# DSC 214
# Topological Data Analysis
## Topic 3: Simplicial Homology

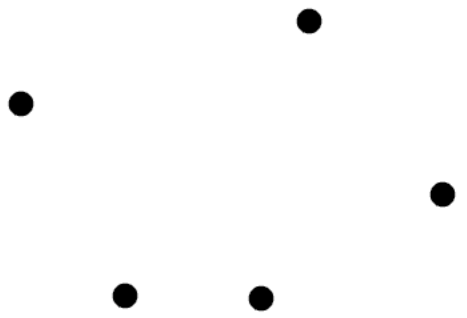
Instructor: Zhengchao Wan

# Overview

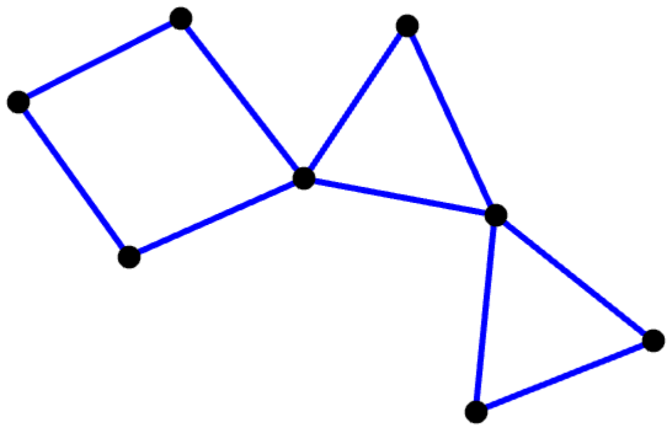
- Review of algebraic tools
- (Simplicial) homology groups
  - a way to quantify topological features
- Notations
  - Chains, cycles, and homology groups
- Matrix view
  - Matrix reduction algorithm

# Motivating examples

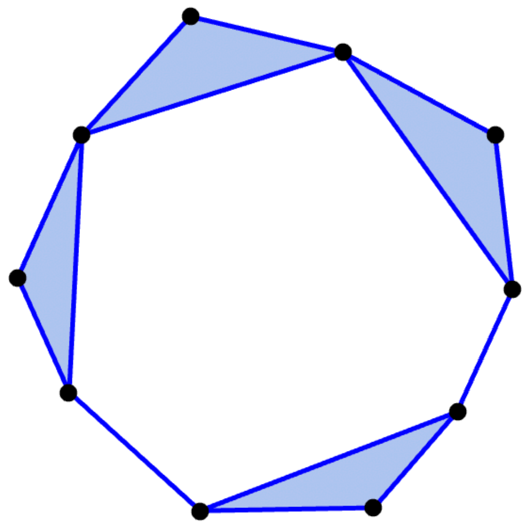
- $i$th homology "counts the number of $i$ dimensional holes" in a topological space



$\dim H_0 = 5$
$\dim H_1 = 0$



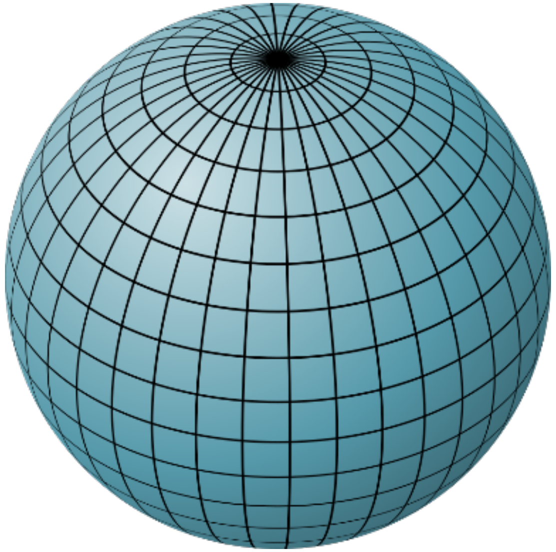
$\dim H_0 = 1$
$\dim H_1 = 3$



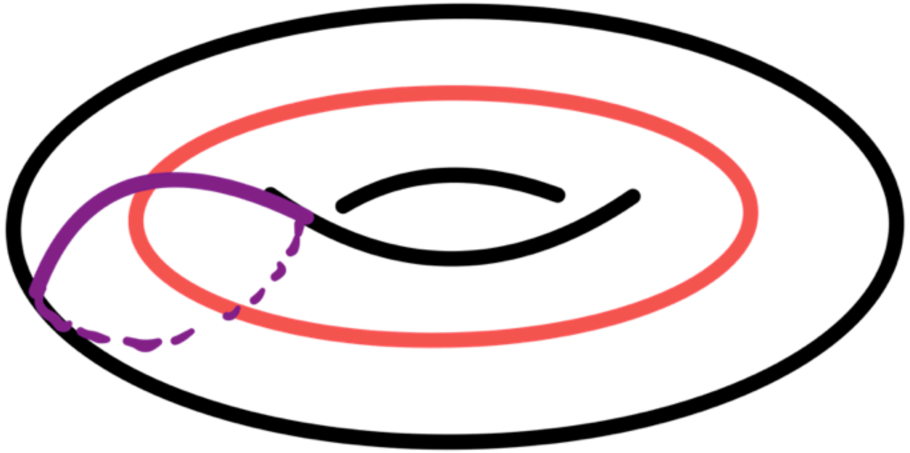
$\dim H_0 = 1$
$\dim H_1 = 1$

# Motivating examples

- $i$th homology "counts the number of $i$ dimensional holes" in a topological space

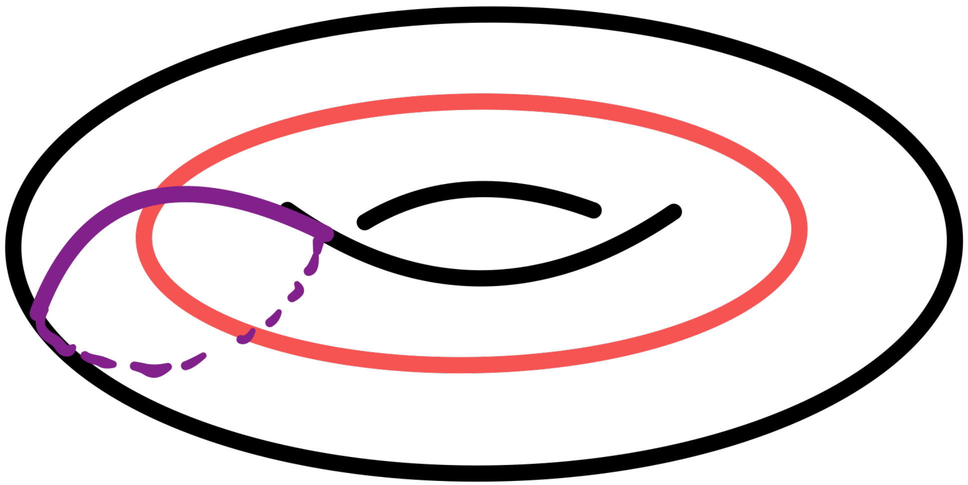


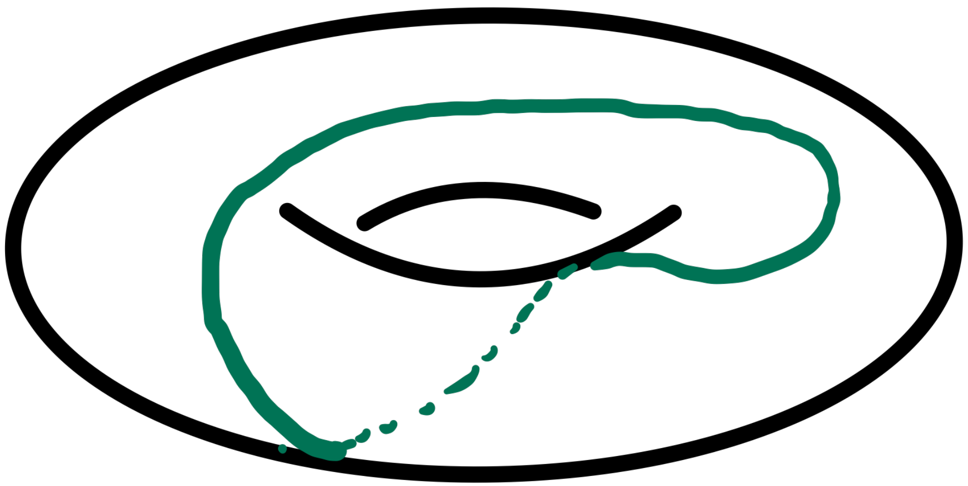
$\dim H_0 = 1$
$\dim H_1 = 0$
$\dim H_2 = 1$



$\dim H_0 = 1$
$\dim H_1 = 2$
$\dim H_2 = 1$

- $i$th homology has a vector space structure!





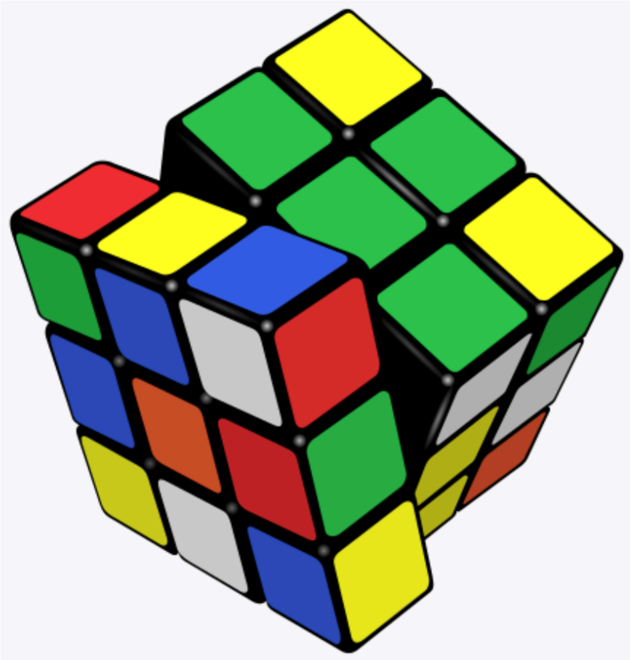
# Part 0:
# Review of algebraic tools

# Group

- A **group** is a tuple $(G, +)$ where $G$ is a set and $+ : G \times G \to G$ is a binary operation
  - **Associativity** $a + (b + c) = (a + b) + c$
  - There exist $0 \in G$ such that $a + 0 = 0 + a = a$
  - For any $a \in G$, there exist $-a \in G$ such that $a + (-a) = 0$
- If $G$ further satisfies the following property, then we call $(G, +)$ an **abelian group**
  - **Commutativity** $a + b = b + a$

# Examples of groups

- $(\mathbb{Z}, +)$ is an abelian group
- $(\mathbb{R}, +)$ is an abelian group
- $(GL_n(\mathbb{R}), \cdot)$ is a non-abelian group
- Rubik's cube group



# Ring and Field

- A **ring** is a tuple $(F, +, \times)$ where $(F, +)$ is an abelian group and $\times : F \times F \to F$ is another binary operation such that
  - **Associativity** $a \times (b \times c) = (a \times b) \times c$
  - There exist 1 in $F$ such that $a \times 1 = a$
  - **Distributivity** $a \times (b + c) = (a \times b) + (a \times c)$
- $(F, +, \times)$ is called a **field** if
  - For any $a \neq 0$ in $F$, there exists $a^{-1} \in F$ such that $a \times a^{-1} = 1$
  - $a \times b = b \times a$

# Examples of fields

- Rational numbers $(\mathbb{Q}, +, \times)$
- Real numbers $(\mathbb{R}, +, \times)$
- Complex numbers $(\mathbb{C}, +, \times)$
- Finite fields
  - For any prime number $p$, $\mathbb{Z}_p = \{0,1,\ldots,p-1\}$
  - $+, \times$ modulo $p$
  - $(\mathbb{Z}_p, +, \times)$ is a field

# The most important example in this class: $\mathbb{Z}_2$

- $\mathbb{Z}_2 = \{0,1\}$ is the smallest field

| $+$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | $\boxed{0}$ |

| $\times$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

# Vector space

- A vector space over a field $F$ is a set $V$ of vectors with operations
  - Vector addition $V \times V \to V \quad (v, w) \mapsto v + w$
  - Scalar multiplication $F \times V \to V \ (\lambda, v) \mapsto \lambda v$
- Satisfying
  - $(V, +)$ is an abelian group
  - $\lambda(u + v) = \lambda u + \lambda v$ and $(\lambda + \mu)v = \lambda v + \mu v$ and $\lambda(\mu v) = (\lambda\mu)v$
  - $1v = v$

# Examples of vector spaces

- $\mathbb{R}^d$ is a vector space over $\mathbb{R}$ with operations
  - $(x_1, \ldots, x_d) + (y_1, \ldots, y_d) = (x_1 + y_1, \ldots, x_d + y_d)$
  - $\lambda(x_1, \ldots, x_d) = (\lambda x_1, \ldots, \lambda x_d)$

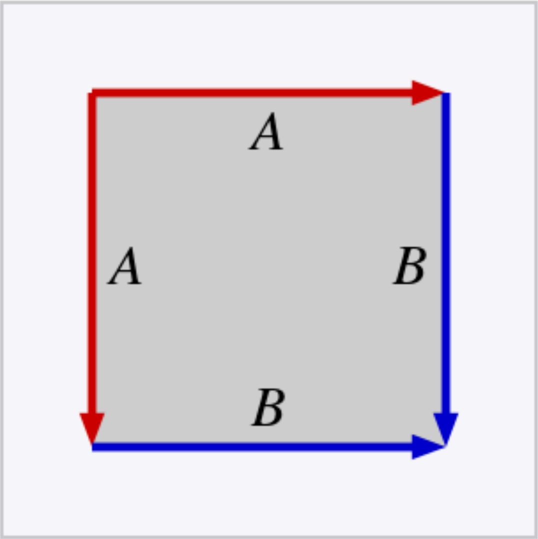
- $\mathbb{Z}_2^d = \{(x_1, \ldots, x_d) \,|\, x_i \in \{0,1\}\}$ is a vector space over $\mathbb{Z}_2$ with operations
  - $(x_1, \ldots, x_d) + (y_1, \ldots, y_d) = (x_1 + y_1, \ldots, x_d + y_d)$ mod 2
  - $\lambda(x_1, \ldots, x_d) = (\lambda x_1, \ldots, \lambda x_d)$

## Basis and Dimension

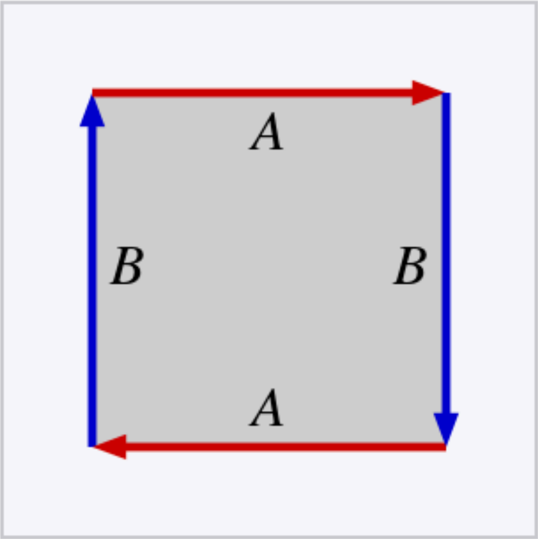
- Let $V$ be a vector space over $F$
- A finite subset $W = \{w_1, \ldots, w_n\} \subset V$ is **linearly independent** if
  - $\lambda_1 w_1 + \cdots + \lambda_n w_n = 0$ iff $\lambda_1 = \cdots = \lambda_n = 0$
- $W$ is **spanning** if for any $v \in V$, there exist $\lambda_1, \ldots, \lambda_n \in F$ such that
  - $\lambda_1 w_1 + \cdots + \lambda_n w_n = v$
- $W$ is a **basis** for $V$ if it is linearly independent and spanning. We call $n$ the dimension of $V$, denoted by $\dim V$

