- For any compact set K in X, the set $Y \setminus K \in \tau$.

To see that this is indeed a topology, note

- $\emptyset$ and $Y = Y \setminus \emptyset$ are in τ;
- $U = \bigcup U_\alpha$ is open in X; and $\bigcup(Y \setminus K_\beta) = Y \setminus \bigcap K_\beta = Y \setminus C$ is in τ since $\bigcap K_\beta$ is a closed subset of the compact set K_{β_0} for some β_0 and hence compact; and $\bigcup U_\alpha \cup \bigcup(Y \setminus K_\beta) = U \cup (Y \setminus C) = Y \setminus (C \setminus U)$ is in τ since $C \setminus U$ is a closed subset of the compact set C and hence compact.
- $U_1 \cap U_2$ is open in X; $(Y \setminus C_1) \cap (Y \setminus C_2) = Y \setminus (C_1 \cap C_2)$ is in τ since $C_1 \cap C_2$ is compact; and $U \cap (Y \setminus C) = U \cap (X \setminus C)$ is open in X since C is closed in X.

Note that the subspace topology on X induced from Y is the same as the original topology on X (note $Y \setminus C \cap X$ is open in X).

To see that Y is compact, note if $\mathcal{U}$ is an open cover of Y, then $\mathcal{U}$ must have an element of the form $\infty \in \setminus K$ where K is a compact set in X. K is then also compact in Y. Hence, there exists a finite subcover of $\mathcal{U}$ that covers K. Together with the element $Y \setminus K$, we have a finite subcover of $\mathcal{U}$ that covers Y.

To see that Y is Hausdorff, note for any $x, x' \in X$, we just use the Hausdorff property of X to separate them. For any $x \in X$ and $y = \infty$, since X is locally compact, there exists some open neighbourhood U of x and some compact set K such that $U \subset K$. Then, U and $Y \setminus K$ are disjoint open neighbourhoods of x and ∞ respectively.

($\Leftarrow$) We still write the extra point in $Y \setminus X$ as ∞. Note since Y is Hausdorff, as a subspace, X is also Hausdorff. For any $x \in X$, since Y is Hausdorff, there exist disjoint open neighbourhoods U of x and V of ∞. Then, $K = Y \setminus V$ is a compact in Y (and hence in X) since it is closed. Note that $U \subset K \subset X$. Therefore, X is locally compact.

It remains to show that Y is unique up to homeomorphism. Let Y' be another compactification of X. Define the map $f : Y \to Y'$ as follows: $h(\infty) = \infty'$ and $h(x) = x$ for any $x \in X$. Then, I'll leave it as an exercise to check that h is a homeomorphism. □

6.4 Paracompact and Partition of Unity

Another generalization of compactness is paracompactness. We will use it to prove the partition of unity theorem which is very useful in differential geometry. We will also prove another metrization theorem using paracompactness.

First recall the definition of compactness.

Definition 6.56. A space X is said to be *compact* if every open cover of X has a finite subcover.

It turns out that we can characterize compactness using refinements of open covers.

Definition 6.57. Let $\mathcal{A}$ be a collection of subsets of a set X. A collection $\mathcal{B}$ of subsets of X is said to be a *refinement* of $\mathcal{A}$ if for each $B \in \mathcal{B}$, there exists some $A \in \mathcal{A}$ such that $B \subset A$.

Proposition 6.58. *A space X is compact iff every open cover of X has a finite open refinement that covers X.*

This can be generalized as follows.

Definition 6.59. A space X is said to be *paracompact* if every open cover $\mathcal{A}$ of X has a locally finite open refinement $\mathcal{B}$ that covers X, i.e., for each $x \in X$, there exists some open neighbourhood U of x such that U intersects only finitely many elements of $\mathcal{B}$.

