DSC 214
Topological Data Analysis

Topic 10: Discrete Morse Theory

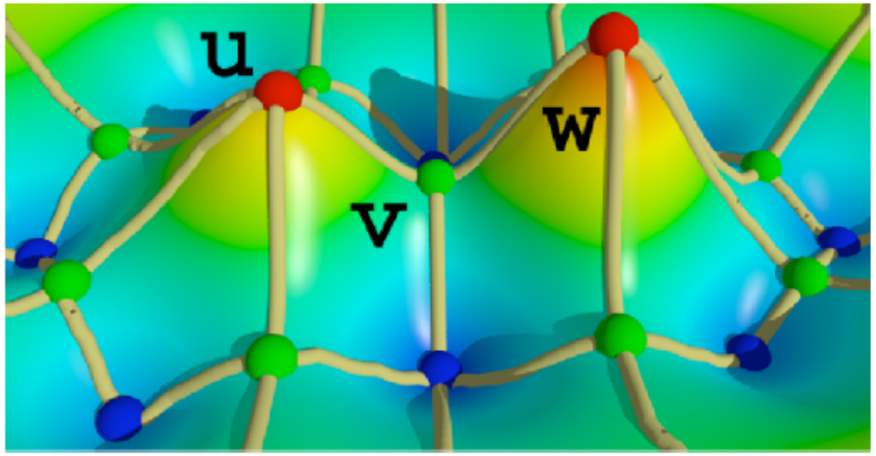
Instructor: Zhengchao Wan

Outline

- Morse theory provides another set of topological skeletons that can characterize structure behind a scalar function
- Discrete Morse theory is a combinatorial analog of Morse theory
  - Potentially more computationally friendly
- Outline:
  - Review Morse theory
  - Introduction of discrete Morse theory
  - DMT + PH

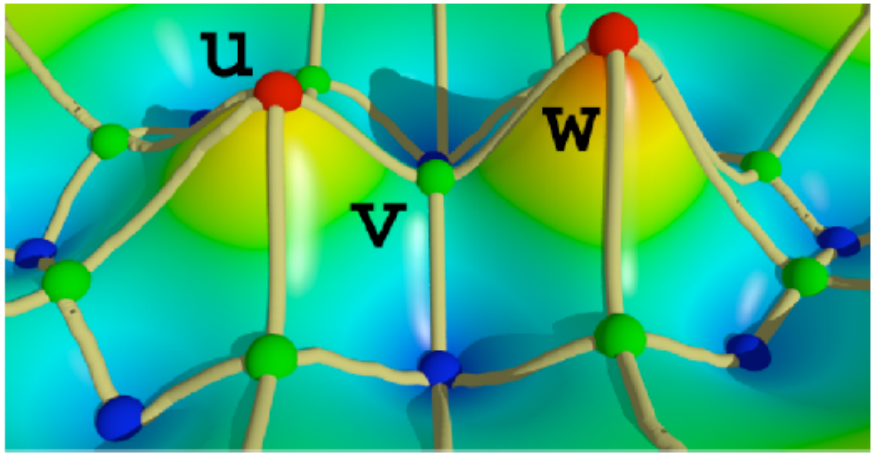
Section 1:
Review of some concepts in Morse theory

Morse Theory: Smooth Case



Morse Theory: Smooth Case

- Let $f: R^d \rightarrow R$ be a Morse function
- Gradient of f at x: $\nabla f(x) = \left[\frac{\partial f}{x_1}, \frac{\partial f}{x_2}, \ldots, \frac{\partial f}{x_d}\right]^T$



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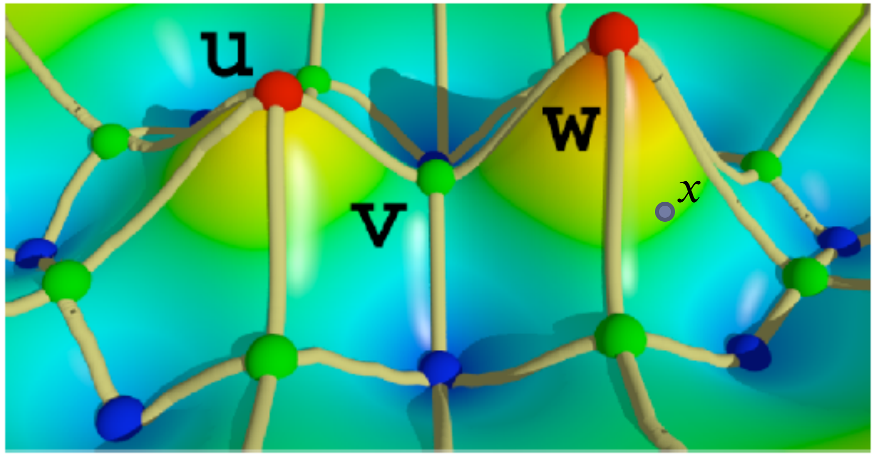

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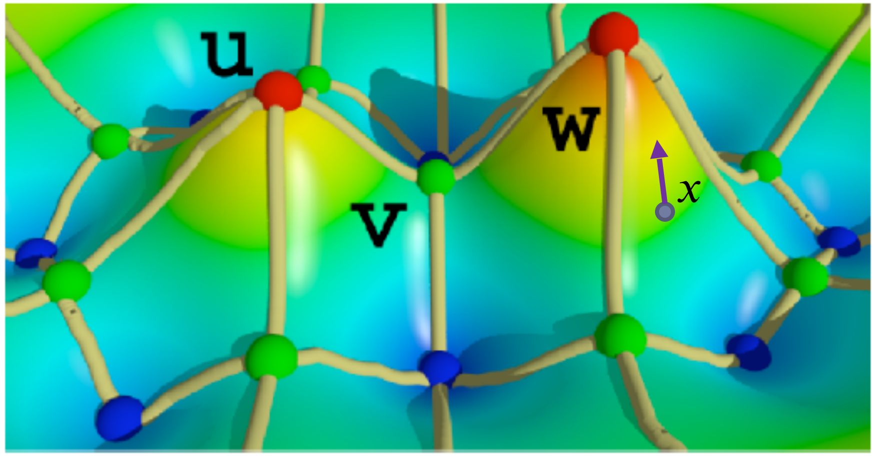

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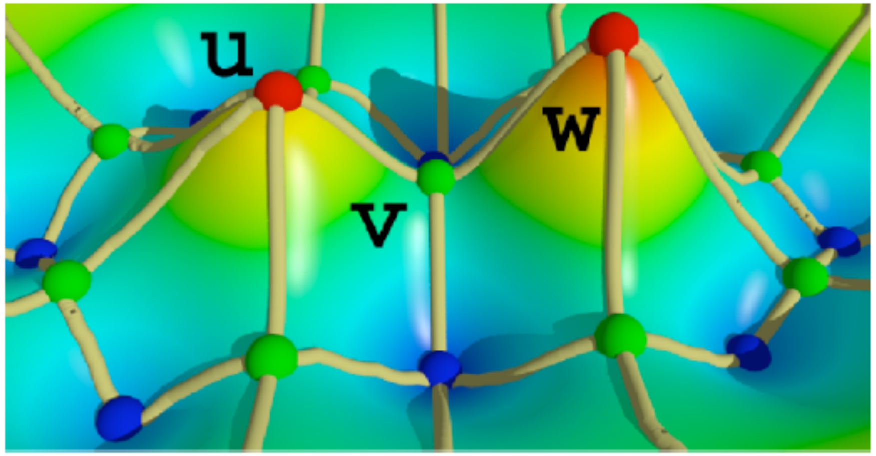
- Gradient of f at x: $\nabla f(x) = \left[\frac{\partial f}{x_1}, \frac{\partial f}{x_2}, \ \ldots, \frac{\partial f}{x_d}\right]^T$
- Critical points of f:
  - $\{ x \in R^d \mid \nabla f(x) = 0 \}$



Index of critical points

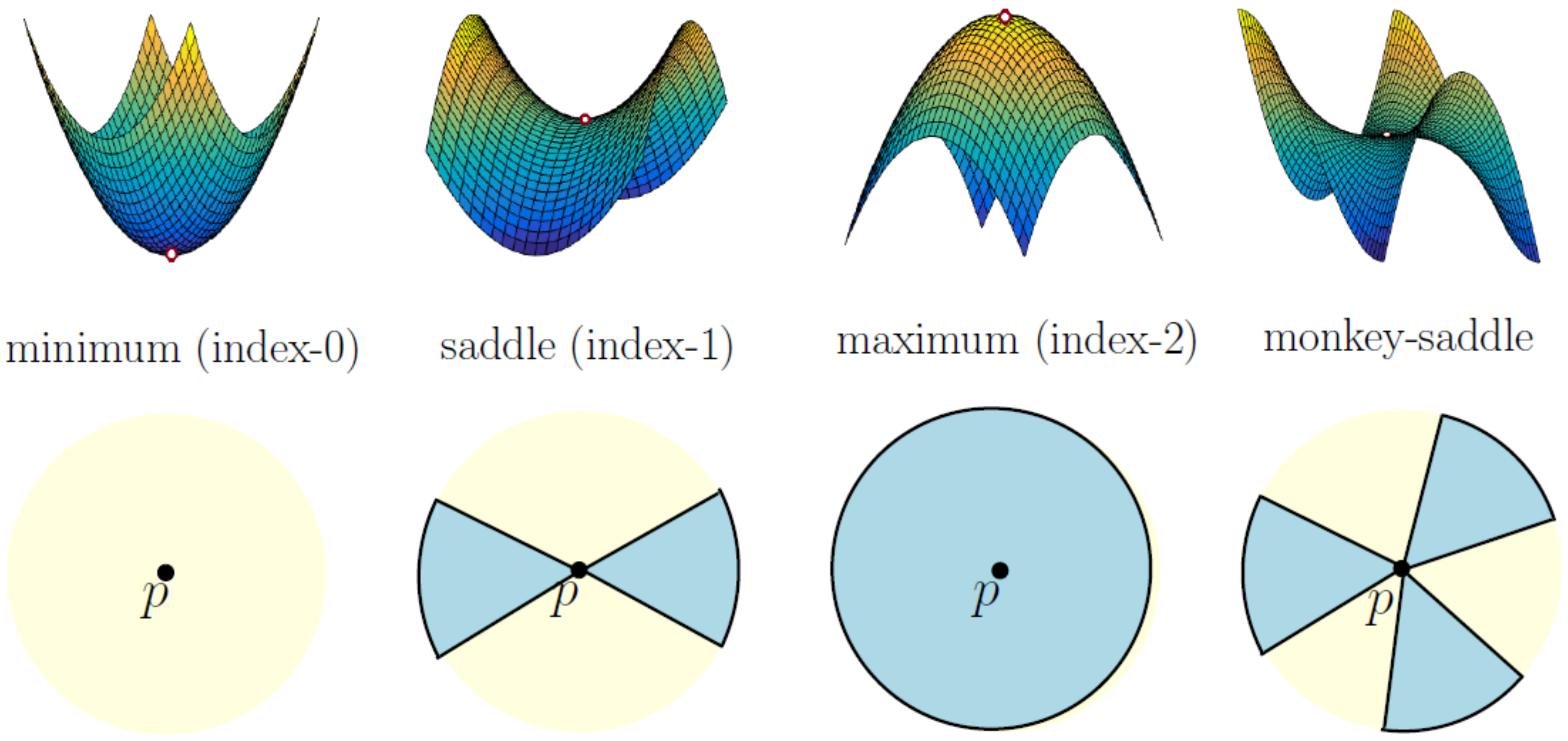
Index of critical points

Proposition 2 (Morse Lemma). *Given a smooth function $f : M \to \mathbb{R}$ defined on a smooth m-manifold M, let p be a non-degenerate critical point of f. Then there is a local coordinate system in a neighborhood $U(p)$ of p so that (i) the coordinate of p is* $(0, 0, \ldots, 0)$, *and (ii) locally for every point $x = (x_1, x_2 ... x_m)$ in neighborhood $U(p)$,*

$$f(x) = f(p) - x_1^2 - ...x_s^2 + x_{s+1}^2 ...x_m^2, \qquad \textit{for some} \;\; s \in [0, m].$$

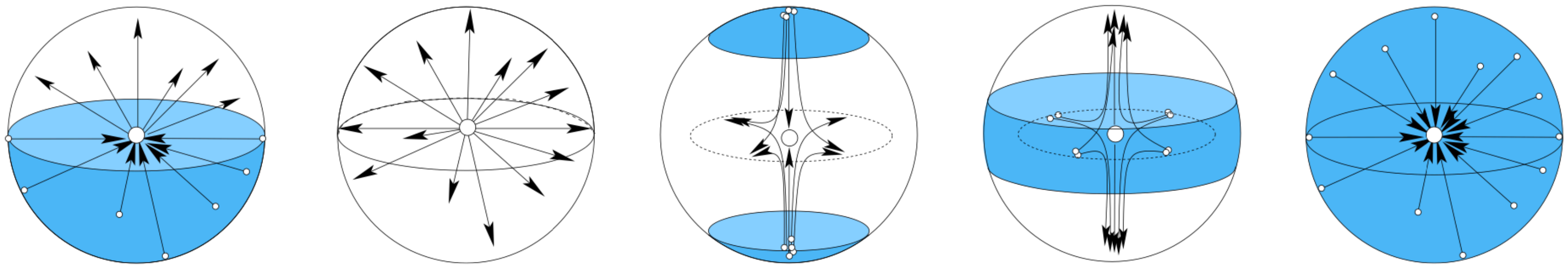
The number s of minus signs in the above quadratic representation of $f(x)$ is called the index of the critical point p.

Critical points and local view



Local View

- For d-dimensional manifold M
- An open neighborhood $B_r(p)$ of p
  - $B_r(p)$ is an m-dimensional open ball
  - Consider the boundary of the closure of $B_r(p)$



Courtesy of Gyulassy 2006

Classical results

Theorem 4. *Given a Morse function $f : M \to \mathbb{R}$ defined on a smooth manifold M, let p be an index-k critical point of f with $\alpha = f(p)$. Assume $f^{-1}([\alpha - \varepsilon, \alpha + \varepsilon])$ is compact for a sufficiently small $\varepsilon > 0$ such that there is no other critical points of f contained in this interval-level set other than p. Then the sublevel set $M_{\leq \alpha+\varepsilon}$ has the same homotopy type as $M_{\leq \alpha-\varepsilon}$ with a k-cell attached to its boundary* $\mathrm{Bd}\, M_{\leq \alpha-\varepsilon}$.

- A compact manifold is homotopy equivalent to a cell complex

0-cell 1-cell 2-cell 3-cell K-cell

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- A compact manifold is homotopy equivalent to a cell complex
- Morse inequalities:
  - (Weak one) # index k critical points = $c_k \geq \beta_k(M)$
  - (Strong one) $\sum_{i=0}^{k} (-1)^{k-i} c_i \geq \sum_{i=0}^{k} (-1)^{k-i} \beta_i$

Morse Theory: Smooth Case

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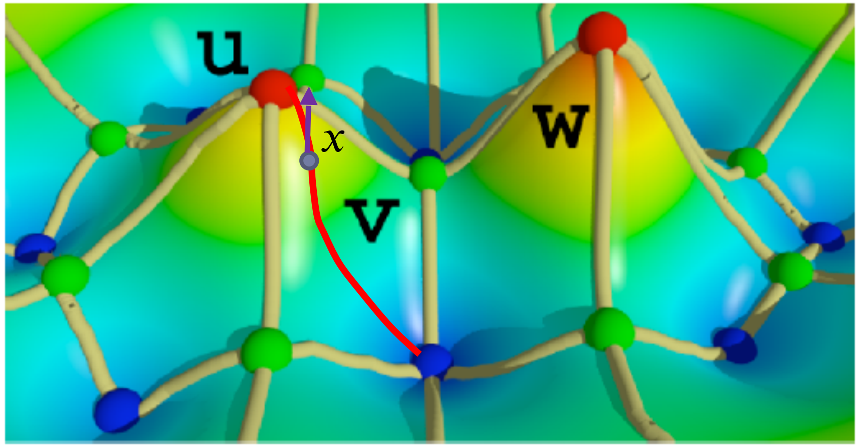
  - a maximal path in R^d whose tangent vectors agree with gradient of f at every point of the path



