From Data to Donations: Optimal Fundraising Campaigns for Non-Profit Organizations

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Non-profit organizations play a crucial role in mitigating social problems around the world, but their financial viability often relies on raising funds from donors through costly campaigns. We collaborate with a major international non-profit to develop and test a data-driven approach to improve the efficiency of their fundraising campaigns. We develop a novel explore-then-exploit algorithm with clustering to learn donor preferences across different campaigns, estimate expected donations, and inform personalized campaign strategies. Besides showing the value of this approach theoretically, we demonstrate its effectiveness empirically on both synthetic and real data from our partner organization. Our approach significantly increases net revenues over both the organization's current strategy as well as standard multi-armed bandit algorithms from the literature, thus demonstrating the potential for data-driven approaches to improve the success of non-profit fundraising campaigns.

Key words: Non-profit organization, optimal campaign, multi-armed bandit, clustering

1. Introduction

We study how non-profit organizations (NPOs) should target their fundraising campaigns to finance their operations. NPOs are pivotal in addressing social issues globally, implementing projects in areas such as human services, healthcare, and education. Much of this activity is financed through philanthropic donations: In the U.S. alone, charitable giving amounted to $560 billion in 2023, corresponding to about 2% of the economy. While some donations originate from corporate or foundation giving, the majority (67%) come from individuals.[1] Efficiently raising funds from donors is thus critical for the sustainable operation of NPOs.

Fundraising from potential donors is, however, costly and subject to significant uncertainty. Our partner organization, a national branch of a major international NPO, is no exception. This branch contributes to thousands of projects around the world every year, helping millions of people. It raises funds mainly from its pool of more than a million

[1] See https://www.nptrust.org/philanthropic-resources/charitable-giving-statistics/.

active individual donors, via channels including mail, email, and internet campaigns, as well as sponsorships. While sponsorships—initiated by phone calls—involve a commitment to fixed, periodic donations, all other channels depend on ad-hoc contributions and are hence subject to significant uncertainty over donor response to campaigns.

Like many non-profit organizations, our partner NPO's fundraising efforts are focused on mail campaigns. These campaigns constitute a major source (around half) of the organization's revenues, but also incur large costs. Each month, the NPO sends out substantial physical brochures with a specific *theme* via paper mail to select donors, bearing significant printing and mailing costs.[2] The theme of the campaign brochure (e.g., type of projects funded) varies each month, but the same theme repeats annually in the same month. Because most donors contribute sporadically and favor specific themes like education, healthcare, or well-being, sending all campaign brochures indiscriminately to all potential donors is not feasible. The NPO's current stated strategy is to contact donors who have made donations in the last three years. Around 8% of campaign letters sent this way attract donations, and mail-campaign costs represent 30% of raised revenues on average. Given that most campaign letters currently do not result in donations, a nn-profit organization such as our partner could potentially greatly improve cost-efficiency and environmental sustainability by carefully selecting the most appropriate recipients for each campaign from its diverse donor pool.

Identifying the right campaigns to send to donors—where the expected donation exceeds costs—is challenging. The non-profit organization must accurately estimate, for each donor, their expected donations for each campaign. Yet the data available to do so are limited. Most donors are seldom contacted: our partner NPO, as per its strategy, only sends each campaign to a fraction of potential donors. Each donor may be interested in only a few specific campaigns, and the annual frequency of campaigns limits the opportunities to learn these preferences over a donor's active lifetime. Ideally, exogenous contextual data such as age, gender, and location could be used to directly infer preferences, but our analysis shows that these data have little predictive power in the case of our partner organization. Alternatively, estimates could be based on donations made by the NPO's existing donors—but these data, in turn, are skewed by selection effects from the organization's current

[2] The focus on mail campaigns is partly driven by demographics: as is common in the non-profit sector (Freund and Blanchard-Fields 2014, Cutler et al. 2021), the majority of donations to our partner NPO are made by older individuals, with an average donor age over 65, who tend to prefer tangible materials (Times 2017, Smith 2022).

campaign strategy.

To address these challenges, we propose a novel method to estimate donor preferences and optimize mail fundraising campaigning strategies for NPOs by deciding, for each campaign, which donors the NPO should contact. We develop a non-parametric approach to estimate donor's campaign preferences from observed donation outcomes. Established methods for learning individual preferences for diverse campaigns require repeatedly observing each donor's responses to all campaigns, and hence either sending out a large number of campaign brochures over many years, or facing high variance in estimates from limited data. Instead, we propose segmenting donors based on their observed similarities in campaign preferences. We categorize donors into several types, with each type showing similar preferences for all campaigns. Instead of estimating campaign preference at individual level, we estimate campaign preferences at the type level, which exploits the collective donation outcomes from all donors within a type to improve accuracy and decrease variance. Shifting from individual to group-based estimation also speeds up the estimation process by relying on fewer observations of the same campaign from each donor.

We study two scenarios, distinguished by the data at a non-profit organization's disposal: 1) The NPO possesses a comprehensive, unbiased dataset of donor behavior and aims to refine or develop new strategies for new donors, and 2) the NPO wishes to develop fundraising strategies but lacks representative donors' donation data like our case. By a data representative, we refer to a dataset containing rich samples that accurately capture the donation behavior of the entire donor population. For both scenarios, we propose an *explore-then-exploit* algorithm with clustering that first sends a certain number of campaign brochures to each donor, and then uses the obtained responses to cluster them into donor types. In the first scenario, the algorithm classifies individual donors to a known set of types following this exploration period. In the second, more challenging, one, it must also learn what the possible donor types themselves are. Based on expected revenues for different campaigns, the algorithm then exploits the learned preferences by taking the same campaign decisions for all donors of each type.

For each scenario, we both derive theoretical guarantees for the algorithm and demonstrate its empirical effectiveness on data. We show that the algorithms are tractable and achieve sub-linear regret. We then conduct a comprehensive numerical study using both synthetic and real data from our partner organization. Our clustering-based approach outperforms both our partner NPO's existing strategy and state-of-the-art multi-armed bandit algorithms (Slivkins 2019, Lattimore and Szepesvári 2020) and recommender system (Jannach et al. 2010) from the literature. We also contribute an open-source dataset to the community, allowing other researchers to further explore the data and contribute to the understanding of donation behavior and other operational or managerial problems of NPOs. In summary, our approach allows for a better match of non-profit organizations' mail campaigns and their recipients operations of NPOs to improve both their financial and environmental sustainability, demonstrating the potential value for data-driven methods to improve the success of non-profit fundraising campaigns.

Notation: We denote $[T] = \{1, 2, \cdots, T\}$ as the set of all positive integers less than or equal to T. Vectors are denoted by bold lowercase letters, and random variables/vectors are indicated by a tilde on top of the letter. We use capital letters to denote constants and calligraphic uppercase letters for sets. For a vector $\boldsymbol{v}$, we refer to the 2-norm $\|\boldsymbol{v}\|_2$ as its norm and v_i its i-th entry. For a matrix $\boldsymbol{A} \in \mathbb{R}^{d_1 \times d_2}$, the l_2 operator norm, also known as the spectral norm, is defined as $\|\boldsymbol{A}\|_2 = \max_{\|\boldsymbol{x}\|_2=1} \|\boldsymbol{A}\boldsymbol{x}\|_2$. The indicator function is denoted by $\mathbb{1}\{.\}$. For any statement involving a random variable, unless specified otherwise, we claim that the statement holds with certainty.

2. Related Literature

Our work is closely related to three streams of literature: (1) operations of non-profit organizations, (2) learning consumer preferences, and (3) advertising campaigns in marketing.

2.1. Operations of Non-Profit Organizations

This paper contributes to the growing body of work on operations management for non-profit organizations (see Berenguer and Shen (2020) for a recent review). This literature can be broadly divided in two categories: resource allocation and fundraising. A common goal in the first category is examining the allocation of funds to projects to maximize designed non-profit objectives (Keshvari Fard et al. 2022, Kotsi et al. 2023, Fangwa et al. 2024). Another stream of recent papers considers the assignment of volunteers to projects in non-profit platforms (Manshadi and Rodilitz 2022, Berenguer et al. 2023, Lo et al. 2024, Ata et al. 2024).

Our work belongs to the literature on non-profit fundraising, where previous work have studied strategies to maximize donations or foster repeated giving. These include identifying important factors for the design of campaigns (Ryzhov et al. 2016, Zhang et al. 2023), optimizing default campaign menus (Castro and Rodilitz 2024), and improving staffing decisions to enhance donor experiences with the NPO (Lin et al. 2023). Most closely related to our work in this stream of literature are Singhvi and Singhvi (2022) and Song et al. (2022), which study the operations of crowdfunding platforms. They consider the optimal selection of a funding campaign to recommend to each arriving viewer on the platform, based on either contextual bandits (Singhvi and Singhvi 2022) or an econometric model of customer utility (Song et al. 2022). In this setting, the platform targets each donor with a single campaign as the donor arrives. In our setting, by contrast, the non-profit organization simultaneously selects many recipients for fixed campaigns that repeat annually, without access to contextual information.

2.2. Learning Consumer Preferences

Our work also relates to the growing literature on learning consumer preferences, which draws from discrete choice modeling (Ben-Akiva and Lerman 1985), machine learning (James et al. 2013), recommender systems (Jannach et al. 2010), and multi-armed bandits (Lattimore and Szepesvári 2020). Consumer preferences are often studied using discrete choice models (see Gallego et al. (2019) for a recent review) for a consumer's probability of choosing a product. Recently, machine learning approaches have also been applied to this task (Aouad and Désir 2022, Chen and Mišić 2022, Wang et al. 2023), as well as considering preferred product sets (Yoganarasimhan 2020). Recommender systems, particularly matrix completion (Koren et al. 2009) and tensor completion (Farias and Li 2019), have achieved considerable success in predicting the preferences of a large number of customers for a large set of products, especially when obtaining the preference of each customer all products is not feasible. These methods, however, rely on the availability of rich unbiased data, which is not the case in our setting, where data on a customer's preferences must be collected over time.

In the literature on learning consumer preferences over time (online learning), contextual multi-armed bandits (Rusmevichientong and Tsitsiklis 2010, Li et al. 2010) are among the most popular techniques for their simplicity combined with their capability to make personalization decisions. They model customer preferences as a function of predictors typically combining characteristics of both the product and the customer. These models have been widely adopted for learning optimal personalized pricing (Chen et al. 2020, Ban and Keskin 2021, Elmachtoub et al. 2021), optimal personalized assortments (Golrezaei et al. 2014, Miao and Chao 2021, 2022), personalized campaign recommendation (Singhvi and Singhvi 2022), among other applications.

These studies, however, assume a homogeneous model, i.e., the parameters of the unknown model are the same across the entire population. Among studies relaxing this assumption, Bernstein et al. (2019) perform assortment optimization assuming that customers form clusters, and the belonging of a customer to clusters is modeled by Dirichlet process. Miao et al. (2022) propose clustering products together whose price sensitivity parameter confidence interval intersect, to address the large estimation variance for low-sale products. To address the large number of design-context pairs in contextual design problems, Li et al. (2024) propose a Gaussian mixture model-based Bayesian framework to accelerate the identification of the best design for each context.

Our work contributes to this line of literature by considering donors as heterogeneous and modeling this heterogeneity through hidden types. The type of each donor is determined by their underlying non-parametric campaign preferences, not specified by mixture models or contextual model parameters which are not available in our application. Additionally, we must quickly identify, for each donor, their corresponding favorable campaigns, rather than focusing on the single most-favored campaign. Moreover, in our case, customers (donors) do not arrive sequentially but in batches as we can contact all donors in each month, rendering the problem context different from those in the literature.

2.3. Advertising Campaigns in Marketing

Advertising campaigns have been studied extensively in the marketing literature. Common goals in this literature are identifying best designs for different campaign locations or customer segments (e.g., Schwartz et al. (2017) and Moazeni et al. (2020)) and testing the effectiveness of campaigns for facilitating market entry or retaining customers (e.g., Bonfrer and Drèze (2009), Zantedeschi et al. (2017), Adena and Huck (2020), Bollinger et al. (2024), Chan et al. (2024)).

More closely related to our work is the stream of literature considering targeting customers with advertising campaigns. Targeting can take place across media and this literature considers questions around when to target repeat customers to avoid attrition (Neslin et al. 2006, Ascarza 2018, Lemmens and Gupta 2020) where to place campaigns (Agarwal et al. 2011, Chan and Park 2015), when to target customers with campaigns (Hong et al.

2024), and how to match viewers with ad campaigns in real-time for internet advertisements (Mookerjee et al. 2017, Agrawal et al. 2023). All these works study targeting a single campaign to customers, in a fixed time period, a common setting in for-profit business operations. This paper, by contrast, considers a non-profit setting where the organization must learn how to target multiple distinct and annually-repeated campaigns to customers who contribute sporadically to campaigns.

3. Donation Behavior and Problem Definition

In this section, we first demonstrate important patterns in donation behavior in our partner organization's donor pool. We then use these observations to define the optimization problem that a non-profit organization faces when seeking to target potential donors with campaign brochures.

3.1. Donation Data and Behavior

We obtained a dataset from our partner NPO capturing the organization's interactions with over 1.5 million donors on thirteen mail campaigns per year over five years (from 2018 to 2022). The data contain information about which donors have been contacted with each campaign, and whether and how much they donated in response. For each campaign, the number of contacts ranges from 100,000 to 700,000, with an average response rate of 8%. Additionally, the data includes basic demographics for each donor: age, gender, and post code. With an average age over 65, the donors are on average significantly older than the general population of the country; see Appendix A.1 for more details of the demographics. We make the following observations from the data to inform our modeling approach.

First, donation amounts across all donors and campaigns almost always come from the set of $\{5, 10, 15, 20, 25, 30, 40, 50, 100\}$ in local currency. Figure 1 demonstrates this observation, with the exception of a significant number of 19 donations, which we group together with 20. We have excluded donation amounts larger than 100—which represent around 0.1% of the number of donations and 5–10% of the total donation amount—from consideration. Since the NPO will typically communicate with donors before large donations, these interactions fall outside the scope of our analysis. Out of the remaining donations in the data, around 92% fall within the set described above, with the rest distributed among other amounts. We conclude that donation outcomes can be accurately represented with a relatively small discrete set of likely donation amounts.

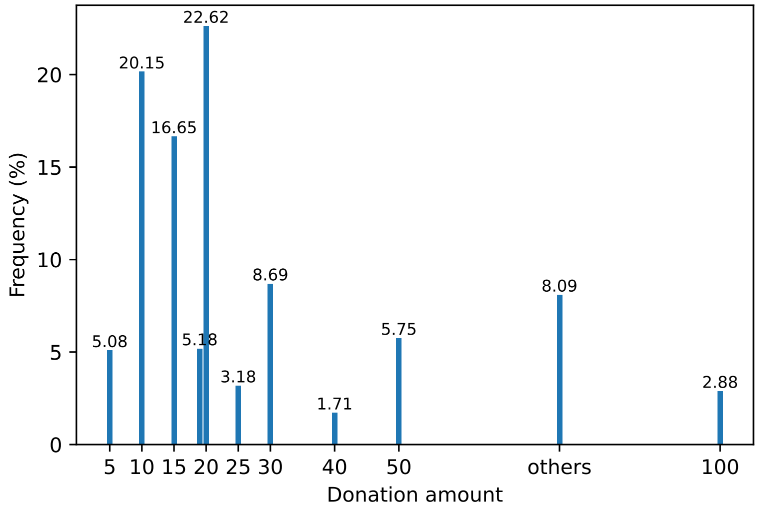


Figure 1 **Frequency of donation amounts in local currency. The value "others" represents all other donation amount, conditional on donations being made.**

Next, and most importantly, donation outcomes across campaigns are clustered. We make this observation based on data from the around 200,000 donors that the NPO has contacted regularly throughout the five years (at least three out of every five campaigns, or 39 out of 65 total campaigns). For each of these donors, we calculated their average donation amount for each of the thirteen campaigns over the five years. We then used K-means clustering on the resulting donor vectors for different values of K, picking the value at the elbow point of the fitting score.

Figure 2 presents the resulting seven clusters as box-and-whisker plots. Each row shows the donation distributions for a cluster for each of the thirteen campaigns, with the size of the cluster at the end of the row. The donation behaviors within the clusters differ significantly in terms of preferences for individual campaigns and overall donation propensity. The donors in the first-row cluster, representing nearly 30% of all donors, donate little to each campaign. The second cluster's donors are similar but show a preference for campaign thirteen, while those in the fifth cluster favor campaigns two and eight. The final cluster contains donors who appear to regularly donate to all campaigns, representing potentially lucrative contributors to the organization. The clusters exhibit consistently different donation behavior. We demonstrate in Appendix A.2 that donations to different campaigns show little correlation within a cluster, and in Appendix A.3 that earlier donations have

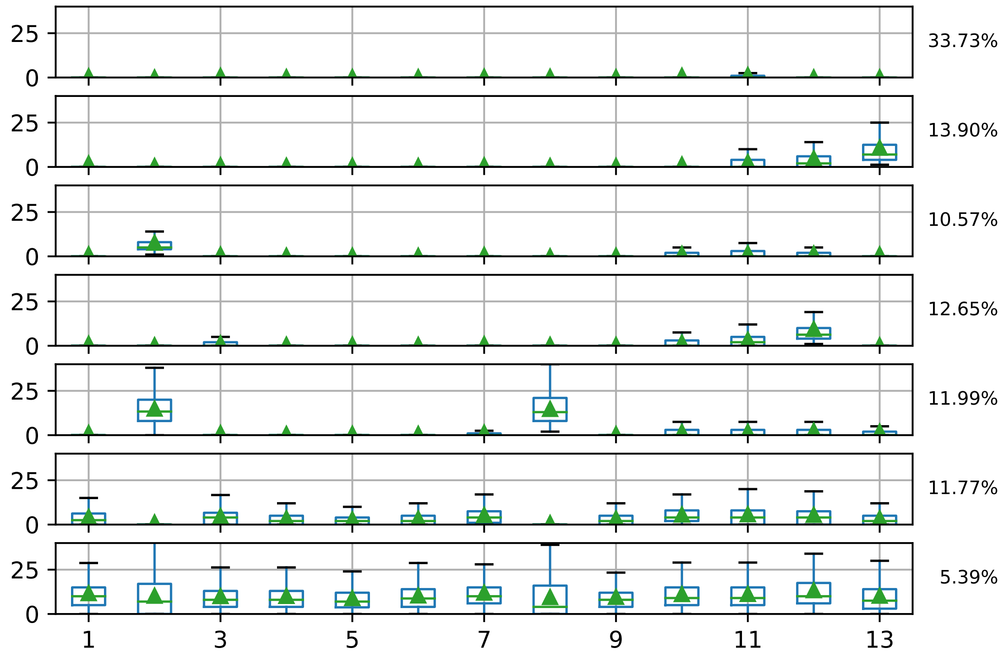


Figure 2 **The results of K-means clustering of average donation amounts. The x-axis shows different campaigns and the y-axis shows the average donation amount in local currency for each campaign. The numbers at the end of each row indicate the relative sizes of clusters. The plot's whiskers extend to the 5th and 95th percentiles and the box encompasses the 25th to 75th percentiles. A green line marks the median and a triangle indicates the mean. Outliers have been omitted to improve the figure's readability.**

little temporal effect on future donations. Thus, we assume that donations from donors are independent of campaigns and time periods.

The significantly different donation behavior across the clusters suggests that it is useful to think about donors in terms of distinct *types* with their own preferences. That is, the organization's diverse project themes and consistent yearly campaign pattern naturally allow donor preferences for different campaigns to show in the donation amounts. Given the presence of donor types, instead of learning individual preferences separately, we can learn their types. We will leverage this idea below when defining the NPO's campaign targeting problem, which will allow an efficient estimation of donation behavior from limited data.

