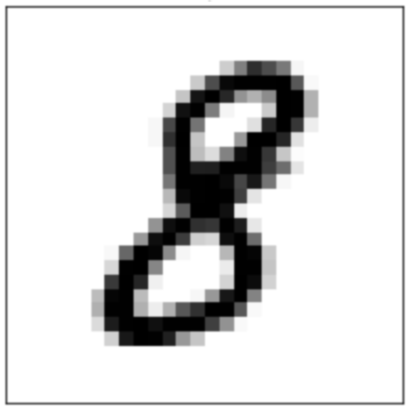
Section 2:
PH induced by Functions

Real-valued Functions

- Input data:
  - $f: X \rightarrow R$

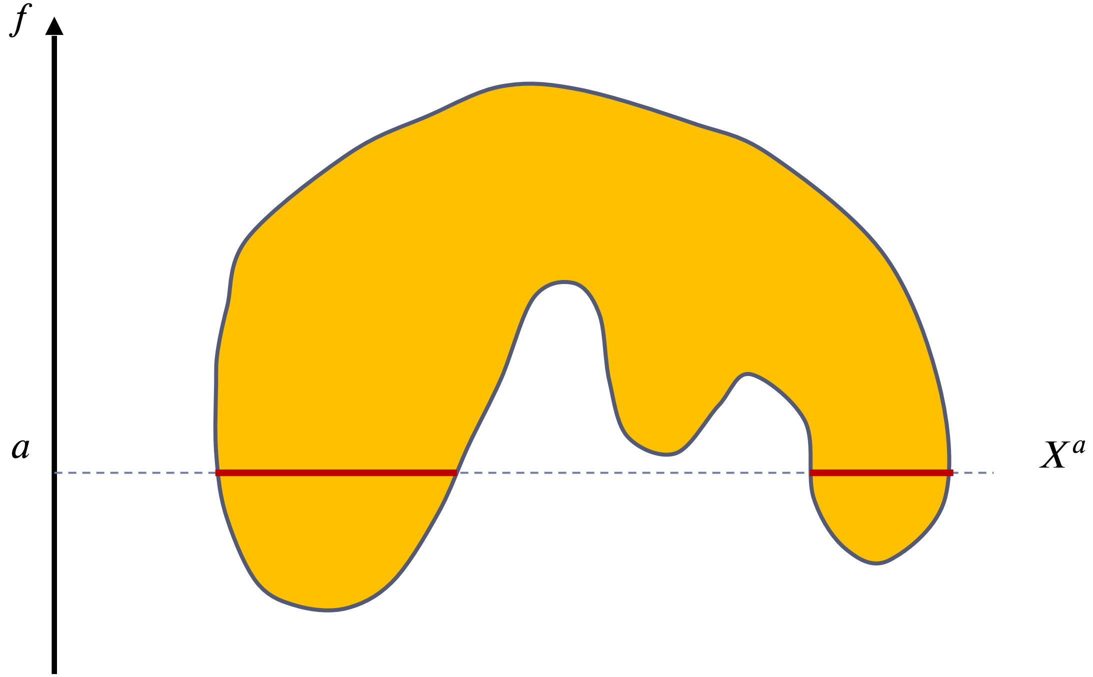






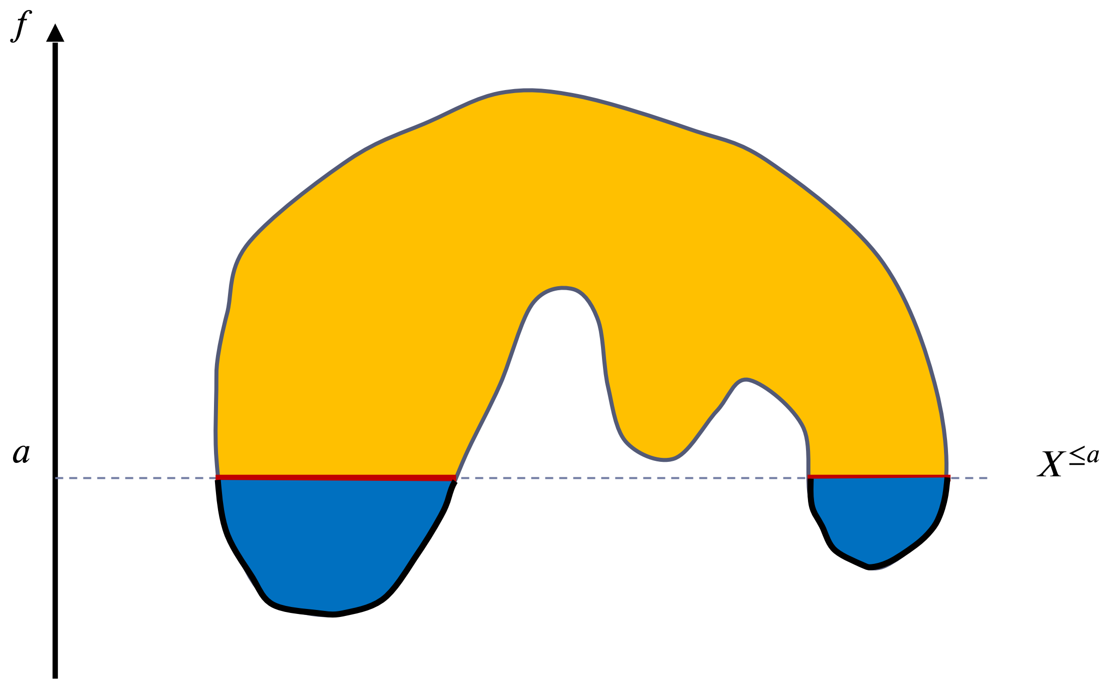
Notations

- Function: $f: X \to R$
- Level set: $X^a = \{x \in X \mid f(x) = a\}$,
- Sub-level set: $X^{\leq a} = \{x \in X \mid f(x) \leq a\}$
  - $X^{\leq a} \subseteq X^{\leq b}$ for any $a \leq b$



Notations

- Function: $f: X \to R$
- Level set: $X^a = \{x \in X \mid f(x) = a\}$,
- Sub-level set: $X^{\leq a} = \{x \in X \mid f(x) \leq a\}$
  - $X^{\leq a} \subseteq X^{\leq b}$ for any $a \leq b$



Function-induced Filtration

- Function: $f: X \to R$
- Level set: $X^a = \left\{x \in X \mid f(x) = a\right\}$,
- Sub-level set: $X^{\leq a} = \{x \in X \mid f(x) \leq a\}$
  - $X^{\leq a} \subseteq X^{\leq b}$ for any $a \leq b$
- For any sequence $a_1 \leq a_2 \leq \cdots \leq a_n$ (with $a_n \geq f_{max}$)
  - Sublevel set filtration of X w.r.t f:
  - $X^{\leq a_1} \subseteq X^{\leq a_2} \subseteq \cdots \subseteq X^{\leq a_n}$

Function-induced Filtration

- Function: $f: X \to R$
- Level set: $X^a = \{x \in X \mid f(x) = a\}$,
- Sub-level set: $X^{\leq a} = \{x \in X \mid f(x) \leq a\}$
  - $X^{\leq a} \subseteq X^{\leq b}$ for any $a \leq b$
- For any sequence $a_1 \leq a_2 \leq \cdots \leq a_n$ (with $a_n \geq f_{max}$)
  - Sublevel set filtration of X w.r.t f:
  - $X^{\leq a_1} \subseteq X^{\leq a_2} \subseteq \cdots \subseteq X^{\leq a_n}$
- Persistence module
  - $H_*(X^{\leq a_1}) \to H_*\left(X^{\leq a_2}\right) \to \cdots \to H_*(X^{\leq a_n})$

Function-induced Filtration

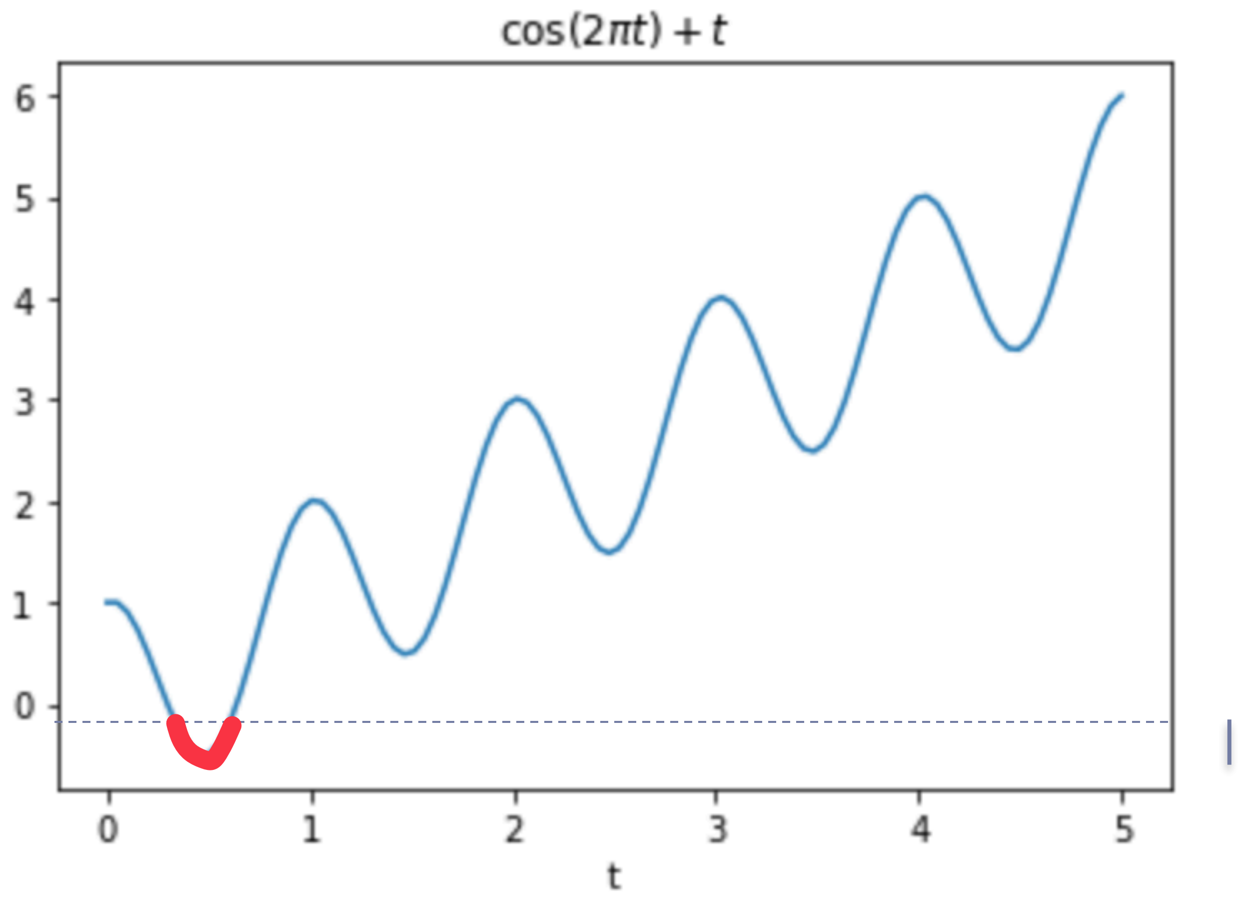
- Function: $f: X \to R$
- Level set: $X^a = \{x \in X \mid f(x) = a\}$,
- Sub-level set: $X^{\leq a} = \{x \in X \mid f(x) \leq a\}$
  - $X^{\leq a} \subseteq X^{\leq b}$ for any $a \leq b$
- For any sequence $a_1 \leq a_2 \leq \cdots \leq a_n$ (with $a_n \geq f_{max}$)
  - Sublevel set filtration of X w.r.t f:
  - $X^{\leq a_1} \subseteq X^{\leq a_2} \subseteq \cdots \subseteq X^{\leq a_n}$
- Persistence module
  - $H_*(X^{\leq a_1}) \to H_*\left(X^{\leq a_2}\right) \to \cdots \to H_*(X^{\leq a_n})$
- General persistence module indexed by real
  - $\mathscr{P}_f = \left\{ H_*\left(X^{\leq a}\right) \to H_*\left(X^{\leq b}\right) \right\}_{a \leq b}$

Thickening is a sub-level set filtration

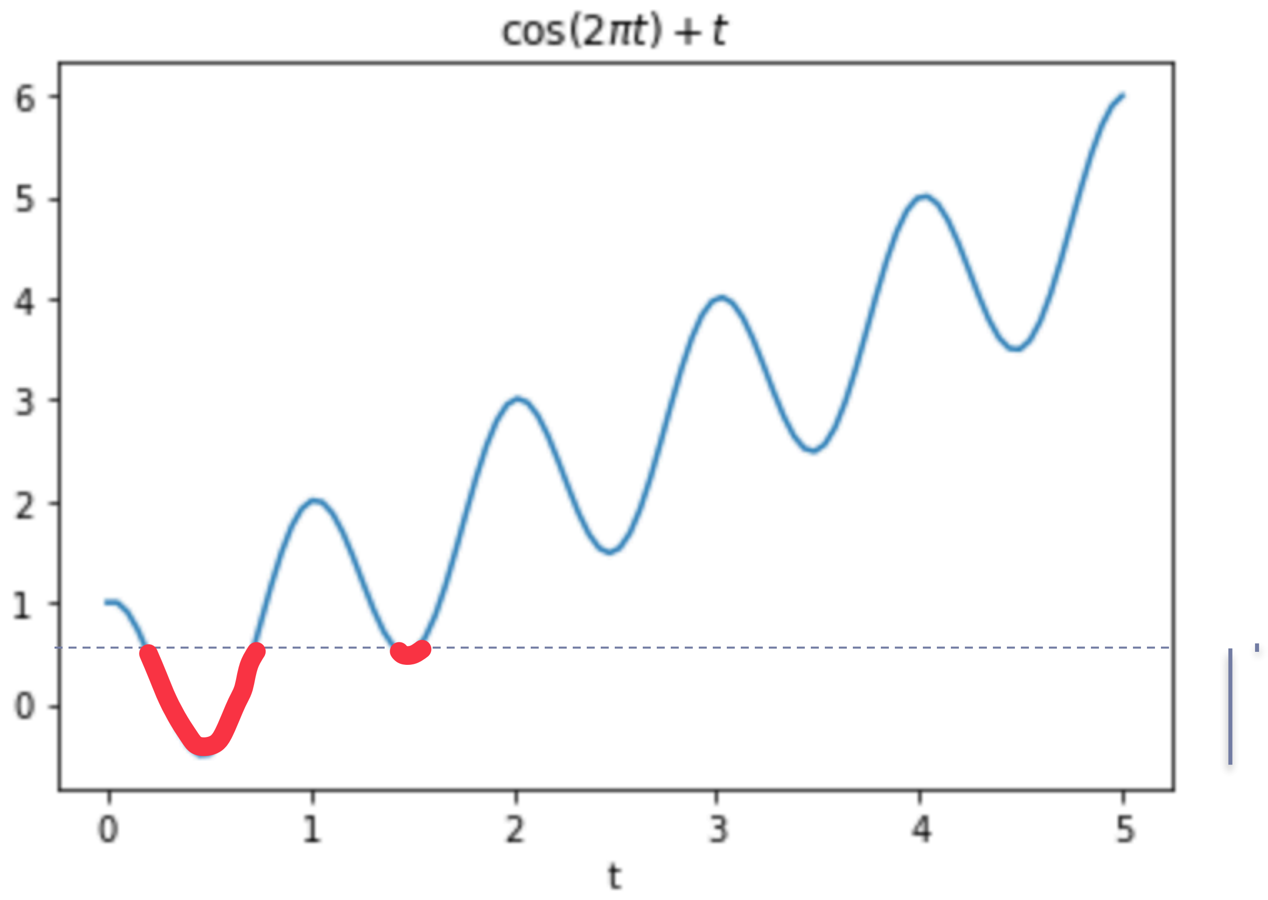
- Given a compact set $X \subset \mathbb{R}^d$
- Function: $f : \mathbb{R}^d \to \mathbb{R}$ defined by $f(x) := d(x, X)$
- Sub-level set:

  $(\mathbb{R}^d)^{\leq a} = \{x \in \mathbb{R}^d : f(x) \leq a\} = \{x \in \mathbb{R}^d : d(x, X) \leq a\} = X^a$

