DSC 214
Topological Data Analysis
Topic 3: Simplicial Homology

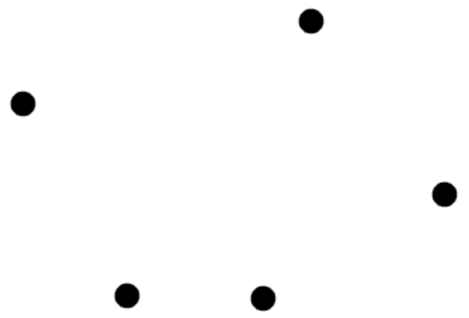
Instructor: Zhengchao Wan

Overview

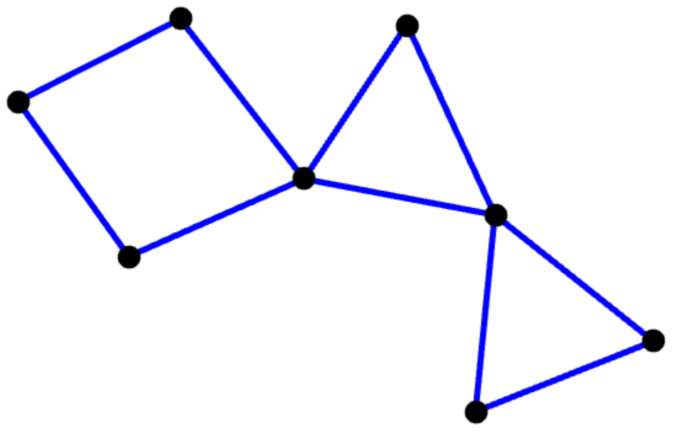
- Review of algebraic tools
- (Simplicial) homology groups
  - a way to quantify topological features
- Notations
  - Chains, cycles, and homology groups
- Matrix view
  - Matrix reduction algorithm

Motivating examples

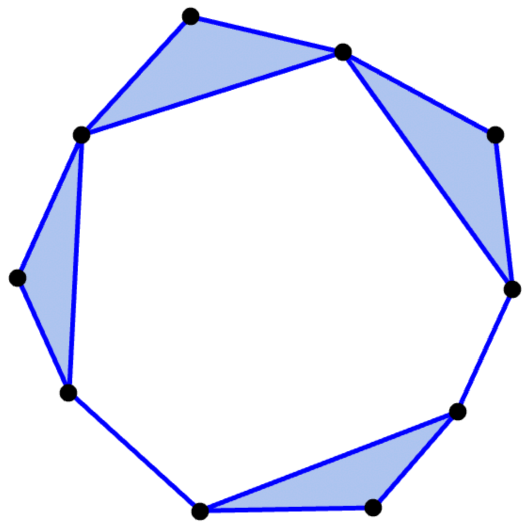
- ith homology "counts the number of i dimensional holes" in a topological space



$\dim H_0 = 5$
$\dim H_1 = 0$



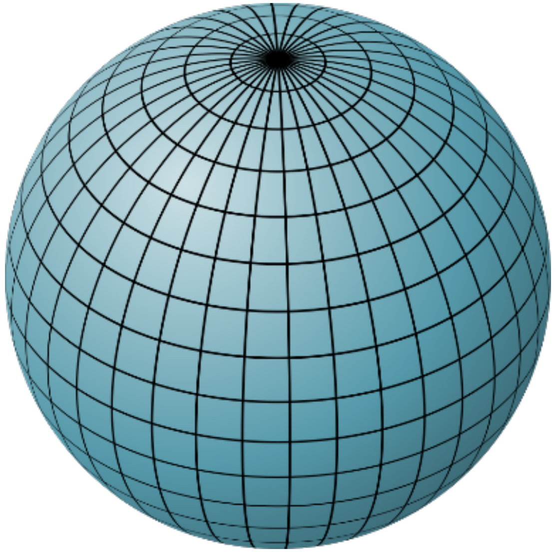
$\dim H_0 = 1$
$\dim H_1 = 3$



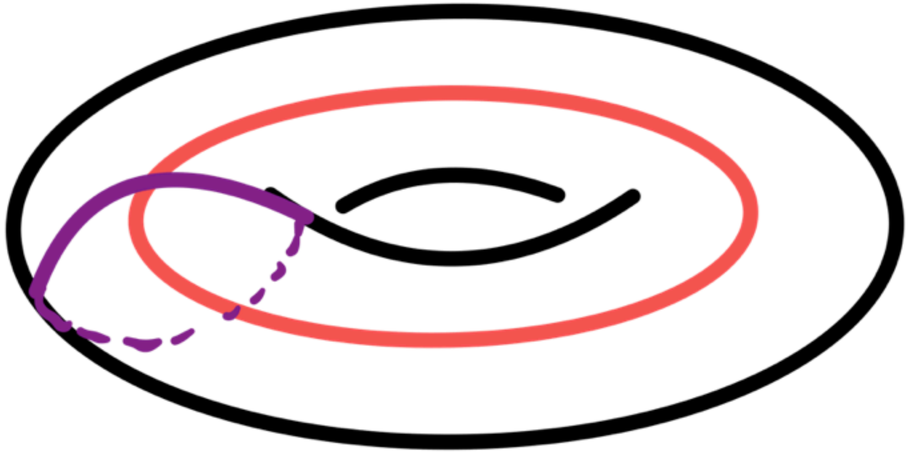
$\dim H_0 = 1$
$\dim H_1 = 1$

Motivating examples

- ith homology "counts the number of i dimensional holes" in a topological space

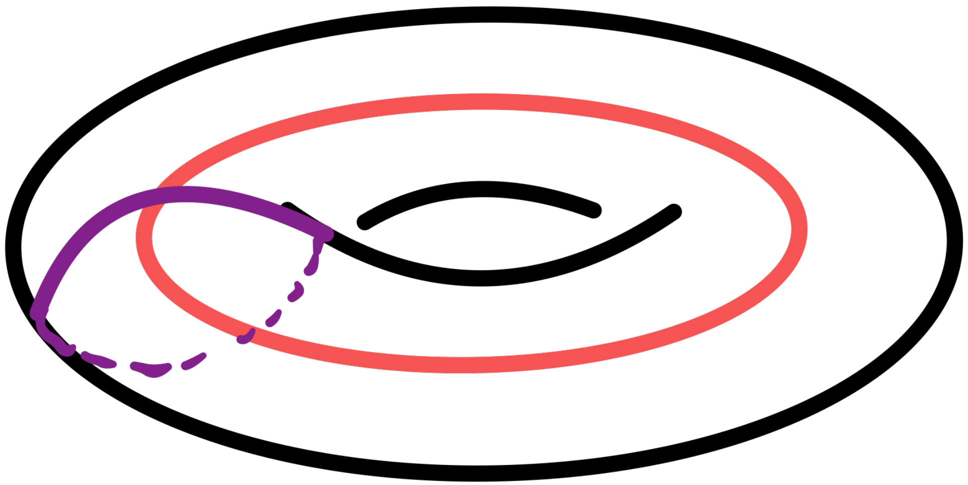


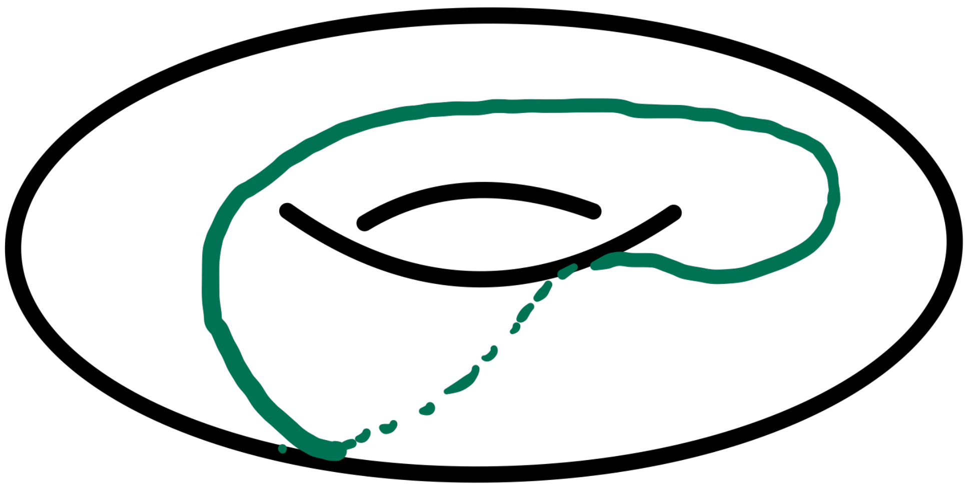
$\dim H_0 = 1$
$\dim H_1 = 0$
$\dim H_2 = 1$



$\dim H_0 = 1$
$\dim H_1 = 2$
$\dim H_2 = 1$

- ith homology has a vector space structure!





Part 0:
Review of algebraic tools

Group

- A **group** is a tuple $(G, +)$ where G is a set and $+ : G \times G \to G$ is a binary operation
  - **Associativity** $a + (b + c) = (a + b) + c$
  - There exist 0 such that $a + 0 = 0 + a = a$
  - For any $a \in G$, there exist $-a \in G$ such that $a + (-a) = 0$
- If G further satisfies the following property, then we call $(G, +)$ an **abelian group**
  - **Commutativity** $a + b = b + a$

Examples of groups

- $(\mathbb{Z}, +)$ is an abelian group
- $(\mathbb{R}, +)$ is an abelian group
- $(GL_n(\mathbb{R}), \cdot)$ is a non-abelian group

Ring and Field

- A **ring** is a tuple $(F, +, \times)$ where $(F, +)$ is an abelian group and $\times : F \times F \to F$ is another binary operation such that
  - **Associativity** $a \times (b \times c) = (a \times b) \times c$
  - There exist 1 in F such that $a \times 1 = a$
  - **Distributivity** $a \times (b + c) = (a \times b) + (a \times c)$
- $(F, +, \times)$ is called a **field** if
  - For any $a \neq 0$ in F, there exists $a^{-1} \in F$ such that $a \times a^{-1} = 1$
  - $a \times b = b \times a$

Examples of fields

- Rational numbers $(\mathbb{Q}, +, \times)$
- Real numbers $(\mathbb{R}, +, \times)$
- Complex numbers $(\mathbb{C}, +, \times)$
- Finite fields
    - For any prime number p, $\mathbb{Z}_p = \{0,1,\ldots,p-1\}$
    - $+, \times$ modulo p
    - $(\mathbb{Z}_p, +, \times)$ is a field

$\mathbb{Z}_2$

- $\mathbb{Z}_2 = \{0,1\}$ is the smallest field

+	0	1
0	0	1
1	1	$\boxed{0}$

$\times$	0	1
0	0	0
1	0	1

Vector space

- A vector space over a field F is a set V of vectors with operations
  - Vector addition $V \times V \to V \quad (v, w) \mapsto v + w$
  - Scalar multiplication $F \times V \to V \ (\lambda, v) \mapsto \lambda v$
- Satisfying
  - $(V, +)$ is an abelian group
  - $\lambda(u + v) = \lambda u + \lambda v$ and $(\lambda + \mu)v = \lambda v + \mu v$ and $\lambda(\mu v) = (\lambda\mu)v$
  - $1v = v$

Examples of vector spaces

- **Major example**: $\mathbb{R}^d$ is a vector space over $\mathbb{R}$ with operations
  - $(x_1, \ldots, x_d) + (y_1, \ldots, y_d) = (x_1 + y_1, \ldots, x_d + y_d)$
  - $\lambda(x_1, \ldots, x_d) = (\lambda x_1, \ldots, \lambda x_d)$

- $\mathbb{Z}_2^d = \{(x_1, \ldots, x_d) \mid x_i \in \{0,1\}\}$ is a vector space over $\mathbb{Z}_2$ with operations
  - $(x_1, \ldots, x_d) + (y_1, \ldots, y_d) = (x_1 + y_1, \ldots, x_d + y_d) \bmod 2$
  - $\lambda(x_1, \ldots, x_d) = (\lambda x_1, \ldots, \lambda x_d)$

Basis and Dimension

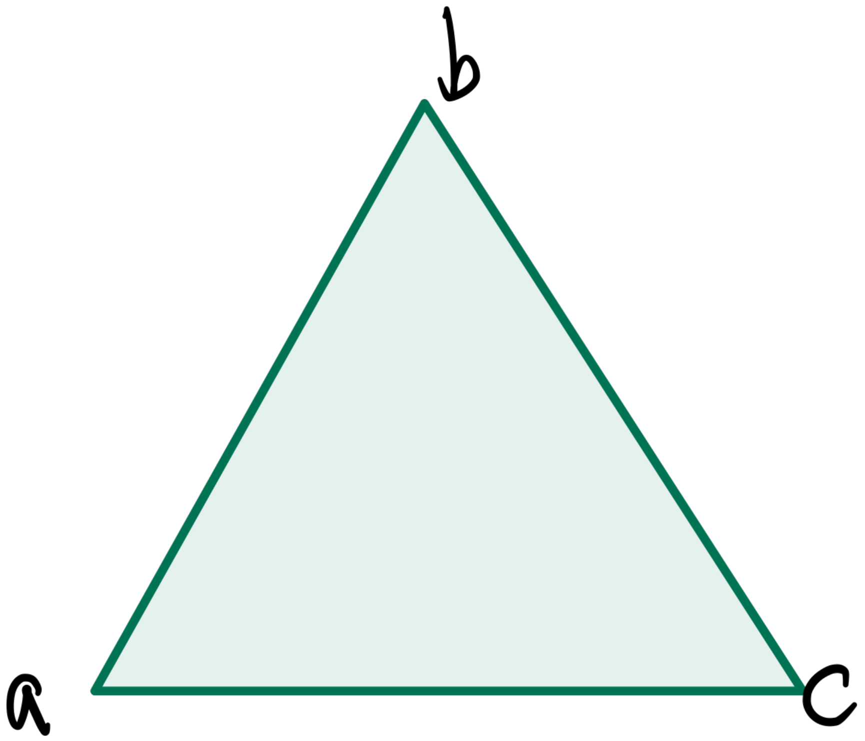
- Let V be a vector space over F
- A finite subset $W = \{w_1, \ldots, w_n\} \subset V$ is **linearly independent** if
  - $\lambda_1 w_1 + \cdots + \lambda_n w_n = 0$ iff $\lambda_1 = \cdots = \lambda_n = 0$
- W is **spanning** if for any $v \in V$, there exist $\lambda_1, \ldots, \lambda_n \in F$ such that
  - $\lambda_1 w_1 + \cdots + \lambda_n w_n = v$
- W is a **basis** for V if it is linearly independent and spanning. We call n the dimension of V, denoted by $\dim V$