# Quotient is a way of identifying/collapsing points

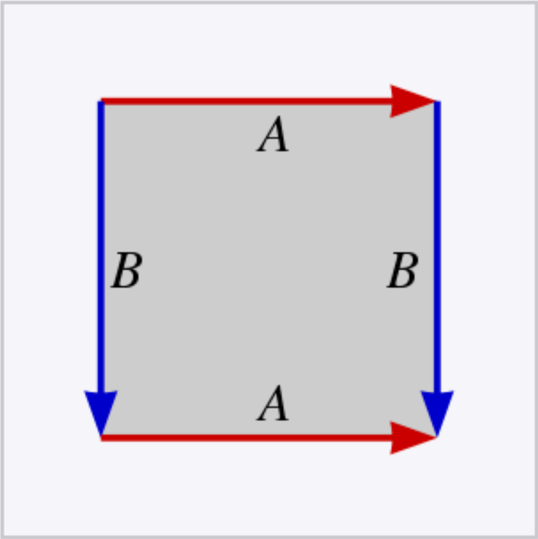
- Quotient topological space



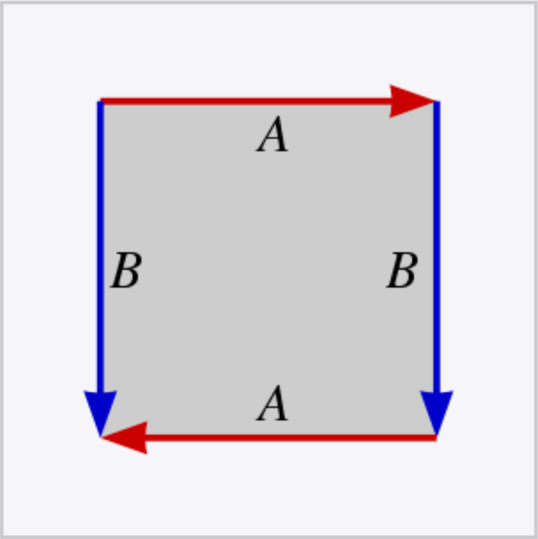

sphere




real projective plane




torus




Klein bottle

# Quotient

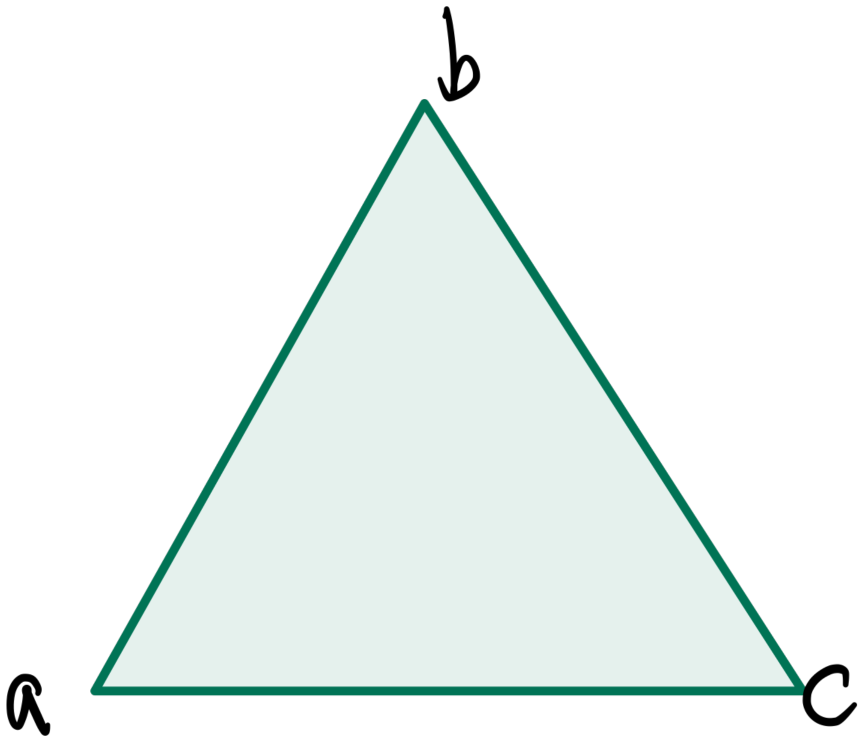
- Let $V$ be a vector space and $W \subset V$ be a linear subspace.
- An equivalence relation $\sim$ on $V$:
  - $v \sim u$ iff $v - u \in W$
  - Equivalence class $[v] = \{u \in V \mid v - u \in W\}$
  - Think of information along directions in $W$ as **redundant**
- The quotient of $V$ by $W$ is the set $V/W = \{[v] \mid v \in V\}$ with
  - Vector addition $[v] + [u] := [v + u]$
  - Scalar multiplication $\lambda[v] := [\lambda v]$

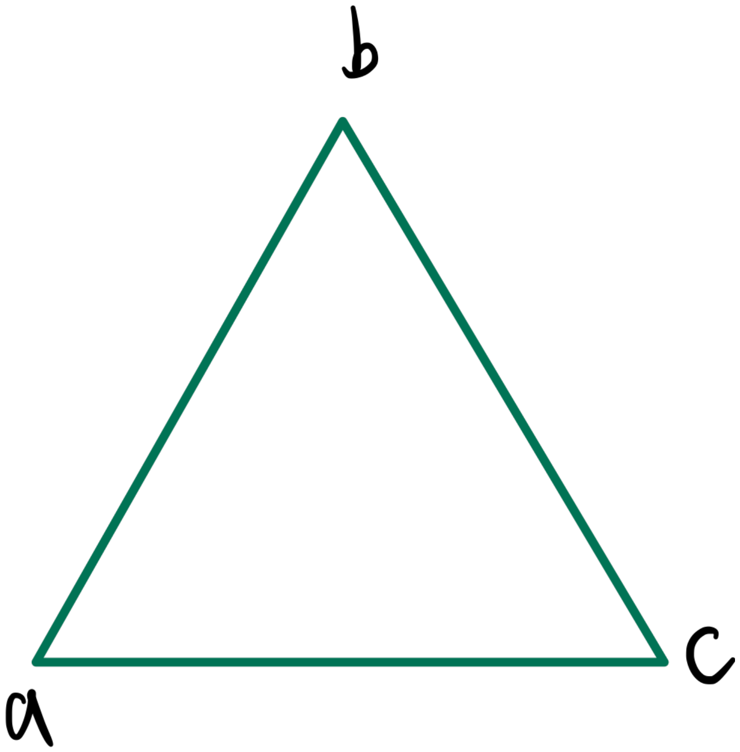
# Quotient

- Let $V$ be a vector space and $W \subset V$ be a linear subspace.
- $\dim V/W = \dim V - \dim W$
- Examples: $\mathbb{R}^3/\mathbb{R} \cong \mathbb{R}^2$, $\mathbb{R}^3/\mathbb{R}^2 \cong \mathbb{R}$

# Part 1:
# Simplicial Homology

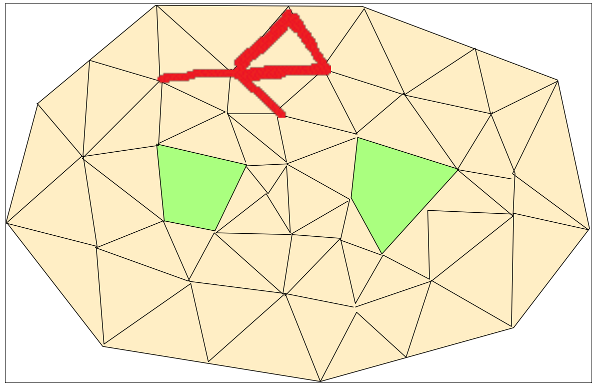
# Chains



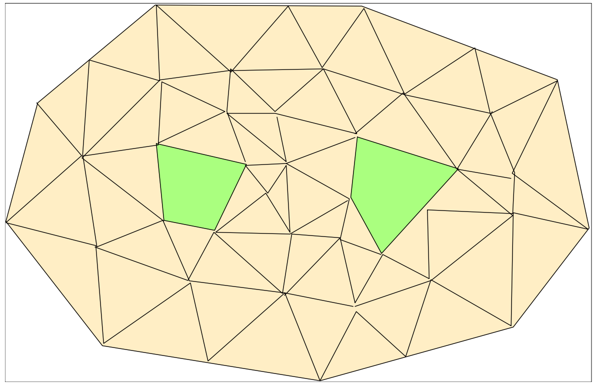
# Chains

- Given a simplicial complex $K$, a p-chain is
  - A formal sum of $p$-simplices $c = \sum c_i \sigma_i$
    - Coefficients $c_i$ come from a ring
  - In what follows, we use $\mathbb{Z}_2$ coefficients
    - i.e, $c_i \in \{0, 1\}$, equipped with *modulo-2* addition
    - thus a $p$-chain is just a **subset** of $p$-simplices!



# Chains

- Given a simplicial complex $K$, a *p-chain* is
  - A formal sum of $p$-simplices $c = \sum c_i \sigma_i$
  - Under $\mathbb{Z}_2$-coefficients: a collection of $p$-simplices
- p-th *chain group* of $K$
  - $C_p(K)$: collection of $p$-chains with operation $+$
    - $c = \sum c_i \sigma_i$, and $c' = \sum c'_i \sigma_i$, then
    - $c + c' = \sum c_i \sigma_i + \sum c'_i \sigma_i = \sum [(c_i + c'_i) \ mod\ 2]\ \sigma_i$



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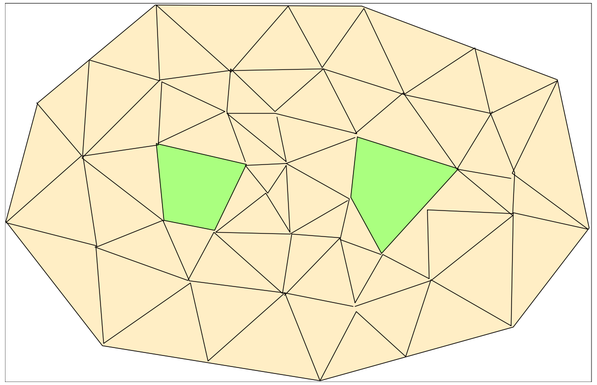

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- Remark: when coefficients comes from $\mathbb{Z}_2$, the chain group $C_p(K)$ is a vector space with basis $\{p - \text{simplices}\ \sigma \in K\}$
  - $\dim C_p(K) = n_p$ (i.e., # $p$-simplices)

# Chains

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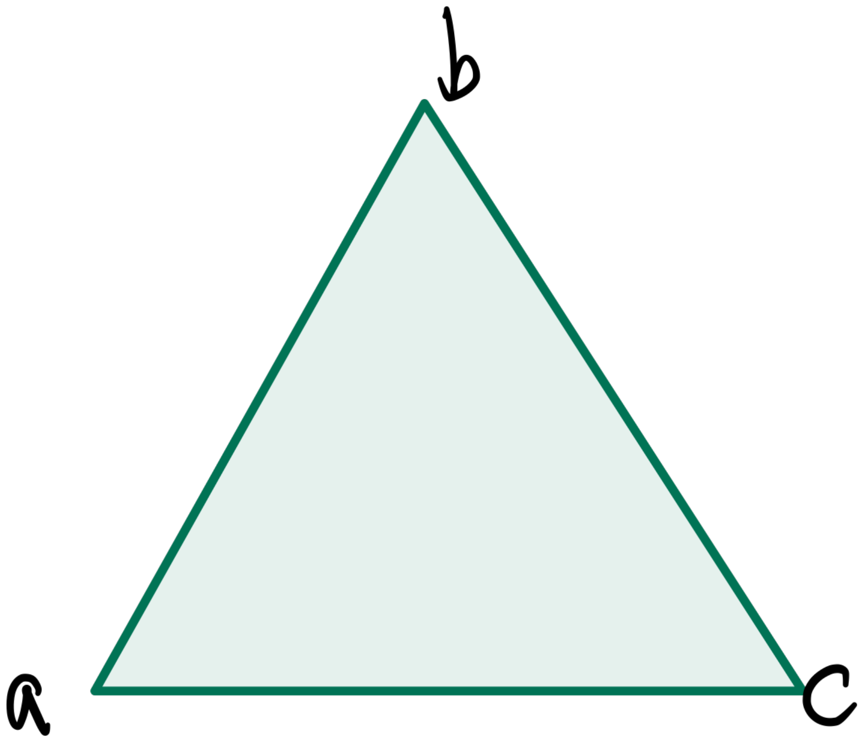
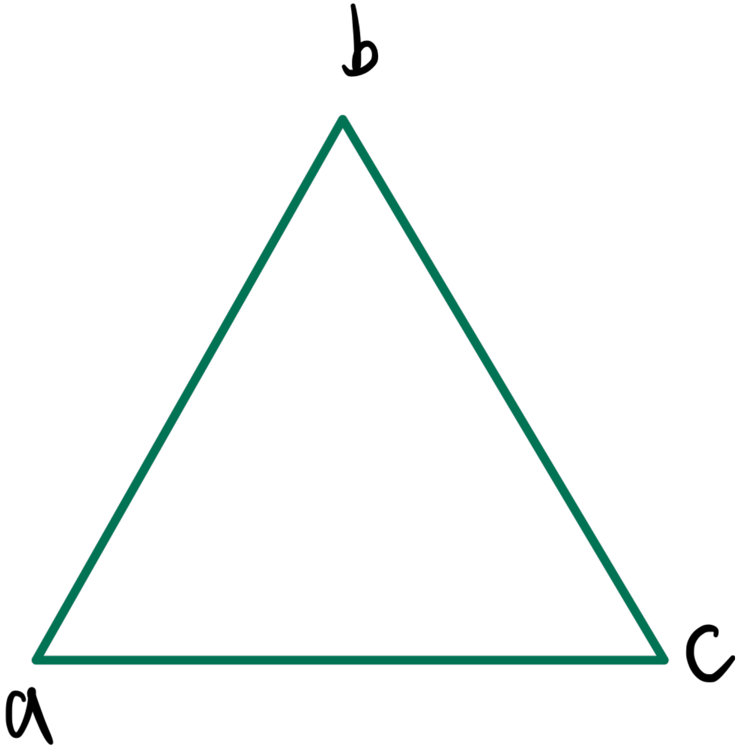
- $C_0(K),\ C_1(K),\ \ldots C_n(K),\ \ldots$
  - Boundary operators to connect them!