Example 6.60. $\mathbb{R}^n$ is paracompact. Let $\mathcal{A}$ be an open cover of $\mathbb{R}^n$. For each $k \in \mathbb{N}$, consider the closed ball $\overline{B_k(0)}$ of radius k centered at the origin. Since $\overline{B_k(0)}$ is compact, there exists a finite subcover $\mathcal{A}_k = \{A_{k,1}, \ldots, A_{k,n_k}\}$ of $\mathcal{A}$ that covers $\overline{B_k(0)}$. Then, let $\mathcal{B}_k = \{A_{k,1} \cap (\mathbb{R}^n \setminus \overline{B_{k-1}(0)}), \ldots, A_{k,n_k} \cap (\mathbb{R}^n \setminus \overline{B_{k-1}(0)})\}$ (Here we let $B_0(0) = \emptyset$). Then, $\mathcal{B} = \bigcup \mathcal{B}_k$ is a locally finite open refinement of $\mathcal{A}$ that covers $\mathbb{R}^n$. This is first of all because $B_k(0)$ is covered by $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$.

Example 6.61. Every compact space is paracompact.

One useful fact about locally finiteness is the following lemma.

Lemma 6.62. *Let $\mathcal{A}$ be a locally finite collection of sets in a topological space X. Then, $\overline{\bigcup_{A \in \mathcal{A}} A} = \bigcup_{A \in \mathcal{A}} \overline{A}$.*

Proof. The inclusion $\supseteq$ is clear. For the other direction, let $x \in \overline{\bigcup_{A \in \mathcal{A}} A}$. Then, for any open neighbourhood U of x, we have $U \cap \bigcup_{A \in \mathcal{A}} A \neq \emptyset$. Since $\mathcal{A}$ is locally finite, there exists some open neighbourhood V of x such that V intersects only finitely many elements of $\mathcal{A}$, say $A_1, \ldots, A_n$. Now, I claim that $x \in \overline{A_i}$ for some i. If not, then $x \in V \setminus \overline{A_1} \cup \cdots \cup \overline{A_n}$, which is a neighbourhood of x that does not intersect $\bigcup_{A \in \mathcal{A}} A$. This is a contradiction. Therefore, $x \in \overline{A_i}$ for some i and hence $x \in \bigcup_{A \in \mathcal{A}} \overline{A}$. $\square$

Theorem 6.63. *Every paracompact Hausdorff space X is normal.*

Proof. We first establish regularity. Let $x \in X$ and let A be a closed set in X such that $x \notin A$. For each $y \in A$, find open neighbourhoods V_y of y such that $x \notin \overline{V_y}$ (Hausdorff). Then, $\mathcal{U} = \{X \setminus A\} \cup \{V_y : y \in A\}$ is an open cover of X. Let $\mathcal{W}$ be a locally finite open refinement of $\mathcal{U}$ that covers X and let $\mathcal{V} = \{W \in \mathcal{W} : W \cap A \neq \emptyset\}$. Then, $V = \cup_{W \in \mathcal{V}} W$ is an open neighbourhood of A.

Since $\mathcal{W}$ is locally finite, we have that

$$\overline{V} = \bigcup_{W \in \mathcal{V}} \overline{W}.$$

Note that for any $W \in \mathcal{V}$, we have that $x \notin \overline{W}$. So $x \notin \overline{V}$. So, we have found disjoint open neighbourhoods of x and A, namely, $X \setminus \overline{V}$ and V respectively. Therefore, X is regular.

The proof of normality is similar. We simply replace where we used Hausdorff with regularity and where we separated a point and a closed set with separating two closed sets. $\square$

It turns out that metric spaces are paracompact. This makes paracompactness a very nice notion that generalizes both compactness and metric spaces (note that metric spaces may not be locally compact).

To prove this result, we need a technical lemma.

Lemma 6.64. *Let X be regular. Then, TFAE: Every open covering $\mathcal{U}$ of X has a refinement that is:*

1. *a locally finite open cover;*
2. *A countably locally finite open cover, i.e., it can be written as a countable union of locally finite collections;*
3. *a locally finite cover;*
4. *a locally finite closed cover.*

Proof. 1$\Rightarrow$ 2: straightforward.