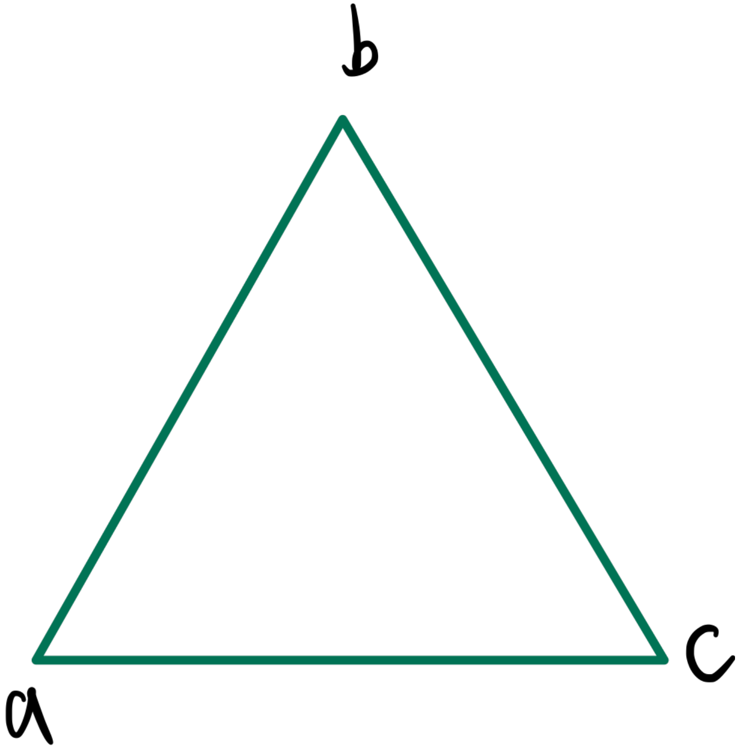
Quotient

- Let V be a vector space and $W \subset V$ be a linear subspace.
- An equivalence relation $\sim$ on V:
  - $v \sim u$ iff $v - u \in W$
  - Equivalence class $[v] = \{u \in V \mid v - u \in W\}$
- The quotient of V by W is the set $V/W = \{[v] \mid v \in V\}$ with
  - Vector addition $[v] + [u] := [v + u]$
  - Scalar multiplication $\lambda[v] := [\lambda v]$

Part 1:
Simplicial Homology

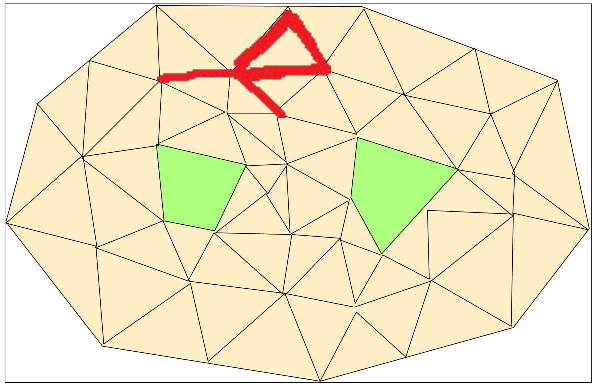
Chains





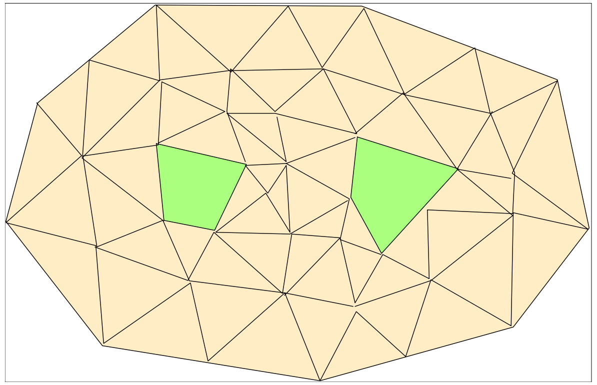
Chains

- Given a simplicial complex K, a p-chain is
  - A formal sum of p-simplices $c = \sum c_i \sigma_i$
    - Coefficients c_i come from a ring
  - In what follows, we use $\mathbb{Z}_2$ coefficients
    - i.e, $c_i \in \{0, 1\}$, equipped with *modulo-2* addition
    - thus a p-chain is just a **subset** of p-simplices!



Chains

- Given a simplicial complex K, a *p-chain* is
  - A formal sum of p-simplices $c = \sum c_i \sigma_i$
  - Under $\mathbb{Z}_2$-coefficients: a collection of p-simplices



- p-th *chain group* of K
  - $C_p(K)$: collection of p-chains with operation $+$
    - $c = \sum c_i \sigma_i$, and $c' = \sum c'_i \sigma_i$, then
    - $c + c' = \sum c_i \sigma_i + \sum c'_i \sigma_i = \sum [(c_i + c'_i) \ mod \ 2] \ \sigma_i$

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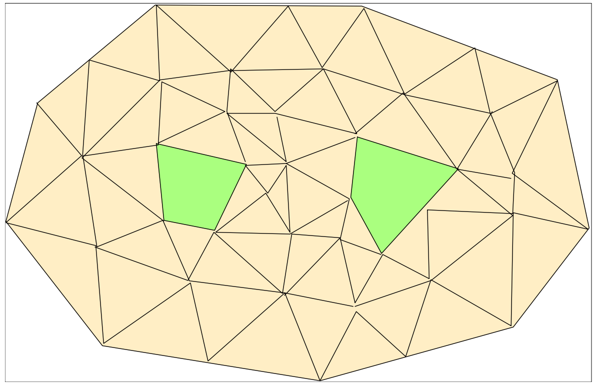

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- Remark: when coefficients comes from $\mathbb{Z}_2$, the chain group $C_p(K)$ is a vector space with basis $\{p - \text{simplices } \sigma \in K\}$
  - $\dim C_p(K) = n_p$ (i.e., # p-simplices)

Chains

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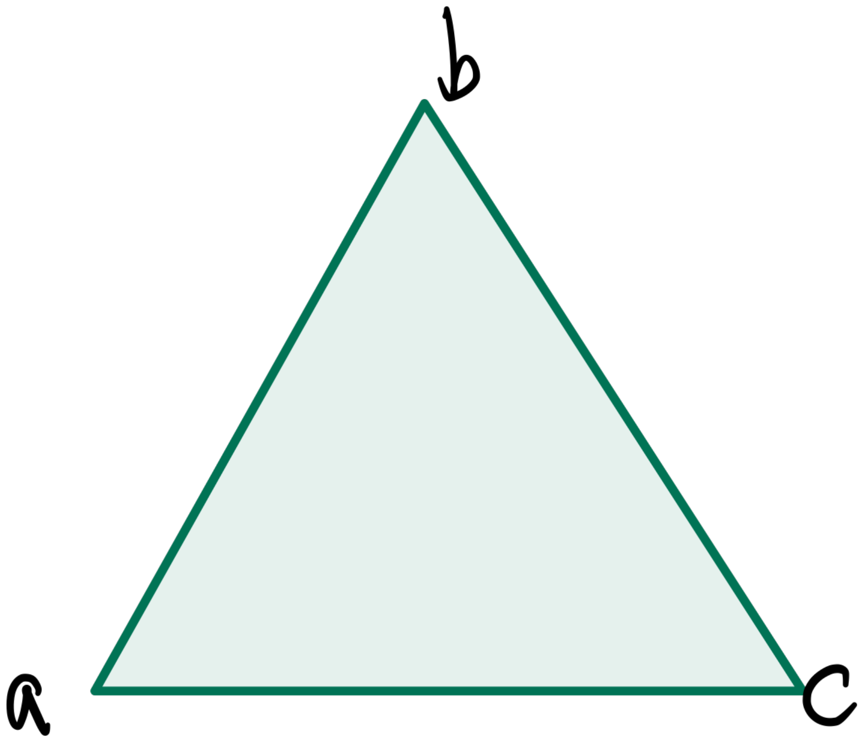
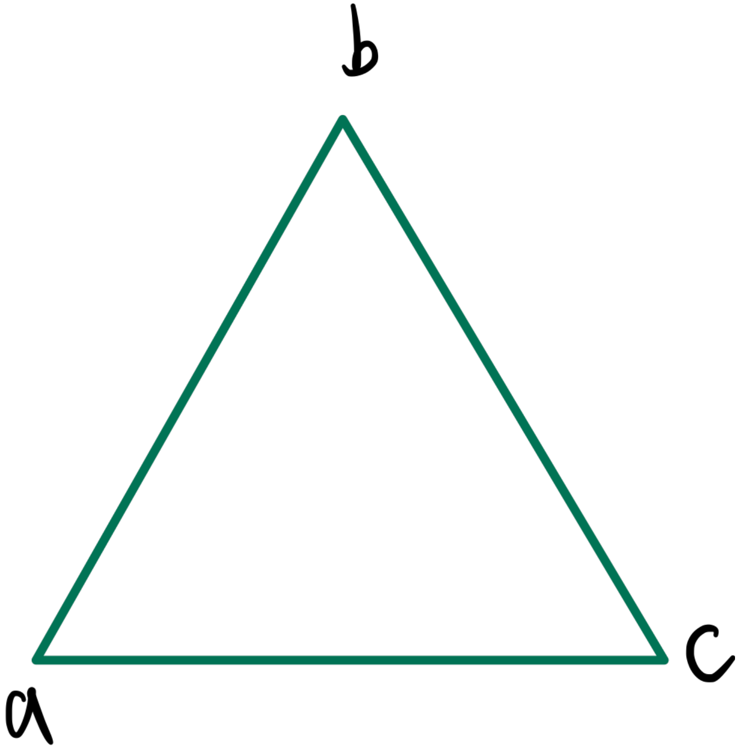
- $C_0(K),\ C_1(K),\ \ldots C_n(K),\ \ldots$
  - Boundary operators to connect them!

Boundary operator





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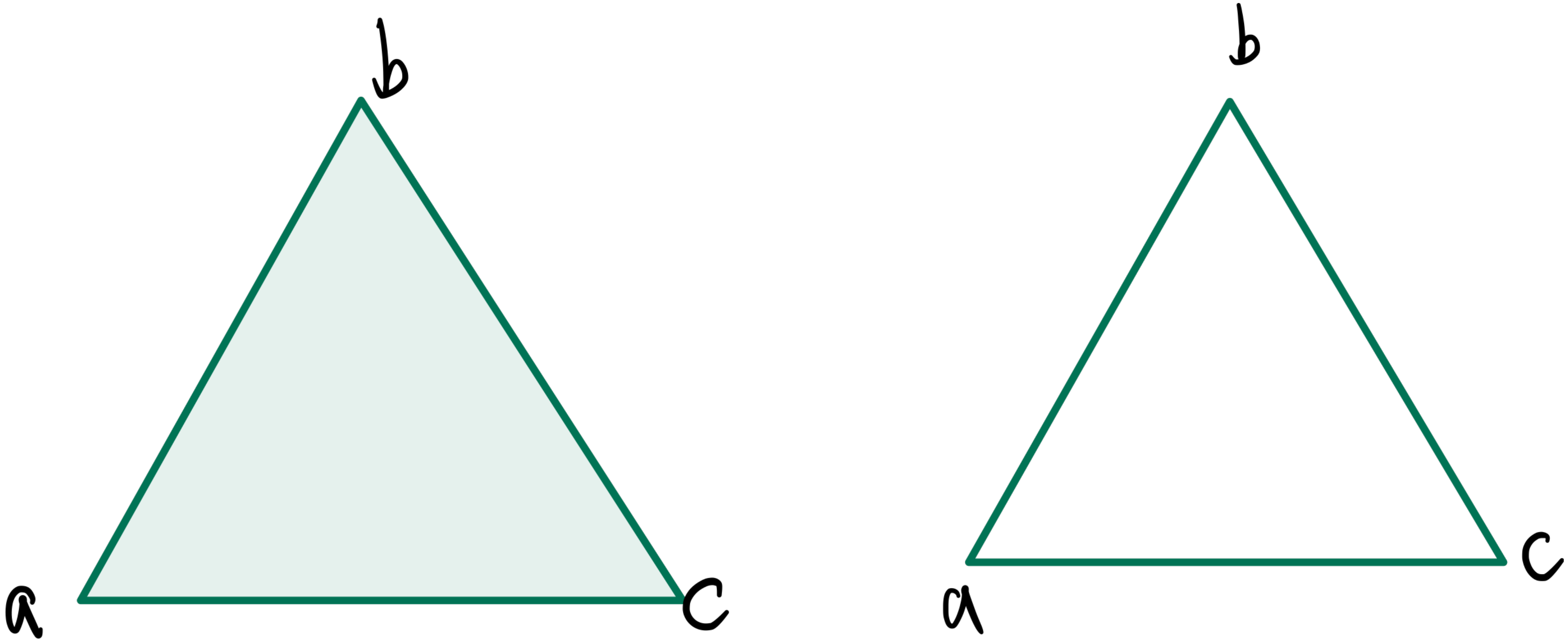
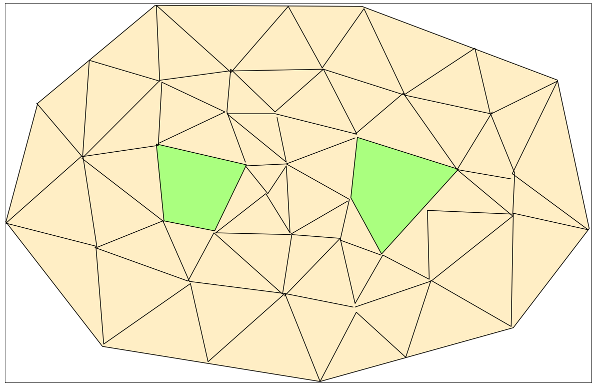


$$\partial\{a, b, c\} = \{a, b\} + \{a, c\} + \{b, c\}$$

Boundary operator

- p-th boundary operator (*a linear map*) $\partial_p : C_p \to C_{p-1}$
  - For a simplex $\sigma = \{v_0, \ldots, v_p\}$
    - $\partial_p(\sigma) = \sum_{i=0}^{p} \{v_0, \ldots, \hat{v}_i, \ldots, v_p\}$
    - $\partial_p(\sigma) =$ set of $(p-1)$-faces of σ
  - $c = \sum c_i \sigma_i \quad \Rightarrow \quad \partial_p(c) = \sum c_i \partial_p(\sigma_i)$



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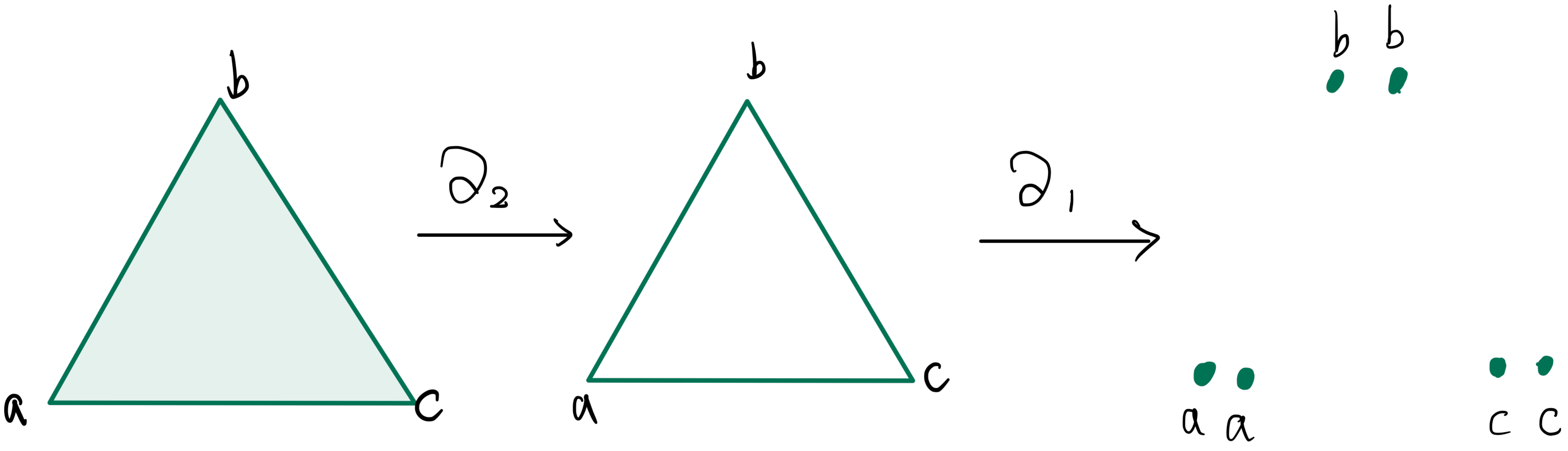
- Chain complex:
  - a sequence of vector spaces connected by linear maps

$$\cdots \xrightarrow{\partial_{p+2}} \mathrm{C}_{p+1} \xrightarrow{\partial_{p+1}} \mathrm{C}_p \xrightarrow{\partial_p} \mathrm{C}_{p-1} \xrightarrow{\partial_{p-1}} \cdots .$$