# Boundary operator



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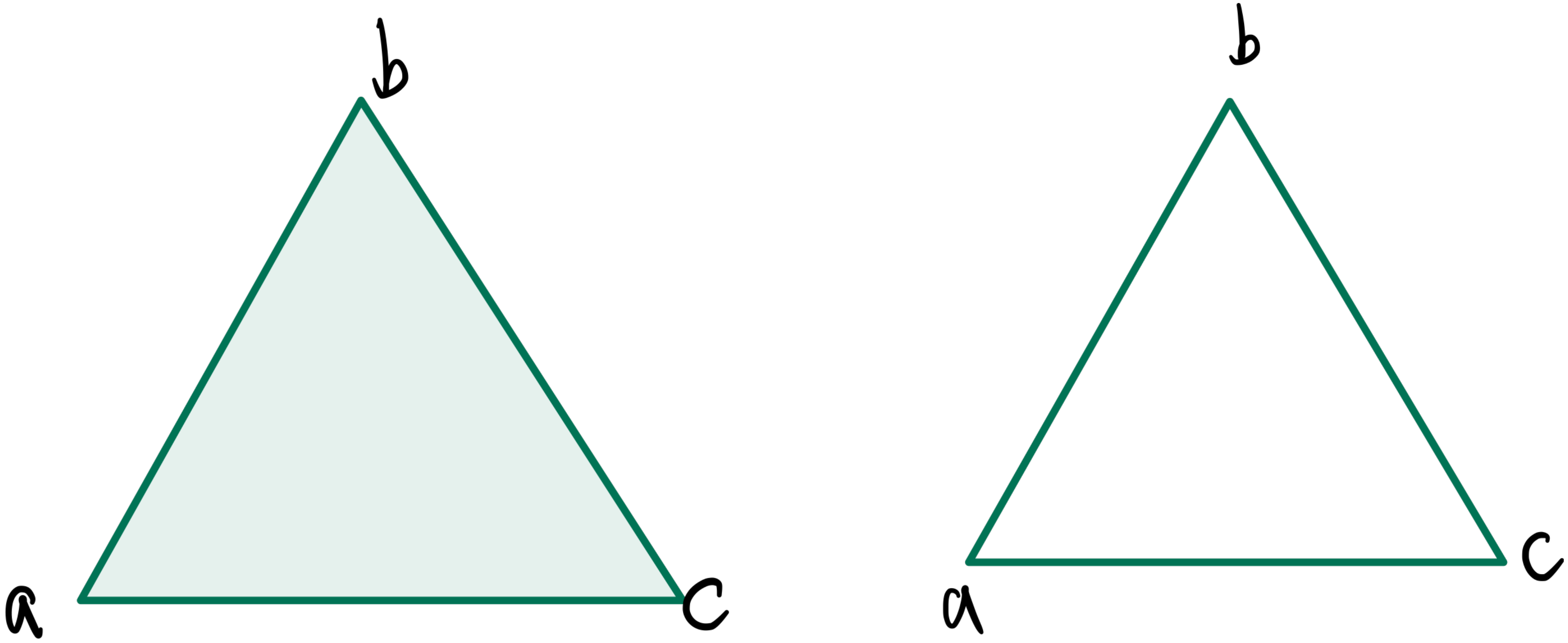


$$\partial\{a,b,c\} = \{a,b\} + \{a,c\} + \{b,c\}$$

# Boundary operator

- $p$-th boundary operator (*a linear map*) $\partial_p : C_p \to C_{p-1}$
  - For a simplex $\sigma = \{v_0, \ldots, v_p\}$
    - $\partial_p(\sigma) = \sum_{i=0}^{p} \{v_0, \ldots, \hat{v}_i, \ldots, v_p\}$
    - $\partial_p(\sigma) =$ set of $(p-1)$-faces of $\sigma$
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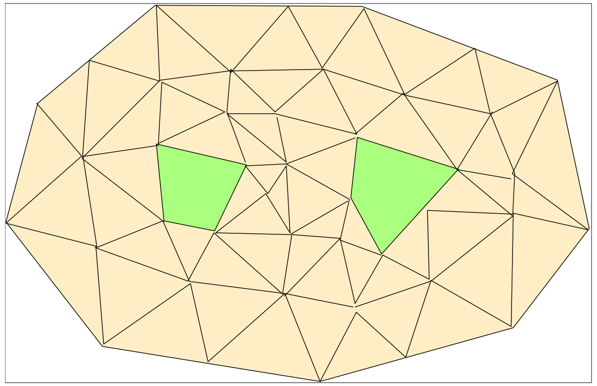

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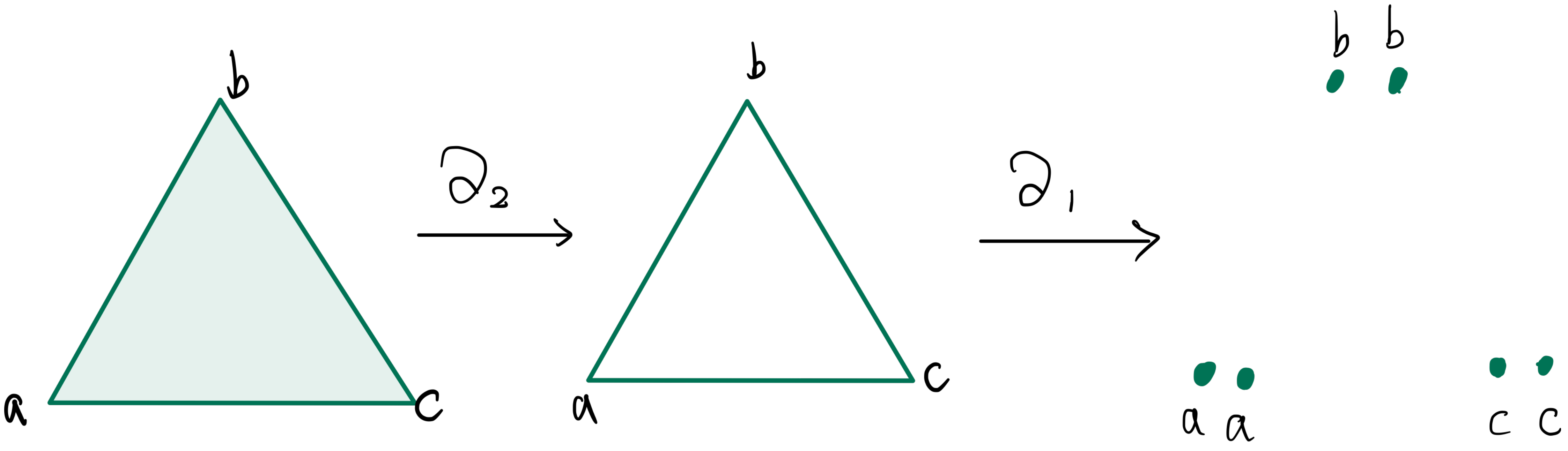
  - $c = \sum c_i \sigma_i \quad \Rightarrow \quad \partial_p(c) = \sum c_i \partial_p(\sigma_i)$
- Chain complex:
  - a sequence of vector spaces connected by linear maps

$$\cdots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \cdots .$$

# Boundary operator



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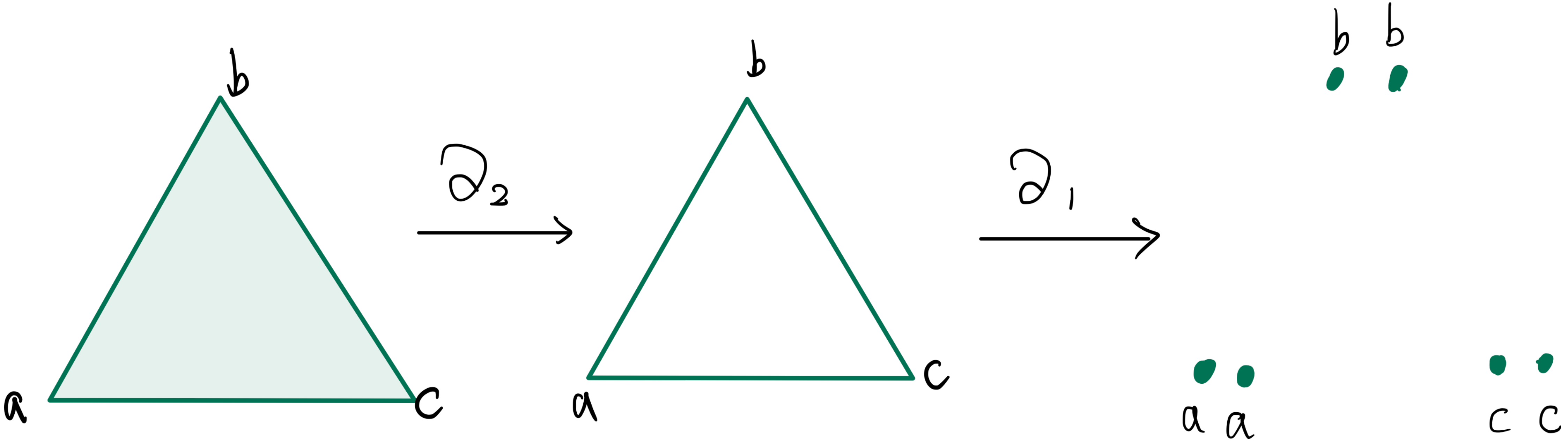


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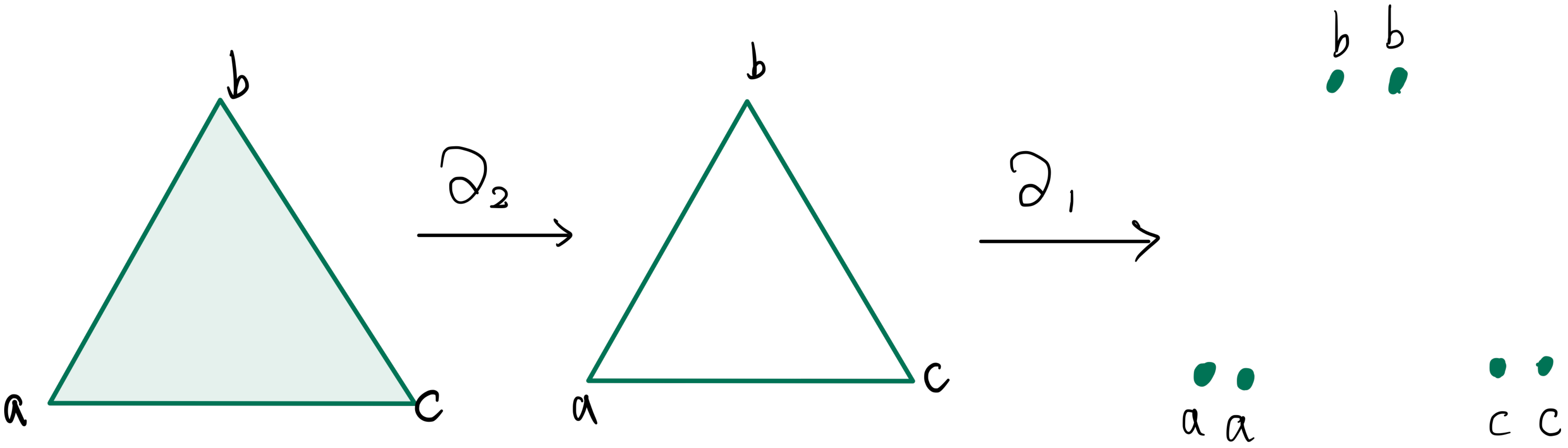


$$\partial\{a,b,c\} = \{a,b\} + \{a,c\} + \{b,c\}$$

$$\partial\partial\{a,b,c\} = \partial(\{a,b\} + \{a,c\} + \{b,c\}) = 2a + 2b + 2c = 0$$

# Boundary operator

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    - $\partial_p(\sigma) = \sum_{i=0}^{p} \{v_0, \ldots, \hat{v}_i, \ldots, v_p\}$
    - $\partial_p(\sigma) =$ set
  - $c = \sum c_i \sigma_i \quad \Rightarrow$

Theorem (Fundamental Boundary Property):
$$\partial_p \circ \partial_{p+1} = 0$$

- Chain complex:
  - a sequence of vector spaces connected by linear maps

$$\ldots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \ldots .$$

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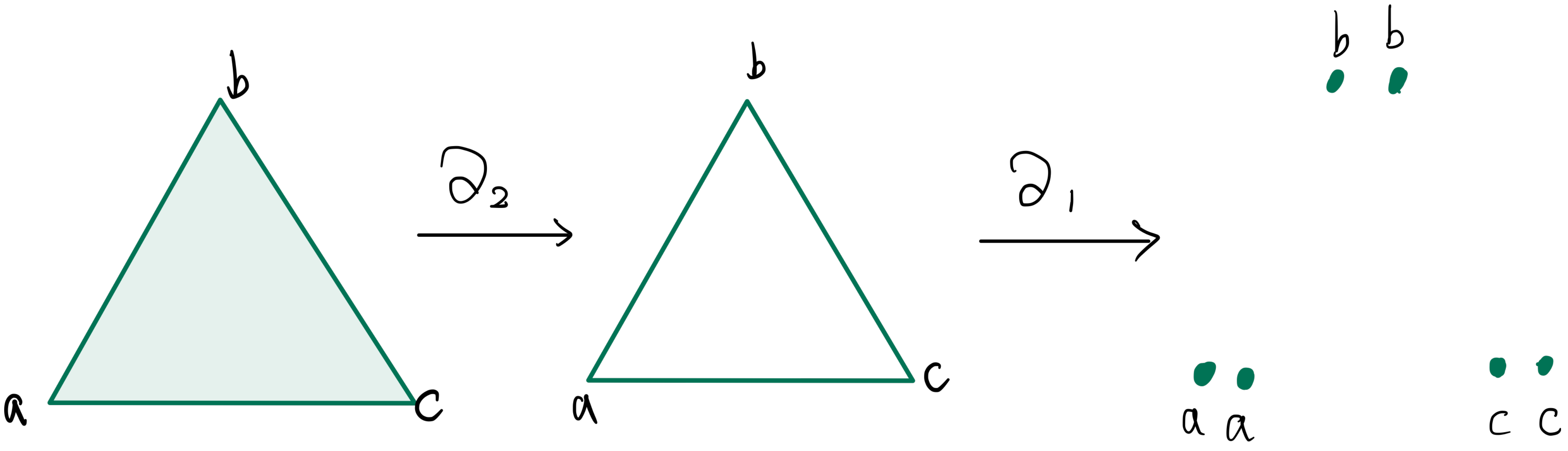

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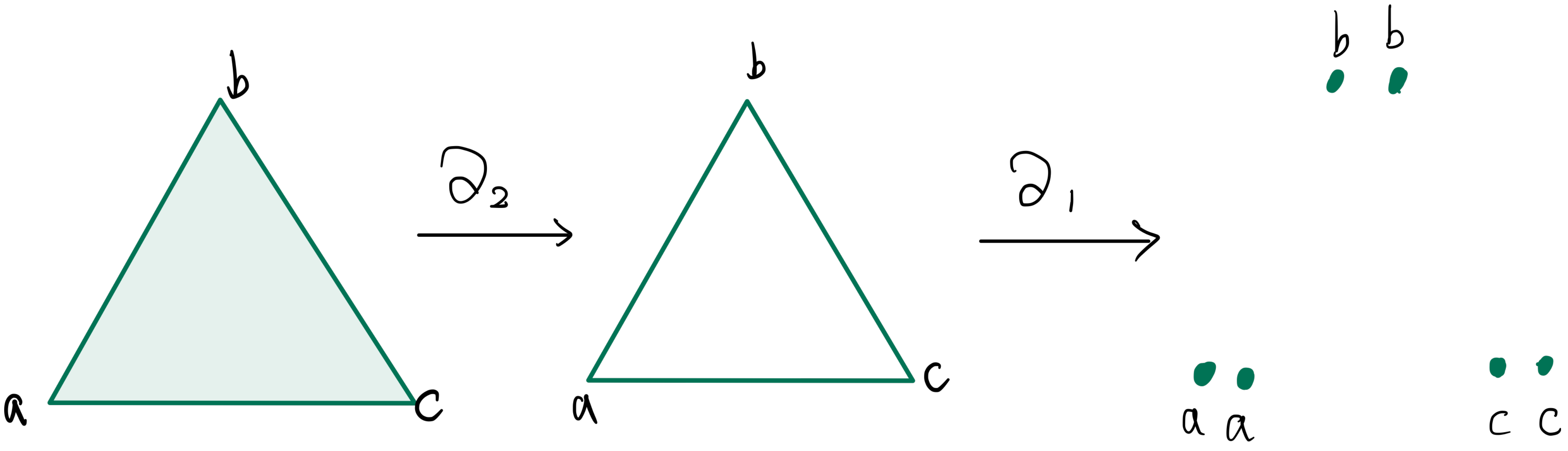


$$\partial\{a, b, c\} = \{a, b\} + \{a, c\} + \{b, c\}$$

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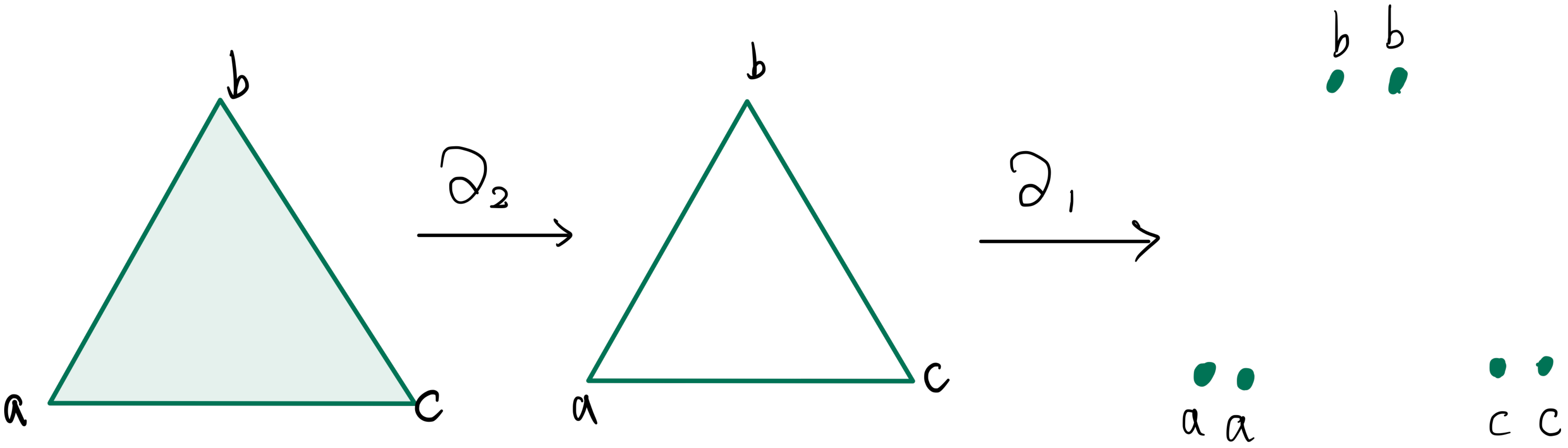