3.2. Abstract Campaign Problem and Performance Metric

In this section, we introduce the campaign problem and describe the metrics used to evaluate campaign algorithms. To abstractly model this problem, let us denote the donor pool by the set $[D]$, the set of campaigns by $[C]$, and the total campaign periods by T. Each campaign $c \in [C]$ incurs an expense e_c for each letter sent. We define $\tilde{a}_{dct} \in \mathcal{A} = \{0 :$

no send, 1 : send$\}$ as the action taken by our partner NPO to donor $d \in [D]$ for campaign $c \in [C]$ at period $t \in [T]$. This action is modeled as a random variable to reflect its dependence on the random donation outcomes from the donor prior to period t.

Upon receiving a letter of campaign c at period t, donor d might choose to donate certain amount via bank transfer, with non-donation represented by 0. We model this donation as a random variable $\tilde{r}_{dct}$, supported on a discrete set $\mathcal{R}$ that includes 0. Without loss of generality, the support for $\tilde{r}_{dct}$ remains consistent across donors, campaigns, and periods. Based on our discoveries from real data described in Section 3.1, $\tilde{r}_{dct}$ is assumed independent across donors, campaigns, and periods, and its distribution is identical through all periods for fixed d and c. Additionally, donors are categorized into K types, where donors of the same type share an identical joint donation distribution across campaigns. Driven by these observations and the discrete nature of $\mathcal{R}$, we define $\{\mathbf{Q}^k\}_{k\in[K]}$ as the set of distributions governing the donation outcomes for all donors across all campaigns, with $\mathbf{Q}^k = (\boldsymbol{q}_c^k)_{c\in[C]}$, where $\boldsymbol{q}_c^k$ is a column vector of dimension $|\mathcal{R}|$ indicating the distribution of a type k donor donating in $\mathcal{R}$ to campaign c. Specifically, $q_{cr}^k = \mathbb{P}[\tilde{r}_{dct} = r]$ for donor d of type $k \in [K]$, $c \in [C]$, $t \in [T]$, and $r \in \mathcal{R}$. For each type $k \in [K]$, we define $\boldsymbol{r}^k = (\mathbb{E}[\tilde{r}_{dct}|\tilde{a}_{dct}])^\top_{c\in[C]}, d \in \mathrm{K}^{-1}(k)$, as the expected donation vector when receiving all campaign letters. Since the donation amount is 0 with certainty when $\tilde{a}_{dct} = 0$, we omit the distribution and expectation of donation under this campaign decision, and assume it is always clear to the reader.

Donor-type relationship is modeled using a mapping $\mathrm{K} : [D] \to [K]$, and denote $\mathrm{K}(d)$ the type of donor $d \in [D]$. For each type $k \in [K]$, we define, $\mathrm{K}^{-1}(k) \subseteq [D]$ as the set, and $N_k = |\mathrm{K}^{-1}(k)|$ as the number, of donors of that type. Donor types are arranged such that $N_1 \geq N_2 \geq \cdots \geq N_K$. We denote the history of campaign actions to and donation responses from donor d up to period M by $\mathcal{H}_M^d = \{\tilde{a}_{dct}, \tilde{r}_{dct}\}_{c\in[C],t\in[M]}$. For any histories $\{\mathcal{H}_M^d\}_{d\in[D]}$ of all donors, the probability measure of donations conditional on being contacted is defined as the product measure across donors, campaigns, and periods.

To maximize the expected total net donation

$$\mathbb{E}[\sum_{d\in[D]}\sum_{c\in[C]}\sum_{t\in[T]}(\tilde{r}_{dct} - e_c)\tilde{a}_{dct}] = \sum_{d\in[D]}\sum_{c\in[C]}\sum_{t\in[T]}\mathbb{E}[(\tilde{r}_{dct} - e_c)\tilde{a}_{dct}]$$
$$= \sum_{d\in[D]}\sum_{c\in[C]}\sum_{t\in[T]}\mathbb{E}[\mathbb{E}[(\tilde{r}_{dct} - e_c)\tilde{a}_{dct}|\tilde{a}_{dct}]] = \sum_{d\in[D]}\sum_{c\in[C]}\sum_{t\in[T]}(r_c^{\mathrm{K}(d)} - e_c)\mathbb{E}[(\tilde{a}_{dct} = 1)],$$

the above equalities show that it is optimal to send campaign c to donors $d \in [D]$ if the expected donation $r_c^{\mathrm{K}(d)}$ exceeds the associated costs, i.e., the optimal action is $a^\star_{dct} = \mathbb{1}\{r_c^{\mathrm{K}(d)} \geq e_c\}$. Deviating from this benchmarkeither sending campaign c to a donor d with $r_c^{\mathrm{K}(d)} < e_c$, or not sending it to a donor with $r_c^{\mathrm{K}(d)} \geq e_c$results in regret. The expected regret is $|r_c^{\mathrm{K}(d)} - e_c|$ conditional on a wrong campaign action. In a multi-donor, multi-campaign, and multi-period setting, the total regret of any campaign algorithm is the sum of the regrets over all donors $[D]$, all campaigns $[C]$, and all periods $[T]$:

$$\mathbb{E}[\sum_{d\in[D]}\sum_{c\in[C]}\sum_{t\in[T]}|r_c^{\mathrm{K}(d)} - e_c| \cdot \mathbb{1}\{\tilde{a}_{dct} \neq \mathbb{1}\{r_c^{\mathrm{K}(d)} \geq e_c\}\}],$$

where $\mathbb{1}\{\tilde{a}_{dct} \neq \mathbb{1}\{r_c^{\mathrm{K}(d)} \geq e_c\}\}$ represents if the action $\tilde{a}_{dct}$ is optimal or not.

As we must explore each donor's donation probability for different campaigns, regret is inevitable to a donor when it is suboptimal to send all campaigns to her. Therefore, it makes sense to focus on the average regret across donors. Ultimately, we adopt the regret formula presented in Equation (1):

$$R(T) = \frac{1}{D}\mathbb{E}[\sum_{d\in[D]}\sum_{c\in[C]}\sum_{t\in[T]}|r_c^{\mathrm{K}(d)} - e_c| \cdot \mathbb{1}\{\tilde{a}_{dct} \neq \mathbb{1}\{r_c^{\mathrm{K}(d)} \geq e_c\}\}]. \tag{1}$$

4. Campaign with Known Donation Distributions

In this section, we address the campaign problem under the assumption that the donation distributions $\{\mathbf{Q}^k\}_{k\in[K]}$ of all donor types are known, but the specific types of the donors under study remain unidentified. The strategy proposed here is beneficial for NPOs who possess a comprehensive, unbiased dataset of donor behavior and aims to refine or develop new strategies for new donors. In such cases, campaign decisions for a donor is fully determined once the donor's type is identified. Given that the inference of donor types is independent in this context, we illustrate our campaign algorithm using a representative donor. In the following, Section 4.1 introduces an explore-then-exploit campaign algorithm with maximum likelihood clustering (*cf.* Algorithm 1) and the associated performance is discussed in Section 4.2. The result is discussed in Section 4.3.

4.1. Campaign Algorithm

In this section, we present an explore-then-exploit campaign algorithm with maximum likelihood clustering to help a non-profit organization campaign efficiently when donation distributions are known. This is detailed in Algorithm 1. Initially, the algorithm exposes the donor to all campaigns over M periods, subsequently cluster the donor into one of the types. During the exploitation phases, decisions regarding all campaigns for the donor are based on the expected donation vector associated with the assigned type. Within the algorithm, given a campaign contact and donation history $\mathcal{H}_M^d$, the log-likelihood of $\mathcal{H}_M^d$ being generated by type k is computed as follows:

$$\ell\left(\mathcal{H}_M^d;\mathbf{Q}^k\right)=\sum_{t\in[M]}\sum_{c\in[C]}\mathbb{1}\{a_{dct}=1\}\Big(\sum_{r\in\mathcal{R}}\mathbb{1}\{r_{dct}=r\}\ln\left(q_{cr}^k\right)\Big).$$

Algorithm 1 Explore-then-exploit algorithm with maximum likelihood clustering

1: Set exploration periods M and the donation distributions $\{\mathbf{Q}^k\}_{k\in[K]}$.

 # exploration

2: Send all campaigns to the donor d for M periods and record all the sends and responses to history $\mathcal{H}_M^d$.

 # clustering

3: Cluster the donor using maximum likelihood, i.e., $\mathrm{K}(d)\in\arg\max_{k\in[K]}\ell(\mathcal{H}_M^d;\mathbf{Q}^k)$.

 # exploitation

4: For each period $t\in\{M+1,\cdots,T\}$, send campaign $c\in[C]$ to the donor if $r_c^{\mathrm{K}(d)}\geq e_c$.

Remark 1. Algorithm 1 can be extended to a dynamic version. Specifically, for each period of the exploitation phase, we iterate over the clustering of the donor (line 3) and making campaign decisions for the next period.

4.2. Performance Analysis

In this section, we present the performance analysis (regret bound) of Algorithm 1. To achieve so, we first establish an uniform upper bound for clustering error probability, i.e., the probability that we allocate the donor to a wrong cluster. As we use a representative donor, we simplify the campaign actions and donation records to $\tilde{a}_{ct}$ and $\tilde{r}_{ct}$, omitting donor-specific notation. Throughout this section, we assume that $q_{cr}^k>0$ for all $c\in[C]$, $r\in\mathcal{R}$, and $k\in[K]$ to streamline the expression. For cases where this assumption is violated, we can adapt our result straightforwardly as discussed after the proof of Proposition 1.

To estimate the clustering error probability, let us introduce some notations. After M periods of exploration, where we send all campaigns to the donors, i.e., $\tilde{a}_{ct}=1$ for all $c\in[C]$ and $t\in[M]$, denote by Ω the sample space of all donation histories, expressed as:

$$\Omega=\{\mathcal{H}:\mathcal{H}\text{ is a donation history from }M\text{ exploration periods}\}=\mathcal{R}^{C\times M},$$

where the last equality is because the support of $\tilde{r}_{ct}$ is $\mathcal{R}$ across campaigns and there are C campaigns at each period. We omit the action history in the sample space as all actions are 1s. Following the exploration phase, upon getting a donation history $\mathcal{H}$, we implement maximum likelihood clustering to allocate the donor to type j if $\ell(\mathcal{H};\mathbf{Q}^j) \geq \ell(\mathcal{H};\mathbf{Q}^k)\ \forall k \in [K]$. We denote the set of all histories that make the donor be allocated to type j as

$$\Omega_j = \left\{\mathcal{H} \in \Omega : \ell(\mathcal{H};\mathbf{Q}^j) \geq \ell(\mathcal{H};\mathbf{Q}^k) \quad \forall k \in [K]\right\}.$$

We can estimate the probability that Algorithm 1 incorrectly clusters donors of type k to type j as $\mathbb{P}[\tilde{\mathcal{H}} \in \Omega_j \mid \tilde{\mathcal{H}}$ is generated by $\mathbf{Q}^k]$. A careful computation of this probability allows us to establish the following bound:

PROPOSITION 1. *Suppose a donor is of type $k \in [K]$. After M periods of exploration, the probability that the donor is erroneously clustered as type $j \neq k$ by Algorithm 1 is upper bounded by* $\exp(-2M\gamma_{kj})$.

In the expression,

$$\gamma_{kj} = (\sum_{c\in[C]} \mathrm{KL}(\boldsymbol{q}_c^k \| \boldsymbol{q}_c^j))^2 / \sum_{c\in[C]} (\max_{r,r'\in\mathcal{R}} \ln(\frac{q_{cr}^j q_{cr'}^k}{q_{cr}^k q_{cr'}^j}))^2, \quad \text{and} \quad \mathrm{KL}(\boldsymbol{q}_c^k \| \boldsymbol{q}_c^j) = \sum_{r\in\mathcal{R}} q_{cr}^k \cdot \ln(\frac{q_{cr}^k}{q_{cr}^j})$$

is the well-known KL-divergence between distributions $\boldsymbol{q}_c^k$ and $\boldsymbol{q}_c^j$.

Next, define $\gamma = \min_{k\neq j\in[K]} \gamma_{kj}$, we invoke the above clustering error probability to upper bound the regret associated with Algorithm 1, which is presented below.

THEOREM 1. *Suppose the donation distribution of all types $\{\mathbf{Q}^k\}_{k\in[K]}$ are known. The regret incurred by Algorithm 1 is upper bounded by*

$$\begin{aligned}\min_{M\in[T]} \max_{k\in[K]} M \cdot (\sum_{c\in[C]} |r_c^k - e_c| \cdot \mathbb{1}\{r_c^k < e_c\}) + \\ (T-M) \sum_{j\in[K]} \sum_{c\in[C]} |r_c^k - e_c| \cdot \mathbb{1}\{\mathbb{1}\{r_c^j \geq e_c\} \neq \mathbb{1}\{r_c^k \geq e_c\}\} \cdot \exp(-2M\gamma_{kj}).\end{aligned} \tag{2}$$

With the exploration period M being $\max\{1, \lceil \frac{1}{2\gamma} \ln(2T\cdot(K-1)\gamma)\rceil\}$ *and defining* $\Delta_{\max}^c = \max_{k\in[K]} |r_c^k - e_c|$, *the upper bound is simplified to,*

$$\sum_{c\in[C]} \Delta_{\max}^c \min\left\{T, \frac{1}{2\gamma}\ln(2T\cdot(K-1)\gamma) + 1 + \frac{1}{2\gamma}\right\}.$$

Theorem 1 presents the regret bound in two forms: a robust optimization form and an analytical form. The analytical regret expression facilitates an easy interpretation of the incurred regret and identifies the parameter γ that is crucial for our algorithm's performance. Although it offers clear insight into the regret, the analytical regret bound becomes trivial, reducing to $\sum_{c\in[C]} \Delta_{\max}^c T$, when γ is small. It is trivial because, in a campaign problem spanning T periods, $\sum_{c\in[C]} \Delta_{\max}^c T$ is the maximum possible regret. We note that this triviality results from the relaxation used to derive the analytical solution. With a more precise representation of the incurred regret value and the clustering error probability, the robust optimization formulation (2) offers a tighter regret bound. For the same reason, solving the optimization problem (2) provides a better choice of exploration periods M than directly using $\max\{1, \lceil \frac{1}{2\gamma} \ln(2T \cdot (K-1)\gamma) \rceil\}$.

4.3. Discussion

In this section, we aim to demonstrate that solving the robust optimization problem (2) is easy. Additionally, by exploring the formulation in (2), we want to highlight why the regret diminishes when γ is small instead of being the trivial value in the analytical form.

The optimization problem (2) is easy to solve as its computational complexity is $\mathcal{O}((T + C|\mathcal{R}|)K^2)$. To solve the problem, we need to calculate the values γ_{kj}, $\sum_{c\in[C]} |r_c^k - e_c| \cdot \mathbb{1}\{r_c^k < e_c\}$, and $\sum_{c\in[C]} |r_c^k - e_c| \cdot \mathbb{1}\{\mathbb{1}\{r_c^j \geq e_c\} \neq \mathbb{1}\{r_c^k \geq e_c\}\}$ once for each $j \neq k \in [K]$ and store them. This computation has $\mathcal{O}(C|\mathcal{R}|K^2)$ operations required by the computation of $\gamma_{kj}, k \neq j \in [K]$. To determine the optimal exploration period M that minimizes (2), we enumerate every feasible $M \in [T]$. For each M, we identify the type $k \in [K]$ that leads to the maximum regret when mis-clustered to all other clusters $j \in [K]$, requiring a computational effort of $\mathcal{O}(TK^2)$. Thus, putting these two parts together confirms the stated computational complexity.

Furthermore, by examining the regret formulation in (2), we demonstrate that the regret is small other than being trivial when γ is small, a case where it is difficult to distinguish some donor types. Suppose, at the optimal M, the maximization over $k \in [K]$ in (2) is attained at j, but $\gamma_{jk} \neq \gamma$ for all $k \in [K]$. In this scenario, the regret upper bound should be related to $\min_{k\in[K]} \gamma_{jk} > \gamma$, thereby refining the analytical regret expression. Conversely, if $\gamma_{jk} = \gamma$ for some $k \in [K]$, we must have $\mathbf{Q}^j = \mathbf{Q}^k + \boldsymbol{\varepsilon}$ for a small element-wise $\boldsymbol{\varepsilon}$. Then, it is very likely $\mathbb{1}\{\mathbb{1}\{r_c^j \geq e_c\} \neq \mathbb{1}\{r_c^k \geq e_c\}\} = 0$ due to the two distributions being very close, implying no regret during exploitation periods from mis-clustering these donor types. Alternatively, if $\mathbb{1}\{\mathbb{1}\{r_c^k \geq e_c\} \neq \mathbb{1}\{r_c^j \geq e_c\}\} = 1$, this implies $\min_{j,k}\{r_c^k, r_c^j\} < e_c \leq \max_{j,k}\{r_c^k, r_c^j\}$. Thus, the incurred regret, $|r_c^j - e_c|$, is bounded by $|r_c^j - r_c^k|$. However, by $\mathbf{Q}^j = \mathbf{Q}^k + \boldsymbol{\varepsilon}$, we can bound $|r_c^j - r_c^k| \leq (\sum_{r\in\mathcal{R}} r) \max \boldsymbol{\varepsilon}$ for every $c \in [C]$. Since $\boldsymbol{\varepsilon}$ is very small element-wise, so is $|r_c^j - r_c^k|$. In all cases, the regret does not escalate excessively when γ is small.

Lastly, it is noteworthy that if a charity has previously contacted a donor but not frequently enough, we can include these past donation records in $\mathcal{H}_M^d$ to either reduce the number of exploration periods or maintain the same number while enhancing clustering accuracy. Either strategy effectively reduces the incurred regret.

5. Campaigning with Unknown Donation Distributions

In this section, we will first introduce in Section 5.1 an algorithm (*cf.* Algorithm 2) for the campaign problem when donation distributions $\{\mathbf{Q}^k\}_{k\in[K]}$ are unknown. The strategy is suitable to NPOs who wish to develop fundraising strategies but lack representative donation data like our case. In such scenarios, since learning the exact distribution of each donor type does not provide additional advantages for making campaign decisions beyond identifying the expected donation vector $\{\boldsymbol{r}^k\}_{k\in[K]}$ of each type, our focus will be solely on learning these expected donation vectors. Subsequently, we will evaluate the performance of Algorithm 2 in Section 5.2 and present an overview of the performance analysis in Section 5.3. In Section 5.4, we discuss the potential adaptation and extension of our algorithm to other settings. To highlight the most significant adaptations of our algorithm when the critical input parameters Δ and ϕ (to be defined) are small or unknown, we provide a separate discussion and analysis in Section 5.5.

In this section, define $\beta = \min\{a : N_j \leq aN_k \text{ for any } k, j \in [K]\}$ to represent the relative size of different types. Here, type size means the number of donors of the type. Additionally, we define $\Delta = \min_{k,j\in[K],k\neq j} \|\boldsymbol{r}^k - \boldsymbol{r}^j\|_2$ as the minimal distance between expected donation vectors. We define an open ball of radius α around center $\boldsymbol{r} \in [0, \max\mathcal{R}]^C$ as the subset $\mathcal{B}(\boldsymbol{r}, \alpha) = \{\boldsymbol{r}' \in [0, \max\mathcal{R}]^C : \|\boldsymbol{r} - \boldsymbol{r}'\|_2 < \alpha\}$.

