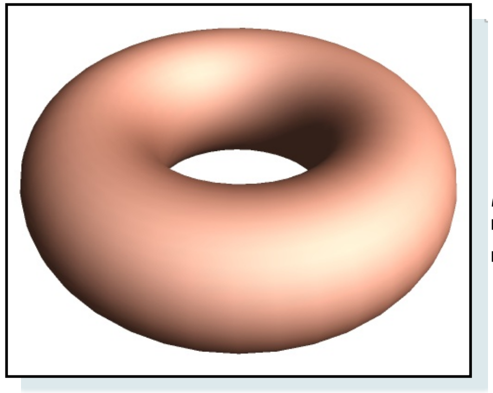
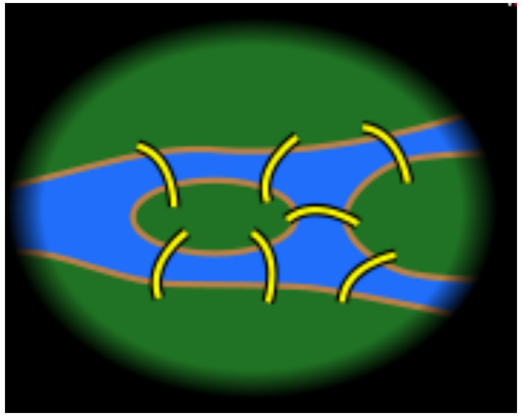
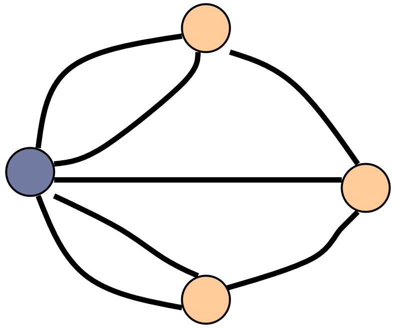
DSC 214
Topological Data Analysis
Topic 1: Basics

Instructor: Zhengchao Wan

Goal

 $\cong$ 

 $\simeq$ 

- Fundamental Questions
  - What is a topological space?
  - What is a "continuous" way of turning one space to another?
  - When can we say two spaces are the "same"?

Overview

- Fundamental concepts
  - Topological space
  - Continuous maps
  - Homeomorphisms and homotopies
  - Manifolds

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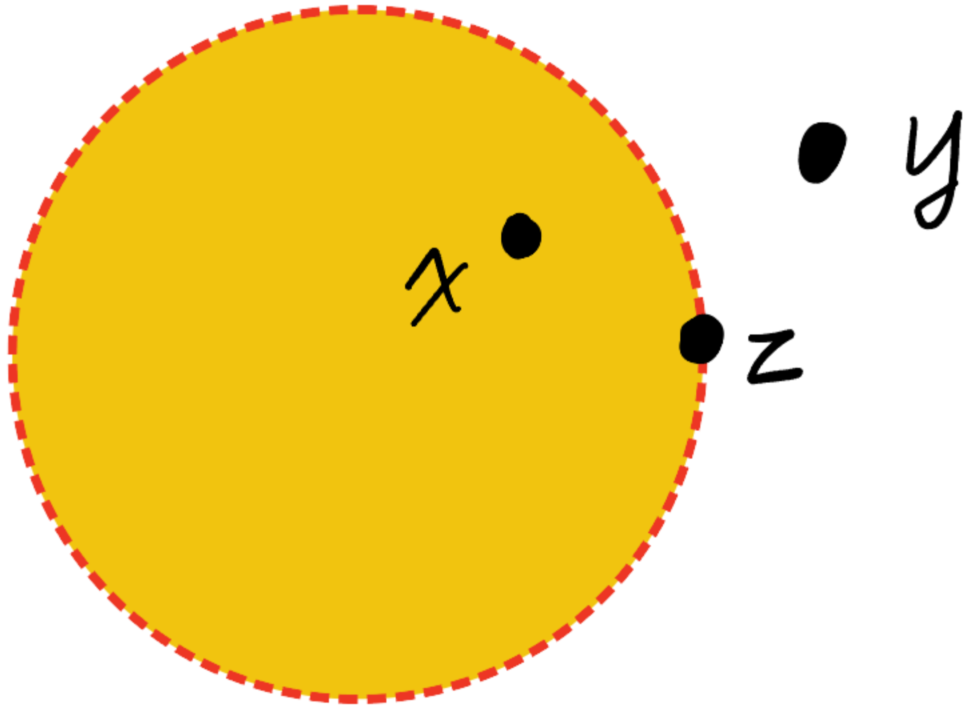
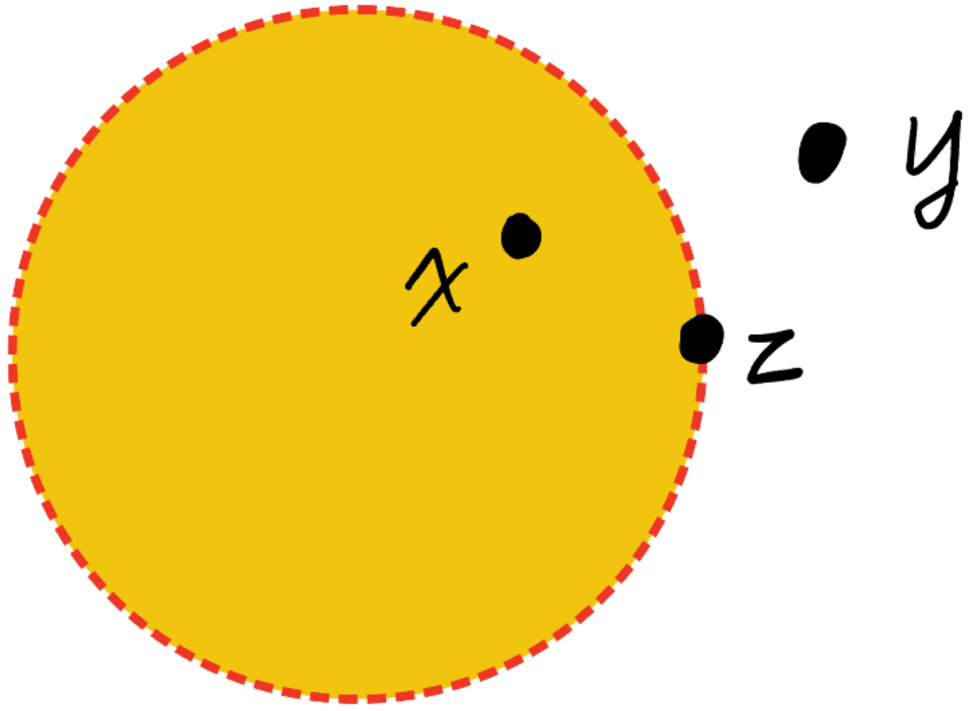
How we mathematically talk about space of interest

Set theory and beyond



Set theory and beyond

- Given a disk D (without boundary)



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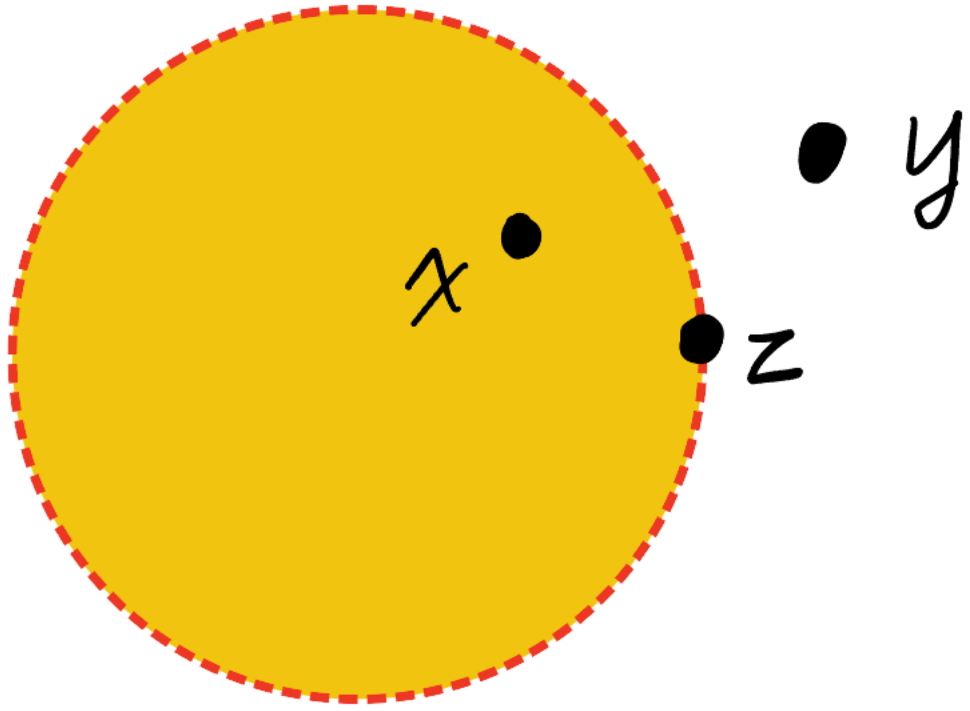

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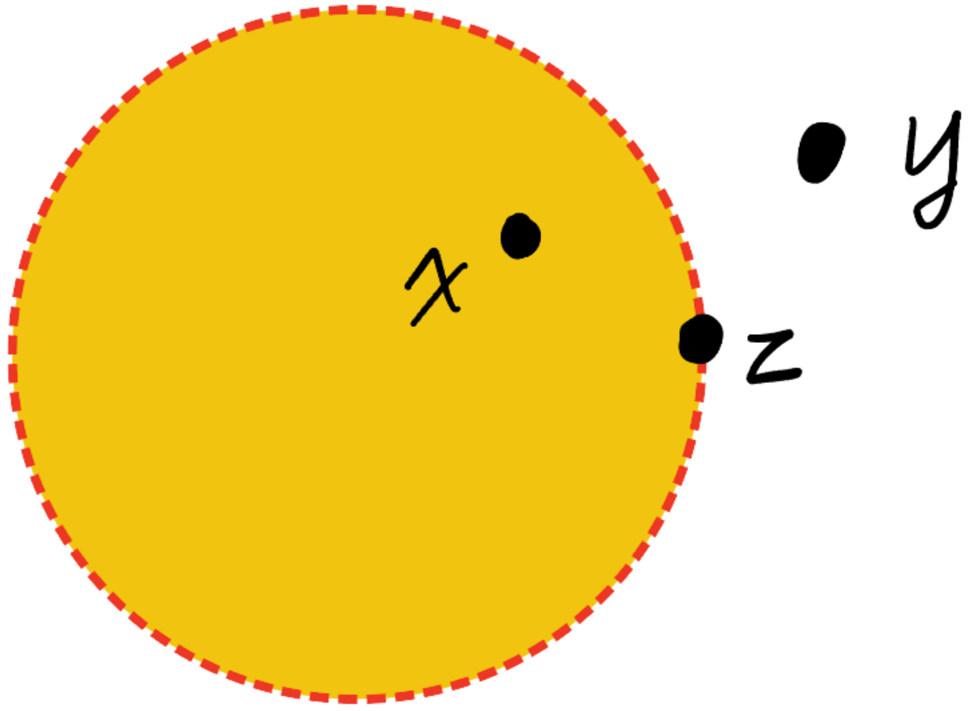

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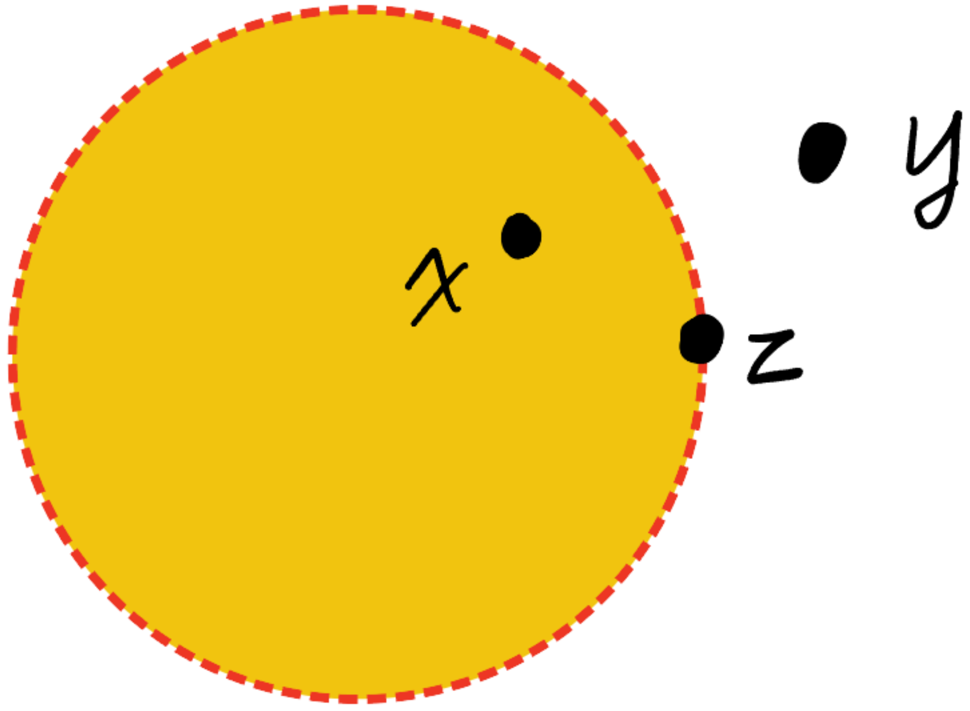

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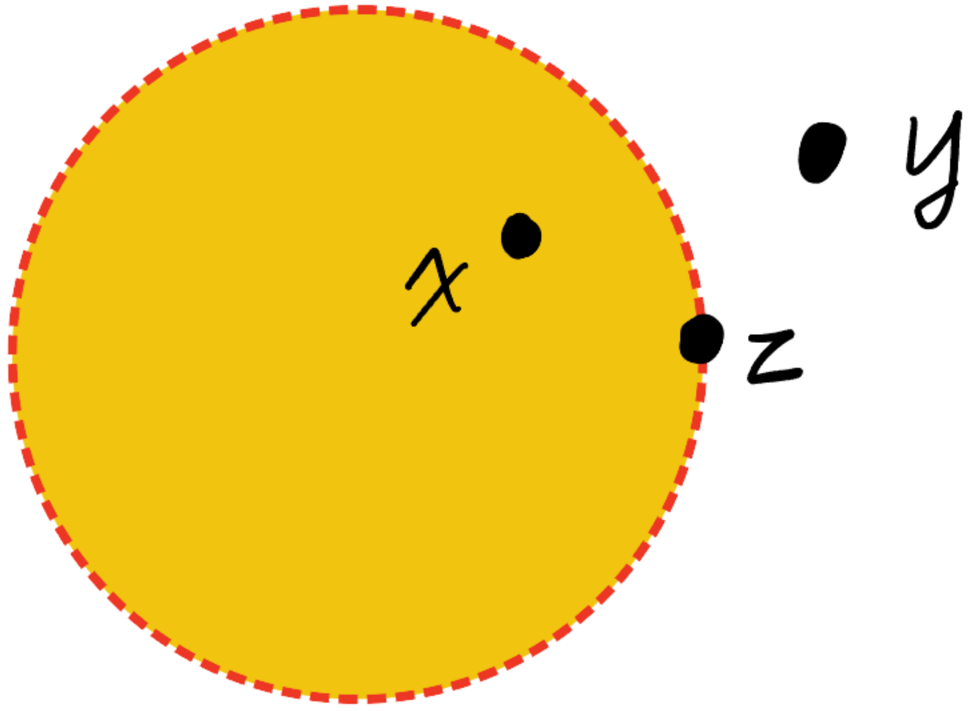

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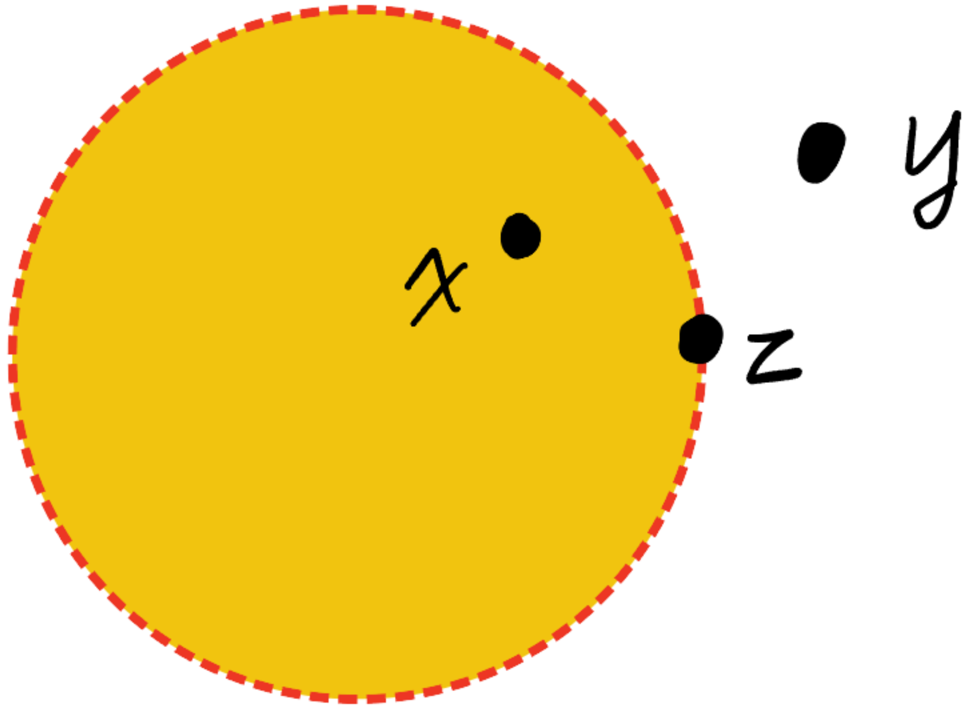
    - x and z are in the "**closure**" of D



Why do we care?

- We want to rigorously define "continuous transformation"
  - A continuous map shouldn't tear things apart
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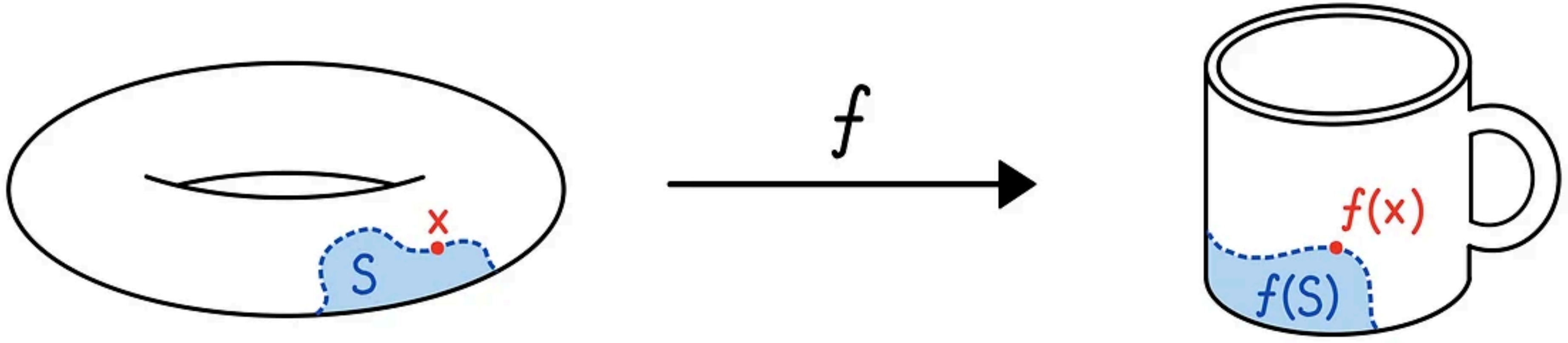

From https://wgyory.wixsite.com/toolatetopologize/post/post-1

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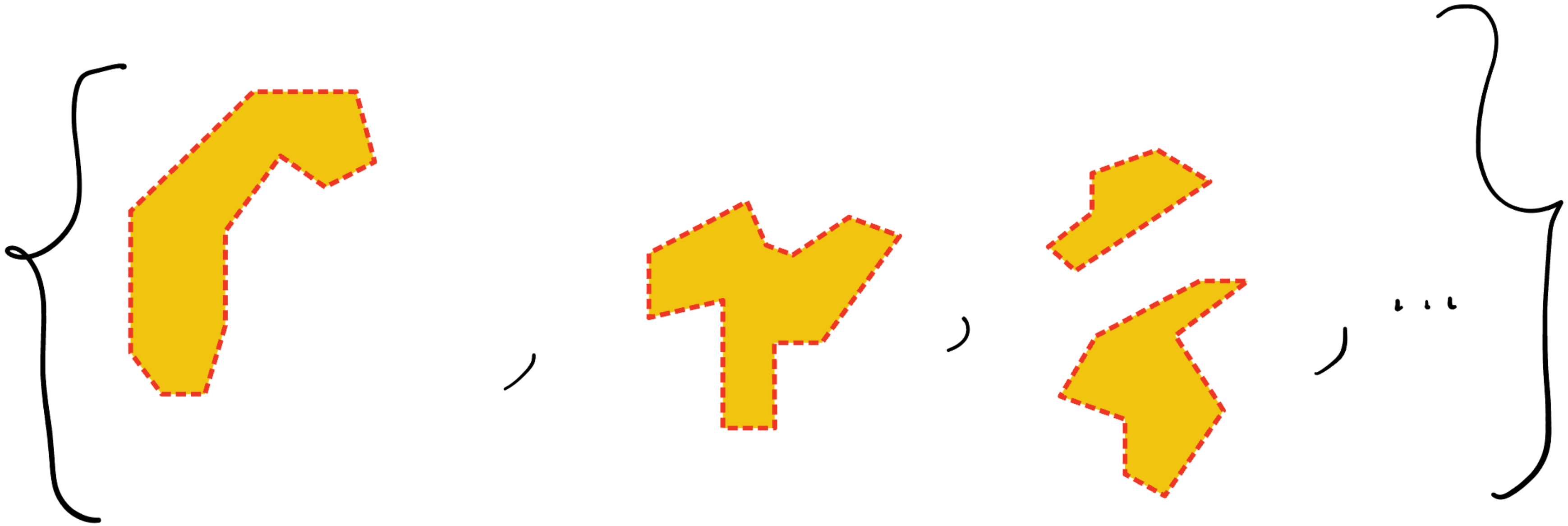
- We keep track of **ALL** the relations "S contacts x" to make the above intuition rigorous!