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- An integral line $L: (0, 1) \rightarrow R^d$:
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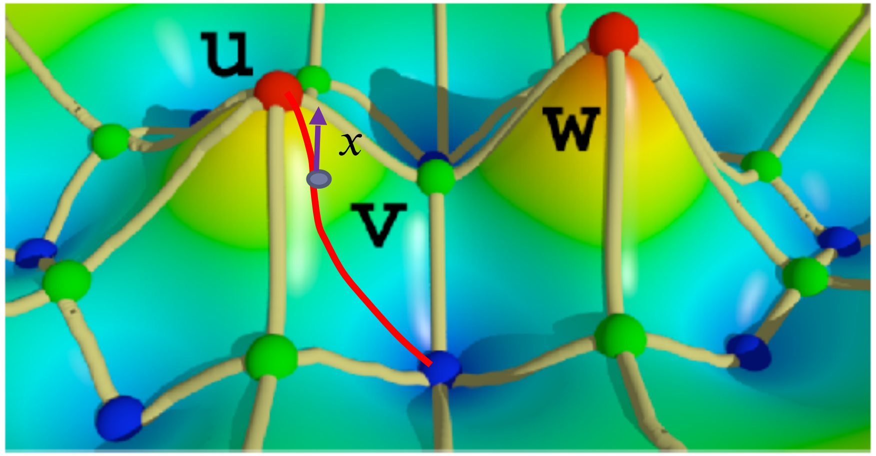
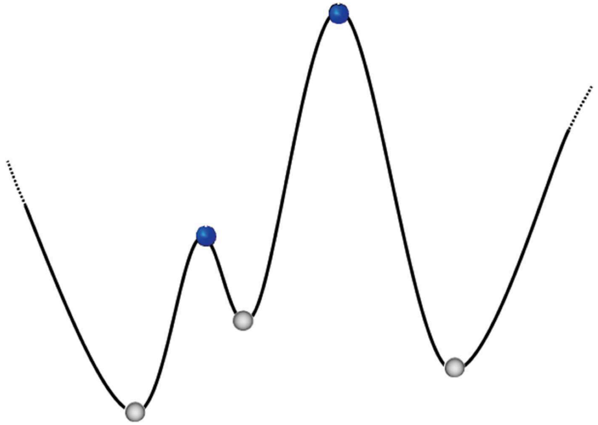
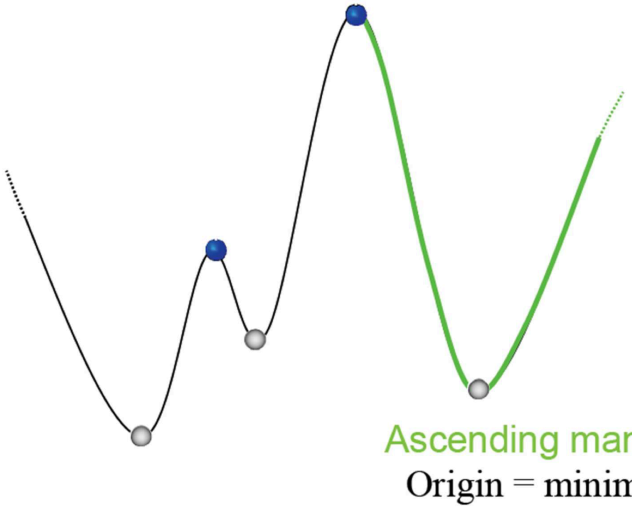
  - has origin and destination at critical points
    - $Dest(L) = \lim_{p \to 1} L(p)$
    - $Ori(L) = \lim_{p \to 0} L(p)$

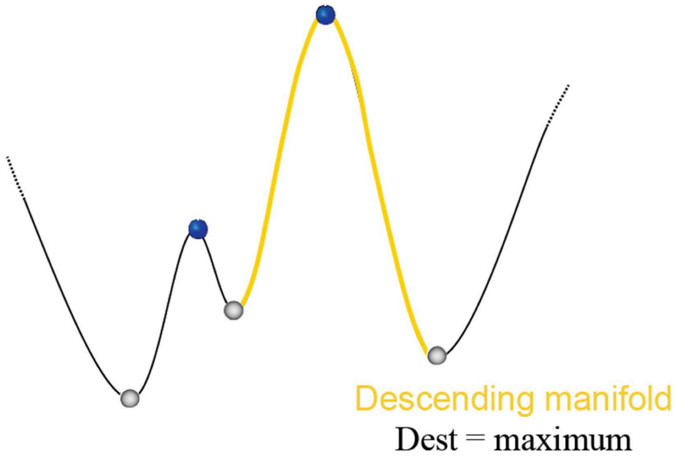


Stable / Unstable Manifolds

- Given a critical point x of f
  - Stable (descending) manifold $S(x) = \left\{ \ y \in R^d \mid dest\left(y\right) = x \ \right\}$
  - Unstable (ascending) manifold $U(x) = \{ \ y \in R^d \mid ori\left(y\right) = x \ \}$
- 1-d example (pictures from Guoning Chen)







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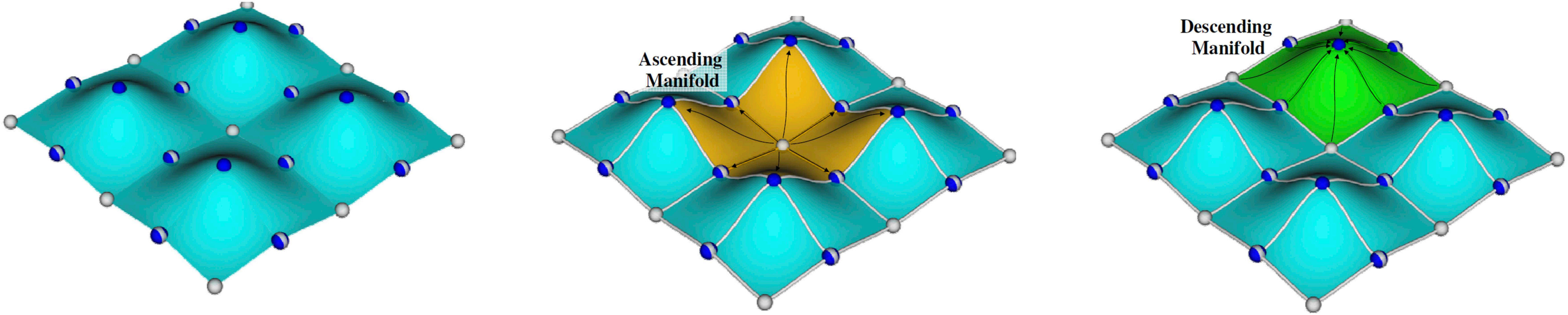
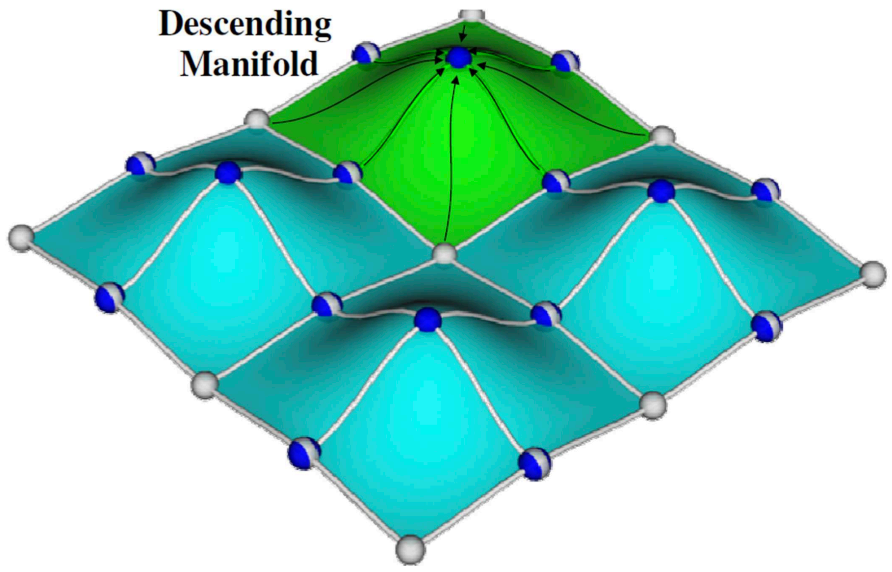
- 2-d example (pictures from Guoning Chen)



Morse Complex

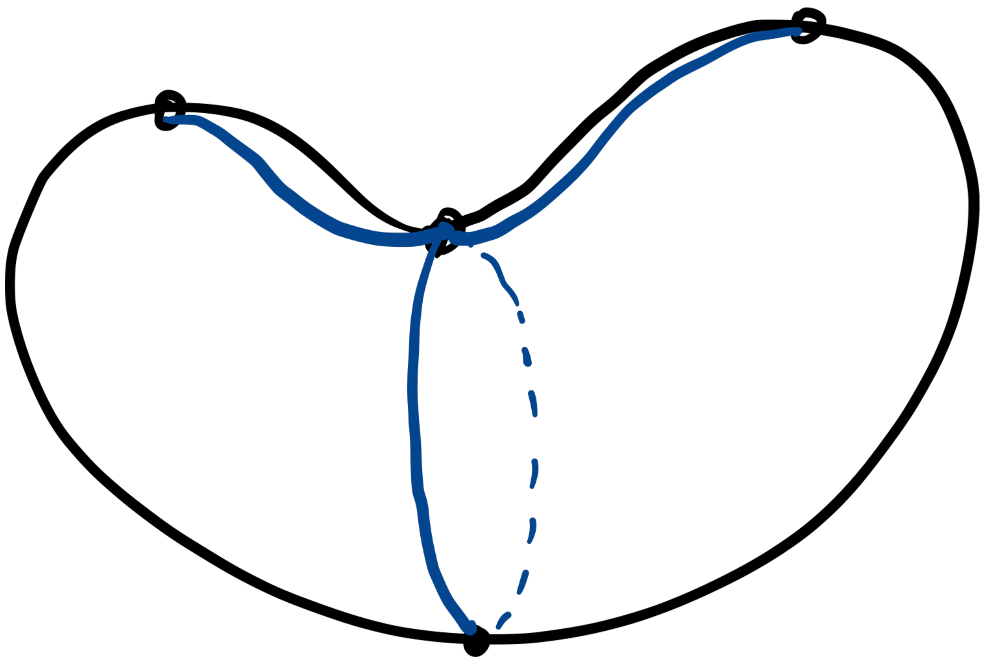
- Descending manifolds give rise to a cell complex structure to the manifold M. We call this the **Morse Complex.**

- Chain complex structure
  - MC_k is the vector space generated by index k critical points
  - $\partial : MC_k \rightarrow MC_{k-1}$ counts the number of connecting orbits
  - Morse homology coincides with singular homology



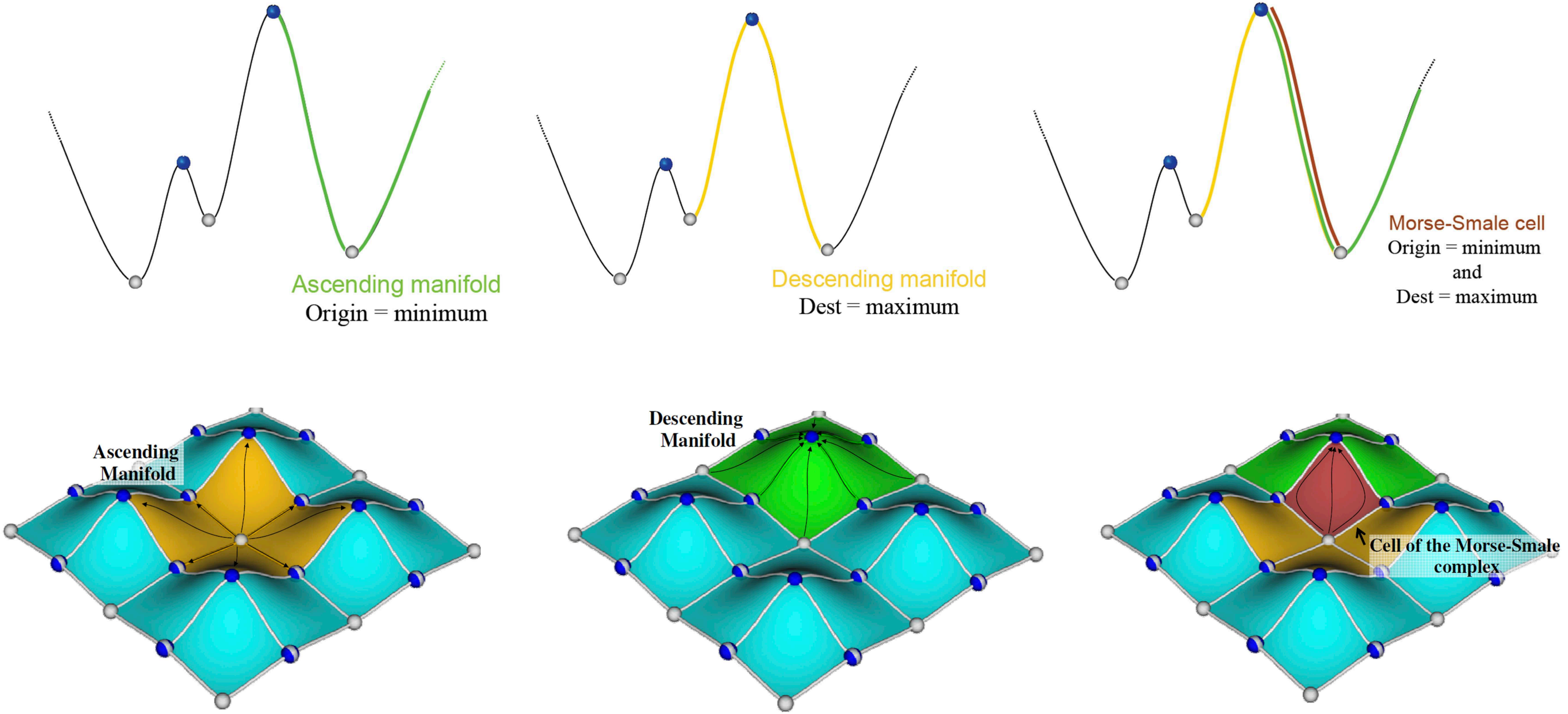
Example

- $0 \to \mathbb{Z}_2^2 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0$



Morse Complex and Morse Smale Complex

- Descending manifolds give rise to a cell complex structure to the manifold M. We call this the Morse Complex.
- When the Morse function $f : M \to \mathbb{R}$ is "good", we consider intersections of ascending and descending manifolds which is a new cell complex structure called **Morse-Smale** Complex