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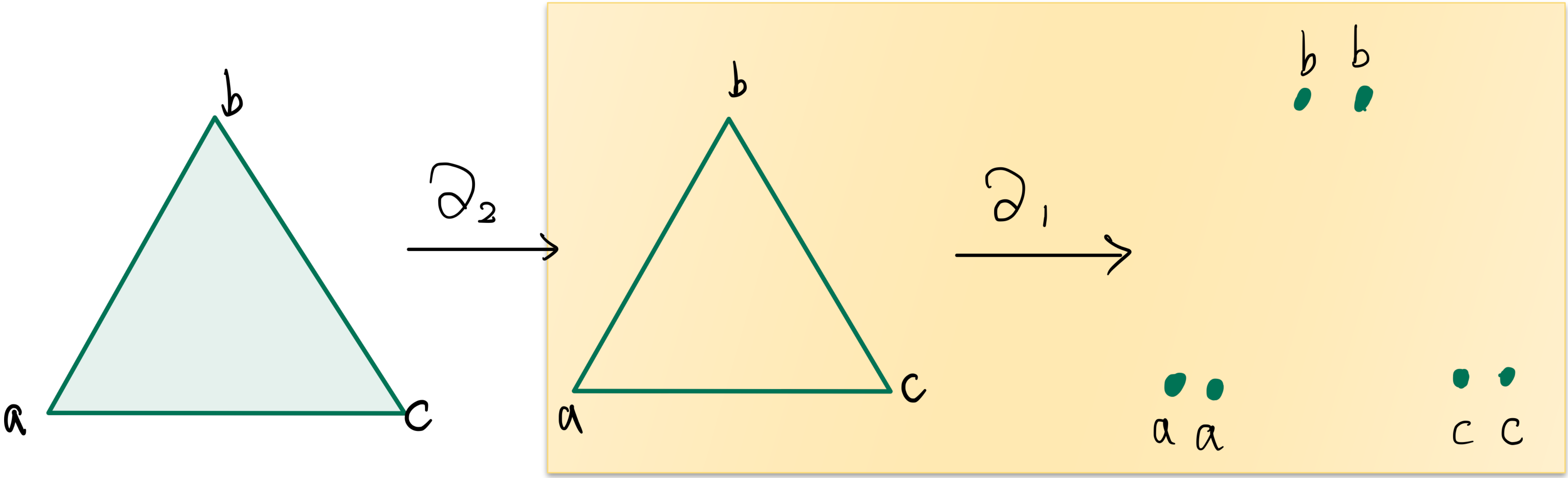


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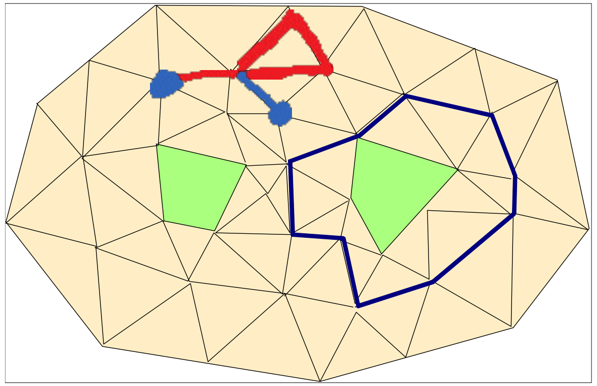
# Cycles and Boundaries

- Cycles:
  - $p$-cycle: a $p$-chain whose boundary is 0
  - $p$-th cycle group $Z_p(K) = \ker(\partial_p)$
  - $Z_p$ is a subgroup of $C_p$, denoted by $Z_p \subseteq C_p$



# Cycles

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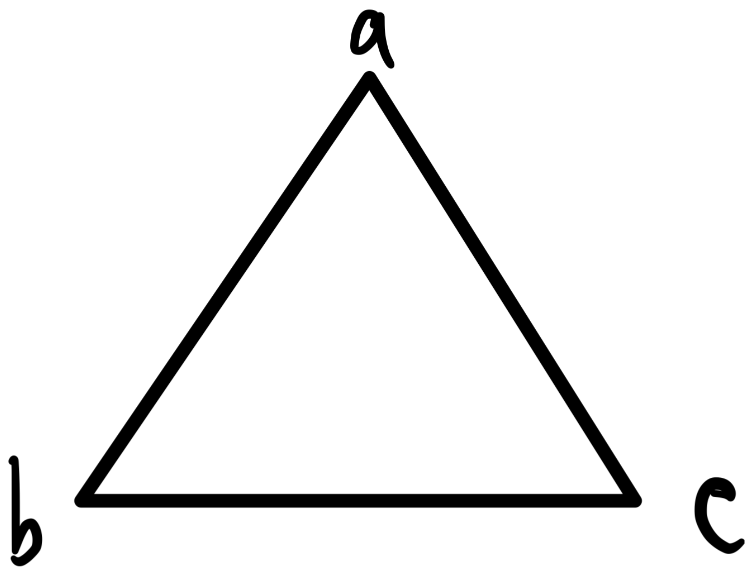
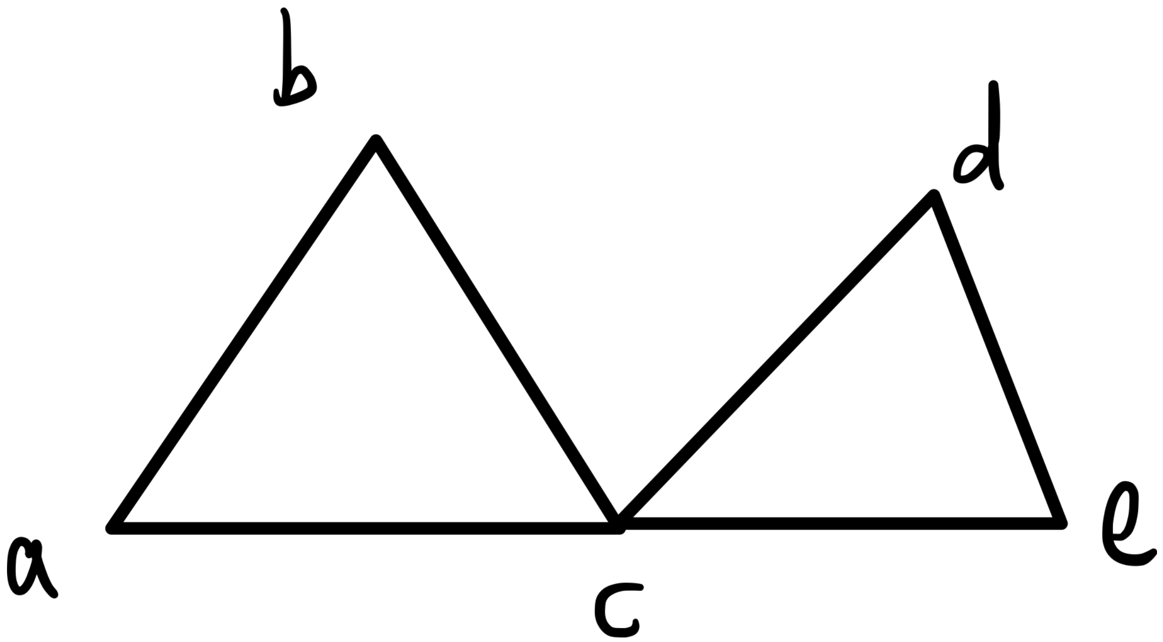



$$Z_1(K) = \langle \{a, b\} + \{b, c\} + \{a, c\} \rangle$$

$$\dim Z_1(K) = 1$$

# Cycles

- Cycles:
  - $p$-cycle: a $p$-chain whose boundary is 0
  - $p$-th cycle group $Z_p(K) = \ker(\partial_p)$

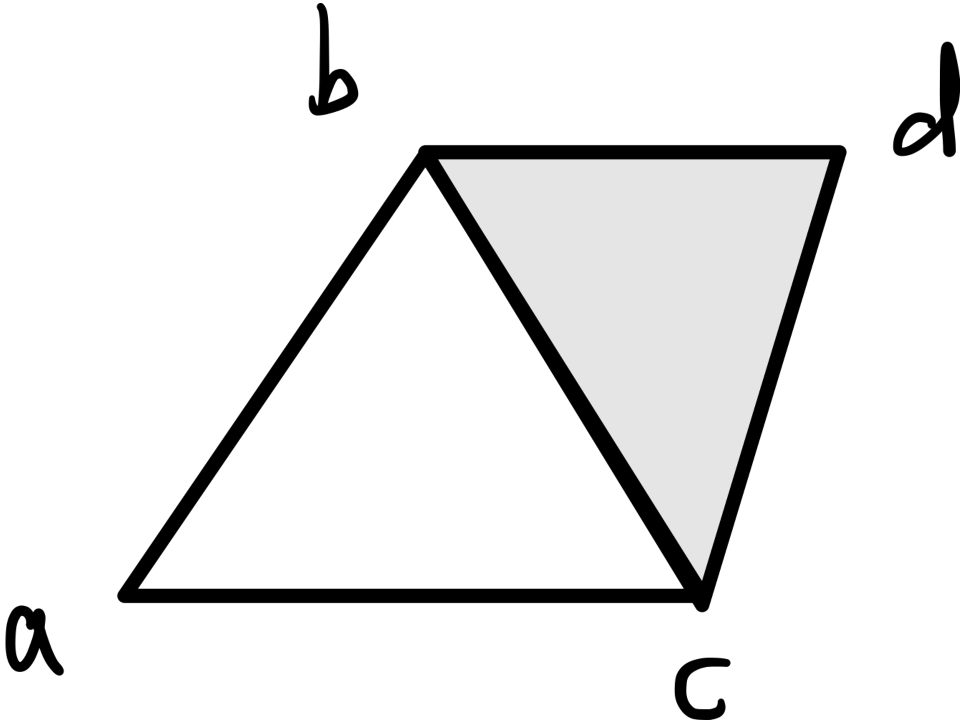



$$Z_1(K) = \langle \{a,b\} + \{b,c\} + \{a,c\}, \{c,d\} + \{d,e\} + \{c,e\} \rangle$$

$$\dim Z_1(K) = 2$$

# Cycles

- Cycles:
  - $p$-cycle: a $p$-chain whose boundary is 0
  - $p$-th cycle group $Z_p(K) = \ker(\partial_p)$

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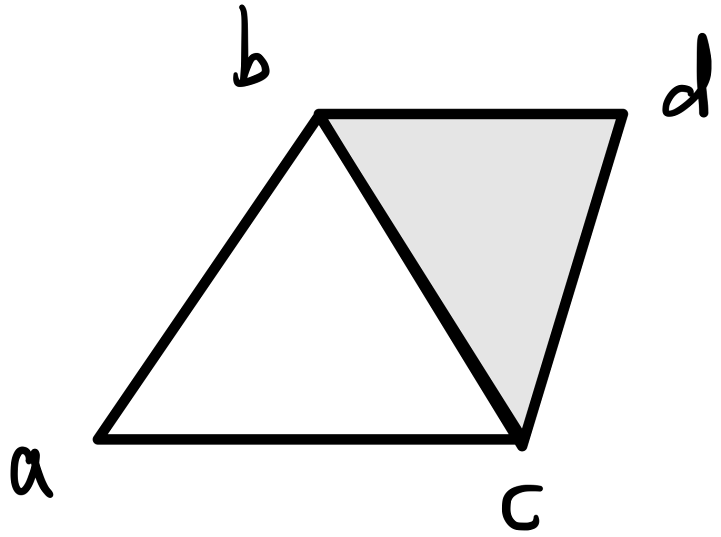
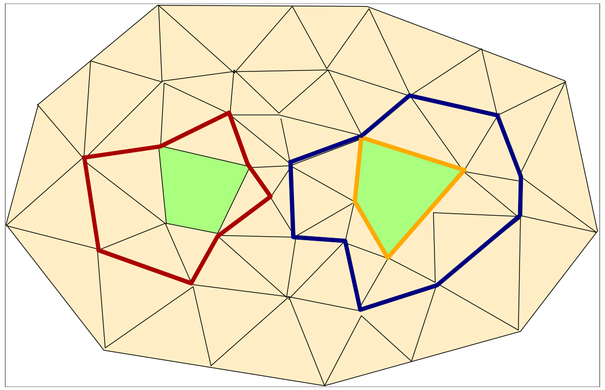
# Cycles




$$(\{a, b\} + \{b, c\} + \{a, c\}) - (\{a, b\} + \{b, d\} + \{c, d\} + \{a, c\}) = \{b, c\} + \{b, d\} + \{c, d\}$$

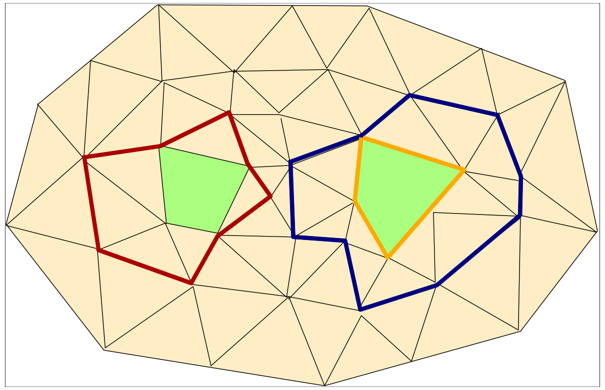
$$\partial_2\{b, c, d\} = \{b, c\} + \{b, d\} + \{c, d\}$$

# Cycles and Boundaries



# Cycles and Boundaries

- Cycles:
  - $p$-cycle: a $p$-chain whose boundary is 0
  - $p$-th cycle group $Z_p(K) = \ker(\partial_p)$
- Boundary cycles:
  - $p$-boundary: a $p$-cycle which is the boundary of some $(p+1)$-chain
  - $p$-th boundary group $B_p(K) = \mathrm{Im}(\partial_{p+1})$
- $\partial_p \circ \partial_{p+1} = 0 \Rightarrow B_p \subseteq Z_p \subseteq C_P$



# Cycles and Boundaries

- Cycles:
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  - $p$-th cycle group $Z_p(K) = \ker(\partial_p)$
- Boundary cycles:
  - $p$-boundary: a $p$-cycle which is the boundary of some $(p+1)$-chain
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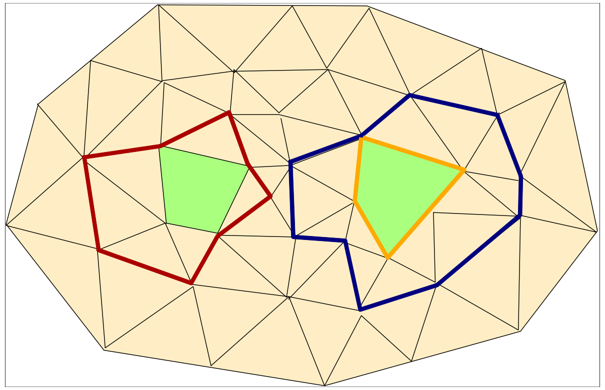
- $\partial_p \circ \partial_{p+1} = 0 \Rightarrow B_p \subseteq Z_p \subseteq C_P$

Under $\mathbb{Z}_2$ coefficients, $B_p,\ Z_p,\ C_p$ are all vector spaces.