5.1. Campaign Algorithm

In this section, we introduce our proposed campaign strategy detailed in Algorithm 2, designed for situations where donation distributions are unknown. This algorithm follows the explore-then-exploit framework and is enhanced with clustering (Lines 4 to 11).

Initially, the algorithm sends all campaigns to all donors over M periods to compute the empirical donation vector $\hat{\boldsymbol{r}}_d$ for each donor $d \in [D]$. Subsequently, we cluster donors based on these empirical donation vectors, leveraging the concentration of measure. According to this principle, the empirical average $\hat{\boldsymbol{r}}_d, d \in [D]$ should be centered around their true values within a specific distance with high probability. This phenomenon indicates that the vectors $\hat{\boldsymbol{r}}_d, d \in [D]$ naturally form clusters of certain diameter (specifically $\frac{\Delta}{3}$ for technical reasons), each corresponding to a distinct donor type. The algorithm's loop identifies these clusters and estimates the donation vector for each type. For donors who do not belong to any established cluster centers, we assign them to the nearest identified type and re-estimate the center of each type (*cf.* lines 11 and 12). In the exploitation phase, campaign decisions for each donor are based on the estimated donation vector of the cluster to which they have been assigned. In line 5 of Algorithm 2, different distance measures can be applied. Here, we utilize the Euclidean distance to showcase the analyzing strategy.

We note that our proposed algorithm is applicable to real applications as the computational complexity of the problem increases only quadratically with the donor size and linearly with the the number of type and number of campaigns.

Proposition 2. *The computational complexity of Algorithm 2 is* $\mathcal{O}((K+C)D^2)$.

Remark 2. We can extend Algorithm 2 in several directions that could potentially yield better performance, though it might require significantly more complex analysis. For instance, we can develop a dynamic version of Algorithm 2 by repeating steps 4 to 12 during each period of the exploitation phase. Additionally, between steps 12 and 13, we could introduce an iterative loop that repeatedly clusters donors and estimates type centers until convergence, mimicking the K-means algorithm. We refer to these as the *dynamic* and the *re-assignment* extensions of Algorithm 2, respectively. We demonstrate the strong performance of these extensions in our numerical studies.

5.2. Performance Analysis

We now present the main result of this section, which provides an upper bound on the regret induced by Algorithm 2. For simplicity of notation, define $\phi = \min_{k \in [K]}\{\frac{\Delta^2}{36(\sigma_k^2 + R_k \Delta/18)}\}$, where $\sigma_k^2 = \sum_{c \in [C]} \operatorname{var}(\tilde{r}_c)$, with $\tilde{r}_c$ being a random variable following the discrete distribution $\boldsymbol{q}_c^k$, and $R_k = \sqrt{\sum_{c \in [C]} \max\{(r_c^k)^2, (\max \mathcal{R} - r_c^k)^2\}}$. Further, define $\xi = C + 1$ and $\kappa = \beta \cdot (K-1) + 1$. The parameter ϕ, inherent to the problem instance, represents the separability of types: larger value of ϕ corresponds to better type separability.

Algorithm 2 Explore-then-exploit campaign algorithm with euclidean clustering.

```
Set number of exploration periods M and the smallest distance between types Δ.
# Exploration
Send all campaigns to all donors for M periods. For each donor d ∈ [D], compute the
    empirical donation vector by r̂_d = (Σ_{t∈[M]} r_dct / M)_{c∈[C]}.
# Clustering donors
Let 𝒟 = [D] be the unclustered donors.
for k ∈ [K] do
    Define 𝒱_d = {d' ∈ 𝒟 : ‖r̂_d − r̂_d'‖_2 < Δ/3} to be neighbors of donor d ∈ [D].
    Find the donor with the largest neighborhood: d_k = arg max_{d∈𝒟} |𝒱_d|. Break ties arbitrarily.
    Let S^k = 𝒱_{d_k} be estimated donors of type k.
    Set 𝒟 = 𝒟 \ S^k. # Remove clustered donors from further consideration.
    Estimate the expected donation vector of type k as r̂^k = 1/|S^k| · (Σ_{d∈S^k} r̂_d).
end for
Assign any remaining donors d ∈ 𝒟 to their closest types S^k, k ∈ [K].
Re-estimate the expected donation vector of type k as r̂^k = 1/|S^k| · (Σ_{d∈S^k} r̂_d).
# Exploitation
For each period t ∈ {M+1, ···, T}, send campaign c ∈ [C] to all donors in S^k, k ∈ [K],
    with r̂^k_c ≥ e_c.
```

Theorem 2. *Fix a campaign problem with parameters K, C, β, T and ϕ. Assume that the size of the smallest type is no smaller than 3, and suppose*

$$T \geq \frac{\left[2\ln\left(3K\exp(\phi)\right) + 3\ln(\kappa+1)\right](\kappa+1)^2 + 3(\kappa+1)}{3.6\kappa\phi}.$$

Choosing $M = \lceil \frac{\ln(2.4\kappa\xi T\phi)}{\phi} \rceil$, and recalling that $\Delta^c_{\max} = \max_{k\in[K]}|r^k_c - e_c|$, the regret incurred by Algorithm 2 is upper bounded by

$$\sum_{c\in[C]}(\Delta^c_{\max} + \max\mathcal{R})\left(\frac{\ln(2.4\kappa\xi T\phi)+\phi}{2\phi} + \frac{1}{2\phi} + \sqrt{\frac{\kappa T[\ln(KN_K) + \ln(2.4\kappa\xi T\phi)+\phi]}{2.4N_K\phi}} + \frac{3}{2}\sqrt{\frac{(\max\mathcal{R})^2\ln(8KN_KTC)T^2\phi}{2N_K\ln(2.4\kappa\xi T\phi)}}\right).$$

Next, let us focus on the components of the regret expression in Theorem 2 and provide managerial insights on why Algorithm 2 is effective. Before diving in, note that the analysis has been simplified to enhance clarity. Specifically, as it is clear in the proof, we do not select our exploration periods M to minimize the overall regret but rather to balance the regret incurred due to exploration and expected clustering errors. We adopt this approach because we assume our clustering algorithm is applied to scenarios with large types, where the regret from all other components becomes negligible once the aforementioned trade-off is achieved. This will become clear as we discuss the regret expression in detail below. In high level, the regret can be divided into three parts: due to *exploration*, *mis-clustering*, and *estimation deviations*. We examine these components one by one below. Since the multiplier $\sum_{c\in[C]}(\Delta_{\max}^c + \max \mathcal{R})$ applies to all parts, we omit it to focus on the terms within the bracket.

The regret due to exploration is $\frac{\ln(2.4\kappa\xi T\phi)+\phi}{2\phi}$, closely related to the number of exploration periods. If the exploration phase is too short, clustering errors can be large, leading to high regret during exploitation. Conversely, if exploration is too long, while clustering accuracy improves and more precise campaign decisions are made during exploitation periods, sending too many letters to uninterested recipients incurs higher regret during exploration periods. Thus, the exploration length must balance the regrets from both exploration and exploitation. Because the goal of exploration is to cluster donors accurately, the number of periods required should be inversely related to the separability ϕ of donor types, aligning with our expression.

The regret from mis-clustering is $\frac{1}{2\phi} + \sqrt{\frac{\kappa T(\ln(KN_K)+\ln(2.4\kappa\xi T\phi)+\phi)}{2.4N_K\phi}}$. This term has two components: the first component, $\frac{1}{2\phi}$, captures the probability of mis-clustering a donor after the chosen M periods. We call it the expected clustering error, or the expected portion of mis-clustered donors in each type. However, because the actual size of each type is finite, the realized mis-clustered portion of donors can deviate from its expectation by η, a random amount. By Hoeffding's inequality, as η increases, the likelihood of such deviations decreases. Thus, we select η that maximize the expected regret from mis-clustered donors, yielding the second component: $\sqrt{\frac{\kappa T(\ln(KN_K)+\ln(2.4\kappa\xi T\phi)+\phi)}{2.4N_K\phi}}$. We call it the deviation of mis-clustering error. To be rigorous, we note that while $\frac{1}{2\phi}$ includes part of regret from mis-clustering deviations, the amount is marginal and omitted for simplicity.

The regret from estimation deviations of the expected donation vector is $\frac{3}{2}\sqrt{\frac{(\max\mathcal{R})^2\ln(8KN_KTC)T^2\phi}{2N_K\ln(2.4\kappa\xi T\phi)}}$. It is important to clarify that this term accounts solely for the estimation error due to finite samples, as the error caused by mis-clustering has already been included in the second part. Because type sizes and exploration periods are finite, our estimates of the expected donation vector, based on all donors in a type, can deviate from the true value by a margin ν. Using a similar approach as for η, we can bound the regret from estimation errors by the given value.

As indicated by their expressions, the regrets incurred by deviations in clustering error and in estimation error are inversely related to the type size N_K. This phenomenon should also be evident from the law of large numbers: the larger the type size, the smaller the deviation in the portion of mis-clustered donors and in the estimation of donation vectors. Intuitively, clustering is most beneficial in cases where each type is large, supporting our simplification in the derivation of the regret in Theorem 2 on selecting M to minimize the regret from only exploration and expected clustering errors.

5.3. Analysis Strategy

In this section, we provide intermediate results that are useful for the proof of Theorem 2, following which allows us to have a clear understanding of the analysis strategy. Since we are interested in the *a-priori* guarantees of the algorithm in this section, we regard the computed values in Algorithm 2 as random variables as they depend on realizations of random donations.

As mentioned, Algorithm 2 relies on the concentration of random vectors, which is established in Lemma 1 and derived from the matrix Bernstein concentration inequality.

LEMMA 1. *Let $\tilde{\boldsymbol{r}}_1, \tilde{\boldsymbol{r}}_2, \ldots, \tilde{\boldsymbol{r}}_m$ be independent and identical distributed C-dimensional vector-valued measurable random variables supported on $[0, \max\mathcal{R}]^C$ with mean $\boldsymbol{r} = (r_1, r_2, \cdots, r_C)$. Define $R = \sqrt{\sum_{c\in[C]}\max\{(r_c)^2, (\max\mathcal{R} - r_c)^2\}}$ and $\sigma^2 = \sum_{c\in[C]}\mathrm{var}(\tilde{x}_{ic})$. Then, for any $\delta \geq 0$, the following inequality holds:*

$$\mathbb{P}\left[\|\frac{1}{m}\sum_{i=1}^{m}\tilde{\boldsymbol{r}}_i - \boldsymbol{r}\|_2 \geq \delta\right] \leq (C+1)\exp\left(\frac{-m\delta^2}{\sigma^2 + R\delta/3}\right).$$

A direct application of Lemma 1 upper bounds the probability of the event that the empirical donation vector $\tilde{\boldsymbol{r}}$ of a donor of type $k\in[K]$ deviates from its true value by more than $\delta = \frac{\Delta}{6}$ by

$$\mathbb{P}[\|\tilde{\boldsymbol{r}} - \boldsymbol{r}^k\|_2 \geq \frac{\Delta}{6}] \leq (C+1)\exp\left(\frac{-M\Delta^2}{36(\sigma_k^2 + R_k\Delta/18)}\right),$$

where $\sigma_k^2 = \sum_{c\in[C]} \mathrm{var}(\tilde{r}_c)$, with $\tilde{r}_c$ being a random variable following the discrete distribution $\boldsymbol{q}_c^k$ supported on $\mathcal{R}$, and $R_k = \sqrt{\sum_{c\in[C]} \max\{(r_c^k)^2, (\max\mathcal{R} - r_c^k)^2\}}$. This result establishes the concentration behavior of donors, rigorously stated in Lemma 2. From this point forward, we define

$$\eth = \max_{k\in[K]} \left\{ (C+1)\exp\left(\frac{-M\Delta^2}{36(\sigma_k^2 + R_k\Delta/18)}\right)\right\} = \xi \exp(-M\phi),$$

where the last equality follows from the definition of ξ and ϕ.

LEMMA 2. *Fix any type $k \in [K]$. Denote by $\tilde{\omega}_k$ the number of donors of type k whose empirical donation vectors $\tilde{\boldsymbol{r}}_d$ differ from $\boldsymbol{r}^k$ by at least $\frac{\Delta}{6}$ after M periods of exploration. For any $\eta \in (0, 1-\eth)$,*

$$\mathbb{P}\left[\frac{\tilde{\omega}_k}{N_k} \geq \eth + \eta\right] \leq \exp\left(\frac{-N_k\eta^2}{2\eth\cdot(1-\eth)}\right).$$

From now on, we fix the hyperparameter $\eta \in [0, 1-\eth]$. Additionally, we define condition **(C)** and a good event **(E)**, and assume both are satisfied throughout this section:

(C) The choice of $\eth$ and η satisfies $\eth + \eta < \frac{1}{\kappa+1}$.

(E) For each donor type $k \in [K]$, at least $(1-\eth-\eta)N_k$ donors have their empirical donation vectors not deviating by more than $\frac{\Delta}{6}$ from their true type centers.

To prove Theorem 2, we categorize the regret incurred by Algorithm 2 into two main phases: exploration and exploitation. The regret during the exploration phase is straightforwardly bounded by its theoretical upper limit, while the regret during the exploitation phase is divided into two parts:

Part (I): The regret incurred when the complement of **(E)** occurs. While this regret can be large, it happens with low probability, resulting in a small expected regret.

Part (II): The regret incurred when **(E)** occurs. Under these conditions, most donors are correctly clustered, allowing accurate estimation of each types donation vector and leading to effective campaign decisions, thus keeping the regret minimal.

The condition **(C)** is inherently assured by the conditions on T specified in Theorem 2 and is mentioned here for clarity of exposition.

In the following, we present a series of results that outline the analysis of **Part (II)**. We begin by showing that each type identified in Line 7 of Algorithm 2 uniquely corresponds to a set $\tilde{\mathcal{D}}^j = \{d \in [D] : \|\tilde{\boldsymbol{r}}_d - \boldsymbol{r}^j\|_2 < \frac{\Delta}{6}\}, j \in [K]$ and includes a substantial number of donors. This correspondence and the property of $\tilde{D}^j$, for $j \in [K]$, which contains most donors from type j and few donors from other types, allow us to correctly identify donor types.

Lemma 3. *The identified clusters $\tilde{S}^k, k \in [K]$ at Line 7 of Algorithm 2*

(i) intersect precisely with one set $\tilde{\mathcal{D}}^j, j \in [K]$, and vice versa.

(ii) Additionally, $|\tilde{S}^k| \geq (1 - \eth - \eta)N_k$ for all $k \in [K]$.

We note that the first result implies a one-to-one mapping (in terms of intersections) between $\{\tilde{S}^k, k \in [K]\}$ and $\{\tilde{\mathcal{D}}^j, j \in [K]\}$. Due to this correspondence, we say that $\tilde{S}^k$ *corresponds to* type j if $\tilde{S}^k \cap \tilde{\mathcal{D}}^j = \emptyset$. We refer to $\tilde{S}^k$ as being of type j because the majority of donors in $\tilde{S}^k$ originate from type j, a result that will be demonstrated later. Furthermore, we call the donors in $\tilde{S}^k$ that do not belong to type j *contaminating donors*. To clarify, saying that a donor d is of type j means $d \in \mathrm{K}^{-1}(j)$. Given the one-to-one mapping between the identified cluster indices and the true indices, we can redefine the indices of the identified clusters to match those of the true types. Therefore, we will assume this alignment to avoid any confusion. Lemma 7 shows that Lemma 3 (ii) still holds after this re-index. We next show that for each set $\tilde{S}^k$, the number of contaminating donors is small.

Lemma 4. *For every $k \in [K]$, the number of contaminating donors in $\tilde{S}^k$ at line 7 is bounded above by $\sum_{j \in [K], j \neq k} (\eth + \eta) N_k$.*

We have now examined the number of contaminating donors in $\tilde{S}^k, k \in [K]$ identified in the loop. However, since $\cup_{k \in [K]} \tilde{S}^k$ may not include all donors, we have not yet accounted for the number of donors who could be mis-clustered, which is addressed in our next result.

Lemma 5. *After line 11:*

(i) For each type, the number of mis-clustered donors is upper bounded by $(\eth + \eta)D$.

(ii) The total number of donors who are mis-clustered is upper bounded by $2(\eth + \eta)D$.

(iii) For each identified cluster, the number of contaminating donors is upper bounded by $(\eth + \eta)D$.

Next, as Lemmas 3 and 4 suggest a good clustering of donors, we dedicate the remainder of this section to bounding the estimation error of $\tilde{\boldsymbol{r}}^k$ in line 12 of Algorithm 2. In this analysis, we introduce another hyper-parameter $\nu \geq 0$, which is used to bound the error between the true donation vector r_c^k and $N_k^{-1} \sum_{d \in \mathrm{K}^{-1}(k)} \tilde{r}_{dc}$, the empirical expected donation vector assuming the true type affiliation of donors is known.

Lemma 6. *Recall $\tilde{\boldsymbol{r}}^k$ is the estimated donation vector in line 12 of Algorithm 2. Then*

$$\mathbb{P}\left[\|\boldsymbol{r}^k - \tilde{\boldsymbol{r}}^k\|_\infty \leq \kappa \cdot (\eth + \eta) \max \mathcal{R} + \nu \quad \forall k \in [K]\right] \geq 1 - \sum_{c \in [C]} \sum_{j \in [K]} 2 \exp\left(-2 N_j M \nu^2\right).$$

We note that the probabilistic nature of this result stems from the trade-off between the estimation error ν and the probability of correctness in our claim. By carefully choosing ν, we achieve a balance between these two competing factors. With these results, we can upper bound the regret incurred during the exploitation periods when **(E)** holds. The full analysis is provided in the proof of Theorem 2 in Appendix C.3.

5.4. Discussion

We dedicate this section to discussing potential adaptations of Algorithm 2 and the corresponding analysis strategies. Specifically, we modify and analyze the algorithm for scenarios with limited sending capacity, small clusters, and no prior knowledge of β, K or ϕ.

While this is not a limitation for our partner NPO, in scenarios where sending campaigns to all donors is infeasible, the algorithm can be adapted by extending the exploration phase. For instance, if there are D donors and the NPO's sending capacity is S campaign letters per period, we can achieve the same exploration duration M for all donors in $\lceil DM/S \rceil$ periods by dispatching letters in each period to S of the D donors who have not yet received the letter M times. The analysis of this adapted approach largely follows our original one, with the only difference being the number of exploration periods increasing from M to $\lceil DM/S \rceil$, which results in an increase in the regret bound.

As evident from Theorem 2, the type size N_K affects the regret through the expression

$$\sqrt{\frac{\kappa T[\ln(KN_K)+\ln(2.4\kappa\xi T\phi)+\phi]}{2.4N_K\phi}}+\frac{3}{2}\sqrt{\frac{(\max\mathcal{R})^2\ln(8KN_KTC)T^2\phi}{2N_K\ln(2.4\kappa\xi T\phi)}}.$$

Omitting the ln terms, the regret scales as $\sqrt{T}$ when $N_K=\mathcal{O}(T)$, and becomes independent of T when $N_K=\mathcal{O}(T^2)$. This implies that the campaign strategy's regret is driven by the trade-off between regret incurred by exploration and the expected clustering error (the remaining regret expressions in Theorem 1) if each type size is at least $\mathcal{O}(T^2)$. If the type size is no smaller than $\mathcal{O}(T)$, focusing solely on the trade-off between the expected clustering error and exploration periods still results in sub-linear regret.

Building on the above understanding, we divide the scenario where small types (smaller than, say, T) exist into two cases. First, when the aggregate number of donors from all small types constitutes a marginal portion of the total donor pool, we can apply our algorithm by excluding these small types. Given their minor proportion, this exclusion has a negligible impact on the overall regret. Second, if all customer types are small or small types account for the majority of donors, the benefits of clustering in reducing estimation variance are less pronounced, and clustering fails to significantly aid decision-making. In such scenario, we might have no choice but use the donor's own responses to make campaign decisions.