Ascending manifold
Origin = minimum
Descending manifold
Dest = maximum
Morse-Smale cell
Origin = minimum
and
Dest = maximum
Ascending
Manifold
Descending
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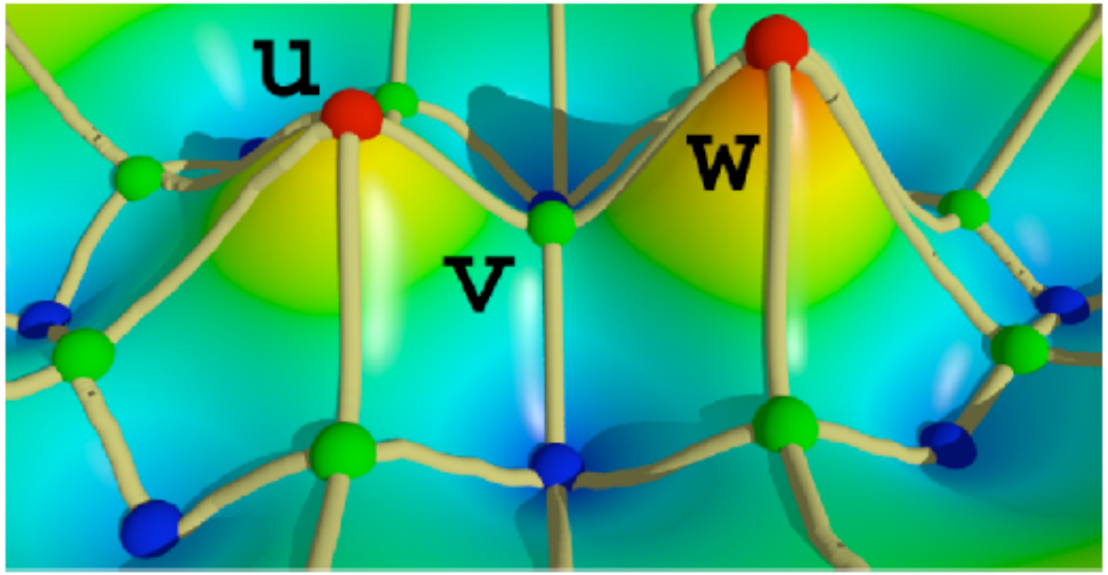
Cell of the Morse-Smale
complex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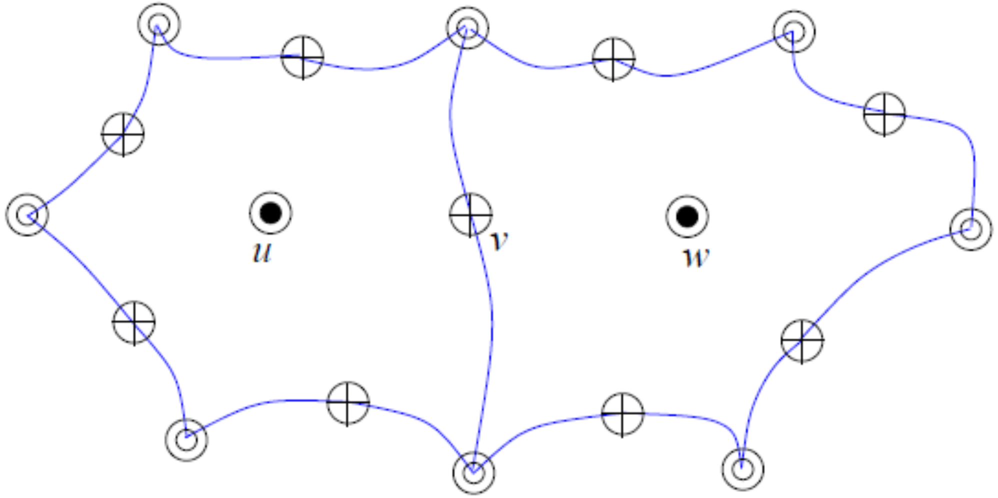
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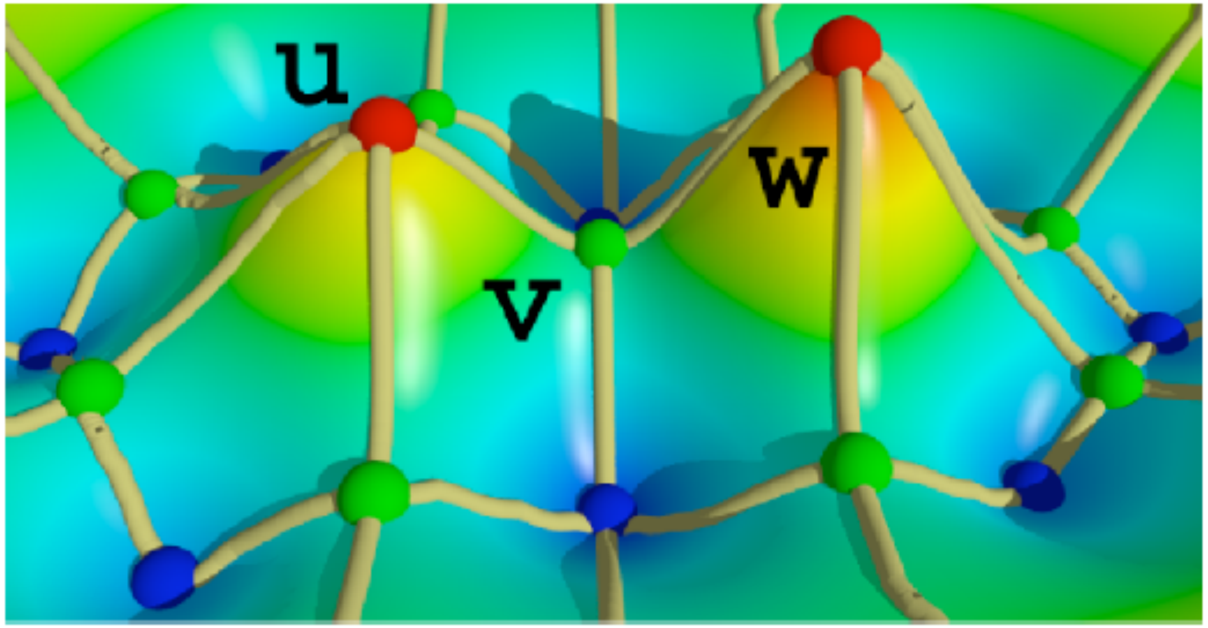
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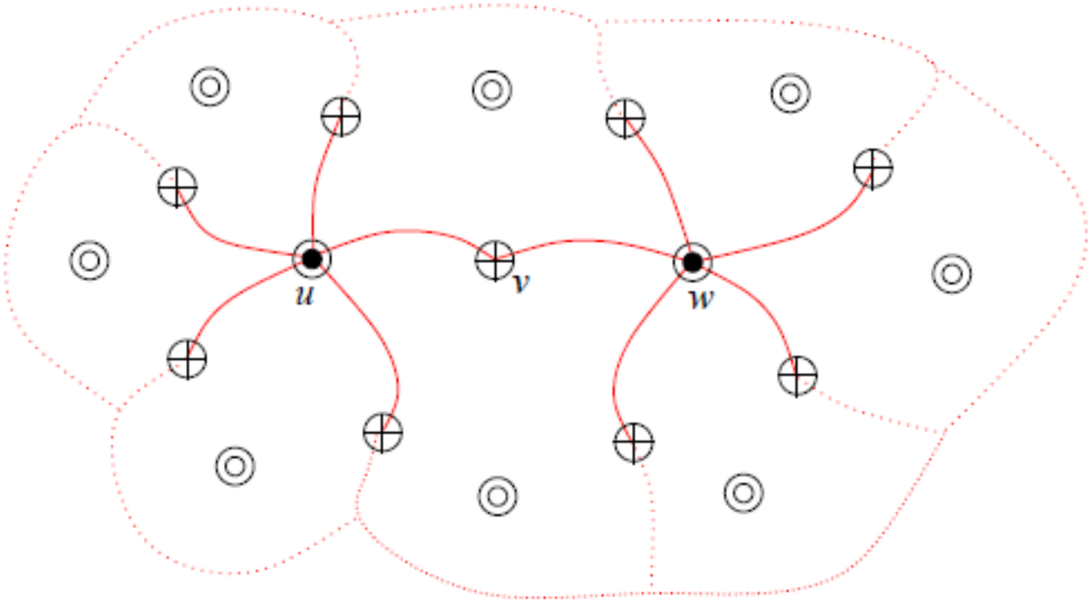
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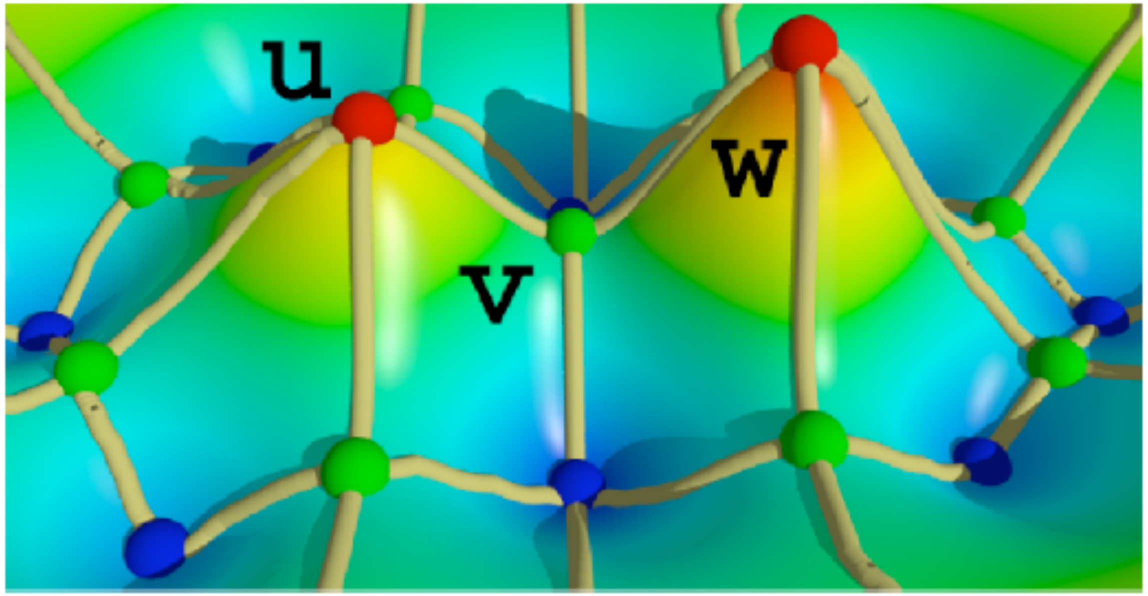

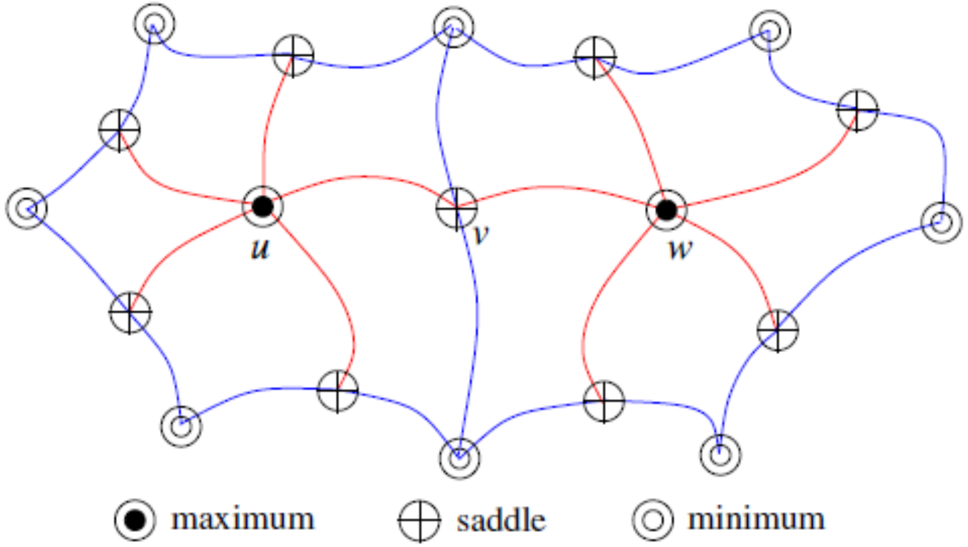

Decomposition into monotonic regions

Remarks

- Critical points capture where topology changes in sub- / super- / level sets
- Stable / non-stable manifolds characterize skeletons of different dimensions
- Morse-Smale complex decompose input space into cells with the same “flow”-behavior of functions
  - can be used e.g, for segmentation via using appropriate descriptor functions

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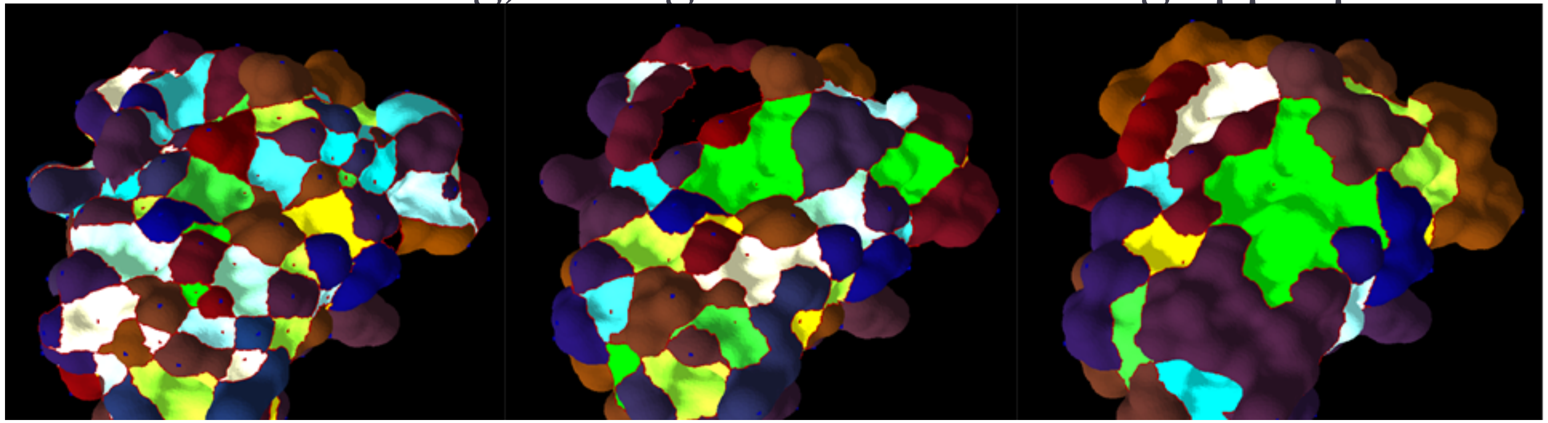

segmentation at different scales via curvature-based descriptor functions

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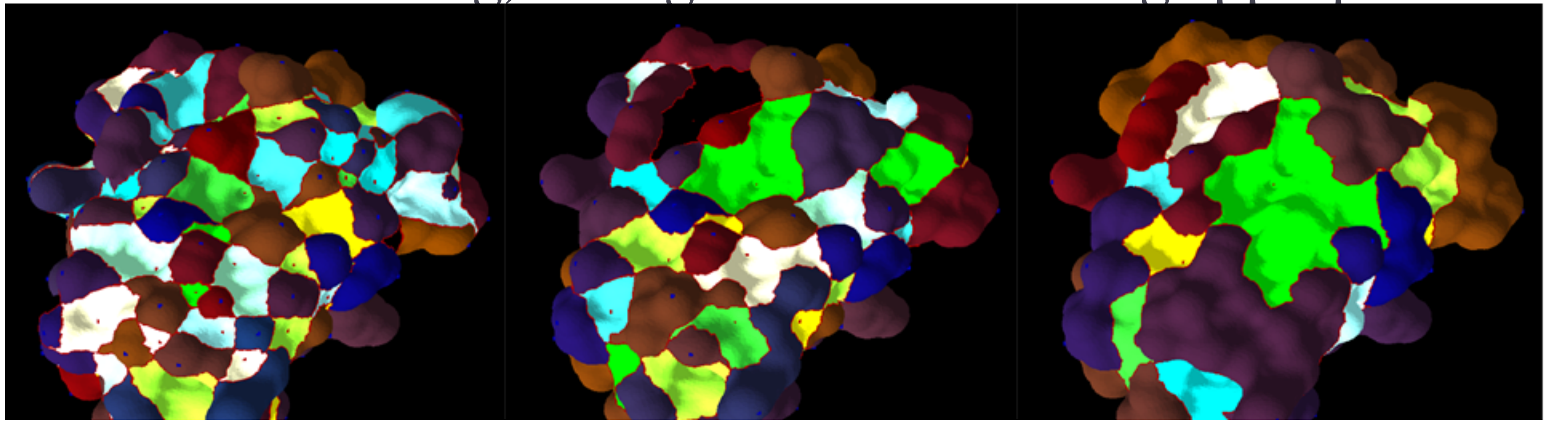

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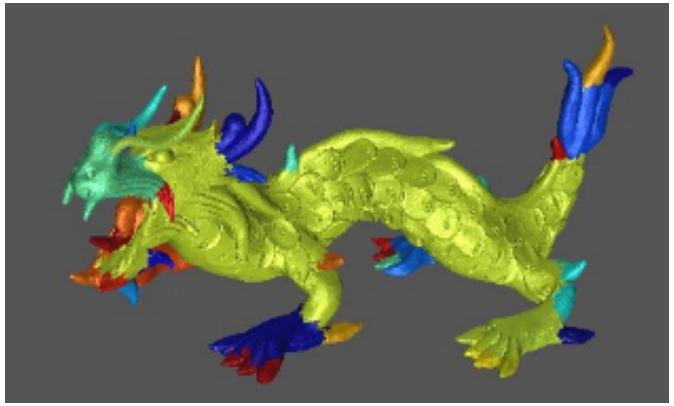
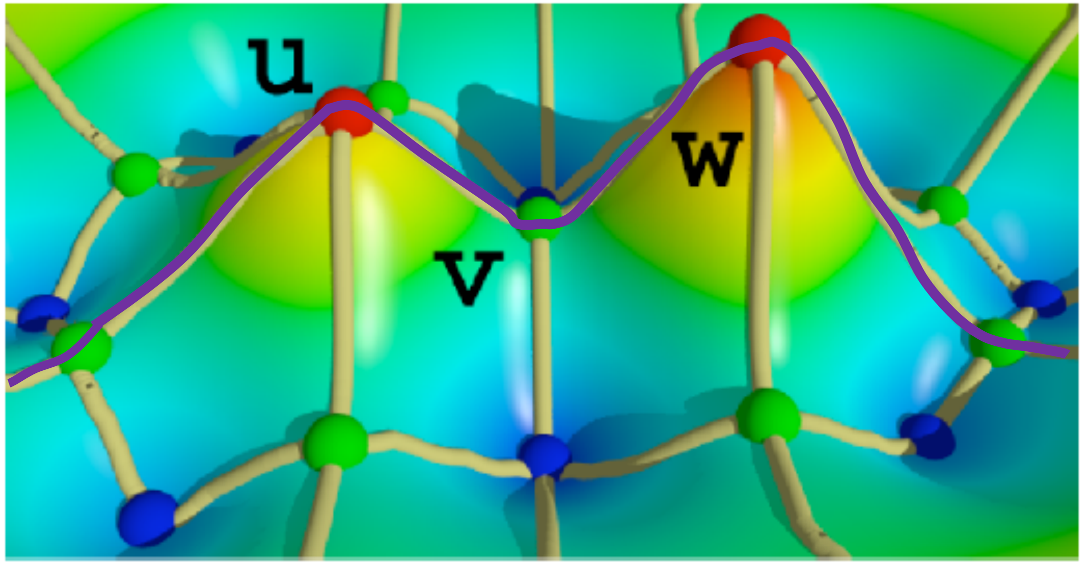
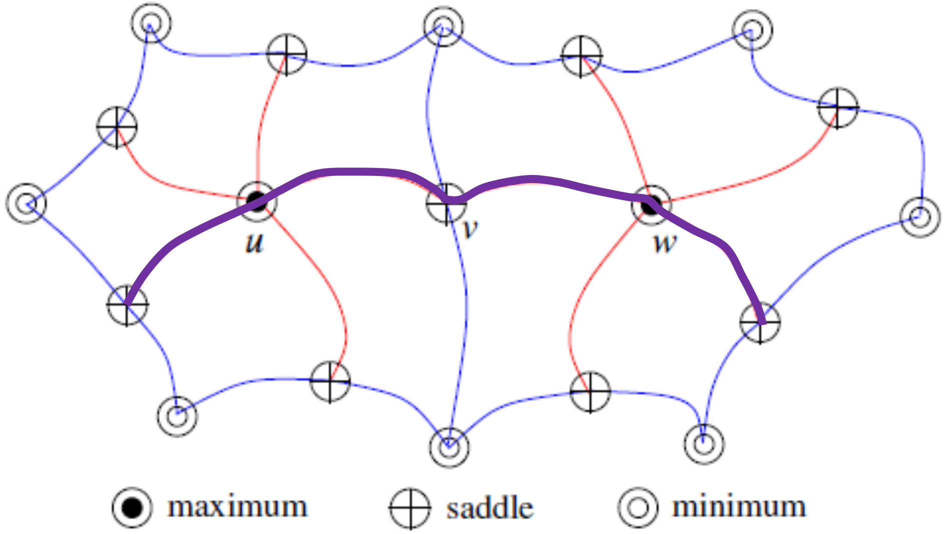