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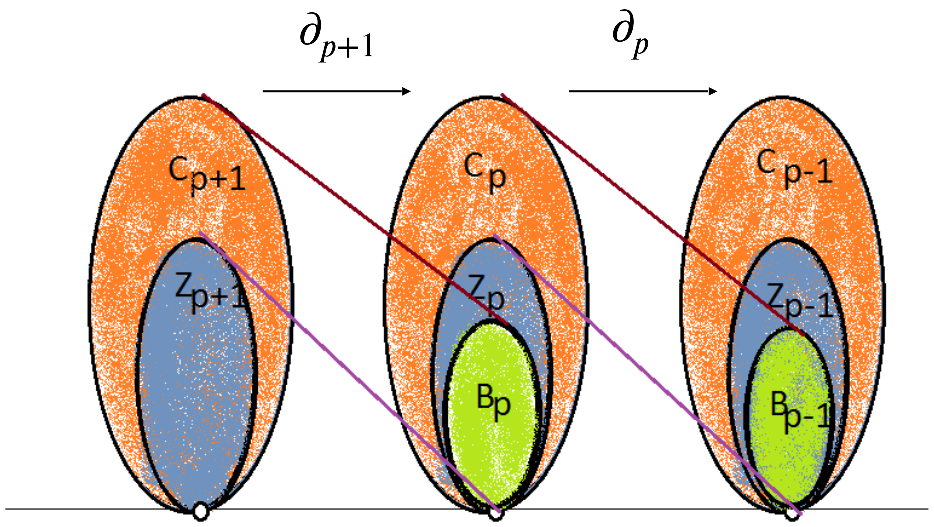
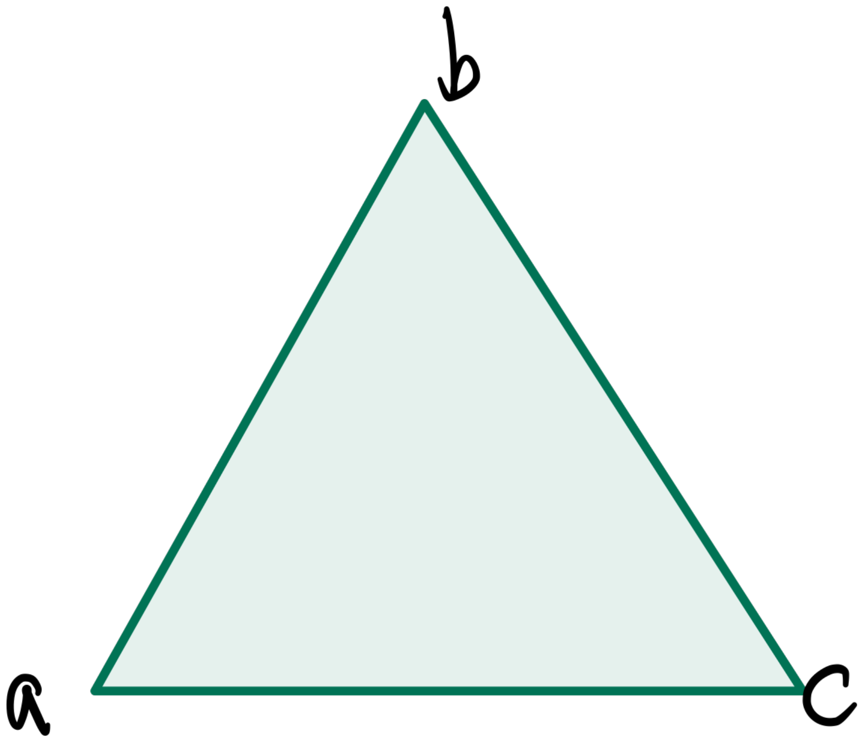
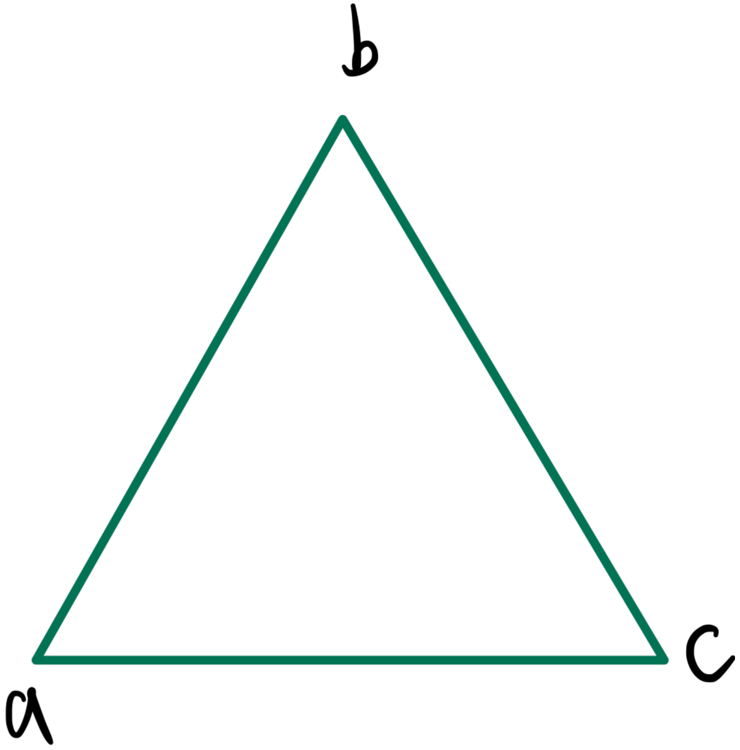
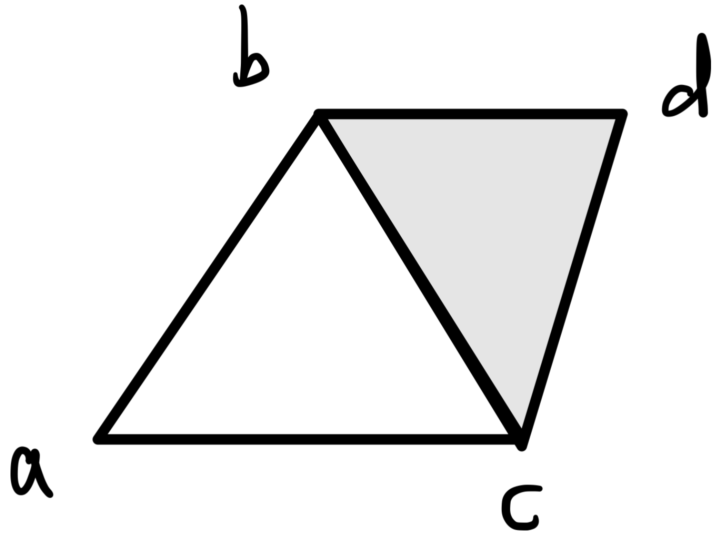


# Cycles and Boundaries



# Cycles and Boundaries




$$(\{a,b\}+\{b,c\}+\{a,c\}) - (\{a,b\}+\{b,d\}+\{c,d\}+\{a,c\}) = \{b,c\}+\{b,d\}+\{c,d\}$$

$$\partial_2\{b,c,d\} = \{b,c\}+\{b,d\}+\{c,d\}$$

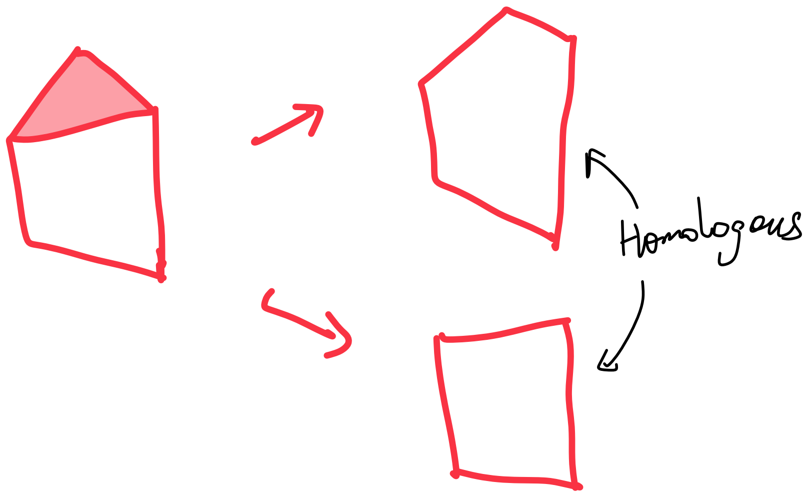
$$\{a,b\}+\{b,c\}+\{a,c\} \sim \{a,b\}+\{b,d\}+\{c,d\}+\{a,c\}$$

# Homology groups

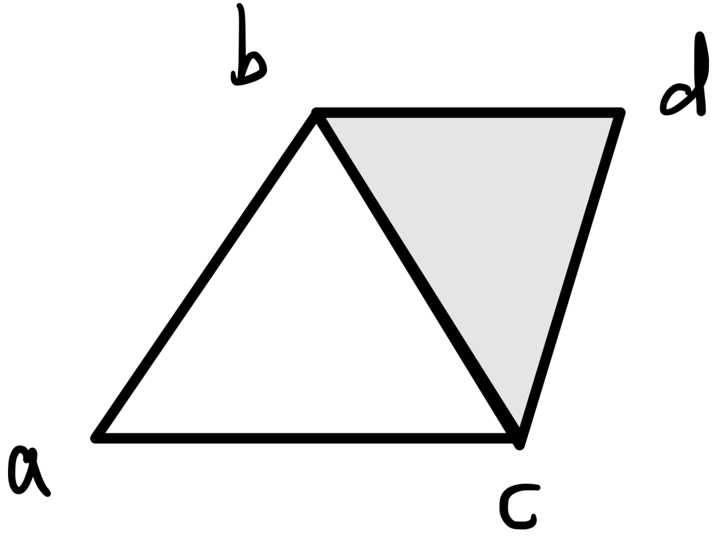
- $p$-th cycle group $Z_p(K) = \ker(\partial_p)$
- $p$-th boundary group $B_p(K) = \mathrm{Im}(\partial_{p+1})$

Under $\mathbb{Z}_2$ coefficients, $C_p$, $B_p$, $Z_p$, $H_p$ are all vector spaces.

- $p$-th *homology group*
  - $H_p(K) = Z_p / B_p$
  - $c_1$ is *homologous to* $c_2$ if
    - $c_1 + c_2 \in B_p$, i.e, $c_1 + c_2$ is a boundary cycle
  - $h = [c] \in H_p$ :
    - the family $p$-cycles homologous to $c$
    - called a *homology class*
  - A cycle is null-homologuous if it is a boundary, and we also say its homology class is trivial.



# Homology



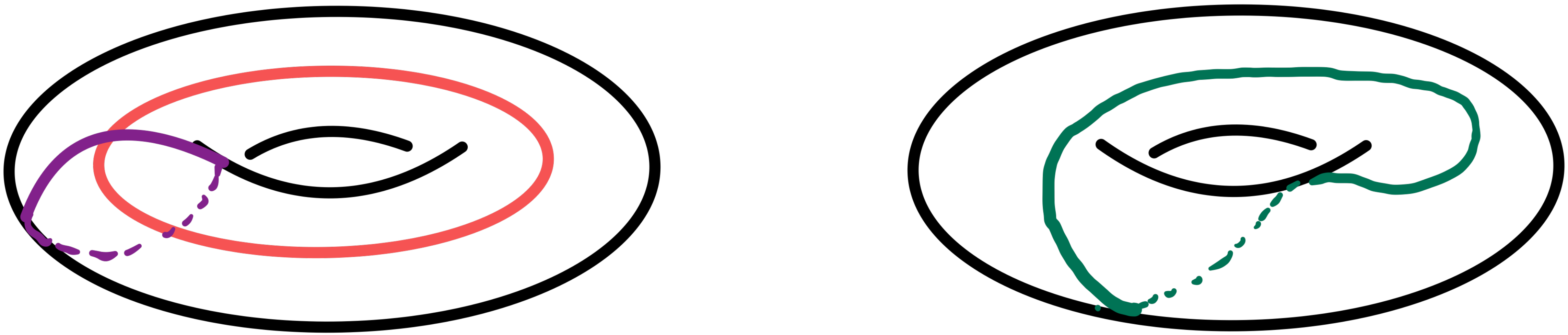

$$Z_1(K) = \langle \{a, b\} + \{b, c\} + \{a, c\}, \{a, b\} + \{b, d\} + \{c, d\} + \{a, c\} \rangle$$

$$B_1(K) = \langle \{b, c\} + \{b, d\} + \{c, d\} \rangle$$

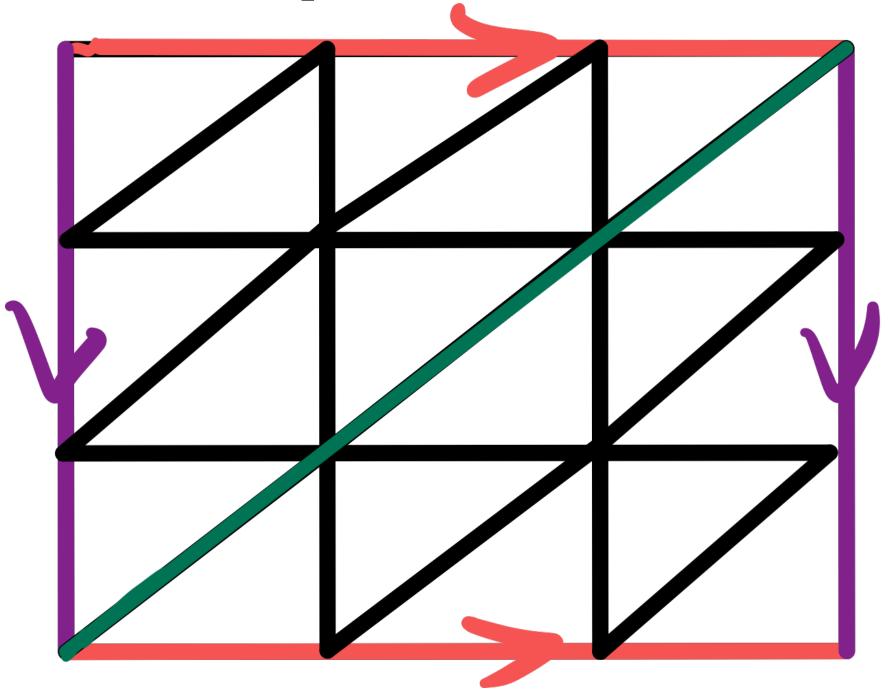
$$H_1(K) = \langle [\{a, b\} + \{b, c\} + \{a, c\}] \rangle$$

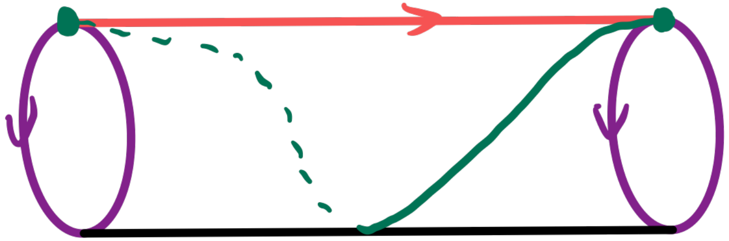
$$\dim H_1(K) = 1$$

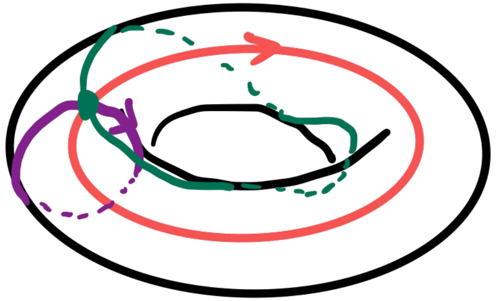
# Torus example

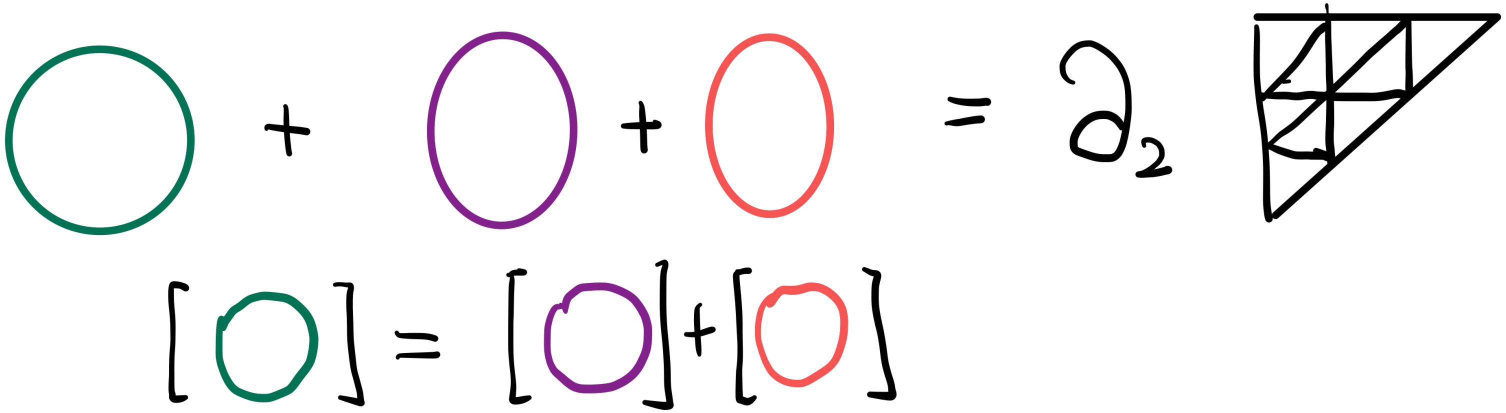


# Torus example









# Homology is homotopy invariant

- If $X$ and $Y$ are homotopy equivalent, then
- $H_n(X) \cong H_n(Y)$

- Hence one can define the homology groups of a manifold through any triangulation
    - Examples:
        - a point
        - a circle
        - a sphere