Topology

- It turns out that we just need to specify all “open” subsets to keep track of all relations “S contacts x” on a given set



Topological space

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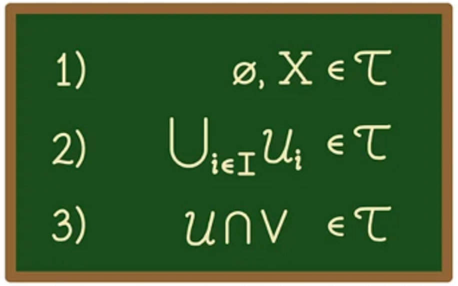

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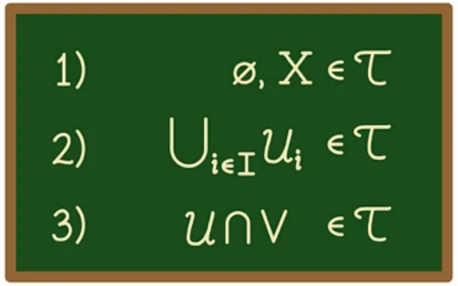

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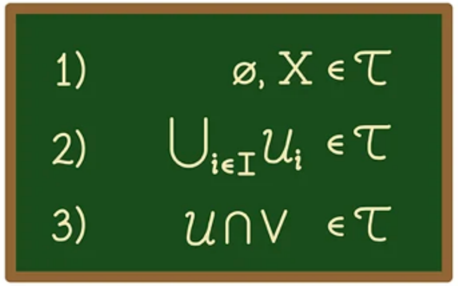

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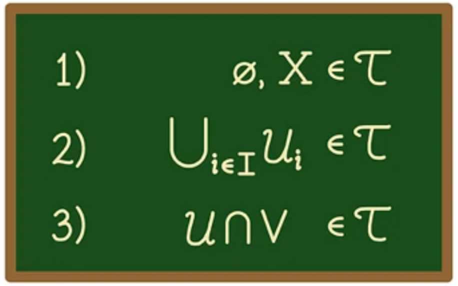

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  - **Metric space topology**

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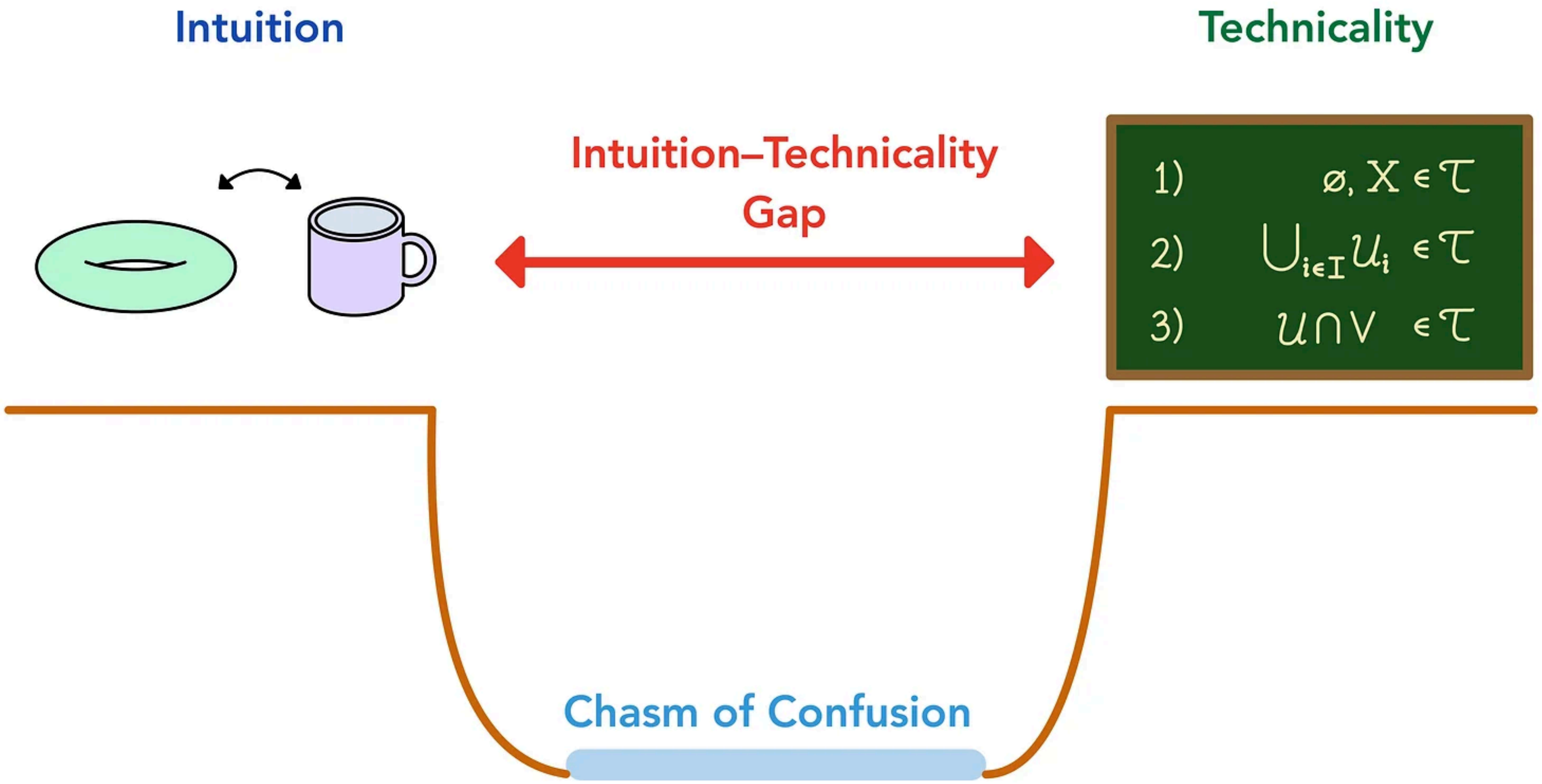

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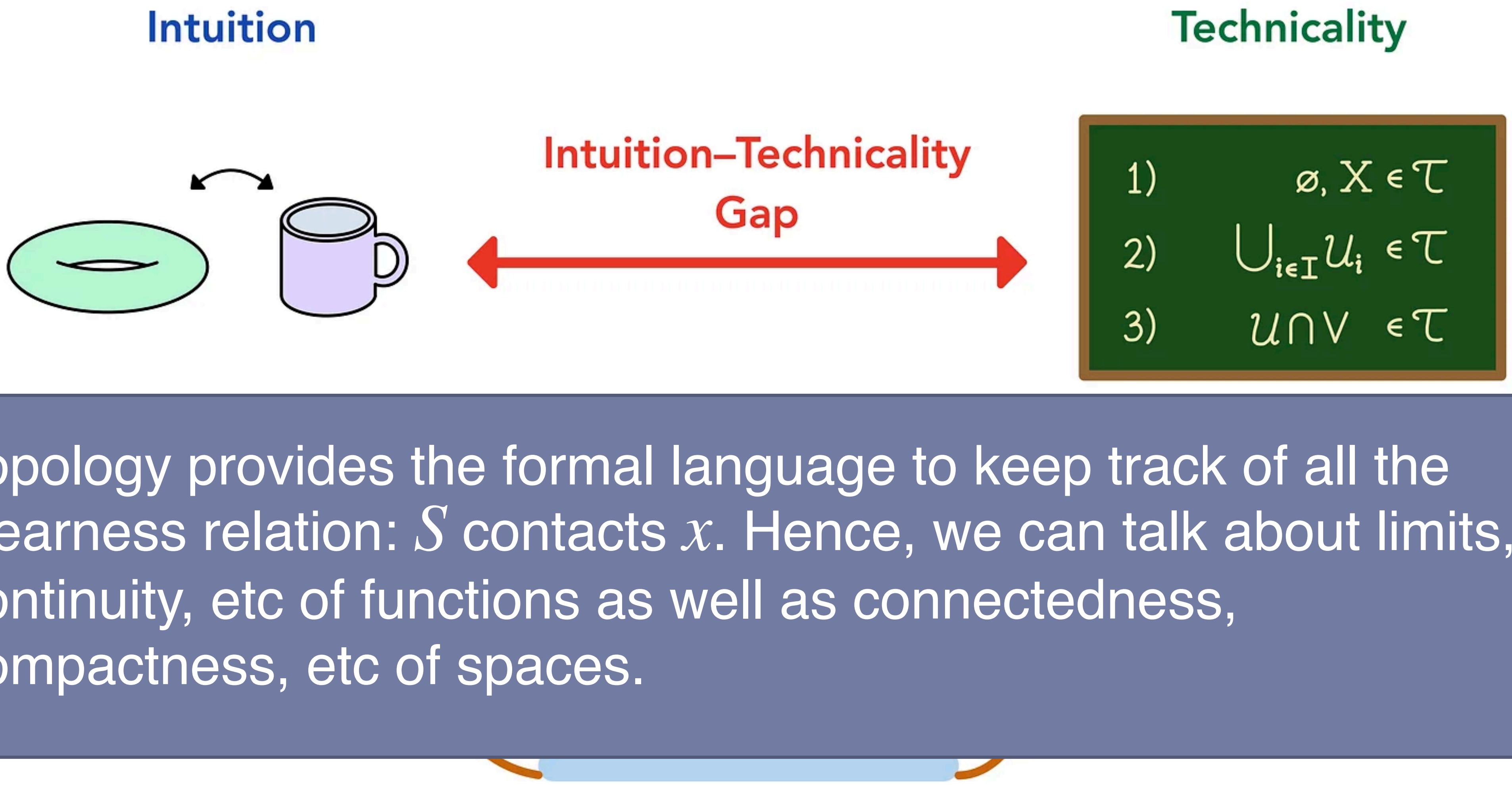

Topology provides the formal language to keep track of all the Nearness relation: S contacts x. Hence, we can talk about limits, continuity, etc of functions as well as connectedness, compactness, etc of spaces.

Open / Closed sets

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- Each set $A \in T$ is called an *open set*
- A set B is *closed* if its complement is open
  - i.e., there exists A such that $B = X \setminus A$

Closure, interior, boundary

Closure, interior, boundary

- Recall a set is closed if its complement is open
- Given a topological space $(\mathsf{X}, \mathcal{T})$ and a subset $A \subseteq \mathsf{X}$:
  - the *closure* of A, denoted by $\bar{A}$, is the smallest closed set containing A.
    - $\bar{A} = \cap_{\text{closed } C \supset A} C$
  - its *interior* A^o is the union of all open subsets of A.
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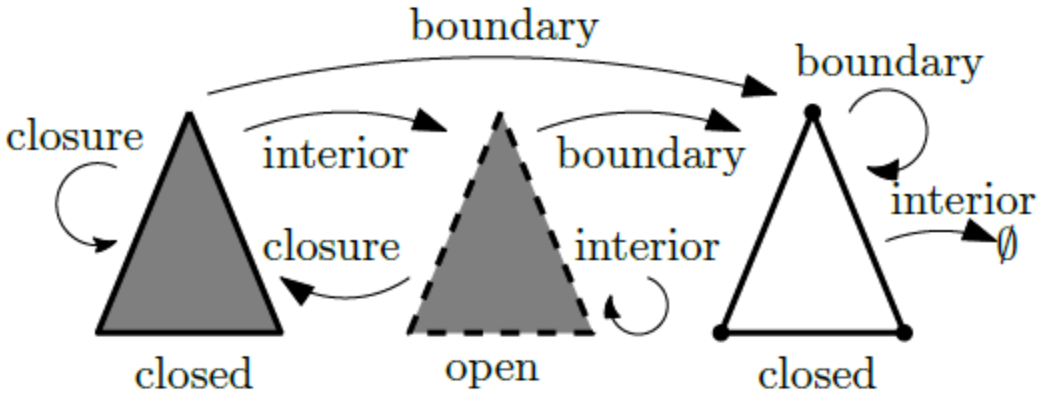

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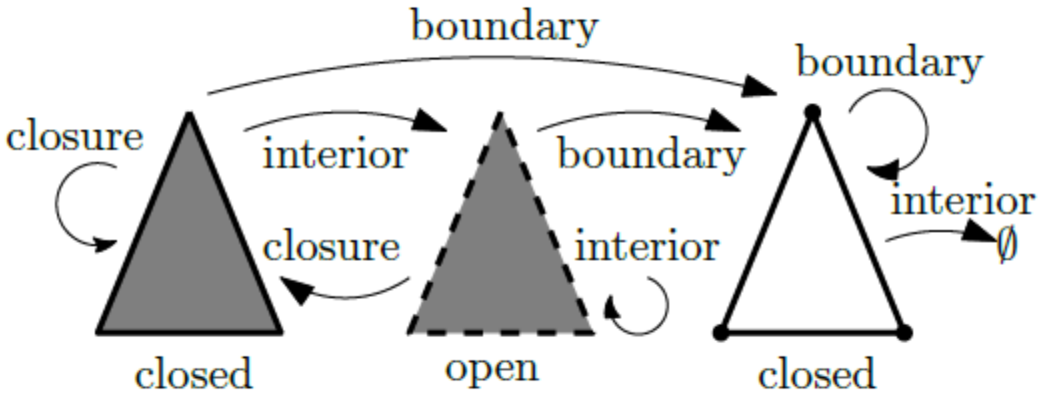
“S contacts x” can be formally defined as $x \in \bar{S}$



Examples in $\mathbb{R}$

- Let $A = [1,2)$
  - $\bar{A} = [1,2]$
  - $A^o = (1,2)$
  - $\partial A = \{1,2\}$

- For any given set X, one can define different topologies on top of that. Some of them can be bizarre, such as the trivial topology

- The most useful topology in this class is the **metric space topology**

Metric space

Definition 2 (Metric space)**.** *A* metric space *is a pair* (X, d) *where* X *is a set and* d *is a distance function* $\mathsf{d} : \mathsf{X} \times \mathsf{X} \to \mathbb{R}$ *satisfying the following properties:*

- $\mathsf{d}(p, q) = 0$ *if and only if* $p = q$
- $\mathsf{d}(p, q) = \mathsf{d}(q, p), \forall p, q \in \mathsf{X}$*;*
- $\mathsf{d}(p, q) \leq \mathsf{d}(p, r) + \mathsf{d}(r, q), \forall p, q, r \in \mathsf{X}$.

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- Examples:
  - $(\mathbb{R}^k, \|\cdot\|_2)$ k-dimensional Euclidean space, equipped with the standard Euclidean distance $d(p, q) = \|p - q\|_2$
  - "Curved" space (manifolds), equipped with geodesic distance
    - e.g, the surface of earth.
  - Space can also be discrete, as very often in data analysis
    - (P, d): a set of points with pairwise distance (or similarity) given.
    - or graphs, equipped with shortest path metric.

Metric space topology

- Open ball:
  - $B_o(c,\ r) = \{x \in X \mid d(c,x) < r\}$

Definition 3 (Metric space topology)**.** *Given a metric space* X*, all metric balls* $\{B_o(c,r) \mid c \in \mathbb{T}$ *and* $0 < r \leq \infty\}$ *and their union constituting the open sets define a topology on* X.

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- The set of metric balls is called a *basis* for this topology on X
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- In general, when we refer to a common metric space, say Euclidean space, we refer to this metric space topology induced by standard metric.

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Exercise: why?

Subspace topology

- A topological space $(\mathsf{X}, \mathcal{T})$, say the Euclidean space
- Given a subset $Y \subseteq \mathsf{X}$, the subspace topology $\left(Y, \mathcal{T}_Y\right)$, (inherited from $(\mathsf{X}, \mathcal{T})$), is such that $\mathcal{T}_Y$ consists of intersection between open sets in $\mathcal{T}$ and Y.

- Common subspaces of Euclidean space
  - Euclidean d-ball: $\mathbb{B}^d = \left\{x \in \mathbb{R}^d \mid \|x\| \leq 1\right\}$
  - Open Euclidean d-ball: $\mathbb{B}_o^d = \left\{x \in \mathbb{R}^d \mid \|x\| < 1\right\}$
  - Euclidean d-sphere: $\mathbb{S}^d = \left\{x \in \mathbb{R}^{d+1} \mid \|x\| = 1\right\}$
  - Euclidean half-space: $\mathbb{H}^d = \left\{x \in \mathbb{R}^d \mid x_d \geq 0\right\}$

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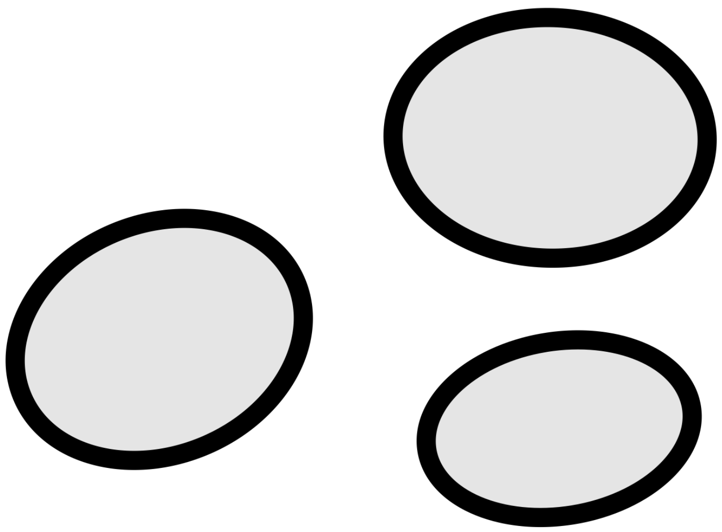

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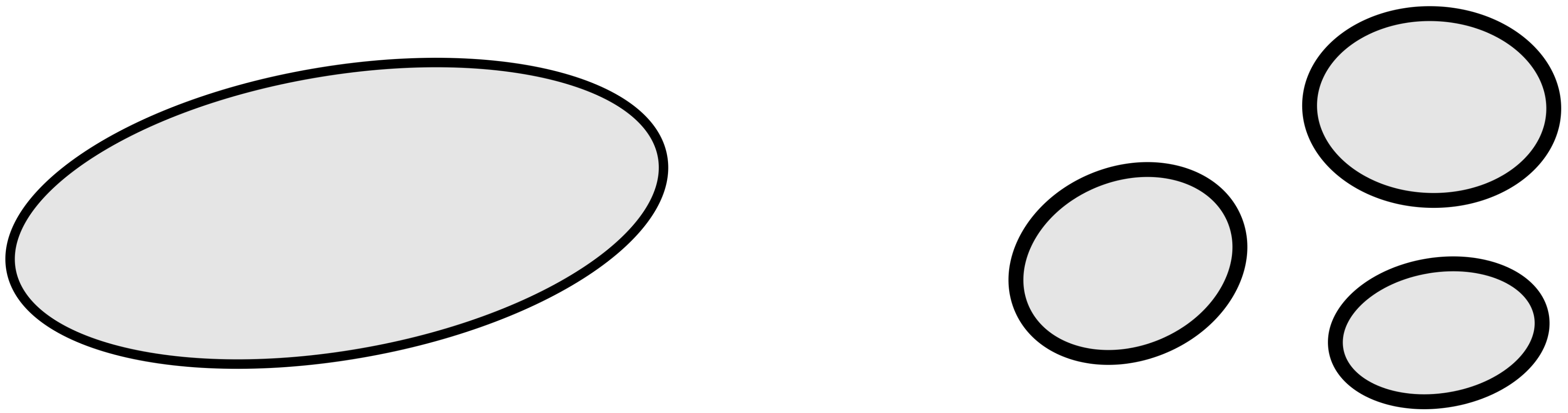

Compactness

- This generalizes the notion of **closed** and **bounded** sets in Euclidean space
- Open cover: $\mathscr{U} = \{U_\alpha\}_{\alpha\in A}$ is an open cover for $(X, \mathscr{T})$ if $U_\alpha \in \mathscr{T}$ and $X = \cup_{\alpha\in A} U_\alpha$
- $(X, \mathscr{T})$ is called **compact** if for any open cover $\mathscr{U} = \{U_\alpha\}_{\alpha\in A}$ there exists a finite subcover, i.e., a finite set $A' \subseteq A$ such that $X = \cup_{\alpha\in A'} U_\alpha$
- Example: $(0,1)$ is not compact but $[0,1]$ is compact

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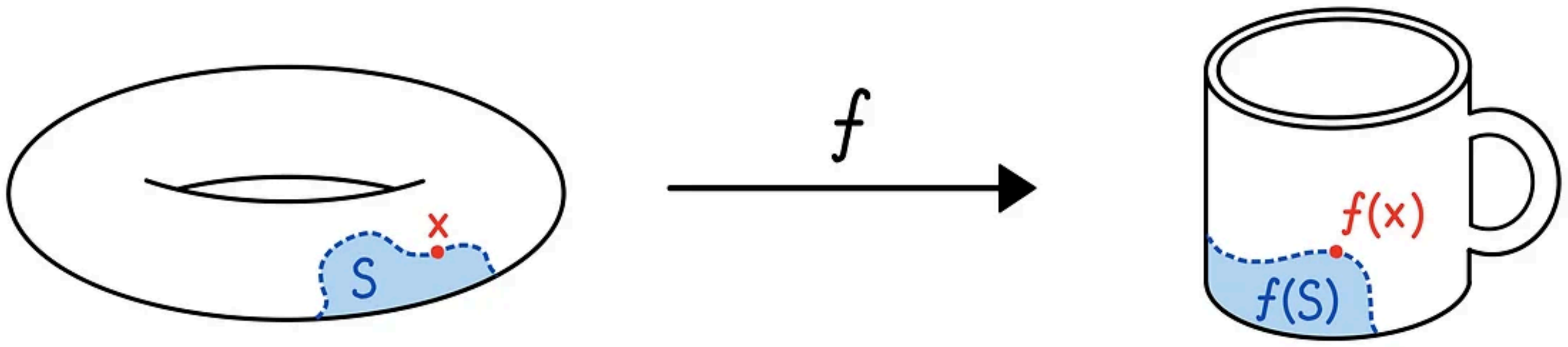
How we mathematically talk about space of interest

Now we need ways to connect different spaces!

Recall

- We want to rigorously define "continuous transformation"
  - A continuous map shouldn't tear things apart
  - If S "contacts" x, under a continuous transformation, we want that $f(S)$ "contacts" $f(x)$



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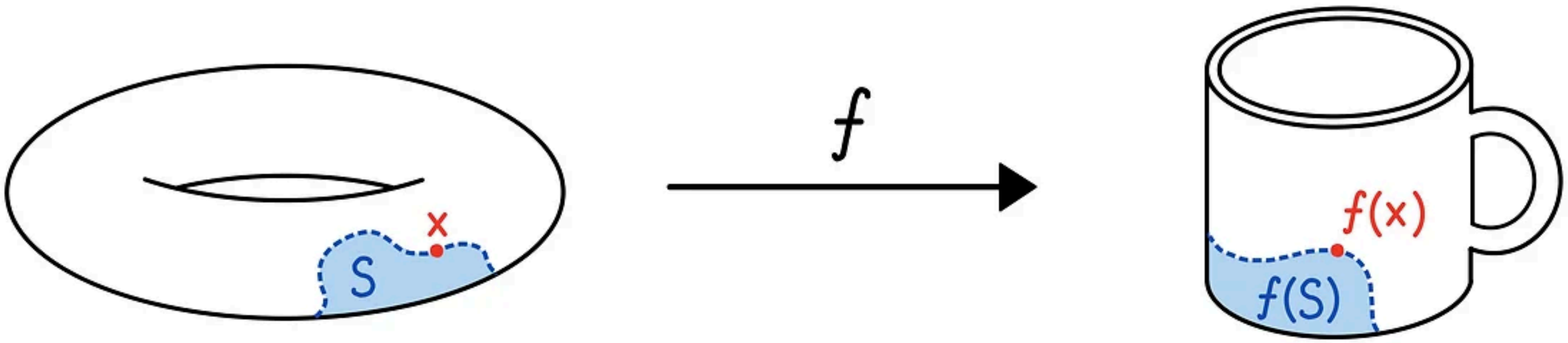
Continuous function

- A function $f : X \to Y$ between two topological spaces is called **continuous** if for any subset $S \subset X$ we have that
  - $f(\bar{S}) \subset \overline{f(S)}$
- A formal way describing "If S contacts x, then $f(S)$ contacts $f(x)$"



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Continuous function

- A function $f : X \to Y$ between two topological spaces is called **continuous** if for any subset $S \subset X$ we have that
  - $f(\bar{S}) \subset \overline{f(S)}$
- A formal way describing "If S contacts x, then $f(S)$ contacts $f(x)$"

1. Closure of image contains image of closure
2. Continuous map does not tear things apart

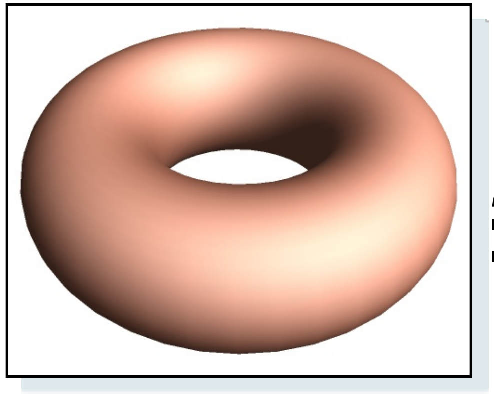
From https://wgyory.wixsite.com/toolatetopologize/post/post-1

Exercise

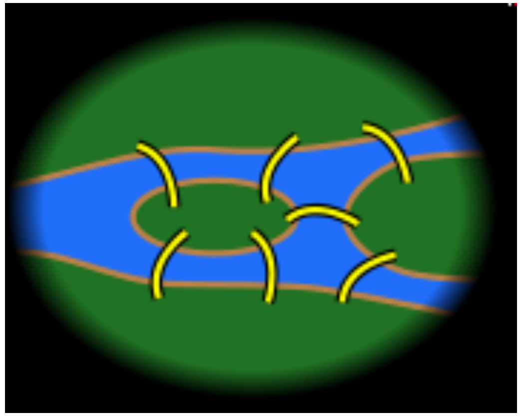
- Please try to prove the notion of continuity in calculus is compatible with the new definition of continuity
  - $\lim_{x \to x_0} f(x) = f(x_0)$ for all x_0

Check-in: Where are we?