segmentation based on heat-kernel signature

Remarks

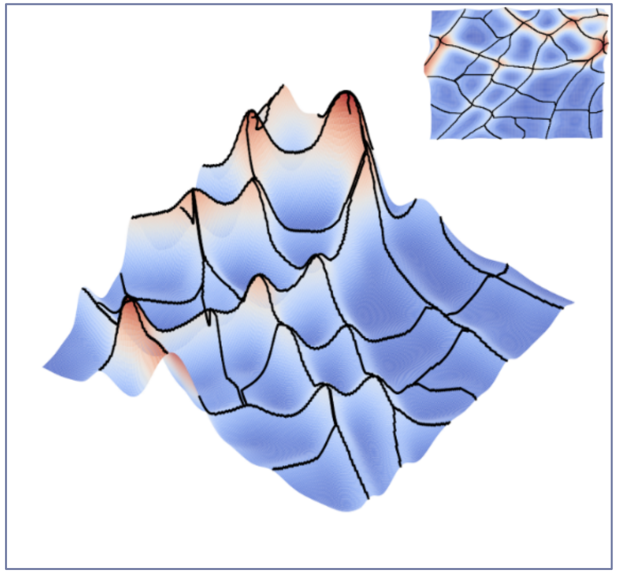




Section 2:
Discrete Morse theory

Discrete Case

- Smooth case
  - 1-unstable manifold from Morse theory



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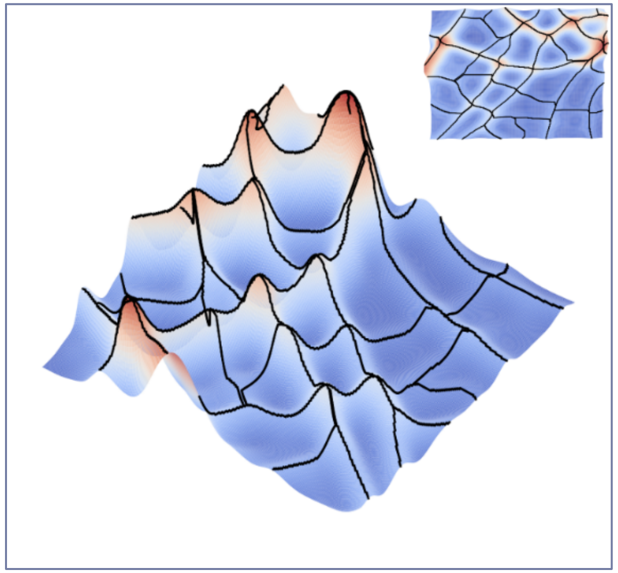
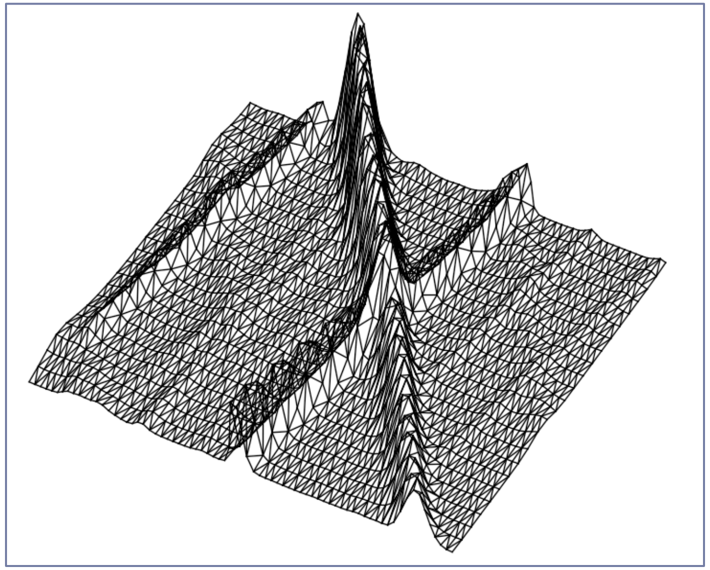


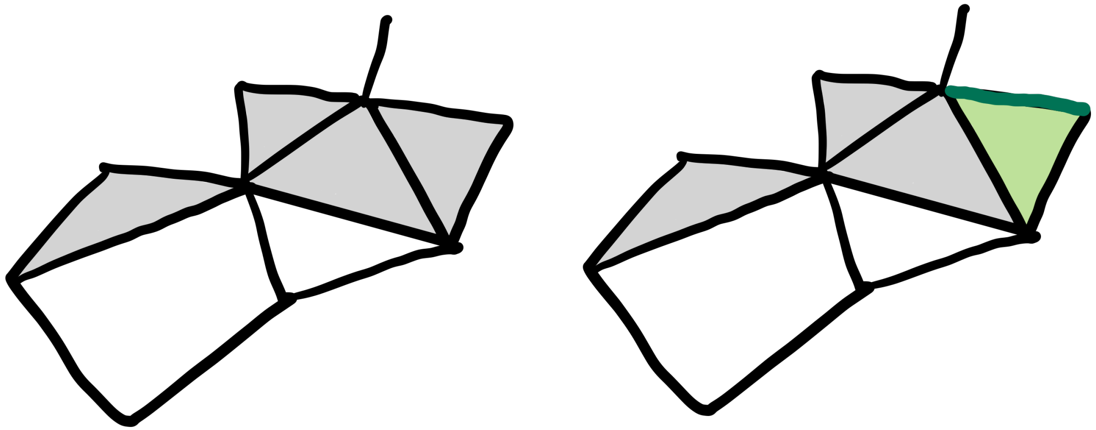
- Discrete case
  - Piecewise-linear (PL) approximation?



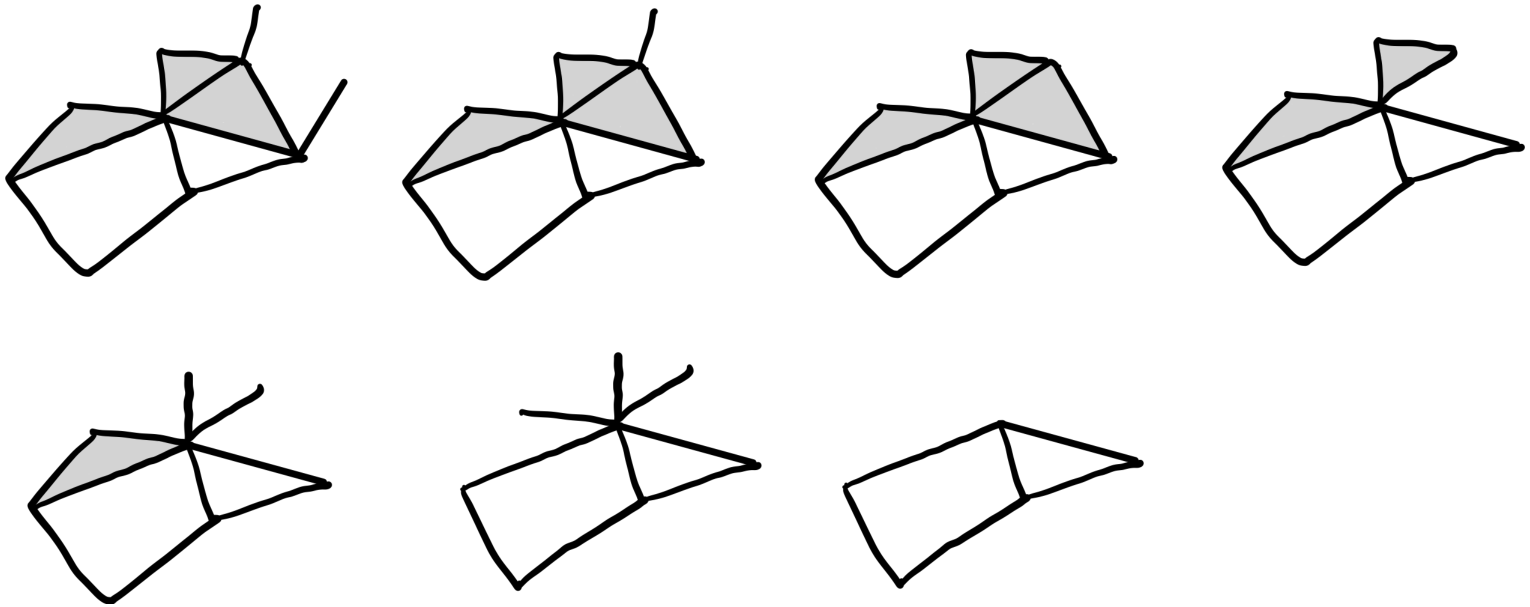
Discrete Morse Theory

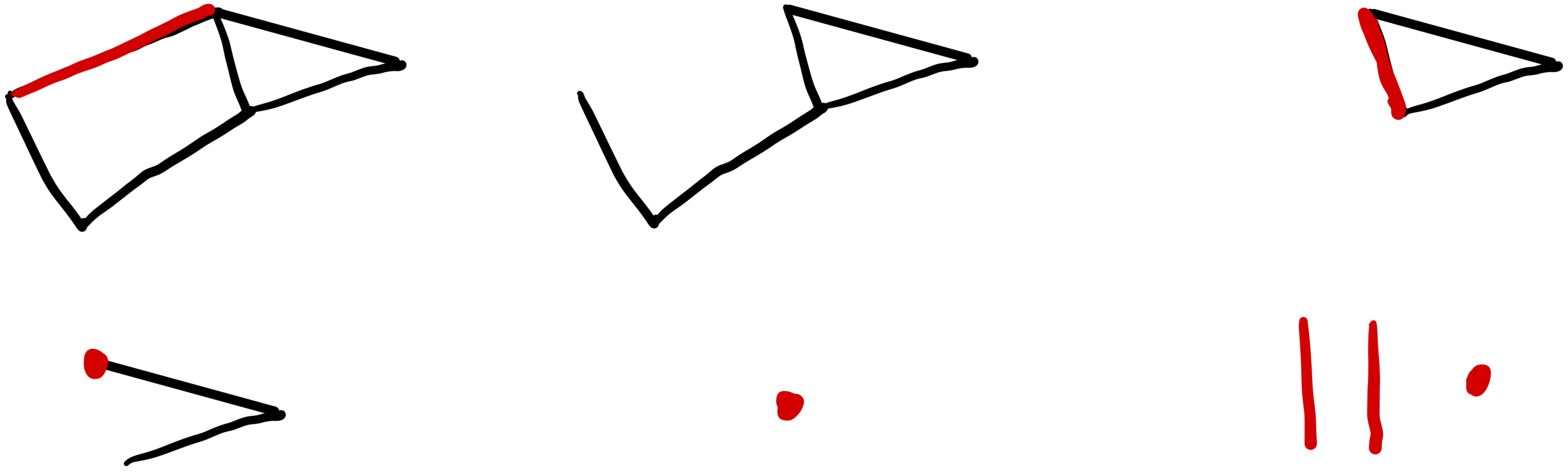
- [Forman 1998, 2002]
- Combinatorial version of Morse theory
- Many results analogous to classical Morse theory
- Works for cell complexes

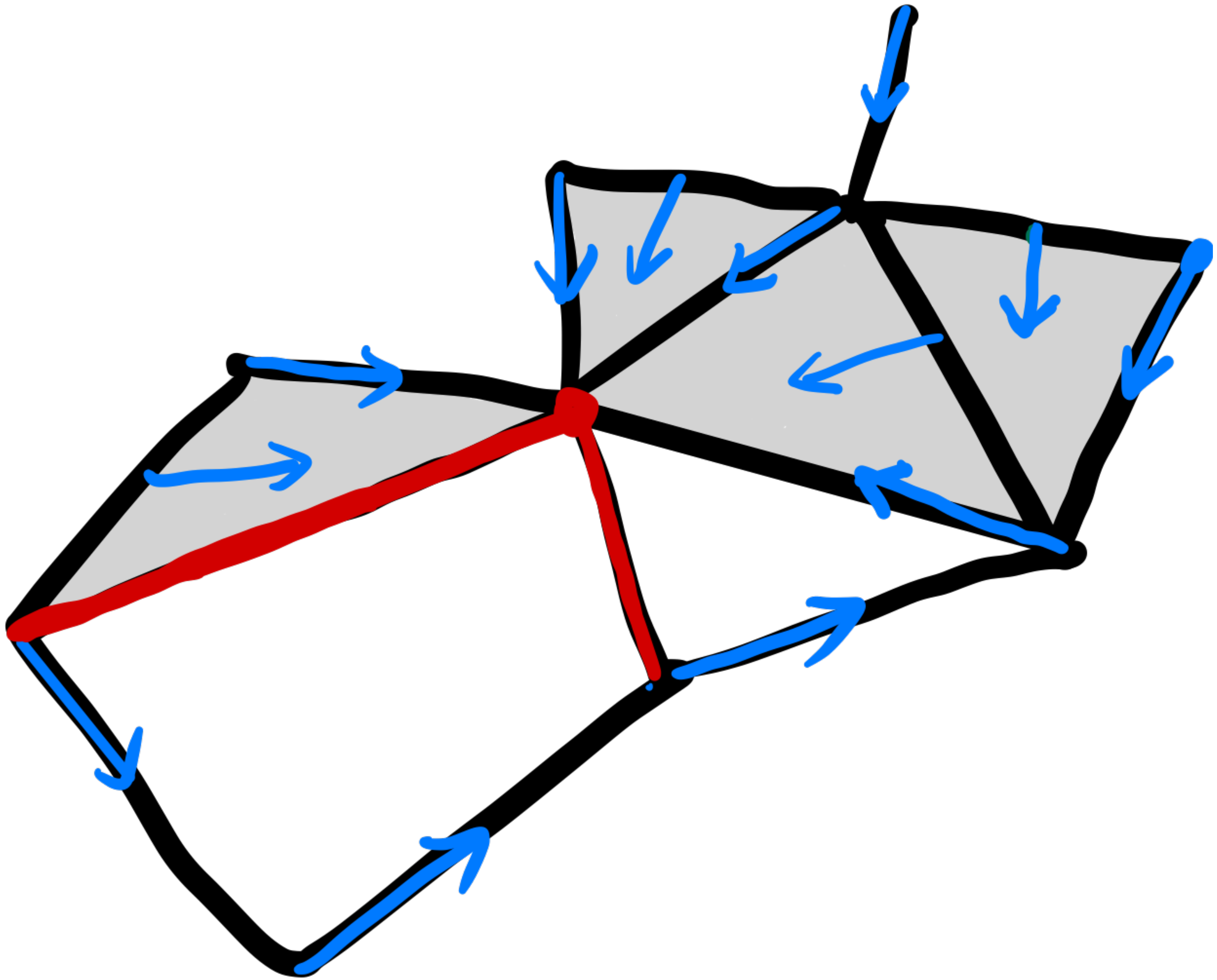
- Two perspectives
  - Discrete Vector fields
  - Discrete Morse functions



- Free pair: $\tau \subset \sigma$ but τ has no other co-face
- Elementary collapse: will not change homotopy type

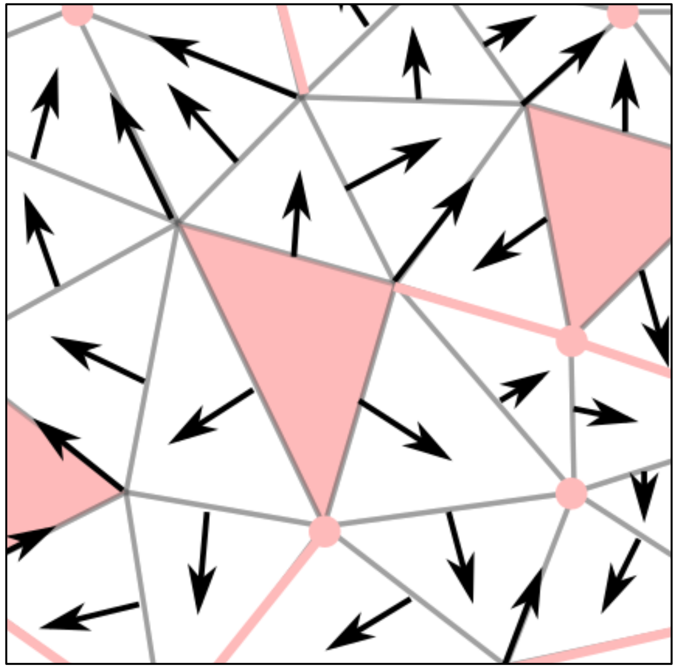


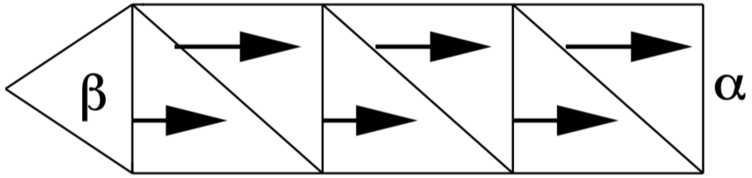


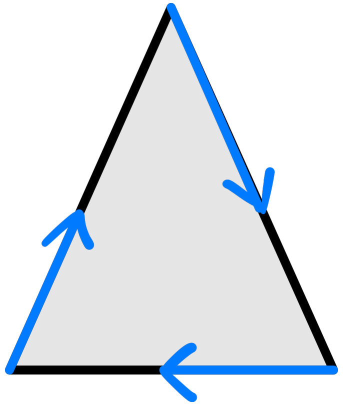


Discrete Gradient Vector Field

- Given a simplicial complex K, a discrete (gradient) vector
  - (σ, τ) s.t. $\sigma < \tau$ (e.g., *vertex-edge or edge-triangle pair*)