Boundary operator



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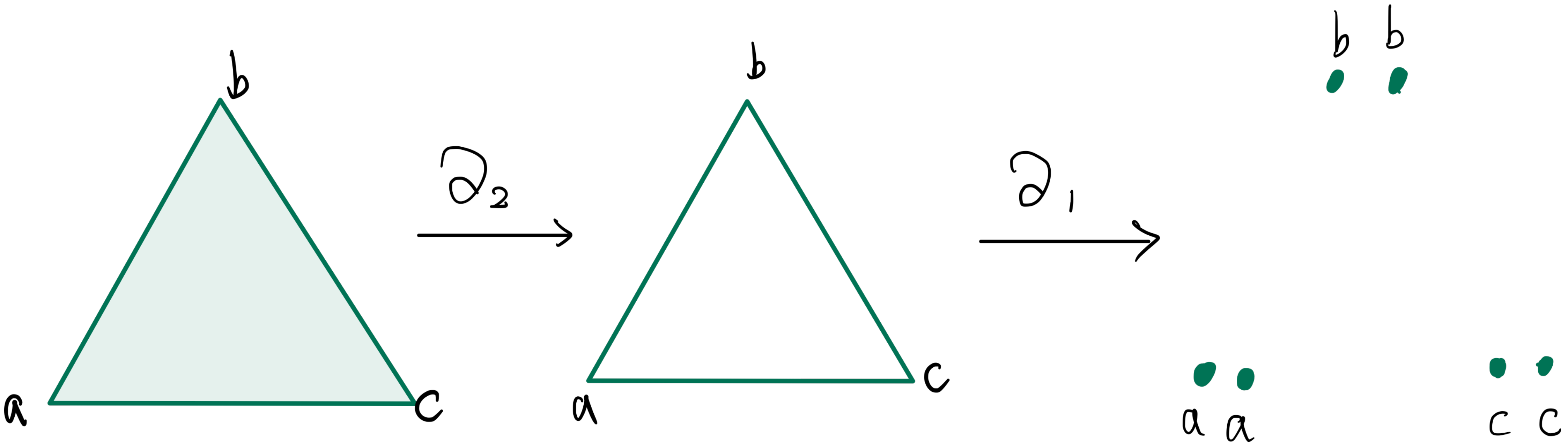

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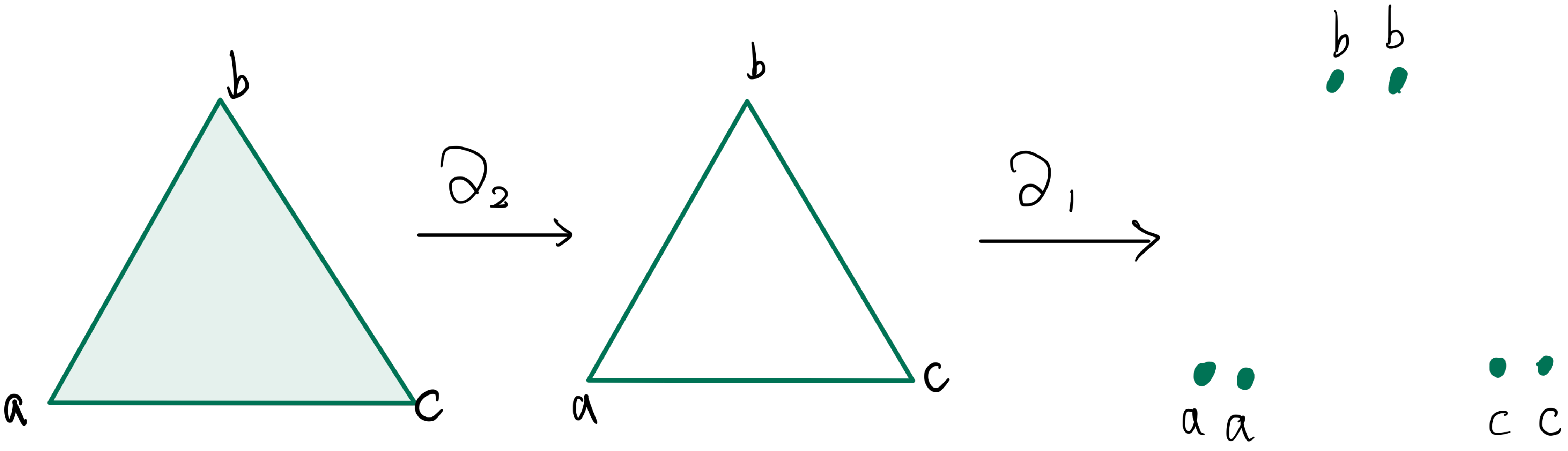

$$\partial\{a,b,c\} = \{a,b\} + \{a,c\} + \{b,c\}$$

$$\partial\partial\{a,b,c\} = \partial(\{a,b\} + \{a,c\} + \{b,c\}) = 2a + 2b + 2c = 0$$

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- Chain complex:
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    - $\partial_p(\sigma) =$ set
  - $c = \sum c_i \sigma_i \quad \Rightarrow$

Theorem (Fundamental Boundary Property):
$\partial_p \circ \partial_{p+1} = 0$

- Chain complex:
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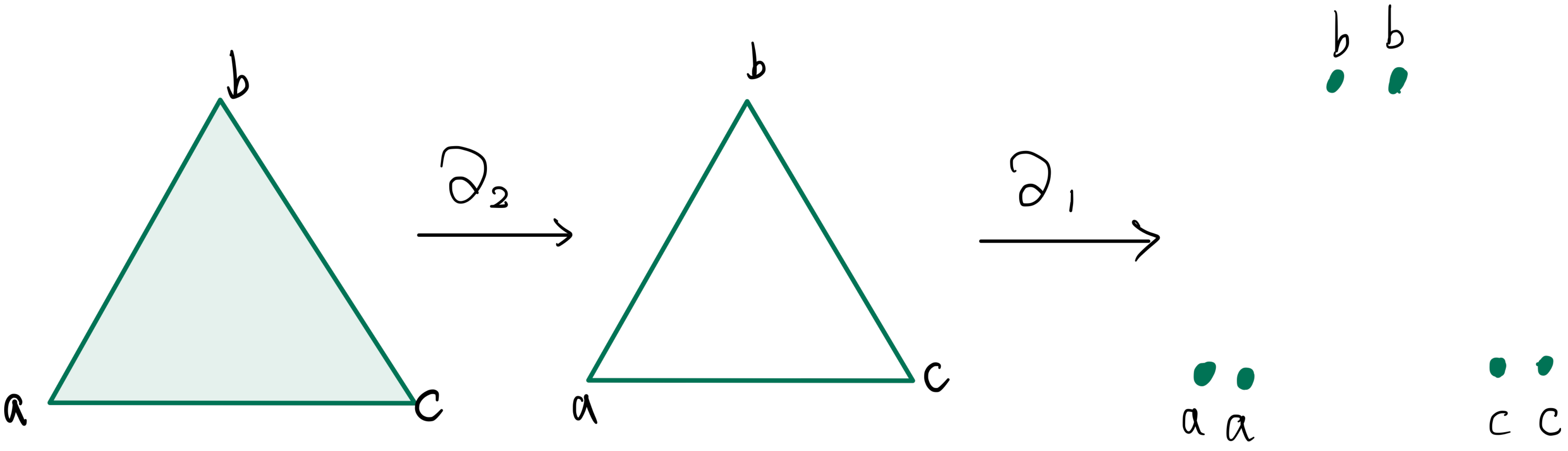

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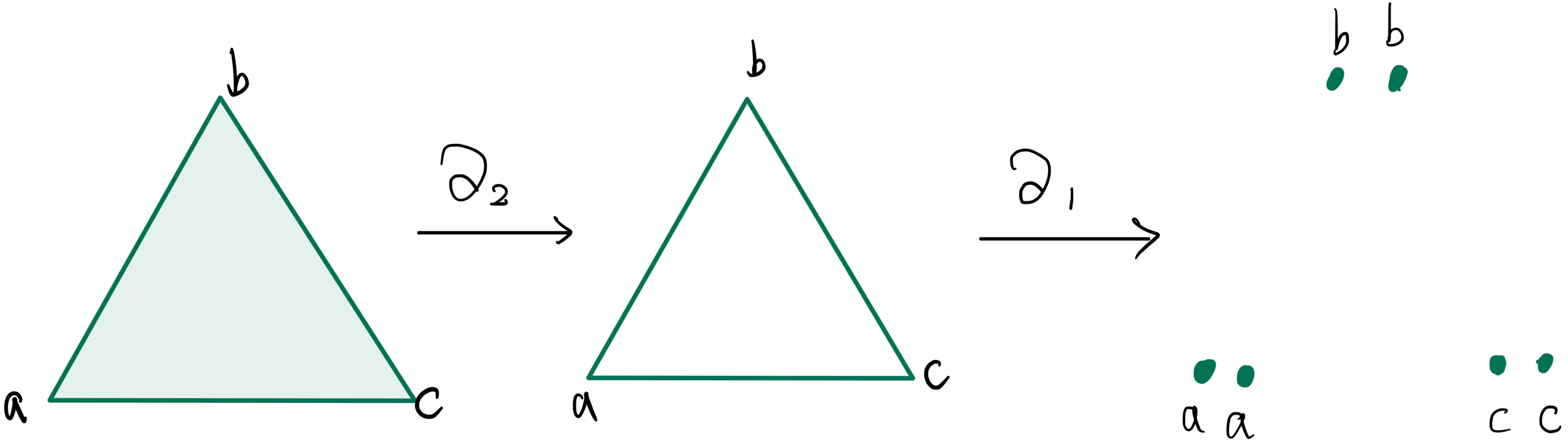

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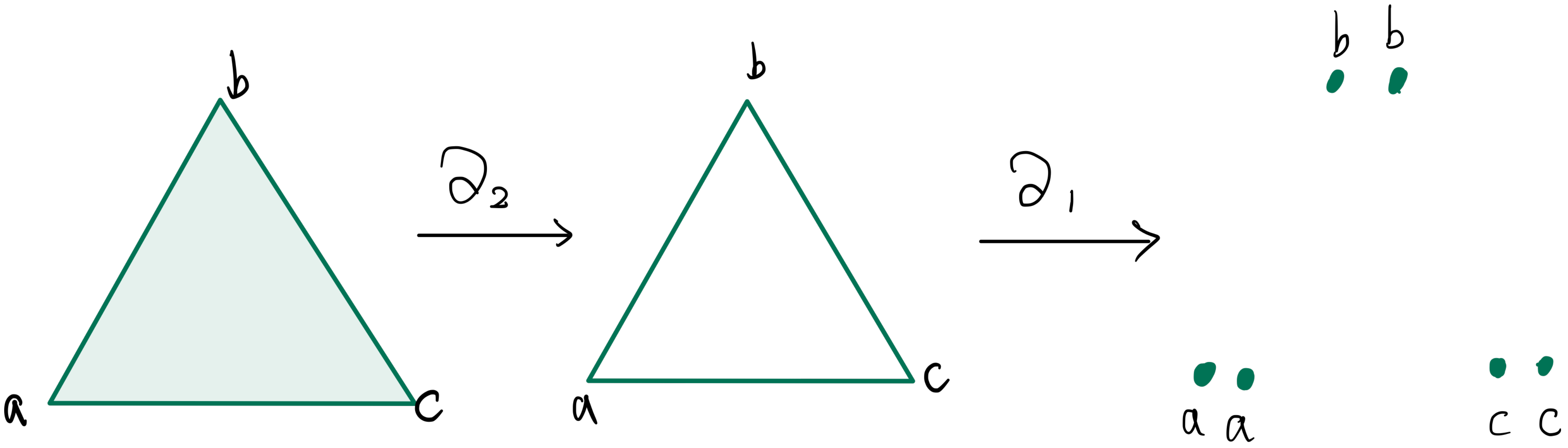

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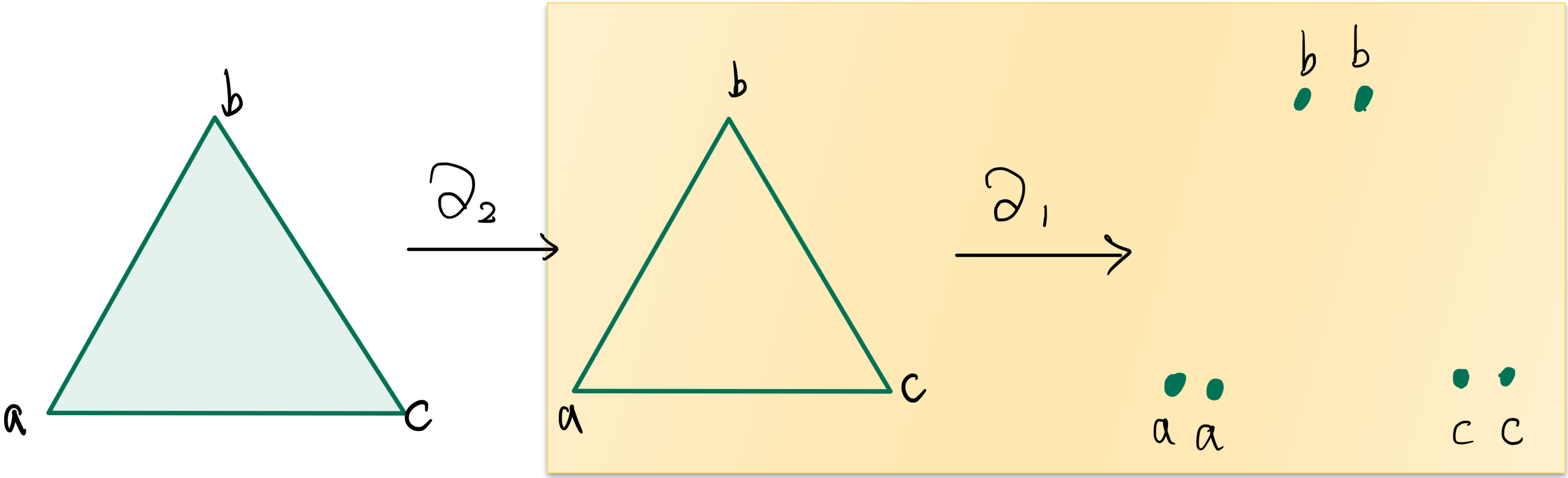

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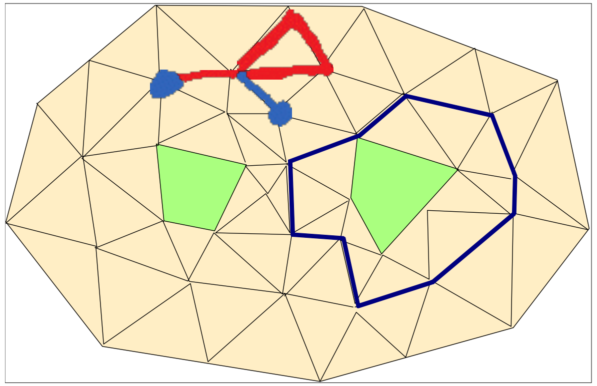
Cycles and Boundaries

- Cycles:
  - p-cycle: a p-chain whose boundary is 0
  - p-th cycle group $Z_p(K) = \ker(\partial_p)$
  - Z_p is a subgroup of C_p, denoted by $Z_p \subseteq C_p$