Next, we demonstrate how to adapt Algorithm 2 to the case where K is unknown. We replace the for loop over $[K]$ (see lines 4 to 10) with a while loop that terminates when the identified type is smaller than, say, T. This choice of T aligns with our earlier discussion: excluding small types has little impact on regret and clustering offers limited benefit to small clusters. The analysis of this adapted algorithm closely mirrors that of Algorithm 2. The only adjustment needed is to modify Lemma 3 to show that each of the first K identified clusters corresponds uniquely to a true donor type, although the proof of this new result follows the same reasoning as Lemma 3. We note that this adjustment is required because, under the while loop, it is possible to identify more than K clusters. However, donors in clusters beyond K are those whose empirical donation vectors deviate significantly from their true values and are considered mis-clustered in the analysis of Algorithm 2. All of these imply that the sets of correctly and incorrectly clustered donors remain the same in both Algorithm 2 and this adaptation, and thus both algorithms share the same regret upper bound. We do not include this more general algorithm and analysis in our main paper because we believe the presented version better illustrates the benefits of clustering within our problem setting.

We note that knowledge of β is used to determine the number of exploration periods M, as outlined in Theorem 2. Given that the total population size is D and the smallest meaningful type size is T, as discussed above, we can estimate $\beta = \frac{D}{T} - 1$. Since the exploration periods M is in logarithmic of β, an upper estimate of β does not significantly affect M or the resulting regret. Moreover, the knowledge of β, through condition **(C)**, is only needed in the proof of Lemma 3 to ensure that our identified clusters correspond to true types. This condition is enforced because we assume that all donors whose empirical donation vectors deviate from the true value by more than $\frac{\Delta}{6}$ are concentrated within a ball of diameter $\frac{\Delta}{3}$. However, this is a significant simplification. A modification of the analysis could eliminate the need for condition **(C)** and, consequently, the knowledge of β. Since this adjustment would significantly complicate the analysis without offering additional insight into the problem, we omit it but are prepared to provide a version if there is interest.

The final and most critical adaptation of Algorithm 2 concerns the knowledge of Δ and ϕ. Since Δ and ϕ are closely linked, our discussion will only mentions Δ. Although it is typical to assume some knowledge of Δ or similar parameters in other learning contexts (Farias and Li 2019, Bastani 2021, Miao and Chao 2021), we examine how to adapt our algorithm under three specific conditions: 1) when Δ is small, 2) when knowledge of Δ is imprecise, and 3) when knowledge of Δ is unavailable. If Δ is small, ϕ will also be small, resulting in large regret as indicated in Theorem 2, which is undesirable. If knowledge of Δ is imprecise, the performance of Algorithm 2 may be compromised. Without fixing the value of Δ, Algorithm 2 becomes unfeasible. These issues can be addressed by setting Δ to a larger value and initiating Algorithm 2 from this new value. Details and analysis of the algorithm with this adjusted Δ are deferred to the next section. Besides this strategy, to address the issues 2) and 3), we propose an additional adaptation of the algorithm assuming that donor types are significant. Details of this adaptation are presented in Section 6.3.

5.5. Campaign with Selected Donor Type Separation Value

In this section, we propose setting the donor type distance Δ to a predetermined value to address the issues outlined previously. This approach aims to mitigate the large regret associated with small real values of Δ, address performance uncertainties due to imprecise knowledge of Δ, and resolve the infeasibility of Algorithm 2 when Δ is unknown. The optimal value for Δ is detailed in Theorem 3. The rationale for this choice is that merging donor types with distances smaller than the selected value does not lead to significant sub-optimality, as demonstrated in Lemma 8. Conversely, for types whose distances exceed the selected threshold, we can have good identification of types, thereby minimizing regret.

For analysis of this adaptation, in order to convey the core idea succinctly, we assume that each type of donor is of significant size, i.e., N_K is large enough so that: 1) the portion of donors whose empirical donation vector falls in certain ball is the same as the specified clustering error probability, i.e., $\eta = 0$ in Lemma 2 with probability 1; and 2) the decision specified by the empirical donation vector of a correctly identified donor type is optimal. These assumptions help us simplify the analysis significantly while retaining the main idea unique to the analysis of this adaptation. One can always follow the proof of Theorem 2 to derive a more precise bound without these simplifications.

We note that large type size assumption is not strong as the algorithm with clustering is designed particularly to such case. The two subsequent assumptions are also not stringent as the regret incurred by these two parts is very small under large donor types as mentioned in the discussion of Theorem 2. In the following, Theorem 3 provides the value of Δ, the number of exploration periods M, and the corresponding performance guarantee.

THEOREM 3. *Consider the same campaign problem as stated in Theorem 2. The regret incurred by the algorithm is upper bounded by*

$$(T\sqrt{C}/2)^{2/3}\left(\sum_{c\in[C]}\Delta^c_{\max}(\ln(T\xi)+1)11C(\max\mathcal{R})^2\right)^{1/3}(2^{-2/3}+2^{1/3})+\sum_{c\in[C]}\Delta^c_{\max},$$

when we set $\Delta=\left(\frac{\sum_{c\in[C]}2\Delta^c_{\max}(\ln(T\xi)+1)11C(\max\mathcal{R})^2}{T\sqrt{C}}\right)^{1/3}$ *and* $M=\lceil\frac{\ln T\xi\phi}{\phi}\rceil$.

We note that $\Delta^c_{\max}, c\in[C]$ is required in the determination of Δ. It would benefit our algorithm if a good estimation of $\Delta^c_{\max}$ is available. However, we can always bound $\Delta^c_{\max}$ by $\max\{e_c,\max\mathcal{R}-e_c\}$ in the worst-case. Additionally, The value of ϕ in the result should be calculated using the chosen Δ. As we can see from the result, the performance of the adapted algorithm is irrelevant to the real separation parameter of donor types anymore.

6. Numerical Studies

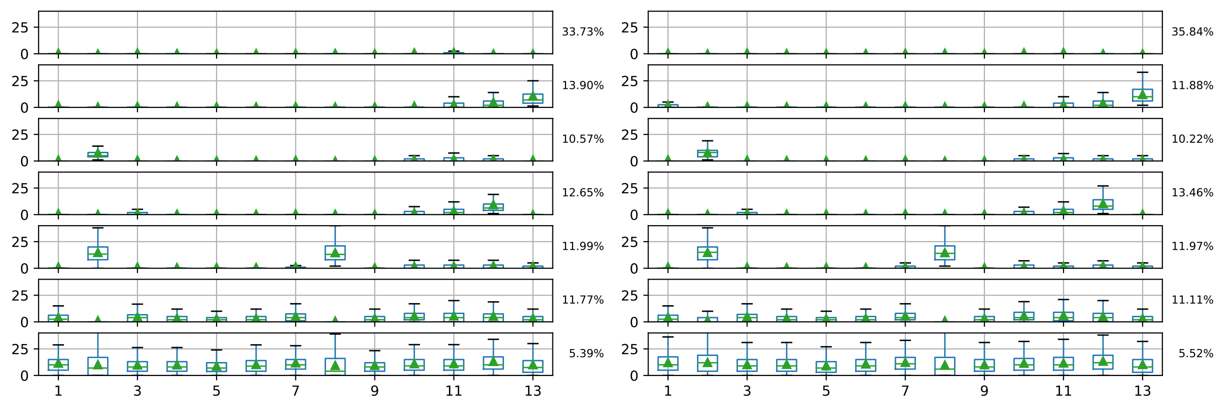
This section presents numerical results utilizing real-world data. In Section 6.1, we compare the performance enhancements provided by our proposed algorithm over other algorithms on real data from our partner NPO. In Sections 6.2 and 6.3, we demonstrate the performance of our algorithms when faced with uncertain knowledge about the input values, i.e., uncertainty in the donation distributions $\{\mathbf{Q}^k\}_{k\in[K]}$ of Algorithm 1 and uncertainty in the smallest distance Δ between expected donation vectors of Algorithm 2, respectively. These robustness checks provide additional insight into when Algorithm 1 can outperform Algorithm 2 under uncertain estimates of the true donation distributions. Furthermore, they offer guidance on how to proceed when a precise estimate of the type separation distance Δ is unavailable for Algorithm 2 All codes were implemented in Python and executed on Intel Xeon 2.20GHz cluster nodes with 64 GB of dedicated main memory in four-core mode.

6.1. Performance of Algorithms on Real Data

We test our algorithms on the real dataset from our partner NPO, comprising contact and donation behaviors of over 1.5 million donors from 2018 to 2022. We select all individuals

campaigns	1	2	3	4	5	6	7	8	9	10	11	12	13
Mean	2.83	7.26	1.98	1.52	1.48	1.50	2.23	3.42	1.43	2.40	3.00	3.95	2.91
	2.61	3.60	1.98	1.38	1.31	1.36	2.03	2.81	1.29	2.39	3.02	3.70	2.88
Std	10.29	12.60	7.68	7.12	7.12	7.07	8.33	9.87	6.81	8.49	9.92	12.24	11.20
	10.01	9.47	7.64	6.73	6.67	6.73	7.96	9.00	6.51	8.51	9.97	11.88	11.13

Table 1 **Mean and variance description of real data and synthetic. For each category, the first entry represents the value from real data while the second entry represents that from synthetic data.**



(a) Real data (b) Synthetic data

Figure 3 **Comparison of clustering results using real data and synthetic data. Based on the analysis of the figures, it is evident that the clustering results are remarkably similar for both the real and synthetic data sets. This similarity underscores the effectiveness of the synthetic data in replicating the clustering characteristics of the real data.**

who had been contacted at least 39 times during 65 total campaigns (13 campaigns annually over 5 years), resulting over 200,000 donors, for this analysis. Donors are clustered into types using the K-means algorithm based on their average donation amounts to all contacted campaigns, and empirical donation distributions are then estimated for each type, serving as the ground truth. Here, the choice of K is the same as the one discussed in Section 3.1. We maintain records of donor contacts and binary donation responses while anonymizing true donation amounts by sampling from the conditional distribution derived from real data. For donor-campaign pairs not previously contacted in the original data, we impute the data by sampling the donation according to the distribution of the type to which the donor is allocated. This procedure preserves donation behavior information (*cf.* Table 1 and Figure 3) and ensures donor privacy. This anonymized dataset is shared publicly to foster further research on NPO operations.

We remark that our algorithm runs well using real data; however, the performance evaluation of our algorithm depends on the outcomes of our campaign decisions, which can not be extracted from real data. Consistent with practices in the online learning literature, we employ the data to estimate a ground truth. This estimated ground truth is then used to simulate the donation outcomes of our actions, enabling us to assess the performance of Algorithm 1 and Algorithm 2 against that of the optimal policy. We compare the performance of our algorithms against the current strategy of our partner NPO (NPO), the explore-then-exploit algorithm (ETE), the upper confidence bound algorithm (UCB), the epsilon greedy algorithm (EPG), a neighborhood-based algorithm (NBA) from recommender system literature, and a simple algorithm that sends all campaigns to all donors (Send_All).

Details on the EPG and UCB algorithms are found in Sections 6 and 7 of Lattimore and Szepesvári (2020), respectively. We run the EPG algorithm under different values of $\epsilon \in \{0.1, 0.2, \dots, 1.0\}$ and select the best result. For the UCB algorithm, the optimism value is too large in our short-horizon campaign problem, leading to bad performance. To mitigate this, we scale the optimism value by factors in $\{0.1, 0.2, \dots, 1.0\}$ and report the best performance. The NBA algorithm, detailed in Section 2.1 of Jannach et al. (2010), is run with neighborhood sizes in $\{500, 800, 1000, 1200, 1500, 1800, 2000, 2500, 3000\}$. The neighborhood is computed using the two-norm distance of their empirical donation vectors, and campaign decisions to a donor are based on the estimated donation vector using the responses of all their neighbors. For this algorithm, we report the best performance across different neighborhood sizes. Additionally, we implement the dynamic versions of Algorithms 1 and 2, as discussed in Remarks 1 and 2, respectively, denoted by "Dy_Alg1" and "Dy_Alg2".

For each algorithm, instead of the absolute regret defined in Section 3, for better interpretation, we report the relative regret:

$$\text{Relative regret} = \frac{\text{regret of the algorithm}}{\text{net donation of the optimal policy}} \times 100\%.$$

This metric quantifies the percentage of the net donation that is lost by different algorithms. We repeat each algorithm for 50 times and the mean of the relative regret is presented in Figure 4a. Here, we do not run for more repetitions because the variance is small. Figure 4b illustrates the outcomes of Algorithm 2 with reassignment of donors as explained in Remark 2. In both diagrams, X-axis is the total period of the campaigning problem and the Y-axis is the relative regret.

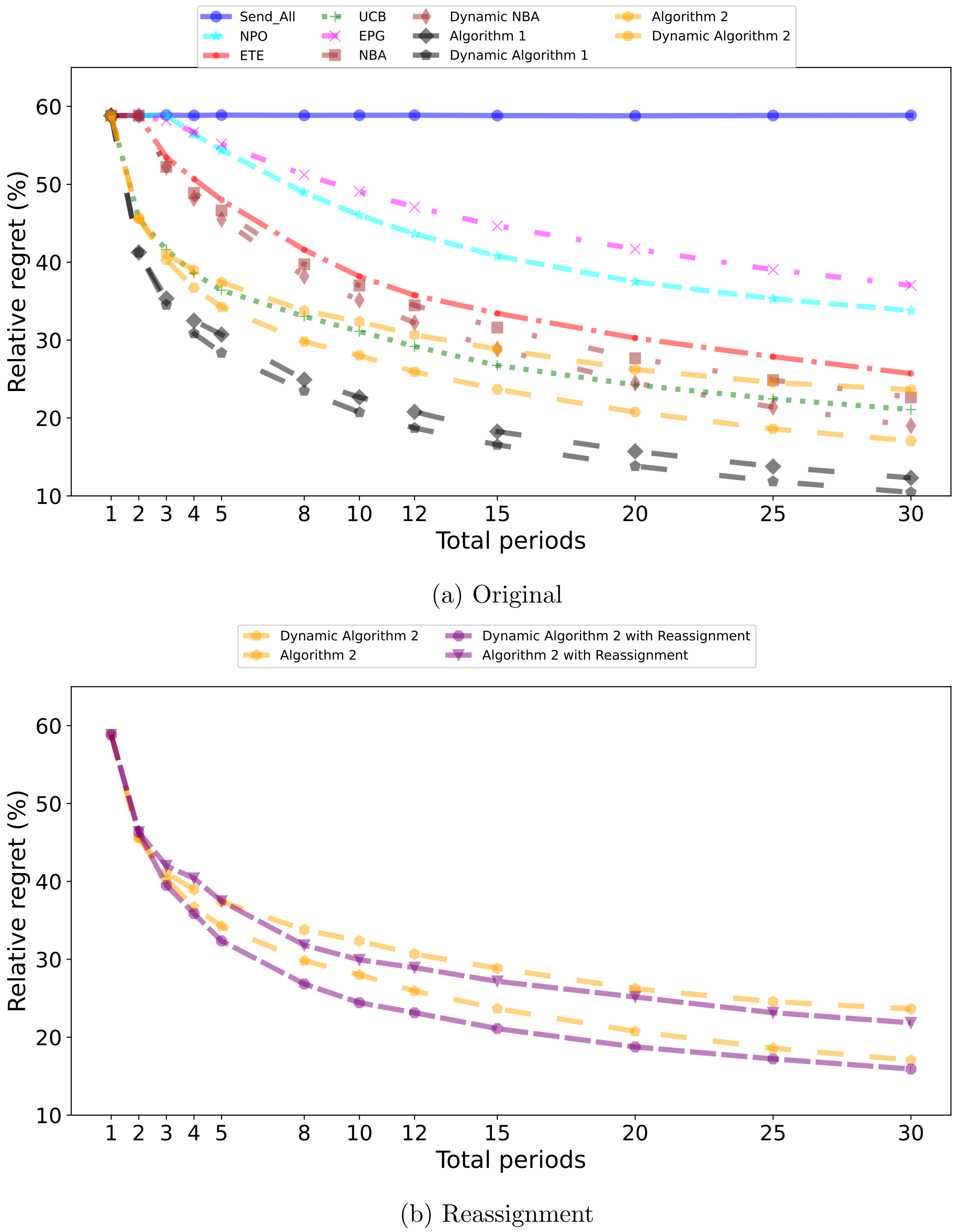


Figure 4 **Comparison of relative regret among different algorithms in log scale. The left diagram compares the performance of Algorithms 1 and 2 and their dynamic version with all other algorithms. The right diagram demonstrates the value of the reassignment modification of Algorithm 2 as discussed in Remark 2.**

Figure 4a clearly demonstrates that Algorithm 1 and its dynamic version perform the best, which is not surprising as both algorithms utilize the true donation distributions while all other algorithms do not. When donation distributions are unknown, the dynamic version of Algorithm 2 consistently outperforms all competing algorithms from the literature, especially under short campaign durations. Additionally, regardless of whether donation distributions are known, dynamic version of our algorithms consistently show improved performance, aligning with our expectations that more information about each donor's behavior enhances clustering accuracy, and hence, campaigning decisions. Furthermore, Figure 4b suggests that the reassignment of donors after clustering can strengthen the performance of our proposed Algorithm 2 and its dynamic variant. If we can only make one modification, dynamic clustering is preferred to the reassignment variant.

6.2. Robustness of Algorithm 1

In this section, we perform a robustness check of Algorithm 1 under erroneous specifications of donation distributions. Through this process, we gain additional insight into the level of uncertainty in our estimation of the ground truth donation distributions under which Algorithm 1 remains preferable to Algorithm 2.

We perform a robustness check by perturbing the underlying donation distributions. Recall that a donation distribution $\mathbf{Q}^k = (\boldsymbol{q}_c^k)_{c\in[K]}$ can be regarded as a matrix with each column $\boldsymbol{q}_c^k$ representing a distribution to campaign c over the support $\mathcal{R}$ of donation amount. For each donation distribution $\boldsymbol{q}_c^k, c \in [C]$ of type $k \in [K]$, the perturbed distribution $\bar{\boldsymbol{q}}_c^k$ is

$$\bar{\boldsymbol{q}}_c^k \propto (\max\{0, q_{cr}^k + \epsilon u_r\})_{r\in\mathcal{R}} \text{ with } u_r \in \{-1, 1\}, \sum_{r\in\mathcal{R}} |u_r| = \Gamma,$$

with $u_r, r \in \mathcal{R}$ being sampled randomly and independently from a uniform distribution. Here, the perturbed distribution is proportional to the given expression as we must normalize it to remain a valid distribution. We can semi-interpret the perturbed distribution as a random element from the budget uncertainty set centered at $\boldsymbol{q}_c^k$ with Γ being the budget and ϵ the scale.

We take the perturbed distributions as if they were the true ones, run Algorithm 1 on the data generated from the true distribution, and record the performance. We repeat this process of generating perturbed distributions and running Algorithm 1 for 250 times, then compute the mean and variance of the relative regret. We report the difference between the mean relative regret and that of Algorithm 2, where no donation distributions are assumed, to better understand which degree of mis-specification of donation distributions remains beneficial or detrimental to our campaign decisions.