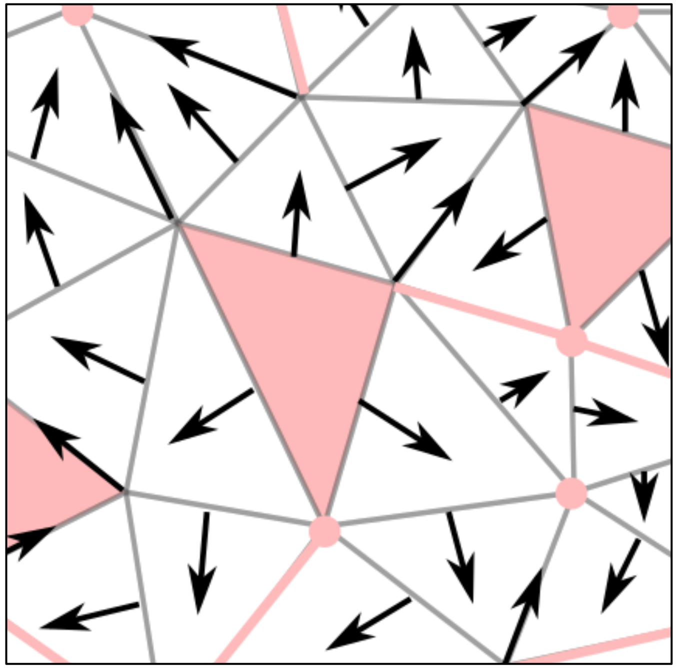


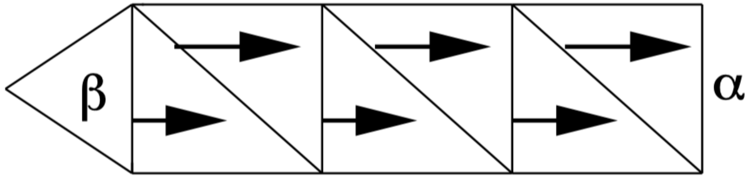


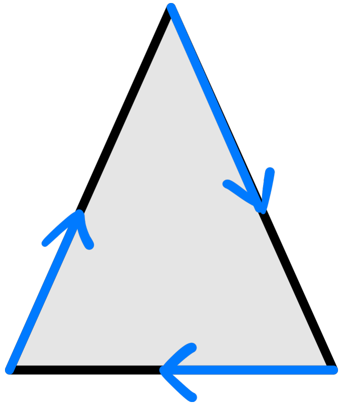


Discrete Gradient Vector Field

- Given a simplicial complex K, a discrete (gradient) vector
  - (σ, τ) s.t. $\sigma < \tau$ (e.g., *vertex-edge or edge-triangle pair*)
- A collection of discrete vectors V is a discrete vector field such that each simplex appears in at most one pair
- V-path: $\sigma_0^{(p)}, \tau_0^{(p+1)}, \ldots, \sigma_{k-1}^{(p)}, \tau_{k-1}^{(p+1)}, \sigma_k^{(p)}$ such that
  - $(\sigma_i^{(p)}, \tau_i^{(p+1)}) \in V$
  - $\tau_i^{(p+1)} \supset \sigma_{i+1}^{(p)}$
  - This path is said to be **closed** if $\sigma_0^{(p)} = \sigma_k^{(p)}$







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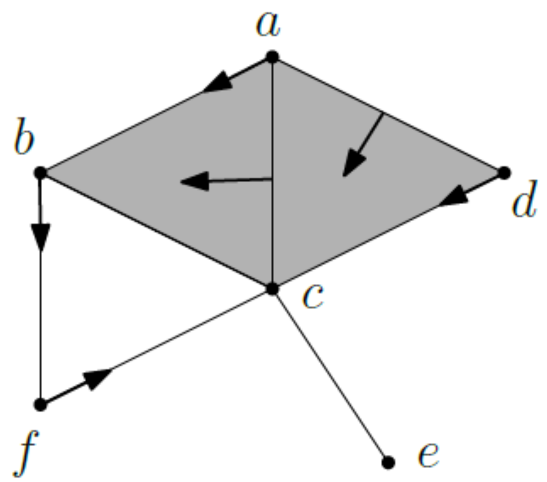

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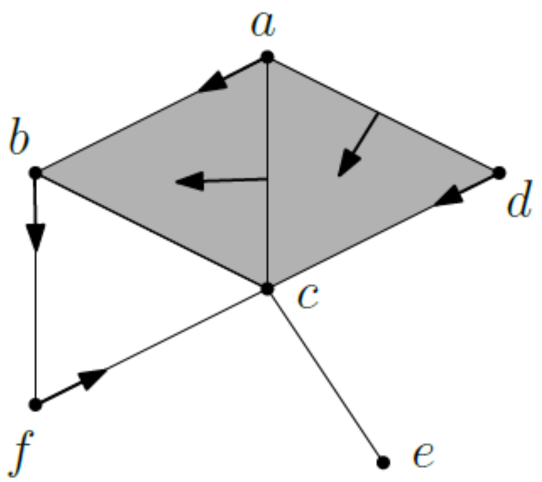

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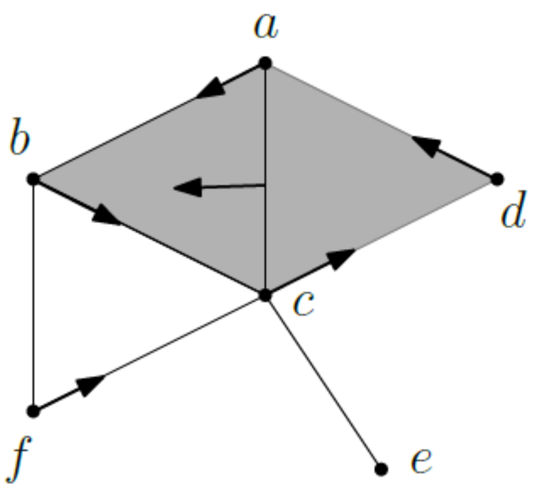

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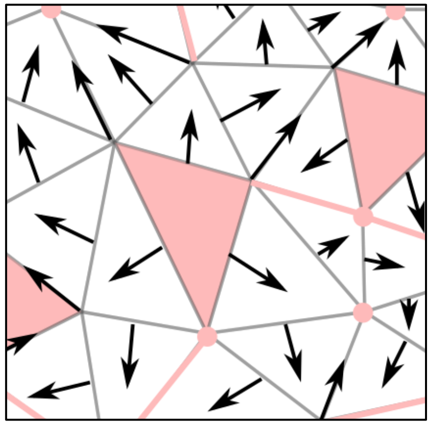
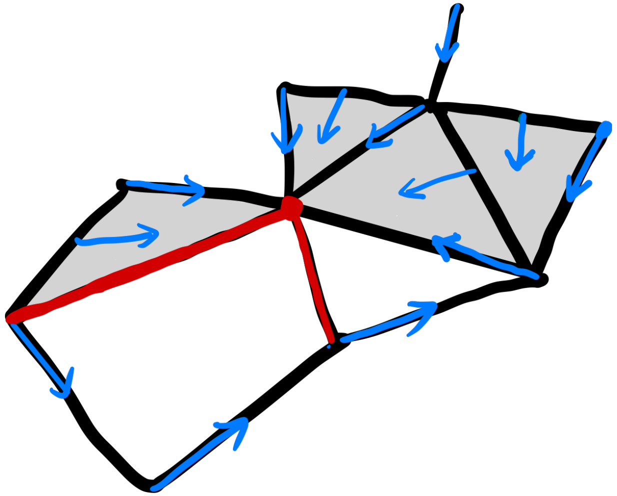
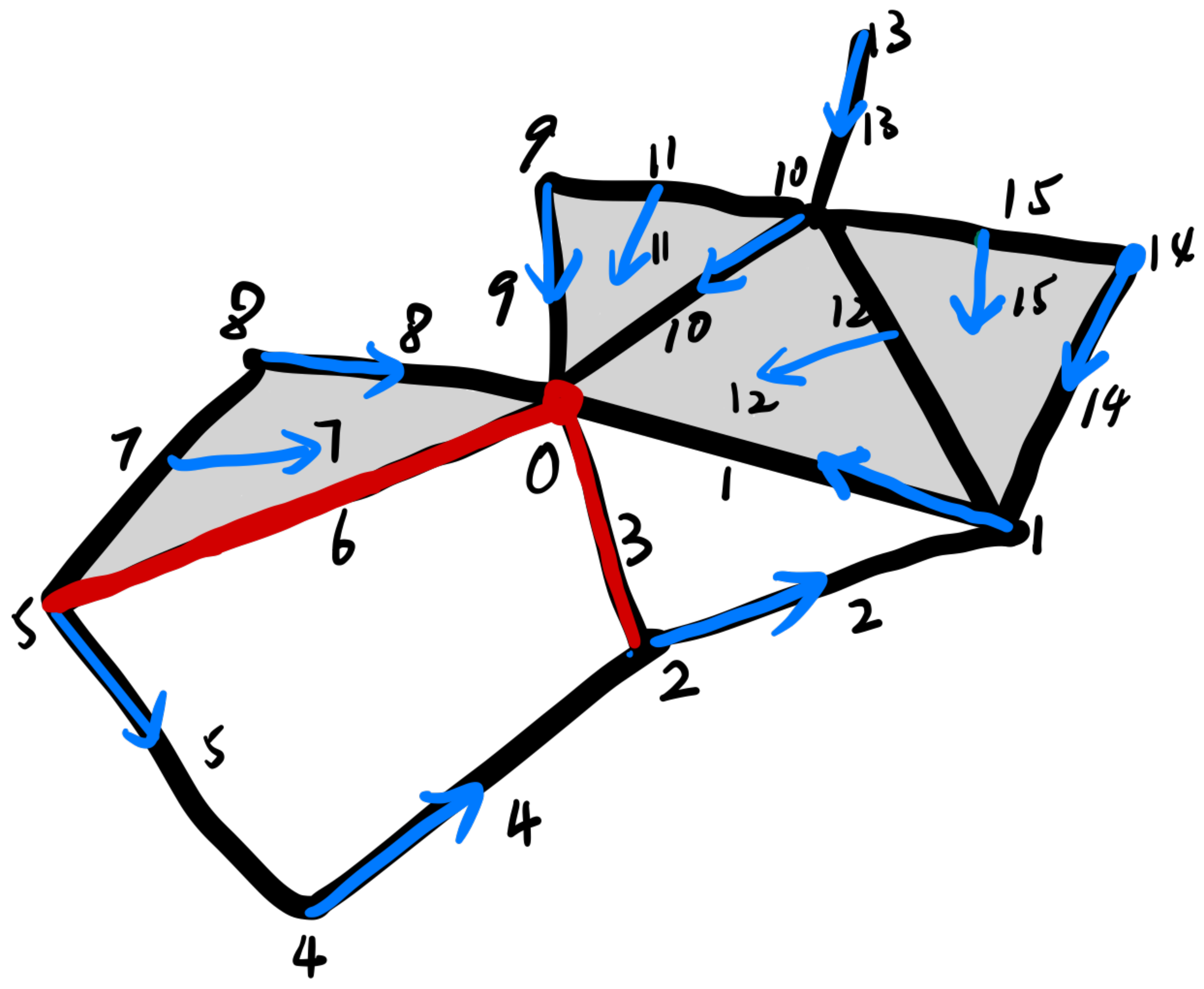


- Discrete gradient vector field $M(K)$:
  - a collection of discrete vectors s.t
    - each simplex appears in at most one vector
    - no cyclic V-path in $M(K)$



- A simplex σ is **critical**, if
  - it does not appear in any vector in $M(K)$



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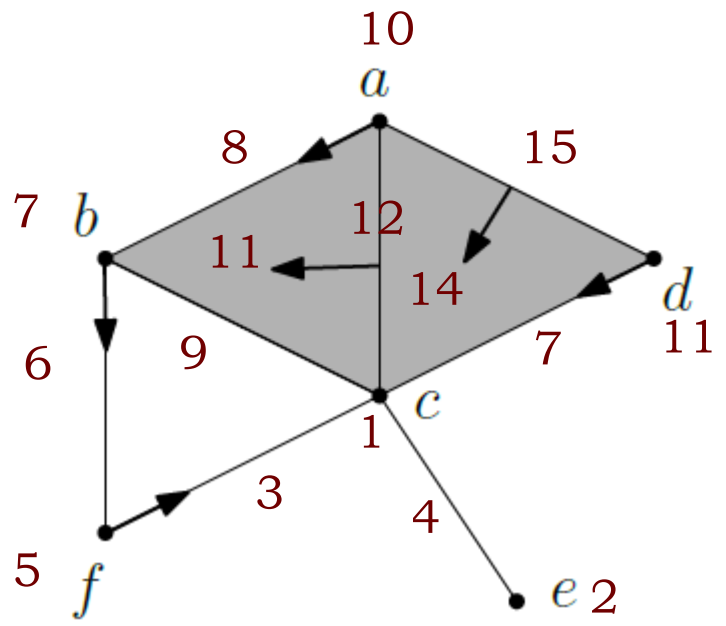
Discrete Morse Function

- Given a simplicial complex K, a discrete Morse function is a simplex-wise function $f\colon K \to R$ that assigns higher numbers to higher dimensional simplices, with at most one exception, locally, at each simplex:
  - (C1) $\#\{\sigma^{p-1} \mid \sigma^{p-1}$ is a face of τ^p and $f(\sigma^{p-1}) \geq f(\tau^p)\} \leq 1$
  - (C2) $\#\{\alpha^{p+1} \mid \alpha^{p+1}$ is a co-face of τ^p and $f(\alpha^{p+1}) \leq f(\tau^p)\} \leq 1$
- It is known that at most one of (C1) and (C2) can have ` $= 1$’



Discrete Morse Function

- Given a discrete Morse function $f: K \to R$, a simplex τ^p is called critical if
  - (C1) $\#\{\sigma^{p-1} \mid \sigma^{p-1}$ is a face of τ^p and $f(\sigma^{p-1}) \geq f(\tau^p)\} = 0$
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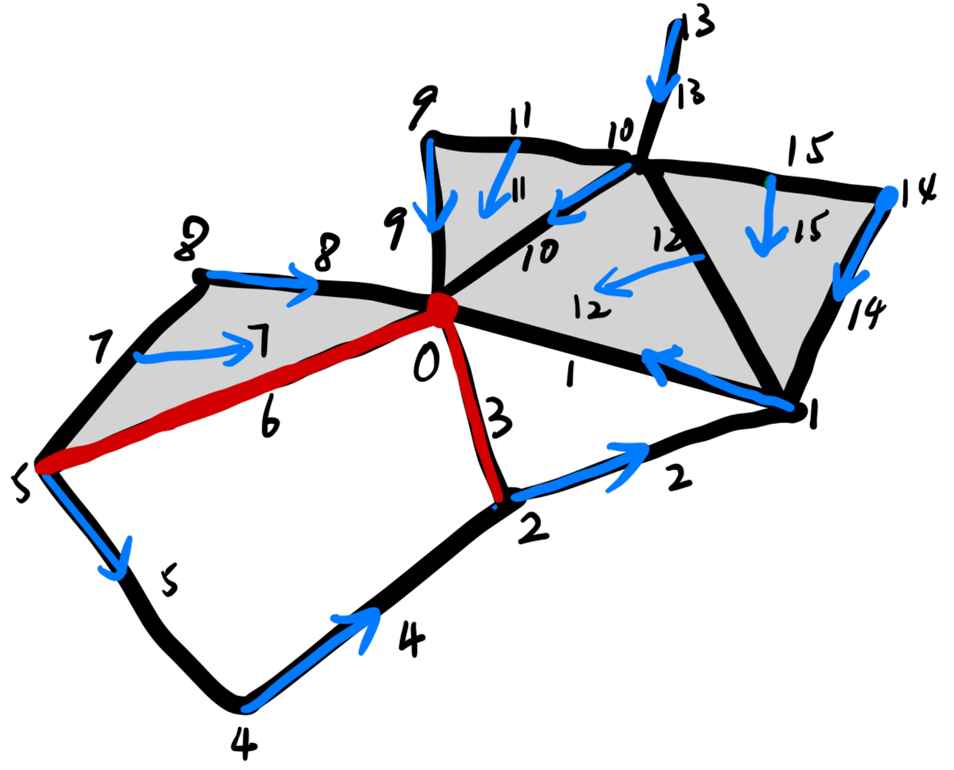

Connecting the two!