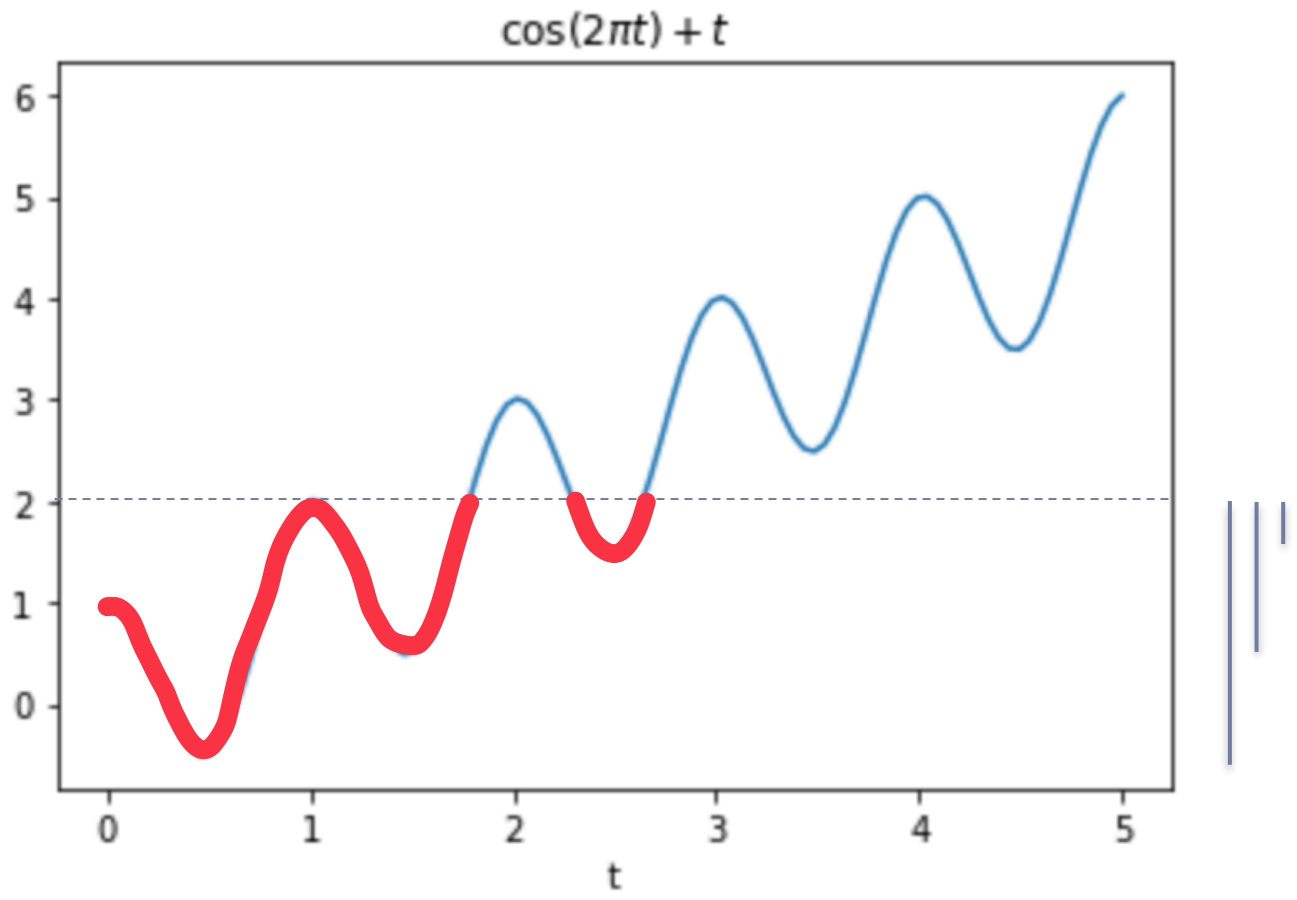
1D example



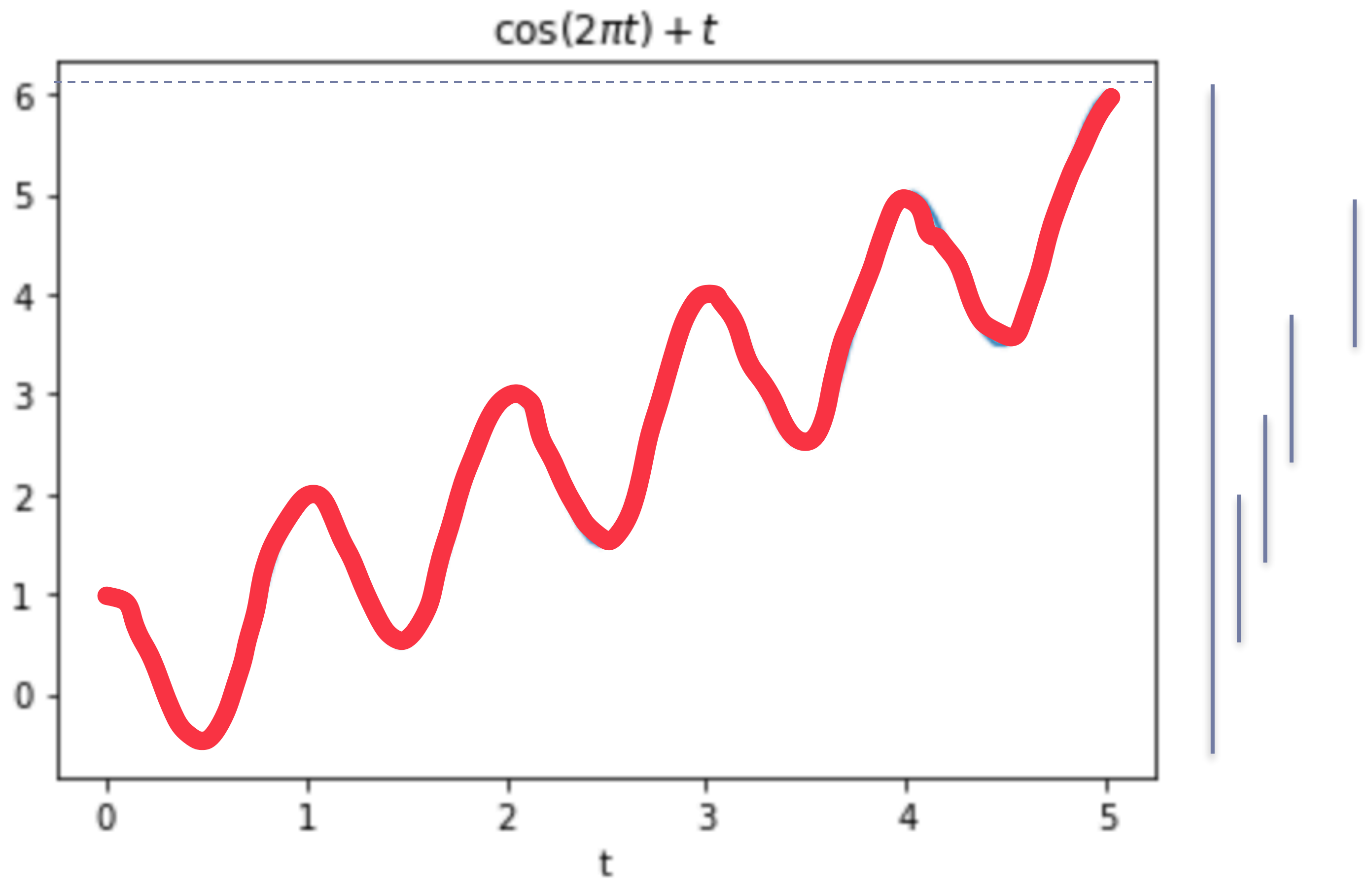
1D example



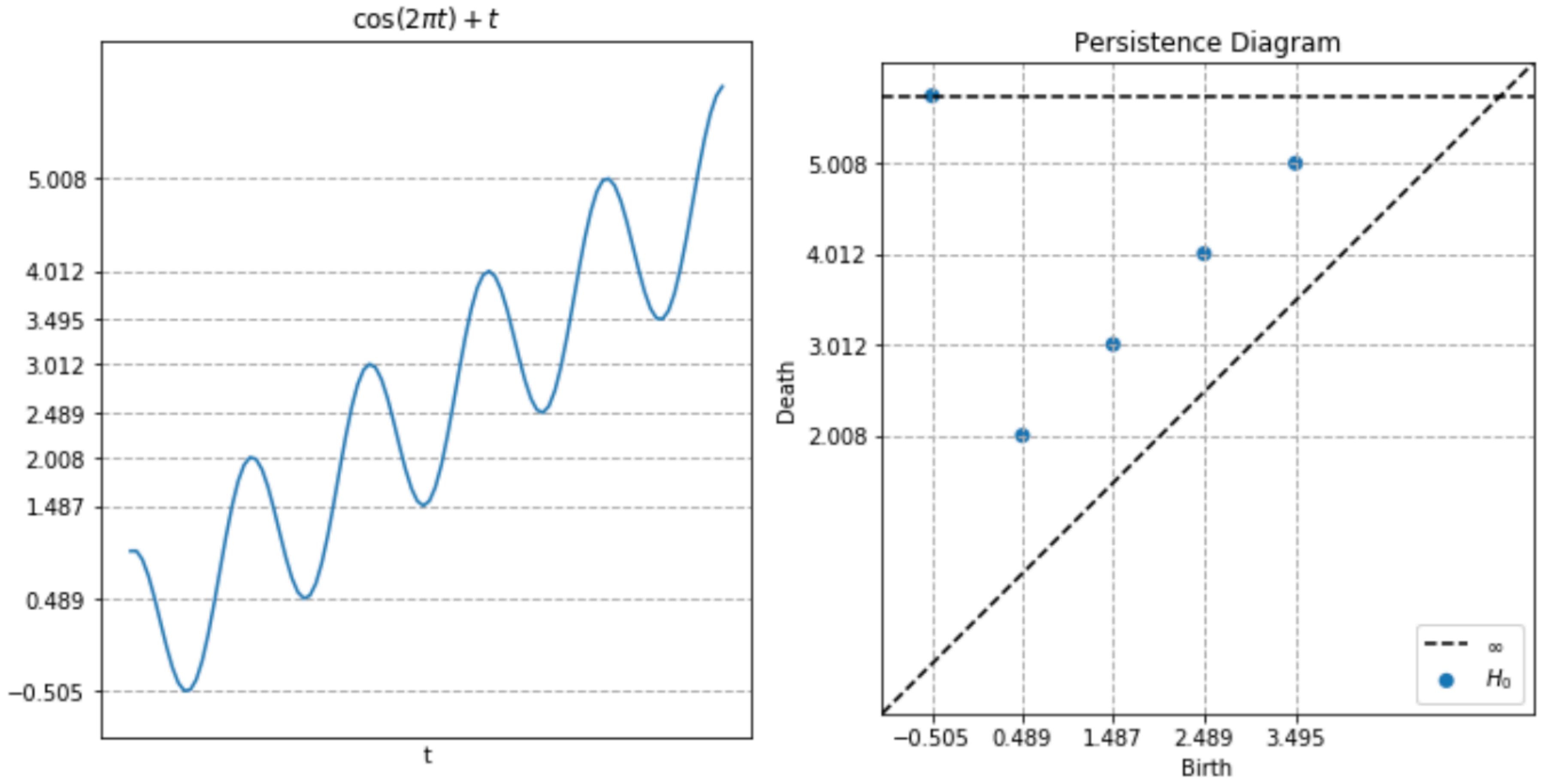
1D example



1D example

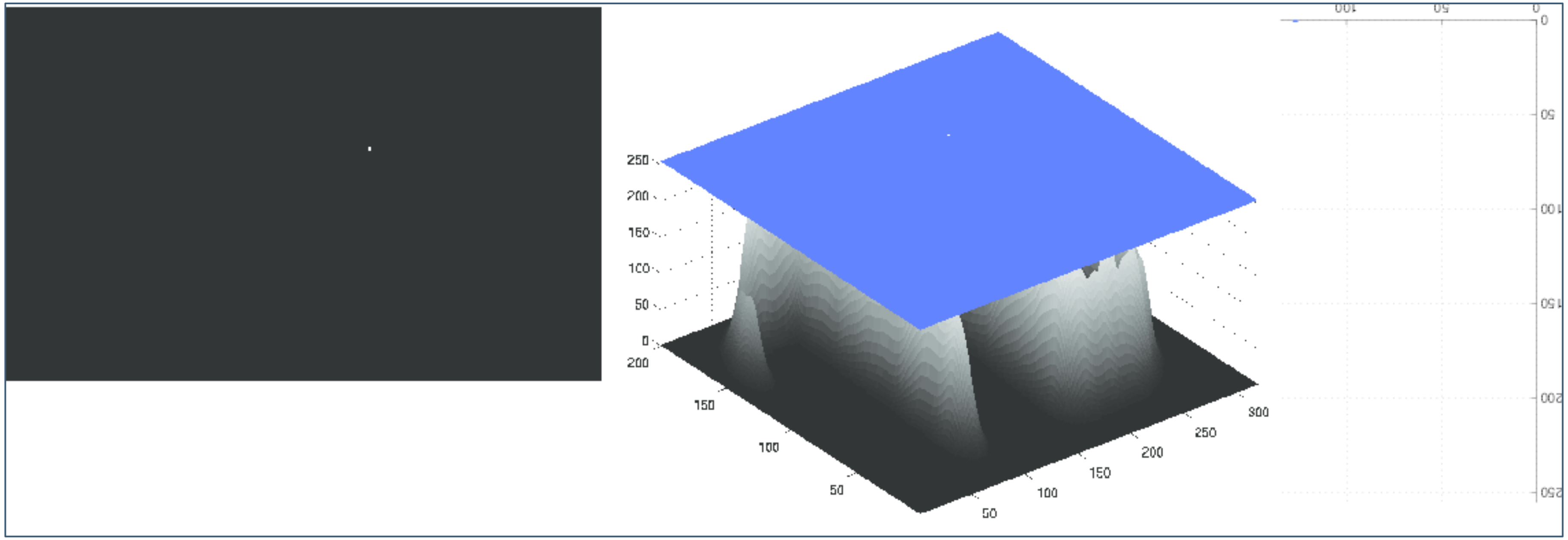


1D example

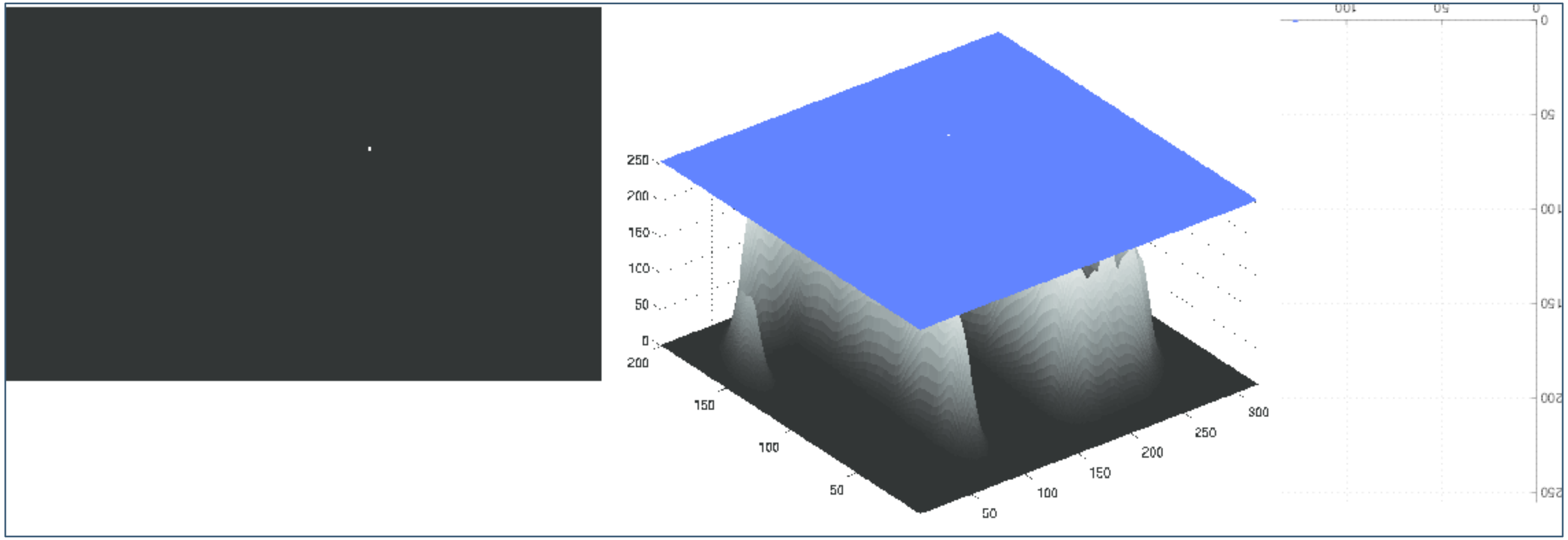


See tutorial from [scikit-tda]

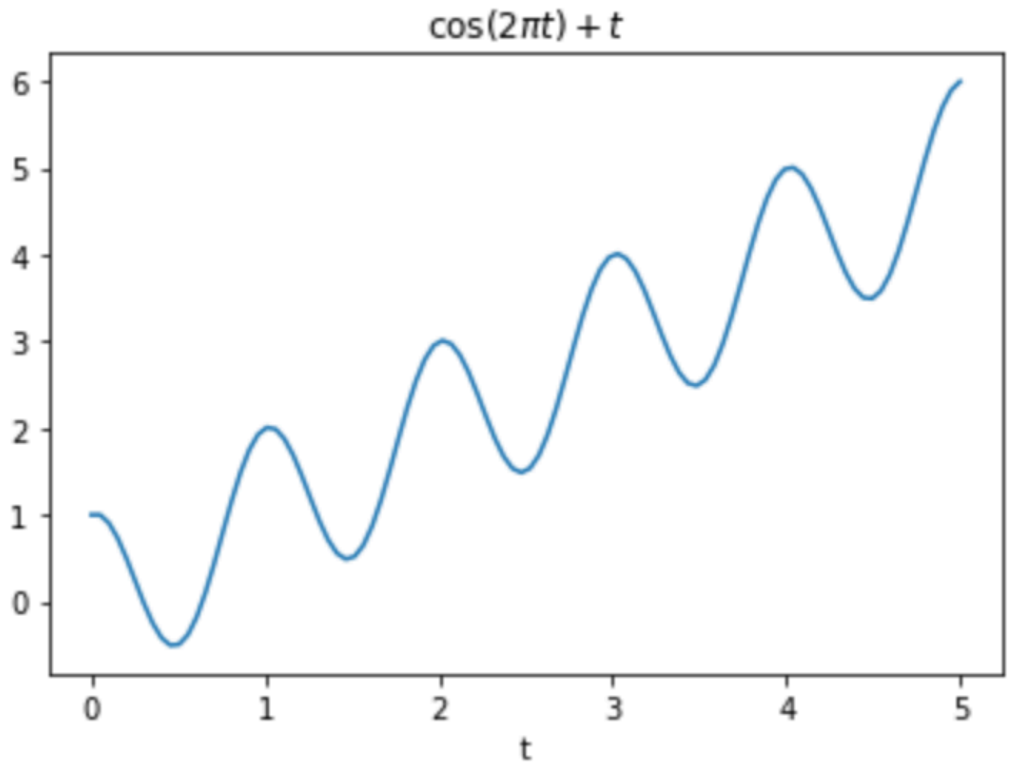
- An example on images
  - By Chao Chen (Stony Brook Univ)

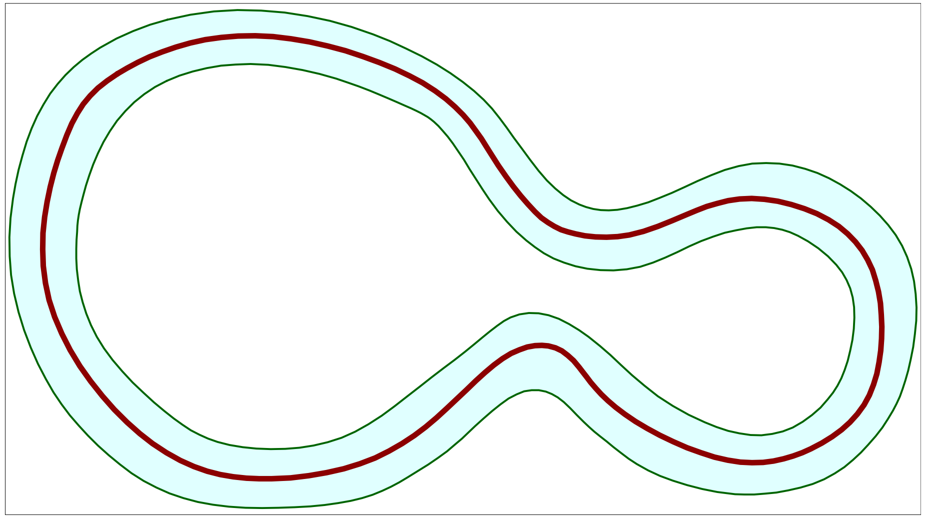


- An example on images
  - By Chao Chen (Stony Brook Univ)



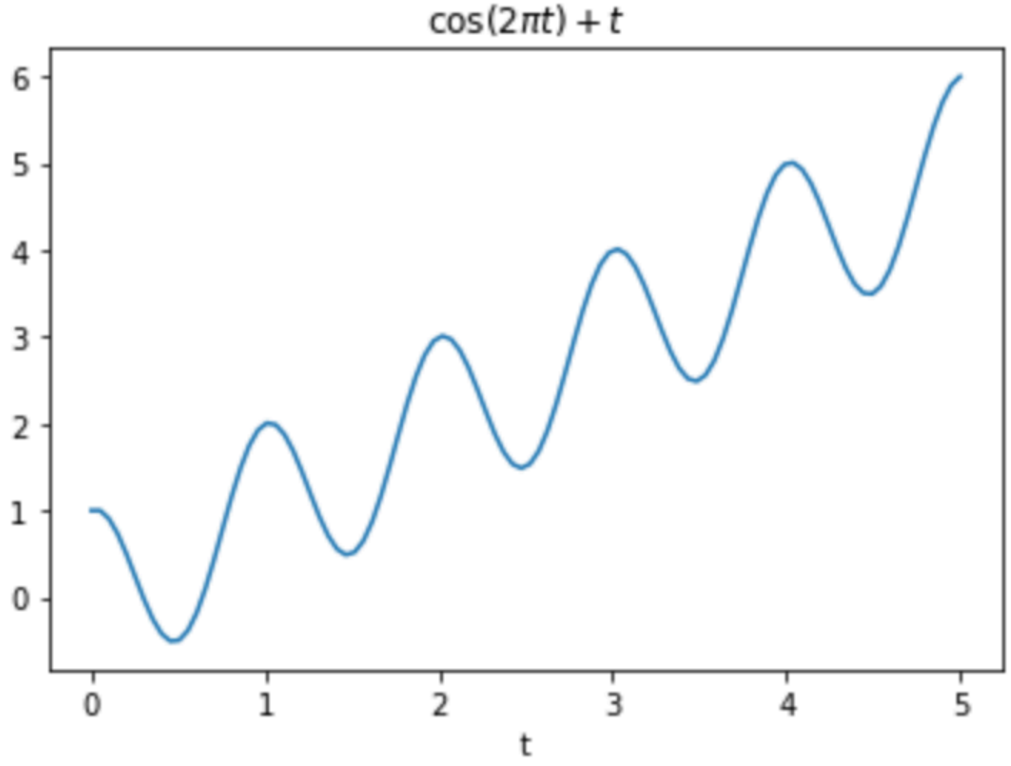
Observations

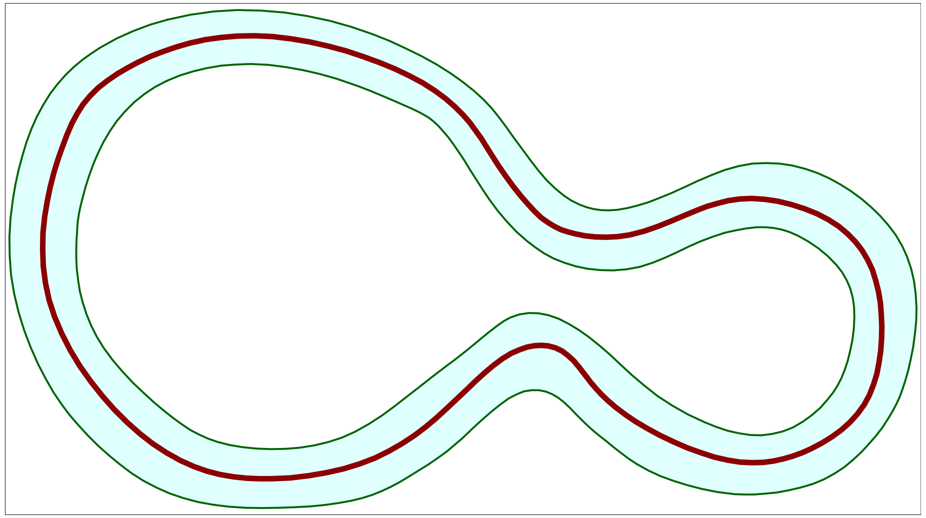




Observations

- Persistence Diagrams of Sublevel set filtration is tracking the shape of the graph of $f : X \to \mathbb{R}$, i.e., $\{(x, f(x)) : x \in X\}$





Observations

Observations

- PDs of Sublevel set filtration is tracking the shape of the graph of $f : X \to \mathbb{R}$

Observations

- PDs of Sublevel set filtration is tracking the shape of the graph of $f : X \to \mathbb{R}$
- $H_*(a)$ only potentially changes at *critical values*

Observations

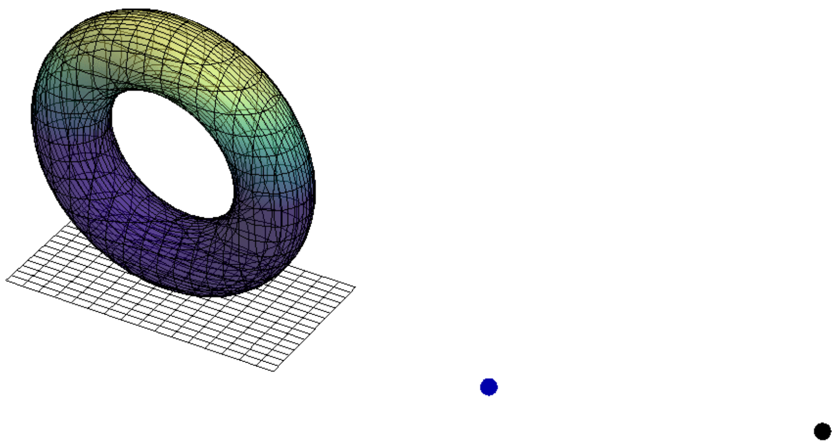
- PDs of Sublevel set filtration is tracking the shape of the graph of $f : X \to \mathbb{R}$
- $H_*(a)$ only potentially changes at *critical values*
  - which, in the case of f is a smooth function defined on a smooth manifold, are exactly the function values of critical points of f

Observations

- PDs of Sublevel set filtration is tracking the shape of the graph of $f : X \to \mathbb{R}$
- $H_*(a)$ only potentially changes at *critical values*
    - which, in the case of f is a smooth function defined on a smooth manifold, are exactly the function values of critical points of f
  - $H_*(a) \to H_*(b)$ is isomorphism for $a < b$ within two consecutive critical values

Observations

- PDs of Sublevel set filtration is tracking the shape of the graph of $f : X \to \mathbb{R}$
- $H_*(a)$ only potentially changes at *critical values*
  - which, in the case of f is a smooth function defined on a smooth manifold, are exactly the function values of critical points of f
  - $H_*(a) \to H_*(b)$ is isomorphism for $a < b$ within two consecutive critical values



https://www-sop.inria.fr/members/Mathijs.Wintraecken/Mathematica.html

Observations

Observations

- PDs of Sublevel set filtration is tracking the shape of the graph of $f : X \to \mathbb{R}$

Observations

- PDs of Sublevel set filtration is tracking the shape of the graph of $f : X \to \mathbb{R}$
- $H_*(a)$ only potentially changes at *critical values*

Observations

- PDs of Sublevel set filtration is tracking the shape of the graph of $f : X \to \mathbb{R}$
- $H_*(a)$ only potentially changes at *critical values*
- Persistence pairings for the sub-level set persistence module
$$\mathscr{P}_f = \left\{ H_*\left(X^{\leq a}\right) \to H_*\left(X^{\leq b}\right) \right\}_{a \leq b}$$

Observations

- PDs of Sublevel set filtration is tracking the shape of the graph of $f : X \to \mathbb{R}$
- $H_*(a)$ only potentially changes at *critical values*
- Persistence pairings for the sub-level set persistence module $\mathscr{P}_f = \left\{ H_*\left(X^{\leq a}\right) \to H_*\left(X^{\leq b}\right) \right\}_{a \leq b}$
  - are of the form $(b,\ d)$, with both b and d being critical values

Observations

- PDs of Sublevel set filtration is tracking the shape of the graph of $f : X \to \mathbb{R}$
- $H_*(a)$ only potentially changes at *critical values*
- Persistence pairings for the sub-level set persistence module $\mathscr{P}_f = \left\{ H_*\left(X^{\leq a}\right) \to H_*\left(X^{\leq b}\right) \right\}_{a \leq b}$
  - are of the form $(b, \ d)$, with both b and d being critical values
  - We sometimes also refer the pairing to between corresponding critical points.