- Fundamental concepts
  - Topological space → How we mathematically talk about space of interest
  - Continuous maps → Now we need ways to connect different spaces!
  - Homeomorphisms and homotopies → Describe relations of spaces
  - Manifolds



$\cong$



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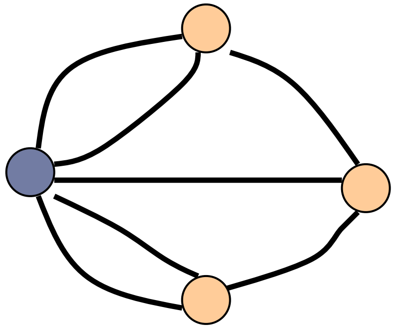

Homeomorphism = homoios + morphē = Similar shapes

Definition 5 (Homeomorphism) *Given two topological spaces* X *and* Y*, a* homeomorphism *between them is a map* $h : \mathsf{X} \rightarrow \mathsf{Y}$ *such that* h *is bijection and the inverse of* h *is also continous.*

Two topological spaces are X *and* Y *are* homeomorphic*, denoted by* $\mathsf{X} \cong \mathsf{Y}$*, if there is a homeomorphism between them.*

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- Homeomorphic spaces are called *topologically equivalent*
  - Note that equivalent relations are transitive.
  - $X \cong Y$ and $Y \cong Z$ implies $X \cong Z$

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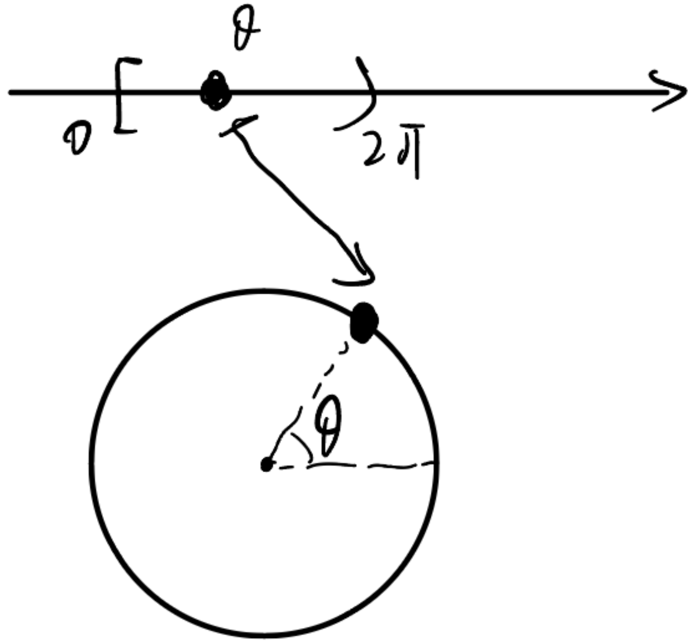
- Homeomorphism preserves all topological quantities: number of connected components, number of holes, voids, etc.

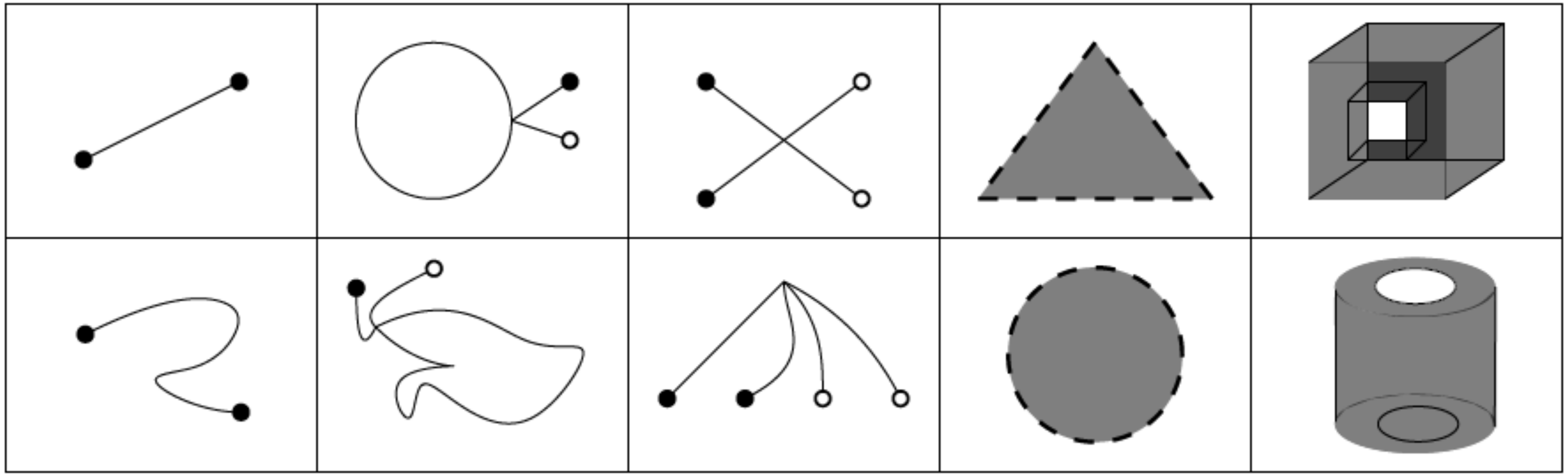
Continuous bijective map may have discontinuous inverse

- $[0,2\pi) \to \mathbb{S}^1$ by $\theta \mapsto e^{i\theta}$



Examples

Examples



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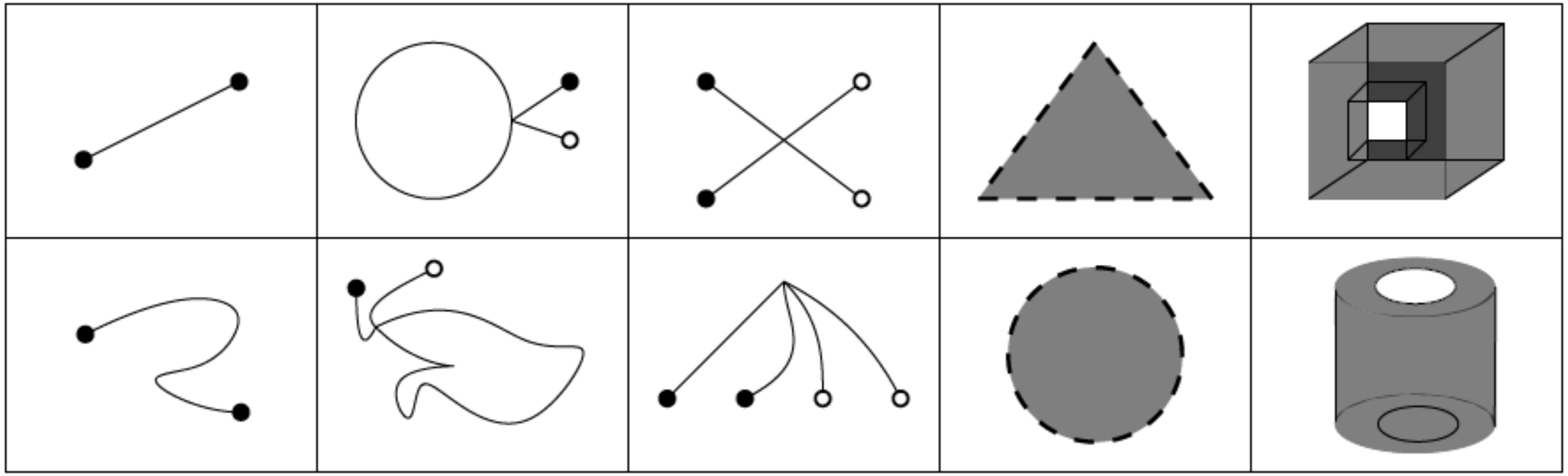

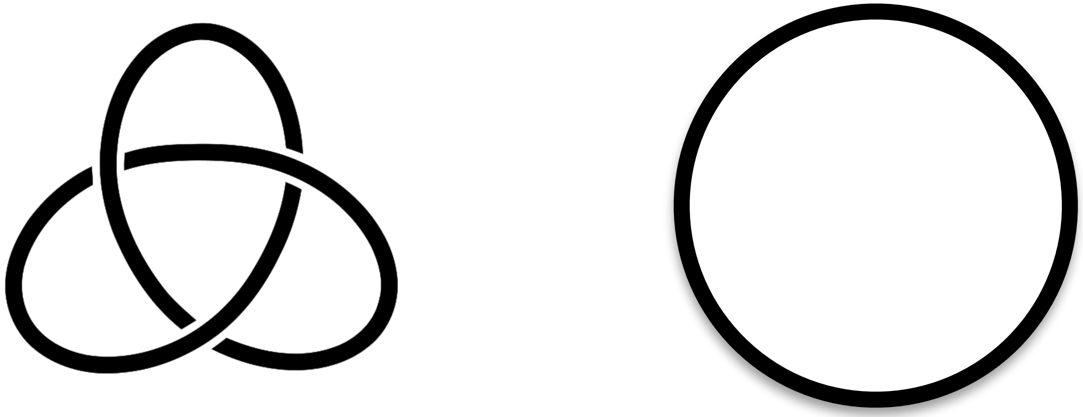

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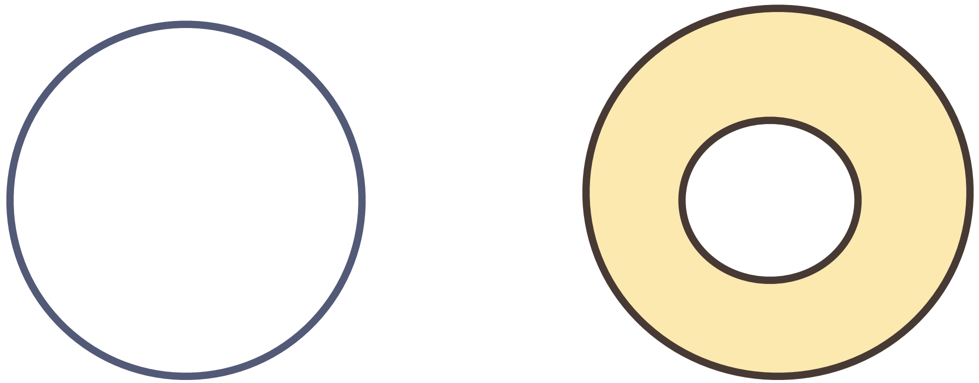
Two topological spaces are X *and* Y *are* homeomorphic, *denoted by* $\mathsf{X} \cong \mathsf{Y}$, *if there is a homeomorphism between them.*

- A trick: remove one point and check connected components
  - Y and I are not homeomorphic; X and Y are not homeomorphic
  - $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic
  - What about $\mathbb{R}^2$ and $\mathbb{R}^3$?

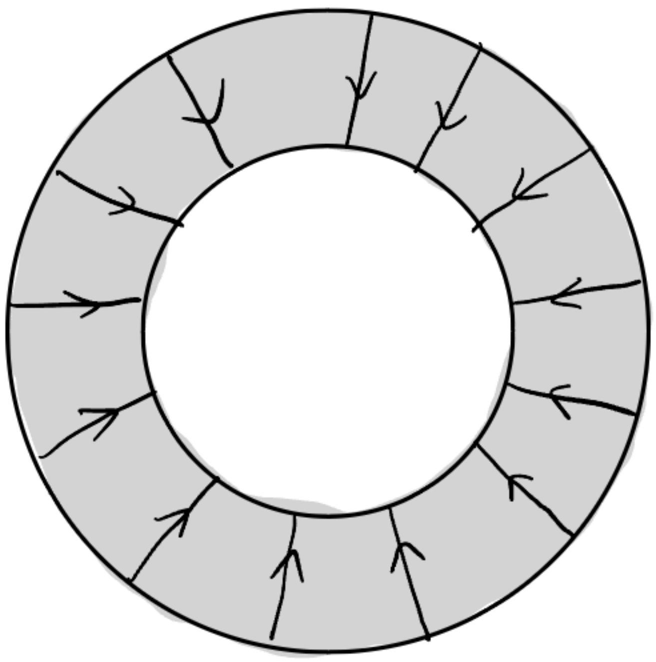
- In general, hard to decide whether two spaces are homeomorphic or not!

Another level of similarity



- They are not homeomorphic (why?)
- But they look very similar

A closer look



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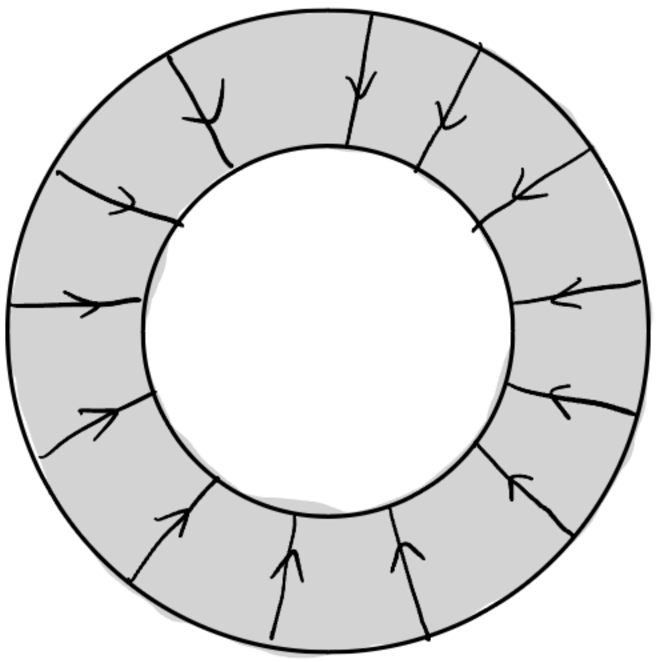
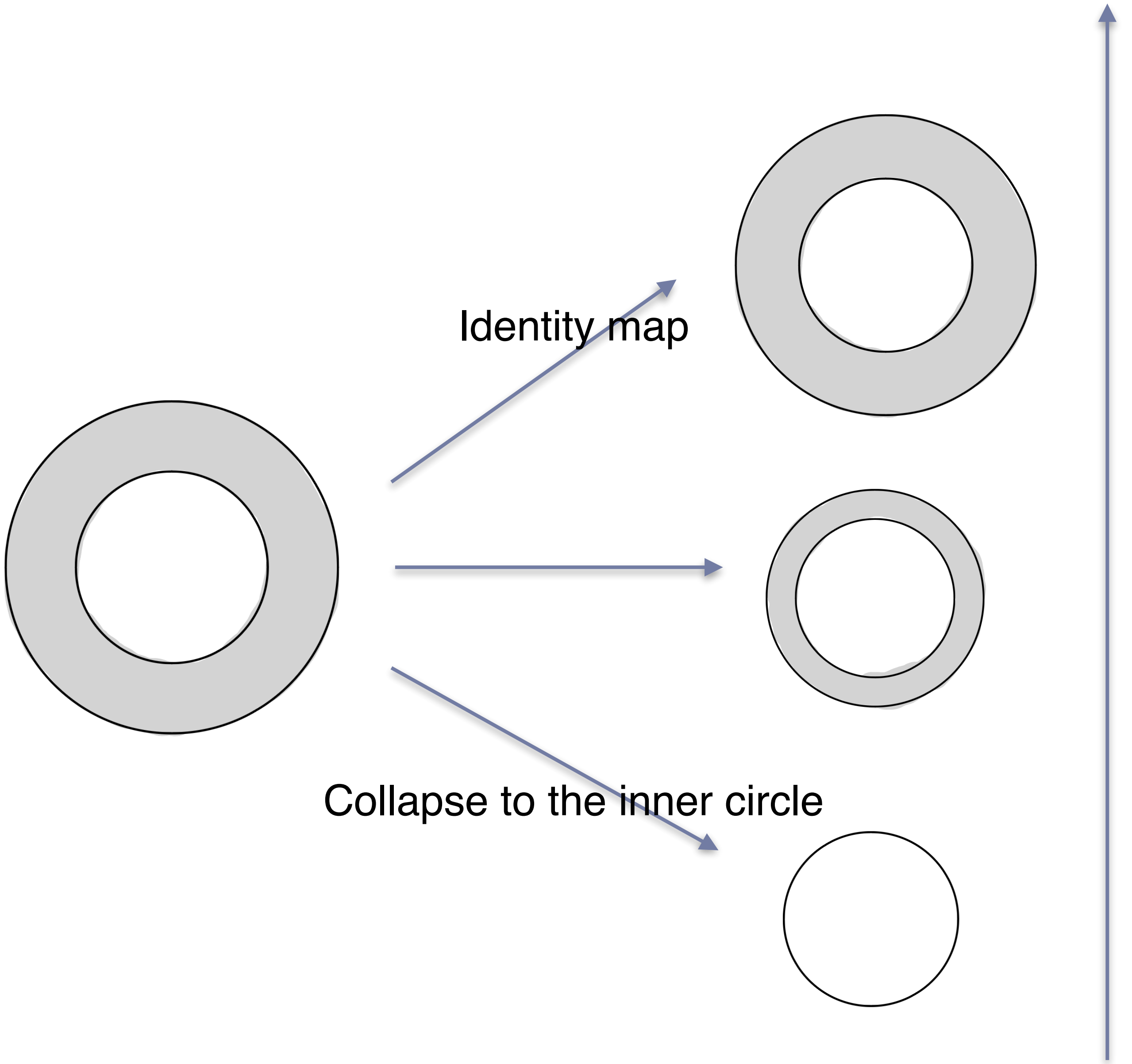
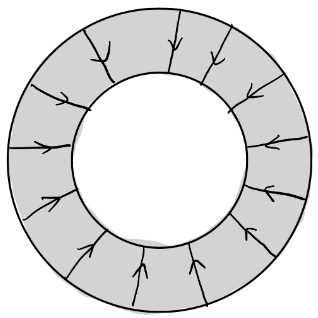
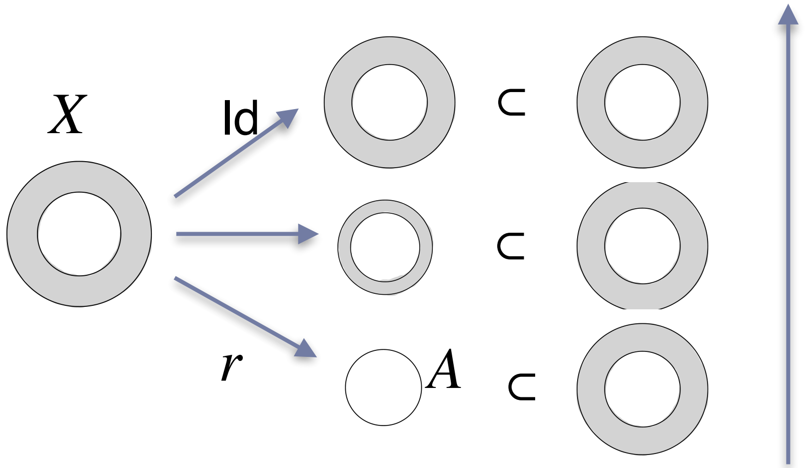




Homotopy and Deformation Retraction

Definition 6 (Homotopy) *First, two $f, g : \mathsf{X} \to \mathsf{Y}$ are* homotopic *if there is a continuous map $H : \mathsf{X} \times [0,1] \to \mathsf{Y}$ such that $H_0 = f$ and $H_1 = g$. This map H is called a* homotopy *connecting f and g.* If H exists, we say f and g are homotopic, denoted by $f \simeq g$





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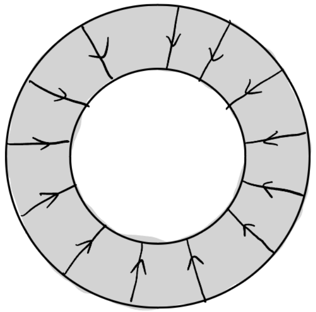

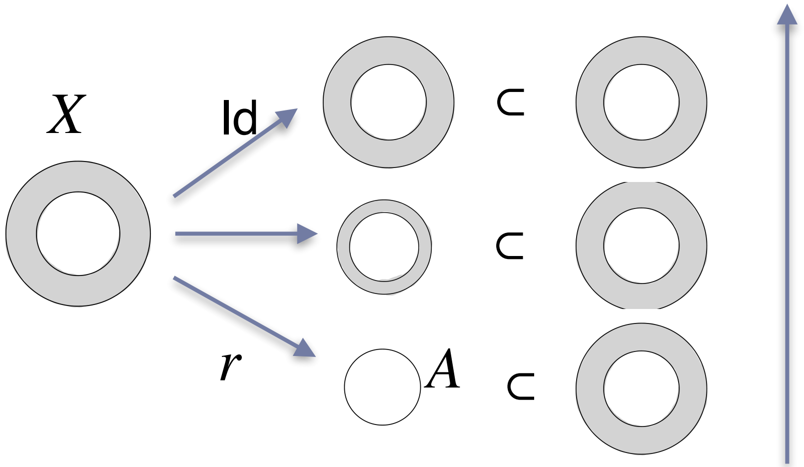

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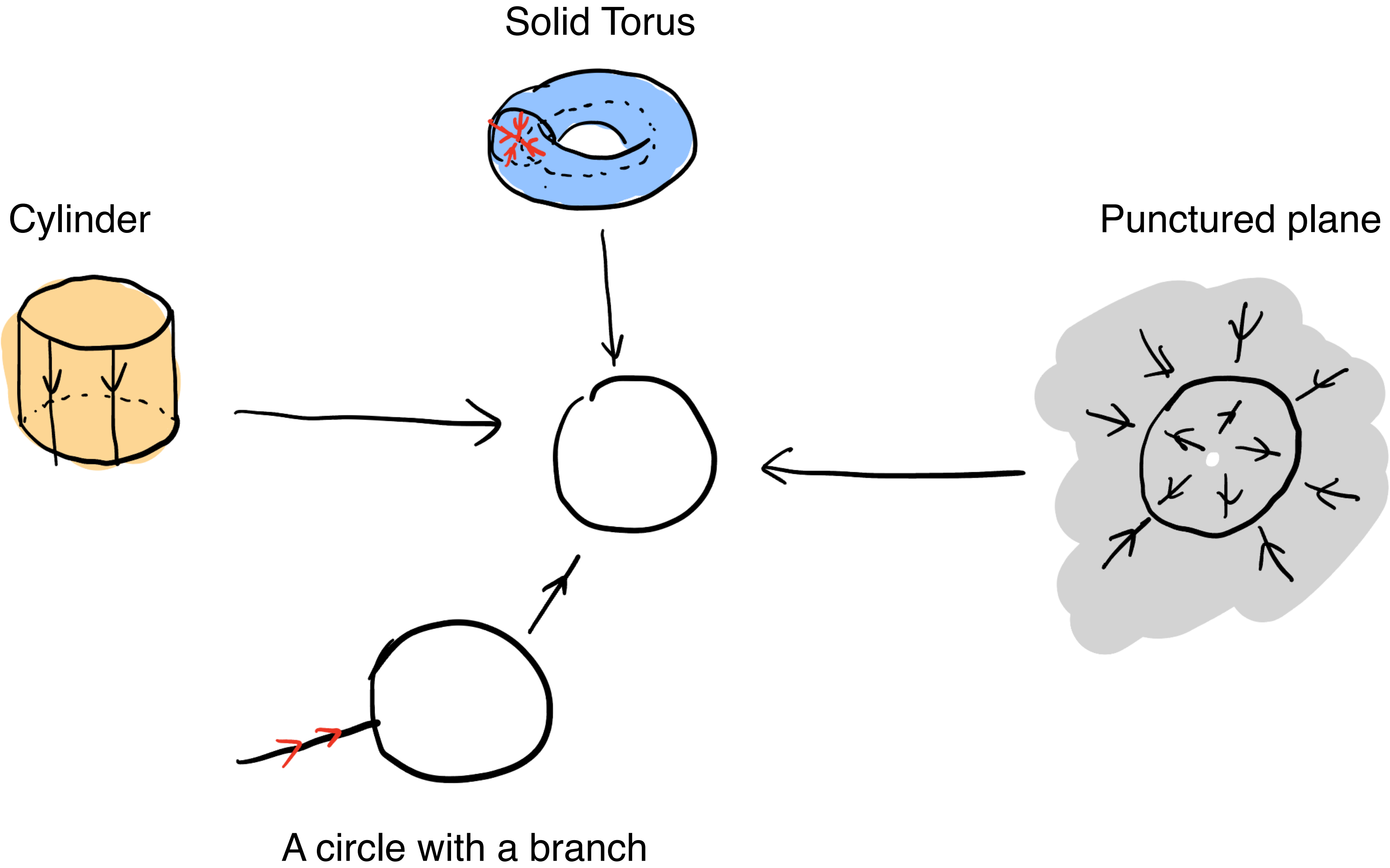
We say that $A \subseteq X$ is a deformation retract of *X if there is a retraction r that is homotopic to the identity map in X. This retraction map is called a* deformation retraction.