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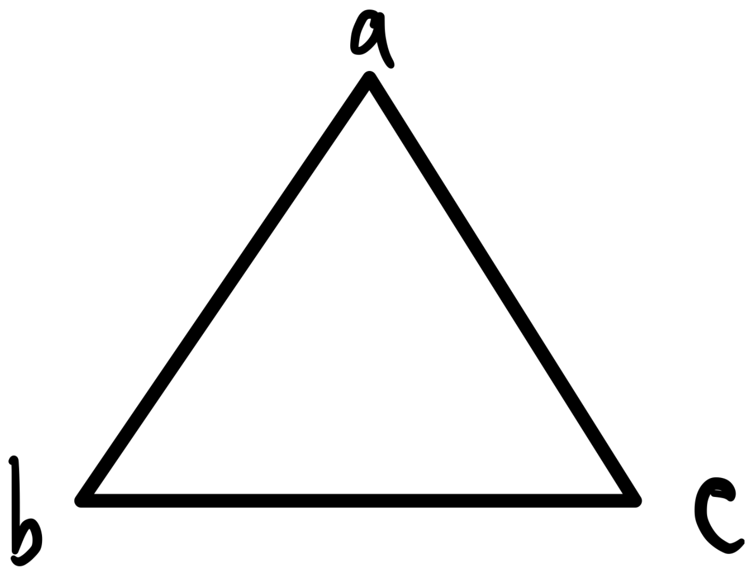

$$Z_1(K) = \langle \{a, b\} + \{b, c\} + \{a, c\} \rangle$$

$$\dim Z_1(K) = 1$$

Cycles

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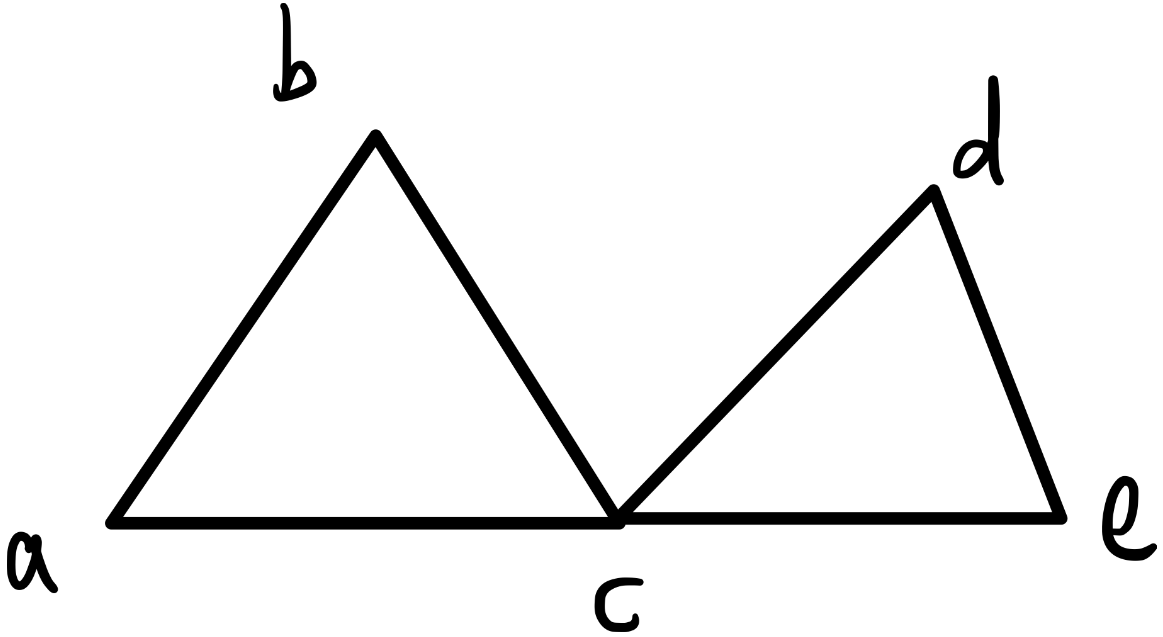

$$Z_1(K) = \langle \{a,b\} + \{b,c\} + \{a,c\}, \{c,d\} + \{d,e\} + \{c,e\} \rangle$$

$$\dim Z_1(K) = 2$$

Cycles

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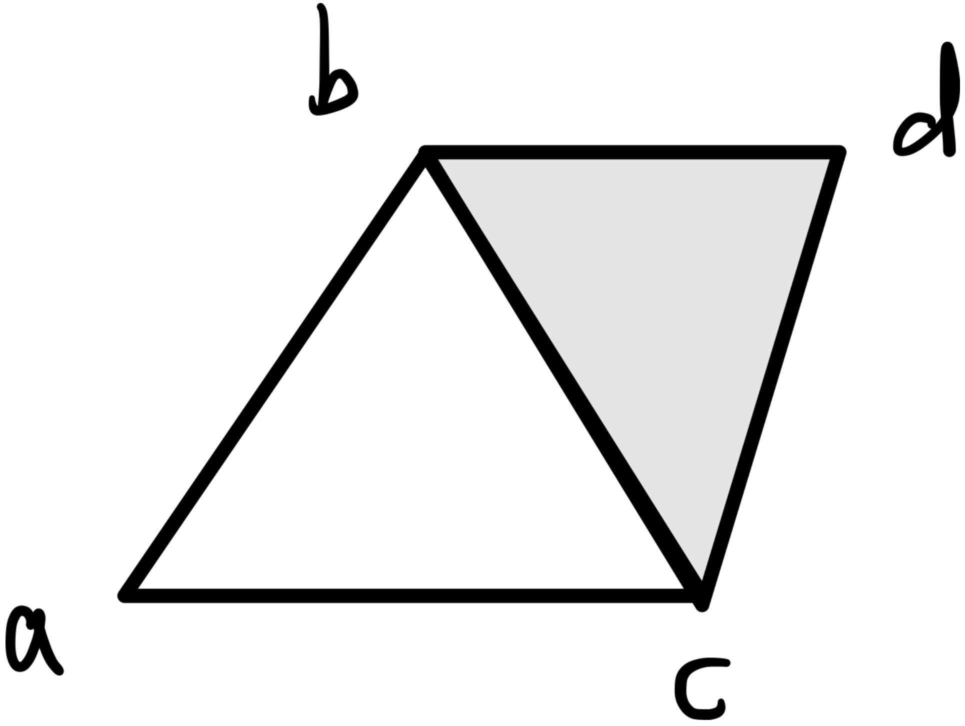
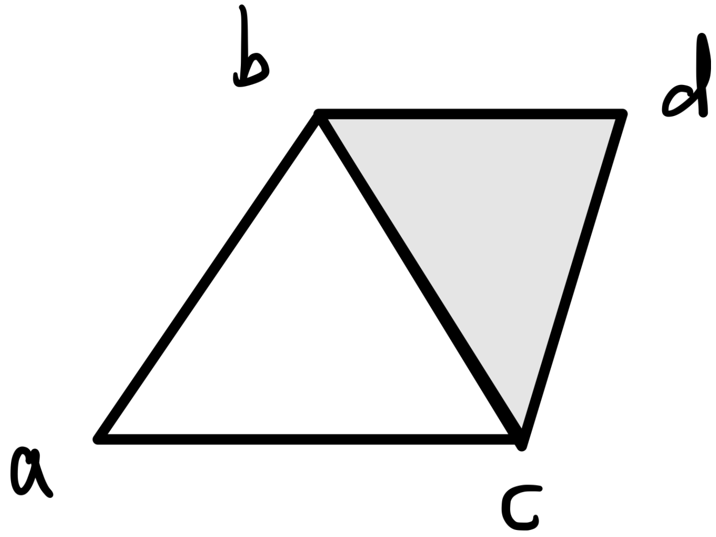


$$Z_1(K) = \langle \{a,b\} + \{b,c\} + \{a,c\}, \{a,b\} + \{b,d\} + \{c,d\} + \{a,c\} \rangle$$

$$\dim Z_1(K) = 2$$

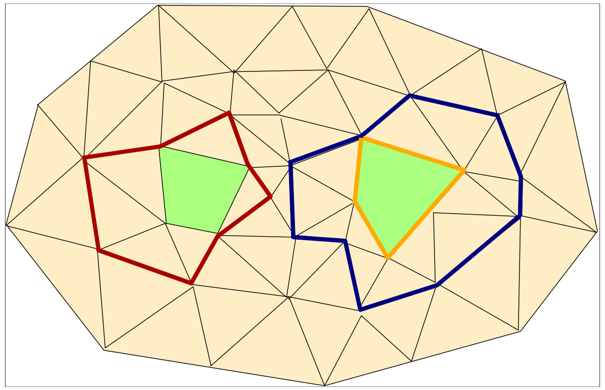
Cycles



$$(\{a,b\}+\{b,c\}+\{a,c\})-(\{a,b\}+\{b,d\}+\{c,d\}+\{a,c\})=\{b,c\}+\{b,d\}+\{c,d\}$$

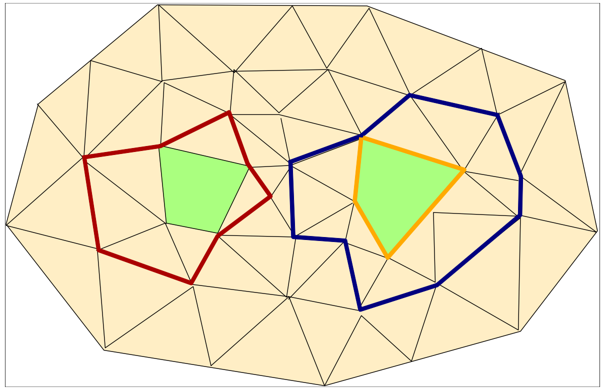
$$\partial_2\{b,c,d\}=\{b,c\}+\{b,d\}+\{c,d\}$$

Cycles and Boundaries



Cycles and Boundaries

- Cycles:
  - p-cycle: a p-chain whose boundary is 0
  - p-th cycle group $Z_p(K) = \ker(\partial_p)$
- Boundary cycles:
  - p-boundary: a p-cycle which is the boundary of some $(p+1)$-chain
  - p-th boundary group $B_p(K) = \mathrm{Im}(\partial_{p+1})$
- $\partial_p \circ \partial_{p+1} = 0 \Rightarrow B_p \subseteq Z_p \subseteq C_P$



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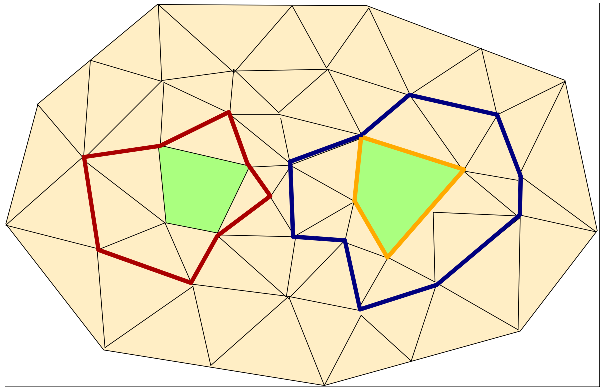
Under $\mathbb{Z}_2$ coefficients, $B_p,\ Z_p,\ C_p$ are all vector spaces.



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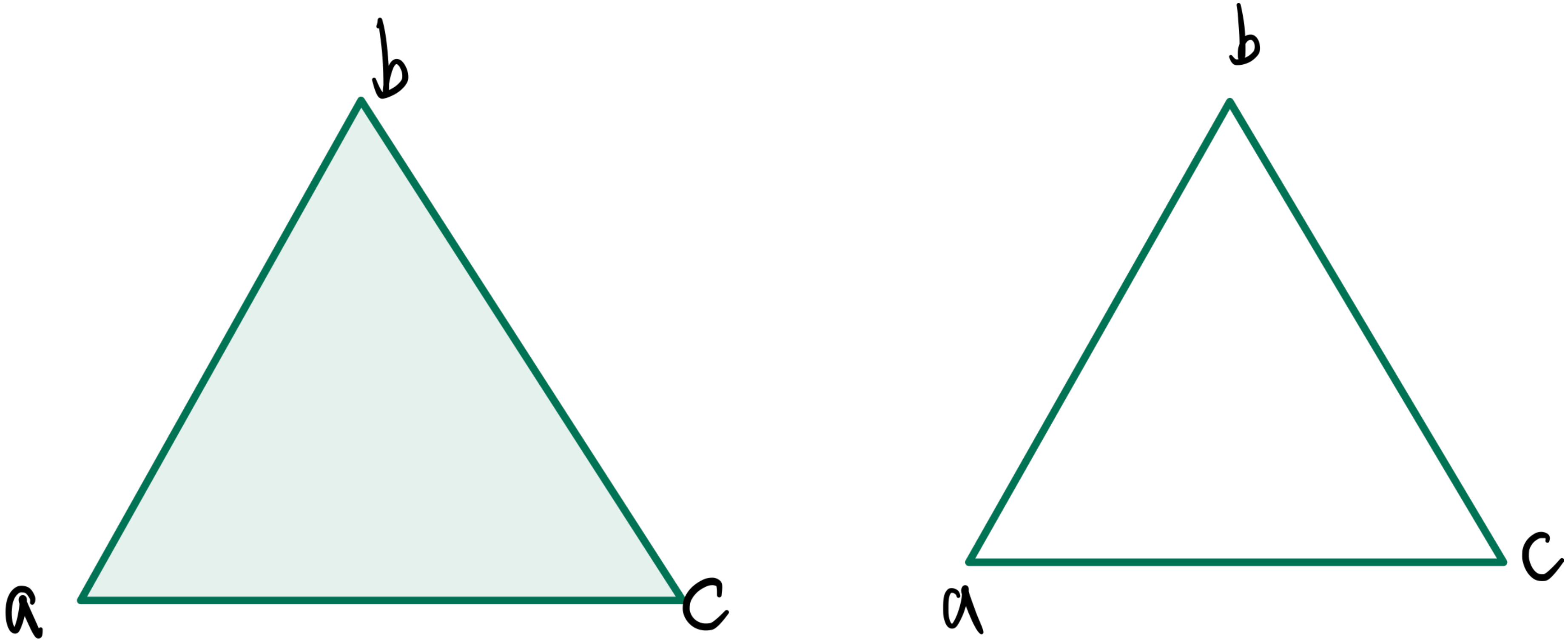