Observations

- PDs of Sublevel set filtration is tracking the shape of the graph of $f : X \to \mathbb{R}$
- $H_*(a)$ only potentially changes at *critical values*
- Persistence pairings for the sub-level set persistence module
  $\mathscr{P}_f = \left\{ H_*\left(X^{\leq a}\right) \to H_*\left(X^{\leq b}\right) \right\}_{a \leq b}$
  - are of the form $(b,\ d)$, with both b and d being critical values
  - We sometimes also refer the pairing to between corresponding critical points.
  - Pairs of the form $(a,\ \infty)$

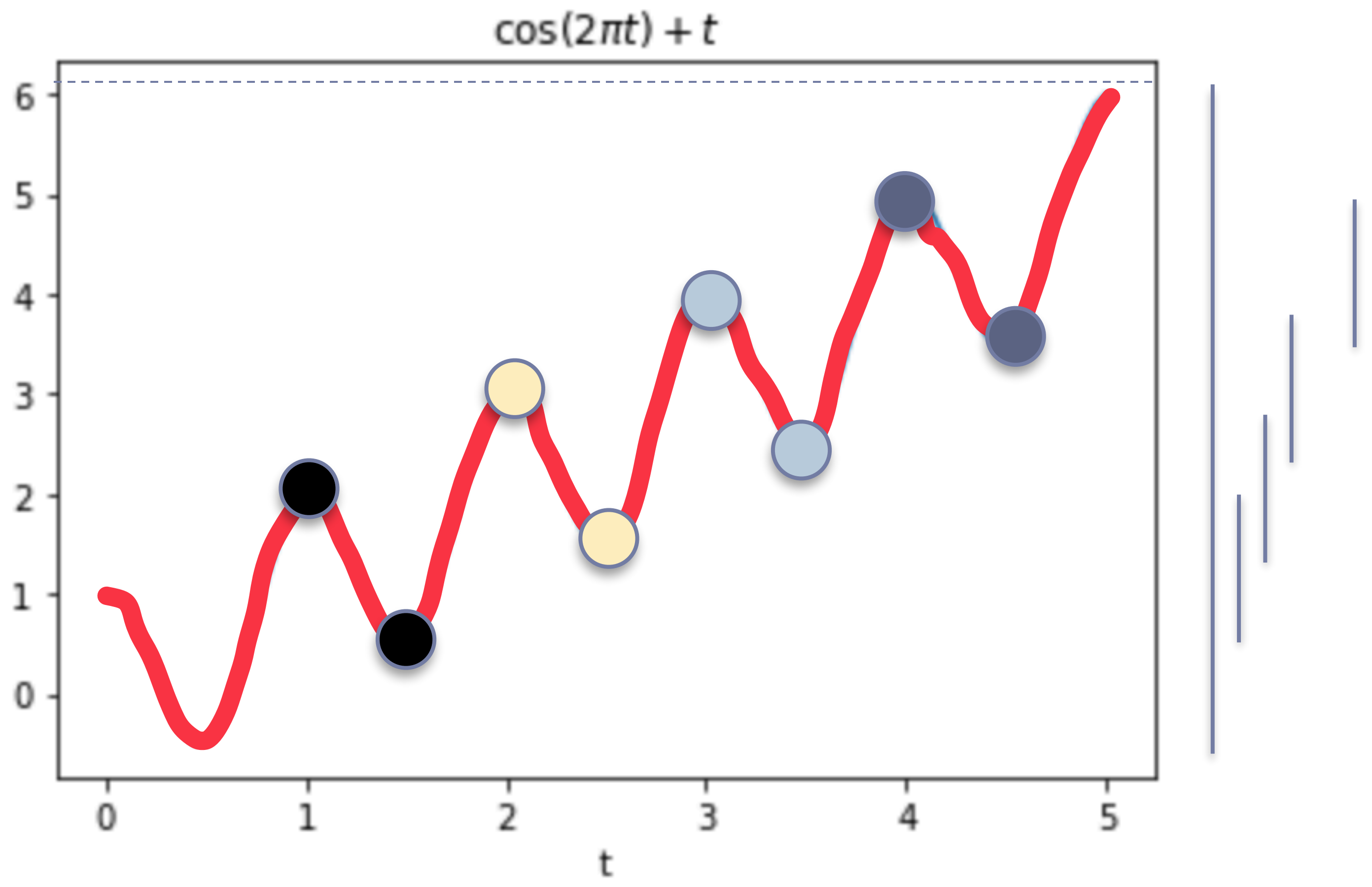
Observations

- PDs of Sublevel set filtration is tracking the shape of the graph of $f : X \to \mathbb{R}$
- $H_*(a)$ only potentially changes at *critical values*
- Persistence pairings for the sub-level set persistence module $\mathscr{P}_f = \left\{ H_*\left(X^{\leq a}\right) \to H_*\left(X^{\leq b}\right) \right\}_{a \leq b}$
  - are of the form $(b,\ d)$, with both b and d being critical values
  - We sometimes also refer the pairing to between corresponding critical points.
  - Pairs of the form $(a,\ \infty)$
    - Correspond to homology of X, i.e, $\mathrm{H}_*(X)$

Observations

- PDs of Sublevel set filtration is tracking the shape of the graph of $f : X \to \mathbb{R}$
- $H_*(a)$ only potentially changes at *critical values*
- Persistence pairings for the sub-level set persistence module
  $\mathscr{P}_f = \left\{ H_*\left(X^{\leq a}\right) \to H_*\left(X^{\leq b}\right) \right\}_{a \leq b}$
  - are of the form $(b, \; d)$, with both b and d being critical values
  - We sometimes also refer the pairing to between corresponding critical points.
  - Pairs of the form $(a, \; \infty)$
    - Correspond to homology of X, i.e, $\mathrm{H}_*(X)$
    - This is because $X^{\leq \infty} = X$

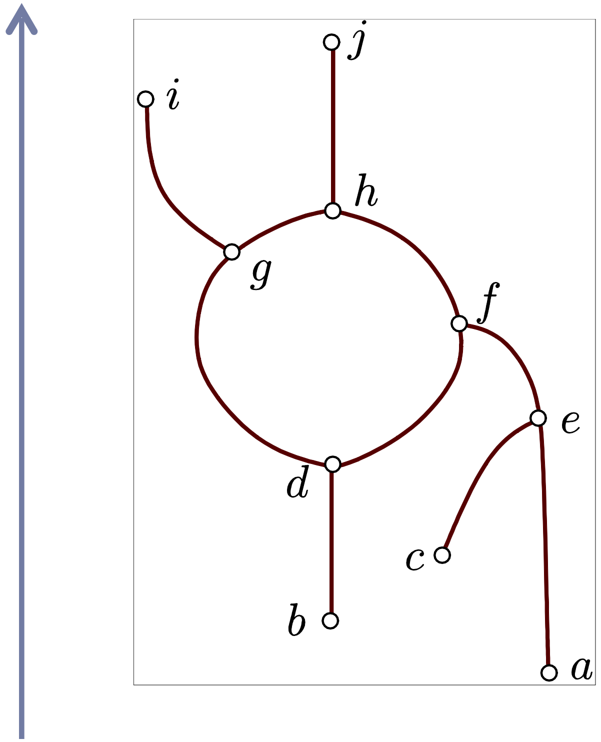
1D example

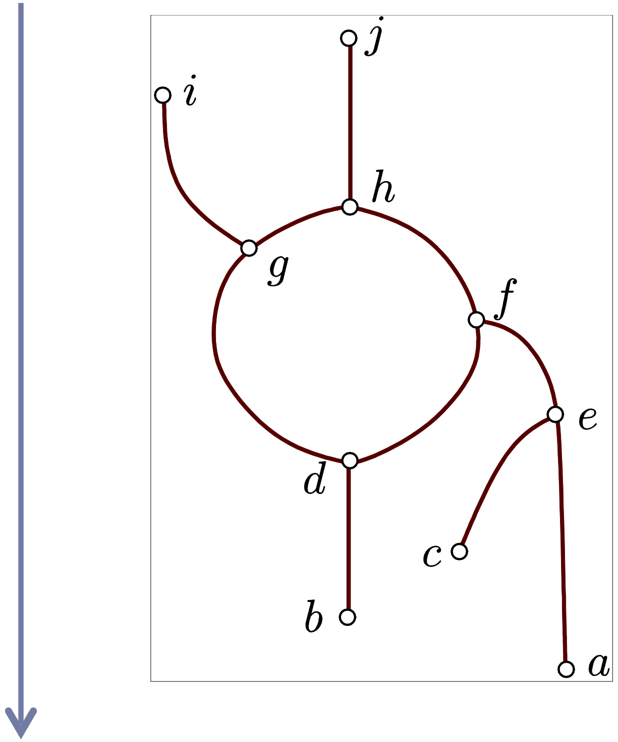


Remark: Super-level set filtration

- Symmetrically, one can also consider super-level set filtration
- Super-level set: $X^{\geq a} = \{x \in X \mid f(x) \geq a\}$
  - $X^{\geq a} \subseteq X^{\geq b}$ for any $a \geq b$
- For any sequence $a_1 \geq a_2 \cdots \geq a_n$ (s.t $a_1 \geq f_{max},\ a_n \leq f_{min}$)
  - Sublevel set filtration of X w.r.t f:
  - $X^{\geq a_1} \subseteq X^{\geq a_2} \subseteq \cdots \subseteq X^{\geq a_n}$
- Persistence module
  - $H_*(X^{\geq a_1}) \to H_*\left(X^{\geq a_2}\right) \to \cdots \to H_*(X^{\geq a_n})$
- General persistence module indexed by real
  - $\mathscr{P}_f = \left\{H_*\left(X^{\geq a}\right) \to H_*\left(X^{\geq b}\right)\right\}_{a \geq b}$

One filtration may not be enough



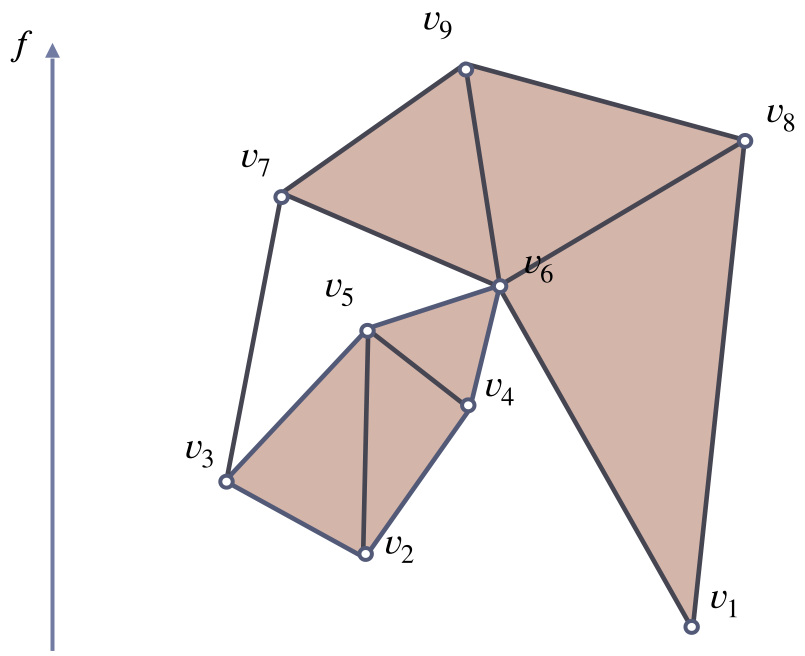


Section 3 :
Computation in the PL-case

- We cannot deal with smooth manifolds or smooth functions directly
  - Triangulate smooth manifolds to be simplicial complexes
  - Approximate smooth functions by piecewise linear functions

Computation – PL Function

- K: a simplicial complex, $|K|$ its underlying space (e.g. a triangulation of a manifold)
- Piecewise linear (PL) function $f: |K| \to R$
  - f defined at vertices (0-simplices) V of K and linearly interpolated within each simplex $\sigma \in K$



Computation – PL Function

- Given PL-function $f : \left| K \right| \to R,$ consider the persistence module induced by its sub-level set filtration
  - $\mathscr{P}_f = \left\{ H_*(|K|^{\leq a}) \to H_*(|K|^{\leq b}) \right\}_{a \leq b}$
- $\{ |K|^{\leq a} \subset |K|^{\leq b} \}$ is still a filtration of topological spaces
- To compute persistence pairings for $\mathscr{P}_f,$ we want to simulate sub-level set filtration by a **filtered simplicial complex**

Lower Star filtration

Lower Star filtration

Lower Star filtration

- Assume vertices $\{v_1, \ldots, v_n\}$ sorted in non-decreasing order by function value f

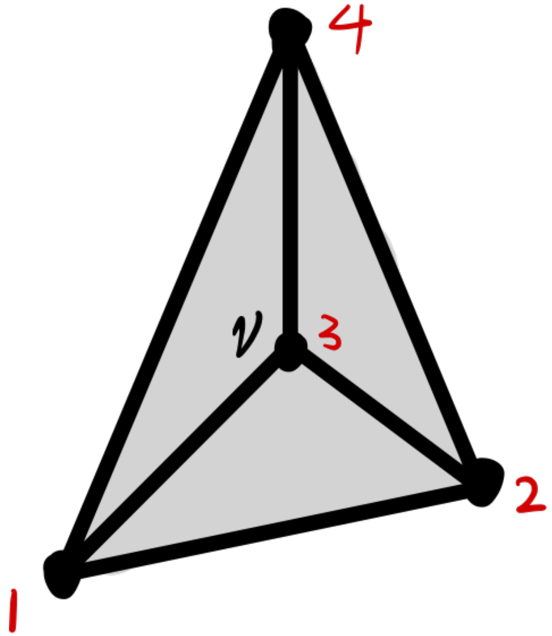
Lower Star filtration

- Assume vertices $\{v_1, \ldots, v_n\}$ sorted in non-decreasing order by function value f
- Consider discrete values $a_1 \leq \cdots \leq a_n$ with $a_i = f\left(v_i\right)$
    - $K_i := \{\sigma \in K \mid f(v) \leq a_i, \ \forall v \in \sigma\}$
    - Consider filtration $\varnothing = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$

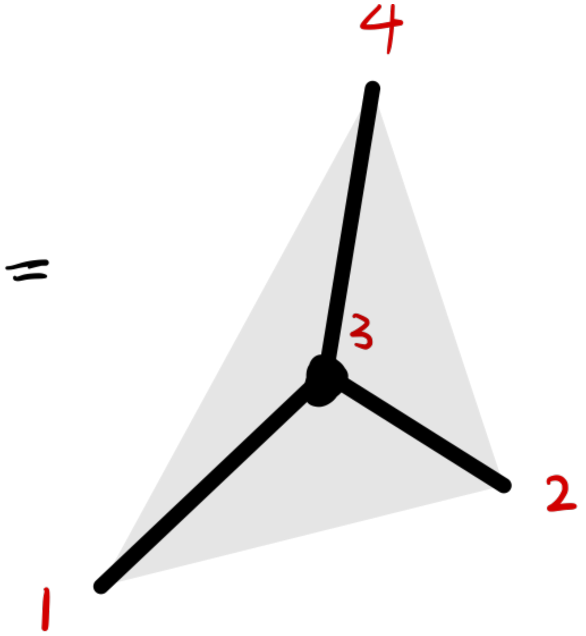
Lower Star filtration

- Assume vertices $\{v_1, \ldots, v_n\}$ sorted in non-decreasing order by function value f
- Consider discrete values $a_1 \leq \cdots \leq a_n$ with $a_i = f(v_i)$
  - $K_i := \{\sigma \in K \mid f(v) \leq a_i, \ \forall v \in \sigma\}$
  - Consider filtration $\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$
- Called lower star filtration as
  - $LowSt\ (v_i) = K_i \setminus K_{i-1}$
    - where $LowSt\ (v) := \{\sigma \in K \mid v \in \sigma\ and\ f(u) \leq f(v)\ for\ any\ u \in \sigma\}$
  - $K_i = \bigcup_{j \leq i} LowSt(v_j)$

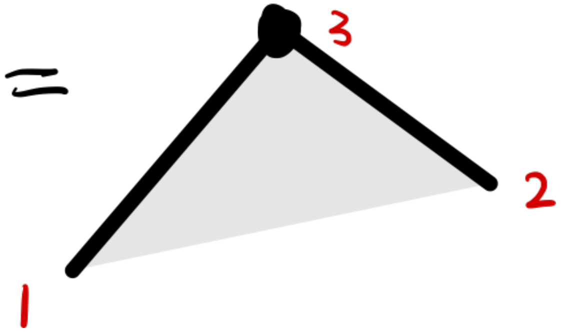
Lower Star vs Star



$St(v) =$

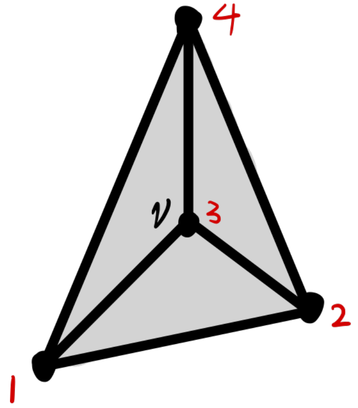


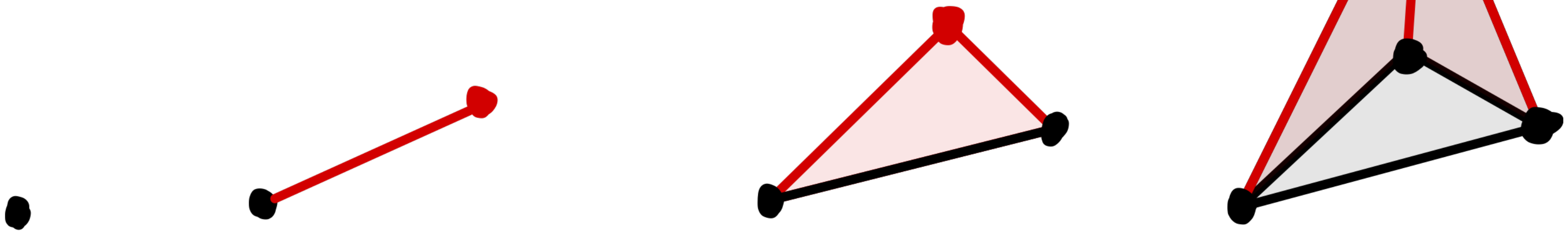
$LowSt(v) =$



Lower Star filtration

- Consider discrete values $a_1 \leq \cdots \leq a_n$ with $a_i = f\left(v_i\right)$
  - $K_i := \{\sigma \in K \mid f(v) \leq a_i,\ \forall v \in \sigma\}$
  - Consider filtration $\varnothing = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$





Computation – PL Function

- Goal: persistence pairings for
  - $\mathcal{P}_f = \left\{ H_*\left(|K|^{\leq a}\right) \to H_*\left(|K|^{\leq b}\right) \right\}_{a \leq b}$

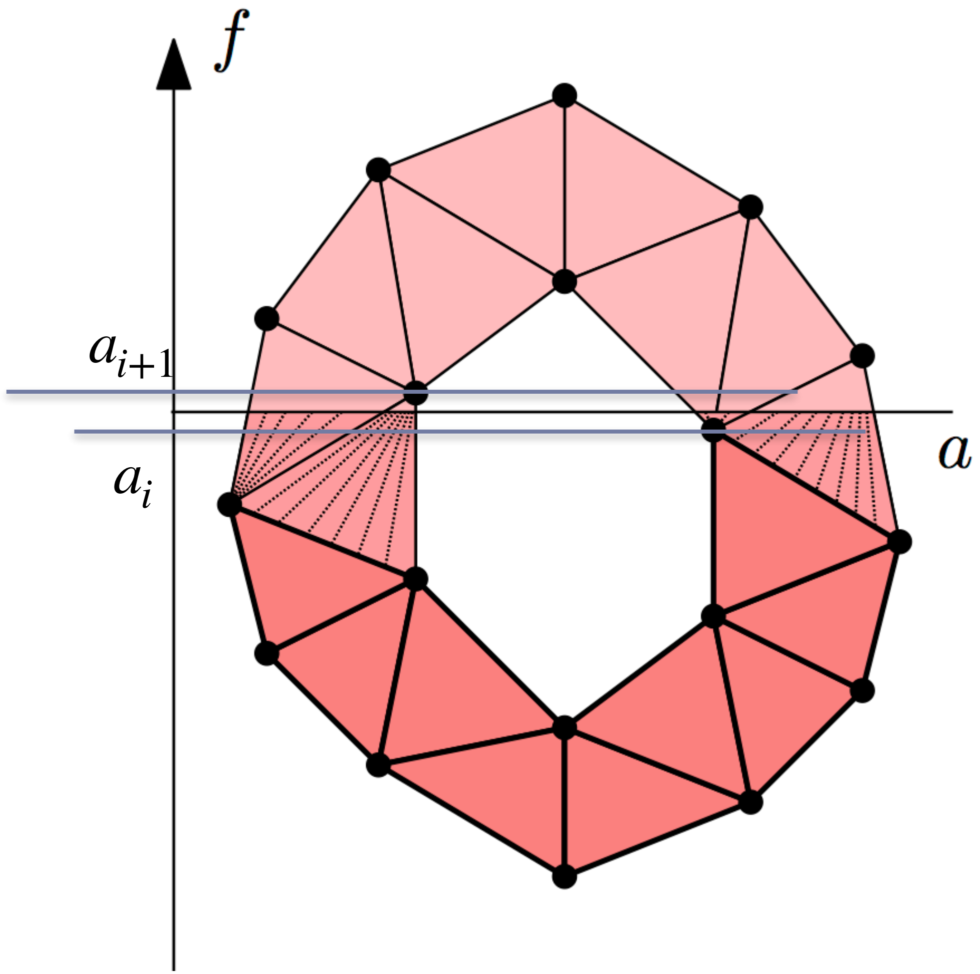
- Simulate sub-level set filtration by ***lower star filtration***
  - Assume vertices $\{v_1, \ldots, v_n\}$ sorted in non-decreasing order by function value f
  - Consider discrete values $a_1 \leq \cdots \leq a_n$ with $a_i = f\left(v_i\right)$
  - $K_i := \{\sigma \in K \mid f(v) \leq a_i, \ \forall v \in \sigma\}$
  - Consider filtration $\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$

Computation – PL Function

- Goal: persistence pairings for
  - $\mathscr{P}_f = \left\{ H_*\left(|K|^{\leq a}\right) \to H_*\left(|K|^{\leq b}\right) \right\}_{a \leq b}$
- Simulate sub-level set filtration by *lower star filtration*
  - $\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$
    - where $K_i \coloneqq \left\{\sigma \in K \mid f(v) \leq a_i,\ \forall v \in \sigma\right\}$
  - $\Rightarrow H_*\left(K_0\right) \to H_*\left(K_1\right) \to H_*\left(K_2\right) \to \cdots \to H_*\left(K_n\right)$

Sub-level set vs Lower Star filtrations

- For any a, if $a_i \leq a < a_{i+1}$, then
- $|K|^{\leq a} \simeq K_i$

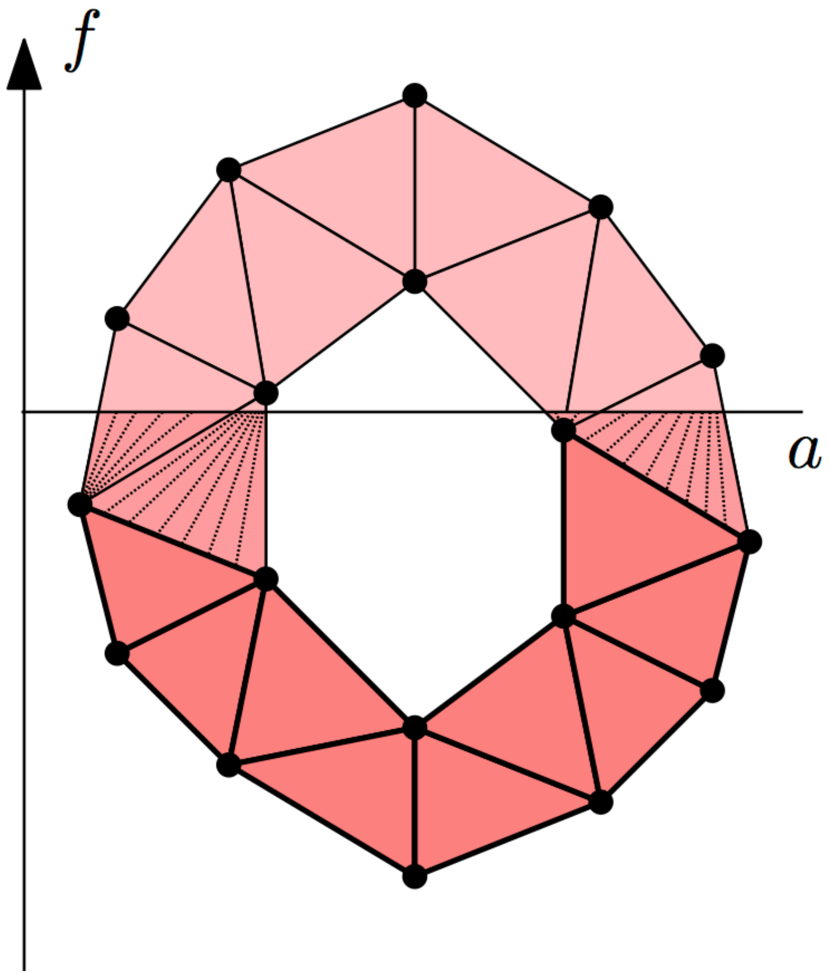


Sub-level set vs Lower Star filtrations

- Lower star filtration
  - $\varnothing \subset K_1 \subset \cdots \subset K_n$
- Sub-level set filtration

  $\varnothing \subset |K|^{\leq a_1} \subset \cdots \subset |K|^{\leq a_n}$

- They induce **isomorphic** persistence modules

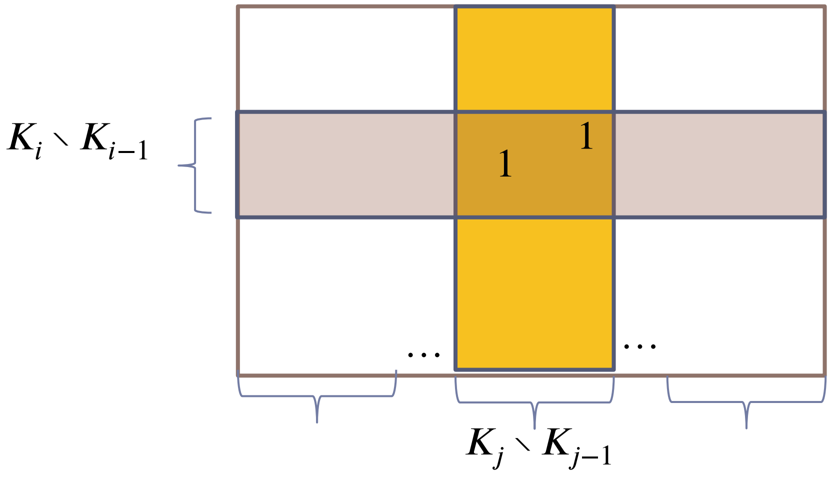


- A simplex-wise filtration *realizes* the lower star filtration if

$$\underbrace{\underbrace{\underbrace{\underbrace{\underbrace{\sigma_1, \dots, \sigma_{I_1}}_{Lst(v_1)}}_{K_1}, \underbrace{\sigma_{I_1+1}, \dots, \sigma_{I_2}}_{\{v_2\} \sqcup Lst(v_2)}}_{K_2}, \dots, \underbrace{\sigma_{I_{j-1}+1}, \dots, \sigma_{I_{j+1}}}_{\{v_j\} \sqcup Lst(v_j)}}_{K_i}, \dots, \underbrace{\sigma_{I_{n-1}+1}, \dots, \sigma_{I_n}}_{\{v_n\} \sqcup Lst(v_n)}}_{K_n}$$

PL-implementation

- Given a PL function $f: K \to R$, perform the persistence algorithm for any simplex-wise lower star filtration.
  - Let P denote the output set of paired simplices
  - Then, $\mu^{i,j} > 0$ if and only if there exists $(\sigma, \ \tau) \in P$ such that $\sigma \in K_i \setminus K_{i-1}$, while $\tau \in K_j \setminus K_{j-1}$
    - $\mu^{i,j}$ equals the cardinality of the set of such pairs.