Tables 2 and 3 present the results of Algorithm 1 and its dynamic variant under different values of scale ϵ and budget Γ when the campaign horizon is 10 years. We have included

		ϵ										
		0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
Γ	1	-7.28	-7.28	-7.27	-7.22	-7.12	-6.84	-5.76	-3.90	-0.23	3.23	5.85
		0.04	0.04	0.04	0.09	0.17	0.30	1.05	2.18	3.70	11.51	17.16
	2	-7.28	-7.28	-7.26	-7.17	-7.09	-6.55	-4.74	-1.51	3.99	9.05	13.00
		0.04	0.04	0.04	0.10	0.16	0.35	1.26	3.37	5.53	16.71	27.45
	3	-7.28	-7.27	-7.24	-7.12	-6.98	-6.10	-3.44	0.78	7.02	12.60	16.57
		0.04	0.04	0.05	0.16	0.22	0.48	1.88	3.74	9.42	34.97	46.43
	4	-7.28	-7.27	-7.24	-7.08	-6.88	-5.67	-2.24	3.15	8.92	15.38	20.23
		0.04	0.04	0.04	0.21	0.29	0.60	2.26	4.59	12.77	45.12	50.76
	5	-7.28	-7.27	-7.22	-7.09	-6.86	-5.22	-0.90	4.56	9.85	16.94	22.44
		0.04	0.04	0.06	0.17	0.24	0.80	2.92	7.90	15.84	43.14	44.35
	6	-7.28	-7.27	-7.17	-7.03	-6.77	-4.78	0.30	5.73	11.79	18.43	23.69
		0.05	0.05	0.09	0.21	0.28	0.96	3.30	7.54	23.30	44.82	39.69
	7	-7.28	-7.27	-7.19	-6.99	-6.72	-4.20	1.46	6.47	13.66	20.11	24.92
		0.04	0.05	0.08	0.26	0.37	1.32	3.32	6.44	31.41	49.16	36.40
	8	-7.28	-7.26	-7.16	-7.01	-6.68	-3.45	2.77	7.11	14.86	21.82	25.80
		0.05	0.05	0.11	0.25	0.37	1.78	3.27	6.89	39.40	42.60	29.96
	9	-7.28	-7.26	-7.15	-7.02	-6.58	-2.78	3.30	7.39	15.23	22.42	26.15
		0.05	0.05	0.10	0.22	0.33	1.95	5.98	5.71	38.58	46.65	28.51
	10	-7.27	-7.26	-7.13	-6.93	-6.40	-2.22	4.06	8.15	16.83	24.33	27.29
		0.05	0.05	0.12	0.27	0.41	2.31	6.48	6.11	48.65	40.35	20.11

Table 2 Relative performance of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 10. For each Γ, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.

results under different campaign horizons in Appendix F. In the table, for each value of $\Gamma - \epsilon$ pair, the first entry represents the difference in relative regret, while the second entry indicates the variance of the relative regret. The results suggest that as the mis-specification of donation distributions increases, the performance of Algorithm 1 deteriorates and shows a larger variance. When the budget for mis-specification is small, Algorithm 1 can tolerate a larger mis-specification scale, and vice versa. In our campaign problem, the tolerance of Algorithm 1 is quite limited, as a difference of two percent can make its performance worse than that of Algorithm 2, where no donation distributions are assumed at all. Based on our understanding, this sensitivity arises from the low donation probability in the ground truth, where even a two-percent deviation in donation probability constitutes a significant change in donation behavior.

6.3. Robustify Algorithm 2

This section first presents an adaptation of Algorithm 2 that can cope with the case where our estimation of Δ is imprecise. Then, we present numerical performance of this adaptation and compare it against that of Algorithm 3, where no value of Δ is required and campaign decisions are made for each donor based on their individual responses,

Γ	ϵ: 0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
1	-3.72	-3.71	-3.70	-3.62	-3.50	-3.11	-1.93	0.14	3.72	7.14	10.01
	0.04	0.04	0.04	0.18	0.40	0.66	1.38	2.50	5.09	13.35	18.28
2	-3.71	-3.71	-3.70	-3.55	-3.48	-2.84	-0.83	2.30	8.16	13.49	17.87
	0.04	0.04	0.04	0.24	0.40	0.64	2.93	4.67	7.44	16.89	25.35
3	-3.71	-3.71	-3.68	-3.47	-3.36	-2.30	0.29	4.82	11.67	17.65	22.28
	0.04	0.04	0.05	0.37	0.52	0.74	3.63	5.19	6.59	20.06	27.78
4	-3.71	-3.71	-3.69	-3.41	-3.23	-1.79	1.51	7.44	14.67	21.01	26.06
	0.04	0.04	0.04	0.53	0.65	0.79	3.74	5.70	8.07	22.41	27.55
5	-3.71	-3.71	-3.65	-3.47	-3.28	-1.28	2.83	9.39	16.49	23.30	28.54
	0.04	0.04	0.09	0.43	0.50	0.95	4.29	5.52	9.03	19.57	21.98
6	-3.72	-3.71	-3.59	-3.37	-3.20	-0.83	4.30	11.46	18.77	25.09	29.72
	0.04	0.04	0.18	0.52	0.57	2.83	4.72	5.15	12.02	19.27	19.42
7	-3.71	-3.70	-3.62	-3.32	-3.11	-0.34	5.61	12.87	20.58	26.58	30.96
	0.04	0.04	0.14	0.64	0.68	2.58	4.58	4.73	13.70	21.61	16.21
8	-3.71	-3.70	-3.58	-3.37	-3.08	0.34	7.21	13.92	22.06	28.30	31.87
	0.04	0.04	0.24	0.63	0.67	3.42	4.60	4.02	16.18	17.41	13.25
9	-3.71	-3.70	-3.58	-3.40	-3.02	0.87	8.33	14.89	23.27	29.41	32.35
	0.04	0.04	0.21	0.55	0.54	3.64	4.01	3.06	16.26	17.40	11.03
10	-3.71	-3.70	-3.54	-3.28	-2.80	1.33	9.60	15.61	24.50	30.76	33.27
	0.04	0.04	0.27	0.64	0.67	3.98	4.12	2.90	20.48	15.73	7.56

Table 3 Relative performance of dynamic version of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 10. For each Γ, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.

i.e., without clustering donors. The approach proposed here also provides an alternative solution the problem that the value of Δ is unavailable.

Let us demonstrate how we can robustify Algorithm 2 to address our imprecise estimation of Δ. In this adaptation, we introduce a type size threshold, which represents the minimum size an identified cluster must have to be considered a valid type. This proposal of this threshold leverages our former discovery that only large donor types contribute meaningfully to decision making and the fact that each donor type is of large size. Using this threshold, we follow Algorithm 2 to perform exploration and identify the first donor type. However, we only accept this identified cluster as a true type if the size of the identified cluster is larger than the threshold. Otherwise, we increase the value of Δ by a factor larger than 1, for instance, 1.1, and re-identify the first cluster using this updated value. This process is repeated until the size of the identified cluster meets or exceeds a predefined threshold, at which point we consider this cluster as a true type. We then repeat this procedure to identify each subsequent true type until all types are identified.

We run this adapted algorithm using the same ground truth as in Section 6.1. Instead of using the precise value of Δ, we multiply it by factors $[0.01, 0.1, 1, 6, 10, 20]$ and treat

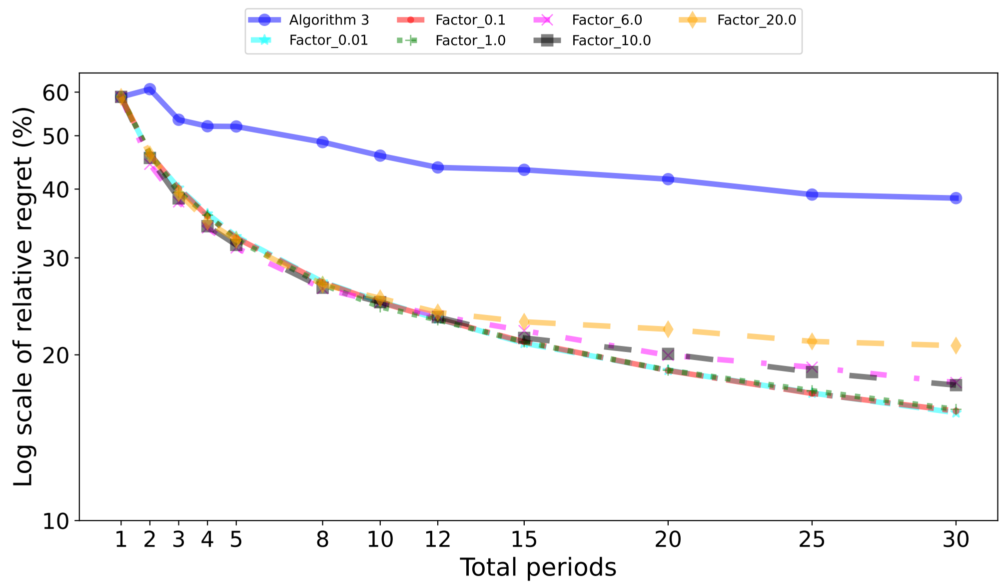


Figure 5 **Performance comparison between incorrect specifications of Δ and Algorithm 3 where no knowledge of Δ is required.**

them as our estimate of Δ, setting the type size threshold at 1000. The choice of factor 6 is because our algorithm divides Δ by 6. Thus, we aim to evaluate the performance of the algorithm when Δ is used as the distance. The performance for each factor under this adaptation is illustrated in Figure 5, with lines labeled 'Factor' followed by the factor number (e.g., the line 'Factor_1.0' corresponds to a factor of 1). These are compared against the performance of Algorithm 3 that requires no knowledge of Δ.

Figure 5 clearly shows that our adaptation of Algorithm 2 consistently outperforms Algorithm 3 irrespective of the scaling applied to Δ. In Algorithm 3, lacking knowledge of Δ results in very conservative optimal exploration periods as outlined in Lemma 14. In our short-horizon campaign problem, these conservative exploration periods are insufficient to reduce the variance in the estimation of expected donation vector based on single individual campaign responses. As a result, campaign decisions made based on the estimated expected donation vector are imprecise, thereby incurring large regret. Conversely, utilizing Δ allows for the clustering of donors within a defined proximity, enhancing the accuracy of donor type estimation and, consequently, campaign decisions. Moreover, starting with a very small scaling of Δ does not significantly impact performance, as it will be incrementally scaled up if the identified cluster is too small. Consequently, for very small initial guesses of Δ, we will ultimately converge to a similar value of Δ when the clustered donors are accepted as a type, resulting in similar clustering outcomes. Thus, their performance does not differ significantly, aligning with the results shown in Figure 5. However, excessive scaling of Δ results in fewer, larger groups, impairing the ability to tailor campaign decisions effectively to each donor and consequently compromising performance, as evidenced by the performance decline at larger factors.

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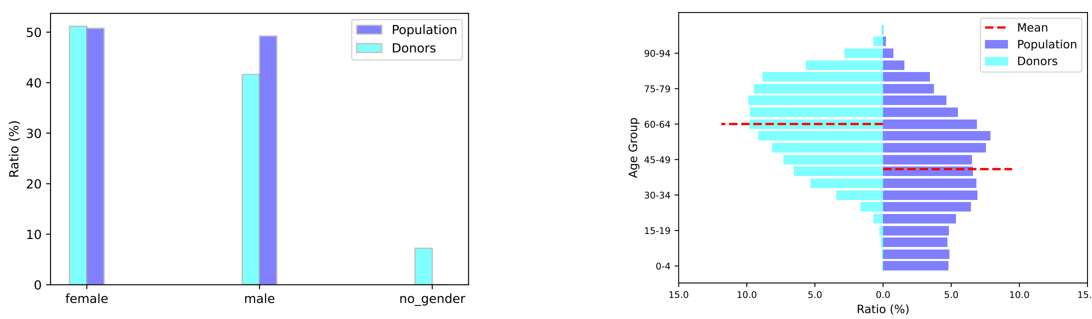
Appendix A: Donation Data Analysis

In this section, we present some data analysis result that would be helpful to understand the donation behavior of the non-profit organization. Below, we first demonstrate that donors of our partner NPO are predominantly from older age groups. Additionally, given the clustering of donors, both the correlation and temporal dependence among donations are negligible.

A.1. Demographic Comparisons of Donors and General Population

Figure 6 presents the age and gender distribution of the donors compared to that of the general population of the country. The proportion of female donors mirrors that of the general population, whereas the proportion of male donors is significantly lower. Notably, a substantial number of donors do not declare their gender, though the figures suggest that most non-declared gender donors are likely male. The age distribution of donors reveals a pronounced skew towards older individuals, with the average age of donors significantly higher than that of the general population. This difference corroborates findings that seniors are more inclined to donate to charitable campaigns (Freund and Blanchard-Fields 2014, Cutler et al. 2021).



(a) Gender

(b) Age distribution

Figure 6 Gender and age distribution of donors compared to the general population in our partner NPO's country

A.2. Correlation of Donations within Clusters

In this section, we study correlations between donations for different campaigns, which is achieved by reporting the Phi coefficient, denoted by Φ, which is a measure of correlation among Bernoulli random variables in the literature.

	$\tilde{B}=1$	$\tilde{B}=0$
$\tilde{A}=1$	a	b
$\tilde{A}=0$	c	d

Table 4 Demonstration table of Phi coefficient.

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	0.01	0.02	0.04	0.03	0.02	0.04	0.01	0.05	0.03	0.02	0.01	0.05
2	0.01	1	0.00	-0.00	0.01	-0.00	0.00	0.05	-0.01	0.01	0.02	0.01	0.01
3	0.02	0.00	1	0.02	0.02	0.02	0.05	0.01	0.03	0.04	0.12	0.04	0.01
4	0.04	-0.00	0.02	1	0.02	0.03	0.05	0.00	0.07	0.02	0.03	0.04	0.05
5	0.03	0.01	0.02	0.02	1	0.05	0.07	-0.01	0.04	0.04	0.06	0.05	0.04
6	0.02	-0.00	0.02	0.03	0.05	1	0.03	0.02	0.06	0.04	0.04	0.04	0.05
7	0.04	0.00	0.05	0.05	0.07	0.03	1	0.02	0.04	0.07	0.04	0.03	0.05
8	0.01	0.05	0.01	0.00	-0.01	0.02	0.02	1	0.01	0.00	0.01	-0.00	0.01
9	0.05	-0.01	0.03	0.07	0.04	0.06	0.04	0.01	1	0.00	0.01	0.04	0.04
10	0.03	0.01	0.04	0.02	0.04	0.04	0.07	0.00	0.00	1	0.03	0.03	0.03
11	0.02	0.02	0.12	0.03	0.06	0.04	0.04	0.01	0.01	0.03	1	0.01	-0.00
12	0.01	0.01	0.04	0.04	0.05	0.04	0.03	-0.00	0.04	0.03	0.01	1	-0.01
13	0.05	0.01	0.01	0.05	0.04	0.05	0.05	0.01	0.04	0.03	-0.00	-0.01	1

Table 5 Phi coefficient of the campaign donation outcomes of cluster 1.

Quantitatively, the Phi coefficient is calculated from a 2×2 contingency table (*cf.* Table 4) using formula:

$$\Phi = \frac{ad - bc}{\sqrt{(a+b)(c+d)(a+c)(b+d)}}.$$

Here, a, b, c, and d represent the counts of the respective combinations of outcomes for random variables $\tilde{A}$ and $\tilde{B}$. This formula is derived from the chi-square statistic (χ^2) for independence in a 2×2 table, where Φ is essentially the square root of the chi-square statistic divided by the sample size, adjusted for the dimensions of the table.

The Phi coefficient is a statistic that measures the correlation between two binary variables. It is similar to the Pearson correlation coefficient but is specifically designed for binary data. The Phi coefficient provides a value between -1 and +1, where:

- $\Phi = 1$ suggests a perfect positive relationship, meaning as one variable increases, the other also increases.
- $\Phi = -1$ indicates a perfect negative relationship, implying that as one variable increases, the other decreases.
- $\Phi = 0$ signifies no association between the variables.

We compute the within-clusters Phi coefficient of different clusters. Clusters are the same as the one presented in Figure 2. Because the difference between the two computation approaches is marginal, we computed the index using all five-year data together. Table 5 to 11 gives the results, which indicate that donation correlation across campaigns is *very small* in each obtained clusters. We note that we do have shading in all the tables. Table 5 to 11 looks very transparent because their correlation is very small.

A.3. Temporal Dependence of Donations

In this section, we explore the impact of a donor's previous donations on their likelihood to contribute in the current period. Specifically, we investigate whether a donor's propensity to donate changes based on the frequency of their donations in the past q periods. This concept of temporal dependence in decision-making has been extensively studied in fields such as operations management and marketing. Specifically, we examine this phenomenon within specific donor clusters rather than across the entire donor set. We aim to determine whether there is a temporal dependence on donation behavior within these clusters.

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	0.02	-0.00	0.02	0.02	0.04	0.03	0.01	0.03	-0.00	0.00	0.02	0.03
2	0.02	1	-0.01	-0.01	-0.01	0.01	0.01	-0.00	0.02	0.01	0.02	0.04	0.02
3	-0.00	-0.01	1	0.01	0.02	0.04	0.02	-0.01	0.03	0.02	0.07	0.01	0.00
4	0.02	-0.01	0.01	1	0.01	0.05	0.02	-0.00	0.03	0.04	0.01	0.03	0.03
5	0.02	-0.01	0.02	0.01	1	0.03	0.04	0.03	0.05	0.04	0.03	0.03	0.03
6	0.04	0.01	0.04	0.05	0.03	1	0.01	0.02	0.05	0.02	0.02	0.05	0.02
7	0.03	0.01	0.02	0.02	0.04	0.01	1	0.02	0.01	0.05	0.01	0.01	0.02
8	0.01	-0.00	-0.01	-0.00	0.03	0.02	0.02	1	-0.00	0.00	-0.03	-0.04	0.02
9	0.03	0.02	0.03	0.03	0.05	0.05	0.01	-0.00	1	-0.00	0.01	0.01	0.02
10	-0.00	0.01	0.02	0.04	0.04	0.02	0.05	0.00	-0.00	1	-0.00	-0.01	-0.01
11	0.00	0.02	0.07	0.01	0.03	0.02	0.01	-0.03	0.01	-0.00	1	-0.04	-0.04
12	0.02	0.04	0.01	0.03	0.03	0.05	0.01	-0.04	0.01	-0.01	-0.04	1	-0.05
13	0.03	0.02	0.00	0.03	0.03	0.02	0.02	0.02	0.02	-0.01	-0.04	-0.05	1

Table 6 **Phi coefficient of the campaign donation outcomes of cluster 2.**

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	0.01	-0.02	0.02	-0.00	0.03	0.00	0.02	0.04	-0.01	-0.03	-0.03	0.00
2	0.01	1	-0.01	-0.01	-0.01	-0.02	-0.01	0.02	-0.01	-0.01	-0.01	-0.01	-0.00
3	-0.02	-0.01	1	-0.02	-0.00	0.01	0.01	0.01	0.01	-0.01	0.02	-0.02	0.00
4	0.02	-0.01	-0.02	1	0.01	0.04	0.03	0.01	0.06	0.00	-0.01	0.02	0.02
5	-0.00	-0.01	-0.00	0.01	1	-0.01	0.01	0.01	0.03	0.01	0.01	0.01	0.00
6	0.03	-0.02	0.01	0.04	-0.01	1	-0.01	0.00	0.07	0.00	-0.00	0.01	0.05
7	0.00	-0.01	0.01	0.03	0.01	-0.01	1	0.00	0.00	0.00	-0.02	-0.02	-0.00
8	0.02	0.02	0.01	0.01	0.01	0.00	0.00	1	-0.02	-0.03	-0.04	-0.02	0.01
9	0.04	-0.01	0.01	0.06	0.03	0.07	0.00	-0.02	1	-0.04	-0.04	-0.00	0.01
10	-0.01	-0.01	-0.01	0.00	0.01	0.00	0.00	-0.03	-0.04	1	-0.07	-0.05	-0.05
11	-0.03	-0.01	0.02	-0.01	0.01	-0.00	-0.02	-0.04	-0.04	-0.07	1	-0.10	-0.07
12	-0.03	-0.01	-0.02	0.02	0.01	0.01	-0.02	-0.02	-0.00	-0.05	-0.10	1	-0.11
13	0.00	-0.00	0.00	0.02	0.00	0.05	-0.00	0.01	0.01	-0.05	-0.07	-0.11	1