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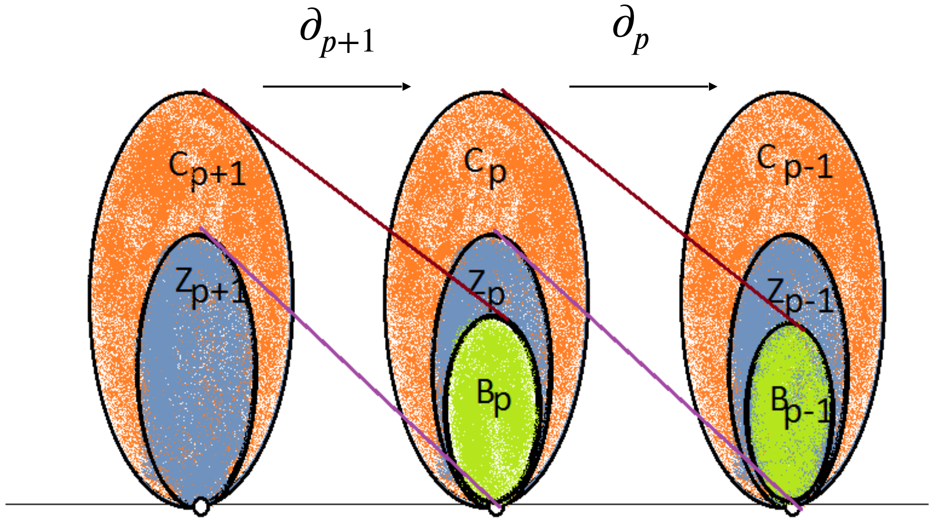
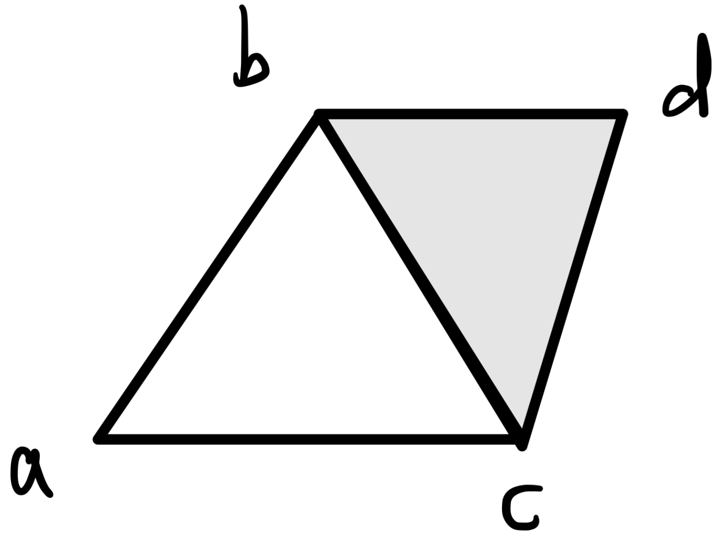
- $\partial_p \circ \partial_{p+1} = 0 \Rightarrow B_p \subseteq Z_p \subseteq C_P$



Cycles and Boundaries



$$(\{a, b\} + \{b, c\} + \{a, c\}) - (\{a, b\} + \{b, d\} + \{c, d\} + \{a, c\}) = \{b, c\} + \{b, d\} + \{c, d\}$$

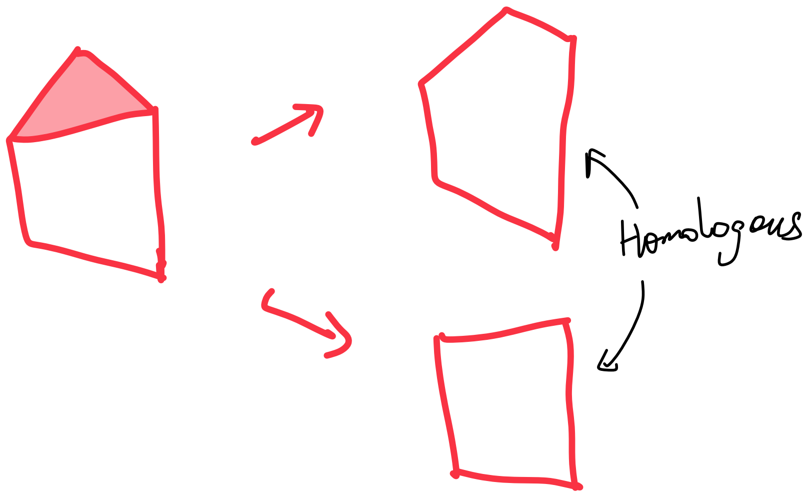
$$\partial_2\{b, c, d\} = \{b, c\} + \{b, d\} + \{c, d\}$$

$$\{a, b\} + \{b, c\} + \{a, c\} \sim \{a, b\} + \{b, d\} + \{c, d\} + \{a, c\}$$

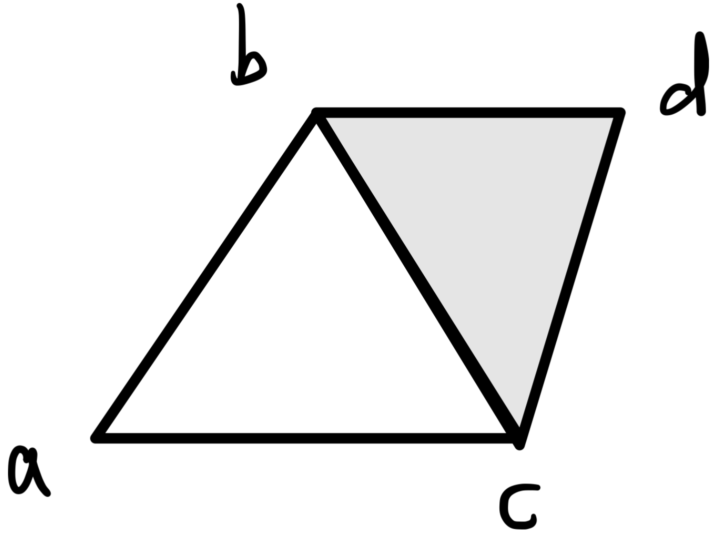
Homology groups

- p-th cycle group $Z_p(K) = \ker(\partial_p)$
- p-th boundary group $B_p(K) = \mathrm{Im}(\partial_{p+1})$
- p-th *homology group*
  - $H_p(K) = Z_p/B_p$
  - c_1 is *homologous to* c_2 if
    - $c_1 + c_2 \in B_p$, i.e, $c_1 + c_2$ is a boundary cycle
  - $h = [c] \in H_p$:
    - the family p-cycles homologous to c
    - called a *homology class*
  - A cycle is null-homologuous if it is a boundary, and we also say its homology class is trivial.

Under $\mathbb{Z}_2$ coefficients, C_p, B_p, Z_p, H_p are all vector spaces.



Homology



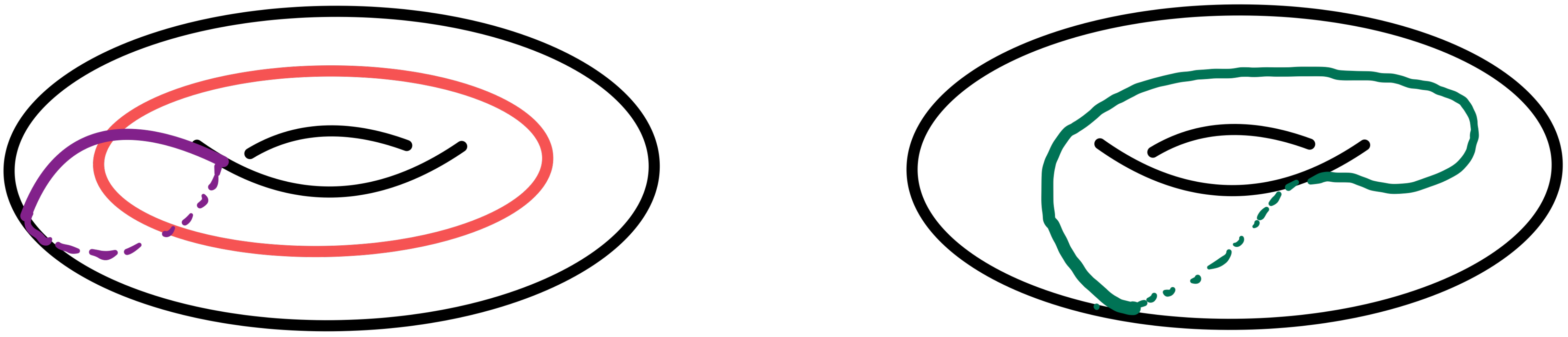
$$Z_1(K) = \langle \{a,b\} + \{b,c\} + \{a,c\}, \{a,b\} + \{b,d\} + \{c,d\} + \{a,c\} \rangle$$

$$B_1(K) = \langle \{b,c\} + \{b,d\} + \{c,d\} \rangle$$

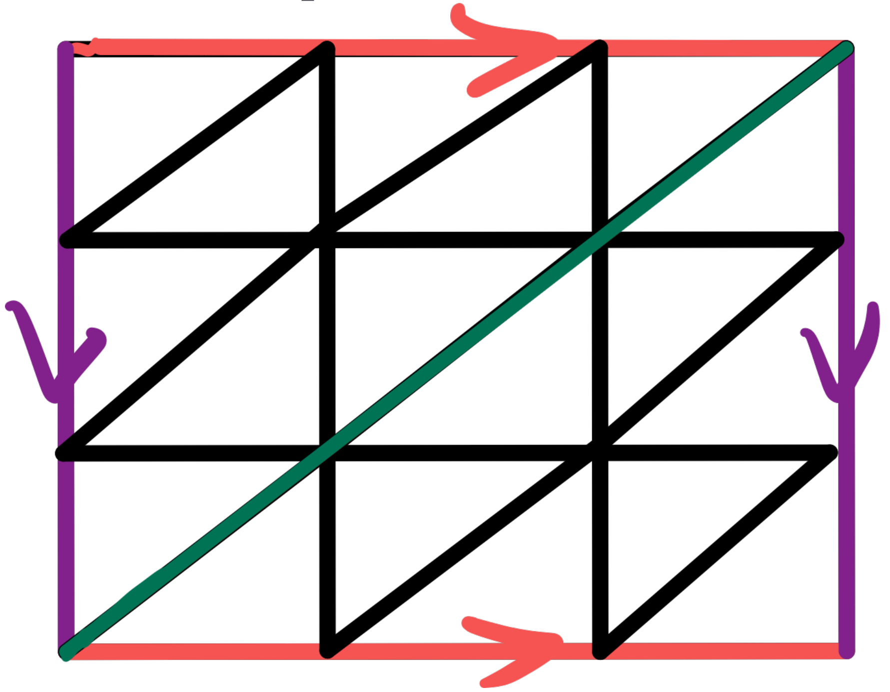
$$H_1(K) = \langle [\{a,b\} + \{b,c\} + \{a,c\}] \rangle$$

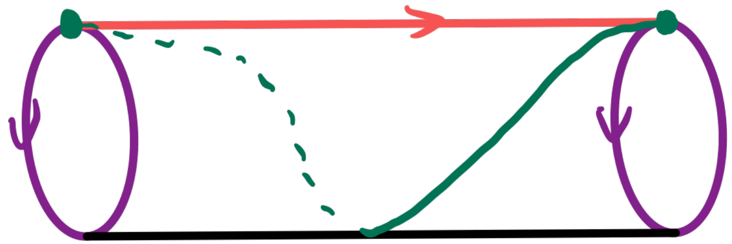
$$\dim H_1(K) = 1$$

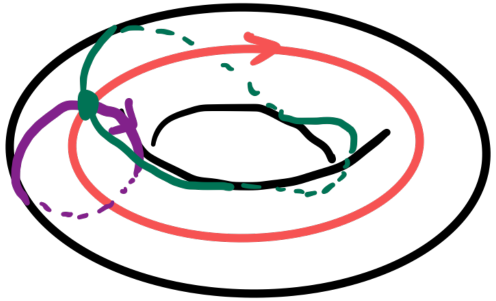
Torus example

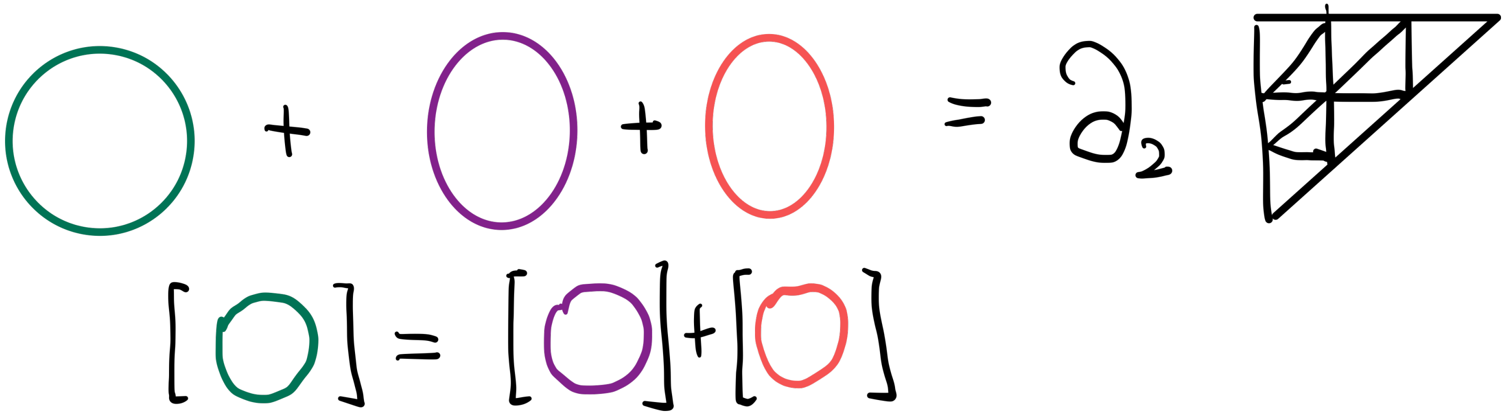


Torus example









Betti numbers

- Betti number: $\beta_p(K) = \dim H_p(K)$
- Theorem:
  - $\beta_p(K) = \dim Z_p - \dim B_p$
- Examples:

Betti numbers

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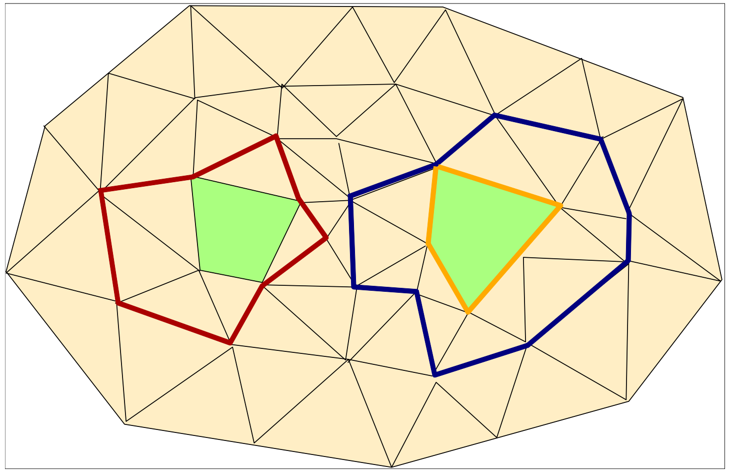


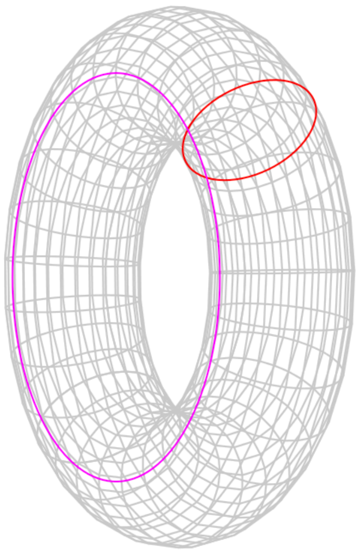
$\beta_0(K) = ?\quad \beta_1(K) = ?$

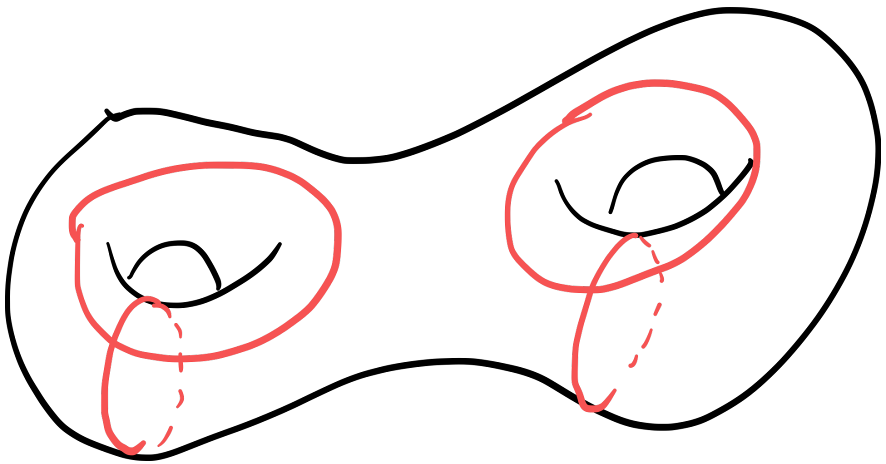
Betti numbers are homotopy invariants

- Fact:
  - Two homotopy equivalent topological spaces have isomorphic homology groups (and thus same Betti numbers).