$\boxed{\mu^{i,j} = 2}$

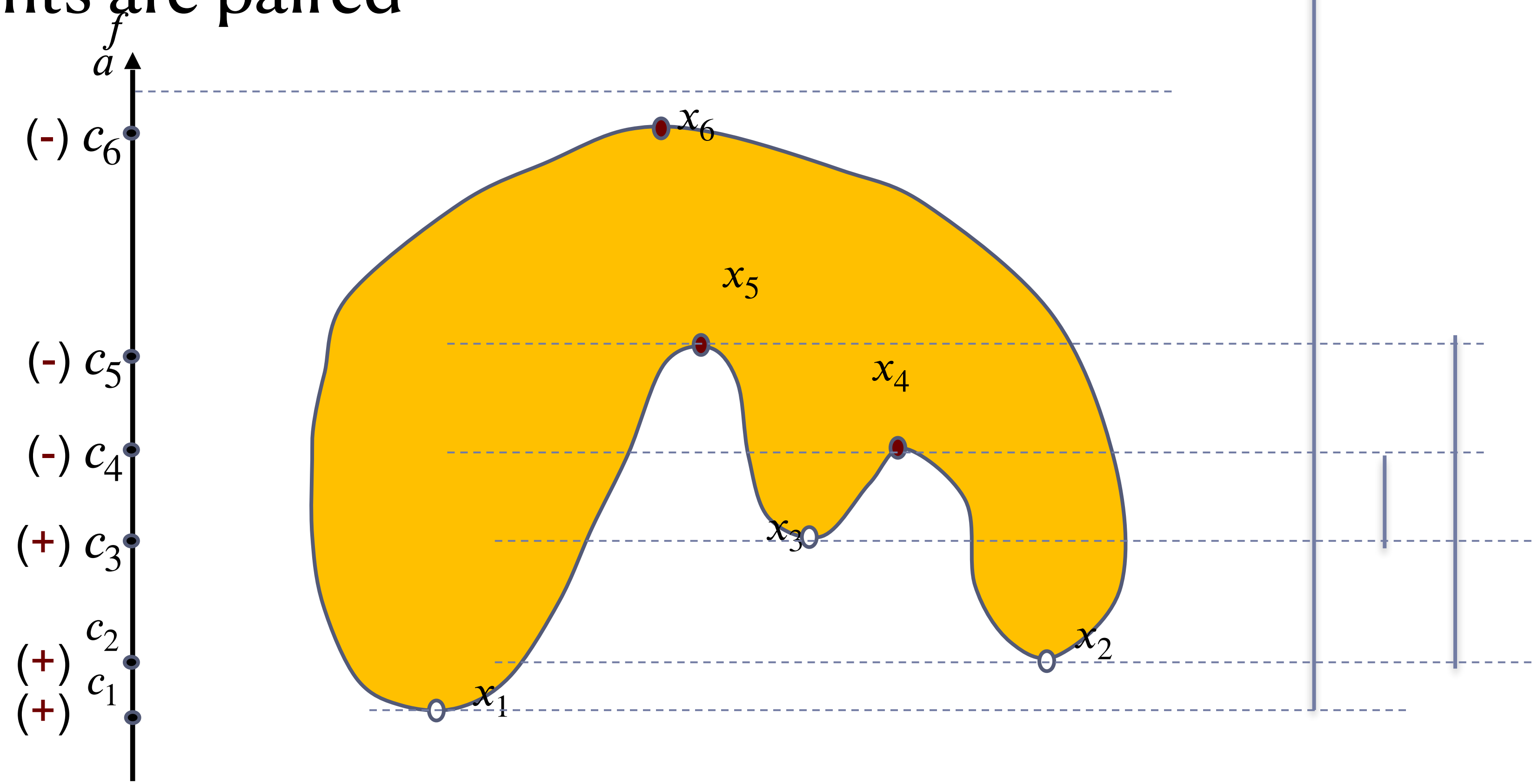
Some online tutorials

- Dionysus
- Scikit-TDA
- A Julia library
- ...

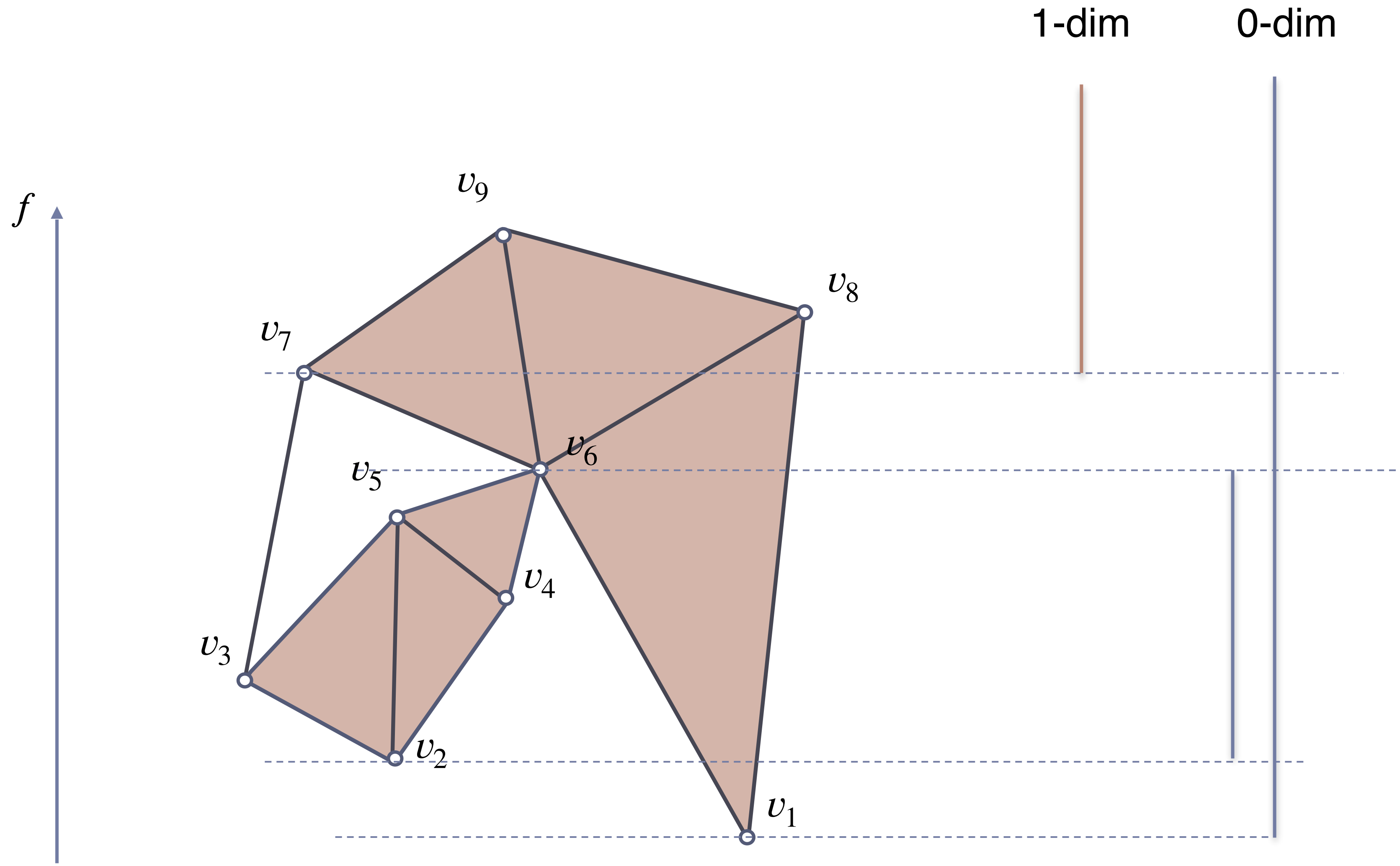
Critical points of PL functions

A Simple Example

- Critical points are paired



Example



- Are we paring any “critical points” for PL functions?

Gradients, critical points

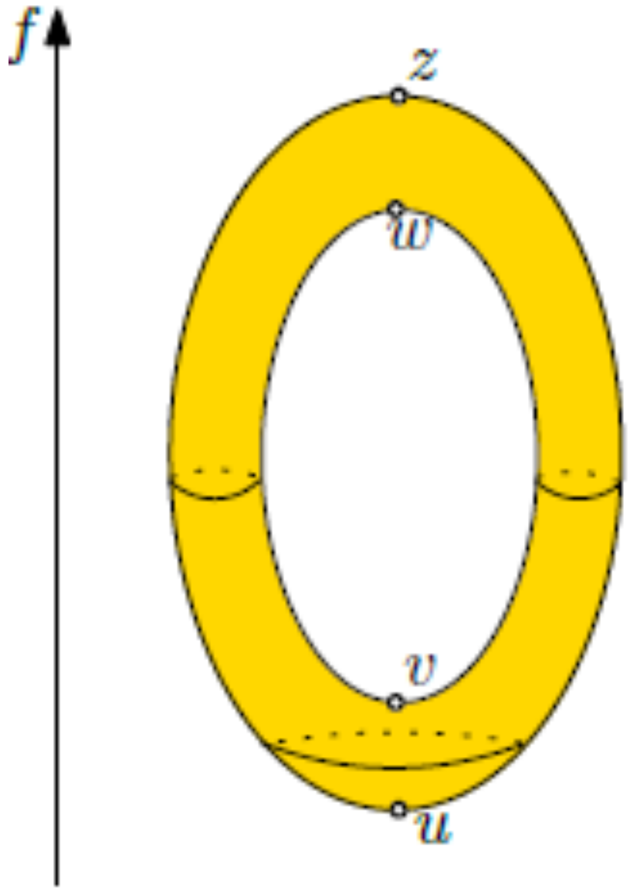
- d-manifold case: $f: M \to R$
- Same intuition, simply within a small neighborhood at each point

Definition 8 (Gradient vector field; Critical points). Given a smooth function $f : M \to \mathbb{R}$ defined on a smooth m-dimensional Riemannian manifold M, the *gradient vector field* $\nabla f : M \to TM$ is defined as follows: for any $x \in M$, let $(x_1, x_2, \ldots, x_m)$ be a local coordinate system in a neighborhood of x with orthonormal unit vectors x_i, the gradient at x is

$$\nabla f(x) = [\frac{\partial f}{\partial x_1}(x), \ \frac{\partial f}{\partial x_2}(x), \ \cdots \ \frac{\partial f}{\partial x_m}(x)]^T.$$

A point $x \in M$ is *critical* if $\nabla f(x)$ vanishes, in which case $f(x)$ is called a *critical value* for f. Otherwise, x is *regular*.

Example



(Non-)degenerate critical points

Definition 9 (Hessian matrix; Non-degenerate critical points)**.** Given a smooth m-manifold M, the *Hessian matrix* of a twice differentiable function $f : M \to \mathbb{R}$ at x is the matrix of second-order partial derivatives,

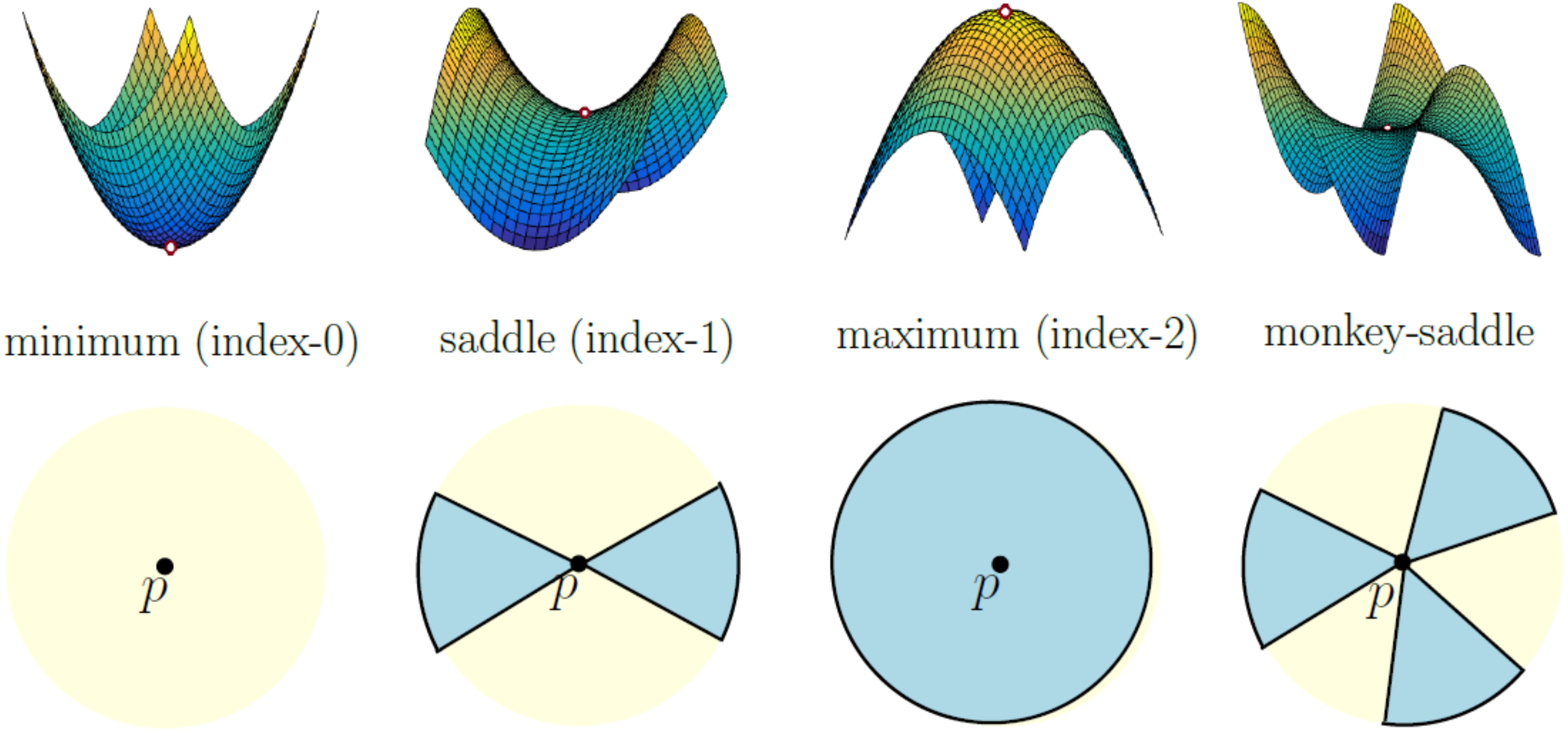
$$Hessian(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}2(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1}(x) & \frac{\partial^2 f}{\partial x_m \partial x_2}2(x) & \cdots & \frac{\partial^2 f}{\partial x_m \partial x_m}(x) \end{bmatrix},$$

where $(x_1, x_2, \ldots, x_m)$ is a local coordinate system in a neighborhood of x.

A critical point x of f is *non-degenerate* if its Hessian matrix $Hessian(x)$ is non-singular (has non-zero determinant); otherwise, it is a *degenerate critical point*.

Number of negative eigenvalues is called the **index** of x

Examples


minimum (index-0)
saddle (index-1)
maximum (index-2)
monkey-saddle
p
p
p
p

Morse Function

Morse Function

- A smooth function is a Morse function if
  - (1) all critical points have distinct function values
  - (2) there is no degenerate critical point.
- Morse functions have well-behaved critical points!

Morse Function

- A smooth function is a Morse function if
  - (1) all critical points have distinct function values
  - (2) there is no degenerate critical point.

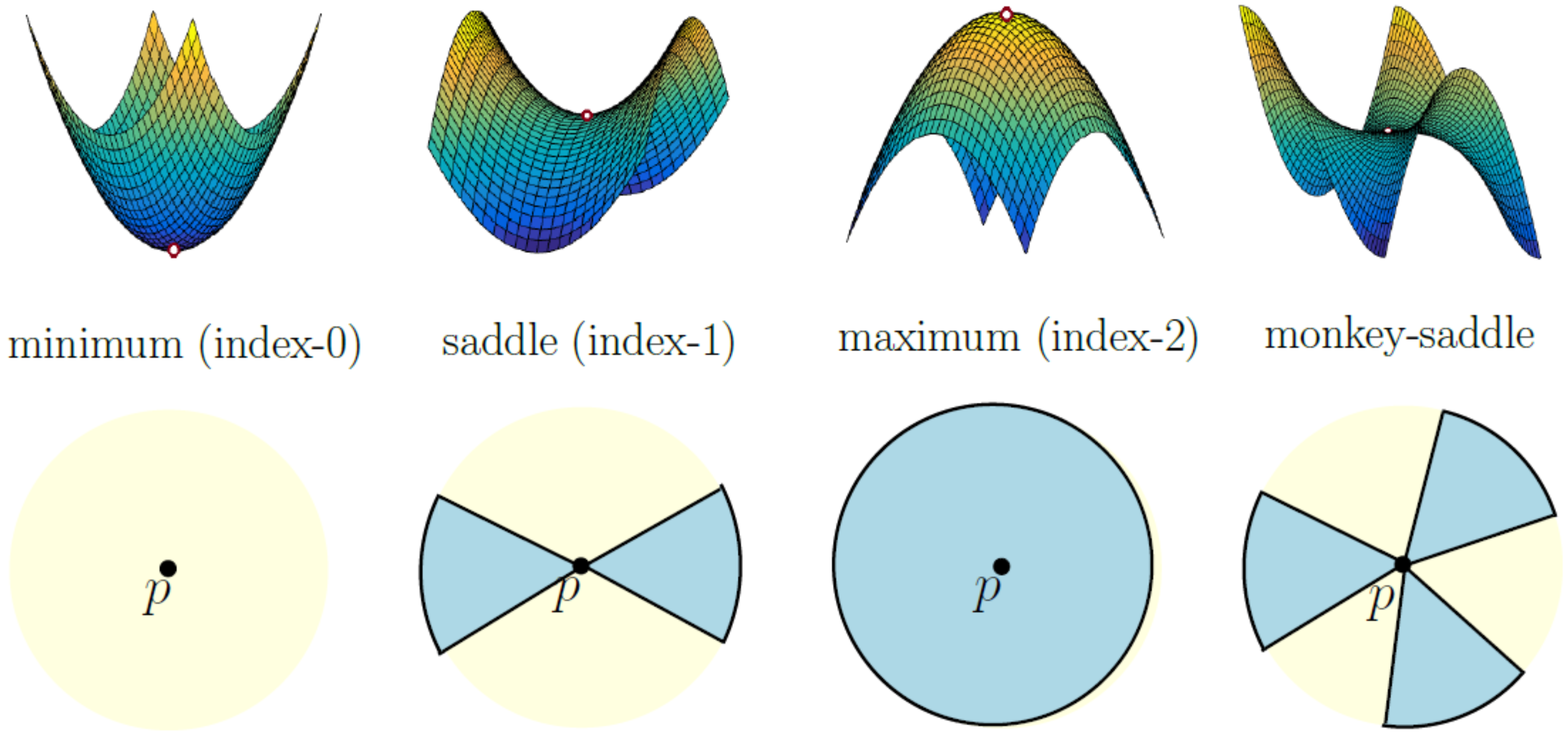
- Morse functions have well-behaved critical points!

Proposition 2 (Morse Lemma). *Given a smooth function $f : M \to \mathbb{R}$ defined on a smooth m-manifold M, let p be a non-degenerate critical point of f. Then there is a local coordinate system in a neighborhood $U(p)$ of p so that (i) the coordinate of p is $(0, 0, \ldots, 0)$, and (ii) locally for every point $x = (x_1, x_2 \ldots x_m)$ in neighborhood $U(p)$,*

$$f(x) = f(p) - x_1^2 - \ldots x_s^2 + x_{s+1}^2 \ldots x_m^2, \qquad \textit{for some} \;\; s \in [0, m].$$

The number s of minus signs in the above quadratic representation of $f(x)$ is called the index of the critical point p.

Examples



minimum (index-0)
saddle (index-1)
maximum (index-2)
monkey-saddle
p
p
p
p

Critical points and topology

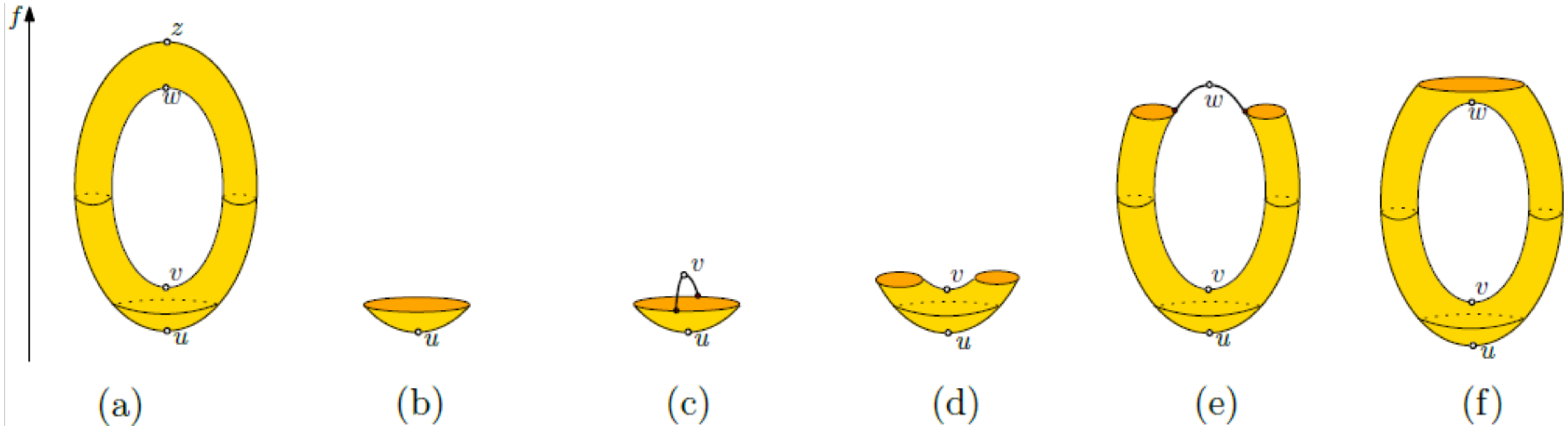
Theorem 3 (Homotopy type of sub-level sets)**.** *Let $f : M \to \mathbb{R}$ be a smooth function defined on a manifold M. Given $a < b$, suppose the interval-level set $M_{[a,b]} = f^{-1}([a,b])$ is compact and contains no critical points of f. Then $M_{\leq a}$ is diffeomorphic to $M_{\leq b}$.*

Furthermore, $M_{\leq a}$ is a deformation retract of $M_{\leq b}$, and the inclusion map $i : M_{\leq a} \hookrightarrow M_{\leq b}$ is a homotopy equivalence.

Critical points and topology

Theorem 3 (Homotopy type of sub-level sets). *Let $f : M \to \mathbb{R}$ be a smooth function defined on a manifold M. Given $a < b$, suppose the interval-level set $M_{[a,b]} = f^{-1}([a,b])$ is compact and contains no critical points of f. Then $M_{\le a}$ is diffeomorphic to $M_{\le b}$.*

Furthermore, $M_{\le a}$ is a deformation retract of $M_{\le b}$, and the inclusion map $i : M_{\le a} \hookrightarrow M_{\le b}$ is a homotopy equivalence.