## Examples

- $H_0(K) \cong \mathbb{Z}_2^k$ where $k$ is the number of connected components
- $H_n(\mathbb{S}^n) = \mathbb{Z}_2$ and $H_m(\mathbb{S}^n) = 0$ for $m \neq 0,n$

# Betti numbers

- Betti number: $\beta_p(K) = \dim H_p(K)$
- Theorem:
  - $\beta_p(K) = \dim Z_p - \dim B_p$
- Examples:

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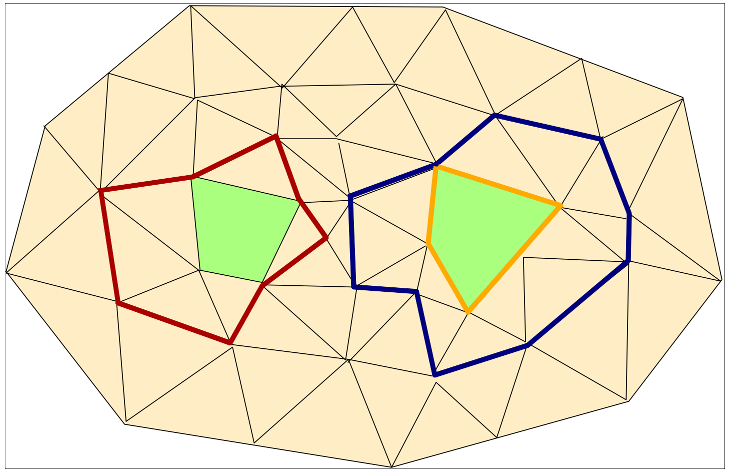

$\beta_0(K) = ?\quad \beta_1(K) = ?$

# Betti numbers are homotopy invariants

- Fact:
  - Two homotopy equivalent topological spaces have isomorphic homology groups and thus same Betti numbers.

- Sometimes in practice, one only cares about Betti numbers instead of explicit structures (bases) of homology groups

# Another definition for Euler characteristic

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- Given a Simplicial complex $K$
  - Recall its Euler characteristic $\chi(K) = \sum_{p=0} (-1)^p n_p(K)$

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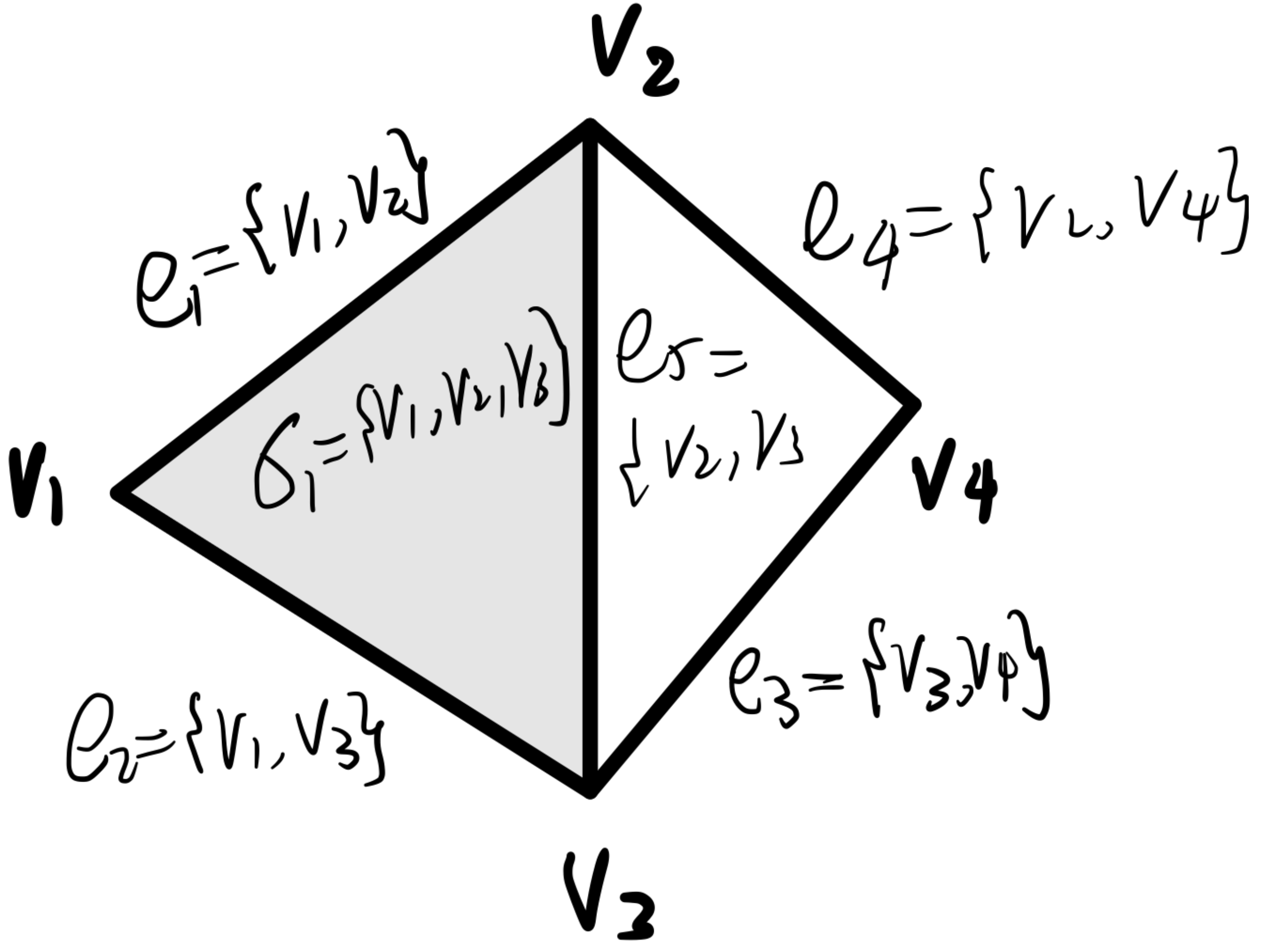
- Theorem (Euler-Poincaré formula)
  - Given a simplicial complexes $K$, one has that
    $$\chi(K) = \sum_{i=0} (-1)^i \beta_i(K)$$

# Part 2:
# Matrix view and computation

# Calculation of Homology



# Boundary Matrix

- $K^p = \left\{\alpha_1, \ldots, \alpha_{n_p}\right\}, \ K^{p-1} = \left\{\tau_1, \ldots, \tau_{n_{p-1}}\right\}$
  - $K^p$ forms a basis for p-th chain group $C_p$
- $\mathrm{n}_{p-1} \times n_p$ boundary matrix $A_p$
  - $A_p[i]\left[j\right] = 1$ iff $\tau_i \subseteq \sigma_j$
  - representing $\partial_p : C_p \to C_{p-1}$ w.r.t. basis $\left\{\alpha_1, \ldots, \alpha_{n_p}\right\}$ and $\left\{\tau_1, \ldots, \tau_{n_{p-1}}\right\}$

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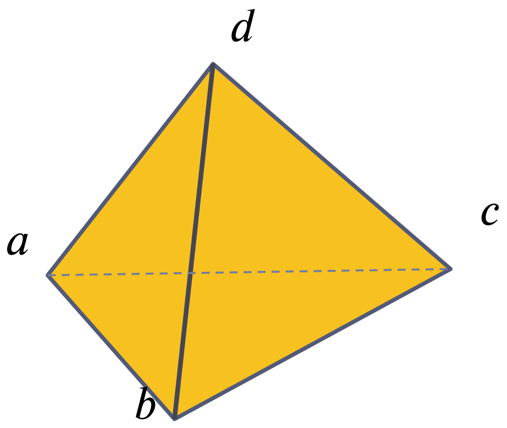


$$A_2 = \begin{array}{c} \\ ab \\ ac \\ ad \\ bc \\ bd \\ cd \end{array} \begin{array}{c} \begin{array}{cccc} abc & abd & acd & bcd \end{array} \\ \begin{pmatrix} 1 & 1 & & \\ 1 & & 1 & \\ & 1 & 1 & \\ 1 & & & 1 \\ & 1 & & 1 \\ & & 1 & 1 \end{pmatrix} \end{array}$$

# Boundary matrix

# Boundary matrix

- Given a p-chain $c = \sum_{i=1}^{n_p} c_i \alpha_i$
  - Under basis $K^p$, vector representation of $c$ is
  - $\vec{c} = \left[c_1,\ c_2, \ldots,\ c_{n_p}\right]^T$
- Boundary $\partial_p c$ is a $(p-1)$-chain with vector representation $A_p \vec{c}$ w.r.t basis $K^{p-1}$

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$$A_p\vec{c} = \begin{bmatrix} a_1^1 & a_1^2 & \cdots & a_1^{n_p} \\ a_2^1 & a_2^2 & \cdots & a_2^{n_p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n_{p-1}}^1 & a_{n_{p-1}}^2 & \cdots & a_{n_{p-1}}^{n_p} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n_p} \end{bmatrix}$$

# Observations

- To compute the cycle space $Z_p = \ker \partial_p$, we simply need to solve the equation $A_p c = 0$ and find a basis for the kernel

- The boundary space $B_p = \mathrm{Im}\partial_{p+1}$ is the space generated by columns of $A_{p+1}$. We need to find a basis for these columns

- We can do both on $A_p$ through matrix reduction

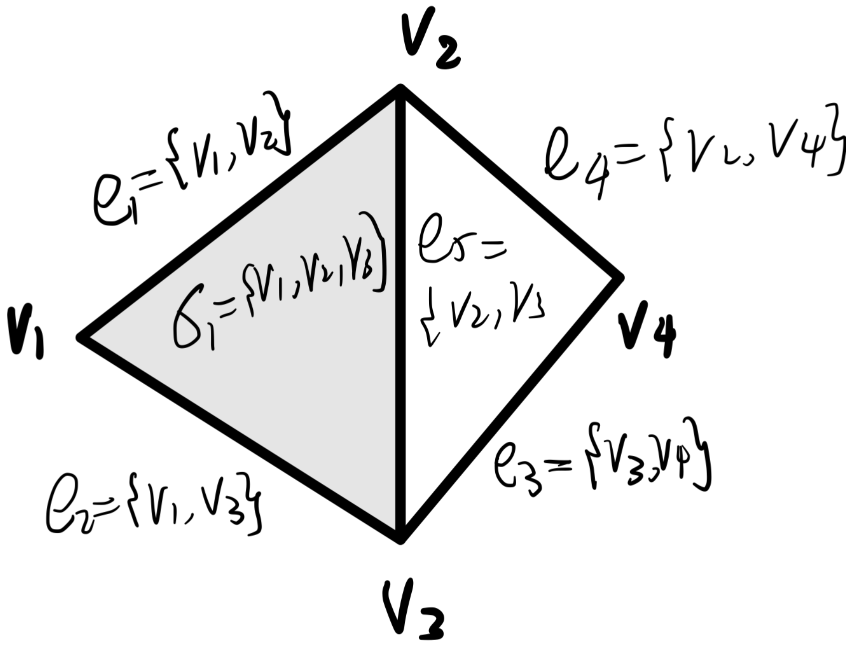
# Matrix reduction

- Turn $A_p$ into the **column reduced form**
  - Each non-zero column has a unique lowID: index of lowest 1-entry
  - Only through adding columns and switching columns
- Read of bases of $B_{p-1} = \mathrm{Im}\partial_p$ and $Z_p = \ker \partial_p$

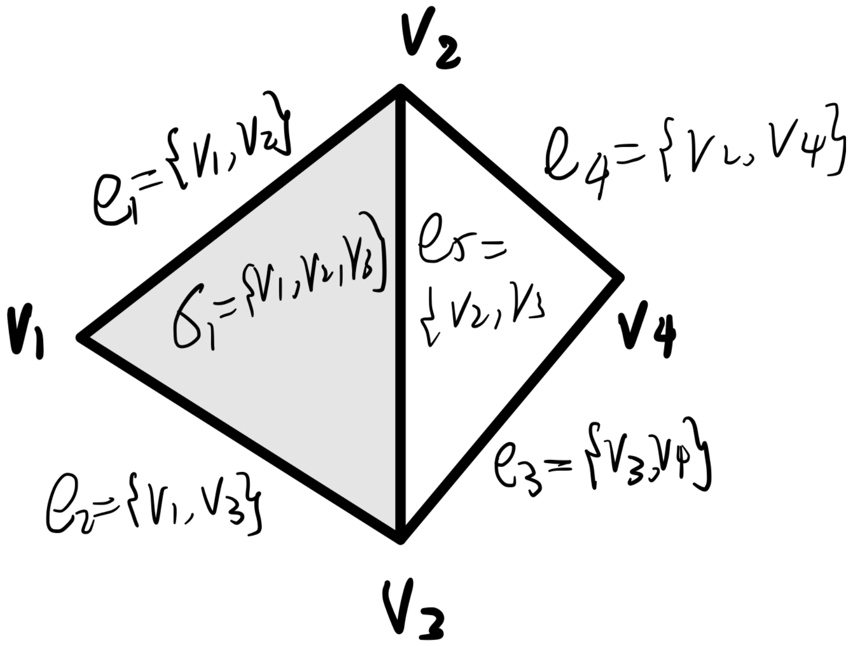
$$\begin{bmatrix} * & * & * & 0 \\ * & 1 & * & 0 \\ 1 & 0 & * & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Column reduced form

$lowId[i] \neq lowId[j]$




| | e1 | e2 | e3 | e4 | e5 |
|---|---|---|---|---|---|
| v1 | 1 | 1 | 0 | 0 | 0 |
| v2 | 1 | 0 | 0 | 1 | 1 |
| v3 | 0 | 1 | 1 | 0 | 1 |
| v4 | 0 | 0 | 1 | 1 | 0 |



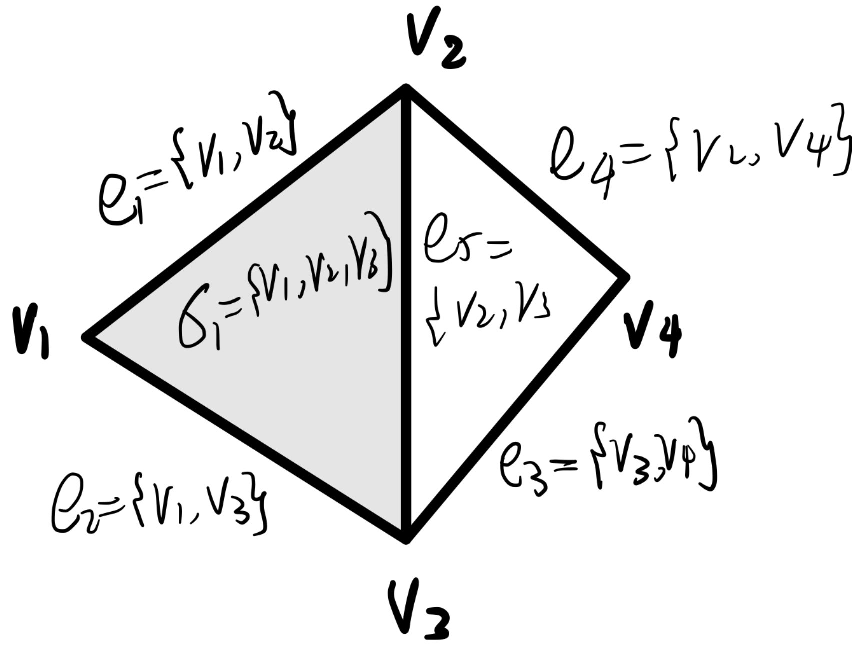

| | e1 | e2 | e3 | e4+e3 | e5 |
|---|---|---|---|---|---|
| v1 | 1 | 1 | 0 | 0 | 0 |
| v2 | 1 | 0 | 0 | 1 | 1 |
| v3 | 0 | 1 | 1 | 1 | 1 |
| v4 | 0 | 0 | 1 | 0 | 0 |