- Now the input is a cell complex – we focus on simplicial complex K from now on.

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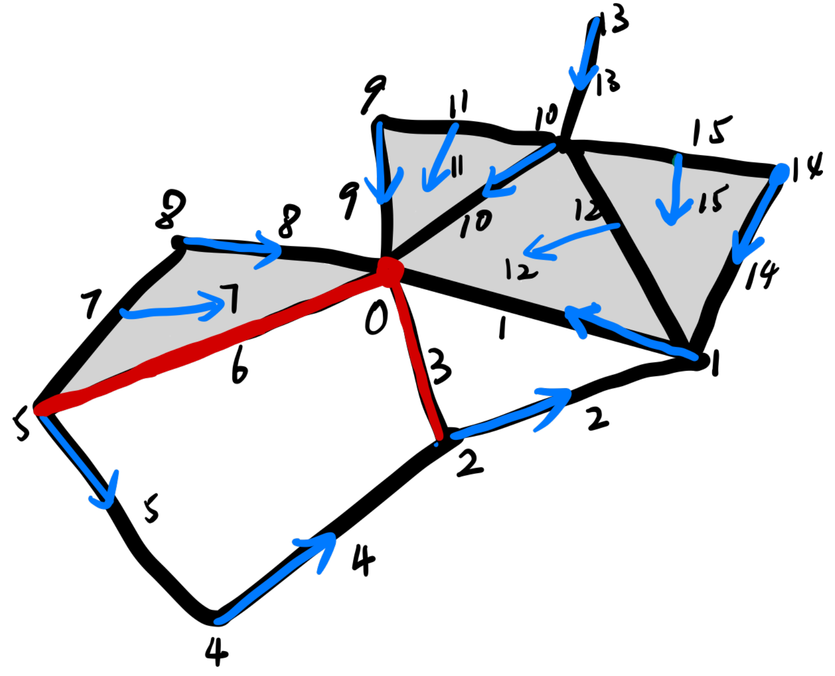

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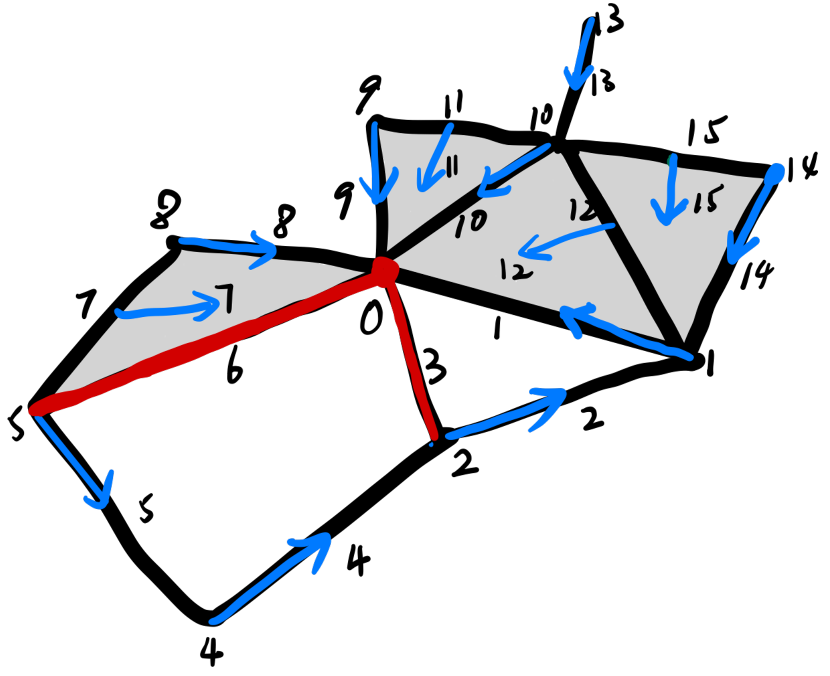
- It is known that at most one of (C1) and (C2) can have ` $= 1$’

- The collection $\{(\sigma,\ \tau)\}$ that makes either (C1) or (C2) ` $= 1$’
  - exactly forms a valid discrete gradient vector field



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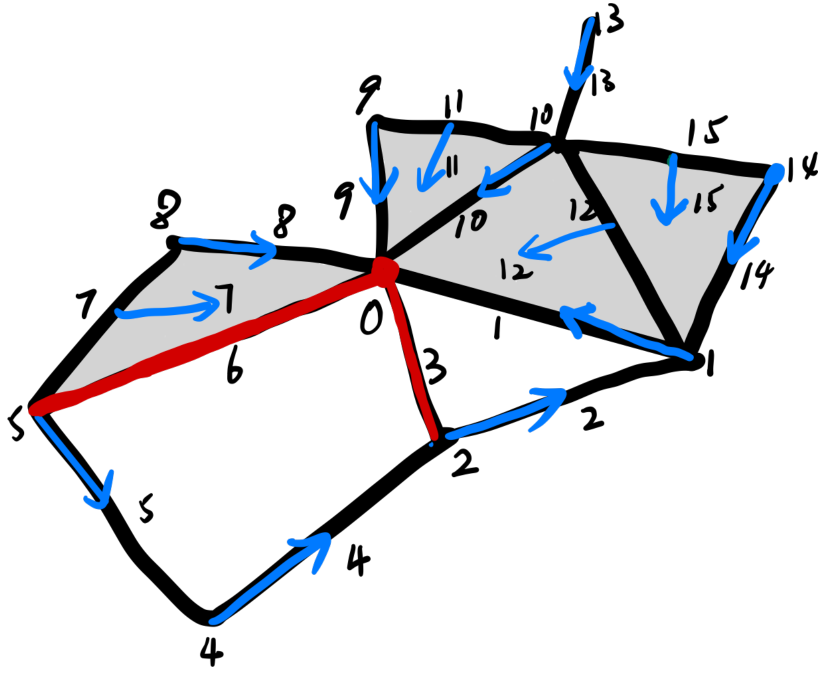
- That is, given a discrete Morse function,
  - Each pair $(\sigma,\ \tau)$ that makes either (C1) or (C2) ` $= 1$’ gives rise to

  a discrete gradient vector



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- On the other hand, given a discrete gradient vector field,
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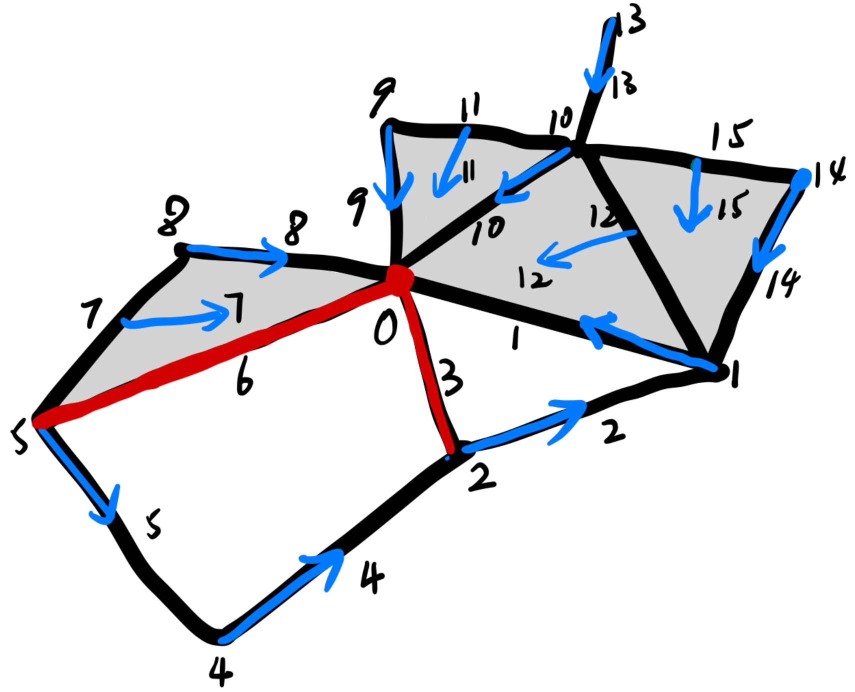

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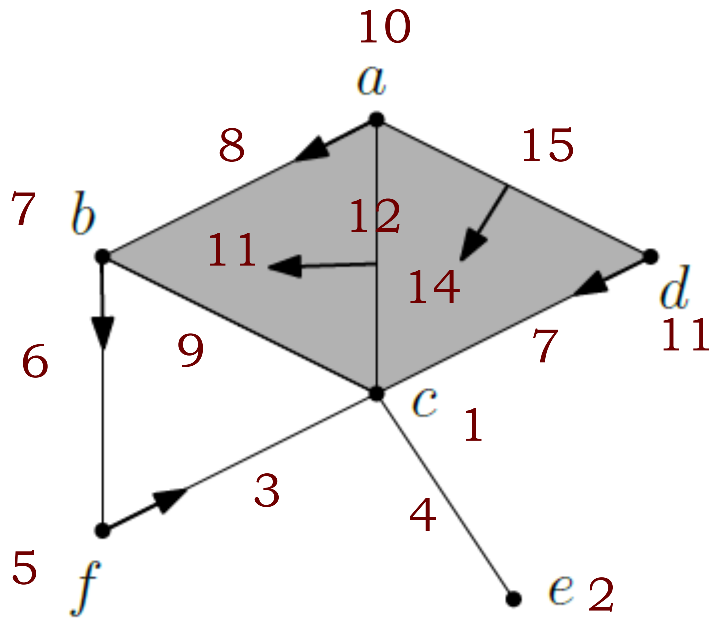

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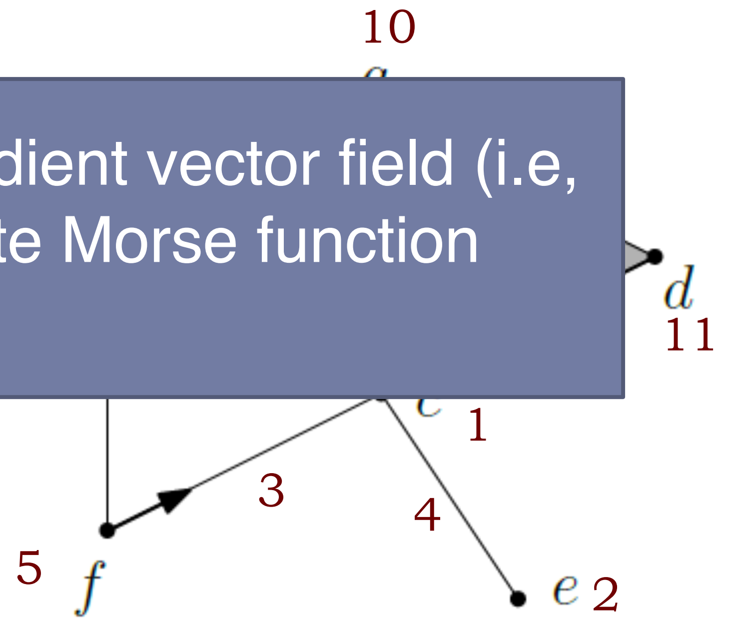
It is often more convenient to talk about just discrete gradient vector field (i.e, collection of matchings) without specifying the discrete Morse function explicitly.



- On the other hand, given a discrete gradient vector field,
  - there is a discrete Morse function to generate this collection

Recall: Critical points and topology

Theorem 3 (Homotopy type of sub-level sets)**.** *Let $f : M \to \mathbb{R}$ be a smooth function defined on a manifold M. Given $a < b$, suppose the interval-level set $M_{[a,b]} = f^{-1}([a,b])$ is compact and contains no critical points of f. Then $M_{\leq a}$ is diffeomorphic to $M_{\leq b}$.*

Furthermore, $M_{\leq a}$ is a deformation retract of $M_{\leq b}$, and the inclusion map $i : M_{\leq a} \hookrightarrow M_{\leq b}$ is a homotopy equivalence.

Recall: Critical points and topology

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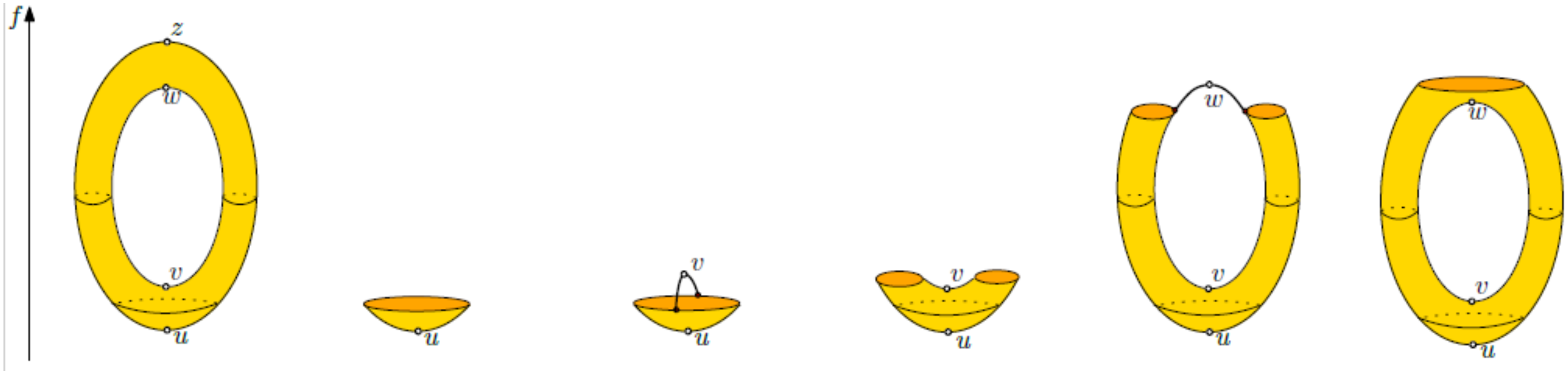
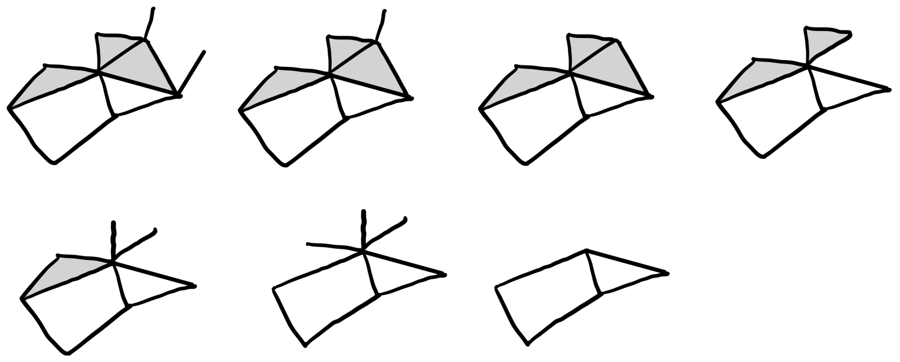


(a) (b) (c) (d) (e) (f)

Analogous results

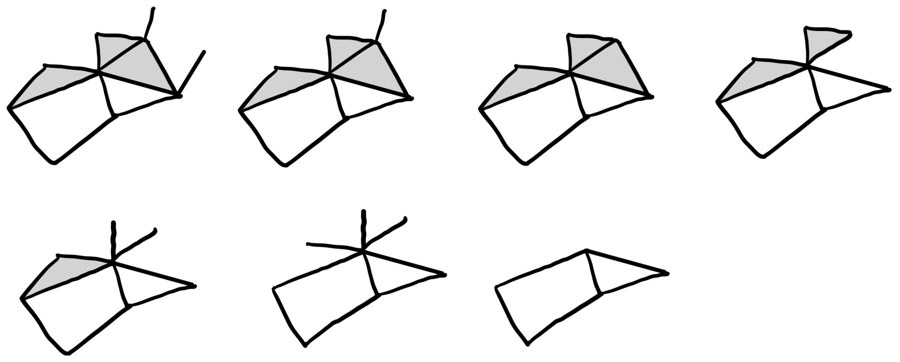
- Given a discrete Morse function $f: K \to R$, the function value that a critical simplex takes is called a critical value.



Analogous results

- Given a discrete Morse function $f: K \to R$, the function value that a critical simplex takes is called a critical value.