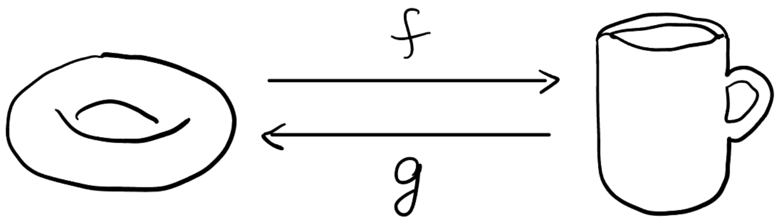
Equivalently, a continuous map $R : X \times [0,1] \to X$ is a deformation retraction of *X* onto *A if for every $x \in X$ and $a \in A$, $R(x,0) = x$; $R(x,1) \in A$ and $R(a,1) = a$.*

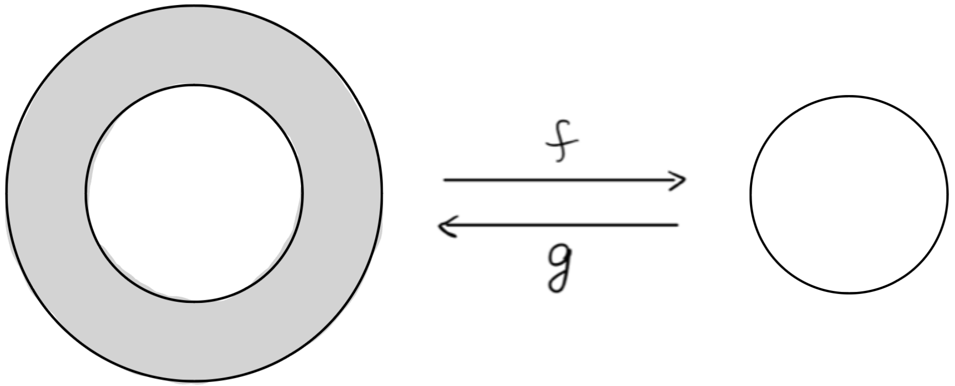
Examples of Deformation Retraction



A weaker notion of similarity: Homotopy equivalent

- When X, Y are homeomorphic, there exists continuous functions $f : X \to Y, g : Y \to X$ such that $f \circ g = Id_Y$ and $g \circ f = Id_X$

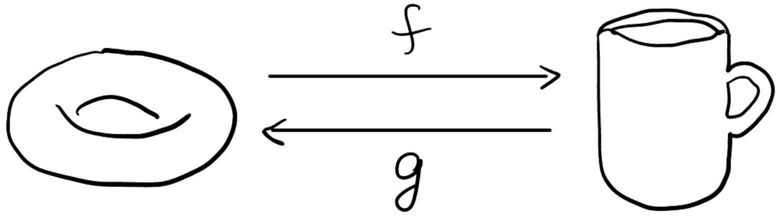


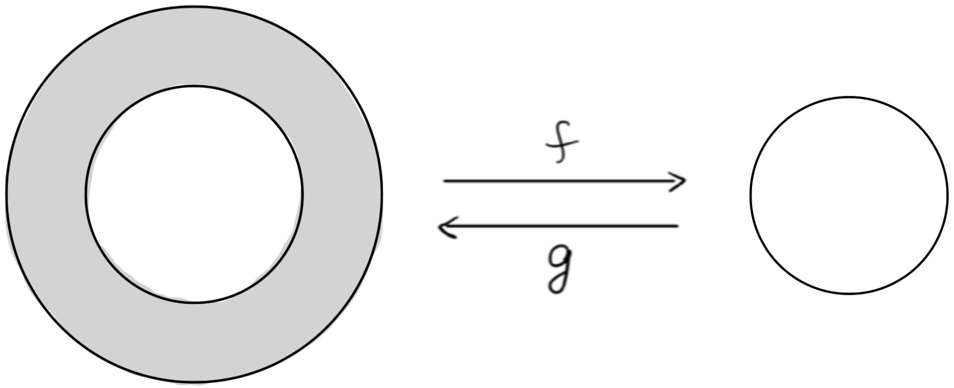


- We say X, Y are **homotopy equivalent**, if there exists continuous functions $f : X \to Y, g : Y \to X$ such that $f \circ g \simeq Id_Y$ and $g \circ f \simeq Id_X$

A weaker notion of similarity: Homotopy equivalent

- Homeomorphism allows stretching and shrinking





- Homotopy allows stretching, shrinking and **crushing/collapsing**

A weaker notion: Homotopy equivalent

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- Theorem:
  - Two homeomorphic spaces X and Y are also homotopy equivalent. But the inverse may not hold.

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- Homotopy equivalent relation is transitive.

- In general, hard to establish homotopy equivalent relation as well

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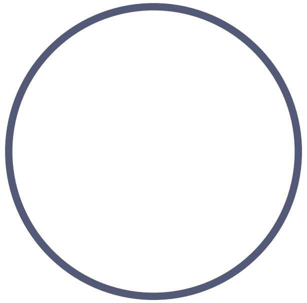
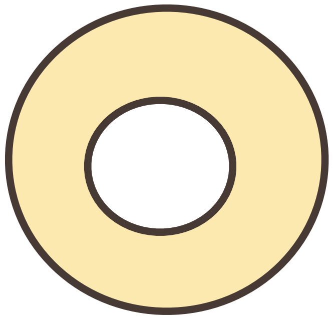
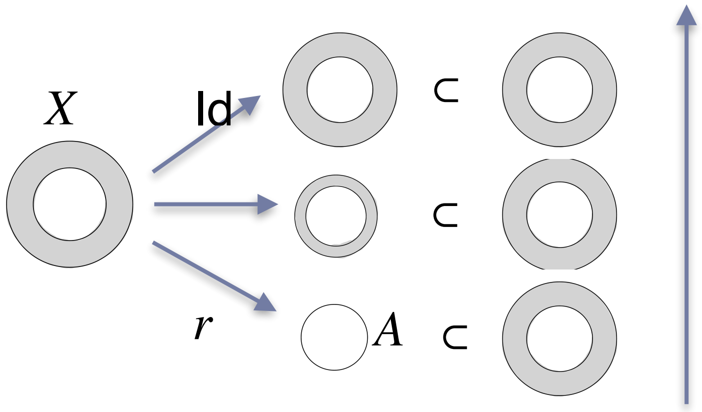
- Theorem:
  - If Y is a deformation retract of X, then X and Y are homotopy equivalent.

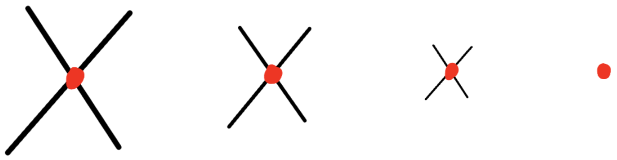




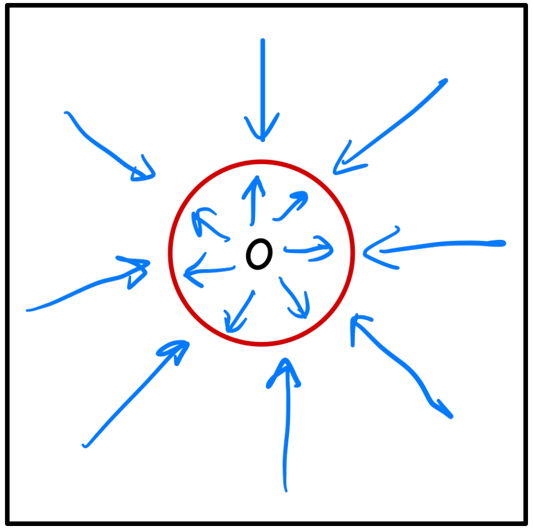


Examples of homotopy equivalence

- X and Y are homotopy equivalent but not homeomorphic



- A disk and a point
- A tree and a point
- A punctured plane and a circle



$$R : [0,1] \times \mathbb{R}^2 \backslash 0 \to \mathbb{R}^2 \backslash 0$$

$$R(t, x) = (1 - t)x + t\frac{x}{\|x\|}$$

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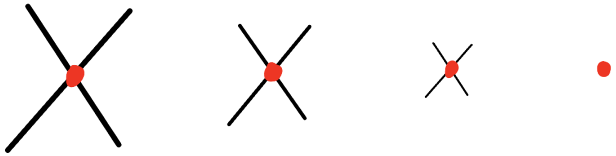

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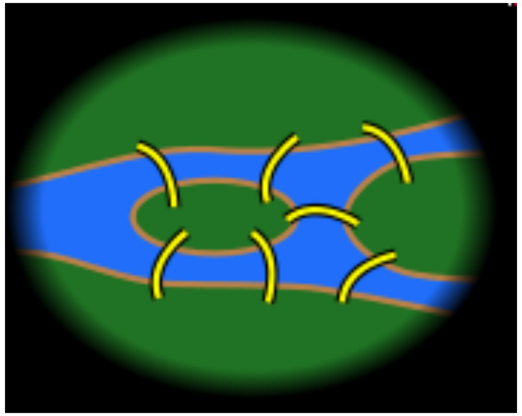

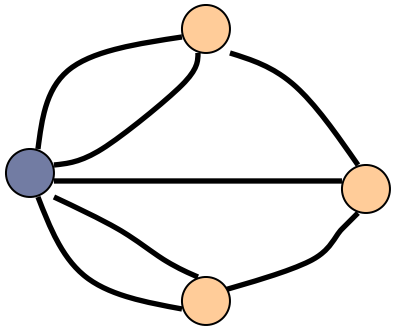

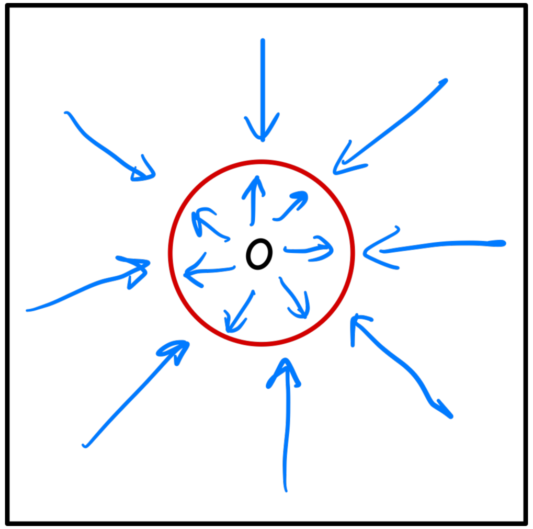

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- A punctured hollow torus ≈?

Examples

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Examples

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- Can we use computer to determine whether two topological spaces are homotopy equivalent or not?

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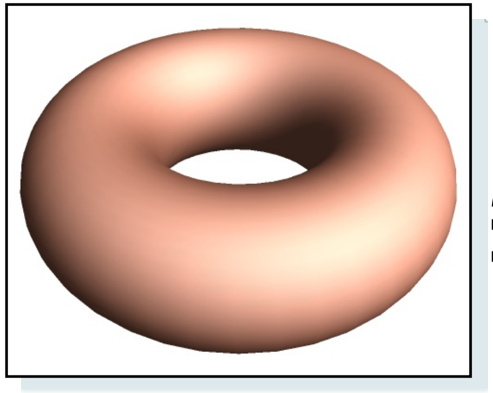
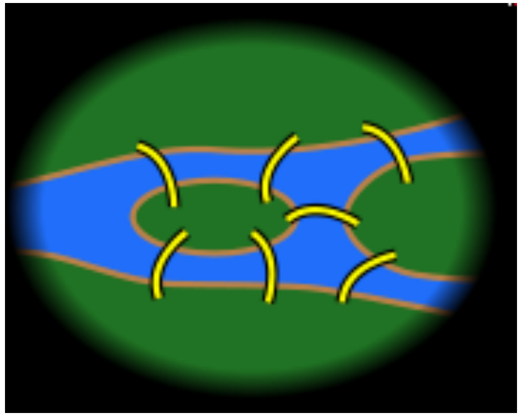
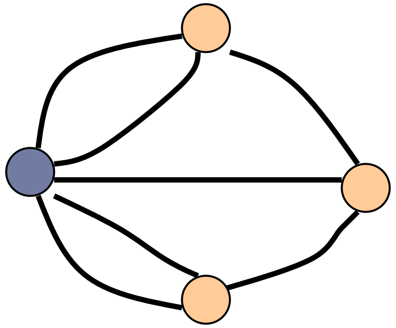
- Can we use computer to determine whether two topological spaces are homotopy equivalent or not?

Both homeomorphism and homotopy equivalence can be hard to detect, and not computationally friendly in general. Later we will focus on homology, which is much easier to compute.

Summary

 $\cong$ 

 $\simeq$ 

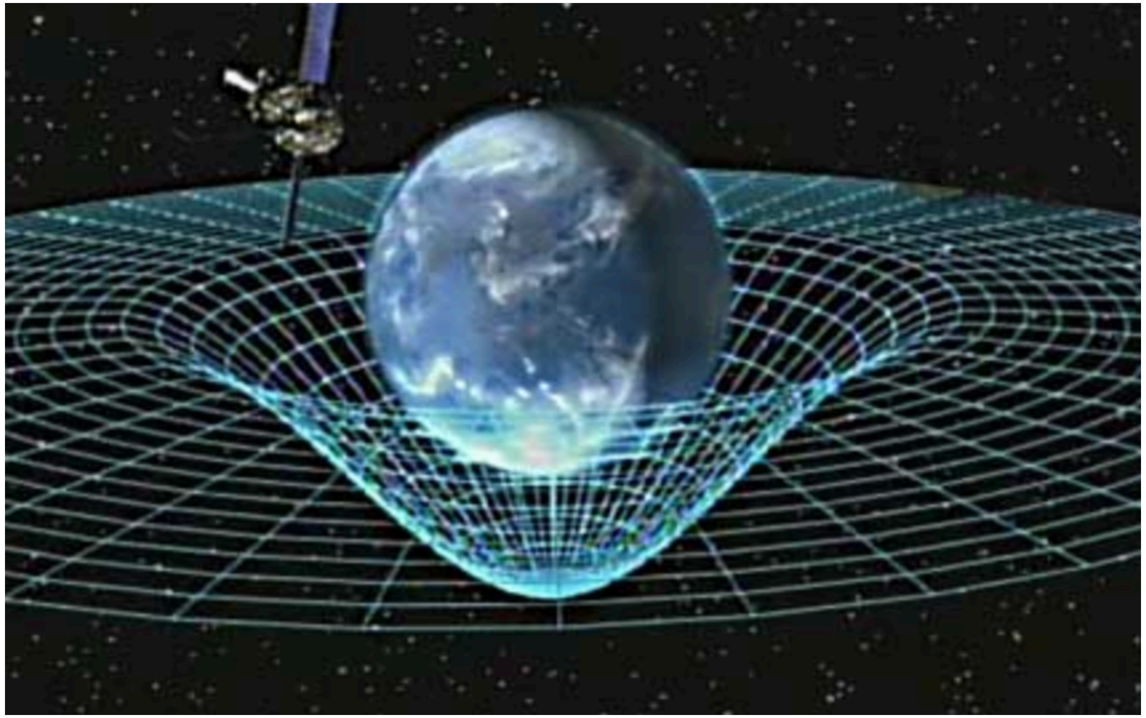
- Fundamental Questions
  - What is a topological space?
  - What is a "continuous" way of turning one space to another?
  - When can we say two spaces are the "same"?

Check-in: Where are we?

- Fundamental concepts
  - Topological space → How we mathematically talk about space of interest
  - Continuous maps → Now we need ways to connect different spaces!
  - Homeomorphisms and homotopies → Describe relations of spaces
  - Manifolds → A generalization of Euclidean space; a special type of topological space

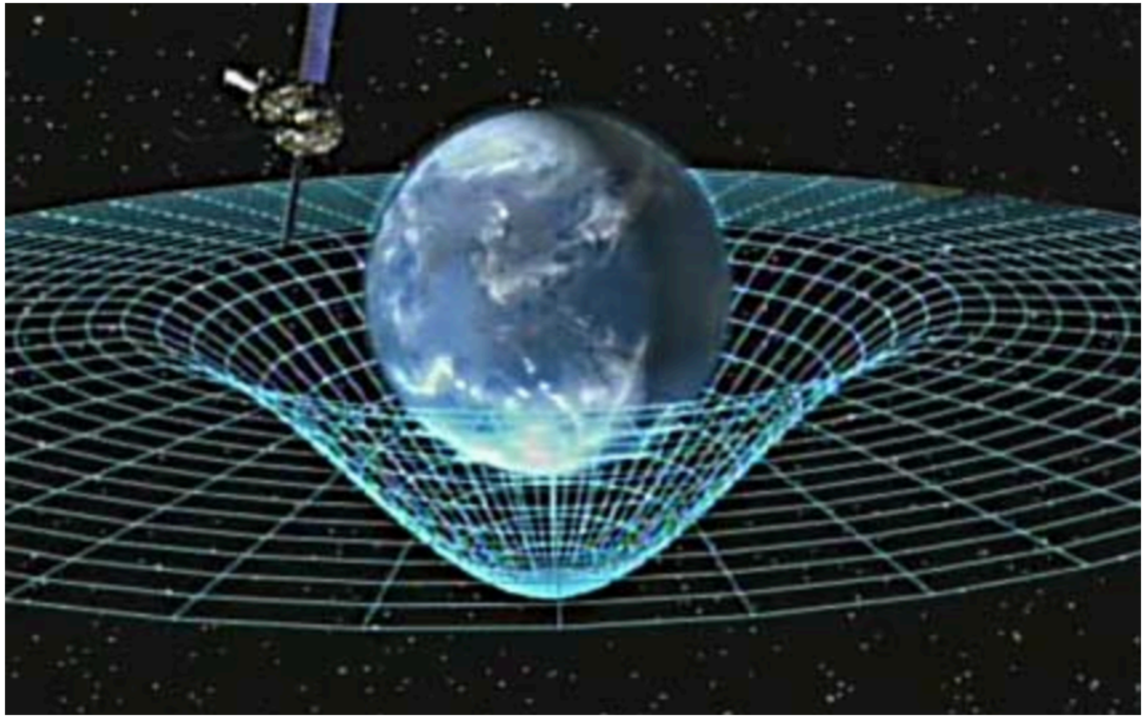
Why manifolds?

- We like Euclidean spaces. Why manifolds?



Why manifolds?

- We like Euclidean spaces. Why manifolds?
  - We live in non-Euclidean space



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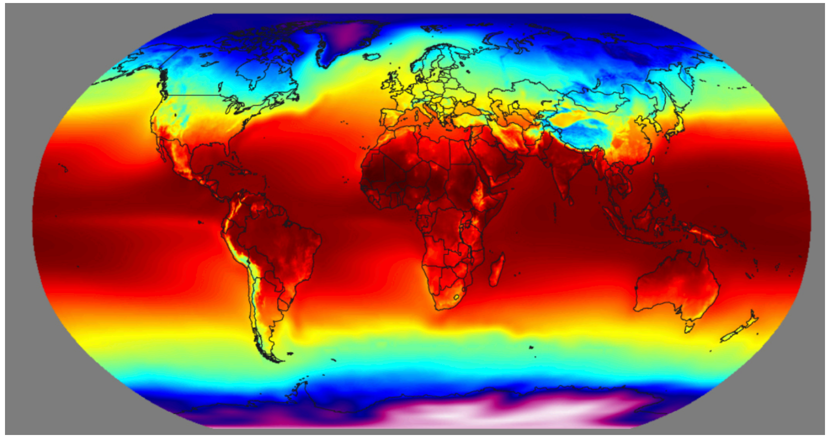

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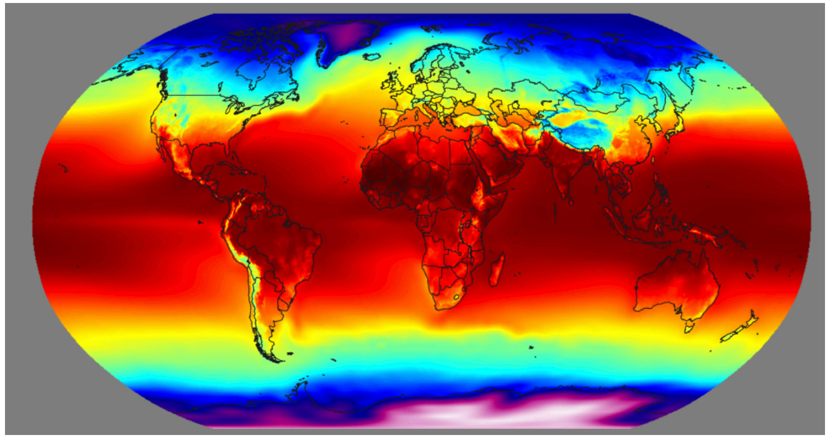

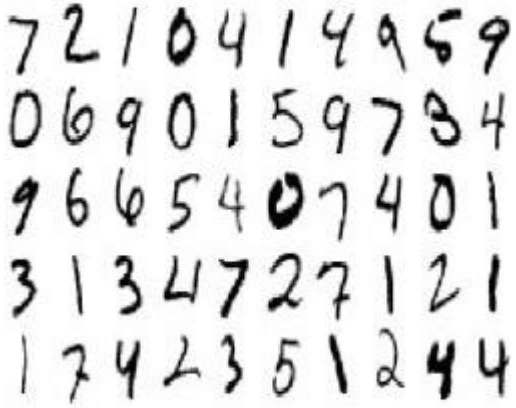

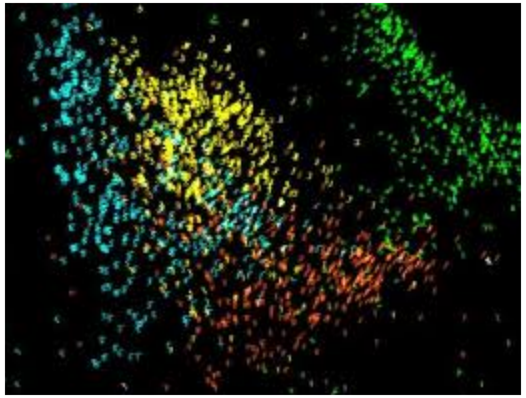

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  - The intrinsic space may not be a linear subspace of the ambient space
    - e.g, the surface of a bunny in $\mathbb{R}^3$

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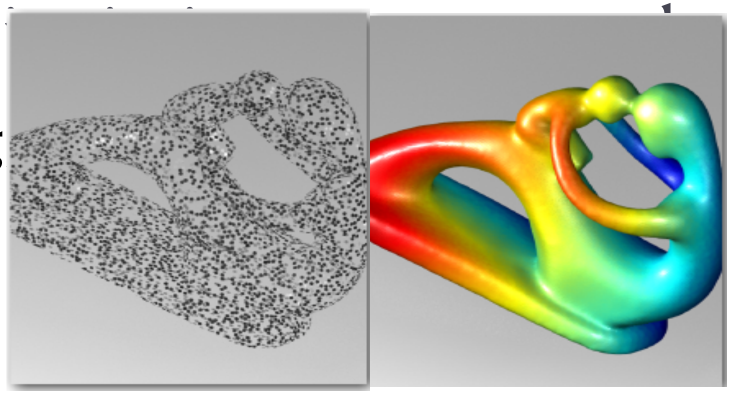
  - The  a linear subspace of the ambient space
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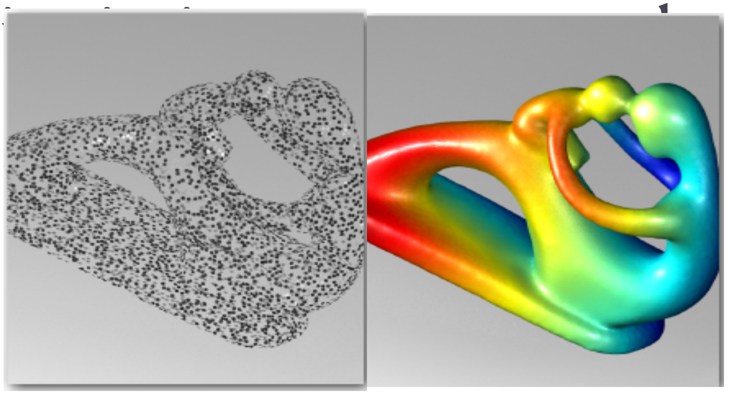
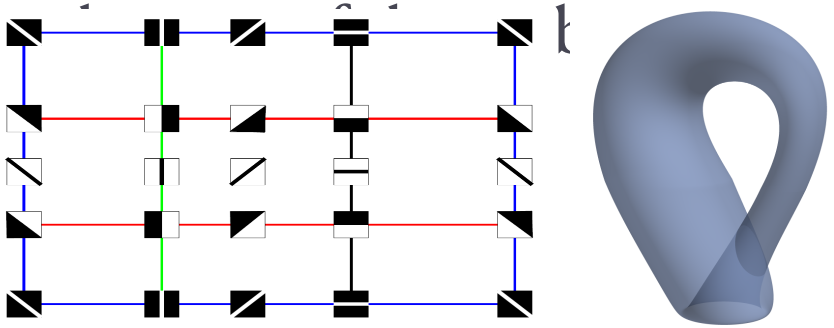
    - e.g





Courtesy of Carlsson et al, *On the local behavior of spaces of natural images*

What are manifolds

- Non-linear, yet still well-behaved spaces!
- In particular, locally it behaves like a linear space.