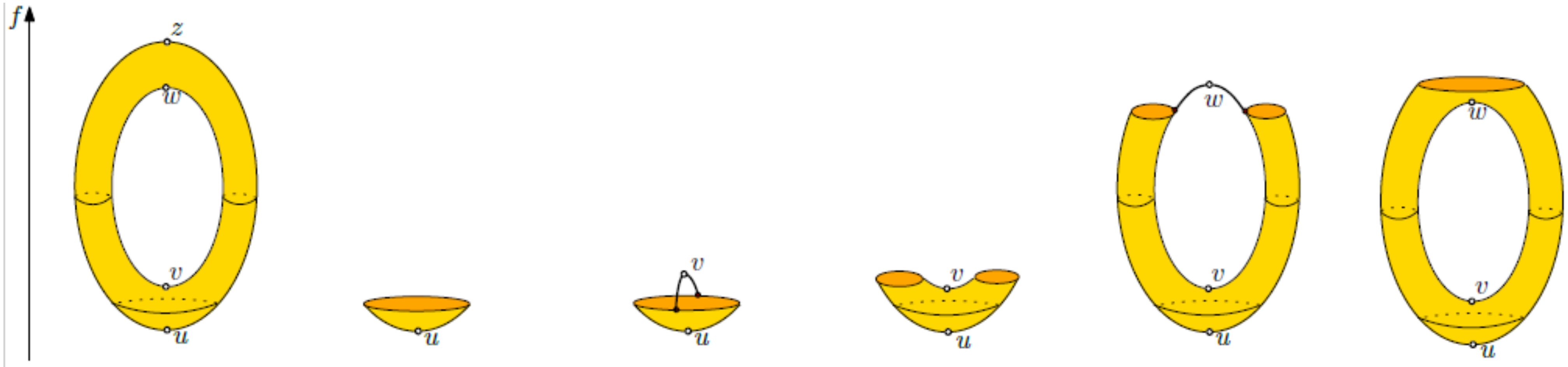
Critical points and topology

Theorem 4. *Given a Morse function $f : M \to \mathbb{R}$ defined on a smooth manifold M, let p be an index-k critical point of f with $\alpha = f(p)$. Assume $f^{-1}([\alpha - \varepsilon, \alpha + \varepsilon])$ is compact for a sufficiently small $\varepsilon > 0$ such that there is no other critical points of f contained in this interval-level set other than p. Then the sublevel set $M_{\leq \alpha + \varepsilon}$ has the same homotopy type as $M_{\leq \alpha - \varepsilon}$ with a k-cell attached to its boundary $\mathrm{Bd}\, M_{\leq \alpha - \varepsilon}$.*

Animation

Critical points and topology

Theorem 4. *Given a Morse function $f : M \to \mathbb{R}$ defined on a smooth manifold M, let p be an index-k critical point of f with $\alpha = f(p)$. Assume $f^{-1}([\alpha - \varepsilon, \alpha + \varepsilon])$ is compact for a sufficiently small $\varepsilon > 0$ such that there is no other critical points of f contained in this interval-level set other than p. Then the sublevel set $M_{\leq \alpha+\varepsilon}$ has the same homotopy type as $M_{\leq \alpha-\varepsilon}$ with a k-cell attached to its boundary* $\mathrm{Bd}\, M_{\leq \alpha-\varepsilon}$.

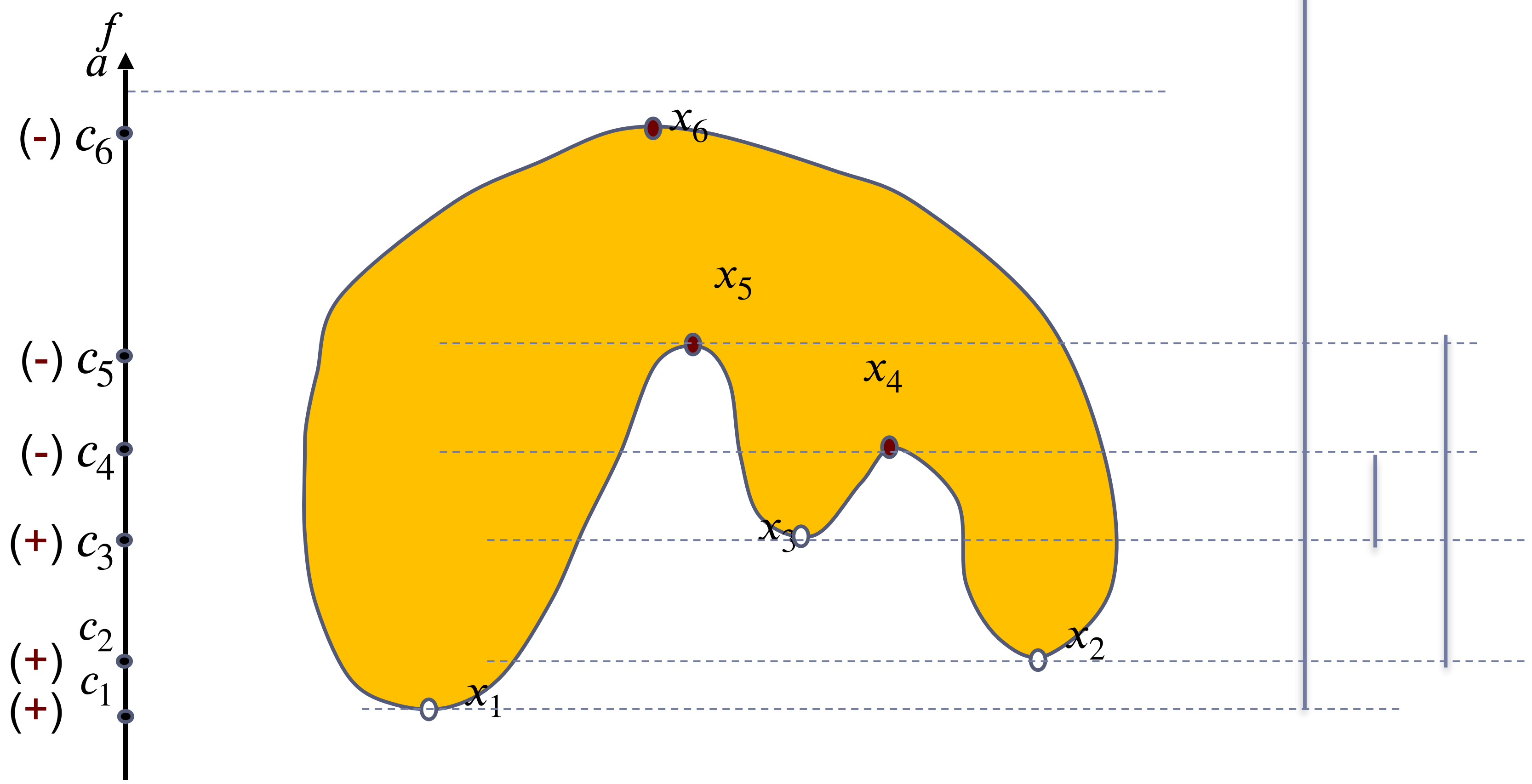


(a) (b) (c) (d) (e) (f)

Animation

A Simple Example

- Critical points are paired
  - The persistence pairings are between critical values, induced by pairings between critical points whose indices differ by 1.

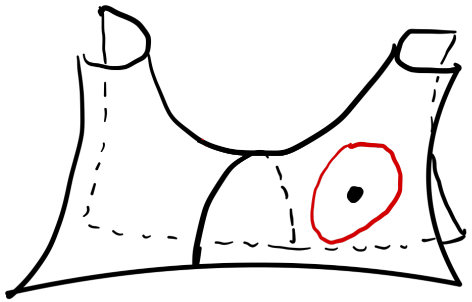


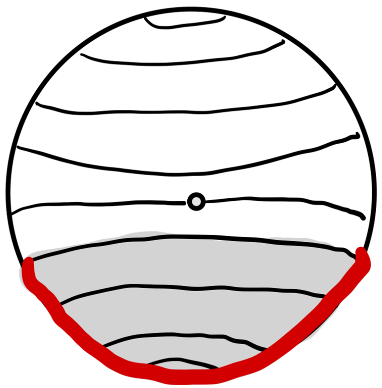
Local View

- For d-dimensional manifold M
- An open neighborhood $B_r(p)$ of p
  - $B_r(p)$ is an m-dimensional open ball
  - Consider the boundary of the closure of $B_r(p)$

Local View

- For d-dimensional manifold M
- An open neighborhood $B_r(p)$ of p
  - $B_r(p)$ is an d-dimensional open ball
  - Consider the boundary of the closure of $B_r(p)$ intersecting the sub-level set $M^{\leq f(p)-\epsilon}$ for some function $f : M \to \mathbb{R}$

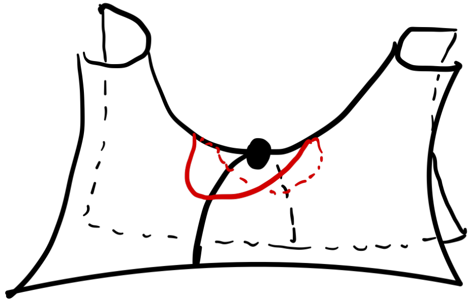


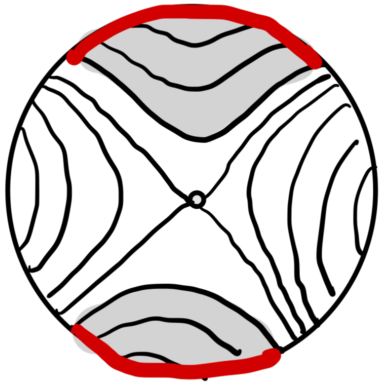


$\tilde{\beta}_0(X) = \beta_0(X) - 1 = 0$
$\beta_p(X) = 0$ for $p \geq 1$
Regular point

Local View

- For d-dimensional manifold M
- An open neighborhood $B_r(p)$ of p
  - $B_r(p)$ is an d-dimensional open ball
  - Consider the boundary of the closure of $B_r(p)$ intersecting the sub-level set $M^{\leq f(p)-\epsilon}$ for some function $f : M \to \mathbb{R}$

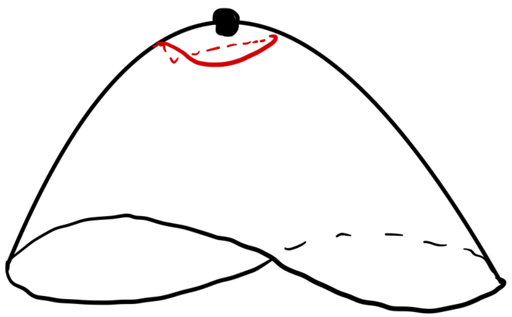


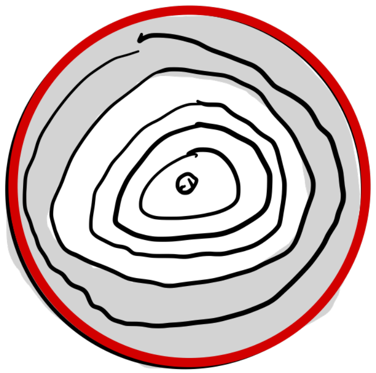


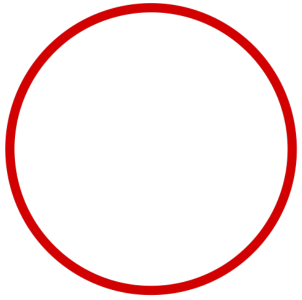
$\tilde{\beta}_0(X) = 1$
$\beta_p(X) = 0$ for $p \geq 1$
Index=1

Local View

- For d-dimensional manifold M
- An open neighborhood $B_r(p)$ of p
  - $B_r(p)$ is an d-dimensional open ball
  - Consider the boundary of the closure of $B_r(p)$ intersecting the sub-level set $M^{\leq f(p)-\epsilon}$ for some function $f : M \to \mathbb{R}$





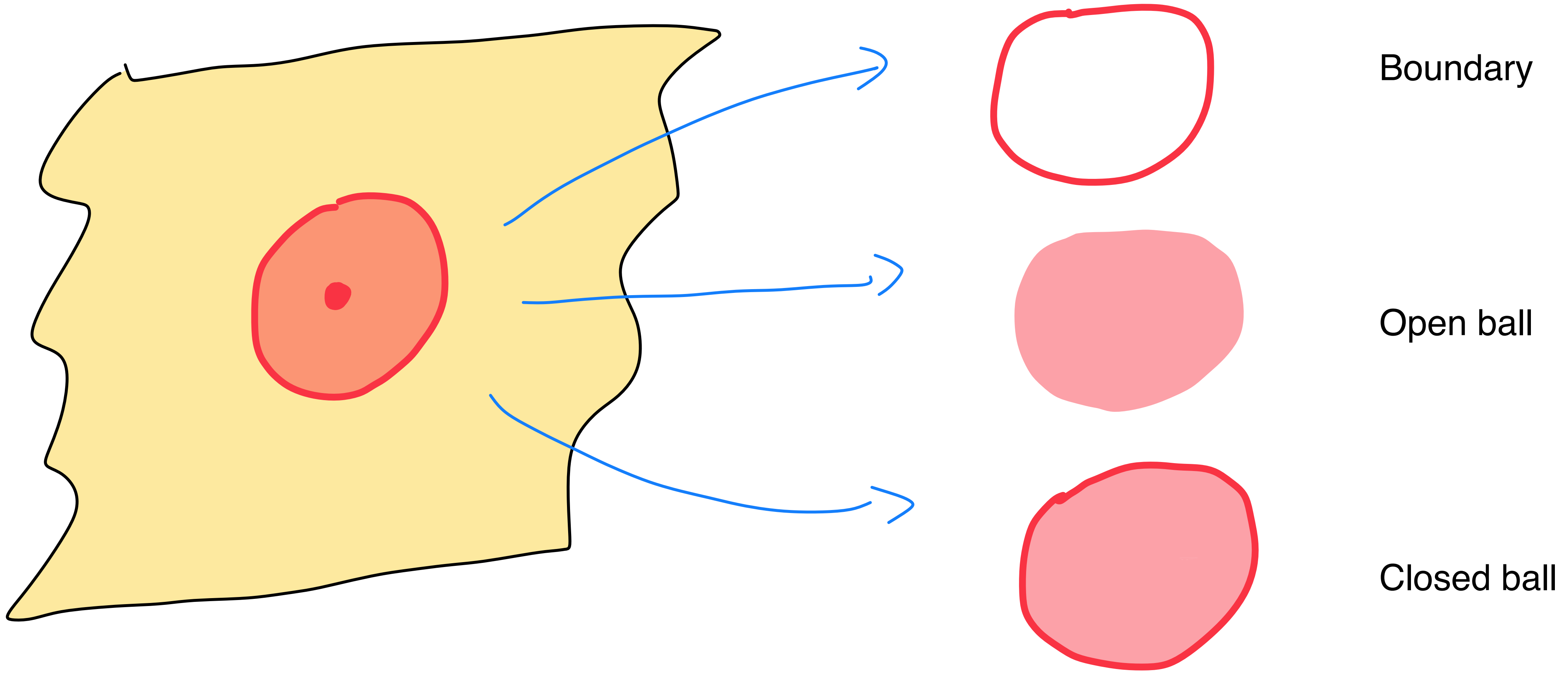


$\tilde{\beta}_0(X) = 0$
$\beta_1(X) = 1$
$\beta_p(X) = 0$ for $p \geq 2$
Index=2

Local View

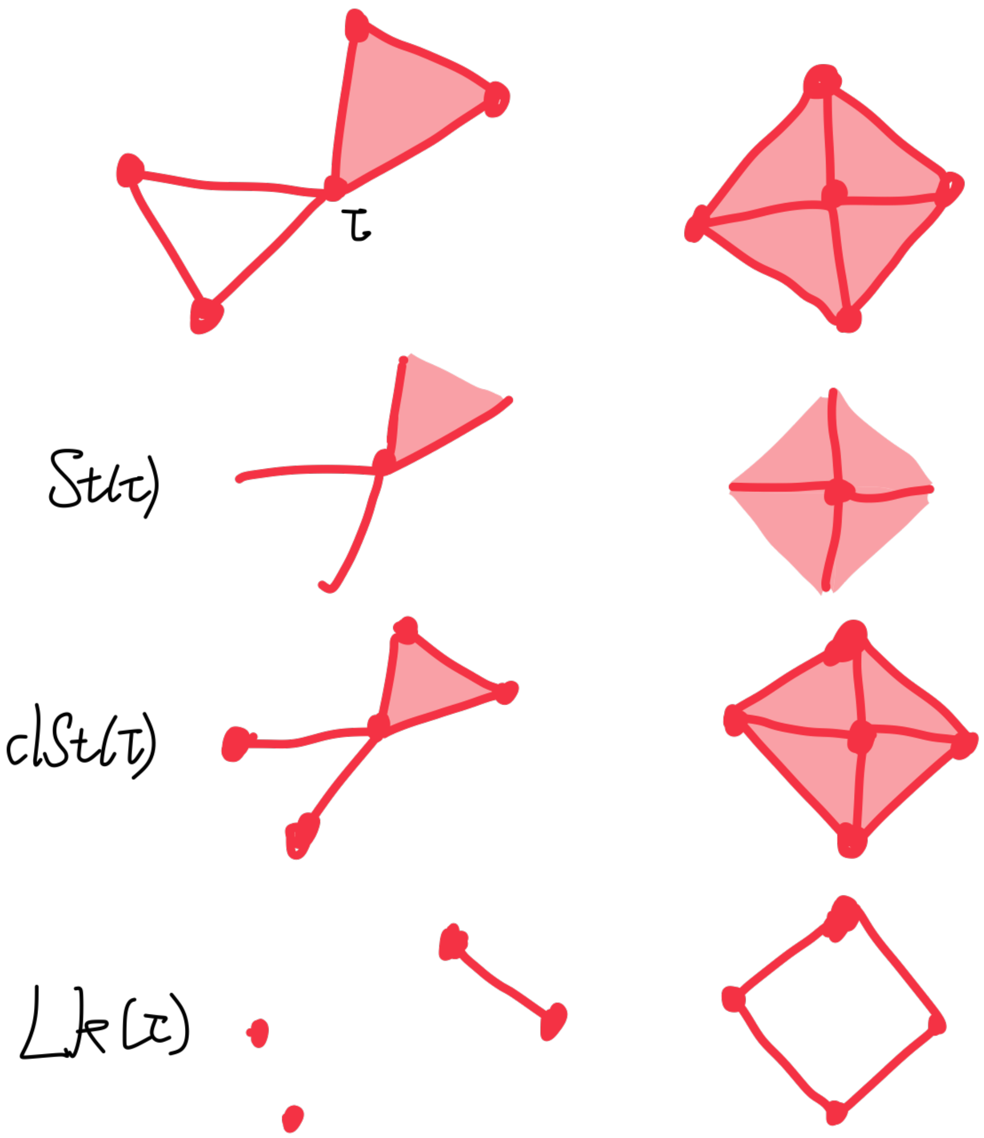
- For d-dimensional manifold M
- An open neighborhood $B_r(p)$ of p
  - $B_r(p)$ is an d-dimensional open ball
  - Consider the boundary of the closure of $B_r(p)$ intersecting the sub-level set $M^{\leq f(p)-\epsilon}$ for some function $f : M \to \mathbb{R}$
  - p is an index k critical point if $\tilde{\beta}_{k-1} > 0$

Star and links



Star and links

- Given a simplex $\tau \in K$
  - Star: $St(\tau) = \{\sigma \in K \mid \tau \subset \sigma\}$
  - Closed star: $cl\,St(\tau) = \bigcup_{\sigma \in St(\tau)} \{\, \sigma' \mid \sigma' \subset \sigma \,\}$
  - Link: $Lk(\tau) = \{\, \sigma \in cl\,St(\tau) \mid \sigma \cap \tau = \varnothing \,\}$

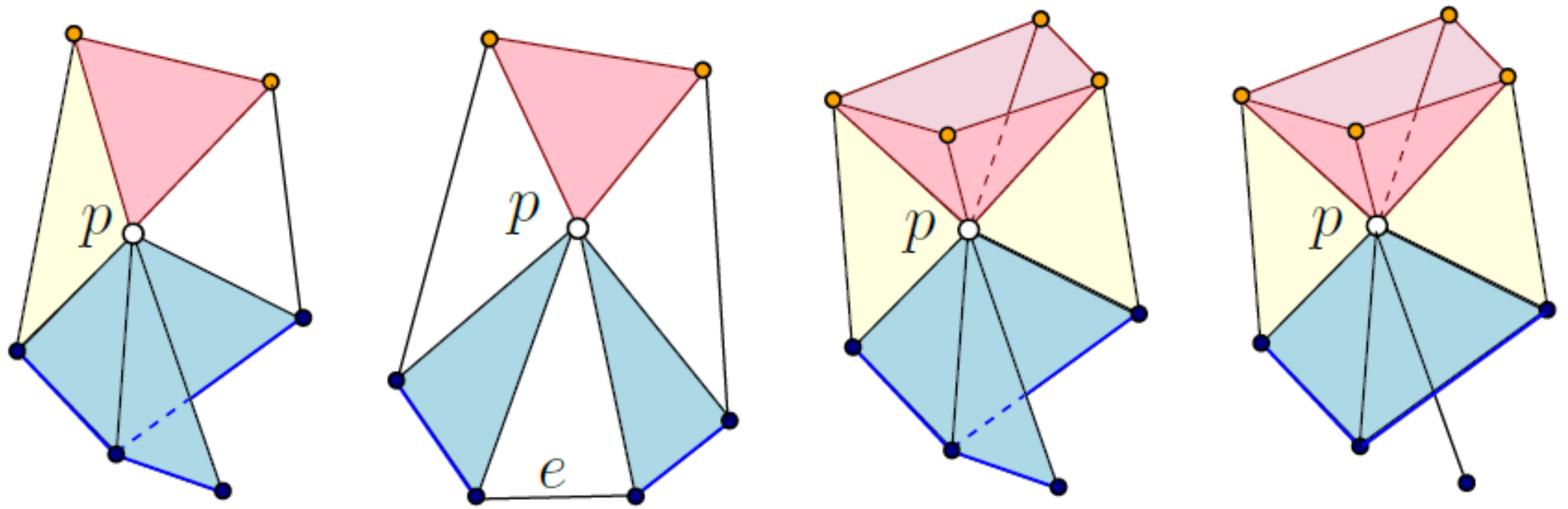


Upper and lower link

- $LowSt\,(v) := \{\sigma \in K \mid v \in \sigma \; and \; f(u) \leq f(v) \; for \; any \; u \in \sigma\}$
- $clLowSt(v) := \{\sigma \in K | \sigma \subset \tau \in LowSt(v)\}$
- $Llk(v) = clLowSt(v) \backslash LowSt(v)$

- $UpSt(v) := \{\sigma \in K | v \in \sigma \text{ and } f(u) \geq f(v), \forall u \in \sigma\}$
- $clUpSt(v) := \{\sigma \in K | \sigma \subset \tau \in UpSt(v)\}$
- $Ulk(v) = clUpSt(v) \backslash UpSt(v)$

Examples



(a) (b) (c) (d)

PL-critical points

- Analogous to critical points in the smooth case.

Definition 1 (Reduced Betti number). $\tilde{\beta}_p(X) = \beta_p(X)$ for $p > 0$. For $p = 0$, $\tilde{\beta}_0(X) = \beta_0(X) - 1$ and $\tilde{\beta}_{-1}(X) = 0$ if X is not empty; otherwise, $\tilde{\beta}_0(X) = 0$ and $\tilde{\beta}_{-1}(X) = 1$.

- $\tilde{\beta}_0(X) =$ number of connected components - 1

PL-critical points

- Analogous to critical points in the smooth case.

Definition 1 (Reduced Betti number). $\tilde{\beta}_p(X) = \beta_p(X)$ for $p > 0$. For $p = 0$, $\tilde{\beta}_0(X) = \beta_0(X) - 1$ and $\tilde{\beta}_{-1}(X) = 0$ if X is not empty; otherwise, $\tilde{\beta}_0(X) = 0$ and $\tilde{\beta}_{-1}(X) = 1$.

PL-critical points

- Analogous to critical points in the smooth case.

Definition 1 (Reduced Betti number). $\tilde{\beta}_p(X) = \beta_p(X)$ for $p > 0$. For $p = 0$, $\tilde{\beta}_0(X) = \beta_0(X) - 1$ and $\tilde{\beta}_{-1}(X) = 0$ if X is not empty; otherwise, $\tilde{\beta}_0(X) = 0$ and $\tilde{\beta}_{-1}(X) = 1$.

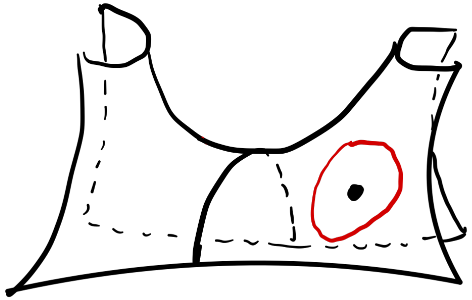
Definition 2 (PL-critical points). Given a PL-function $f : |K| \to \mathbb{R}$, we say that a vertex $v \in K$ is a *regular* vertex or point if $\tilde{\beta}_p(\mathrm{Llk}(v)) = 0$ and $\tilde{\beta}_p(\mathrm{Ulk}(v)) = 0$ for any $p \geq -1$. It is called *PL-critical* (or simply *critical*) vertex or point otherwise.

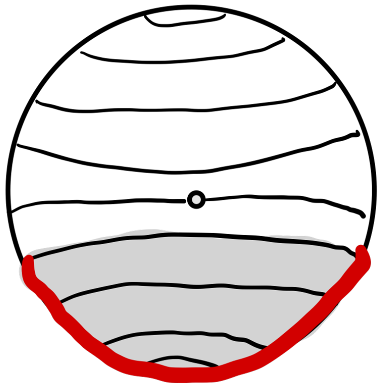
Furthermore, we say that v has lower-link-index p if $\tilde{\beta}_{p-1}(\mathrm{Llk}(v)) > 0$. Similarly v has upper-link-index p if $\tilde{\beta}_{p-1}(\mathrm{Ulk}(v)) > 0$. The function value of a critical point is a *critical value for* f.