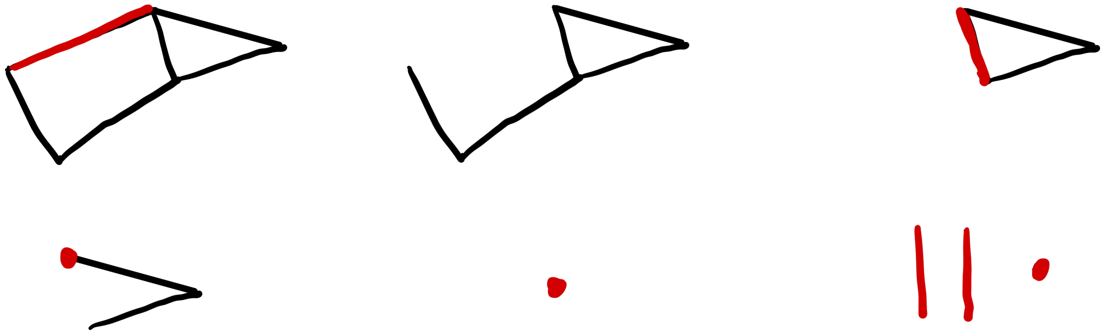
- Theorem [Forman]
  - Let $f: K \to R$ be a discrete Morse function. For any $a \in R$, let $K_{\leq a} := \{\sigma \; and \; its \; faces \mid \sigma \in K \; with \; f(\sigma) \leq a\}$. Then $K_{\leq b}$ deformation retracts to $K_{\leq a}$ (thus $K_{\leq a} \hookrightarrow K_{\leq b}$ is a homotopy equivalence) if there is no critical value within the range $[a, b]$.

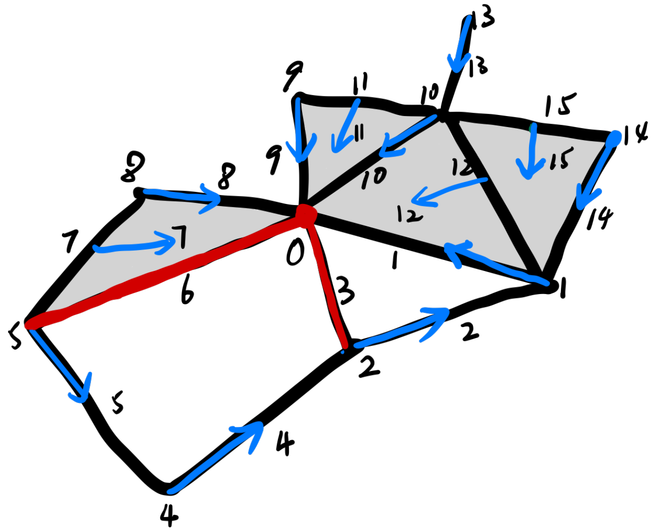


Analogous results

- Given a discrete Morse function $f: K \to R$, the function value that a critical simplex takes is called a critical value.

Theorem 2.5. *Suppose K is a simplicial complex with a discrete Morse function. Then K is homotopy equivalent to a CW complex with exactly one cell of dimension p for each critical simplex of dimension p.*





Morse Inequalities

- Let c_i denote the number of critical i-simplices

Proposition 10.1. *Given a Morse function f on K with its induced Morse matching M, let c_is and β_is defined as above. We have:*

- *(weak Morse inequality)*

  (i) $c_i \geq \beta_i$ *for all* $i \geq 0$.

  (ii) $c_p - c_{p-1} + \cdots \pm c_0 = \beta_p - \beta_{p-1} + \cdots \pm \beta_0$ *where K is p-dimensional.*

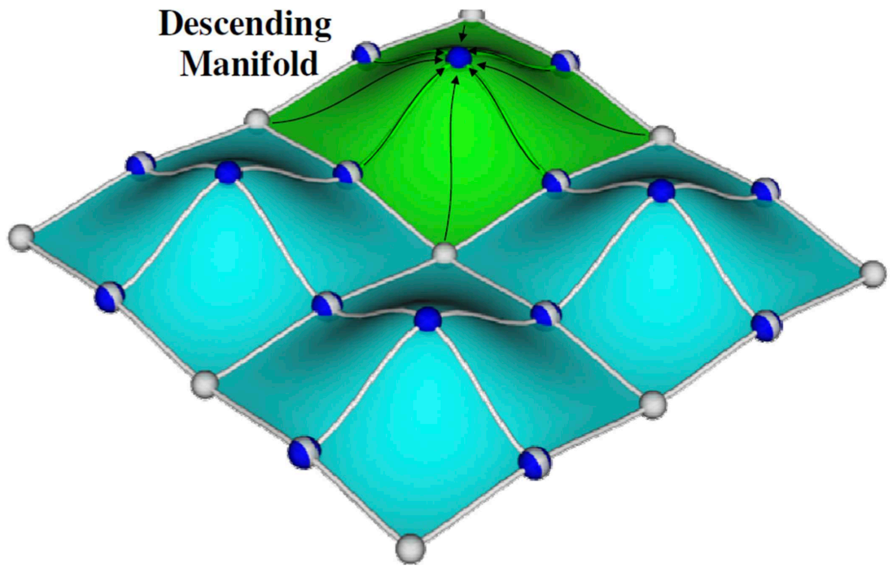
- *(strong Morse inequality)*

$$c_i - c_{i-1} + c_{i-2} - \cdots \pm c_0 \geq \beta_i - \beta_{i-1} + \beta_{i-2} \cdots \pm \beta_0 \text{ for all } i \geq 0.$$

Morse Complex

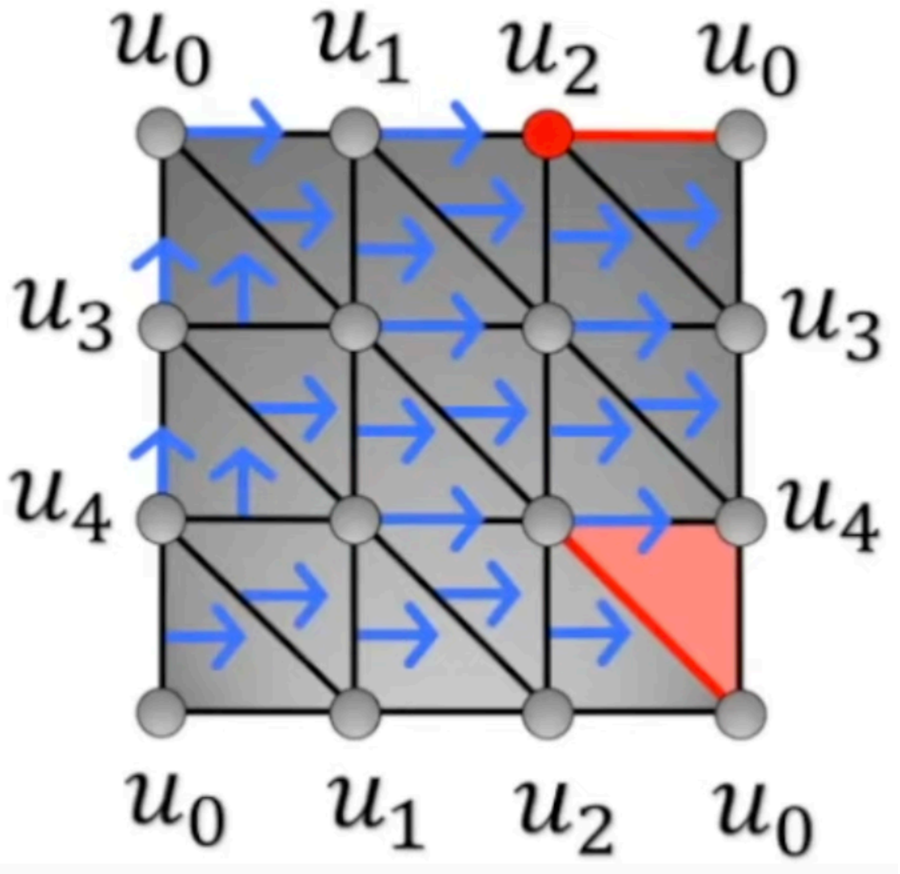
- Descending manifolds give rise to a cell complex structure to the manifold M. We call this the **Morse Complex.**

- Chain complex structure
  - MC_k is the vector space generated by index k critical points
  - $\partial : MC_k \rightarrow MC_{k-1}$ counts the number of connecting orbits
  - Morse homology coincides with singular homology



Morse Complex

- Chain complex structure
  - MC_k is the vector space generated by dimension k critical simplices
  - $\partial : MC_k \to MC_{k-1}$ counts the number (mod 2) of connecting V-paths
  - Morse homology coincides with singular/simplicial homology



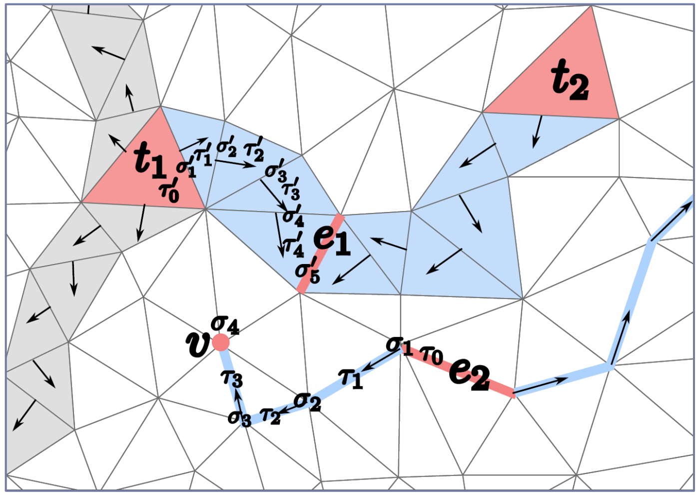
Picture by Nick Scoville

- $0 \to \mathbb{Z}_2^{18} \to \mathbb{Z}_2^{27} \to \mathbb{Z}_2^{9} \to 0$

- Morse Complex
- $0 \to \mathbb{Z}_2 \to \mathbb{Z}_2^{2} \to \mathbb{Z}_2 \to 0$

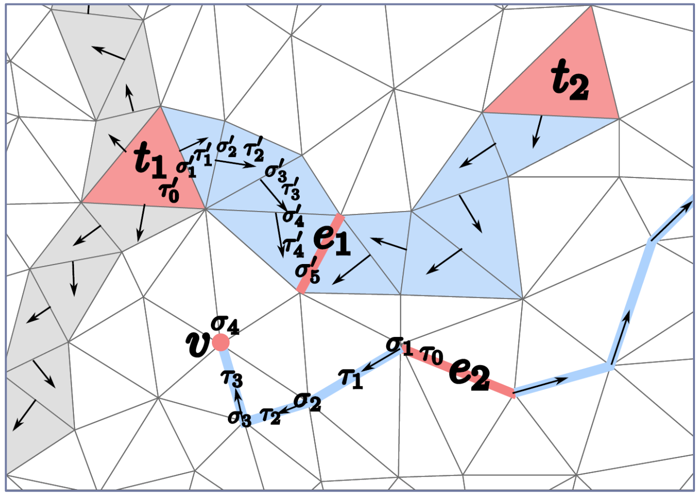
Discrete Gradient Vector Field

- Stable (descending) manifolds of a critical σ



Discrete Gradient Vector Field

- Stable (descending) manifolds of a critical σ
  - All the cells of a path whose destination is σ
- Unstable (ascending) manifolds of a critical σ
  - All the cells of a path whose origin is σ

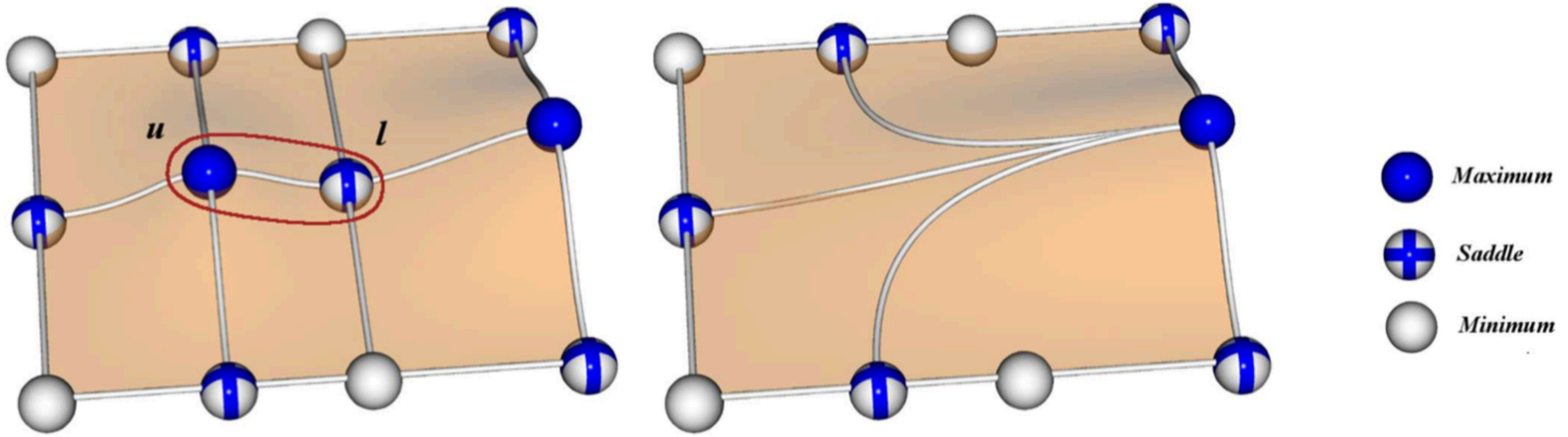


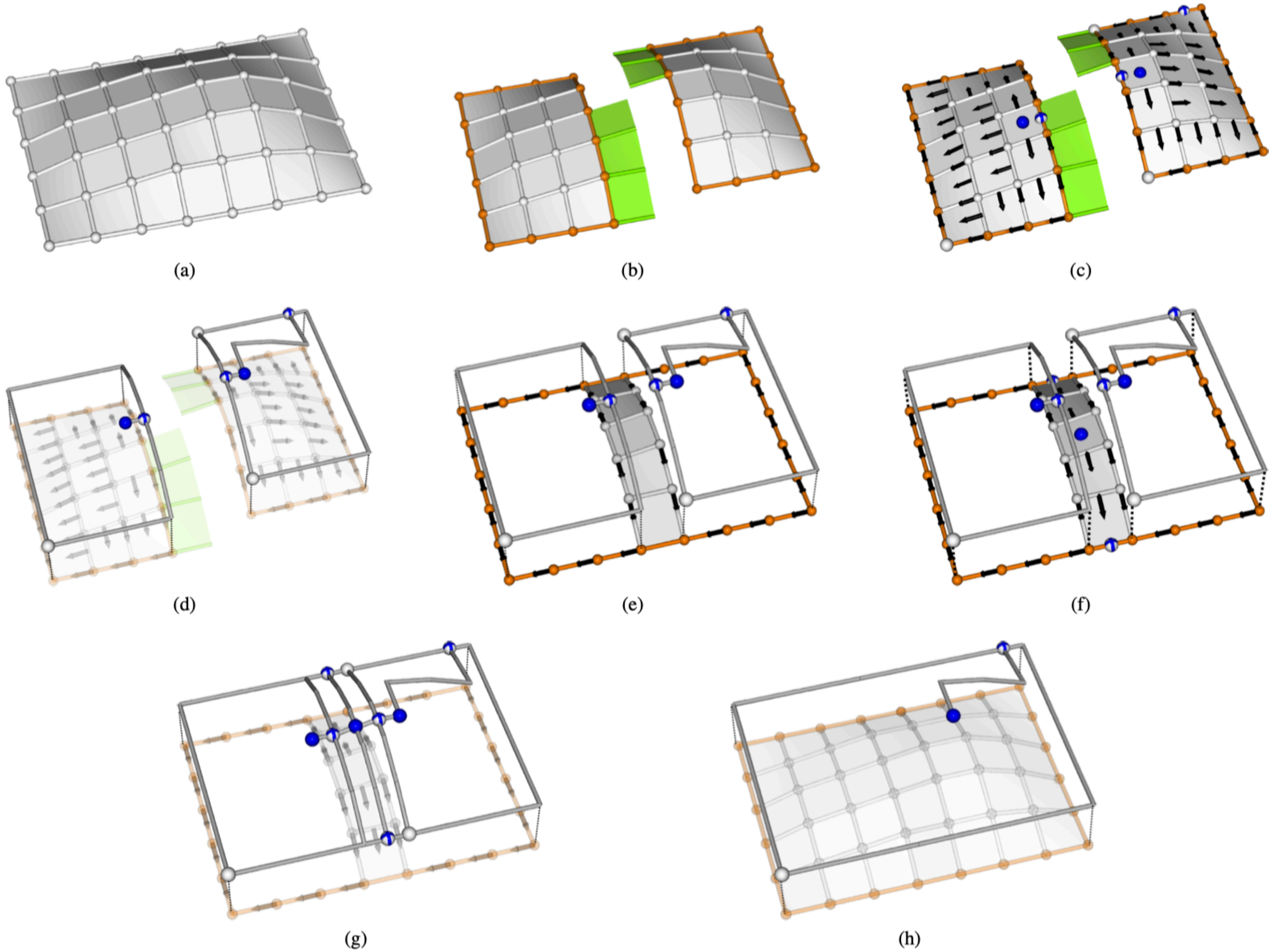
Section 3:
DMT and PH

- Topological simplification for MS construction
- Simplification for PH computation
- Persistence guided Morse cancelation for graph reconstruction