More examples

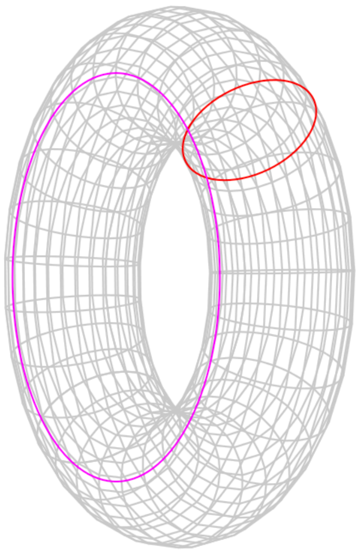
- Meaning of $\beta_0, \beta_1, \beta_2, \ldots$

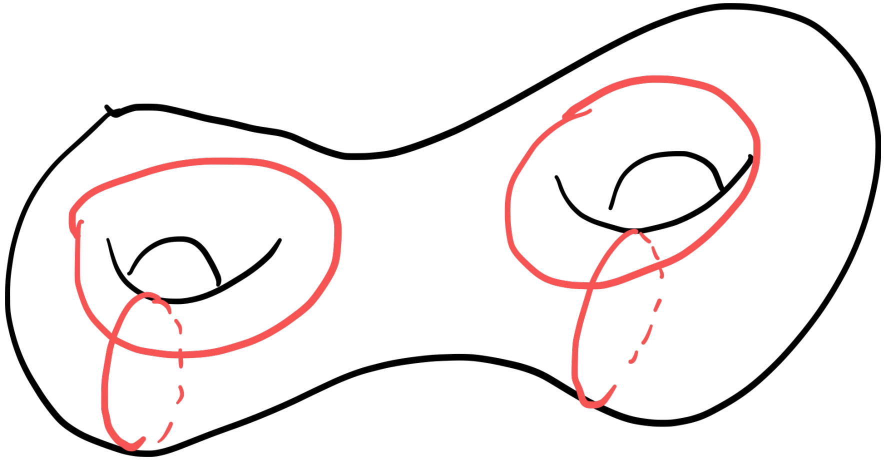




More examples

- Meaning of $\beta_0, \beta_1, \ \beta_2, \ldots$
- For connected, compact orientable 2-manifolds, recall the classification results
  - If it has genus g, then $\beta_0 = 1$, $\beta_1 = 2g$, and $\beta_2 = 1$





Another definition for Euler characteristic

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- Given a Simplicial complex K
  - Recall its Euler characteristic $\chi(K) = \sum_{p=0} (-1)^p n_p(K)$

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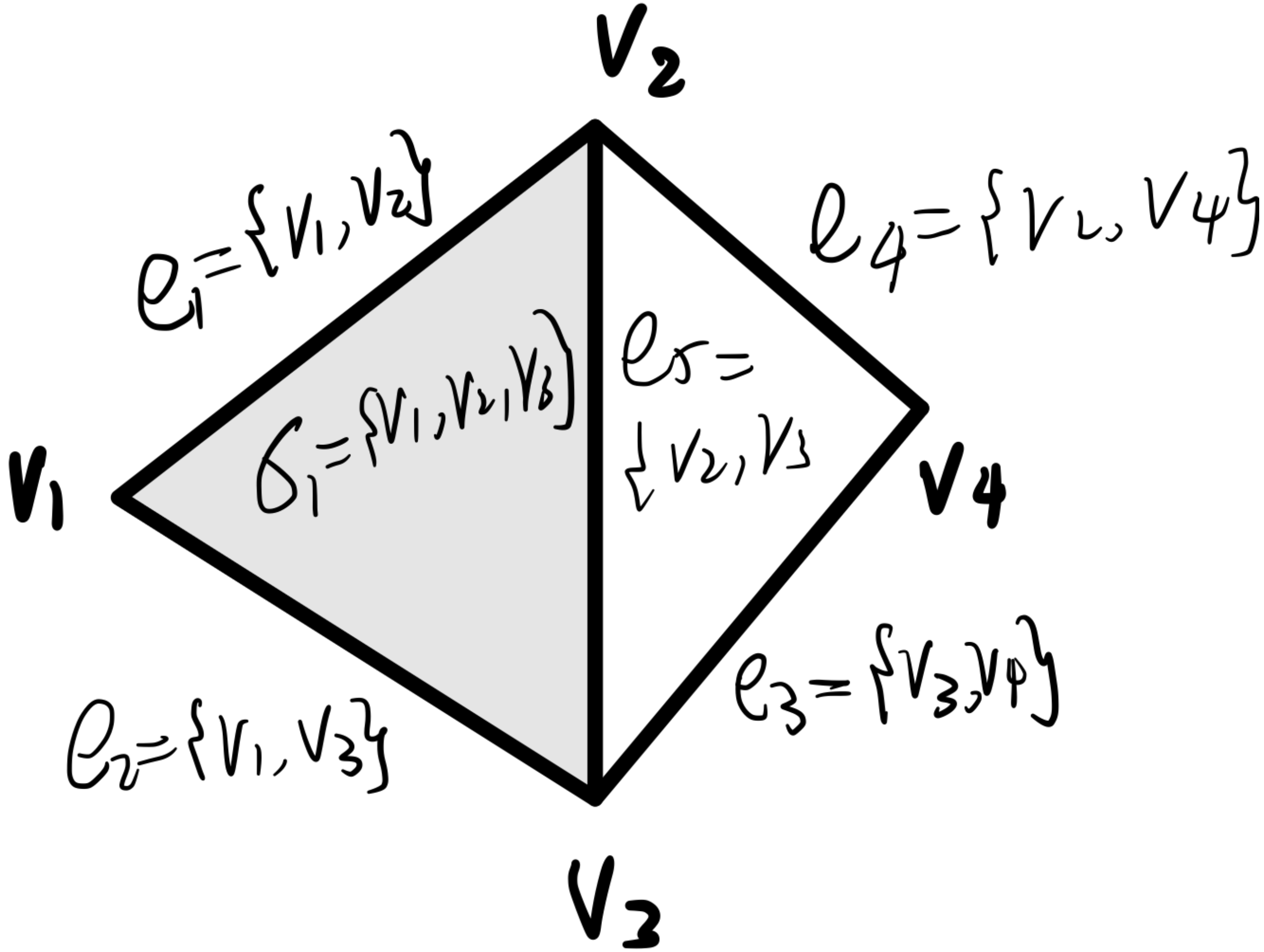
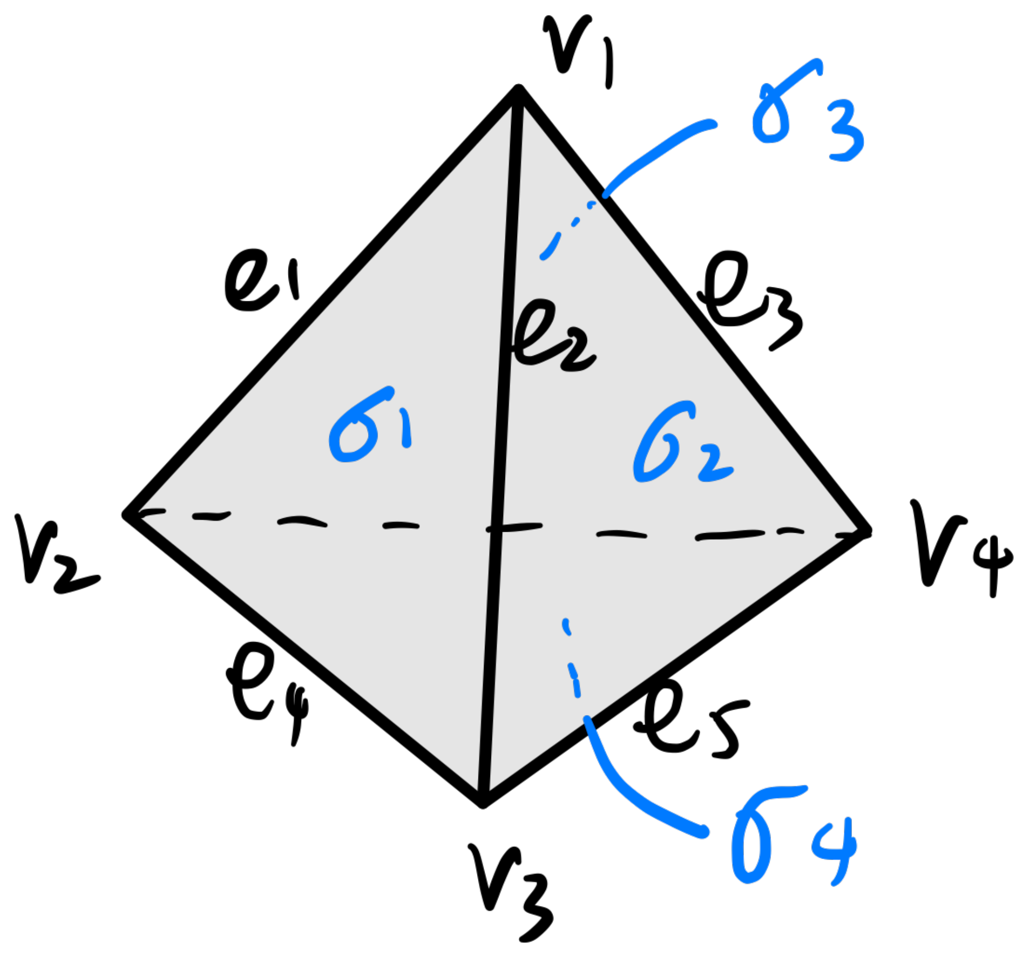
- Theorem (Euler-Poincaré formula)
  - Given a simplicial complexes K, one has that
    $$\chi(K) = \sum_{i=0} (-1)^i \beta_i(K)$$

Part 2:
Matrix view and computation

Calculation of Homology



Calculation of Homology



Boundary Matrix

- $K^p = \left\{\alpha_1, \ldots, \alpha_{n_p}\right\},\ K^{p-1} = \left\{\tau_1, \ldots, \tau_{n_{p-1}}\right\}$
  - K^p forms a basis for p-th chain group C_p
- $\mathrm{n}_{p-1} \times n_p$ boundary matrix A_p
  - $A_p[i]\left[j\right] = 1$ iff $\tau_i \subseteq \sigma_j$
  - representing $\partial_p : C_p \to C_{p-1}$ w.r.t. basis $\left\{\alpha_1, \ldots, \alpha_{n_p}\right\}$ and $\left\{\tau_1, \ldots, \tau_{n_{p-1}}\right\}$

Boundary Matrix

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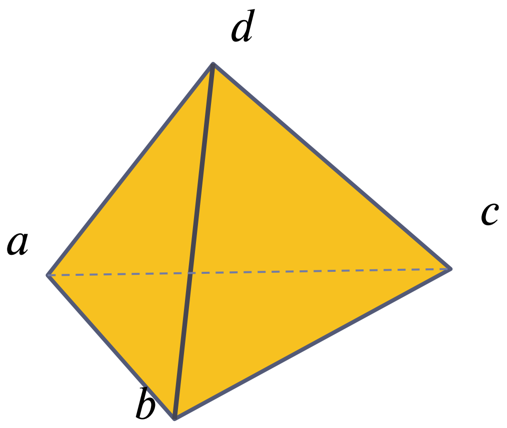
- $n_{p-1} \times n_p$ boundary matrix A_p s.t.
  - $A_p[i][j] = 1$ iff $\tau_i \subseteq \sigma_j$
  - representing $\partial_p\colon C_p \to C_{p-1}$ w.r.t. basis $\left\{ \alpha_1, \ldots, \alpha_{n_p} \right\}$ and $\left\{ \tau_1, \ldots, \tau_{n_{p-1}} \right\}$



$$A_2 = \begin{array}{c} \\ ab \\ ac \\ ad \\ bc \\ bd \\ cd \end{array} \begin{array}{c} \begin{array}{cccc} abc & abd & acd & bcd \end{array} \\ \begin{pmatrix} 1 & 1 & & \\ 1 & & 1 & \\ & 1 & 1 & \\ 1 & & & 1 \\ & 1 & & 1 \\ & & 1 & 1 \end{pmatrix} \end{array}$$

- Given a p-chain $c = \sum_{i=1}^{n_p} c_i \alpha_i$
  - Under basis K^p, vector representation of c is
  - $\overrightarrow{c} = \left[c_1,\ c_2, \ldots,\ c_{n_p}\right]^T$
- Boundary $\partial_p c$ is a $(p-1)$-chain with vector representation $A_p \overrightarrow{c}$ w.r.t basis K^{p-1}

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$$A_p\vec{c} = \begin{bmatrix} a_1^1 & a_1^2 & \cdots & a_1^{n_p} \\ a_2^1 & a_2^2 & \cdots & a_2^{n_p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n_{p-1}}^1 & a_{n_{p-1}}^2 & \cdots & a_{n_{p-1}}^{n_p} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n_p} \end{bmatrix}$$

Boundary Matrix

- Recall $\partial_p : C_p \to C_{p-1}$
  - $Z_p = \ker \partial_p$
  - $B_{p-1} = \mathrm{Im}\partial_p$
- Let $n_p,\ z_p,\ b_p$ denote the dimension of $C_p,\ Z_p,$ and B_p
- $\beta_p = \dim H_p(K)$
- Claim: $(i) \quad n_p = z_p + b_{p-1}$; This follows from

  $\dim V = \dim \ker f + \dim \mathrm{Im} f$ for general linear maps $f : V \to W$

  $(ii)\ \ \beta_p = z_p - b_p$; This follows from $\dim V/W = \dim V - \dim W$

Boundary Matrix

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- $\beta_p = \dim H_p(K)$
- Claim: $(i)\quad n_p = z_p + b_{p-1};$

  $(ii)\ \ \beta_p = z_p\ - b_p$

- Consider A_p
  - Each columns of A_p corresponds to a boundary cycle of dimension $p-1$
  - Rank of A_p gives $b_{p-1} = \dim B_{p-1}$

Simple Alg to compute Betti numbers

- Given a simplicial complex
  - (1) Compute boundary matrix A_p for each dimension p
  - (2) For each p, compute rank of A_p, which is b_{p-1}
  - (3) Use formula $n_p = z_p + b_{p-1}$ to compute all z_ps from b_ps
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  - we only need boundary matrix A_{p+1} and A_p, thus only the set of simplices of dimension $p-1,\ p,\ p+1$

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- Note, this gives a simple algorithm for computing all β_p's via Gaussian elimination

Another alg: Right-reduction algorithm

$$\begin{bmatrix} a & * & * & * & * & * & * & * & * \\ 0 & 0 & b & * & * & * & * & * & * \\ 0 & 0 & 0 & c & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & d & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