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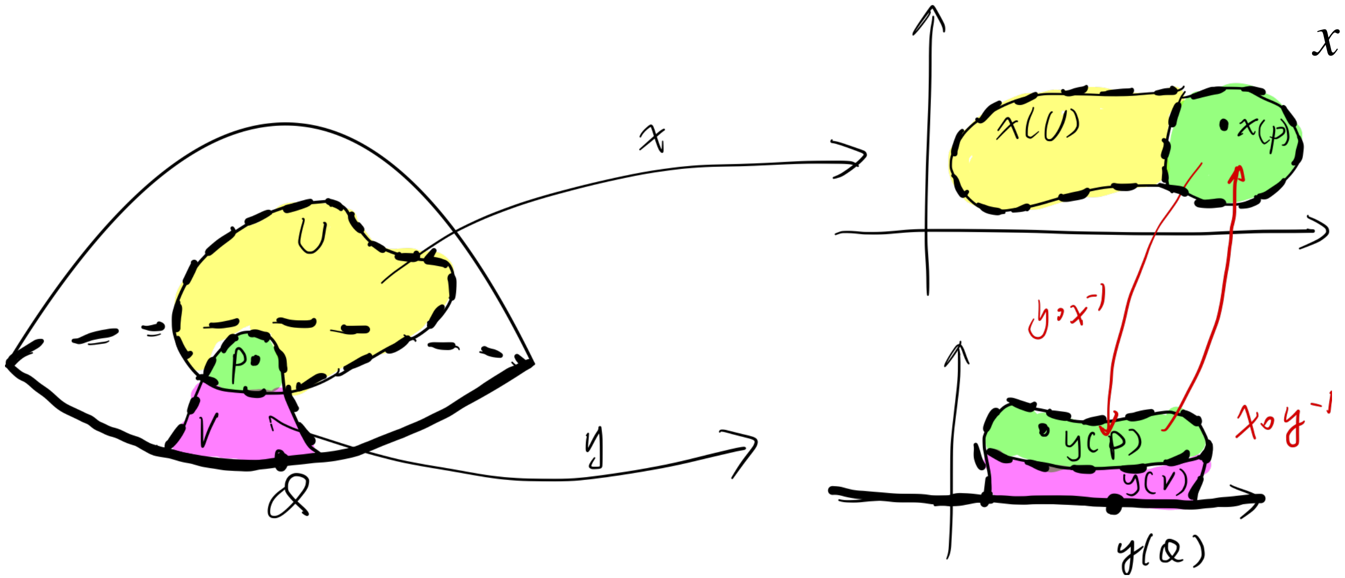

$x : U \to x(U) \subseteq \mathbb{R}^2$ is called a **coordinate chart**

$x \circ y^{-1}$ is a homeomorphism. If $x \circ y^{-1}$ is smooth, then M is called a **differentiable manifold**

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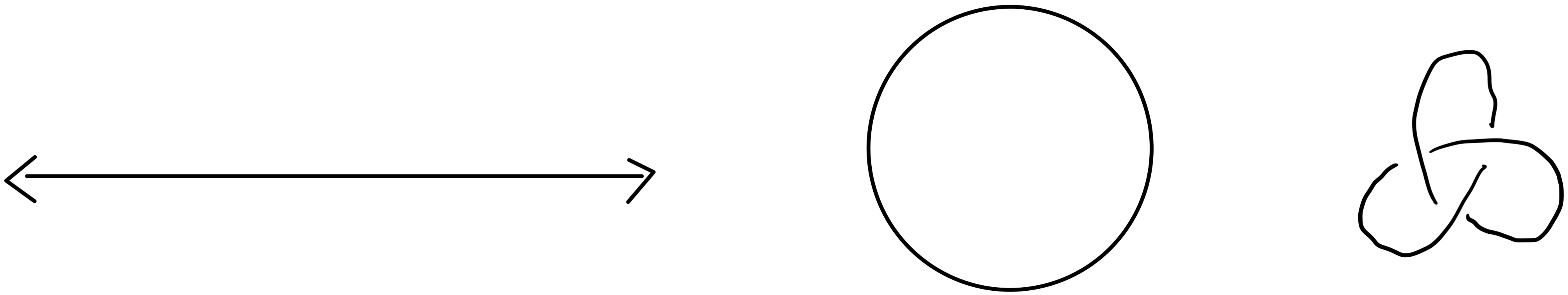
- Interior of M:
  - those points with a neighborhood homeomorphic to $\mathbb{B}_o^d$
- Otherwise, boundary of M

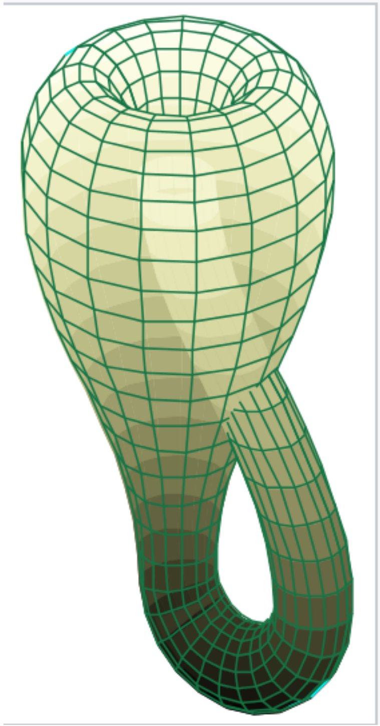
Examples

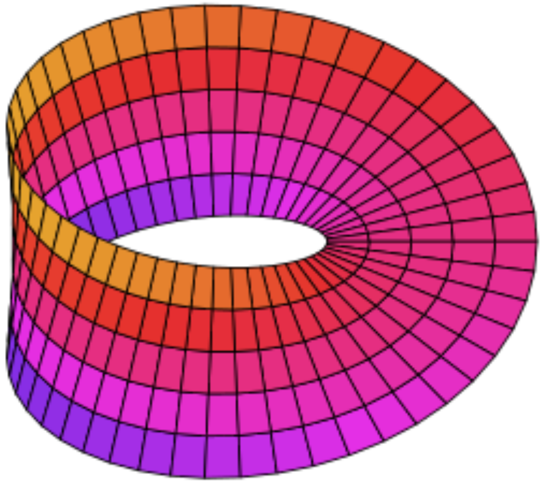
- 1-manifolds



Examples

- 1-manifolds
- 2-manifolds

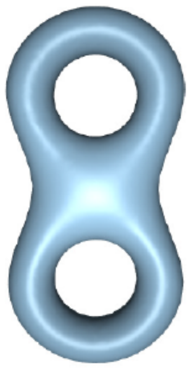
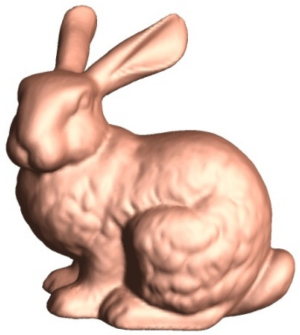


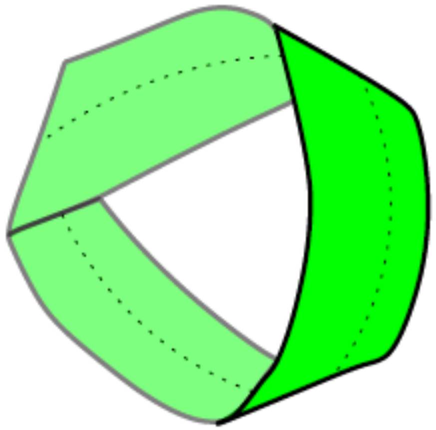


Are these manifolds?

Surfaces

- A surface is a 2-manifold
  - Locally, it looks like either an open disk, or a half-disk



- A 2-manifold M is non-orientable if
  - Starting from some point $p \in M$, one can walk on one side of M and end up on the opposite side of M upon returning to p
- Otherwise, it is orientable.

Surfaces

- While in general it is hard to decide whether two spaces are homeomorphic or not, it turns out that we can characterize compact surfaces (without boundary) completely

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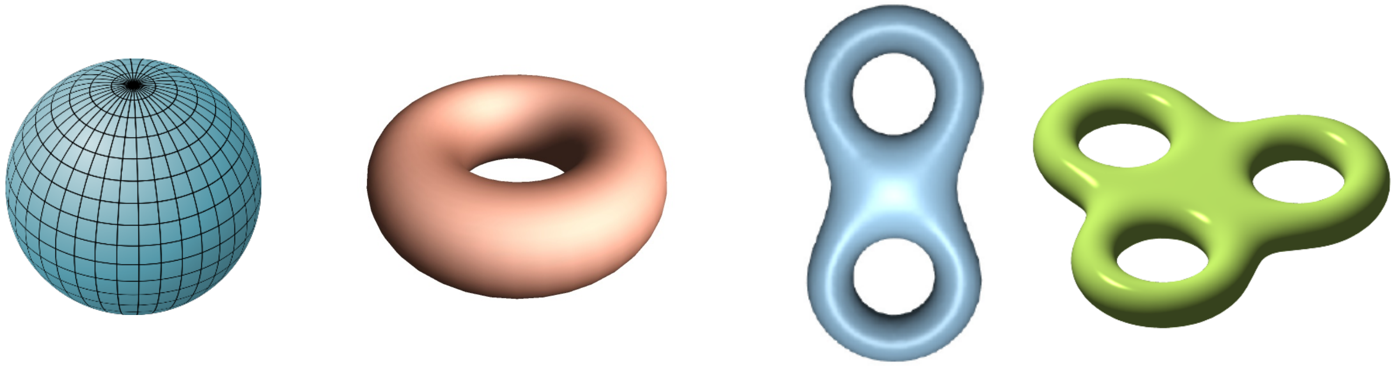

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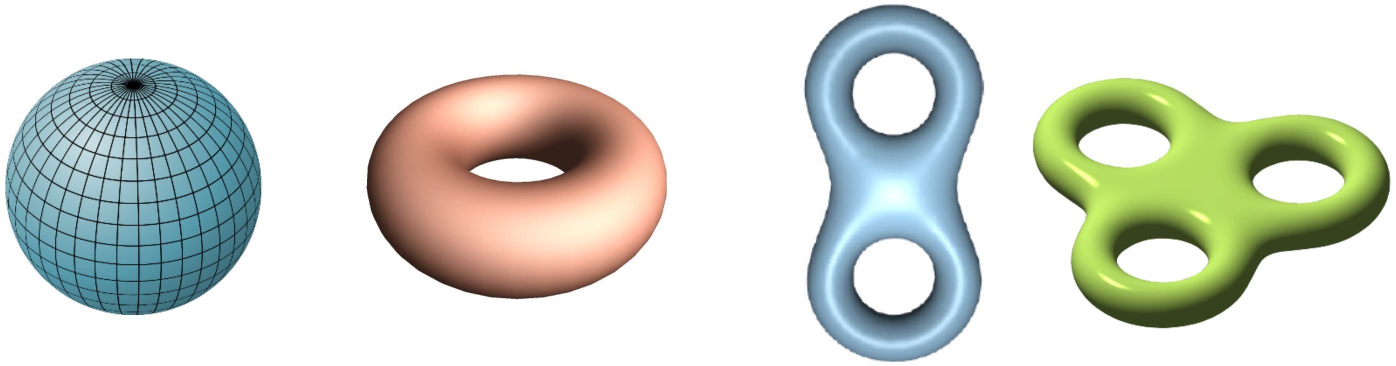

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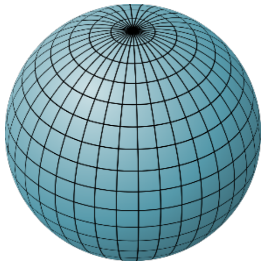
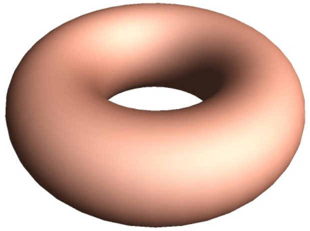
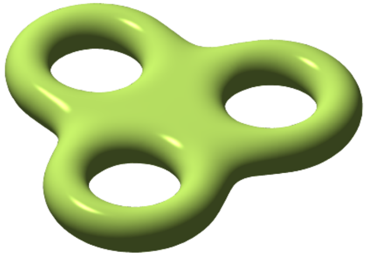
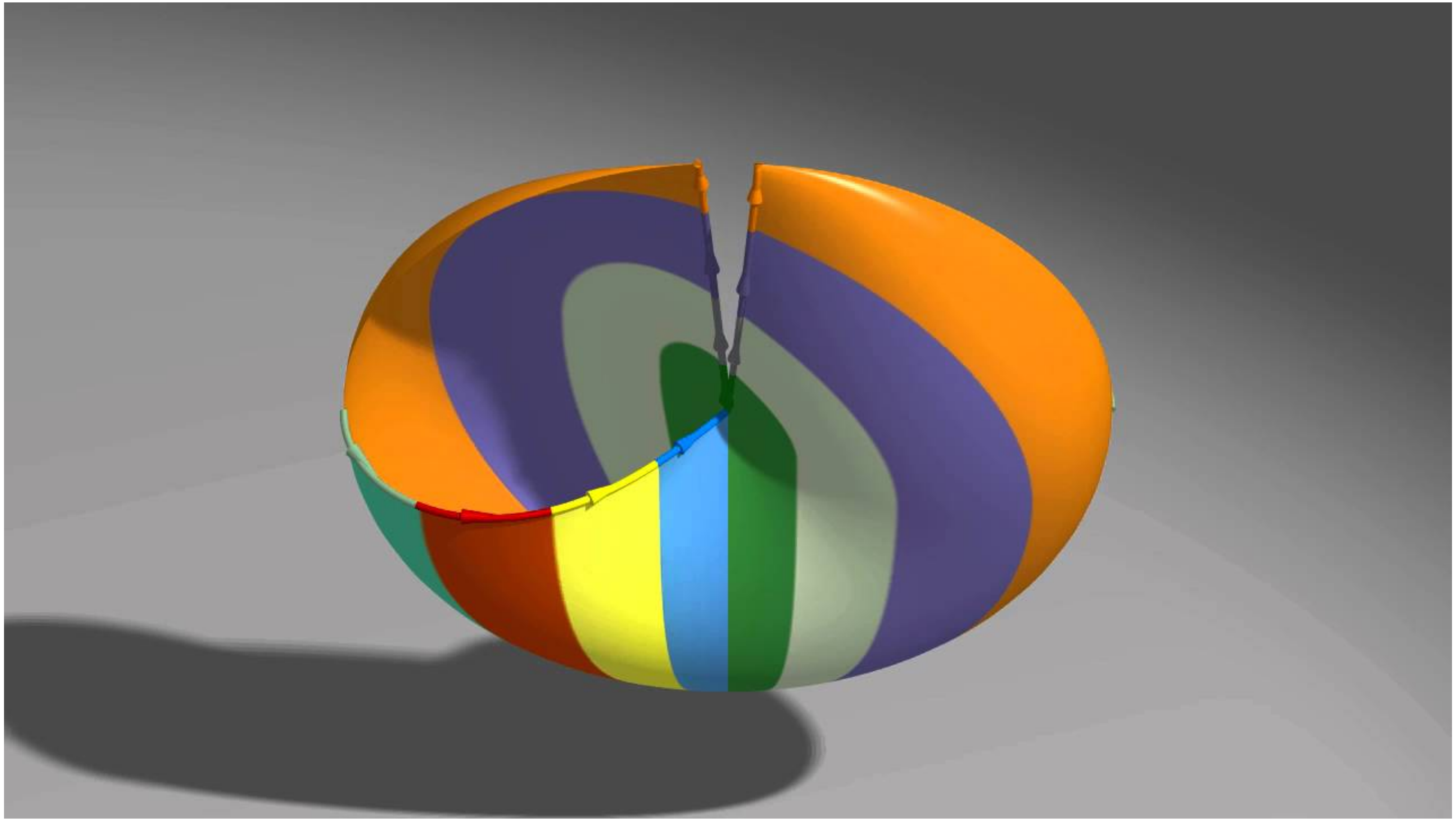






Visualization of projective plane



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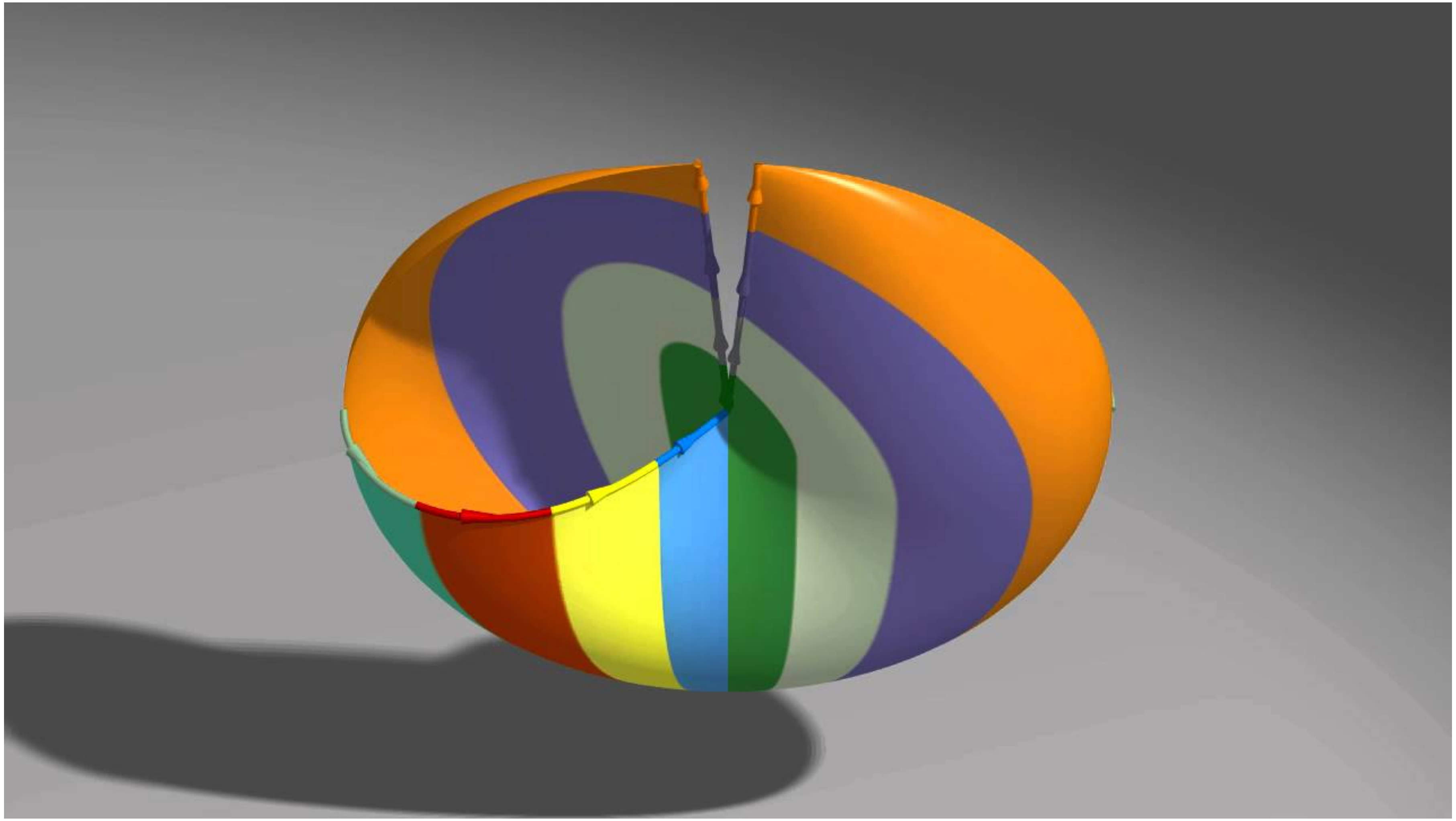
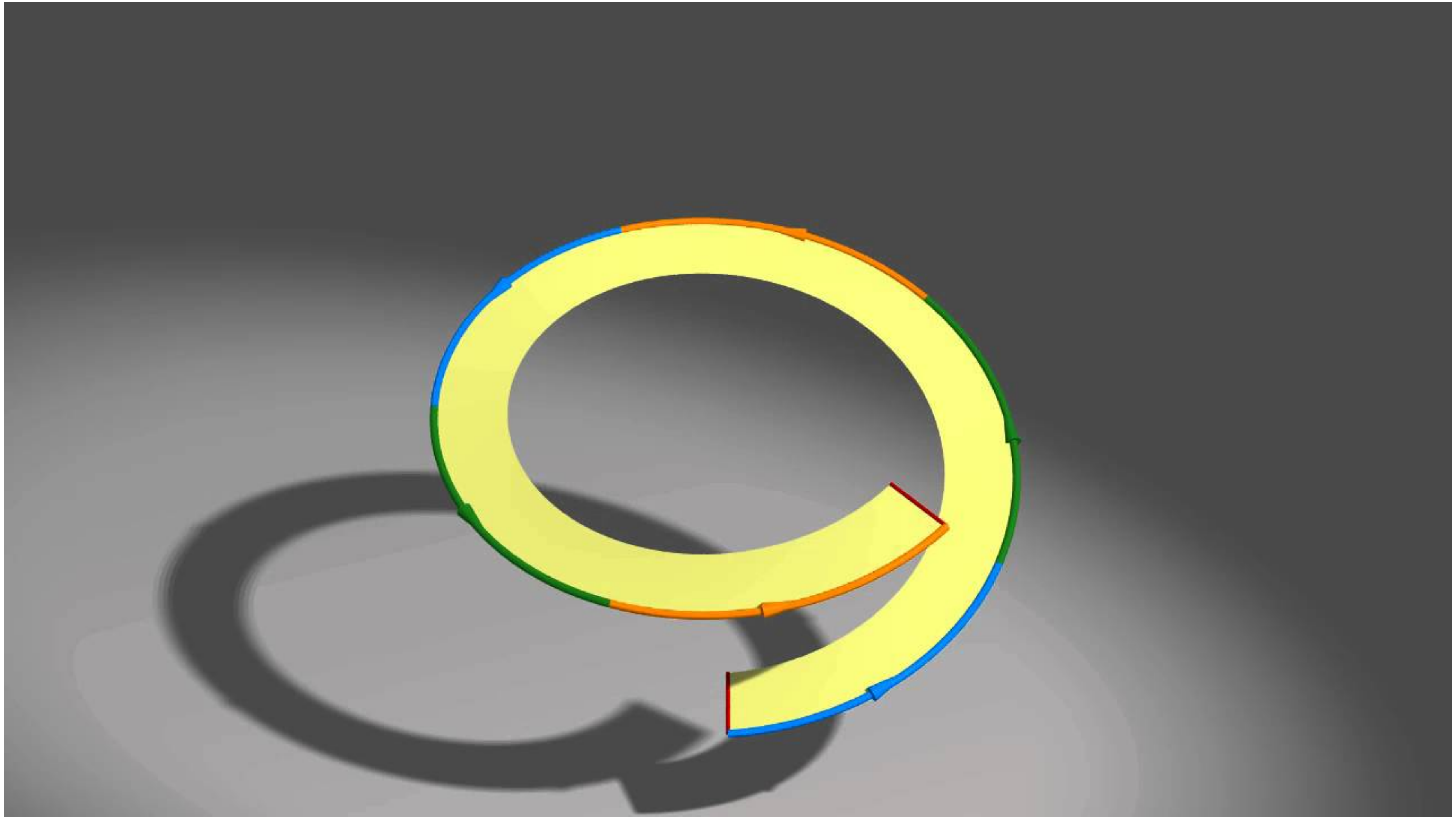