# Right-reduction algorithm

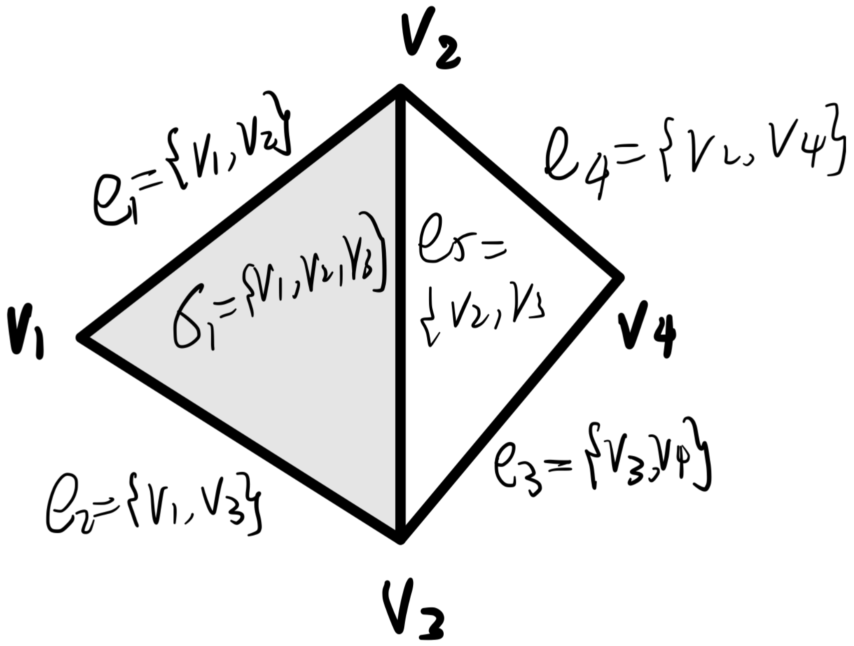
- Starting with boundary matrix $M = A_p$
  - For the $i$-th column corresponding to p-simplex $\sigma_i$,
    - associate a $p$-chain $\Gamma_i$ initialized to $\sigma_i$
  - AddColumn($j$, $i$): add column j to column i
    - $Col_M[i] = Col_M[i] + Col_M[j];\ \ \Gamma_i = \Gamma_i + \Gamma_j$

**Algorithm 1** Right-Reduction($M$)

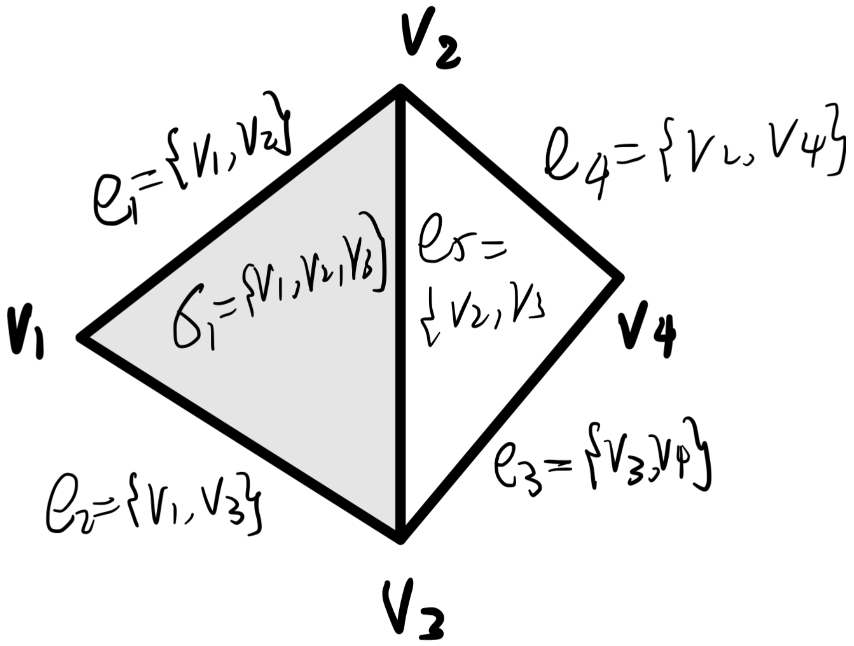
```
for i = 2 to n_p do
    while ∃j < i s.t. lowId[j] = lowId[i] do
        AddColumn(j, i);
    end while
end for
Return(M)
```



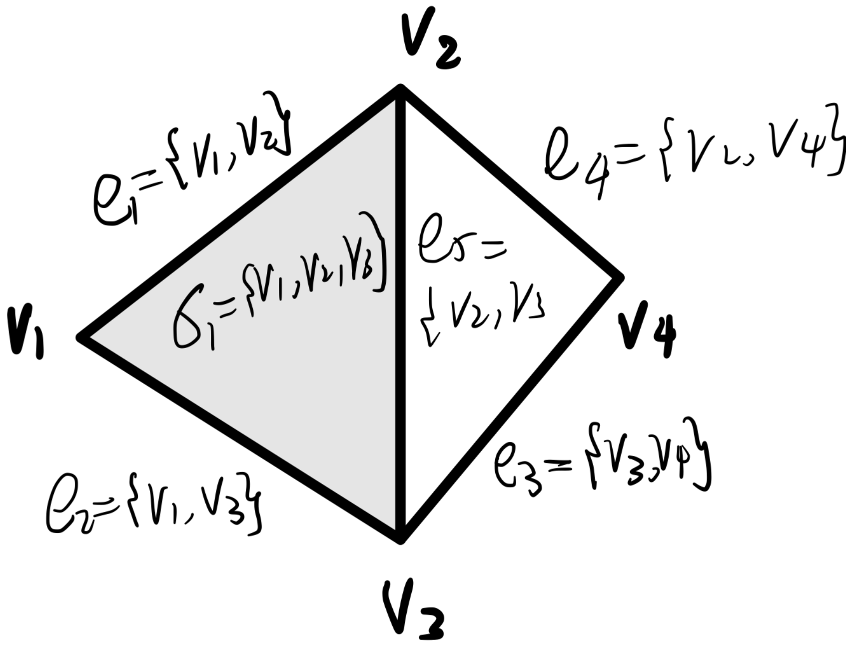

| | e1 | e2 | e3 | e4 | e5 |
|---|---|---|---|---|---|
| v1 | 1 | 1 | 0 | 0 | 0 |
| v2 | 1 | 0 | 0 | 1 | 1 |
| v3 | 0 | 1 | 1 | 0 | 1 |
| v4 | 0 | 0 | 1 | 1 | 0 |



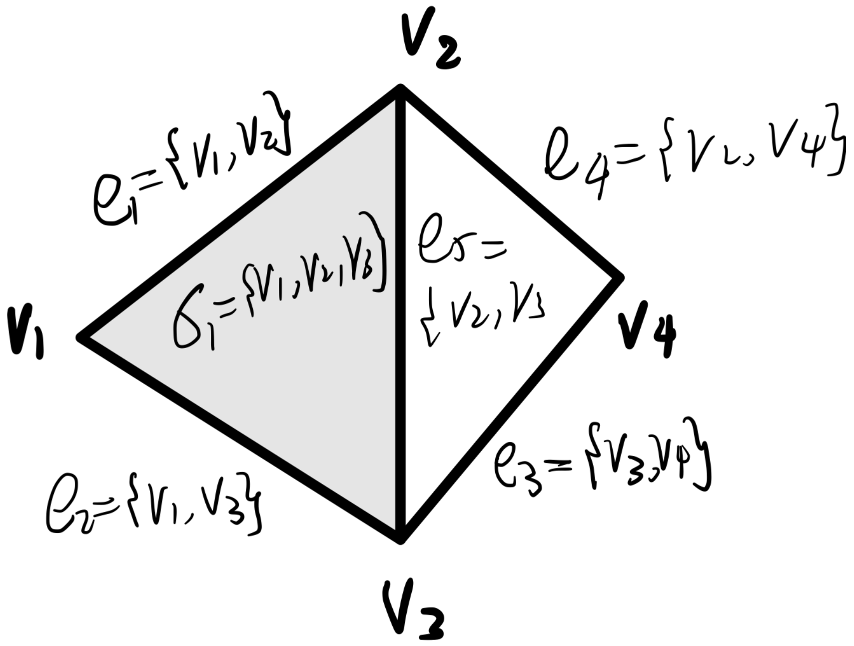

| | e1 | e2 | e3 | e4+e3 | e5 |
|---|---|---|---|---|---|
| v1 | 1 | 1 | 0 | 0 | 0 |
| v2 | 1 | 0 | 0 | 1 | 1 |
| v3 | 0 | 1 | 1 | 1 | 1 |
| v4 | 0 | 0 | 1 | 0 | 0 |



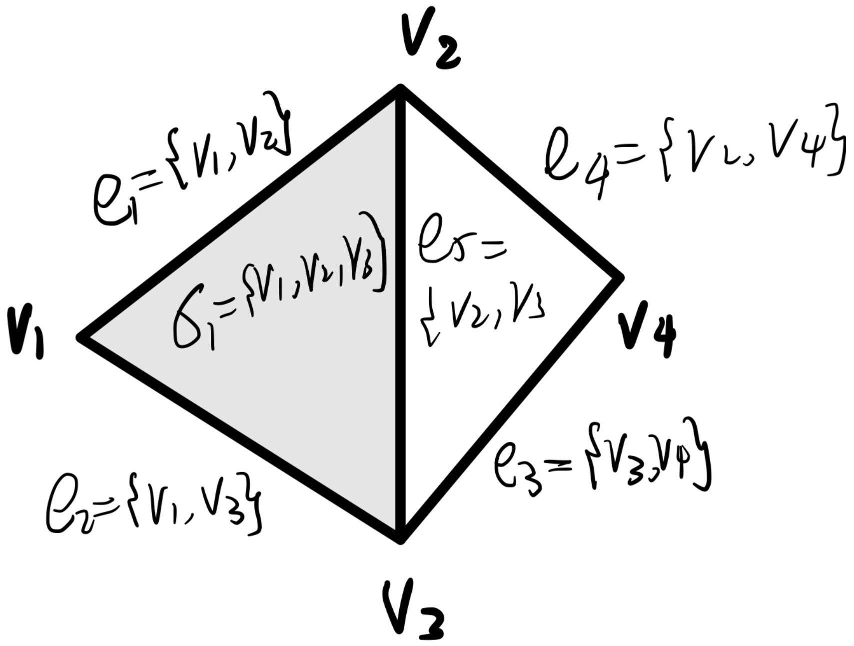

| | e1 | e2 | e3 | e4+e3+e2 | e5 |
|---|---|---|---|---|---|
| v1 | 1 | 1 | 0 | 1 | 0 |
| v2 | 1 | 0 | 0 | 1 | 1 |
| v3 | 0 | 1 | 1 | 0 | 1 |
| v4 | 0 | 0 | 1 | 0 | 0 |



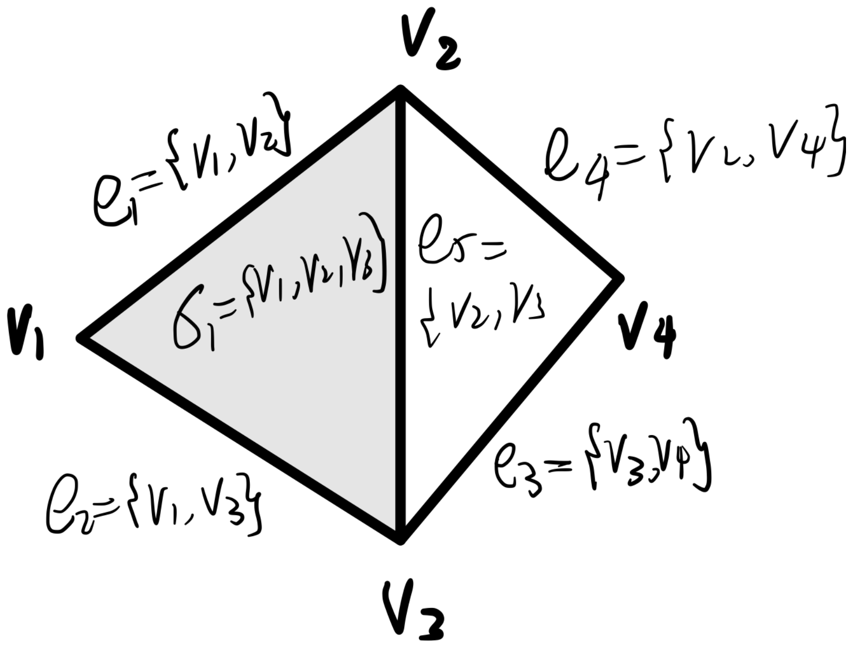

| | e1 | e2 | e3 | e4+e3+e2+e1 | e5 |
|---|---|---|---|---|---|
| v1 | 1 | 1 | 0 | 0 | 0 |
| v2 | 1 | 0 | 0 | 0 | 1 |
| v3 | 0 | 1 | 1 | 0 | 1 |
| v4 | 0 | 0 | 1 | 0 | 0 |




| | e1 | e2 | e3 | e4+e3+e2 +e1 | e5+e2 |
|---|---|---|---|---|---|
| v1 | 1 | 1 | 0 | 0 | 1 |
| v2 | 1 | 0 | 0 | 0 | 1 |
| v3 | 0 | 1 | 1 | 0 | 0 |
| v4 | 0 | 0 | 1 | 0 | 0 |




| | e1 | e2 | e3 | e4+e3+e2+e1 | e5+e2+e1 |
|---|---|---|---|---|---|
| v1 | 1 | 1 | 0 | 0 | 0 |
| v2 | 1 | 0 | 0 | 0 | 0 |
| v3 | 0 | 1 | 1 | 0 | 0 |
| v4 | 0 | 0 | 1 | 0 | 0 |




| | e1 | e2 | e3 | e4+e3+e2+e1 | e5+e2+e1 |
|---|---|---|---|---|---|
| v1 | 1 | 1 | 0 | 0 | 0 |
| v2 | 1 | 0 | 0 | 0 | 0 |
| v3 | 0 | 1 | 1 | 0 | 0 |
| v4 | 0 | 0 | 1 | 0 | 0 |

- $\dim B_0 = 3, \dim Z_1 = 2$, etc —> One can determine Betti numbers
  - $\beta_p = \dim Z_p - \dim B_p$
- Bases for $B_0$ and $Z_1$

# Properties

- Theorem:
  - Procedure Right-Reduction($M$) terminates in $O(n_p n_{p-1}^2)$ time
  - The output matrix $M$ is in column reduced form
  - The set of non-zero columns in $M$ form a basis for $B_{p-1}$
  - The set $\{\Gamma_i \mid col_M[i] = 0 \}$ form a basis for $Z_p$

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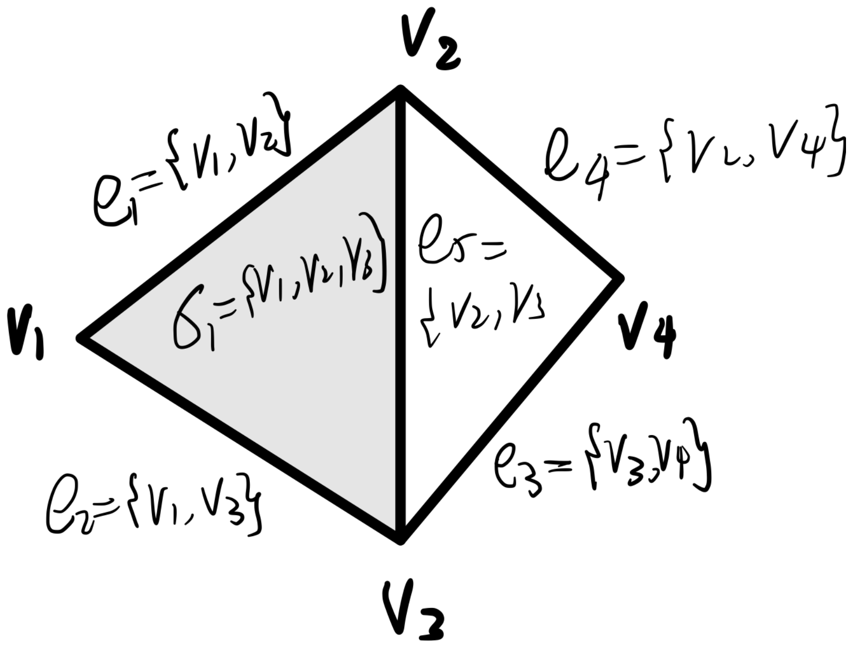
This is not the only reduction algorithm!! Any elimination via row/column additions to convert a matrix into a reduced form works!