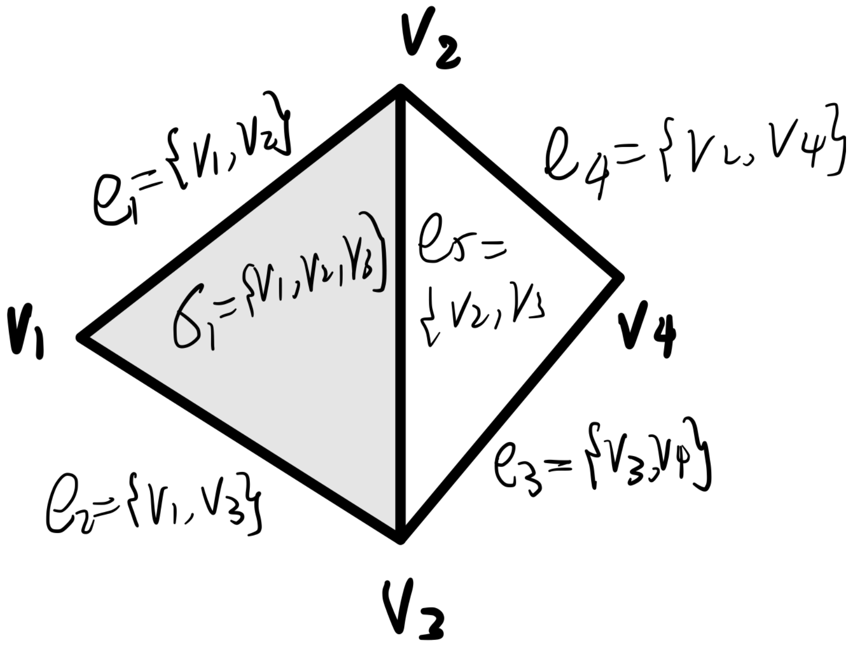
Row echelon form

$$\begin{bmatrix} * & * & * & 0 \\ * & 1 & * & 0 \\ 1 & 0 & * & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Column reduced form

$$lowId[i] \neq lowId[j]$$

Another alg: Right-reduction algorithm



	e1	e2	e3	e4	e5
v1	1	1	0	0	0
v2	1	0	0	1	1
v3	0	1	1	0	1
v4	0	0	1	1	0

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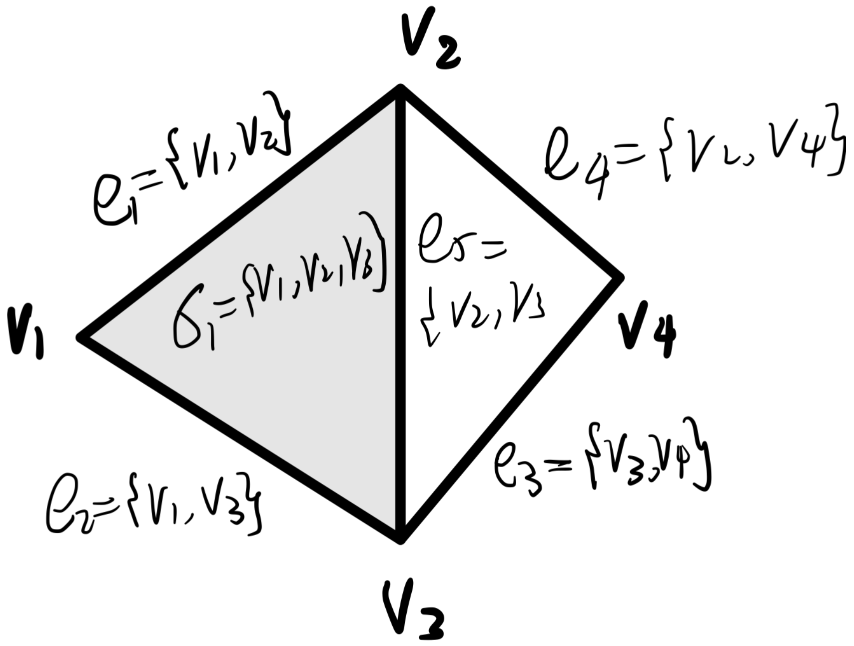

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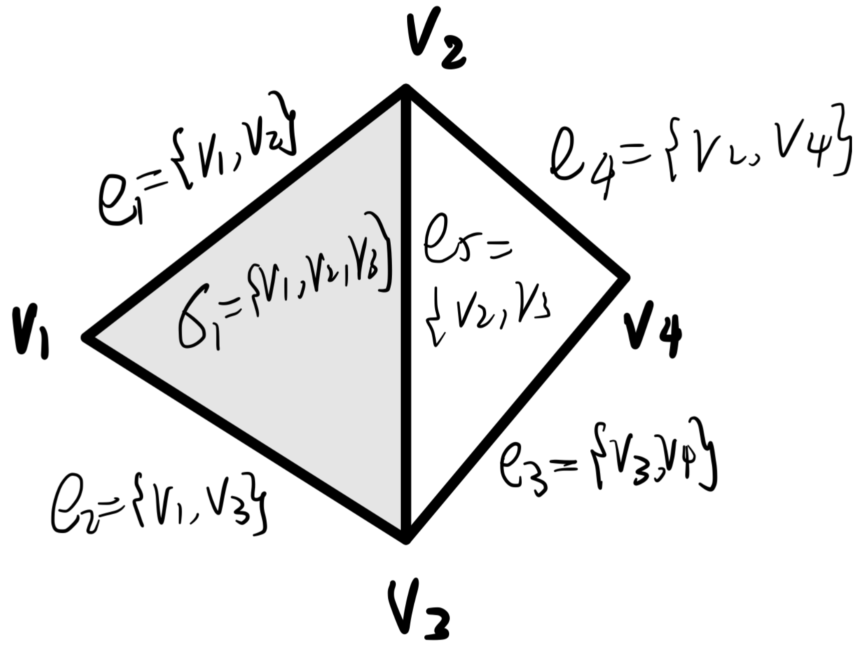

	e1	e2	e3	e4+e3	e5
v1	1	1	0	0	0
v2	1	0	0	1	1
v3	0	1	1	1	1
v4	0	0	1	0	0

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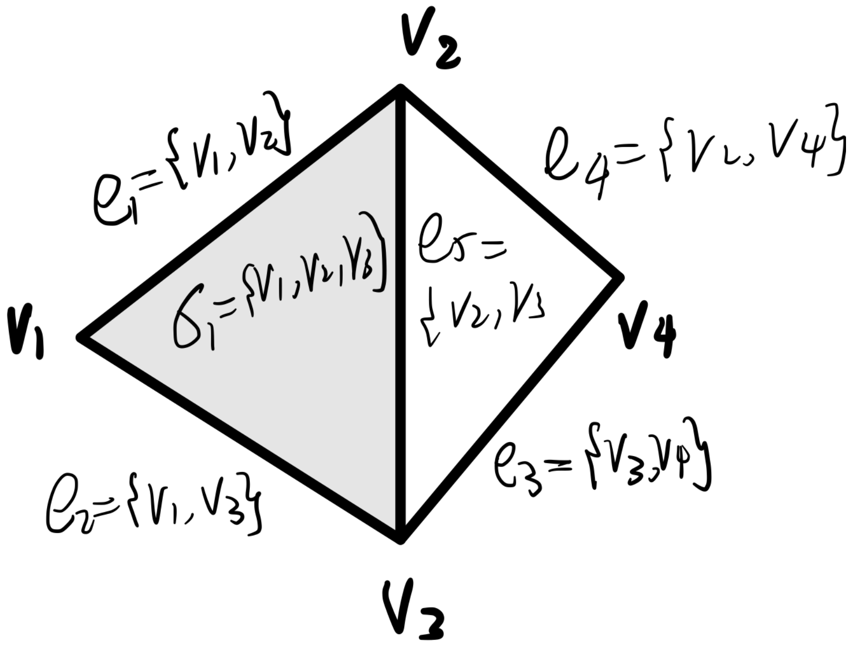

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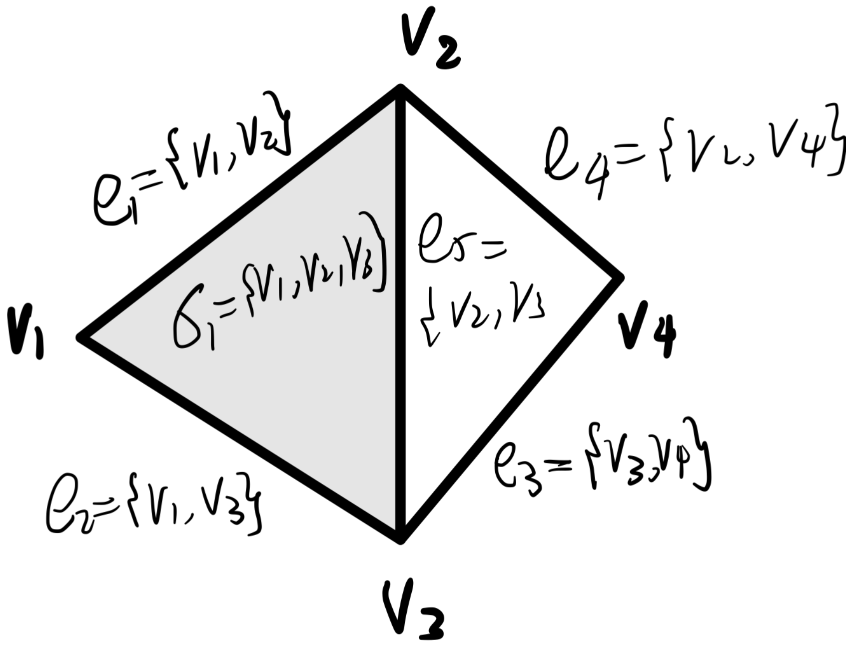

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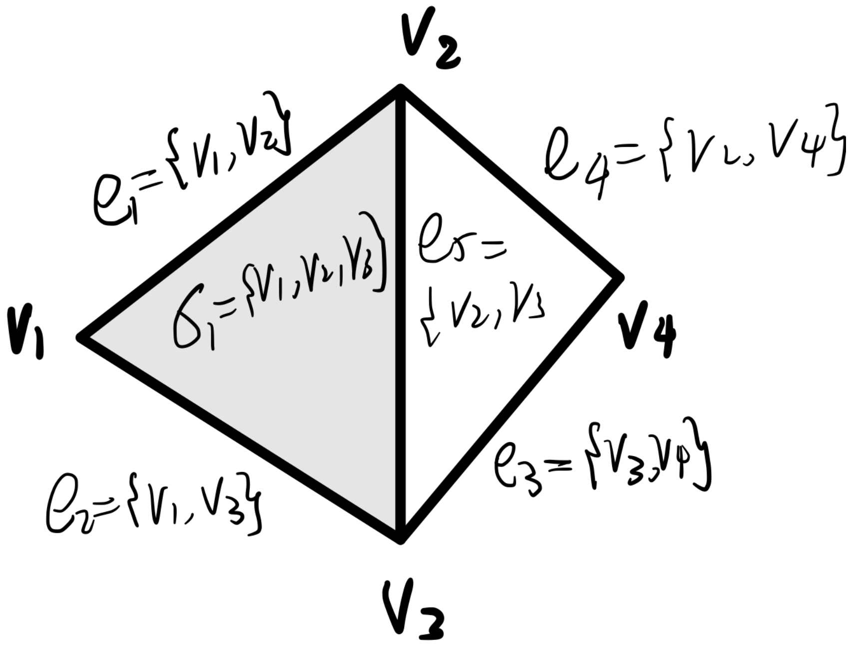

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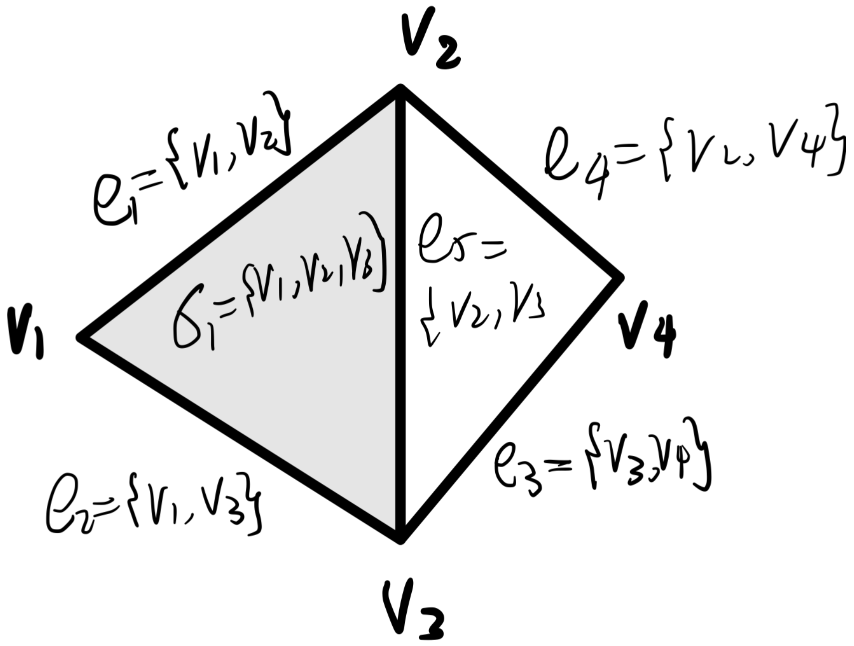

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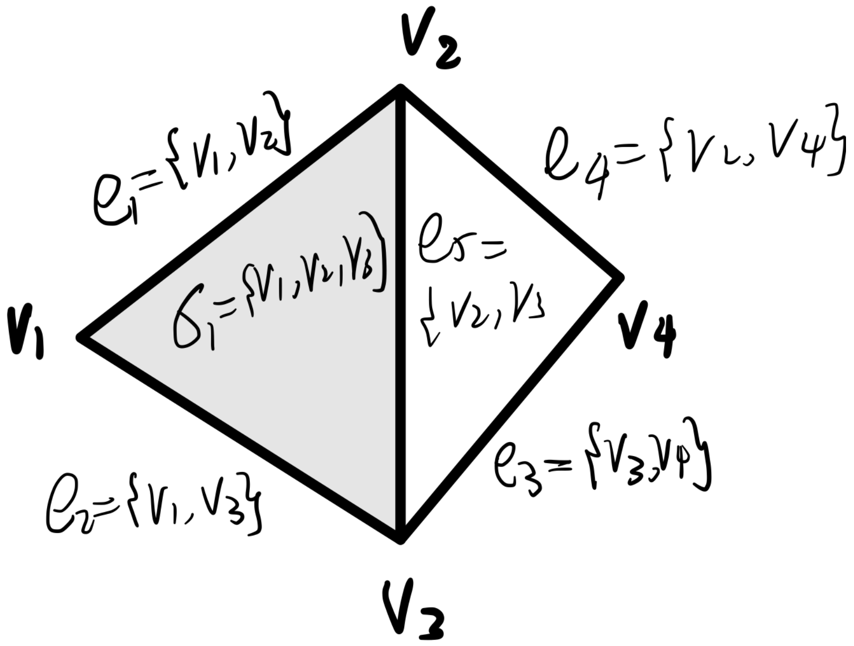