Projective plane = Mobius strip + a disk



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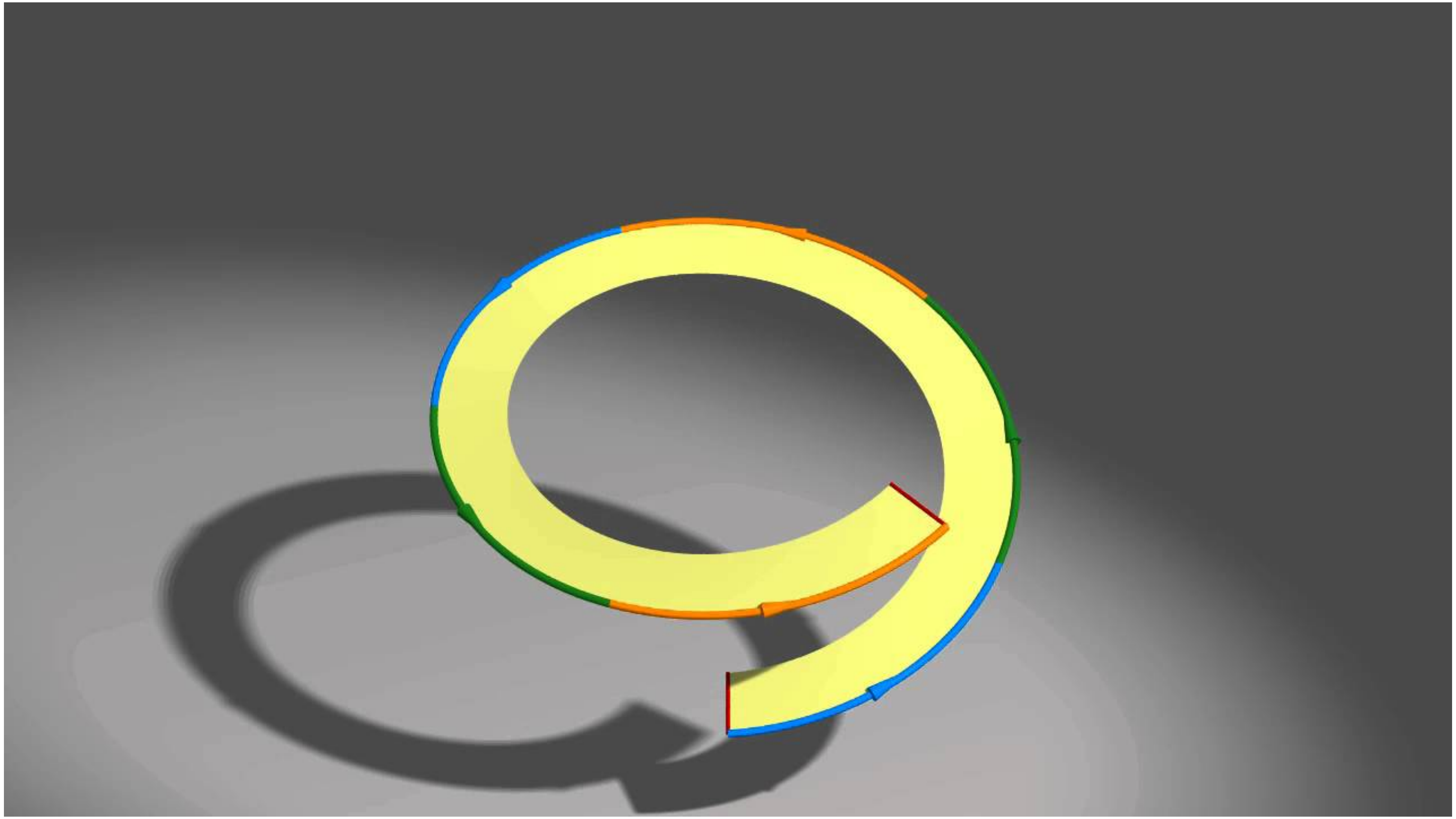
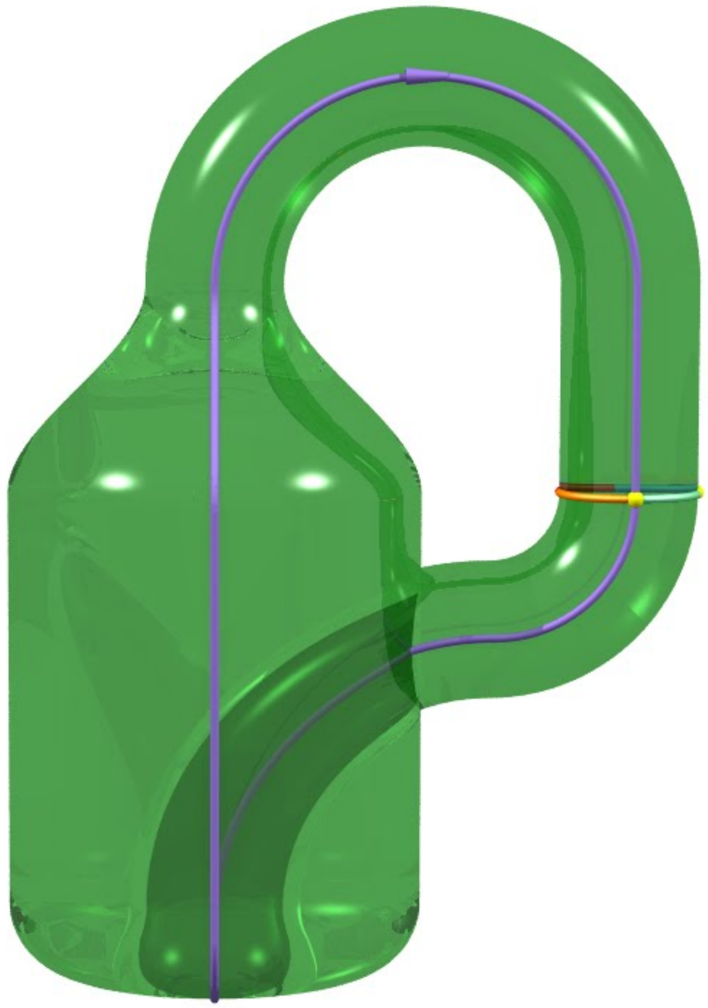


Visualization of Klein bottle



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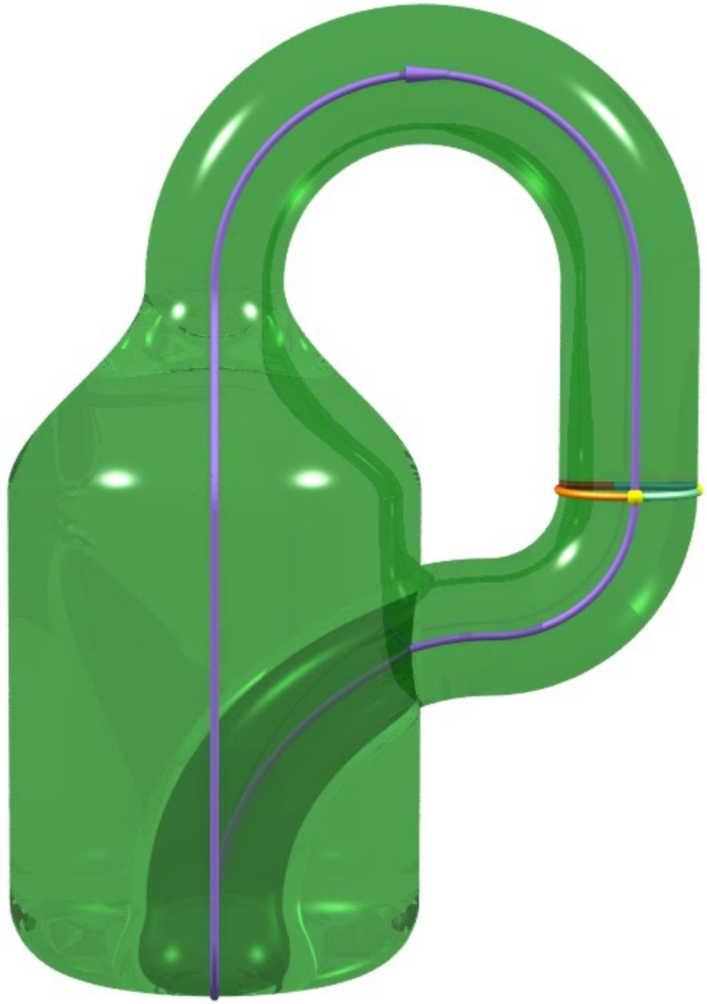
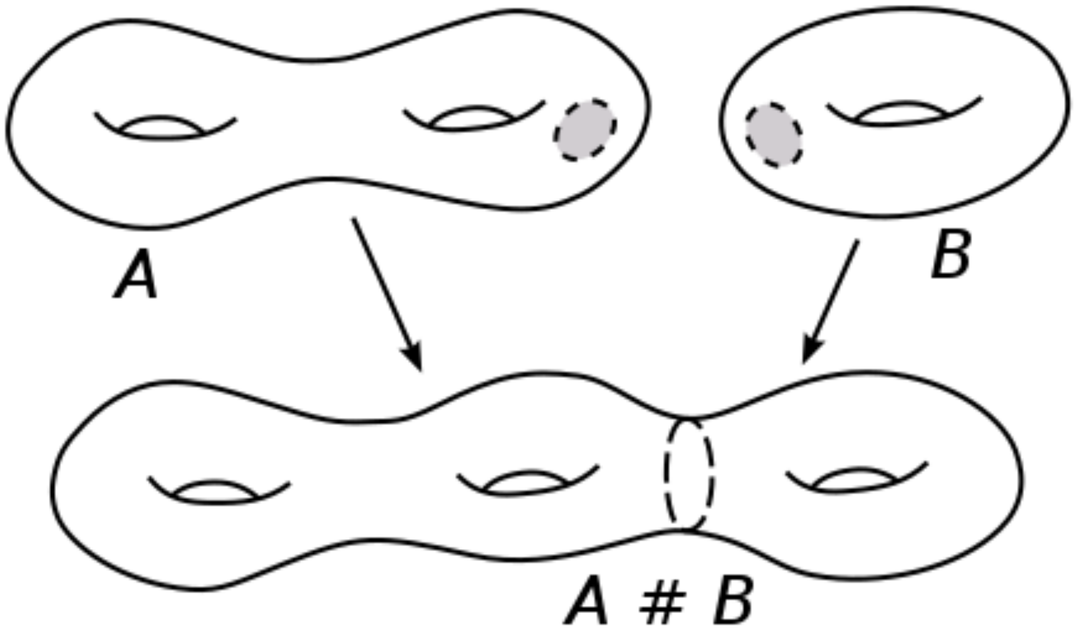


Connected sum operation

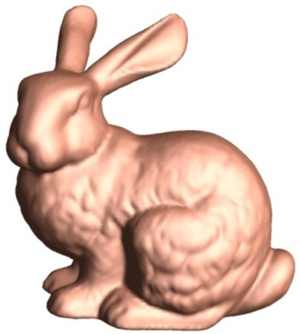
- Given two compact surface M and S, the connected sum $M\#S$ intuitively "merge" the two by cutting off a small disk (cap) from each surface, and the glue the remaining of the two surfaces along the boundary after the cutting.

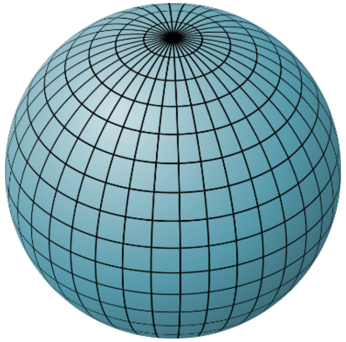


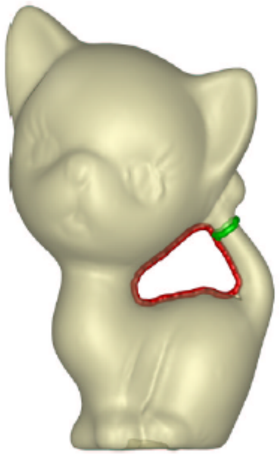
Wikipedia

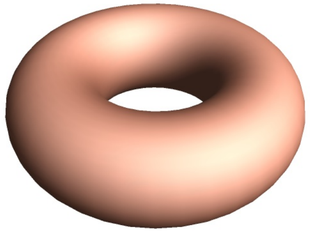
Classification of compact surfaces

Theorem 2 (Classification Theorem) *The two infinite families* $\mathbb{S}$, $\mathbb{T}$, $\mathbb{T}\#\mathbb{T}$, $\ldots$, *and* $\mathbb{P}$, $\mathbb{P}\#\mathbb{P}$, $\ldots$, *exhaust the family of compact 2-manifold without boundary (upto homeomorphism). The first family of surfaces are all orientable; while the second family are all non-orientable. Furthermore, no two surfaces in these sequences are homeomorphic.*

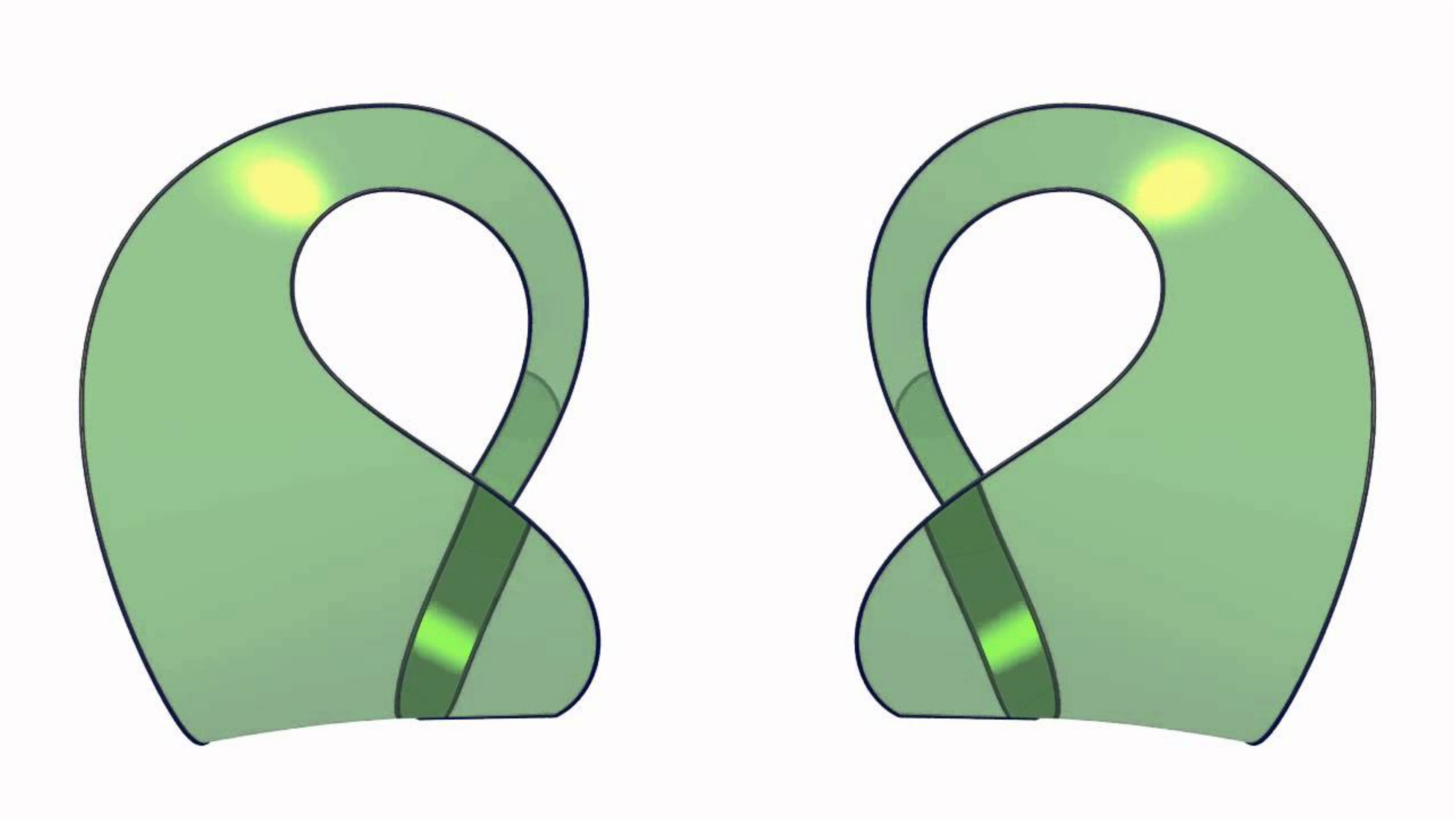








How does $\mathbb{P}\#\mathbb{P}$ become the Klein bottle?



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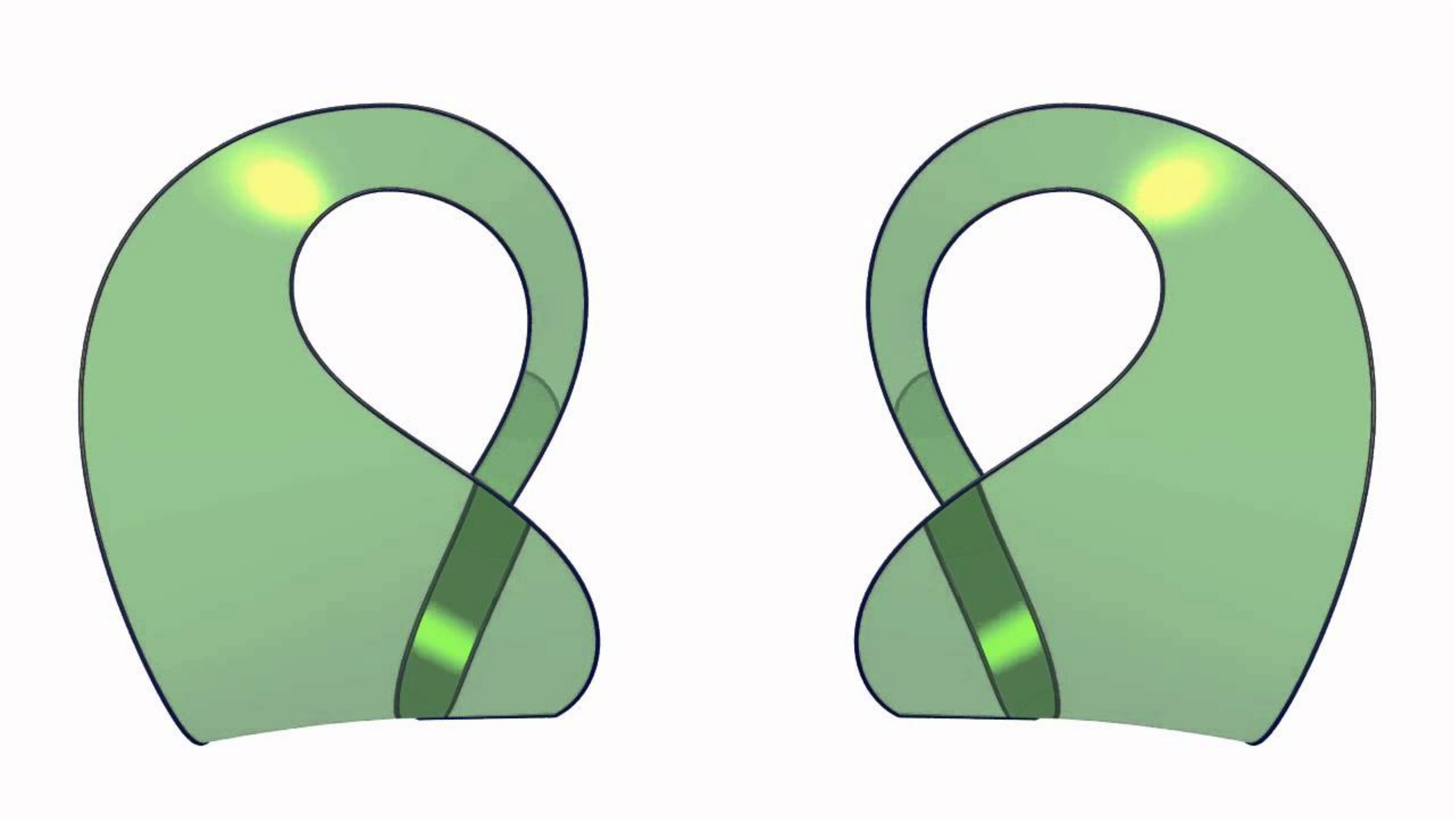

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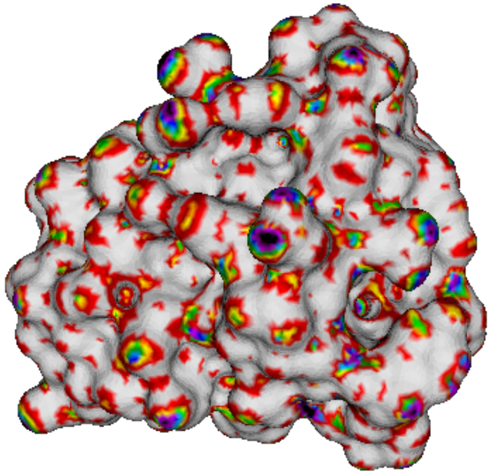
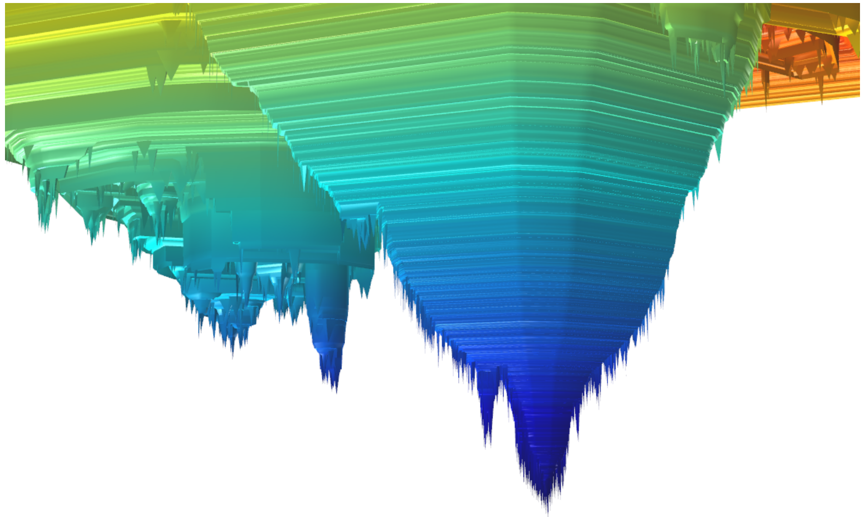
- Intuitively
  - all orientable surfaces without boundaries can be generated by gluing handles to a sphere
- The number of $\mathbb{T}$ or $\mathbb{P}$ needed is called the genus g of the surface M
  - Sphere has genus 0, torus has genus 1, double-torus has genus 2.
- Hence the genus of a surface completely decides its topology upto homeomorphism
  - Any two compact surfaces with the same genus are homeomorphic

Functions on spaces

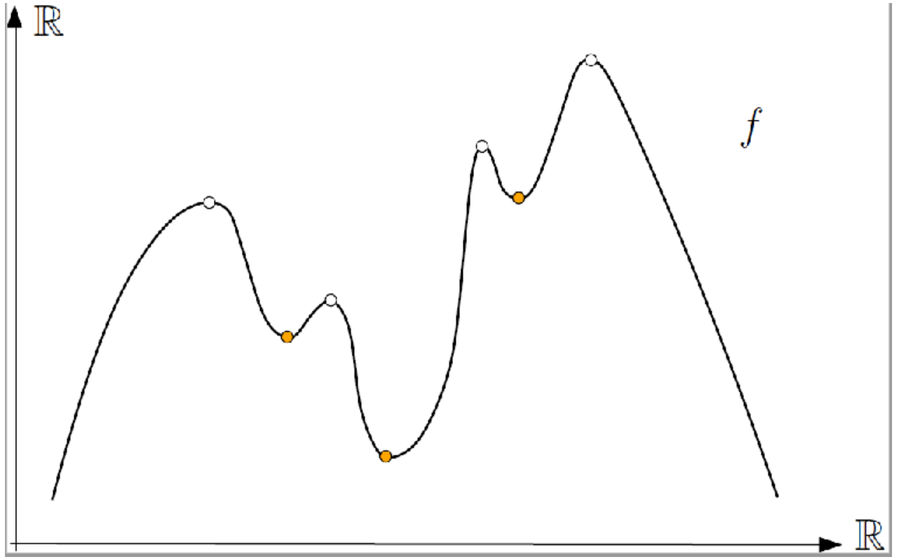
- Properties / attributes of data can often be modeled as functions
- Characterize / summarize functions, as well as summarizing data themselves via this "function" perspective





Gradients and critical points

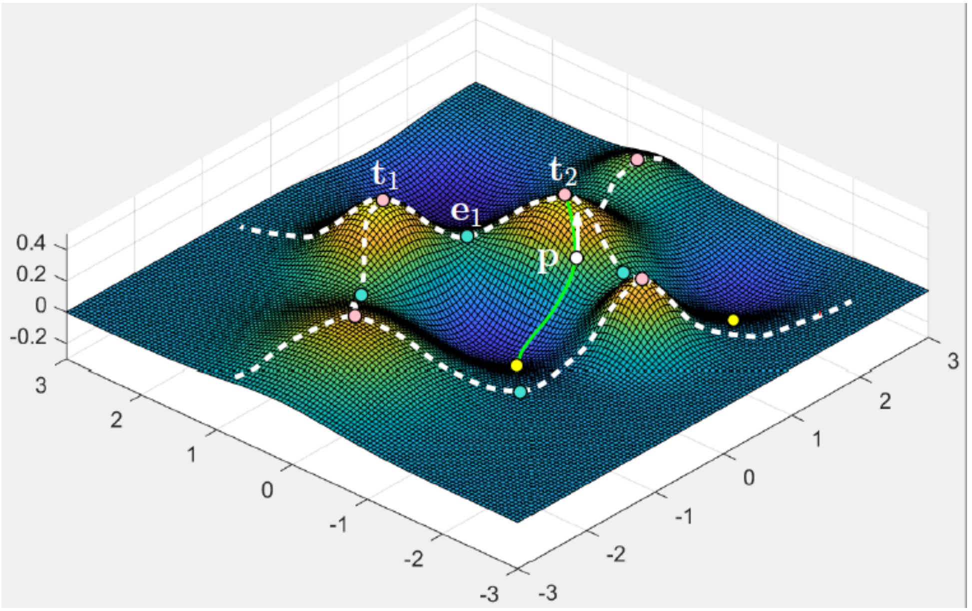
- 1D case: $f\colon \mathbb{R} \to \mathbb{R}$



- Derivative
  - $\nabla f(x) = \lim_{t \to 0} \dfrac{f(x+t) - f(x)}{t}$
  - measures rate of change
- Critical points:
  - A point $x \in \mathbb{R}$ is a *critical point* w.r.t. f if $\nabla f(x) = 0$
  - That is, critical points are where this derivative vanishes.
- A non-critical point is called a *regular point*.