2$\Rightarrow$ 3: Let $\mathcal{V} = \cup_n \mathcal{V}_n$ is a countably locally finite open refinement of $\mathcal{U}$ where each $\mathcal{V}_n$ is locally finite. For each n, define $W_n = \bigcup \mathcal{V}_n$ which is open. Then, $\{W_1, \ldots\}$ covers X. Define $A_n = W_n \setminus \cup_{i<n} W_i$. Then, $\{A_n\}$ is a locally finite refinement of $\{W_n\}$ that covers X. Now, define

$$\mathcal{W} = \{V \cap A_n : V \in \mathcal{V}_n, n \in \mathbb{N}\}.$$

Then, $\mathcal{W}$ is a locally finite refinement of $\mathcal{V}$ and hence $\mathcal{U}$ that covers X.

3 $\Rightarrow$ 4: Let $\mathcal{B}$ be the collection of all open sets U such that $\overline{U}$ is contained in some element of $\mathcal{U}$ (I just want something smaller so that I can take closure). By regularity, $\mathcal{B}$ is an open cover of X. Then, it has a locally finite refinement $\mathcal{C}$ that covers X by (3). Let $\mathcal{D} = \{\overline{C} | C \in \mathcal{C}\}$. Then, $\mathcal{D}$ still covers X which refines $\mathcal{U}$. It is still locally finite which can be checked directly.

4 $\Rightarrow$ 1: Let $\mathcal{V}$ be a locally finite closed refinement of $\mathcal{U}$ that covers X. For each $x \in X$, let W_x be a neighborhood of x meeting only finitely many elements of $\mathcal{V}$. Then, $\{W_x\}$ is an open cover of X. Apply 4, there exists a locally finite closed refinement $\mathcal{A}$ of $\{W_x\}$ that covers X.

For each $V \in \mathcal{V}$, define

$$V^* = X \setminus \bigcup_{\substack{A \in \mathcal{A} \\ A \cap V = \emptyset}} A \supset V.$$

Then, V^* is open as it is the complement of a closed set. Furthermore, $\{V^* | V \in \mathcal{V}\}$ is an open cover of X. Now, we show that $\{V^* | V \in \mathcal{V}\}$ is locally finite. Given any $x \in X$, there is a neighborhood U of x meeting finitely many $A_1, \ldots, A_n$ (they also cover U). But whenever $U \cap V^* \neq \emptyset$, there exists some A_i such that $A_i \cap V^* \neq \emptyset$ which implies that $A_i \cap V \neq \emptyset$. Since each A_i meets only finitely many V's due to the definition, we have that U meets only finitely many V^*'s. Therefore, $\{V^* | V \in \mathcal{V}\}$ is locally finite.

Now, for each $V \in \mathcal{V}$, pick $U \in \mathcal{U}$ such that $V \subset U$, and form $W = V^* \cap U$. Then, the collection of all such W's is an open refinement of $\mathcal{U}$ that covers X and is locally finite. □

Theorem 6.65. *Every metric space is paracompact.*

Proof. □

Example 6.66. This theorem gives us a bunch of examples of paracompact spaces, e.g. $\mathbb{R}^n$, $[0, 1)$ etc.

Partition of Unity

Definition 6.67. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of a topological space X. A *partition of unity* subordinated to $\mathcal{U}$ is a collection of continuous functions $\{\varphi_\alpha : X \to [0, 1]\}_{\alpha \in A}$ such that:

- $\operatorname{supp} \phi_\alpha \subset U_\alpha$; here the support of a function $f : X \to \mathbb{R}$ is defined as $\operatorname{supp} f = \overline{\{x \in X : f(x) \neq 0\}}$
- $\{\operatorname{supp} \varphi_\alpha\}$ is locally finite; and
- For every $x \in X$, $\sum_{\alpha \in A} \varphi_\alpha(x) = 1$.