PL-critical points

Definition 2 (PL-critical points). Given a PL-function $f : |K| \to \mathbb{R}$, we say that a vertex $v \in K$ is a *regular* vertex or point if $\tilde{\beta}_p(\mathrm{Llk}(v)) = 0$ and $\tilde{\beta}_p(\mathrm{Ulk}(v)) = 0$ for any $p \geq -1$. It is called *PL-critical* (or simply *critical*) vertex or point otherwise.

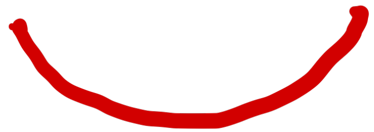
Furthermore, we say that v has lower-link-index p if $\tilde{\beta}_{p-1}(\mathrm{Llk}(v)) > 0$. Similarly v has upper-link-index p if $\tilde{\beta}_{p-1}(\mathrm{Ulk}(v)) > 0$. The function value of a critical point is a *critical value for* f.





$$\tilde{\beta}_0(X) = \beta_0(X) - 1 = 0$$

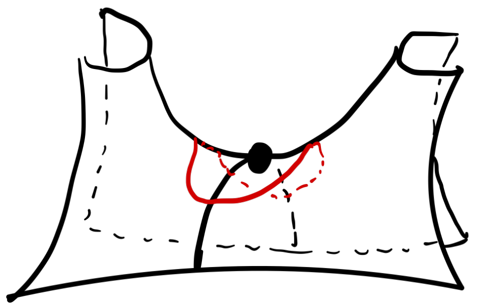
$$\beta_p(X) = 0 \text{ for } p \geq 1$$

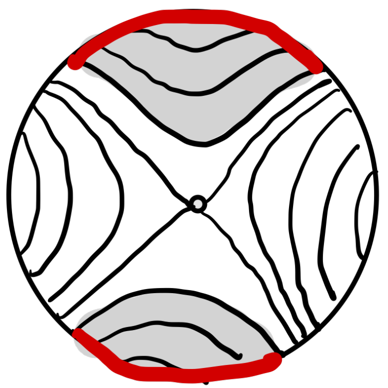


PL-critical points

Definition 2 (PL-critical points)**.** Given a PL-function $f : |K| \to \mathbb{R}$, we say that a vertex $v \in K$ is a *regular* vertex or point if $\tilde{\beta}_p(\mathrm{Llk}(v)) = 0$ and $\tilde{\beta}_p(\mathrm{Ulk}(v)) = 0$ for any $p \geq -1$. It is called *PL-critical* (or simply *critical*) vertex or point otherwise.

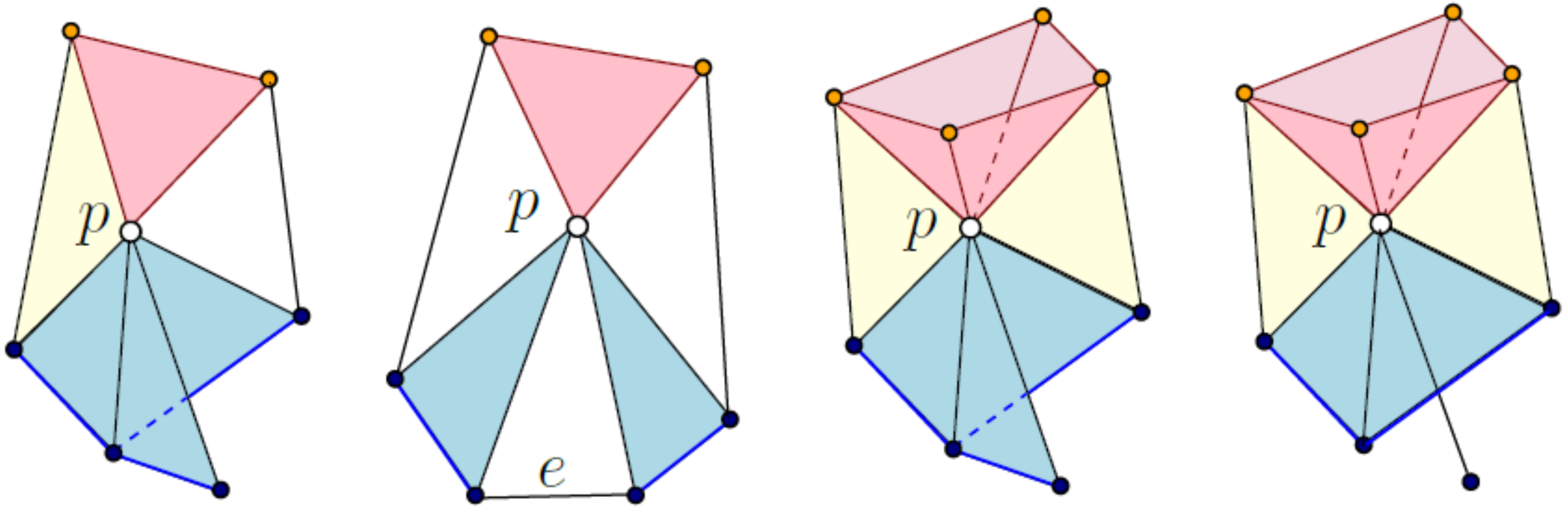
Furthermore, we say that v has lower-link-index p if $\tilde{\beta}_{p-1}(\mathrm{Llk}(v)) > 0$. Similarly v has upper-link-index p if $\tilde{\beta}_{p-1}(\mathrm{Ulk}(v)) > 0$. The function value of a critical point is a *critical value for* f.





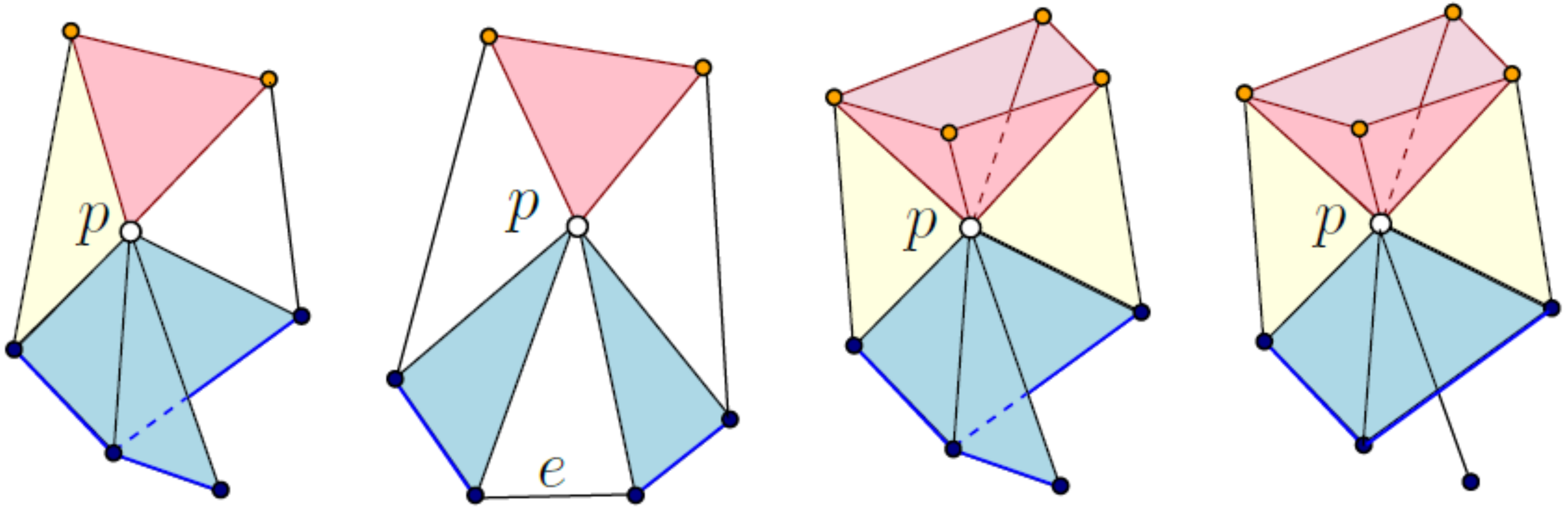
$\tilde{\beta}_0(X) > 0$
$\beta_p(X) = 0$ for $p \geq 1$
Index=1

Examples



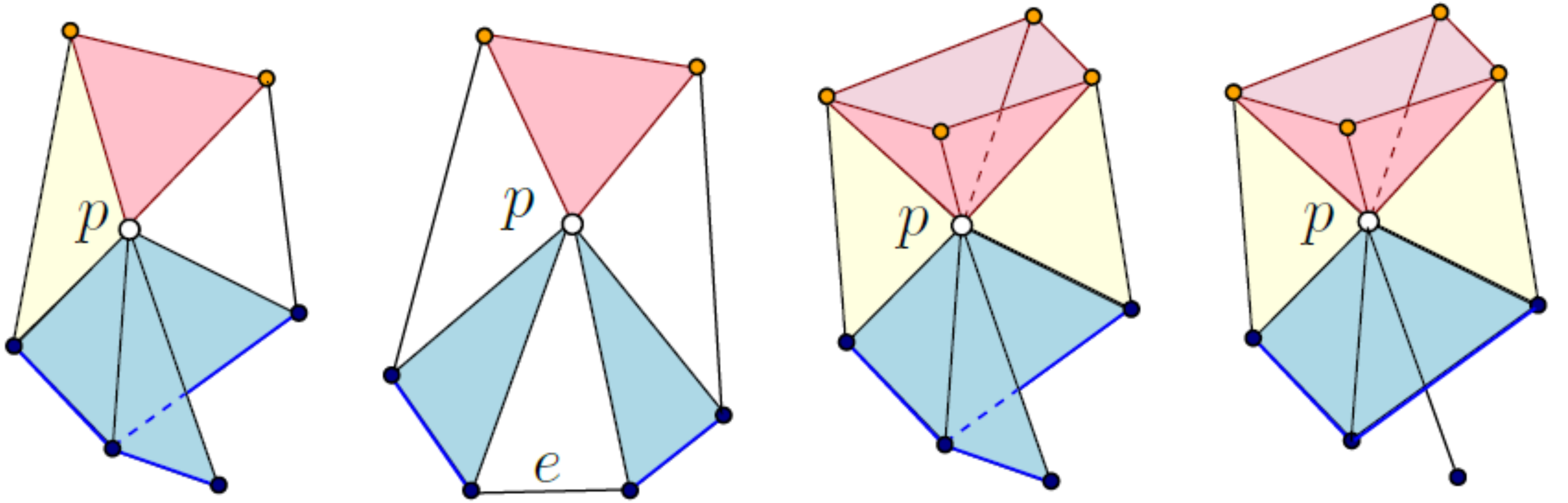
(a) (b) (c) (d)

Examples



(a) (b) (c) (d)

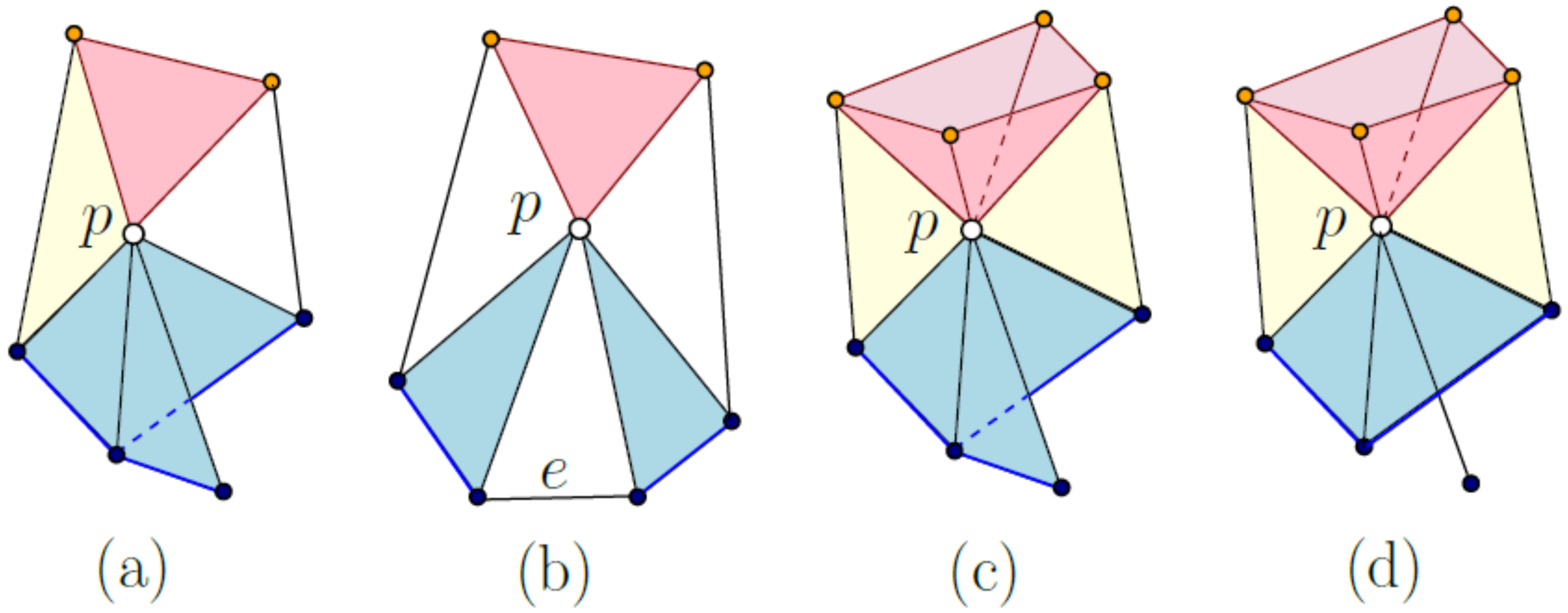
Examples



(a) (b) (c) (d)

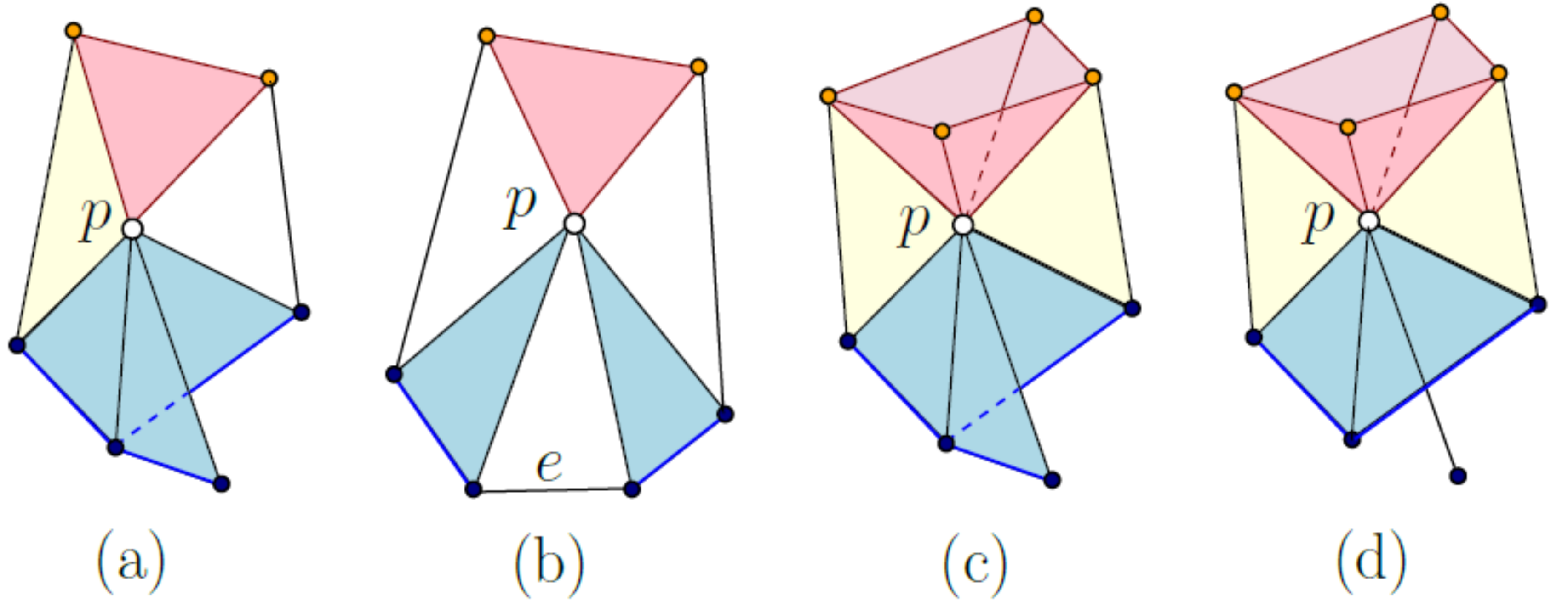
lower-link-index
Upper-link-index

Examples



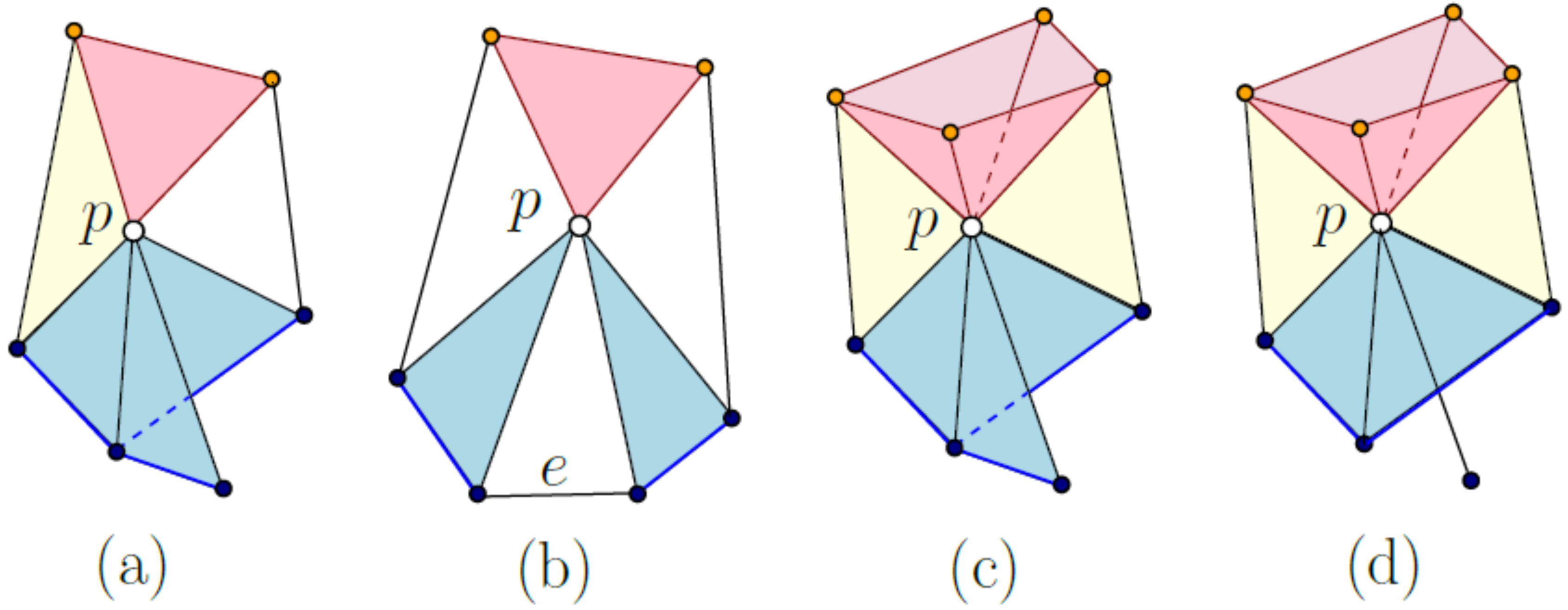
	A
lower-link-index	NA
Upper-link-index	NA

Examples



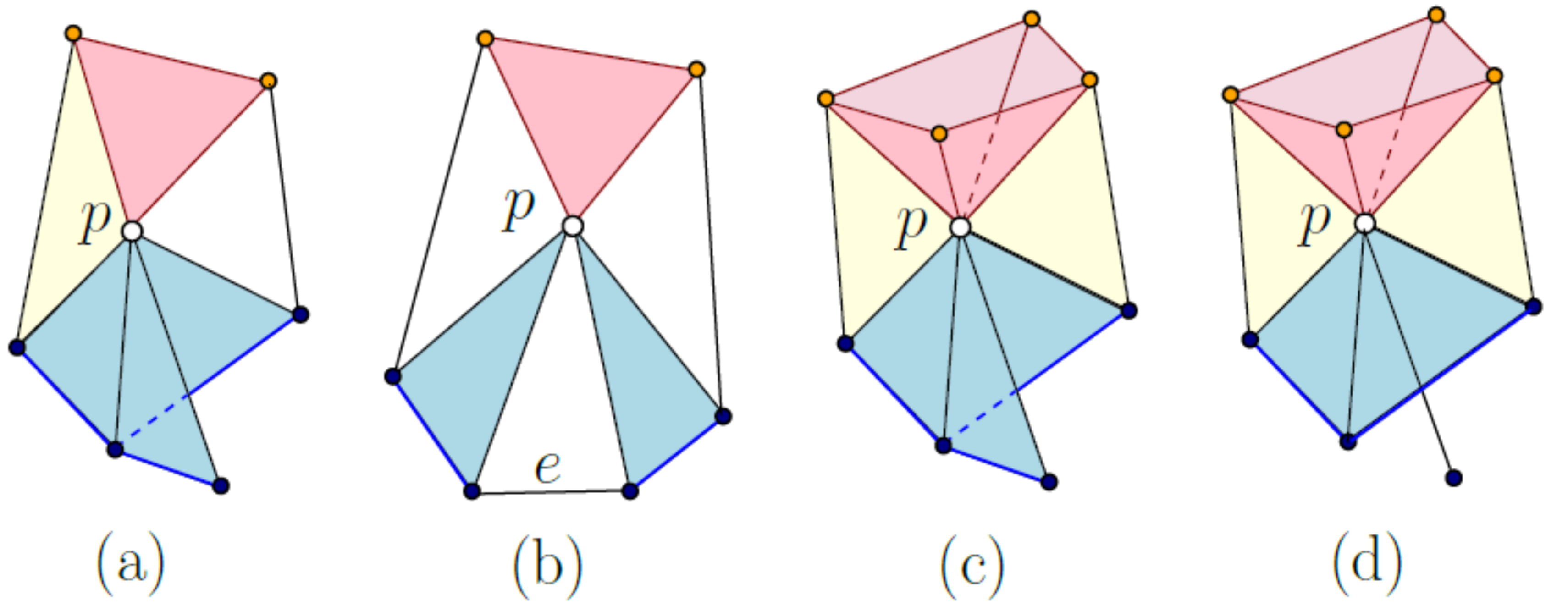
	A	B
lower-link-index	NA	1
Upper-link-index	NA	NA