Gradients, critical points

- dD case: $f\colon \mathbb{R}^d \to \mathbb{R}$



Gradients, critical points

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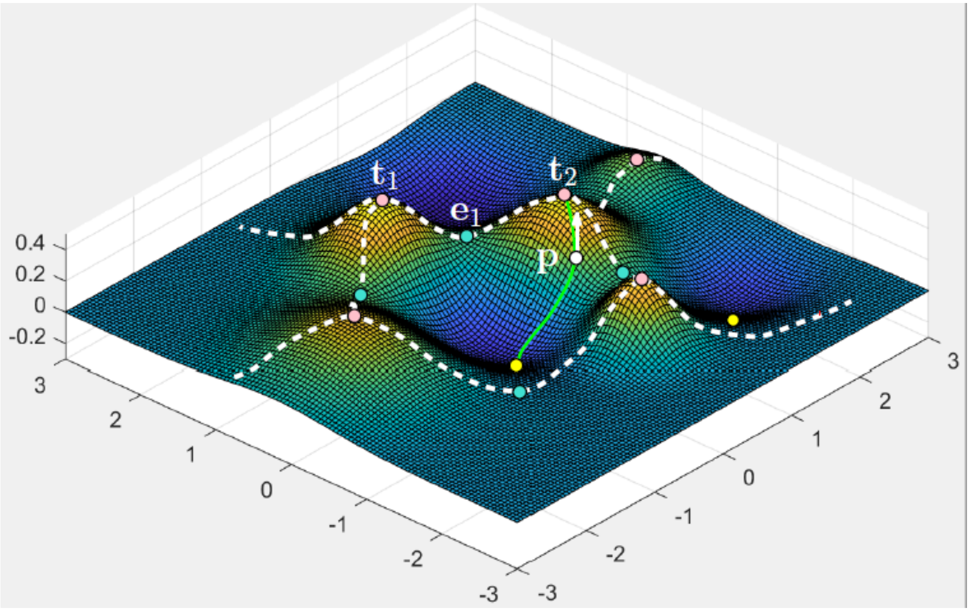
  - $D_v f(p) = \lim_{t\to 0} \frac{f(p+tv) - f(p)}{t}$
  - measures rate of change in direction v



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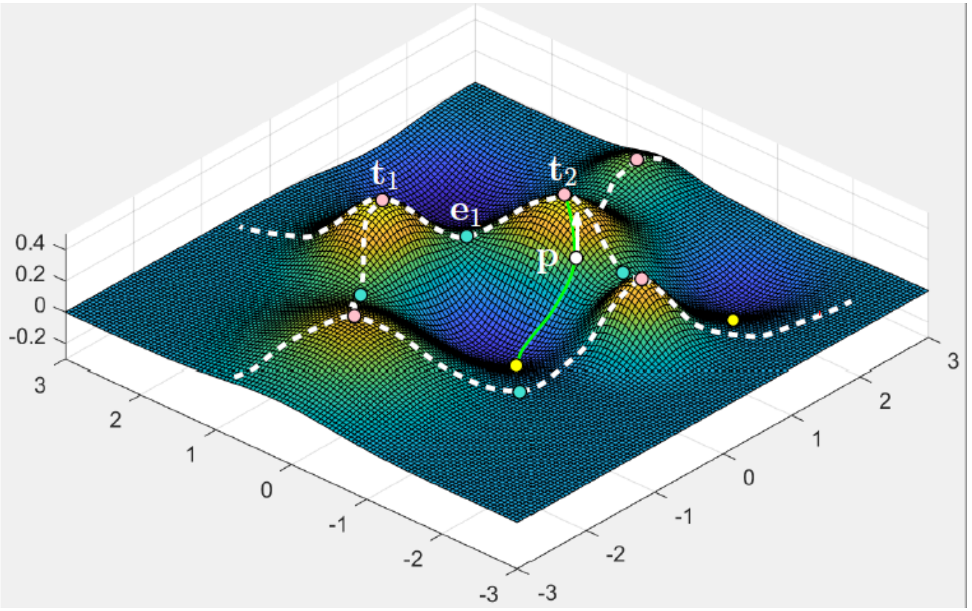
- *Gradient vector* at p
  - $\nabla f(p) = \left[\dfrac{\partial f}{x_1}, \ldots, \dfrac{\partial f}{x_d}\right]^T$, where $x_1, .. x_d$ form an orthonormal coordinate system
    - It is in the direction with largest directional derivative (with steepest rate of increase)
    - The magnitude is that largest rate of increase.



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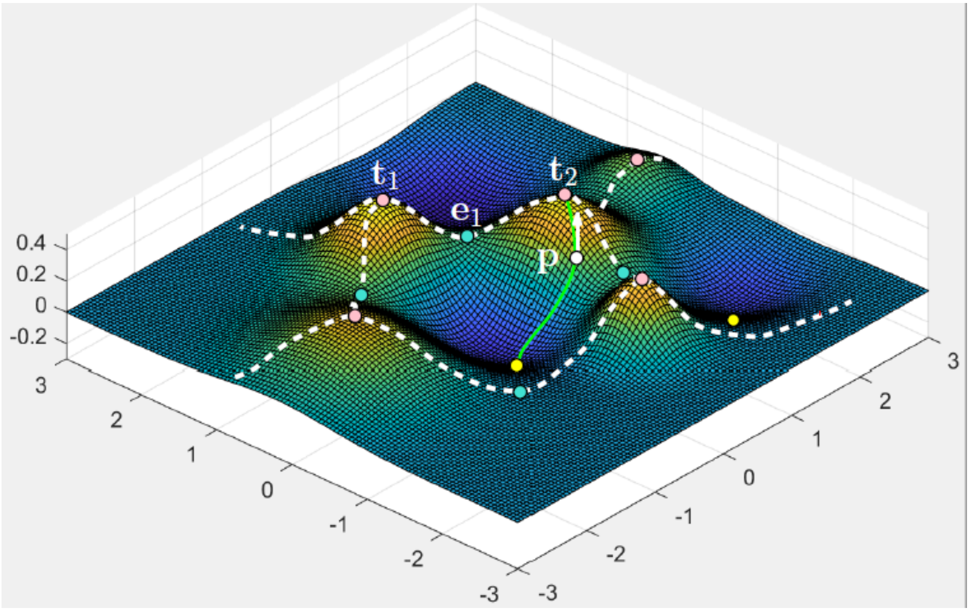

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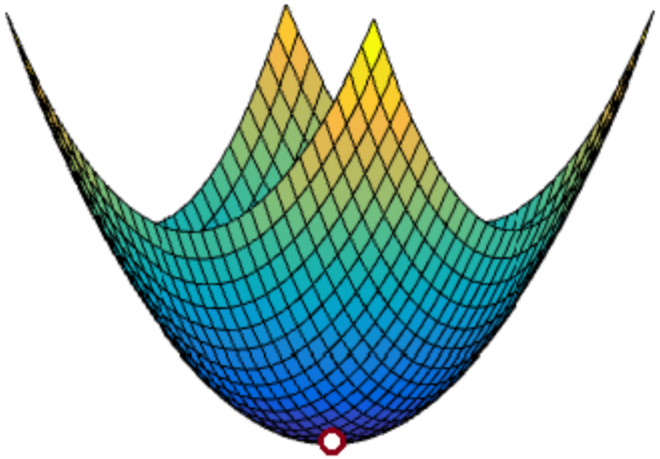
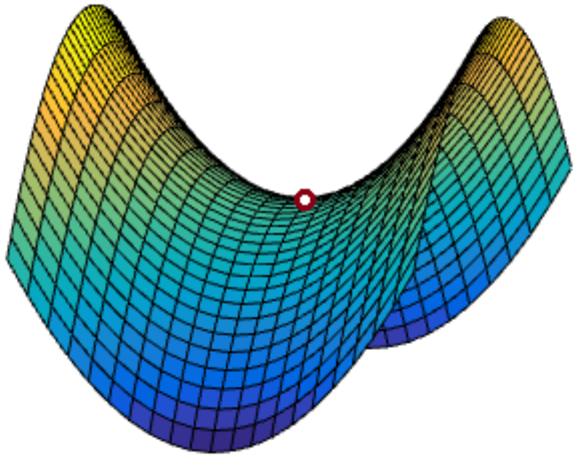
- Critical points:
  - A point p is *critical* if $\nabla f(p) = [0, \ldots, 0]^T$; that is, where gradient vanishes.

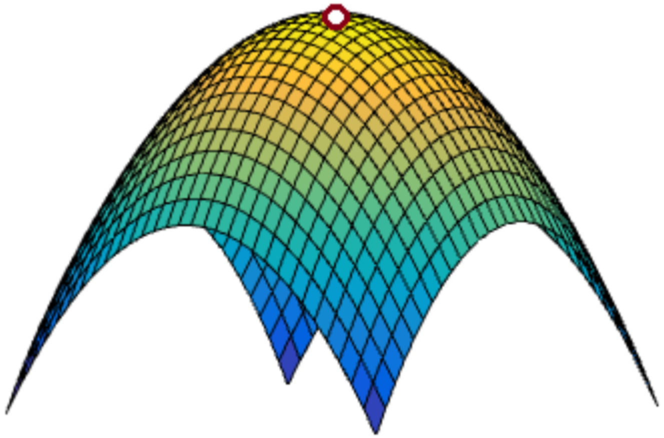
Examples for a 2D function



minimum

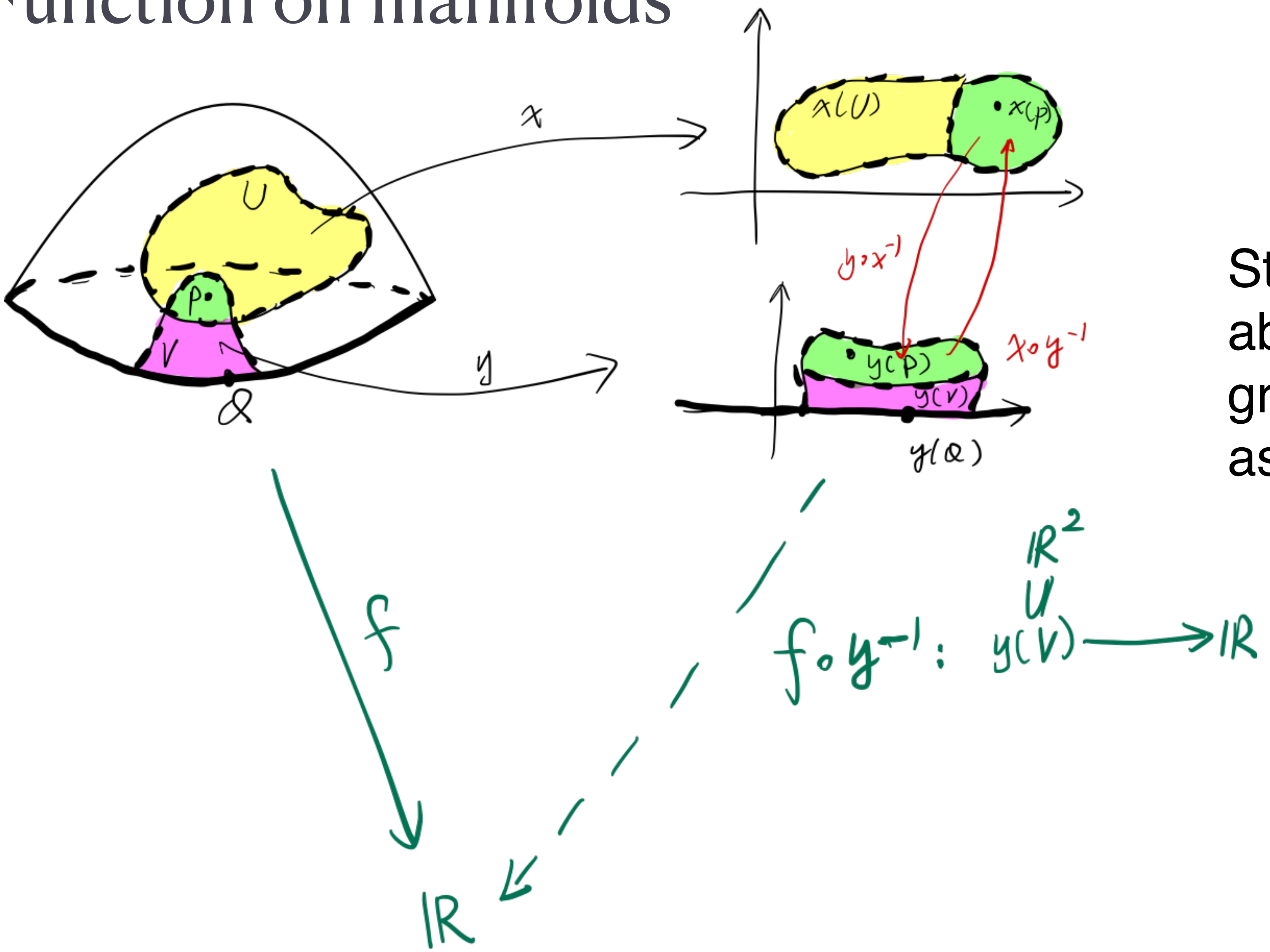


saddle



maximum

Function on manifolds



Still makes sense to talk about derivatives, gradients, critical points as they are local concepts

Gradients, critical points

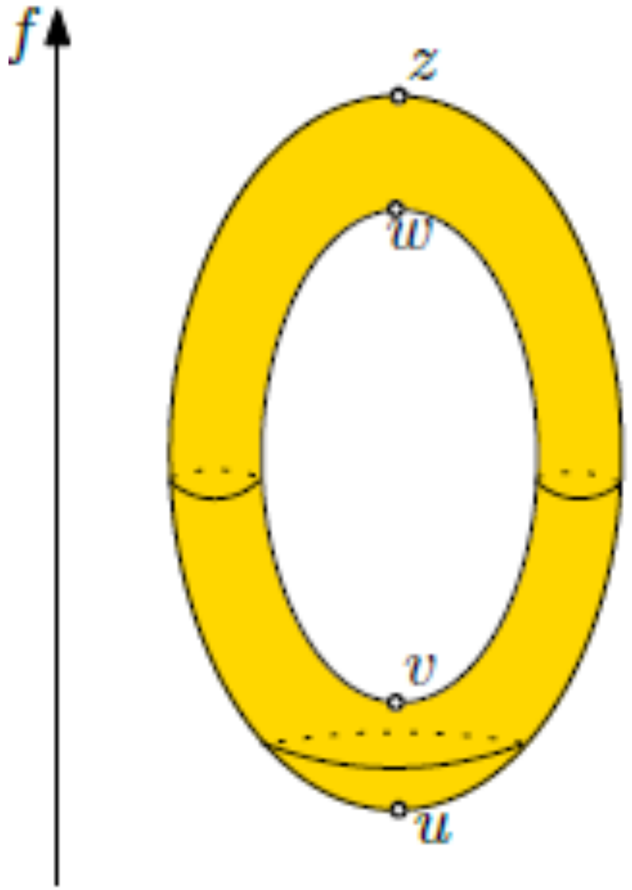
- d-manifold case: $f\colon M \to R$
- Same intuition, simply within a small neighborhood at each point

Definition 8 (Gradient vector field; Critical points)**.** Given a smooth function $f : M \to \mathbb{R}$ defined on a smooth m-dimensional Riemannian manifold M, the *gradient vector field* $\nabla f : M \to TM$ is defined as follows: for any $x \in M$, let $(x_1, x_2, \ldots, x_m)$ be a local coordinate system in a neighborhood of x with orthonormal unit vectors x_i, the gradient at x is

$$\nabla f(x) = [\frac{\partial f}{\partial x_1}(x),\ \frac{\partial f}{\partial x_2}(x),\ \cdots\ \frac{\partial f}{\partial x_m}(x)]^T.$$

A point $x \in M$ is *critical* if $\nabla f(x)$ vanishes, in which case $f(x)$ is called a *critical value* for f. Otherwise, x is *regular*.

Example



(Non-)degenerate critical points

Definition 9 (Hessian matrix; Non-degenerate critical points)**.** Given a smooth m-manifold M, the *Hessian matrix* of a twice differentiable function $f : M \to \mathbb{R}$ at x is the matrix of second-order partial derivatives,

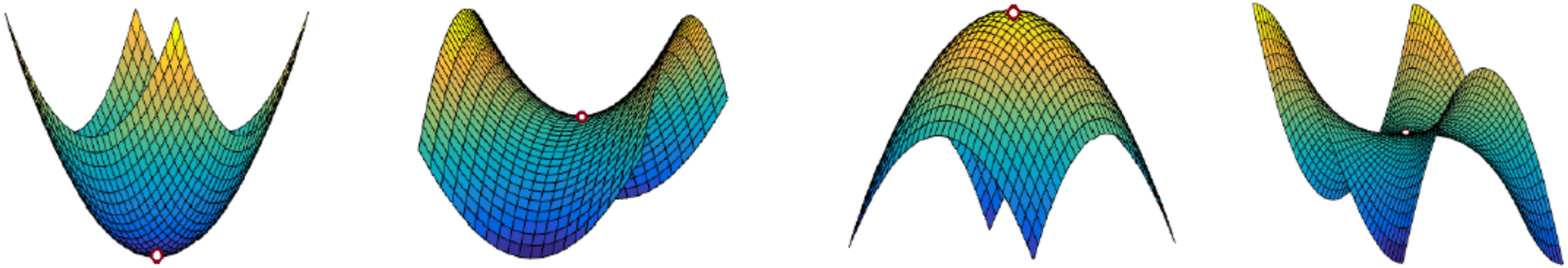
$$Hessian(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}2(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1}(x) & \frac{\partial^2 f}{\partial x_m \partial x_2}2(x) & \cdots & \frac{\partial^2 f}{\partial x_m \partial x_m}(x) \end{bmatrix},$$

where $(x_1, x_2, \ldots, x_m)$ is a local coordinate system in a neighborhood of x.

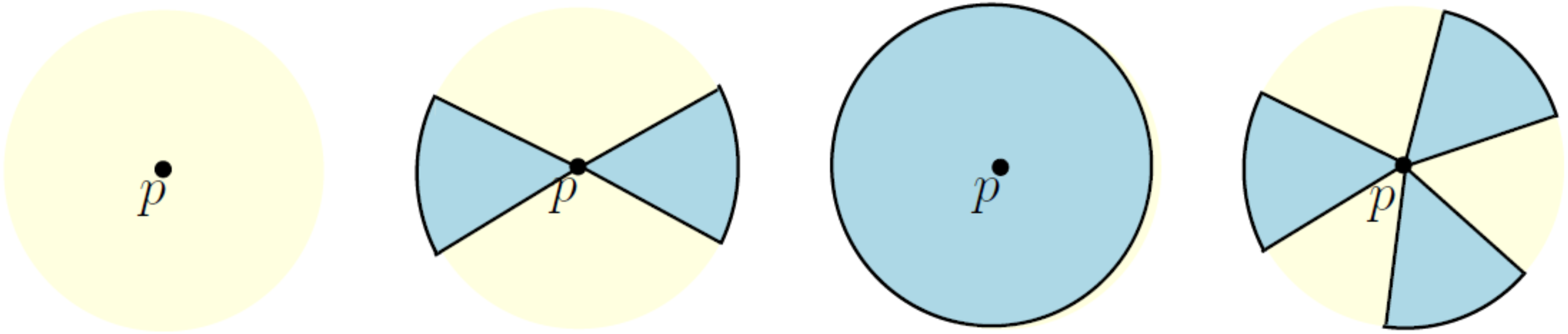
A critical point x of f is *non-degenerate* if its Hessian matrix $Hessian(x)$ is non-singular (has non-zero determinant); otherwise, it is a *degenerate critical point*.

Number of negative eigenvalues is called the **index** of x

Examples



minimum (index-0) saddle (index-1) maximum (index-2) monkey-saddle



Morse Function

Morse Function

- A smooth function is a Morse function if
  - (1) all critical points have distinct function values
  - (2) there is no degenerate critical point.
- Morse functions have well-behaved critical points!

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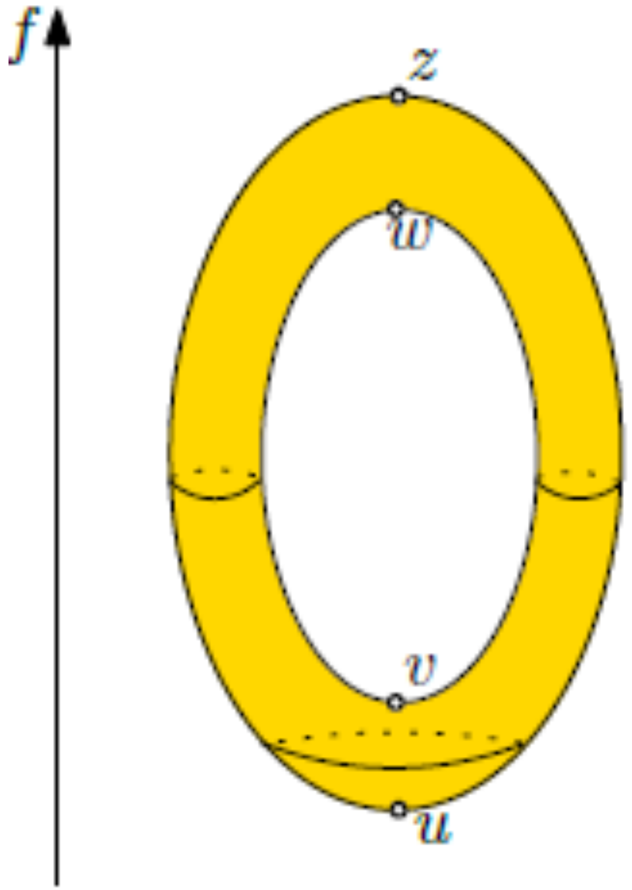
- Morse functions have well-behaved critical points!

Proposition 2 (Morse Lemma). *Given a smooth function $f : M \to \mathbb{R}$ defined on a smooth m-manifold M, let p be a non-degenerate critical point of f. Then there is a local coordinate system in a neighborhood $U(p)$ of p so that (i) the coordinate of p is $(0, 0, \ldots, 0)$, and (ii) locally for every point $x = (x_1, x_2 \ldots x_m)$ in neighborhood $U(p)$,*

$$f(x) = f(p) - x_1^2 - \ldots x_s^2 + x_{s+1}^2 \ldots x_m^2, \qquad \text{for some} \quad s \in [0, m].$$

The number s of minus signs in the above quadratic representation of $f(x)$ is called the index of the critical point p.