Topological simplification for MS complex

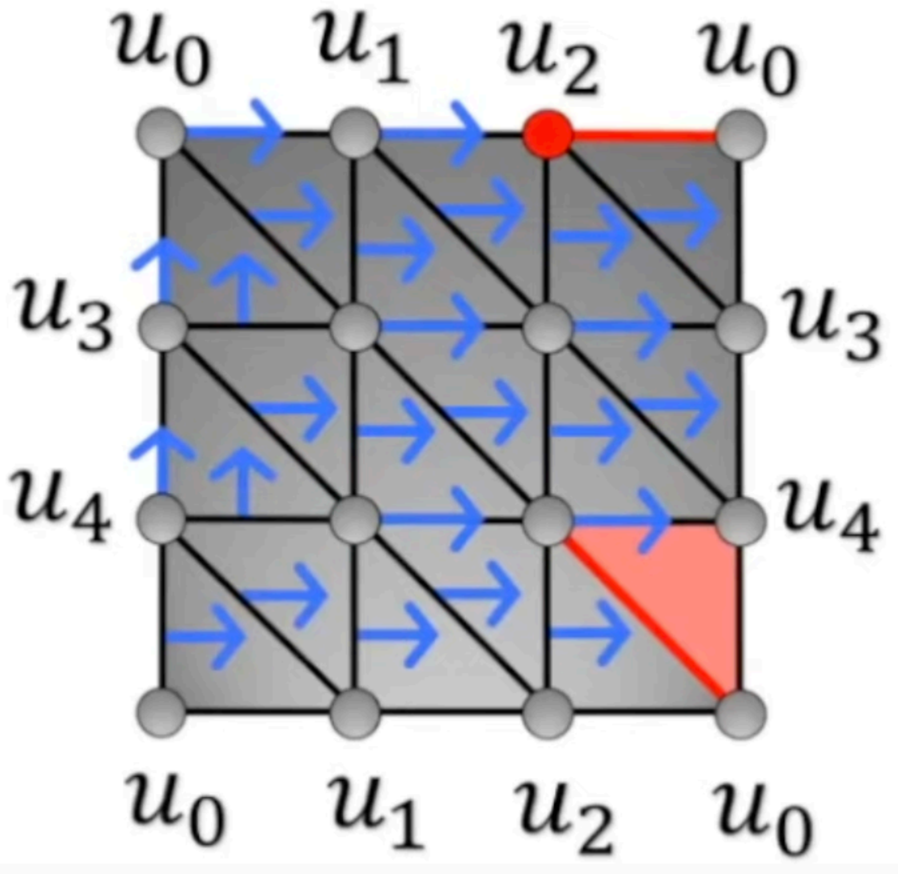
- [Gyulassy et al. 2008]





(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)

Homology computation through Morse complex



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- $0 \to \mathbb{Z}_2^{18} \to \mathbb{Z}_2^{27} \to \mathbb{Z}_2^{9} \to 0$
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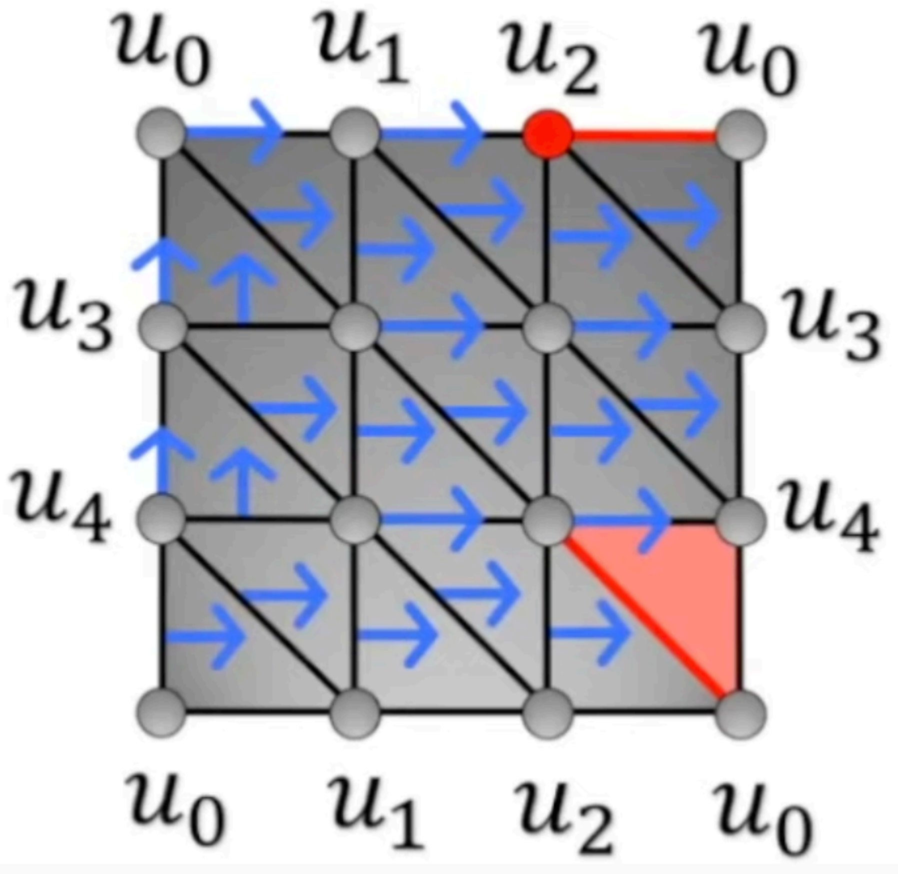
- [Harker et al 2013]
- [Günther et al 2012]
- [Mischaikow et al 2013]

Homology computation through Morse complex

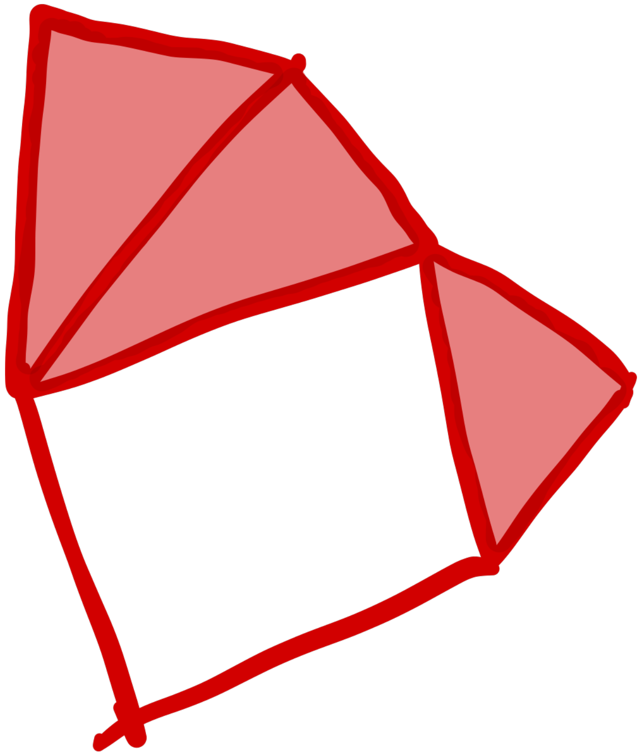


Picture by Nick Scoville

	P0001	P0050	P0100
dim	3	3	3
size in millions	75.56	73.36	71.64
H_0	$\mathbb{Z}^7$	$\mathbb{Z}^2$	$\mathbb{Z}$
H_1	$\mathbb{Z}^{6554}$	$\mathbb{Z}^{2962}$	$\mathbb{Z}^{1057}$
H_2	$\mathbb{Z}^2$		
Linbox::Smith	> 600	> 600	> 600
RedHom::Shave+**Linbox**::Smith	> 600	> 600	> 600
ChomP	400	360	310
RedHom::CR	36	34	33
ChomP::DMT	110	110	100
ChomP::CR+DMT	45	43	42
RedHom::CR+DMT	26	25	24

Simplification via Morse Cancellation

- Trivial Discrete Gradient Vector Field: Every simplex is critical!



- $c_0 = 7 > 1 = \beta_0$
- $c_1 = 10 > 1 = \beta_1$

Simplification via Morse Cancellation

- Morse cancellation operation (to simplify the vector field):
  - A pair of critical simplices $\langle \sigma, \ \tau \rangle$ can be cancelled
    - if there is a unique gradient path between them
    - By reverting that gradient path

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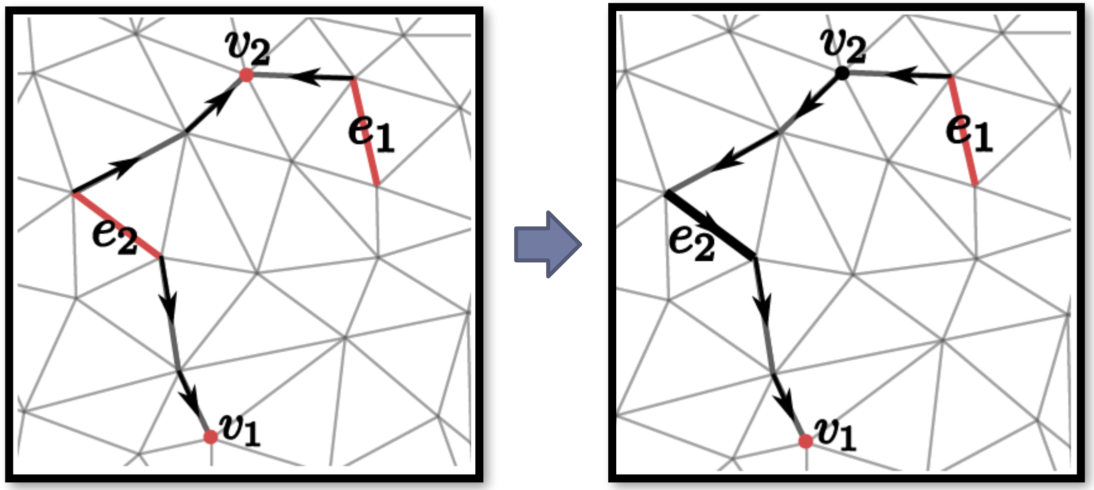

canceling $\langle v_2, e_2 \rangle$

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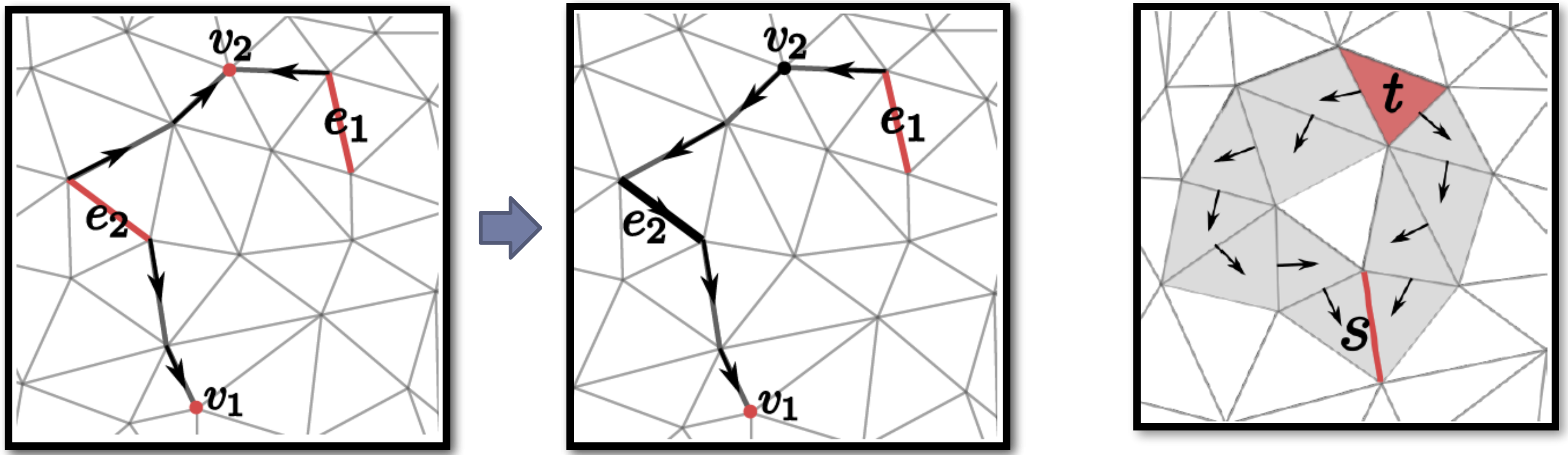

canceling $\langle v_2, e_2 \rangle$

$\langle s, t \rangle$ not cancellable

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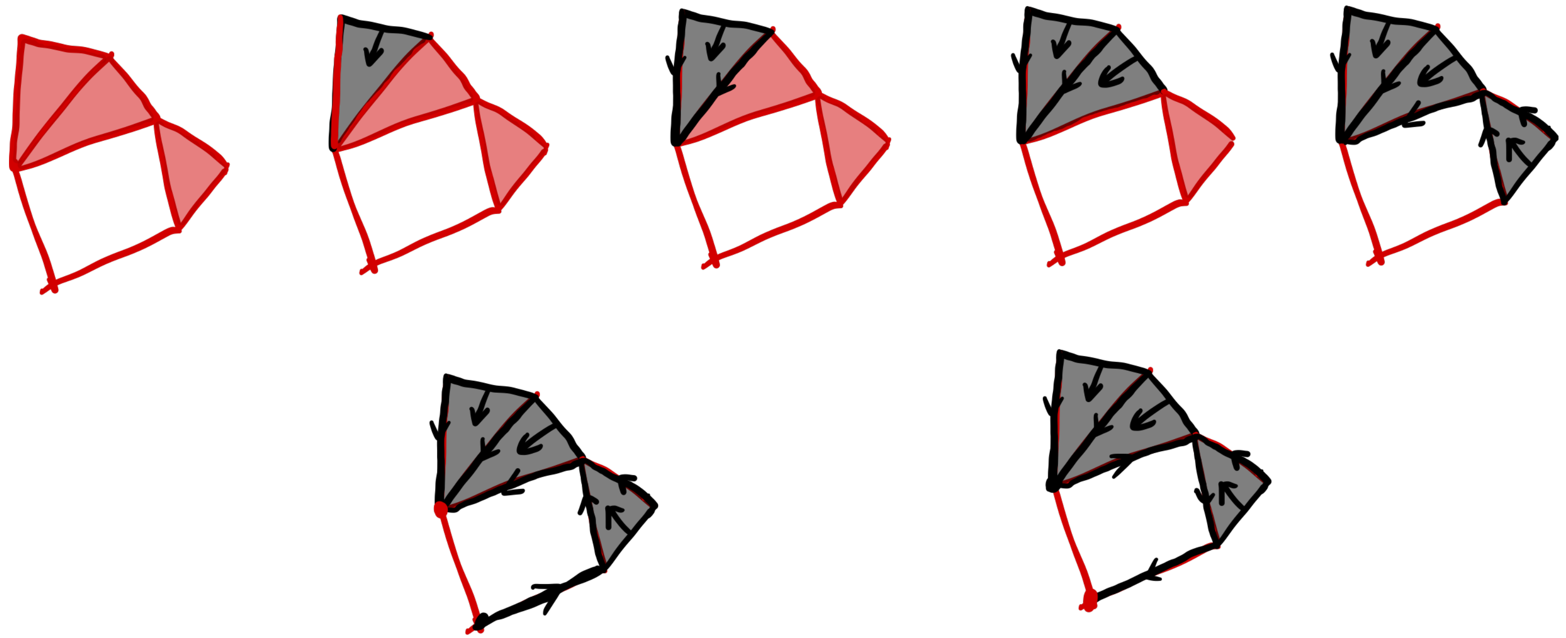
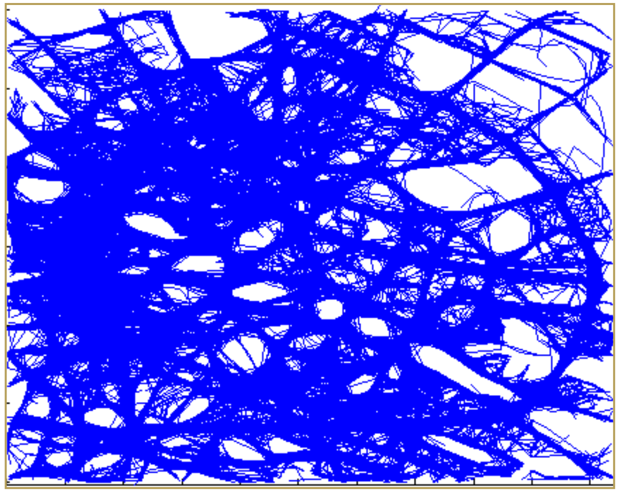


Persistence guided cancelation

Proposition 10.3. *Let* $(v_1, e_1), (v_2, e_2), \cdots, (v_n, e_n)$ *be the sequence of all non-essential persistence pairs of vertices and edges sorted in increasing order of the appearance of the edges* e_i*'s in a filtration of a* 1*-complex* K*. Let* V_0 *be the DMVF in* K *with all simplices being critical. Suppose DMVF* V_{i-1} *can be obtained by cancelling successively* $(v_1, e_1), (v_2, e_2), \cdots (v_{i-1}, e_{i-1})$*. Then,* (v_i, e_i) *can be cancelled in* V_{i-1} *providing a DMVF* V_i *for all* $i \geq 1$*.*

- There is a theoretical guarantee for graph construction

Effect of Simplification



Berlin, 27189 trajectories

(a) Persistent 0.0001 (b) Persistent 0.001 (c) Persistent 0.01 (d) Persistent 0.1

FIN