Examples



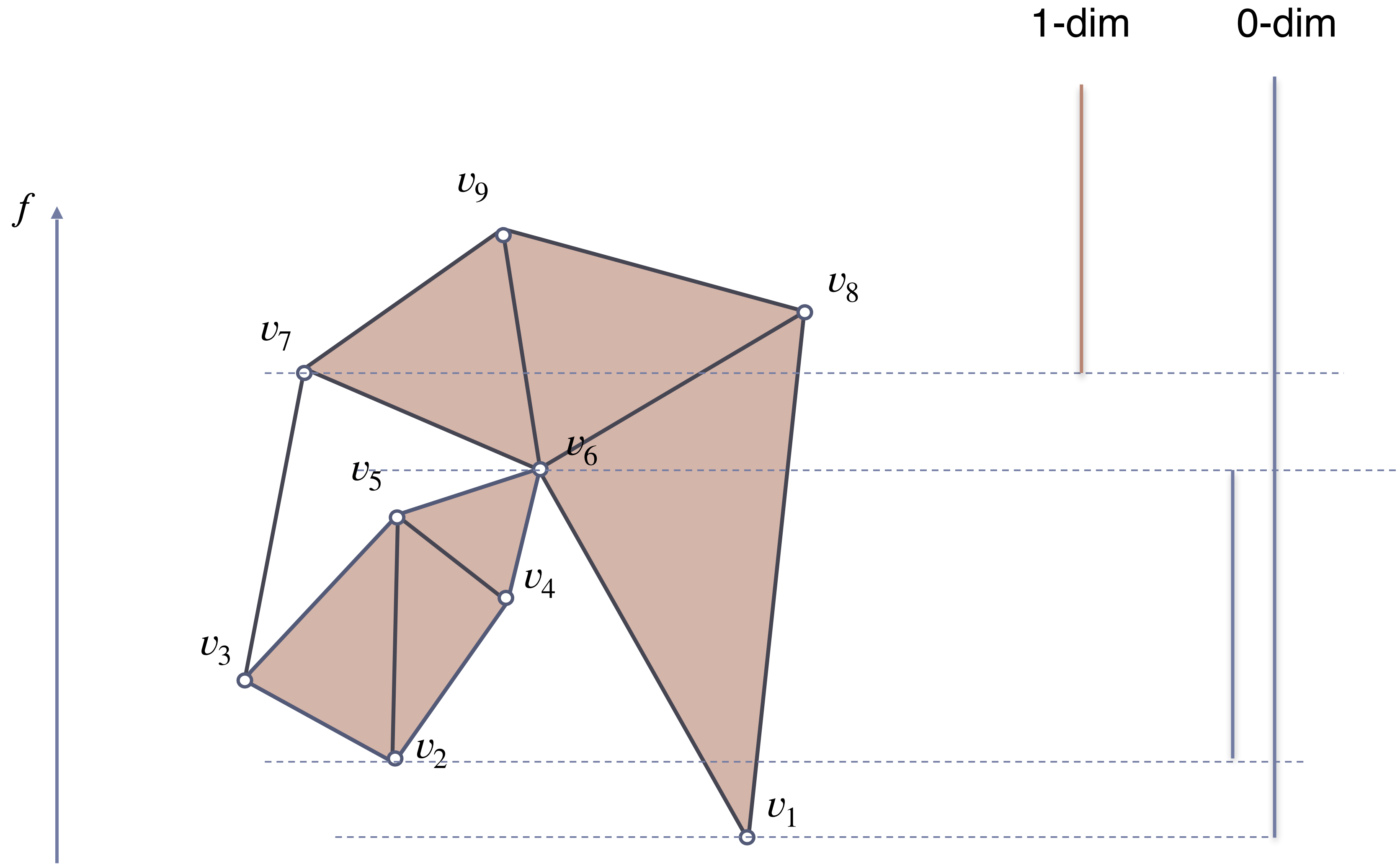
	A	B	C
lower-link-index	NA	1	NA
Upper-link-index	NA	NA	2