Example

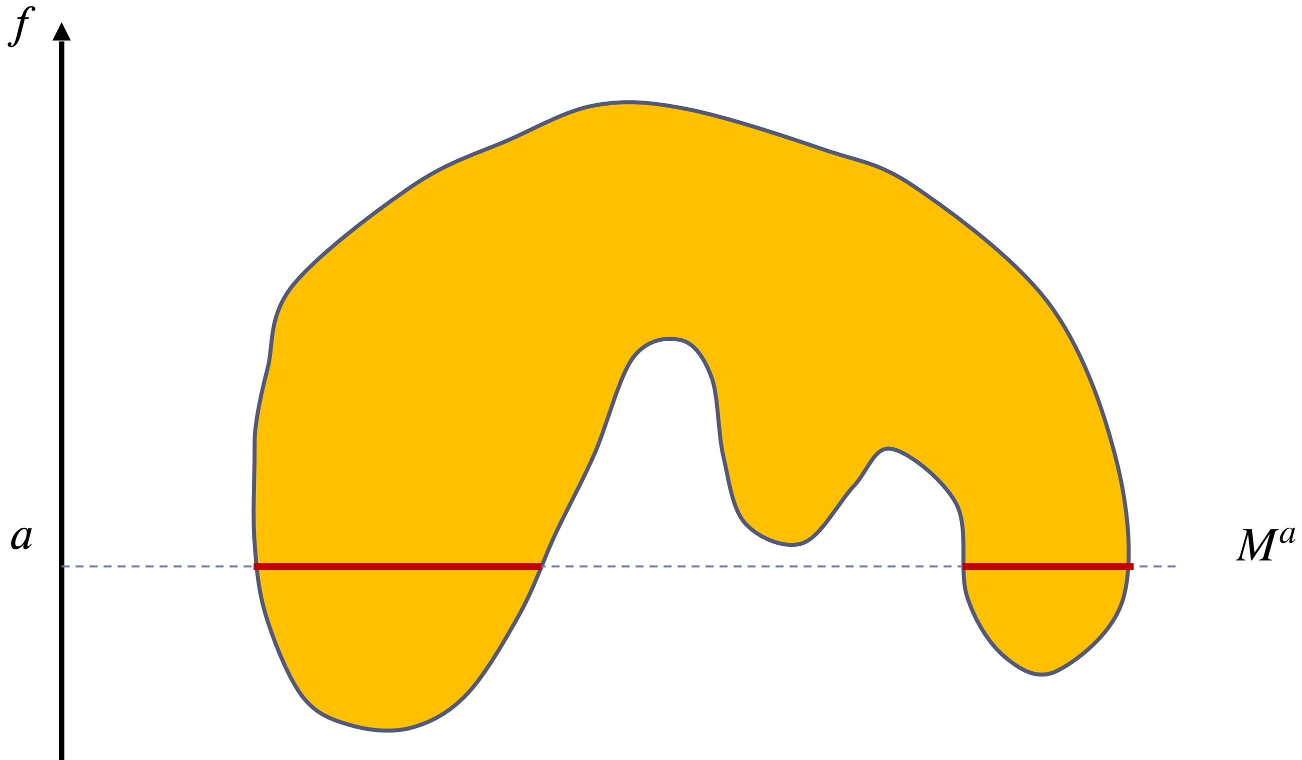


Why do we care about critical points?

- Intuitively, if we sweep the domain w.r.t. the function, this is where the topology of the swept portion changes.

Notations

- Function: $f: M \to R$
- Level set: $M^a = \{x \in M \mid f(x) = a\}$
- Sub-level set: $M^{\leq a} = \{x \in M \mid f(x) \leq a\}$
    - $M^{\leq a} \subseteq M^{\leq b}$ for any $a \leq b$



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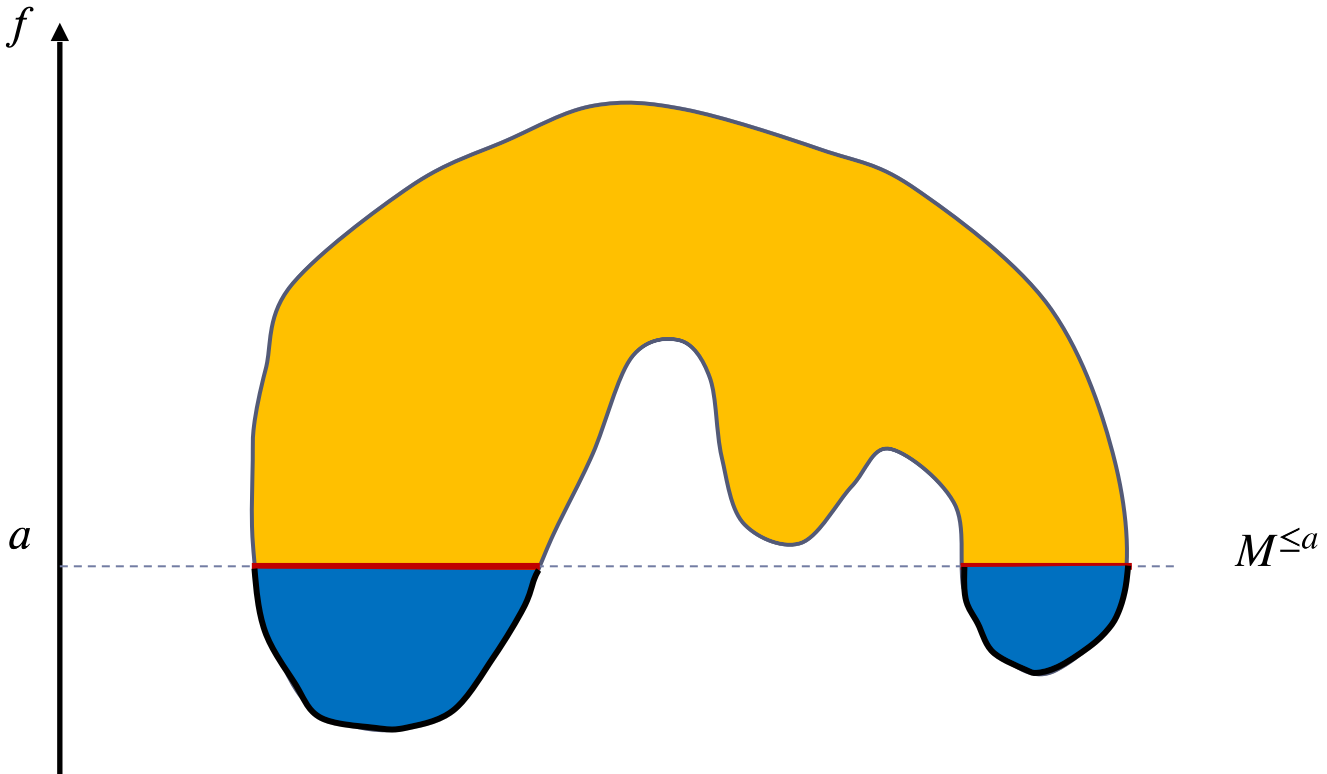
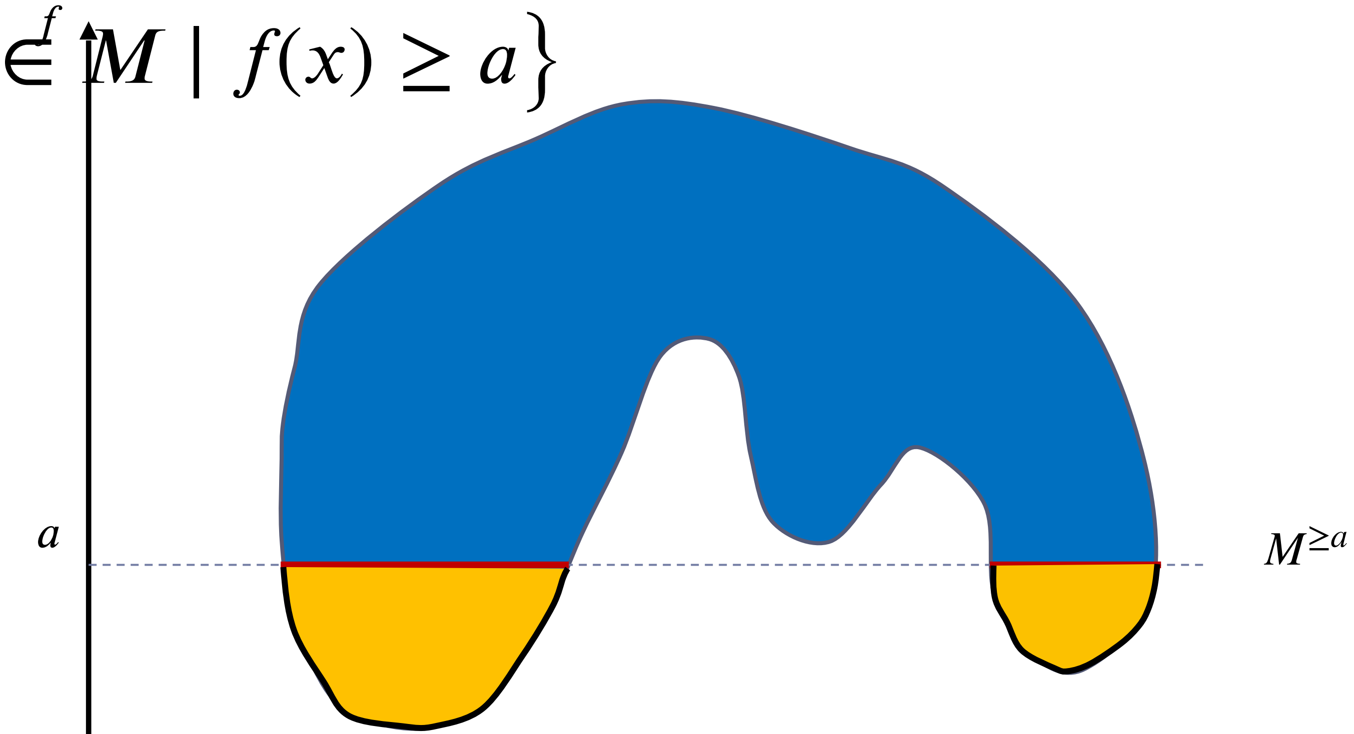


Notations

- Function: $f: M \to R$
- Level set: $M^a = \{x \in M \mid f(x) = a\}$,
- Sub-level set: $M^{\leq a} = \{x \in M \mid f(x) \leq a\}$
- Super-level set: $M^{\geq a} = \{x \in M \mid f(x) \geq a\}$
  - $M^{\geq a} \supseteq M^{\geq b}$ for any $a \leq b$



Critical points and topology

Theorem 3 (Homotopy type of sub-level sets)**.** *Let $f : M \to \mathbb{R}$ be a smooth function defined on a manifold M. Given $a < b$, suppose the interval-level set $M_{[a,b]} = f^{-1}([a, b])$ is compact and contains no critical points of f. Then $M_{\leq a}$ is diffeomorphic to $M_{\leq b}$.*

Furthermore, $M_{\leq a}$ is a deformation retract of $M_{\leq b}$, and the inclusion map $i : M_{\leq a} \hookrightarrow M_{\leq b}$ is a homotopy equivalence.

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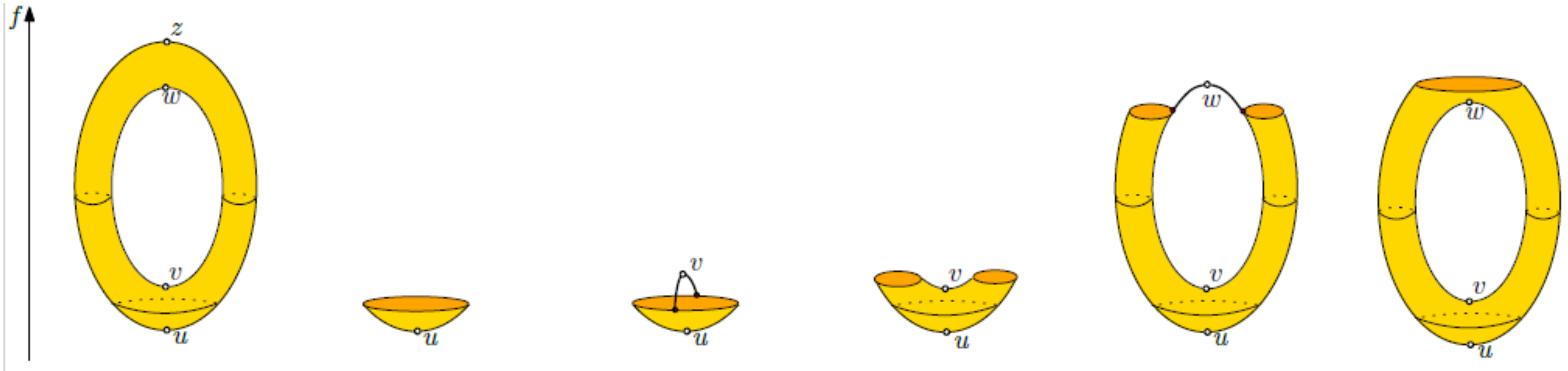

(a) (b) (c) (d) (e) (f)

Critical points and topology

Theorem 4. *Given a Morse function $f : M \to \mathbb{R}$ defined on a smooth manifold M, let p be an index-k critical point of f with $\alpha = f(p)$. Assume $f^{-1}([\alpha - \varepsilon, \alpha + \varepsilon])$ is compact for a sufficiently small $\varepsilon > 0$ such that there is no other critical points of f contained in this interval-level set other than p. Then the sublevel set $M_{\leq \alpha + \varepsilon}$ has the same homotopy type as $M_{\leq \alpha - \varepsilon}$ with a k-cell attached to its boundary* $\mathrm{Bd}\, M_{\leq \alpha - \varepsilon}$.

Animation

Critical points and topology

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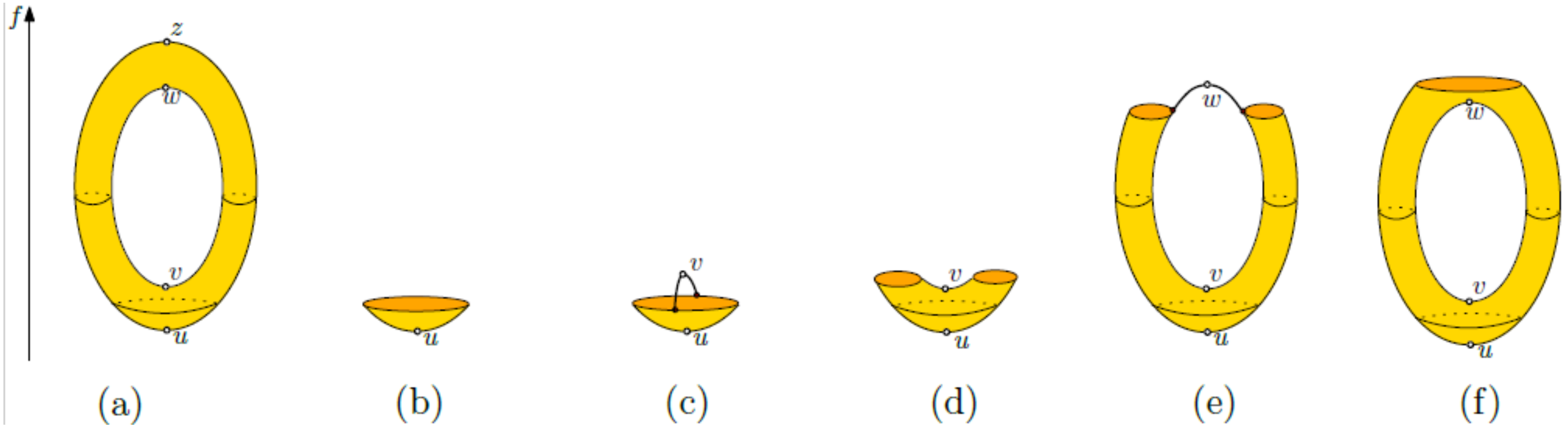
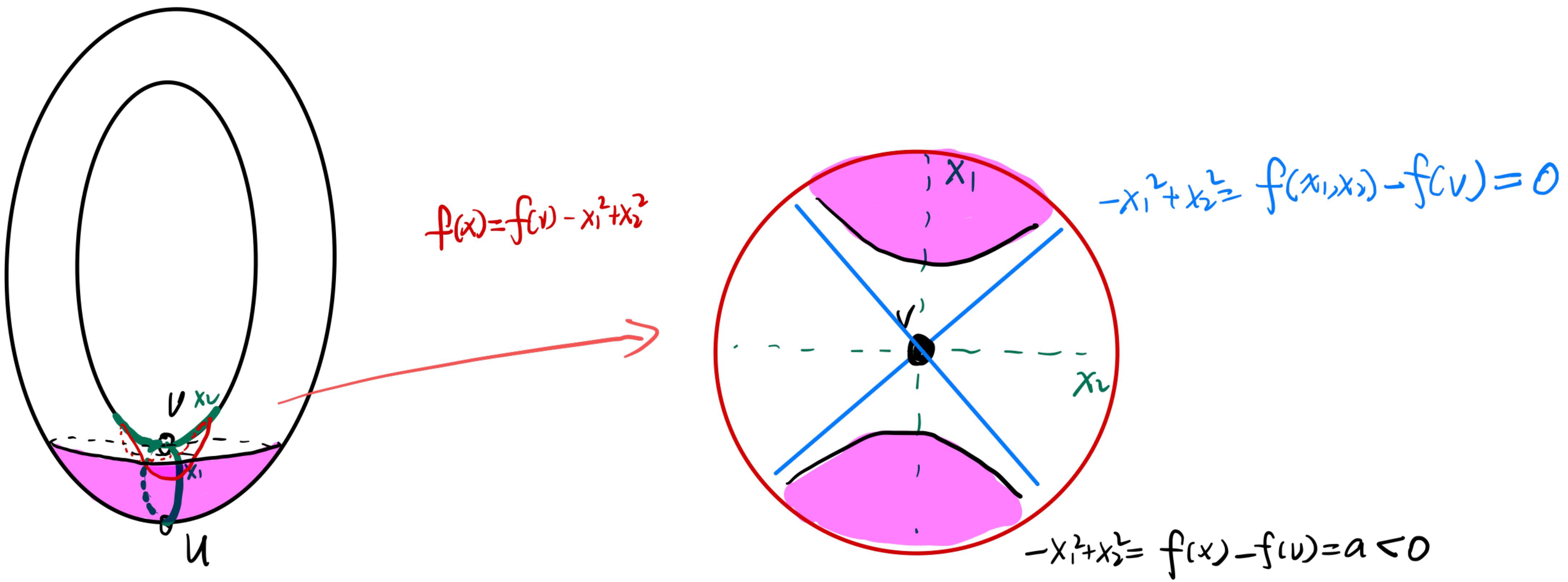
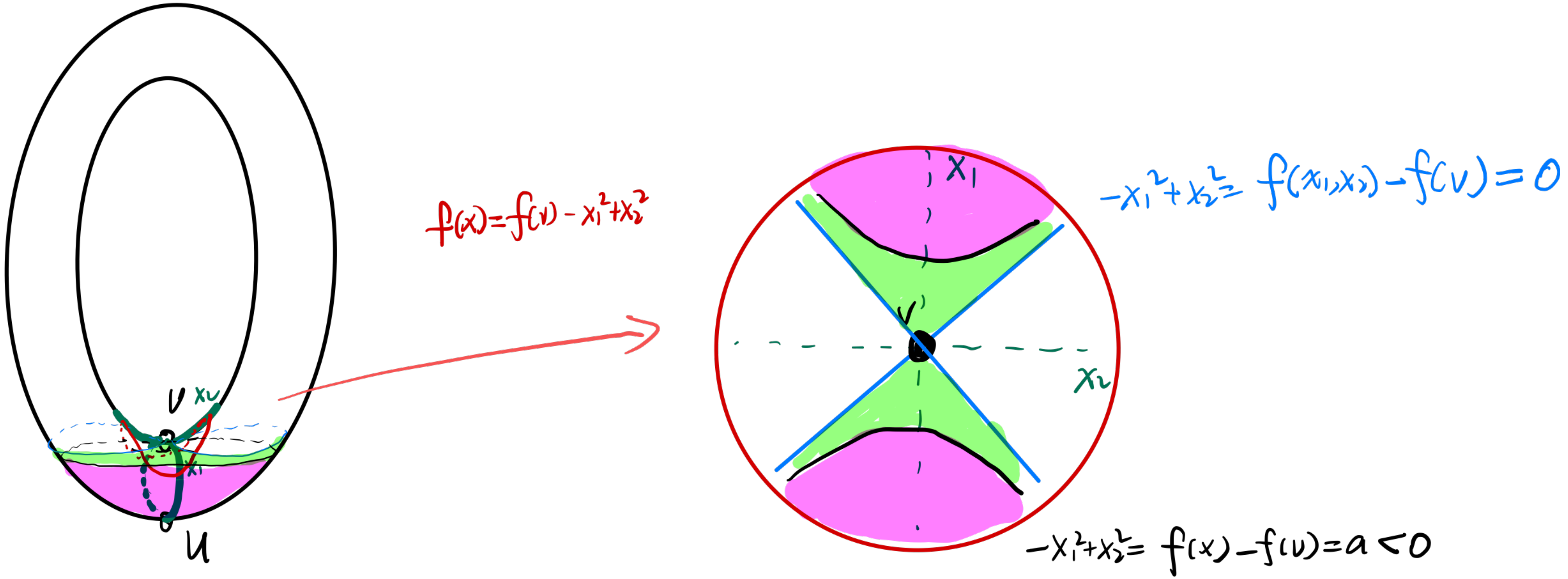


Animation

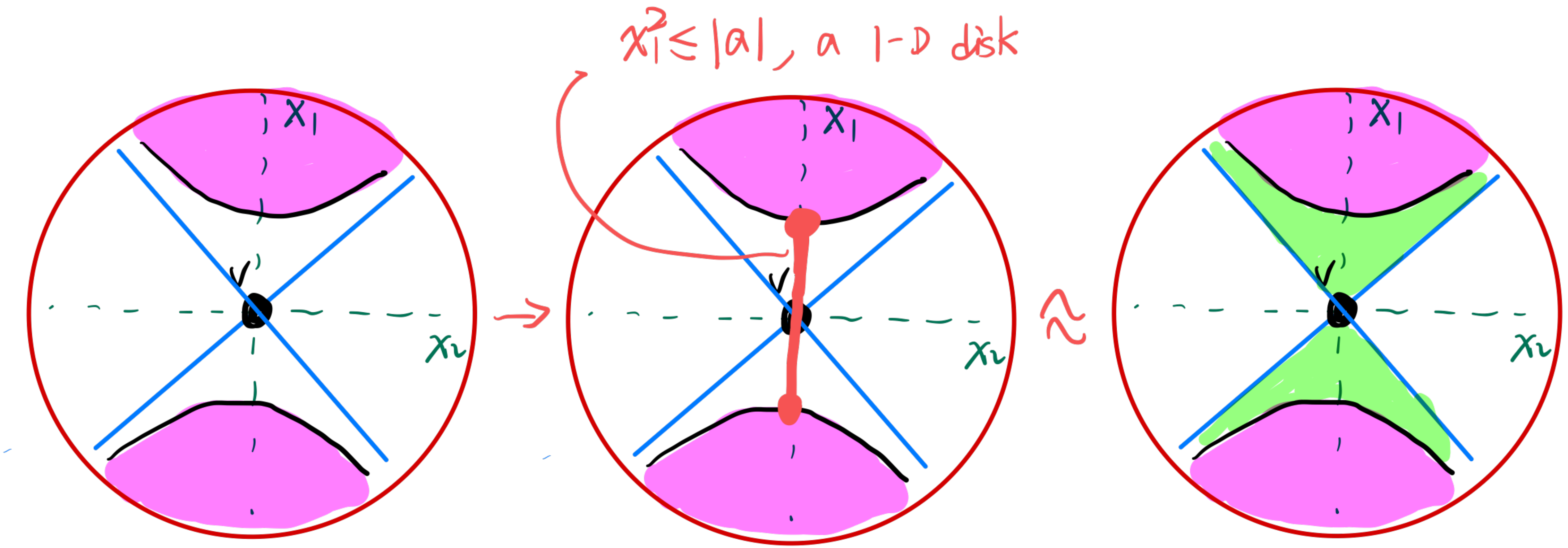
Proof by picture



Proof by picture



Proof by picture



FIN