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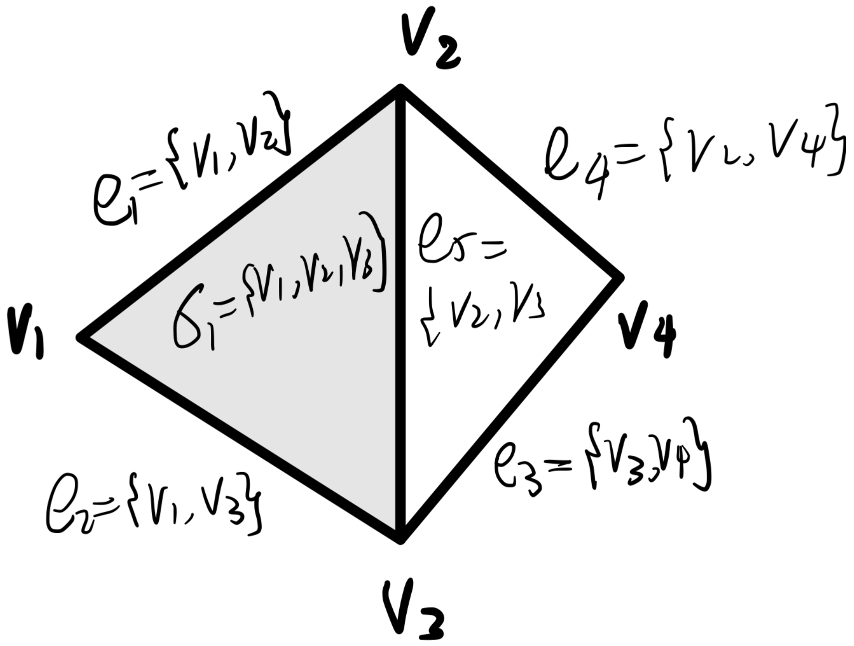

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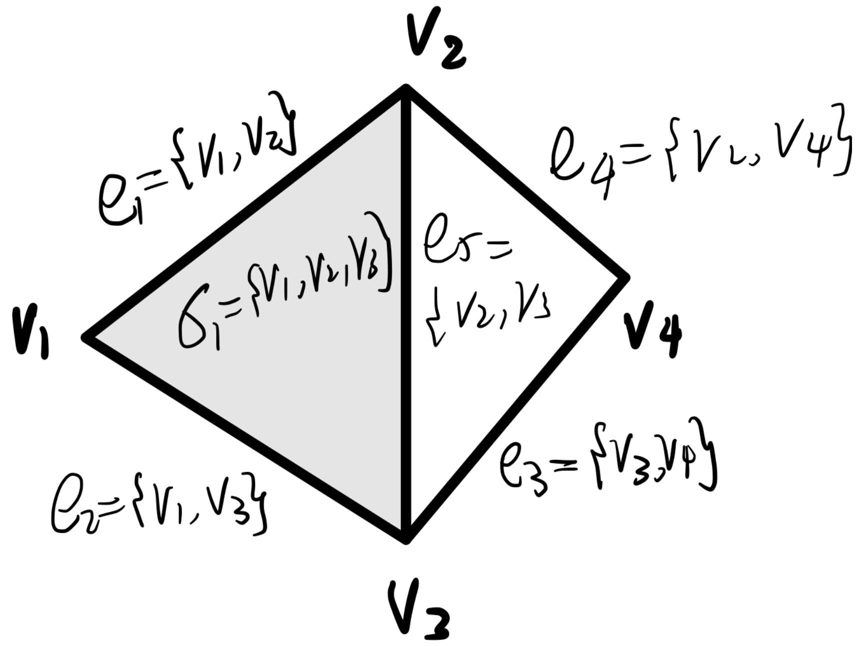

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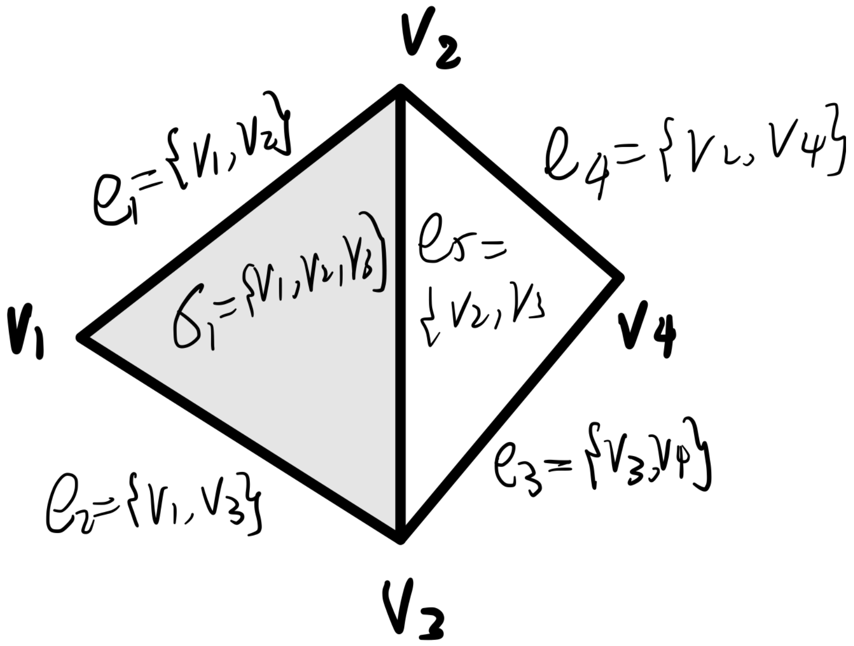

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Another alg: Right-reduction algorithm

- Starting with boundary matrix $M = A_p$
  - For the i-th column corresponding to p-simplex σ_i,
    - associate a p-chain Γ_i initialized to σ_i
  - AddColumn(j, i): add column j to column i
    - $Col_M[i] = Col_M[i] + Col_M[j]; \;\; \Gamma_i = \Gamma_i + \Gamma_j$

Algorithm 1 Right-Reduction(M)

```
for i = 2 to n_p do
    while ∃j < i s.t. lowId[j] = lowId[i] do
        AddColumn(j, i);
    end while
end for
Return(M)
```

Properties

- Lemma:
  - Each reduction (column addition) step maintains the following invariance: After k-th stages, $M^{(k)}$ has the same rank as A_p, and $\partial_p \Gamma_j^{(k)} = col_M[j]$ for any j.

Properties

- **Lemma:**
  - Each reduction (column addition) step maintains the following invariance: After k-th stages, $M^{(k)}$ has the same rank as A_p, and $\partial_p \Gamma_j^{(k)} = col_M[j]$ for any j.

- **Lemma:**
  - At the end of the reduction algorithm, each non-zero column has a unique low-ID .

- Reduced form:
  - A matrix M is in reduced form if each non-zero column has a unique low-ID.

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- Lemma:
  - If a matrix is in reduced form, then all its non-zero columns are linearly independent.

Properties

- Theorem:
  - Procedure Right-Reduction(M) terminates in $O(n_p n_{p-1}^2)$ time
  - The output matrix M is in column reduced form
  - The set of non-zero columns in M form a basis for B_{p-1}
  - The set $\{\Gamma_i \mid col_M[i] = 0\}$ form a basis for Z_p

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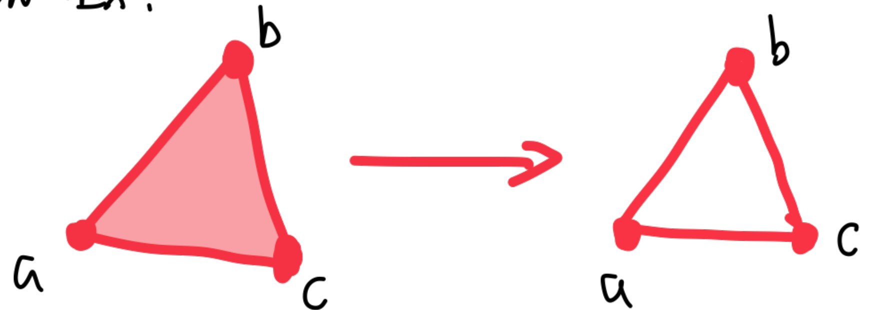
This is not the only reduction algorithm!! Any elimination via row/column additions to convert a matrix into a reduced form works!

Part 3:
Functoriality of Homology

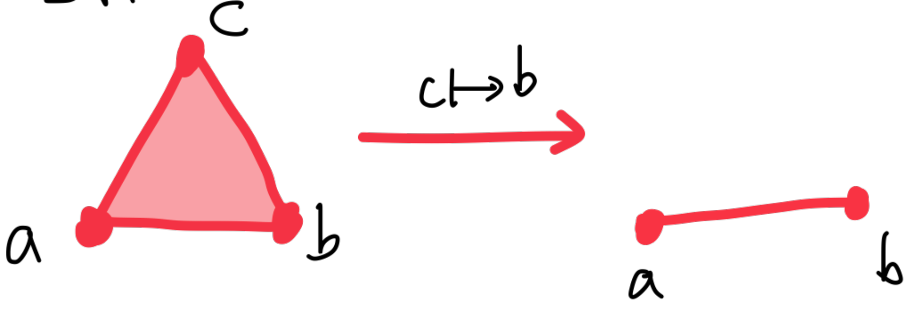
Simplicial map

- Intuitively, analogous to continuous maps between topological spaces
- Given simplicial complexes K and L
  - a function $f : V(K) \to V(L)$ is called a simplicial map if
    - for any $\sigma = \{p_0, \ldots, p_d\} \in \Sigma(K)$, $f(\sigma) = \left\{ f\left(p_0\right), \ \ldots, \ f\left(p_d\right) \right\}$ spans a simplex in L, i.e., $f(\sigma) \in \Sigma(L)$.
  - A simplicial map is also denoted $f : K \to L$

NON-EX:



EX:



Functoriality of Simplicial Homology

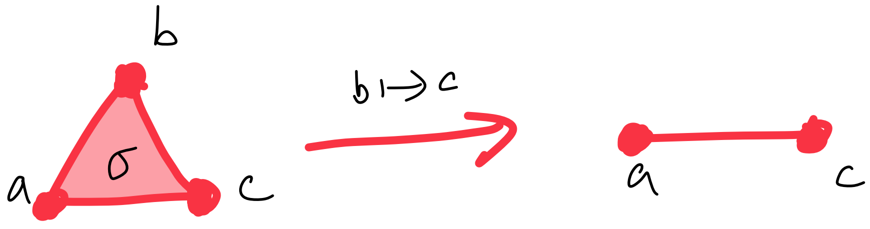
- Let $K = (V, \Sigma)$ and $K' = (V', \Sigma')$ and let $f : V \to V'$ be a simplicial map. Then,
  - f induces a linear map on homology groups: $f_p : H_p(K) \to H_p(K')$
  - If there exist $K'' = (V'', \Sigma'')$ and another simplicial map $g : V' \to V''$, then
    - $(g \circ f)_p = g_p \circ f_p$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & V' \arrow[r, "g"] & V'' \end{tikzcd}$$

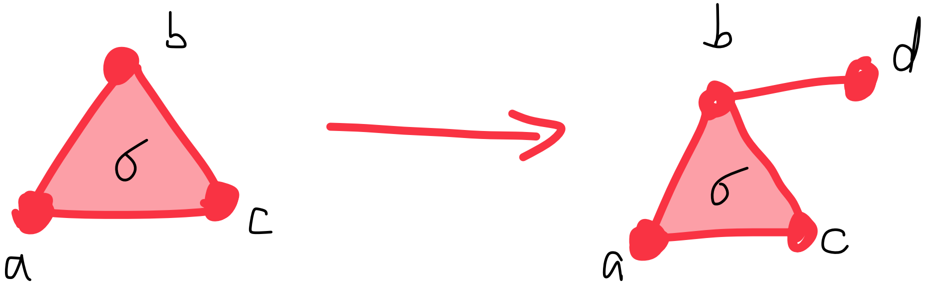
$$\begin{tikzcd} H_p(K) \arrow[r, "f_p"] \arrow[rr, bend right, "(g \circ f)_p = g_p \circ f_p"'] & H_p(K') \arrow[r, "g_p"] & H_p(K'') \end{tikzcd}$$

Construction of f_p

- Define $\bar{f}_p : C_p(K) \to C_p(K')$
  - $\bar{f}_p(\sigma) = \begin{cases} f(\sigma) & \text{if } f(\sigma) \text{ is } p-\text{dimensional} \\ 0 & \text{otherwise} \end{cases}$
  - Define $f_p : H_p(K) \to H_p(K')$
    - $f_p([c]) := [\bar{f}_p(c)]$

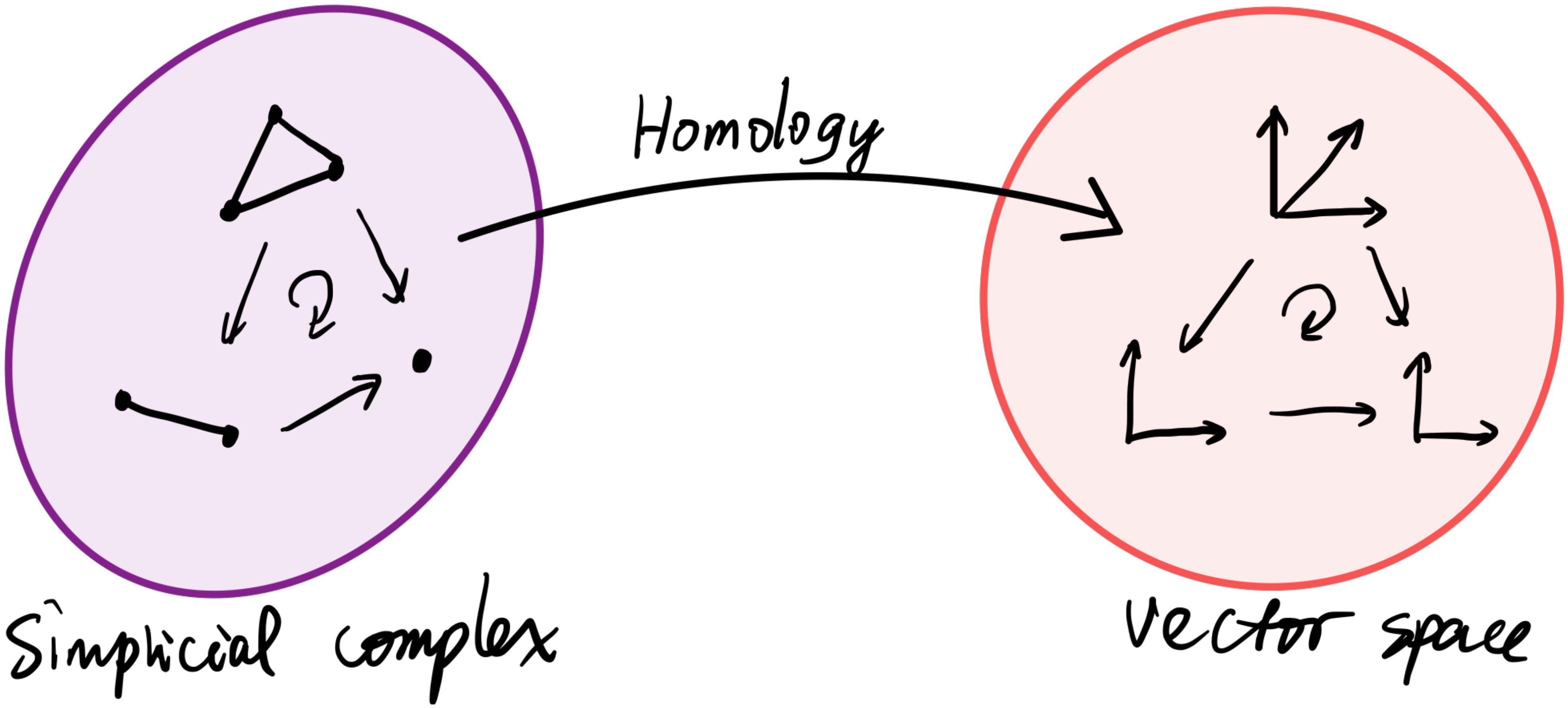


$\bar{f}_2(\sigma) = 0$ since $f(\sigma) = \{a, c\}$ is 1-dim



$\bar{f}_2(\sigma) = \sigma$

Mind picture of functoriality



FIN