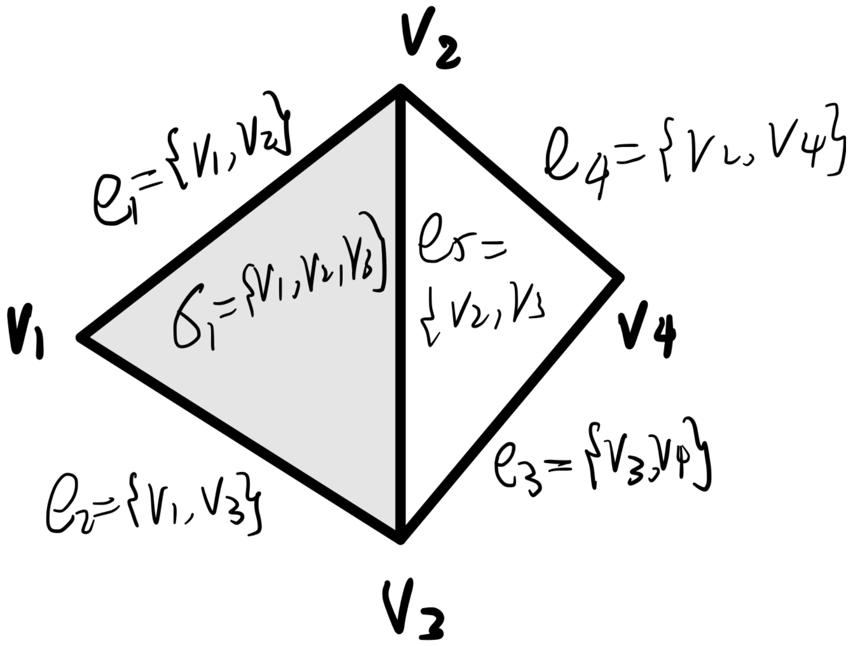

| | e1 | e2 | e3 | e4+e3+e2 +e1 | e5+e2+e1 |
|---|---|---|---|---|---|
| v1 | 1 | 1 | 0 | 0 | 0 |
| v2 | 1 | 0 | 0 | 0 | 0 |
| v3 | 0 | 1 | 1 | 0 | 0 |
| v4 | 0 | 0 | 1 | 0 | 0 |

- $\dim B_0 = 3, \dim Z_1 = 2$, etc —> One can determine Betti numbers
- Bases for $B_0$ and $Z_1$
- Can we obtain a basis for $H_1$?

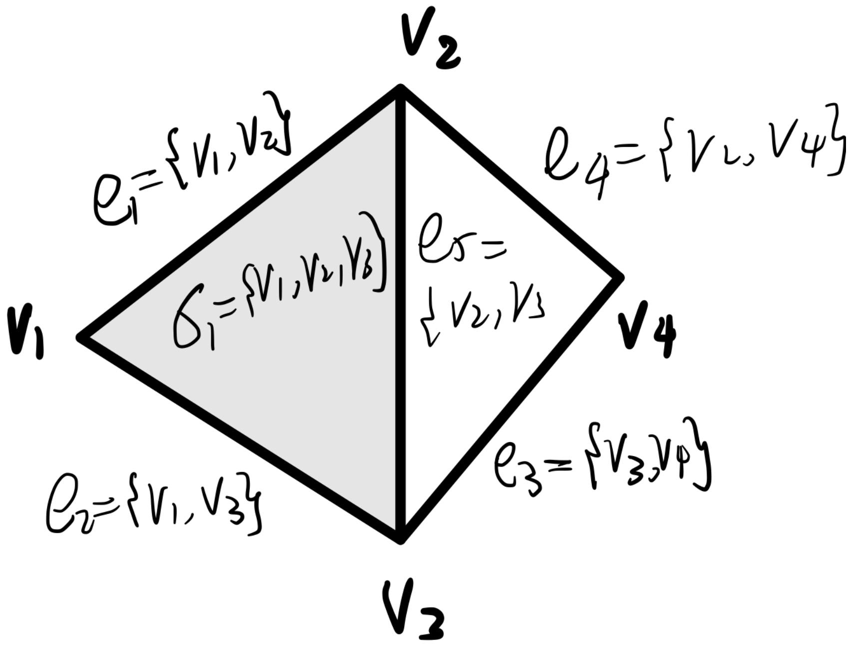
# Computing a basis for homology

$$\left[\begin{array}{c|c} \text{Basis of } B_p & \text{Basis of } Z_p \end{array}\right]$$

- Left part is already column reduced
- Apply Right Reduction to the above matrix to obtain basis of $H_p$




| | e5+e2+e1 | e4+e3+e2+e1 | e5+e2+e1 |
|---|---|---|---|
| **E1** | 1 | 1 | 1 |
| **E2** | 1 | 1 | 1 |
| **E3** | 0 | 1 | 0 |
| **E4** | 0 | 1 | 0 |
| **E5** | 1 | 0 | 1 |



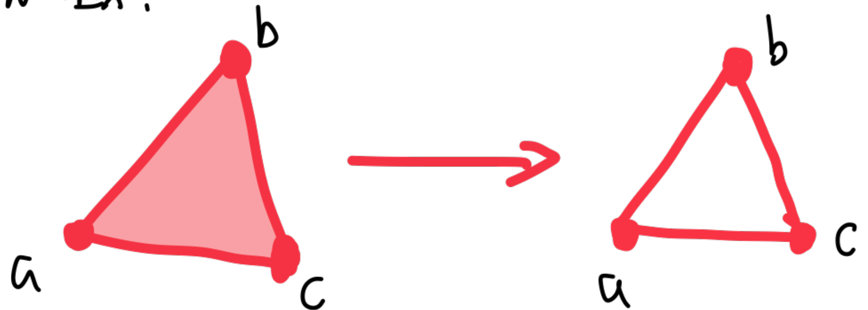

| | e5+e2+e1 | [redacted] | e4+e3+e2+e1 | 0 |
|---|---|---|---|---|
| E1 | 1 | [redacted] | 1 | 0 |
| E2 | 1 | [redacted] | 1 | 0 |
| E3 | 0 | [redacted] | 1 | 0 |
| E4 | 0 | [redacted] | 1 | 0 |
| E5 | 1 | [redacted] | 0 | 0 |

# Part 3:
# Functoriality of Homology

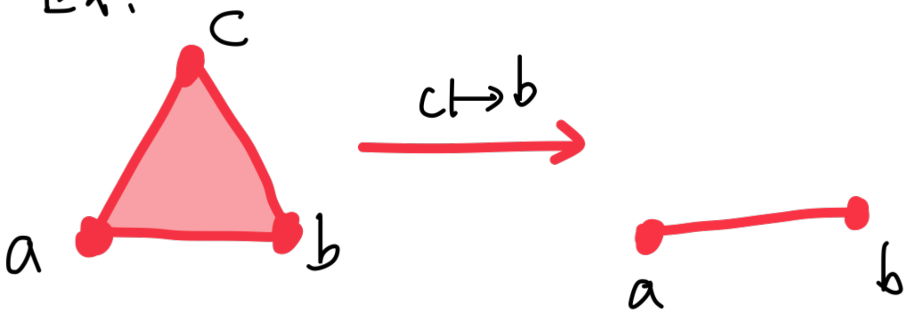
# Simplicial map

- Intuitively, analogous to continuous maps between topological spaces
- Given simplicial complexes $K$ and $L$
  - a function $f : V(K) \to V(L)$ is called a simplicial map if
    - for any $\sigma = \{p_0, \ldots, p_d\} \in \Sigma(K)$, $f(\sigma) = \left\{ f\left(p_0\right), \ \ldots, \ f\left(p_d\right) \right\}$ spans a simplex in $L$, i.e., $f(\sigma) \in \Sigma(L)$.
  - A simplicial map is also denoted $f : K \to L$

NON-EX:




EX!



# Functoriality of Simplicial Homology

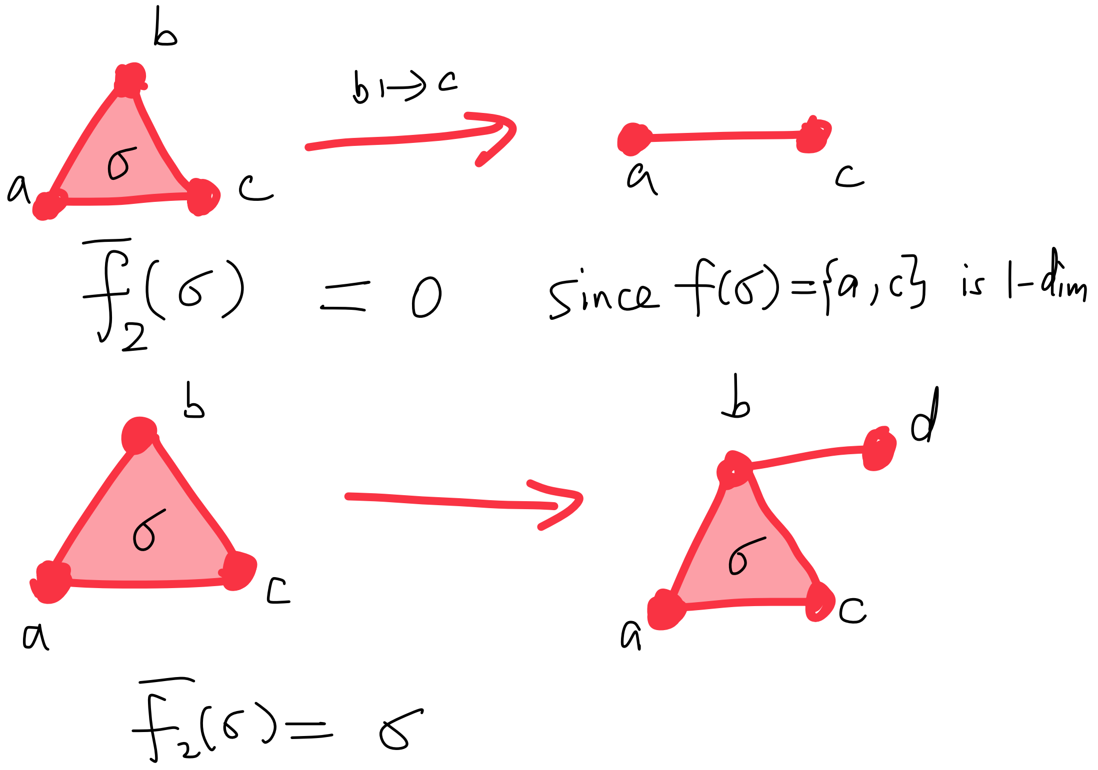
- Let $K = (V, \Sigma)$ and $K' = (V', \Sigma')$ and let $f : V \to V'$ be a simplicial map. Then,
  - $f$ induces a linear map on homology groups: $f_p : H_p(K) \to H_p(K')$
  - If there exist $K'' = (V'', \Sigma'')$ and another simplicial map $g : V' \to V''$, then
    - $(g \circ f)_p = g_p \circ f_p$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & V' \arrow[r, "g"] & V'' \end{tikzcd}$$

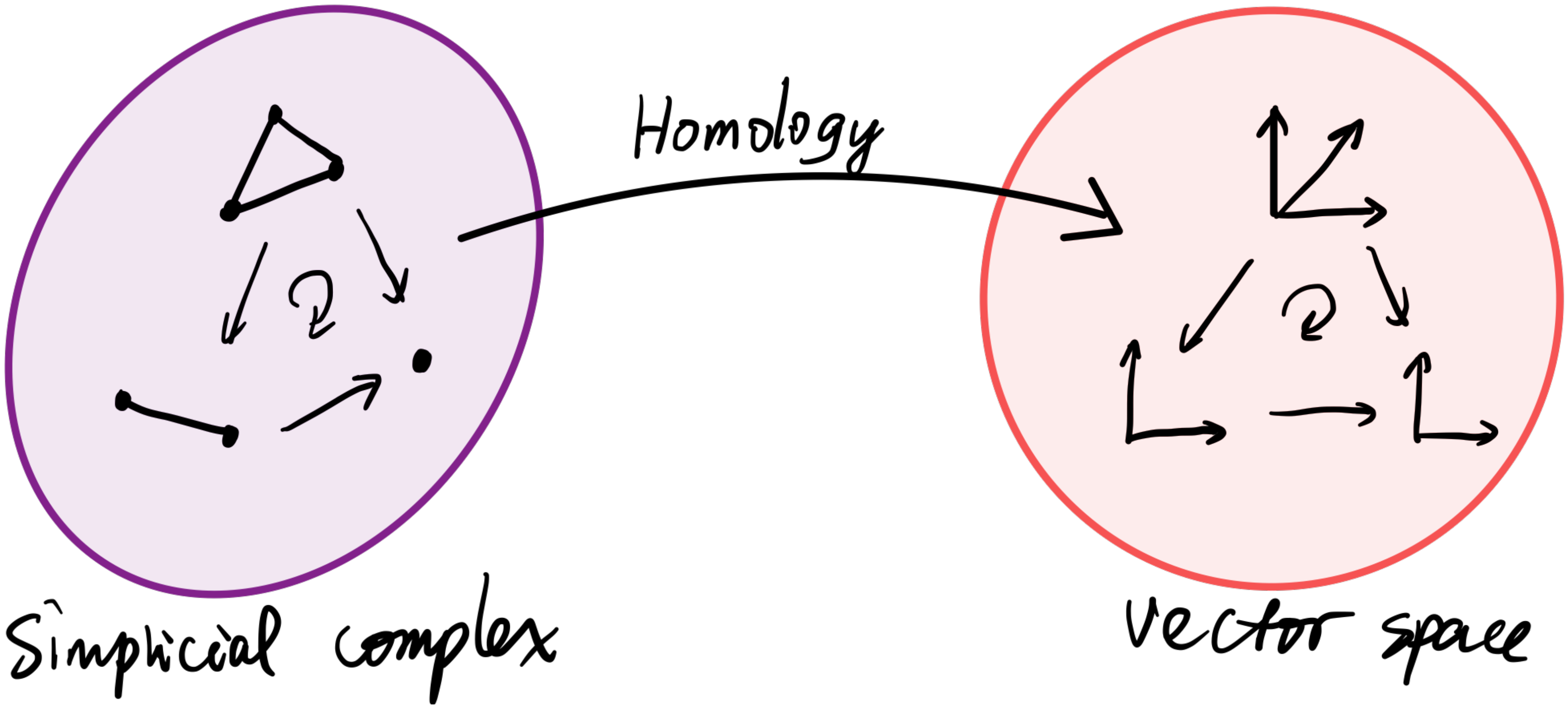
$$\begin{tikzcd} H_p(K) \arrow[r, "f_p"] \arrow[rr, bend right, "(g \circ f)_p = g_p \circ f_p"'] & H_p(K') \arrow[r, "g_p"] & H_p(K'') \end{tikzcd}$$

# Construction of $f_p$

- Define $\bar{f}_p : C_p(K) \to C_p(K')$
  - $\bar{f}_p(\sigma) = \begin{cases} f(\sigma) & \text{if } f(\sigma) \text{ is } p-\text{dimensional} \\ 0 & \text{otherwise} \end{cases}$
  - Define $f_p : H_p(K) \to H_p(K')$
    - $f_p([c]) := [\bar{f}_p(c)]$



# Mind picture of functoriality



FIN