Table 7 **Phi coefficient of the campaign donation outcomes of cluster 3.**

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	0.01	-0.07	0.03	0.03	0.05	0.04	-0.00	0.02	0.01	-0.01	0.01	0.01
2	0.01	1	-0.03	-0.01	0.02	0.01	0.00	0.09	0.00	0.02	0.00	0.02	0.02
3	-0.07	-0.03	1	-0.04	0.02	0.01	0.01	0.00	0.02	-0.01	0.07	0.02	0.01
4	0.03	-0.01	-0.04	1	-0.01	0.04	0.03	0.01	0.05	0.01	-0.00	0.04	0.06
5	0.03	0.02	0.02	-0.01	1	0.01	0.07	0.03	0.05	0.00	0.01	0.03	0.05
6	0.05	0.01	0.01	0.04	0.01	1	-0.03	0.01	0.05	0.03	-0.01	0.03	0.05
7	0.04	0.00	0.01	0.03	0.07	-0.03	1	0.02	-0.04	0.02	0.00	0.03	0.04
8	-0.00	0.09	0.00	0.01	0.03	0.01	0.02	1	-0.02	-0.01	-0.00	0.00	0.01
9	0.02	0.00	0.02	0.05	0.05	0.05	-0.04	-0.02	1	-0.06	-0.03	0.03	0.03
10	0.01	0.02	-0.01	0.01	0.00	0.03	0.02	-0.01	-0.06	1	-0.08	-0.02	0.01
11	-0.01	0.00	0.07	-0.00	0.01	-0.01	0.00	-0.00	-0.03	-0.08	1	-0.06	-0.02
12	0.01	0.02	0.02	0.04	0.03	0.03	0.03	0.00	0.03	-0.02	-0.06	1	-0.04
13	0.01	0.02	0.01	0.06	0.05	0.05	0.04	0.01	0.03	0.01	-0.02	-0.04	1

Table 8 **Phi coefficient of the campaign donation outcomes of cluster 4.**

Following methodologies similar to those found in the literature (Kamakura 2014, Zhang et al. 2017, Sudhir et al. 2019), we employ a utility-based model to assess whether such temporal dependencies significantly

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	0.04	-0.05	0.12	0.05	0.09	0.04	0.01	0.11	-0.00	-0.02	-0.01	0.01
2	0.04	1	0.04	0.01	0.01	0.02	0.01	0.16	0.01	0.04	0.03	0.01	0.01
3	-0.05	0.04	1	-0.01	0.05	0.01	0.06	0.03	0.01	0.06	0.14	0.03	0.02
4	0.12	0.01	-0.01	1	-0.02	0.11	0.03	0.02	0.14	0.00	-0.01	0.02	0.05
5	0.05	0.01	0.05	-0.02	1	0.03	0.09	0.03	0.03	0.04	0.05	0.05	0.06
6	0.09	0.02	0.01	0.11	0.03	1	-0.04	0.05	0.14	0.01	-0.01	0.02	0.05
7	0.04	0.01	0.06	0.03	0.09	-0.04	1	0.06	0.00	0.08	0.04	0.05	0.05
8	0.01	0.16	0.03	0.02	0.03	0.05	0.06	1	0.01	0.05	0.00	0.02	0.01
9	0.11	0.01	0.01	0.14	0.03	0.14	0.00	0.01	1	-0.05	-0.03	0.02	0.03
10	-0.00	0.04	0.06	0.00	0.04	0.01	0.08	0.05	-0.05	1	-0.01	0.02	0.00
11	-0.02	0.03	0.14	-0.01	0.05	-0.01	0.04	0.00	-0.03	-0.01	1	-0.05	0.00
12	-0.01	0.01	0.03	0.02	0.05	0.02	0.05	0.02	0.02	0.02	-0.05	1	-0.04
13	0.01	0.01	0.02	0.05	0.06	0.05	0.05	0.01	0.03	0.00	0.00	-0.04	1

Table 9 Phi coefficient of the campaign donation outcomes of cluster 5.

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	0.00	-0.04	0.08	0.01	0.07	-0.02	0.02	0.08	0.00	0.02	-0.00	-0.01
2	0.00	1	-0.01	-0.00	-0.01	0.01	-0.00	0.16	0.03	-0.01	0.02	0.02	-0.01
3	-0.04	-0.01	1	0.00	0.07	0.03	0.10	0.01	0.01	0.12	0.12	0.13	0.04
4	0.08	-0.00	0.00	1	-0.07	0.14	0.01	0.02	0.12	0.05	0.02	0.02	-0.02
5	0.01	-0.01	0.07	-0.07	1	-0.00	0.12	0.02	-0.01	0.07	0.12	0.09	0.08
6	0.07	0.01	0.03	0.14	-0.00	1	-0.05	0.02	0.15	0.03	0.05	0.03	0.03
7	-0.02	-0.00	0.10	0.01	0.12	-0.05	1	0.01	-0.02	0.13	0.13	0.10	0.08
8	0.02	0.16	0.01	0.02	0.02	0.02	0.01	1	0.01	-0.02	-0.02	-0.02	0.02
9	0.08	0.03	0.01	0.12	-0.01	0.15	-0.02	0.01	1	-0.07	0.03	0.03	0.03
10	0.00	-0.01	0.12	0.05	0.07	0.03	0.13	-0.02	-0.07	1	0.01	0.06	0.04
11	0.02	0.02	0.12	0.02	0.12	0.05	0.13	-0.02	0.03	0.01	1	0.02	0.04
12	-0.00	0.02	0.13	0.02	0.09	0.03	0.10	-0.02	0.03	0.06	0.02	1	0.02
13	-0.01	-0.01	0.04	-0.02	0.08	0.03	0.08	0.02	0.03	0.04	0.04	0.02	1

Table 10 Phi coefficient of the campaign donation outcomes of cluster 6.

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	-0.02	0.06	0.12	0.07	0.12	0.11	-0.01	0.13	0.03	0.07	0.04	0.04
2	-0.02	1	-0.05	-0.02	-0.03	-0.03	-0.01	0.48	-0.03	-0.01	-0.02	-0.01	-0.03
3	0.06	-0.05	1	0.06	0.09	0.06	0.06	-0.00	0.06	0.06	0.13	0.06	0.05
4	0.12	-0.02	0.06	1	0.08	0.13	0.08	-0.01	0.11	0.08	0.08	0.05	0.03
5	0.07	-0.03	0.09	0.08	1	0.08	0.11	-0.01	0.10	0.11	0.07	0.06	0.11
6	0.12	-0.03	0.06	0.13	0.08	1	0.07	0.01	0.15	0.09	0.07	0.09	0.10
7	0.11	-0.01	0.06	0.08	0.11	0.07	1	0.02	0.12	0.12	0.15	0.07	0.07
8	-0.01	0.48	-0.00	-0.01	-0.01	0.01	0.02	1	-0.03	-0.00	-0.01	-0.02	-0.00
9	0.13	-0.03	0.06	0.11	0.10	0.15	0.12	-0.03	1	0.07	0.10	0.09	0.08
10	0.03	-0.01	0.06	0.08	0.11	0.09	0.12	-0.00	0.07	1	0.10	0.10	0.07
11	0.07	-0.02	0.13	0.08	0.07	0.07	0.15	-0.01	0.10	0.10	1	0.09	0.06
12	0.04	-0.01	0.06	0.05	0.06	0.09	0.07	-0.02	0.09	0.10	0.09	1	0.08
13	0.04	-0.03	0.05	0.03	0.11	0.10	0.07	-0.00	0.08	0.07	0.06	0.08	1

Table 11 Phi coefficient of the campaign donation outcomes of cluster 7.

influence donor behavior within a cluster. For a given cluster $k \in [K]$, we define the utility U_{kdt} of the campaign for donor d in cluster k at time t as follows:

$$U_{kdt} = \beta_k + \sum_{c\in[C]\setminus\{1\}} \beta_{kc}\mathbb{1}\{c_t = c\} + \sum_{y=0}^{q} \beta_{ky}\mathbb{1}\{X_{dt} = y\} + \beta_{kr} \times \text{year} + \varepsilon_{dt}, \tag{3}$$

where β_k represents the intrinsic willingness of a donor within cluster k to donate, β_{kc} denotes the preference of a donor in cluster k for campaign c relative to a baseline campaign, designated as campaign 1, and year $= t$ floor division C. The parameter β_{ky} quantifies the influence of the number of donations made from time $t-q$ to the present on the utility of the current campaign. $X_{dt} = \sum_{i=1}^{q} Y_{dt-i}$, where $Y_{dt} = 1$ if donor d donates at time t; $= 0$ otherwise. This formulation of temporal dependence assumes that donors remember only the previous q periods, and that the impact on their current decision to donate is influenced by the count of past donations. We note that, with the independent predictor "year", we assume that donors' donation probability increasing or decreasing with years depends on the sign of the parameter β_{kr}. The dependence of utility on the year might contradict our assumption of stationarity in the donation probability. However, we posit that this variation could be due to the data spanning from 2018 to 2022, covering periods before, during, and after the COVID-19 pandemic. Given the dramatic societal changes during this time, a slight deviation in donation probabilities is unsurprising.

When estimating the parameters of the model (3), it is important to compare the utility of donating, U_{kdt}, to the utility of not donating, expressed as:

$$U_{kdt0} = 0 + \varepsilon_{dt0}.$$

Assuming that both ε_{dt} and ε_{dt0} are independent Gumbel random variables, we can derive the probability that donor d in cluster k will make a donation at time t as follows:

$$\begin{aligned} \mathbb{P}[Y_{dt} = 1] &= \frac{\exp(\beta_k + \sum_{c\in[C]\setminus\{1\}} \beta_{kc}\mathbb{1}\{c_t = c\} + \sum_{y=0}^{q} \beta_{ky}\mathbb{1}\{X = y\} + \beta_{kr} \times \text{year})}{1 + \exp(\beta_k + \sum_{c\in[C]\setminus\{1\}} \beta_{kc}\mathbb{1}\{c_t = c\} + \sum_{y=0}^{q} \beta_{ky}\mathbb{1}\{X = y\} + \beta_{kr} \times \text{year})} \\ &= \frac{1}{1 + \exp(-(\beta_k + \sum_{c\in[C]\setminus\{1\}} \beta_{kc}\mathbb{1}\{c_t = c\} + \sum_{y=0}^{q} \beta_{ky}\mathbb{1}\{X = y\} + \beta_{kr} \times \text{year}))}, \end{aligned} \tag{4}$$

From this formulation, it is evident that we can estimate the parameters using logistic regression.

Additionally, to explore how significantly past donation behavior impacts a donor's current donation probability, we propose building another model, which is:

$$U_{kdt}^{n} = \beta_k^{n} + \sum_{c\in[C]} \beta_{kc}^{n}\mathbb{1}\{c_t = c\} + \varepsilon_{dt} + \beta_{kr}^{n} \times \text{year}. \tag{5}$$

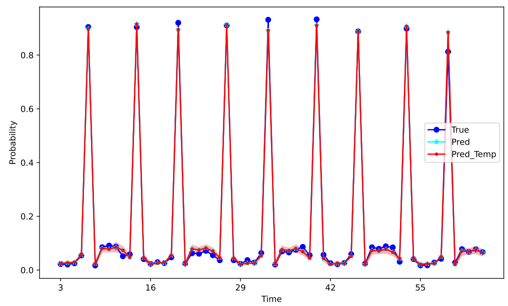
Comparing with model (3), this model omits any potential impact of past donation behavior on the current campaign. If past donation behavior significantly influences current donation probability, we would expect the predicted donation probability

$$\mathbb{P}[Y_{dt} = 1]' = \frac{1}{1 + \exp\left(-\left(\beta_k + \sum_{c\in\{C\}} \beta_{kc}\mathbb{1}\{c_t = c\} + \beta_{kr}^{n} \times \text{year}\right)\right)}, \tag{6}$$

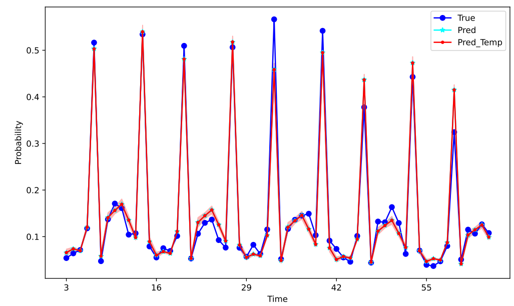
to differ significantly from $\mathbb{P}[Y_{dt} = 1]$.

Setting $q = 1, 2, \cdots, 6$, we perform logistic regressions (4) and (6) on the five-year data to obtain the corresponding parameters. Using these estimated parameters, we can predict the donation probability of each donor to each campaign in the dataset. We then compute the average predicted donation probability for each cluster and compare it against the true average donation probability computed from the data. We denote this type of result as the first-kind and by the second-kind the result where we can compute the difference between the predicted average donation probabilities and that of the true values.

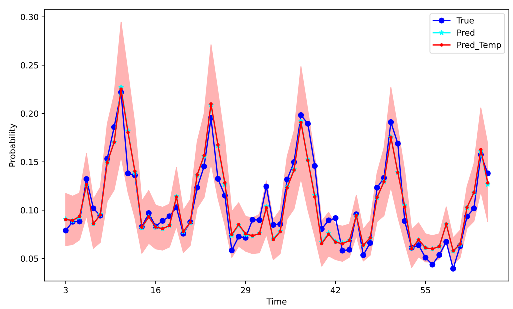
Figures 7 and 8 give the first-kind results under $q = 2$ and $q = 6$. We could have reported this type of results for other q values. However, as the results are quite similar, we omit it here. In the figures, blue dots (labeled “True”) represent the true average donation probability for the campaign in the cluster, cyan stars (labeled “Pred”) are the averages of predictions based on model (5) where temporal dependence is not considered, and red dots (labeled “Pred_Temp”) are the averages of predictions based on model (3) where temporal dependence is considered. Additionally, the shaded area represents the confidence region of the predictions, currently set at 1.96 times of the empirical standard deviation of the corresponding data. Here, the prediction in model (5) does not have any variance because the prediction of the model will be the same for all donors in a cluster. Figures 9 and 10 give the second-kind results under $q = 2$ and $q = 6$. From these figures, we conclude that both models (with and without temporal dependence) fit the data well, and the addition of a temporal dependence does not observably improve the prediction power of the model.



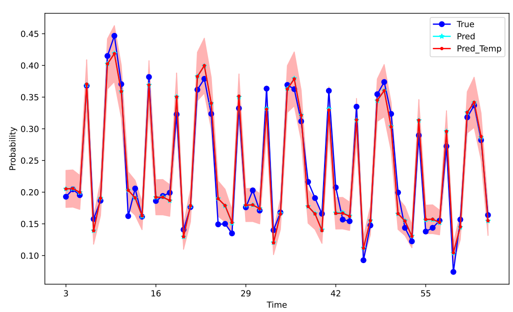
(a) Cluster 1



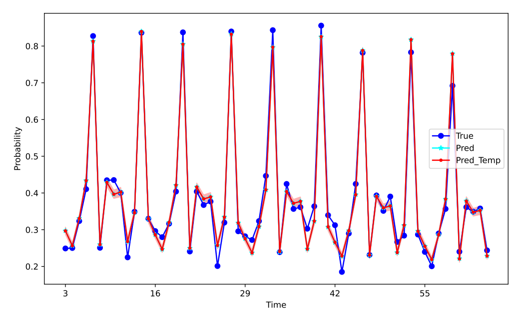
(b) Cluster 2



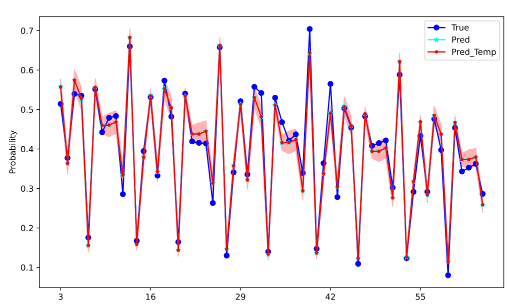
(c) Cluster 3



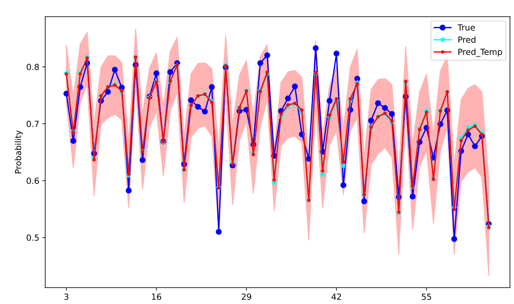
(d) Cluster 4



(e) Cluster 5

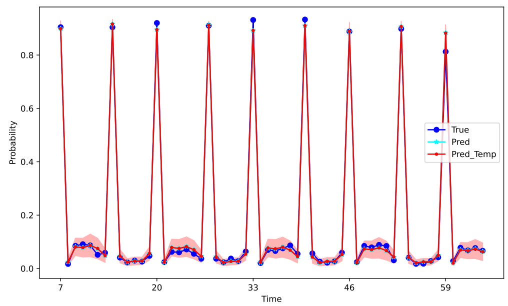


(f) Cluster 6

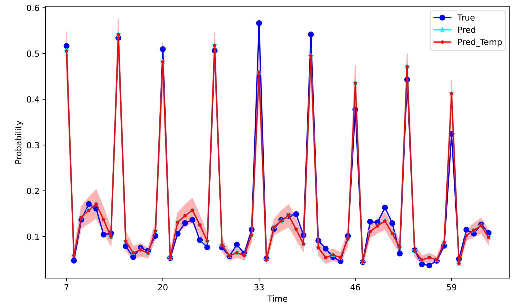


(g) Cluster 7

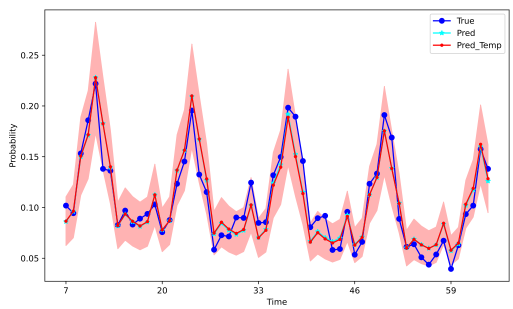
Figure 7 **The average donation probability within each cluster: predictions from models with $q = 2$ and from the data.**