Examples

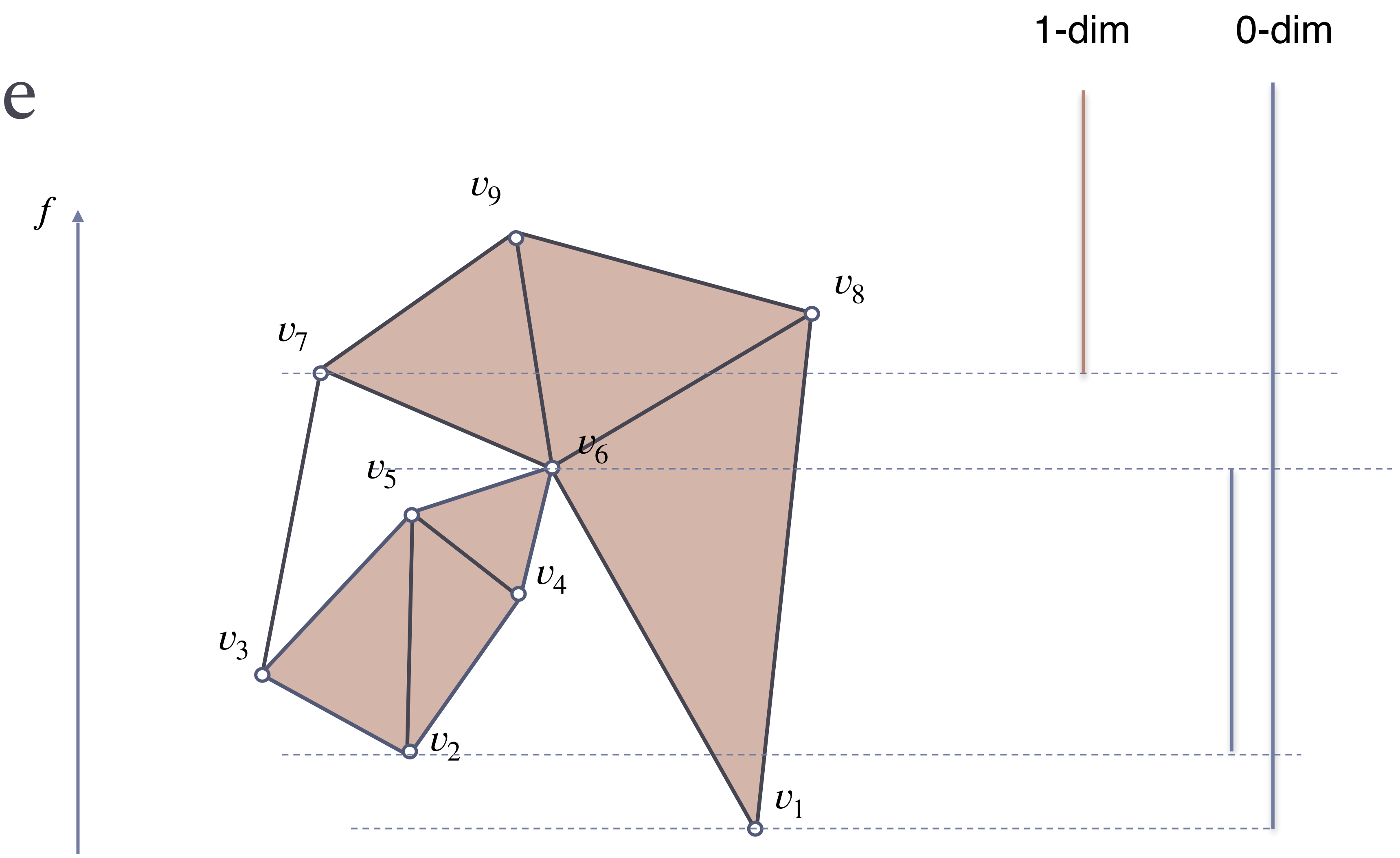


	A	B	C	D
lower-link-index	NA	1	NA	1
Upper-link-index	NA	NA	2	2

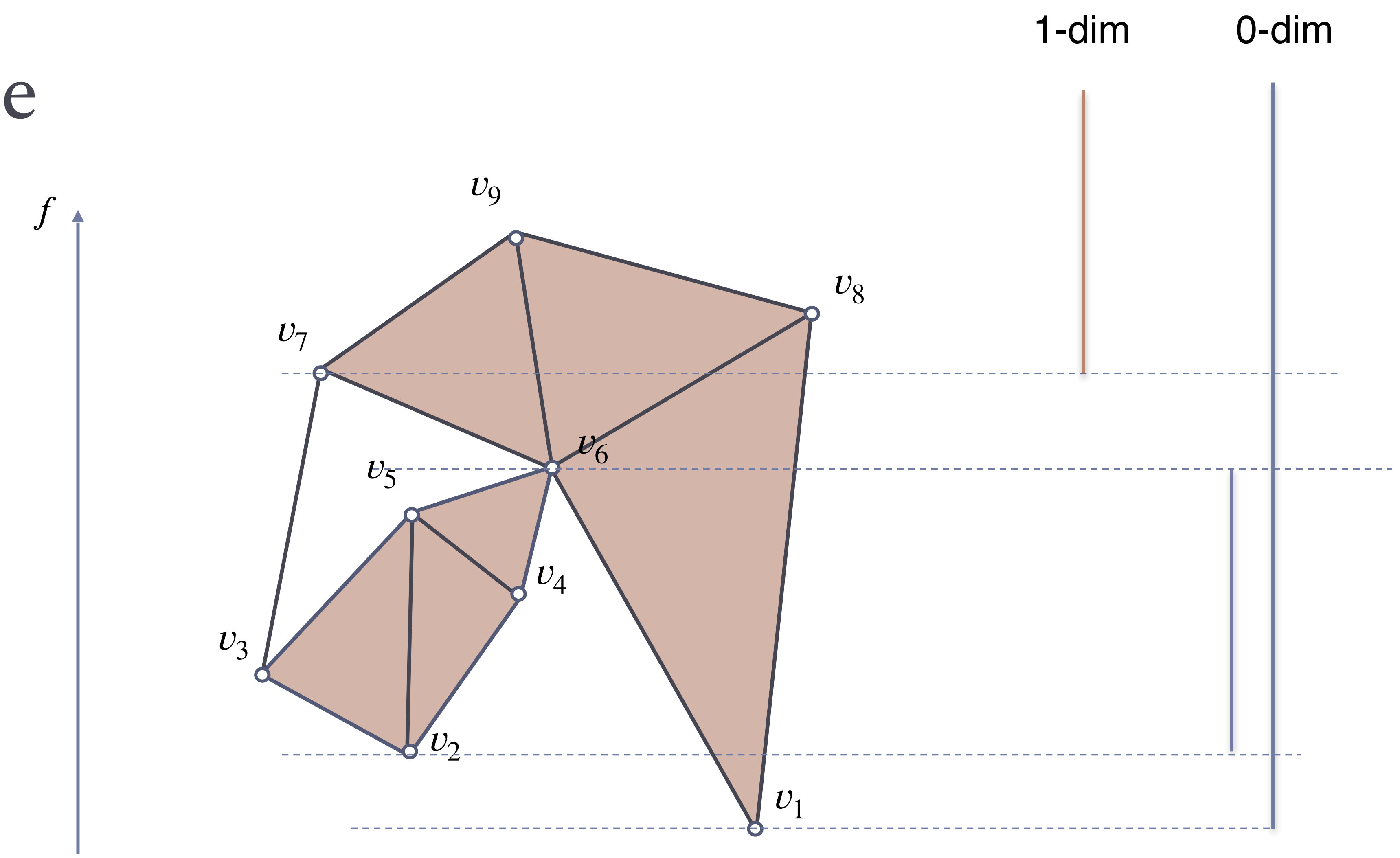
Example



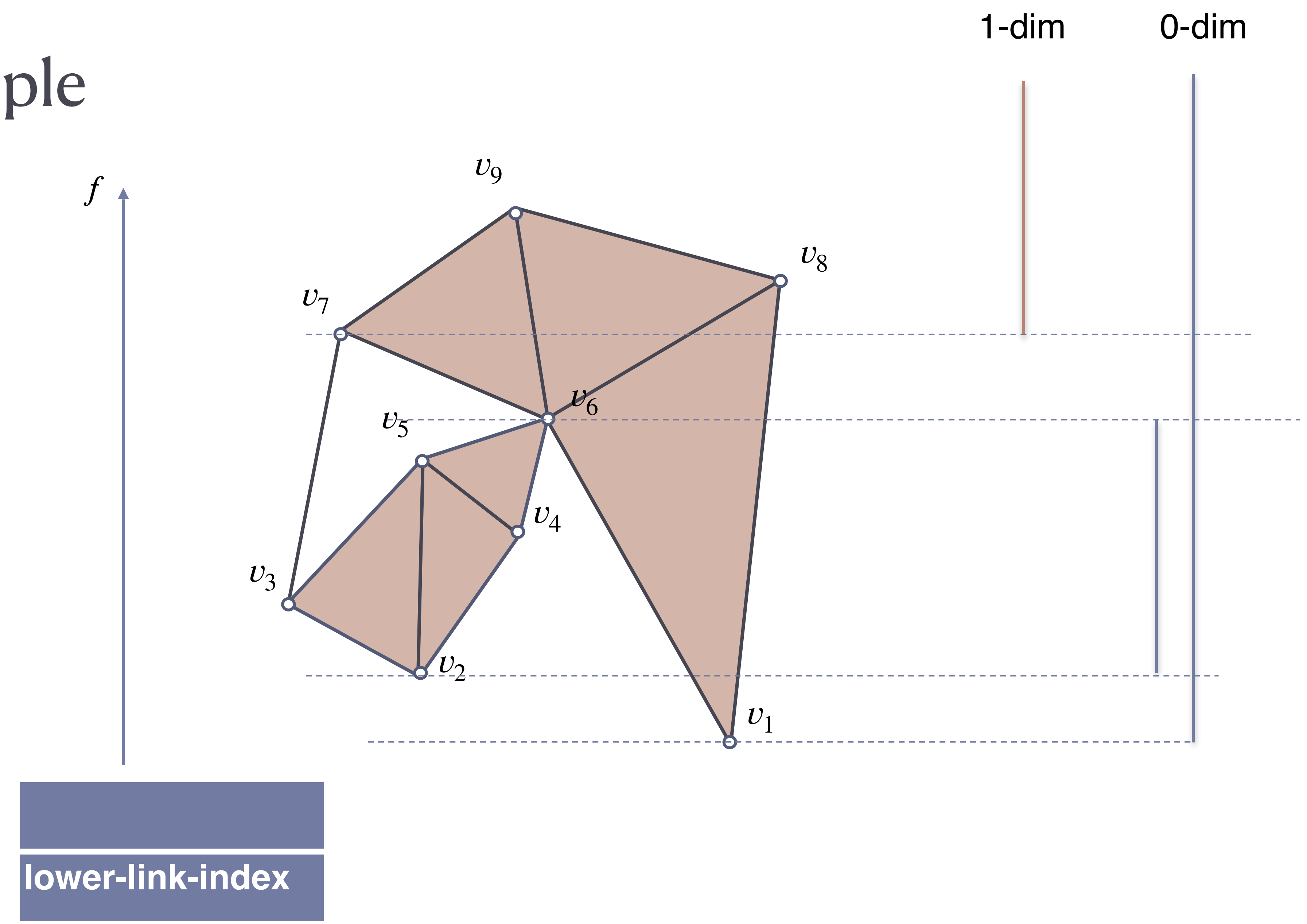
Example



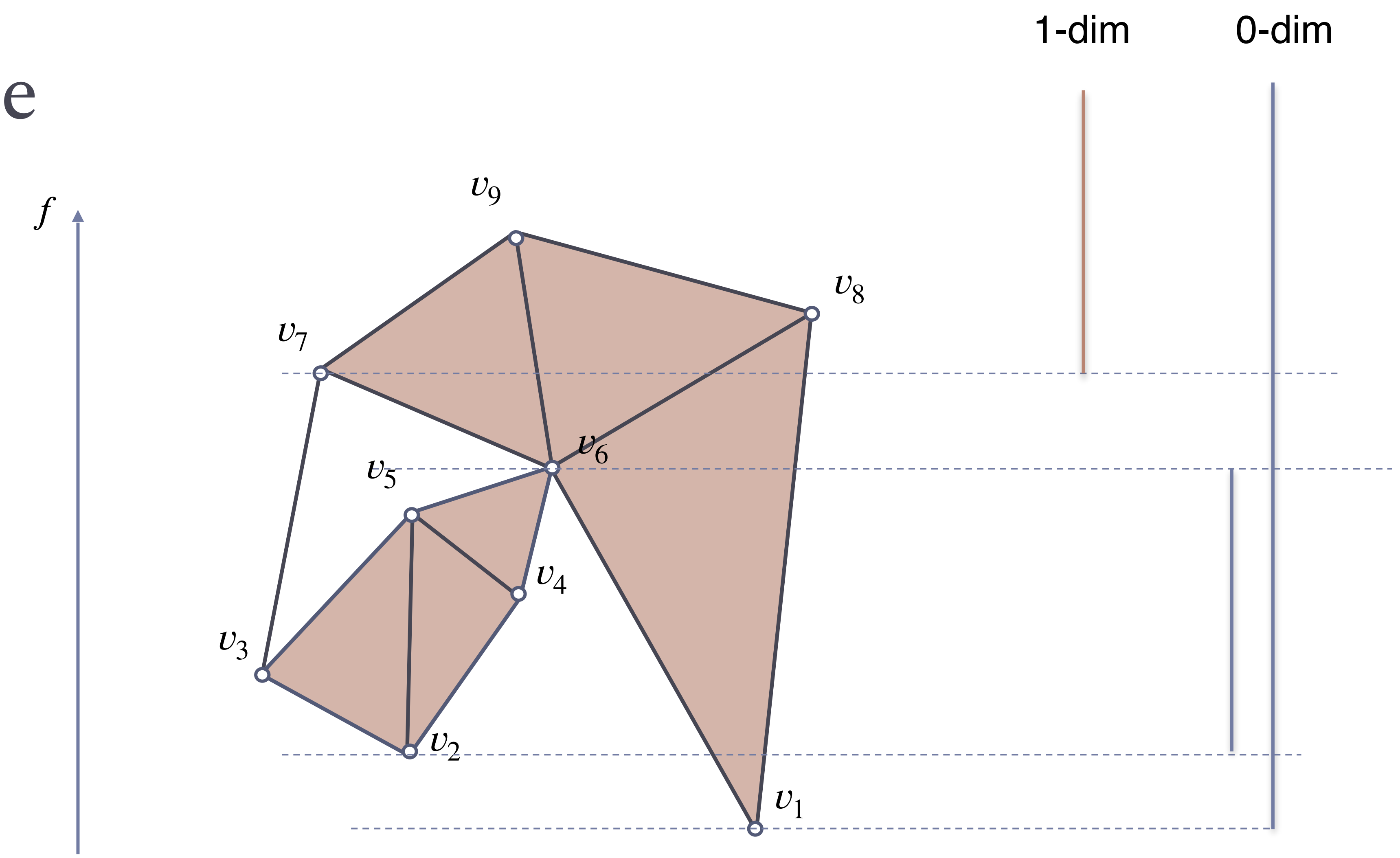
Example



Example

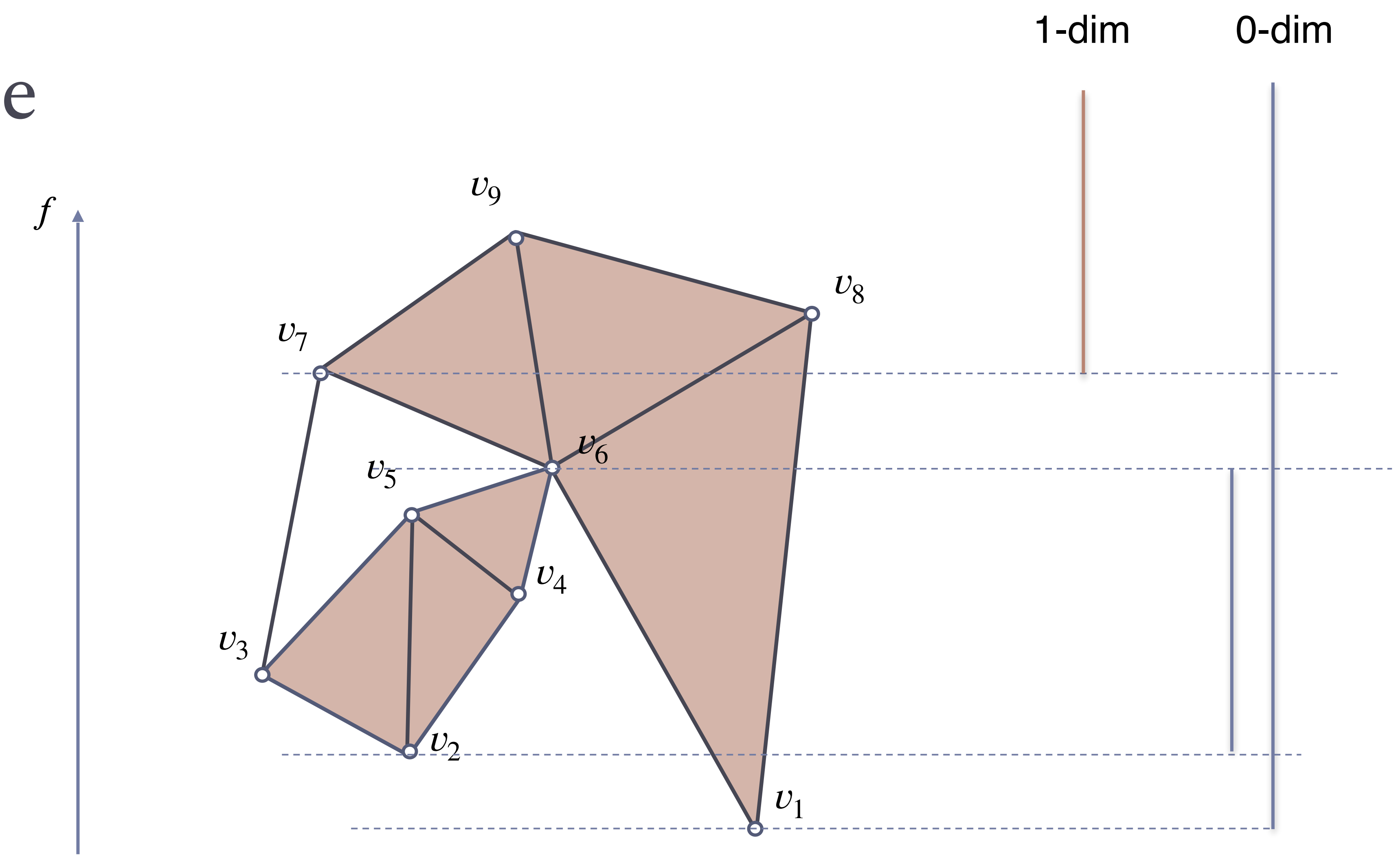


Example



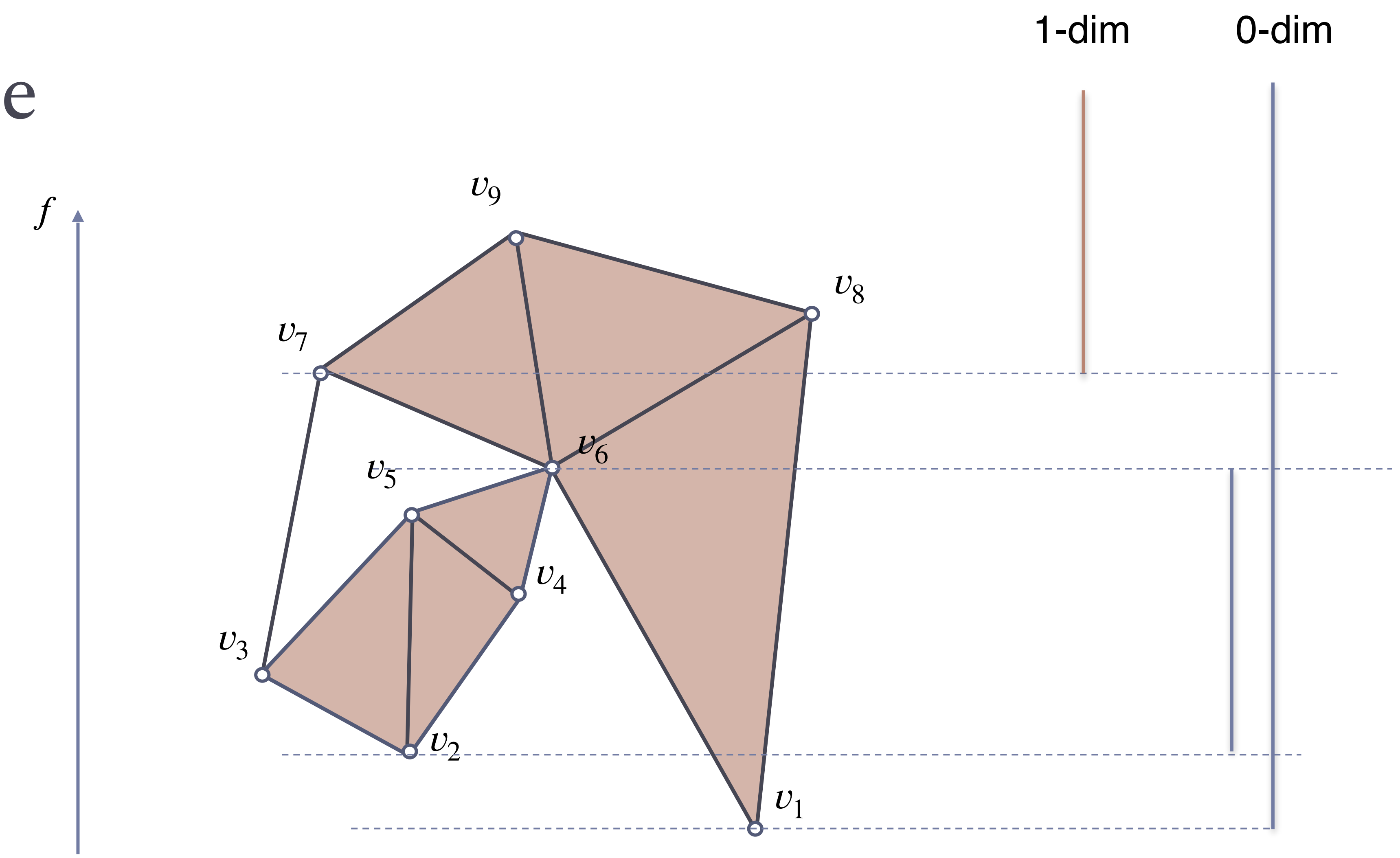
	v1
lower-link-index	0

Example



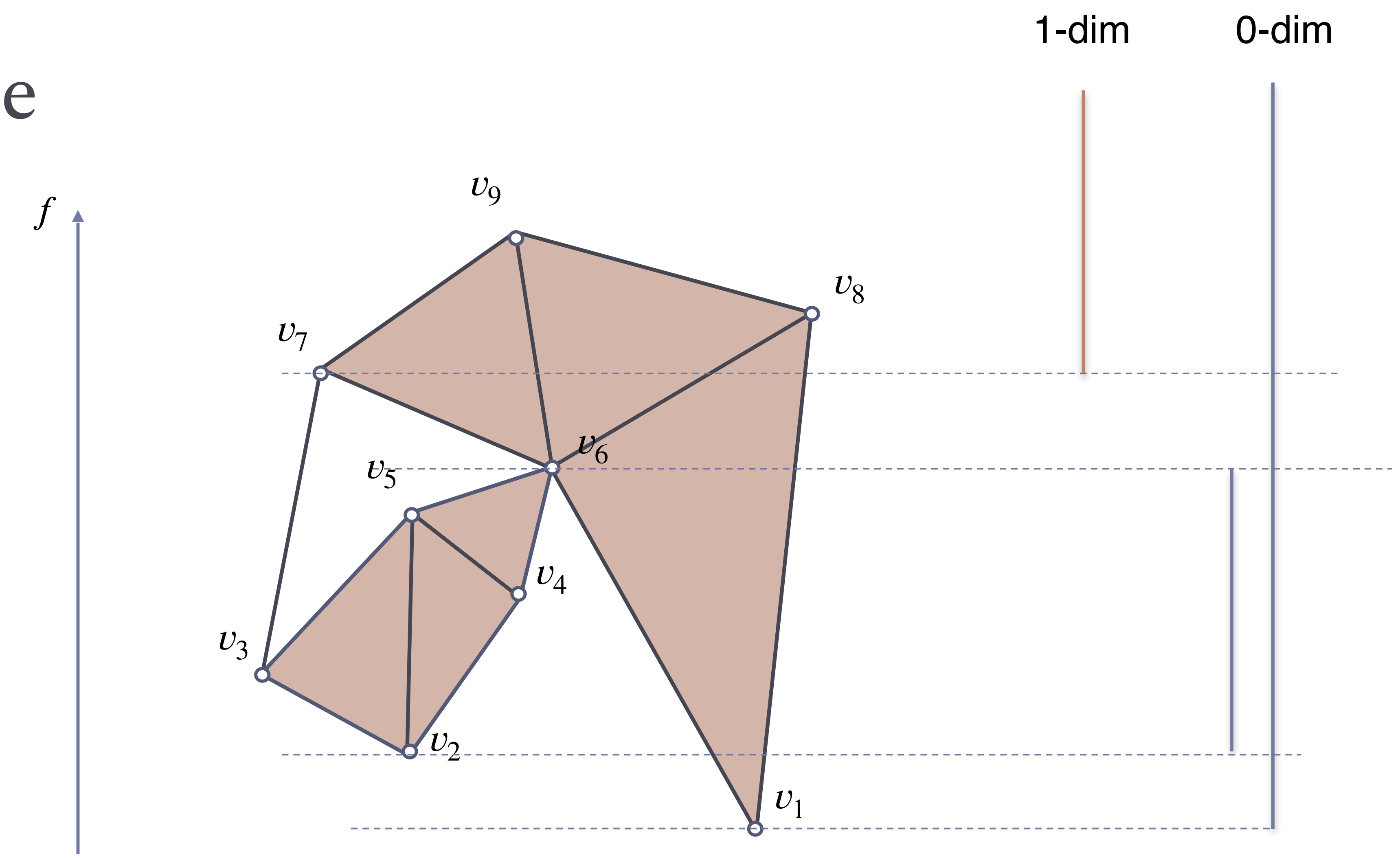
	v1	v2
lower-link-index	0	0

Example



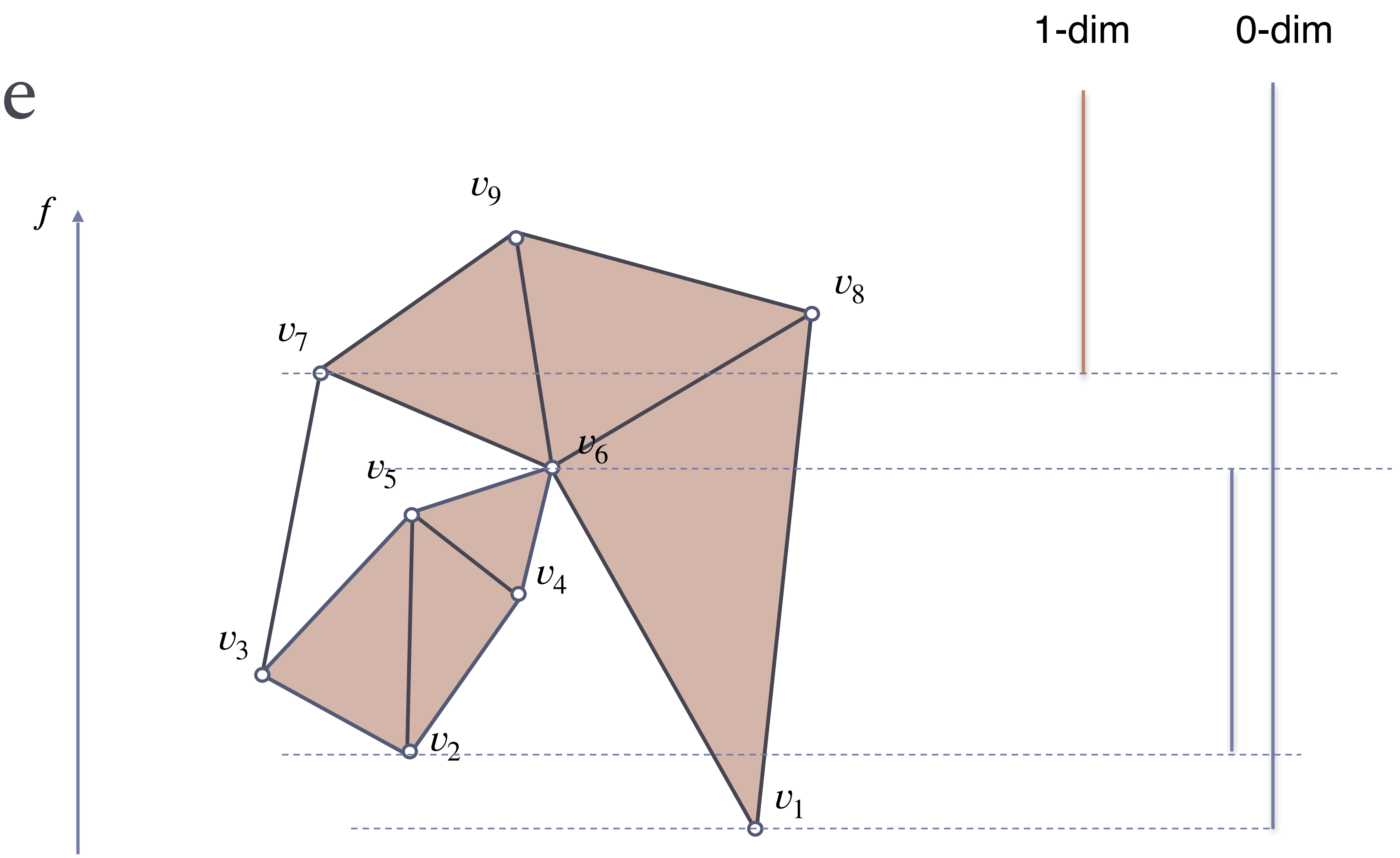
	v1	v2	v3
lower-link-index	0	0	NA

Example



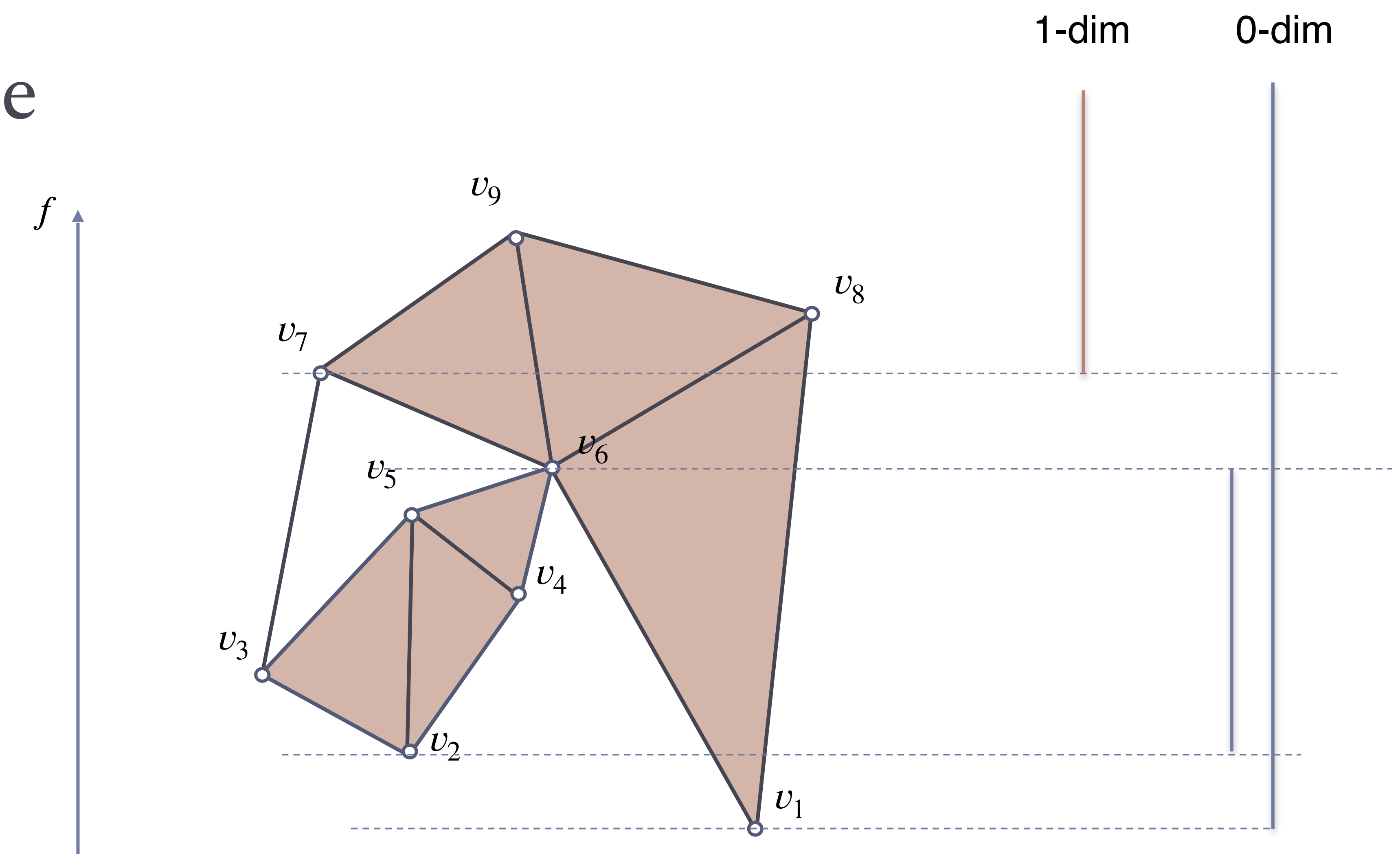
	v1	v2	v3	v4
lower-link-index	0	0	NA	NA

Example



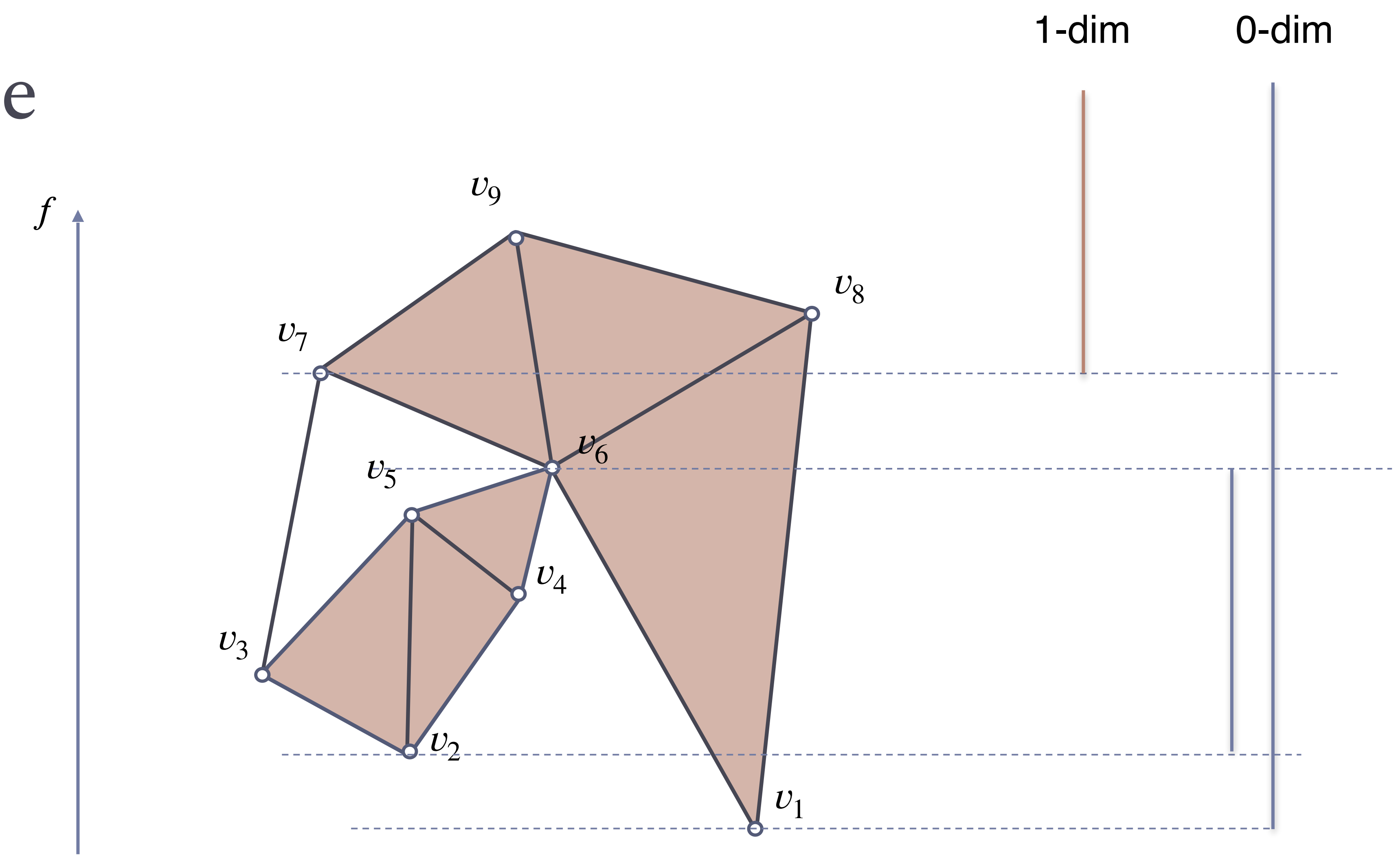
	v1	v2	v3	v4	v5
lower-link-index	0	0	NA	NA	NA

Example



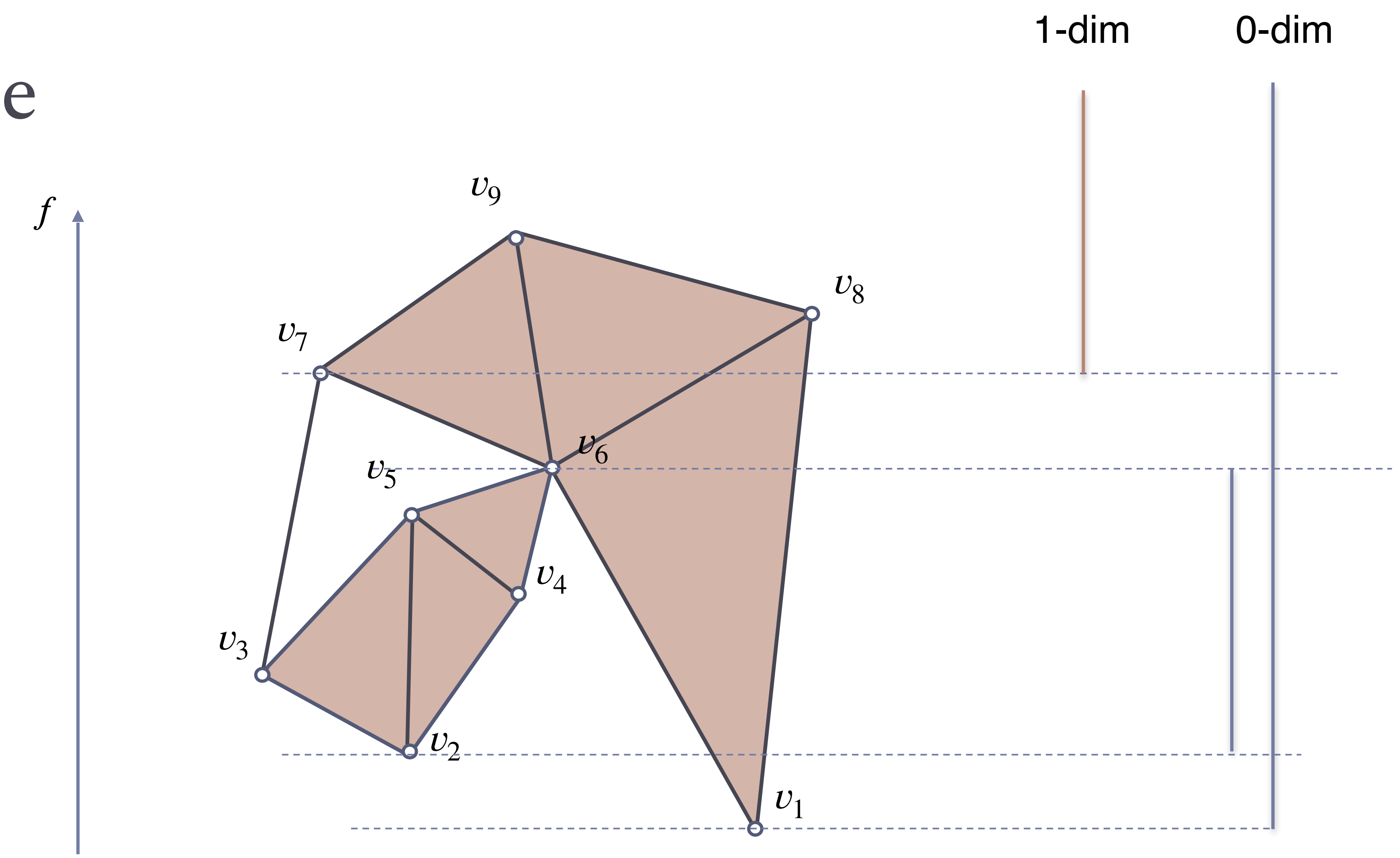
	v1	v2	v3	v4	v5	v6
lower-link-index	0	0	NA	NA	NA	1

Example



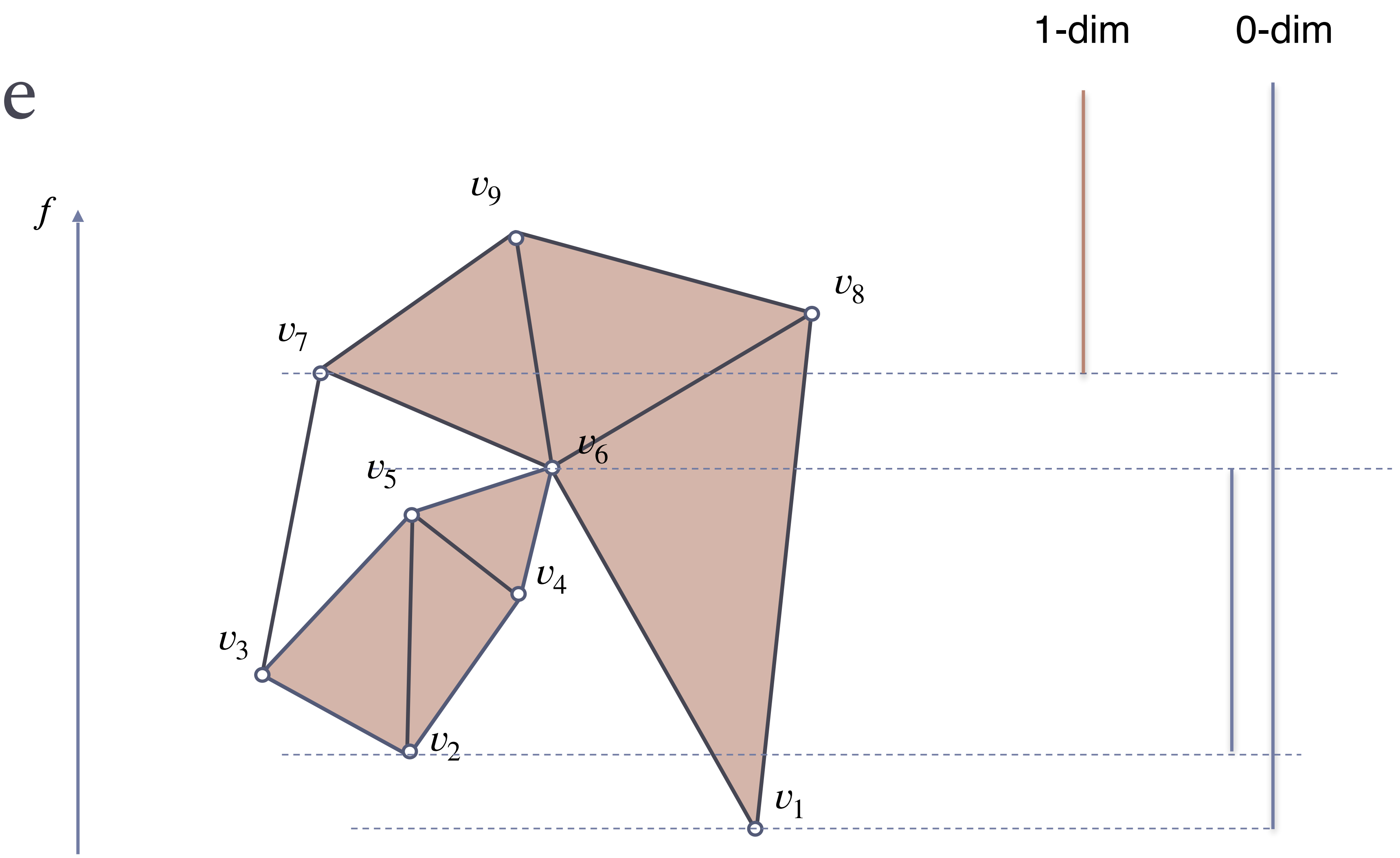
	v1	v2	v3	v4	v5	v6	v7
lower-link-index	0	0	NA	NA	NA	1	1

Example



	v1	v2	v3	v4	v5	v6	v7	v8
lower-link-index	0	0	NA	NA	NA	1	1	NA

Example



	v1	v2	v3	v4	v5	v6	v7	v8	V9
lower-link-index	0	0	NA	NA	NA	1	1	NA	NA

- Theorem
  - Given a PL function $f: |K| \to \mathbb{R}$, for any $2 \leq r \leq n$ and dimension p, $K_{r-1} \subset K_r$ induces an isomorphism $H_p(K_{r-1}) \cong H_p(K_r)$ unless v_r has lower-link-index p or $p+1$

- **Theorem**
  - Given a PL function $f : |K| \to \mathbb{R}$, for any $2 \leq r \leq n$ and dimension p, $K_{r-1} \subset K_r$ induces an isomorphism $H_p(K_{r-1}) \cong H_p(K_r)$ unless v_r has lower-link-index p or $p+1$

- **Theorem**
  - Given a PL function $f : |K| \to \mathbb{R}$ and compute its persistent homology as described. Then for each persistence pair $(i,\ j)$, both v_i and v_j must be PL-critical.

FIN