Lemma 6.68. *Let X be Hausdorff and paracompact. Then, every open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ of X has a shrinking, i.e., an open refinement $\{V_\alpha\}_{\alpha \in J}$ covering X such that $\overline{V_\alpha} \subset U_\alpha$ for each α.*

Proof. Let $\mathcal{A} = \{A \in \tau : \overline{A} \subset U_\alpha, \text{ for some } \alpha \in J\}$. Since X is normal and hence regular, $\mathcal{A}$ is an open cover of X. Let $\mathcal{B}$ be a locally finite open refinement of $\mathcal{A}$ that covers X (exists since X is paracompact). Assume that $\mathcal{B} = \{B_\beta\}_{\beta \in K}$ for some index set K. For each $\beta \in K$, pick some $\alpha \in J$ such that $\overline{B_\beta} \subset U_\alpha$ and define a function $f : K \to J$ by $\beta \mapsto \alpha$.

Then, for each $\alpha \in J$, define

$$V_\alpha = \bigcup_{\beta \in f^{-1}(\alpha)} B_\beta.$$

By locally finiteness of $\mathcal{B}$, we have that

$$\overline{V_\alpha} = \bigcup_{\beta \in f^{-1}(\alpha)} \overline{B_\beta} \subset U_\alpha$$

It is clear that $\{V_\alpha\}_{\alpha \in J}$ is an open refinement of $\mathcal{U}$ that covers X (since $\mathcal{B}$ covers X). We now check that $\{V_\alpha\}_{\alpha \in J}$ is locally finite. Given any $x \in X$, there exists some open neighbourhood W of x that meets only finitely many elements of $\mathcal{B}$, say $B_{\beta_1}, \dots, B_{\beta_n}$. Now, whenever $W \cap V_\alpha \neq \emptyset$, there exists some β_i such that $B_{\beta_i} \subset V_\alpha$, i.e., $f(\beta_i) = \alpha$. So W meets only V_α 's for $\alpha \in \{f(\beta_1), \dots, f(\beta_n)\}$. Therefore, $\{V_\alpha\}_{\alpha \in J}$ is locally finite. □

Theorem 6.69 (Partition of Unity)**.** *Let X be Hausdorff and paracompact. Then, every open cover of X has a partition of unity subordinated to it.*

Proof. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ be an open cover. We apply the shrinking lemma twice to find locally finite open covers $\{V_\alpha\}_{\alpha \in J}$ and $\{W_\alpha\}_{\alpha \in J}$ such that

$$\overline{W_\alpha} \subset V_\alpha \quad \text{and} \quad \overline{V_\alpha} \subset U_\alpha$$

for each α.

Since X is normal, for each α, by Urysohn's lemma (valid in normal spaces) there exists a continuous $h_\alpha : X \to [0,1]$ such that $h_\alpha \equiv 1$ on $\overline{W_\alpha}$ and $h_\alpha \equiv 0$ on $X \setminus V_\alpha$. This implies that

$$supp h_\alpha \subset \overline{V_\alpha} \subset U_\alpha.$$

Since $\{V_\alpha\}$ is locally finite, $\{\overline{V_\alpha}\}$ is also locally finite. Hence, $\{supp h_\alpha\}$ is locally finite.

Set $h = \sum_{\alpha \in J} h_\alpha$. For each $x \in X$, there is some neighborhood U of x that meets only finitely many $supp h_\alpha$'s, say $supp h_{\alpha_1}, \dots, supp h_{\alpha_n}$. Therefore, $h|_U = \sum_{i=1}^n h_{\alpha_i}|_U$. Since each $h_{\alpha_i}|_U$ is continuous, $h|_U$ is continuous. Then, h is continuous. Furthermore, there is W_α containing the point since it is a cover. Hence, $h_\alpha(x) = 1 > 0$ and hence $h(x) > 0$.

Define $\varphi_\alpha = h_\alpha / h : X \to [0,1]$. Then, $\{\varphi_\alpha\}_{\alpha \in J}$ is a locally finite partition of unity subordinated to $\mathcal{U}$. □