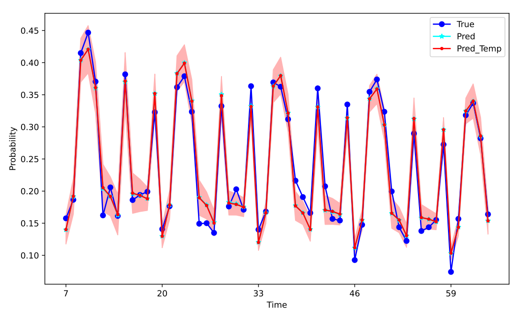
(a) Cluster 1



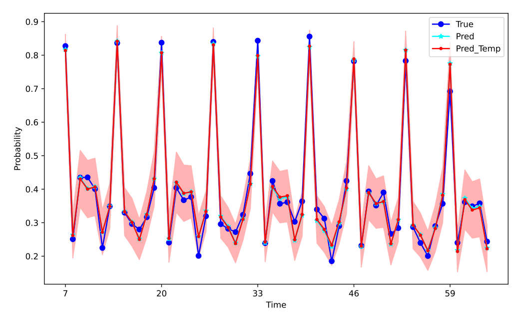
(b) Cluster 2



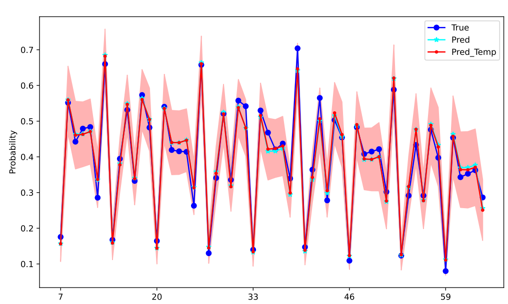
(c) Cluster 3



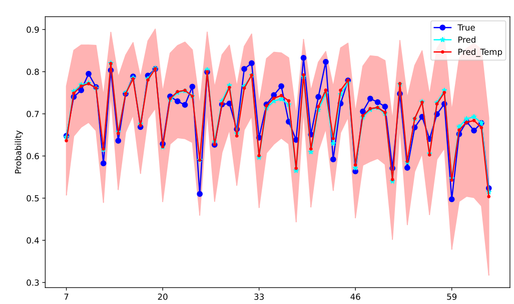
(d) Cluster 4



(e) Cluster 5

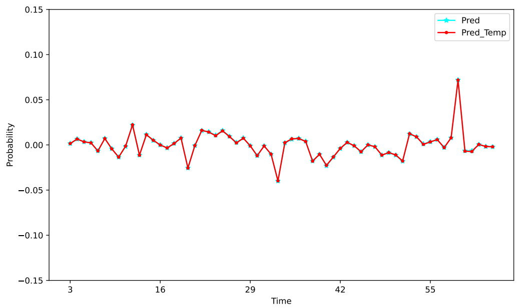


(f) Cluster 6

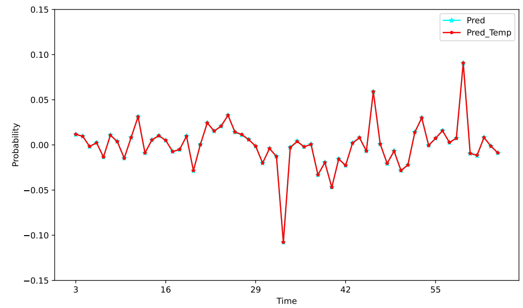


(g) Cluster 7

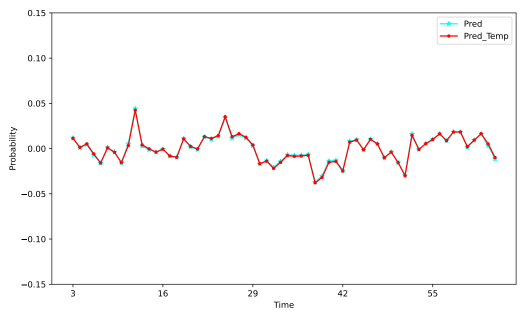
Figure 8 **The average donation probability within each cluster: predictions from models with $q = 6$ and from the data.**



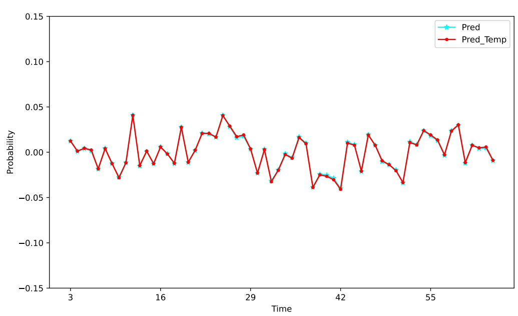
(a) Cluster 1



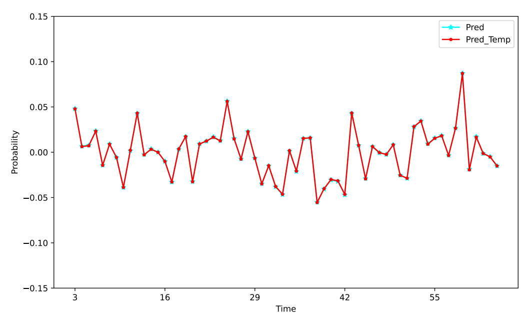
(b) Cluster 2



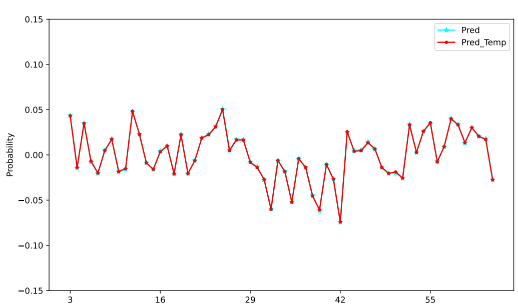
(c) Cluster 3



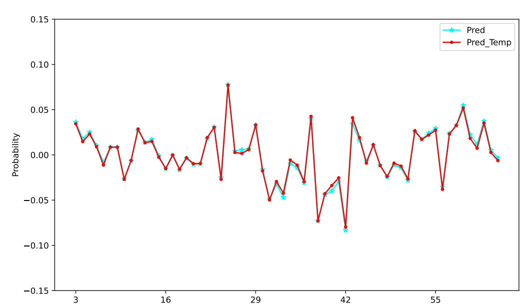
(d) Cluster 4



(e) Cluster 5

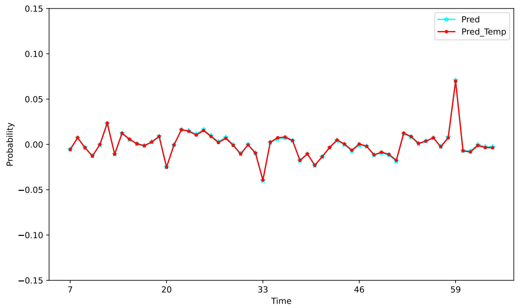


(f) Cluster 6

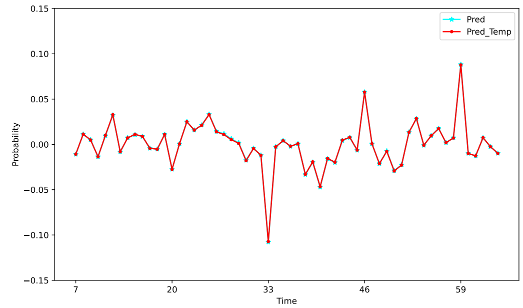


(g) Cluster 7

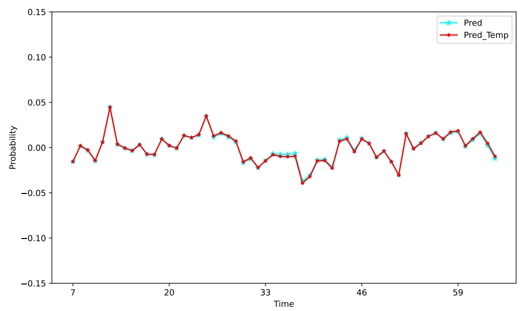
Figure 9 Difference between the predicted and the true average donation probabilities. Results from $q = 2$.



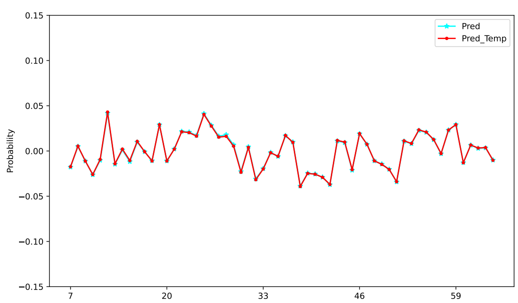
(a) Cluster 1



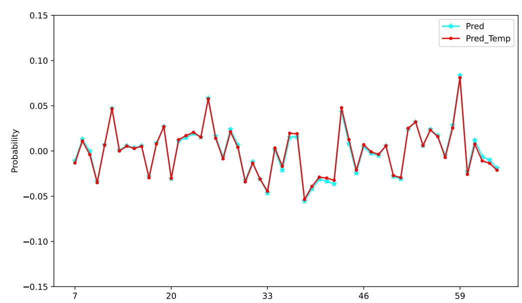
(b) Cluster 2



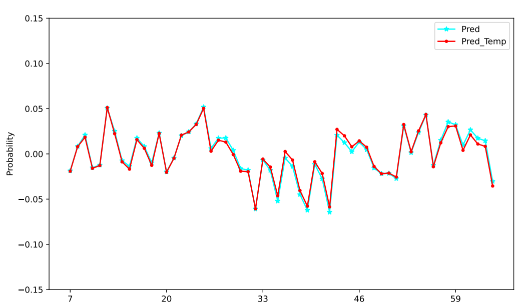
(c) Cluster 3



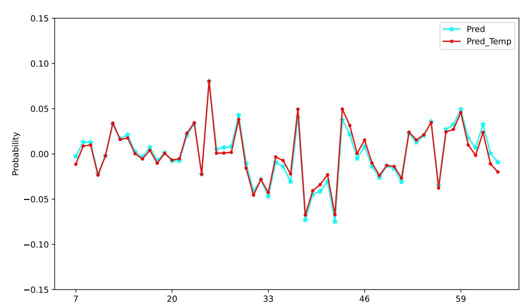
(d) Cluster 4



(e) Cluster 5



(f) Cluster 6



(g) Cluster 7

Figure 10 **Difference between the predicted and the true average donation probabilities. Results from $q = 6$.**

Appendix B: Proof of Results in Section 4

B.1. Proof of the Result in Section 4.2

Recall that, in this problem setting, we look at the representative agent. Thus, we use $\tilde{a}_{ct}$ to represent our prescribed action and $\tilde{r}_{ct}$ to denote the donation from the donor to campaign c at time t.

Proof of Proposition 1: As discussed in the main text, the probability of assigning donor of type k to type $j \neq k$ is

$$
\begin{aligned}
&\mathbb{P}\left[\ell(\tilde{\mathcal{H}};\mathbf{Q}^j) \geq \ell(\tilde{\mathcal{H}};\mathbf{Q}^i) \quad \forall i \in [K] | \tilde{\mathcal{H}} \text{ is generated by } \mathbf{Q}^k\right] && \text{(7a)}\\
\leq &\mathbb{P}\left[\ell(\tilde{\mathcal{H}};\mathbf{Q}^j) - \ell(\tilde{\mathcal{H}};\mathbf{Q}^i) \geq 0 \quad \forall i \in [K] | \tilde{\mathcal{H}} \sim \mathbf{Q}^k\right] && \text{(7b)}\\
\leq &\mathbb{P}\left[\ell(\tilde{\mathcal{H}};\mathbf{Q}^j) - \ell(\tilde{\mathcal{H}};\mathbf{Q}^k) \geq 0 | \tilde{\mathcal{H}} \sim \mathbf{Q}^k\right] && \text{(7c)}\\
= &\mathbb{P}\left[\sum_{c\in[C]}\sum_{t\in[M]}\sum_{r\in\mathcal{R}} \mathbb{1}\{\tilde{r}_{ct}=r\}\left(\ln(q^j_{cr}) - \ln(q^k_{cr})\right) \geq 0 | \tilde{\mathcal{H}} \sim \mathbf{Q}^k\right] && \text{(7d)}\\
= &\mathbb{P}\left[\sum_{c\in[C]}\sum_{t\in[M]}\sum_{r\in\mathcal{R}} \mathbb{1}\{\tilde{r}_{ct}=r\}\ln\left(\frac{q^j_{cr}}{q^k_{cr}}\right) \geq 0 | \tilde{\mathcal{H}} \sim \mathbf{Q}^k\right] && \text{(7e)}\\
= &\mathbb{P}\left[\sum_{c\in[C]}\sum_{t\in[M]}\sum_{r\in\mathcal{R}} (\mathbb{1}\{\tilde{r}_{ct}=r\} - q^k_{cr} + q^k_{cr})\ln\left(\frac{q^j_{cr}}{q^k_{cr}}\right) \geq 0 | \tilde{\mathcal{H}} \sim \mathbf{Q}^k\right] && \text{(7f)}\\
= &\mathbb{P}\left[\sum_{c\in[C]}\sum_{t\in[M]}\sum_{r\in\mathcal{R}} (\mathbb{1}\{\tilde{r}_{ct}=r\} - q^k_{cr})\ln\left(\frac{q^j_{cr}}{q^k_{cr}}\right) \geq \sum_{c\in[C]}\sum_{t\in[M]}\sum_{r\in\mathcal{R}} -q^k_{cr}\ln\left(\frac{q^j_{cr}}{q^k_{cr}}\right) | \tilde{\mathcal{H}} \sim \mathbf{Q}^k\right] && \text{(7g)}\\
= &\mathbb{P}\left[\sum_{c\in[C]}\sum_{t\in[M]}\sum_{r\in\mathcal{R}} (\mathbb{1}\{\tilde{r}_{ct}=r\} - q^k_{cr})\ln\left(\frac{q^j_{cr}}{q^k_{cr}}\right) \geq \sum_{c\in[C]}\sum_{t\in[M]} \left(\boldsymbol{q}^k_c || \boldsymbol{q}^j_c\right) | \tilde{\mathcal{H}} \sim \mathbf{Q}^k\right] && \text{(7h)}\\
= &\mathbb{P}\left[\sum_{c\in[C]}\sum_{t\in[M]}\sum_{r\in\mathcal{R}} (\mathbb{1}\{\tilde{r}_{ct}=r\} - q^k_{cr})\ln\left(\frac{q^j_{cr}}{q^k_{cr}}\right) \geq M\sum_{c\in[C]} \mathrm{KL}\left(\boldsymbol{q}^k_c || \boldsymbol{q}^j_c\right) | \tilde{\mathcal{H}} \sim \mathbf{Q}^k\right] && \text{(7i)}\\
\leq &\exp\left(\frac{-2M^2\left(\sum_{c\in[C]}\mathrm{KL}\left(\boldsymbol{q}^k_c||\boldsymbol{q}^j_c\right)\right)^2}{\sum_{c\in[C]}\sum_{t\in[M]}\left(\max_{r,r'\in\mathcal{R}}\ln\left(\frac{q^j_{cr}q^k_{cr'}}{q^k_{cr}q^j_{cr'}}\right)\right)^2}\right) = \exp\left(-2M\gamma_{kj}\right). && \text{(7j)}
\end{aligned}
$$

In the above derivations, the following steps are taken: The initial expression defines the event corresponding to a clustering error. Here, we denote the history as a random variable because we are interested in the *a priori* probability guarantee of the event which is defined by the possible random generation of history. The equality (7b) shifts the log-likelihood term from the right-hand side to the left. We also use "$\tilde{\mathcal{H}} \sim \mathbf{Q}^k$" to denote that $\tilde{\mathcal{H}}$ is generated by distribution $\mathbf{Q}^k$. The inequality (7c) follows from the enlargement of the event from holding for every $i \in [K]$ to for only k. Equality (7d) represents the log-likelihood function in terms of the underlying data and distribution. Log terms are combined in (7e). Following this, we introduce and then eliminate a term q^k_{cr} in (7f). We note that the expectation of $\mathbb{1}\{\tilde{r}_{ct}=r\}$ is q^k_{cr} as the data is generated from type k. In (7g), terms are reorganized to ensure the left-hand side meets the prerequisites for applying the Hoeffding inequality stated in Lemma 12, and the right-hand side is the KL-divergence which leads to (7h).

In (7i), we recognize the irrelevance of the KL divergence to the summation over i and transform the summation into multiplication. Note that $\tilde{r}_{ct}$ is a discrete random variable with distribution $\boldsymbol{q}_c^k$, we can then regard $\sum_{r\in\mathcal{R}}(\mathbb{1}\{\tilde{r}_{ct}=r\}-q_{cr}^k)\ln\left(\frac{q_{cr}^j}{q_{cr}^k}\right)=\sum_{r\in\mathcal{R}}\mathbb{1}\{\tilde{r}_{ct}=r\}\ln\left(\frac{q_{cr}^j}{q_{cr}^k}\right)-\sum_{r\in\mathcal{R}}q_{cr}^k\ln\left(\frac{q_{cr}^j}{q_{cr}^k}\right)=\sum_{r\in\mathcal{R}}\mathbb{1}\{\tilde{r}_{ct}=r\}\ln\left(\frac{q_{cr}^j}{q_{cr}^k}\right)+\mathrm{KL}\left(\boldsymbol{q}_c^k||\boldsymbol{q}_c^j\right)$ as a new random variable, whose support is in the interval $[\min_{r\in\mathcal{R}}\ln\left(\frac{q_{cr}^j}{q_{cr}^k}\right)+\mathrm{KL}\left(\boldsymbol{q}_c^k||\boldsymbol{q}_c^j\right),\max_{r\in\mathcal{R}}\ln\left(\frac{q_{cr}^j}{q_{cr}^k}\right)+\mathrm{KL}\left(\boldsymbol{q}_c^k||\boldsymbol{q}_c^j\right)]$. By invoking the Hoeffding inequality (*cf.* Lemma 12) and noting that $b_i-a_i=\max_{r\in\mathcal{R}}\ln\left(\frac{q_{cr}^j}{q_{cr}^k}\right)-\min_{r\in\mathcal{R}}\ln\left(\frac{q_{cr}^j}{q_{cr}^k}\right)=\max_{r,r'\in\mathcal{R}}\ln\left(\frac{q_{cr}^j}{q_{cr}^k}\right)-\ln\left(\frac{q_{cr'}^j}{q_{cr'}^k}\right)=\max_{r,r'\in\mathcal{R}}\ln\left(\frac{q_{cr}^j q_{cr'}^k}{q_{cr}^k q_{cr'}^j}\right)$, we obtain the inequality in (7j). The equality refines the probability expression by observing the independence of the summation term from the overall summation over t and the definition of γ_{kj} in the main text. $\square$

We note that, if $\boldsymbol{q}_c^j$ and $\boldsymbol{q}_c^k$ do not share the same support $\mathcal{R}$, the summation over $r\in\mathcal{R}$ in the equality (7d) should be replaced by the summation $r\in\mathcal{R}_c^{kj}$ which is the sub-support where both $\boldsymbol{q}_c^j$ and $\boldsymbol{q}_c^k$ are non-zero. For any $r\in\mathcal{R}$ with $q_{cr}^j=0$ while $q_{cr}^k>0$, we have $\ln(q_{cr}^j)=-\infty$, which invalidates the inequality ≥ 0. This implies the history $\tilde{\mathcal{H}}$ cannot be assigned to type j and contradict the assumption. For any $r\in\mathcal{R}$ with $q_{cr}^k=0$, it always has $\mathbb{1}\{\tilde{r}_{ct}=r\}=0$ because $\tilde{r}_{cr}\sim\boldsymbol{q}_c^k$ has zero probability of being r. Therefore, we should focus on the set $\mathcal{R}_c^{kj}$ for this expansion to make the equality sensible.

Proof of Theorem 1: To prove the result, let us assume for now that the donor is of type k. Hence, the expected regret of this donor can be written as:

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{c\in[C]}\sum_{t\in[T]}|r_c^k-e_c|\cdot\mathbb{1}\{\tilde{a}_{ct}\neq\mathbb{1}\{r_c^k\geq e_c\}\}\right]\\
=&\mathbb{E}\left[\sum_{c\in[C]}\sum_{t\in[M]}|r_c^k-e_c|\cdot\mathbb{1}\{r_c^k<e_c\}+\sum_{c\in[C]}\sum_{t>M}|r_c^k-e_c|\cdot\mathbb{1}\{\tilde{a}_{ct}\neq\mathbb{1}\{r_c^k\geq e_c\}\}\right] \qquad (8a)\\
=&M\cdot\left(\sum_{c\in[C]}|r_c^k-e_c|\cdot\mathbb{1}\{r_c^k<e_c\}\right)+\mathbb{E}\left[\sum_{c\in[C]}\sum_{t>M}|r_c^k-e_c|\cdot\mathbb{1}\{\tilde{a}_{ct}\neq\mathbb{1}\{r_c^k\geq e_c\}\}\right] \qquad (8b)
\end{aligned}
$$

We first split the summation over $t\in[T]$ into $t\in[M]$ and $t>M$. We use $t>M$ to represent $t\in\{M+1,M+2,\cdots,T\}$ for notation simplicity. According to Algorithm 1, we send all campaigns M times to the donor, which implies that $\tilde{a}_{ct}=1$ for all $c\in[C]$ and $t\in[M]$. Consequently, the indicator function $\mathbb{1}\{\tilde{a}_{ct}\neq\mathbb{1}\{r_c^k\geq e_c\}\}$ simplifies to $\mathbb{1}\{r_c^k<e_c\}$, leading to the equality (8a). The equality (8b) applies the linearity of the expectation operator and rearrange terms.

Next, we bound the expectation term in (8b) in as follows:

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{c\in[C]}\sum_{t>M}|r_c^k-e_c|\cdot\mathbb{1}\{\tilde{a}_{ct}\neq\mathbb{1}\{r_c^k\geq e_c\}\}\right]\\
=&\sum_{j\in[K]}\mathbb{E}\left[\sum_{c\in[C]}\sum_{t>M}|r_c^k-e_c|\cdot\mathbb{1}\{\tilde{a}_{ct}\neq\mathbb{1}\{r_c^k\geq e_c\}\}|\text{allocate to type } j\right]\mathbb{P}[\text{allocate to type } j] \qquad (9a)\\
=&\sum_{j\in[K]}\left(\sum_{c\in[C]}\sum_{t>M}|r_c^k-e_c|\cdot\mathbb{1}\{\mathbb{1}\{r_c^j\geq e_c\}\neq\mathbb{1}\{r_c^k\geq e_c\}\}\right)\mathbb{P}[\text{allocate to type } j] \qquad (9b)\\
\leq&\sum_{j\in[K]}\left(\sum_{c\in[C]}\sum_{t>M}|r_c^k-e_c|\cdot\mathbb{1}\{\mathbb{1}\{r_c^j\geq e_c\}\neq\mathbb{1}\{r_c^k\geq e_c\}\}\right)\cdot\exp(-2M\gamma_{kj}) \qquad (9c)
\end{aligned}
$$

$$=(T-M)\sum_{j\in[K]}\sum_{c\in[C]}|r_c^k-e_c|\cdot\mathbb{1}\{\mathbb{1}\{r_c^j\geq e_c\}\neq\mathbb{1}\{r_c^k\geq e_c\}\}\cdot\exp(-2M\gamma_{kj}) \tag{9d}$$

The first equality applies the property of conditional expectation. The equality (9b) comes from the fact that if the donor is allocated to type j, according to the campaign decision rule during exploitation periods, we have $\tilde{a}_{ct}=\mathbb{1}\{r_c^j\geq e_c\}$ for all $c\in[C]$. The inequality in (9c) leverages the bound in Proposition 1, and the equality in (9d) uses the fact that the summation term is independent of t.

Put the above bounds together, we have

$$\begin{aligned}&\mathbb{E}\left[\sum_{c\in[C]}\sum_{t\in[T]}|r_c^k-e_c|\cdot\mathbb{1}\{\tilde{a}_{ct}\neq\mathbb{1}\{r_c^k\geq e_c\}\}\right]\\ \leq& M\cdot(\sum_{c\in[C]}|r_c^k-e_c|\cdot\mathbb{1}\{r_c^k<e_c\})+(T-M)\sum_{j\in[K]}\sum_{c\in[C]}|r_c^k-e_c|\cdot\mathbb{1}\{\mathbb{1}\{r_c^j\geq e_c\}\neq\mathbb{1}\{r_c^k\geq e_c\}\}\cdot\exp(-2M\gamma_{kj}).\end{aligned}$$

As we are not sure the true type of the donor, the regret can be bounded by

$$\max_{k\in[K]}\left[M\cdot(\sum_{c\in[C]}|r_c^k-e_c|\cdot\mathbb{1}\{r_c^k<e_c\})+(T-M)\sum_{j\in[K]}\sum_{c\in[C]}|r_c^k-e_c|\cdot\mathbb{1}\{\mathbb{1}\{r_c^j\geq e_c\}\neq\mathbb{1}\{r_c^k\geq e_c\}\}\cdot\exp(-2M\gamma_{kj})\right].$$

By picking M from its admissible values to minimize the regret expressed above, we conclude the first upper bound.

In order to derive an analytical regret expression, we upper bound both $\mathbb{1}\{r_c^k<e_c\}$ and $\mathbb{1}\{\mathbb{1}\{r_c^j\geq e_c\}\neq\mathbb{1}\{r_c^k\geq e_c\}\}$ by 1 for $j\neq k$ and bound $T-M$ by T, simplifying the bound to

$$\begin{aligned}&\min_{M\in[T]}\max_{k\in[K]}M\cdot\sum_{c\in[C]}|r_c^k-e_c|+T\sum_{j\in[K]\setminus\{k\}}\sum_{c\in[C]}|r_c^k-e_c|\cdot\exp(-2M\gamma_{kj})\\ \leq&\min_{M\in[T]}M\cdot\sum_{c\in[C]}\Delta_{\max}^c+T\cdot(K-1)\sum_{c\in[C]}\Delta_{\max}^c\cdot\exp(-2M\gamma)\\ =&(\sum_{c\in[C]}\Delta_{\max}^c)\min_{M\in[T]}(M+T\cdot(K-1)\exp(-2M\gamma)).\end{aligned}$$

The first inequality applies $\Delta_{\max}^c:=\max_{k\in[K]}|r_c^k-e_c|$ and $\gamma=\min_{k\neq j\in[K]}\gamma_{kj}$. The multiplier $K-1$ comes from the summation $j\in[K]\setminus\{k\}$. The final equality simplifies and rearranges the terms accordingly.

Next, we minimize the term $M+T\cdot(K-1)\cdot\exp(-2M\gamma)$ over M. This is done by taking the derivative of $M+T\cdot(K-1)\cdot\exp(-2M\gamma)$ with respect to M and setting it to zero, which gives:

$$1+T\cdot(K-1)\exp(-2M\gamma)\cdot(-2\gamma)=0\Longleftrightarrow M^\star=\frac{1}{2\gamma}\ln(2T\cdot(K-1)\gamma).$$

When $2T\cdot(K-1)\gamma\leq 1$, i.e., $T\leq\frac{1}{2(K-1)\gamma}$, $M^\star\leq 0$. We can take $M=1$ and our regret is upper bounded by $T\sum_{c\in[C]}\Delta_{\max}^c$. This upper bound, however, does not come from the $M+T\cdot(K-1)\cdot\exp(-2M\gamma)$ but from (8b) by setting $\mathbb{1}\{r_c^k<e_c\}=1$, $\mathbb{1}\{\tilde{a}_{ct}\neq\mathbb{1}\{r_c^k\geq e_c\}\}=1$ and $|r_c^k-e_c|\leq\Delta_{\max}^c$. Otherwise, we take $M=\lceil M^\star\rceil$ and the regret is upper bounded by

$$\sum_{c\in[C]}\Delta_{\max}^c\left(\frac{1}{2\gamma}\ln(2T\cdot(K-1)\gamma)+1+\frac{1}{2\gamma}\right)$$

as $M=\lceil M^\star\rceil\leq\frac{1}{2\gamma}\ln(2T\cdot(K-1)\gamma)+1$ and $T\cdot(K-1)\exp(-2M\gamma)\leq T\cdot(K-1)\exp(-2M^\star\gamma)=T\cdot(K-1)\frac{1}{2T\cdot(K-1)\gamma}=\frac{1}{2\gamma}$. $\square$

Appendix C: Proof of the Results in Section 5

In this section, we prove the results in Section 5. Recall that $\phi = \min_{k\in[K]}\{\frac{\Delta^2}{36(\sigma_k^2+R_k\Delta/18)}\}$, where $\sigma_k^2 = \sum_{c\in[C]} \mathrm{var}(\tilde{r}_c)$, with $\tilde{r}_c$ being a random variable following the discrete distribution $\boldsymbol{q}_c^k$, and $R_k = \sqrt{\sum_{c\in[C]} \max\{(r_c^k)^2, (\max\mathcal{R} - r_c^k)^2\}}$. Additionally, recall that $\xi = C+1$, $\kappa = \beta\cdot(K-1)+1$, and $\eth = \xi\exp(-M\phi)$. Moreover, in this section, let $[0,\max\mathcal{R}]^C$ be a metric space equipped with the two-norm distance. Define the distance between a point $\boldsymbol{r}$ and a subset S as $d(\boldsymbol{r},S) = \min_{\boldsymbol{r}'\in S}\|\boldsymbol{r}-\boldsymbol{r}'\|_2$; the distance between two subsets S,S' as $d(S,S') = \min_{\boldsymbol{r}\in S} d(\boldsymbol{r},S') = \min_{\boldsymbol{r}\in S,\boldsymbol{r}'\in S'}\|\boldsymbol{r}-\boldsymbol{r}'\|_2$; and the radius of a subset S as $\alpha(S) = \max_{\boldsymbol{r},\boldsymbol{r}'\in S}\|\boldsymbol{r}-\boldsymbol{r}'\|_2/2$.

C.1. Proof of the Result in Section 5.1

Proof of Proposition 2: Here, we define an operation as either a summation or a multiplication of two floating-point numbers. By this definition, it is easy to show that line 2 demands $\mathcal{O}(MCD)$ operations. In the loop for clusters identification, calculating the distance between each donor once and storing it simplifies the process, which requires $\mathcal{O}(CD^2)$ operations. During each iteration in the loop to identify a cluster, we need to pinpoint a donor with the largest neighborhood and identify the respective neighbors, requiring $\mathcal{O}(D)$ operations. To exclude the identified cluster from further considerations, we update the pairwise distances among donors, requiring $\mathcal{O}(D^2)$ operations. Consequently, the total computational effort sums to $\mathcal{O}(K(D+D^2))$ operations in the clustering identification process. For lines 11 to 13, the number of operations is of lower magnitude which is omitted. Neglecting smaller order terms, the computational complexity of Algorithm 2 is $\mathcal{O}((K+C)D^2)$. $\square$

We first prove results in Section 5.3 before proving the result in Theorem 2.

C.2. Proof of Results in Section 5.3

Proof of Lemma 1: Regard the random vector $\tilde{\boldsymbol{r}}_i$ as a matrix of dimension $C\times 1$ and note that $\tilde{\boldsymbol{r}}_i \in [0,\max\mathcal{R}]^C$. Define $\tilde{\boldsymbol{a}}_i = \tilde{\boldsymbol{r}}_i - \boldsymbol{r}$, which has an expectation of $\mathbf{0}$. We can show that $\|\tilde{\boldsymbol{a}}_i\|_2$ is bounded by $R = \sqrt{\sum_{c\in[C]} \max\{(r_c)^2, (\max\mathcal{R} - r_c)^2\}}$ almost surely, as follows:

$$\max_{\|\boldsymbol{x}\|_2=1}\|\tilde{\boldsymbol{a}}_i^\top \boldsymbol{x}\|_2 \le \max_{\|\boldsymbol{x}\|_2=1}\|\tilde{\boldsymbol{a}}_i\|_2\|\boldsymbol{x}\|_2 = \|\tilde{\boldsymbol{a}}_i\|_2 \le R.$$

The first inequality applies the Cauchy-Schwartz inequality, the equality uses $\|\boldsymbol{x}\|_2 = 1$, and the last inequality follows from the support of $\tilde{\boldsymbol{r}}_i$, which implies $|\tilde{a}_{ic}| \le \max\{r_c, \max\mathcal{R} - r_c\}$ almost surely.

Additionally,

$$\|\frac{1}{m}\sum_{i=1}^m \mathbb{E}[\tilde{\boldsymbol{a}}_i\tilde{\boldsymbol{a}}_i^\top]\|_2 = \|\mathbb{E}[\tilde{\boldsymbol{a}}_i\tilde{\boldsymbol{a}}_i^\top]\|_2 = \|\mathrm{diag}(\mathrm{var}(\tilde{r}_{i1}),\mathrm{var}(\tilde{r}_{i2}),\cdots,\mathrm{var}(\tilde{r}_{iC}))\|_2 = \max_{c\in[C]}\mathrm{var}(\tilde{r}_{ic}).$$

The first equality holds because $\tilde{\boldsymbol{a}}_i, i\in[m]$ are identically distributed, and the second equality follows because $\mathbb{E}[\tilde{a}_{ij}\tilde{a}_{ik}] = \mathrm{cov}(\tilde{r}_{ij},\tilde{r}_{ik}) = \mathrm{var}(\tilde{r}_{ij})$ if $j=k$ and 0 otherwise by independence. The last equality uses the definition of the l_2 norm of a matrix.

Moreover,

$$\|\frac{1}{m}\sum_{i=1}^m \mathbb{E}[\tilde{\boldsymbol{a}}_i^\top\tilde{\boldsymbol{a}}_i]\|_2 = \|\mathbb{E}[\tilde{\boldsymbol{a}}_i^\top\tilde{\boldsymbol{a}}_i]\|_2 = \|\sum_{c\in[C]}\mathbb{E}[\tilde{a}_{ic}^2]\|_2 = \sum_{c\in[C]}\mathrm{var}(\tilde{r}_{ic}).$$

The first equality again applies the identical distribution of $\tilde{\boldsymbol{a}}_i, i \in [m]$, the second equality reformulates and applies the linearity of expectation, and the last equality uses $\|a\|_2 = a$ for any positive constant.

Clearly, we have

$$\sigma^2 = \max\{\|\frac{1}{m}\sum_{i=1}^{m}\mathbb{E}[\tilde{\boldsymbol{a}}_i\tilde{\boldsymbol{a}}_i^\top]\|_2, \|\frac{1}{m}\sum_{i=1}^{m}\mathbb{E}[\tilde{\boldsymbol{a}}_i^\top\tilde{\boldsymbol{a}}_i]\|_2\} = \sum_{c\in[C]}\mathrm{var}(\tilde{r}_{ic}).$$

Invoking Lemma 13 and substituting the corresponding values gives the result. □

Next, recall that $\eth = \xi\exp(-M\phi)$, we prove the result in Lemma 2.

Proof of Lemma 2: For each donor d from cluster k, denote by $\tilde{X}_{\frac{1}{6}d} = \mathbb{1}\{\|\tilde{\boldsymbol{r}}_d - \boldsymbol{r}^k\|_2 \geq \frac{\Delta}{6}\}$. Note that $\tilde{X}_{\frac{1}{6}d}$ is a Bernoulli random variable. Invoking Lemma 1 with $\delta = \frac{\Delta}{6}$, we have $p := \mathbb{P}[\tilde{X}_{\frac{1}{6}d} = 1] = \mathbb{P}[\|\tilde{\boldsymbol{r}}_d - \boldsymbol{r}^k\|_2 \geq \frac{\Delta}{6}] \leq \eth \leq \frac{1}{2}$ as we have shown in the main text and the last inequality is from condition **(C)**. Additionally, $\tilde{\omega}_k = \sum_{d\in \mathrm{K}^{-1}(k)}\tilde{X}_{\frac{1}{6}d}$ by definition. The result follows from

$$\mathbb{P}\left[\frac{\tilde{\omega}_k}{n} \geq \eth + \eta\right] \leq \mathbb{P}\left[\frac{\tilde{\omega}_k}{n} \geq p + \eta\right] \leq \exp\left(\frac{-2N_k\eta^2}{2p(1-p)}\right) \leq \exp\left(\frac{-2N_k\eta^2}{2\eth(1-\eth)}\right),$$

where the first and the third inequalities hold since $p \leq \eth \leq \frac{1}{2}$, the second inequality is due to the Chernoff inequality in Lemma 9 by noticing that $\tilde{X}_{\frac{1}{6}d} \in \{0, 1\}$. □

A direct application of this result gives us the following bound on probability of the event **(E)**.

Corollary 1. *Event **(E)** happens with probability at least* $1 - \sum_{k\in[K]}\exp\left(\frac{-N_k\eta^2}{2\eth\cdot(1-\eth)}\right)$.

Proof of Corollary 1: The result follows directly from inequalities below:

$$\begin{aligned}
\mathbb{P}[\mathbf{E}] =& \mathbb{P}\left[|d \in \mathrm{K}^{-1}(k) \,:\, \|\tilde{\boldsymbol{r}}_d - \boldsymbol{r}^k\|_2 < \frac{\Delta}{6}| \geq (1 - \eth - \eta)N_k \;\; \forall k \in [K]\right] \\
=& \mathbb{P}\left[|d \in \mathrm{K}^{-1}(k) \,:\, \|\tilde{\boldsymbol{r}}_d - \boldsymbol{r}^k\|_2 \geq \frac{\Delta}{6}| \leq (\eth + \eta)N_k \;\; \forall k \in [K]\right] \\
=& 1 - \mathbb{P}\left[\exists k \in [K], |d \in \mathrm{K}^{-1}(k) \,:\, \|\tilde{\boldsymbol{r}}_d - \boldsymbol{r}^k\|_2 \geq \frac{\Delta}{6}| \geq (\eth + \eta)N_k\right] \\
\geq& 1 - \sum_{k\in[K]}\mathbb{P}\left[|d \in \mathrm{K}^{-1}(k) \,:\, \|\tilde{\boldsymbol{r}}_d - \boldsymbol{r}^k\|_2 \geq \frac{\Delta}{6}| \geq (\eth + \eta)N_k\right] \\
\geq& 1 - \sum_{k\in[K]}\exp\left(\frac{-N_k\eta^2}{2\eth\cdot(1-\eth)}\right).
\end{aligned}$$

The first equality follows from the definition of event **(E)**, while the second equality arises because $|d \in \mathrm{K}^{-1}(k) \,:\, \|\tilde{\boldsymbol{r}}_d - \boldsymbol{r}^j\|_2 < \frac{\Delta}{6}| \geq (1 - \eth - \eta)N_k$ holds precisely when $|d \in \mathrm{K}^{-1}(k) \,:\, \|\tilde{\boldsymbol{r}}_d - \boldsymbol{r}^j\|_2 \geq \frac{\Delta}{6}| \leq (\eth + \eta)N_k$, given that $|\mathrm{K}^{-1}(k)| = N_k$. The final equality uses the principle $\mathbb{P}[A] = 1 - \mathbb{P}[A^c]$, where A^c denotes the complement of A. The first inequality employs the union bound. The last inequality leverages the first result in Lemma 2. □

Let $\tilde{\mathcal{P}}^k$ to be the smallest open ball such that $\tilde{\boldsymbol{r}}_d \in \tilde{\mathcal{P}}^k$ for every $d \in \tilde{S}^k$. Define $\tilde{\mathcal{E}}^j = \{d \in \mathrm{K}^{-1}(j) \,:\, \tilde{\boldsymbol{r}}_d \in \mathcal{B}(\boldsymbol{r}^j, \frac{\Delta}{6})\}, j \in [K]$. Recall $\tilde{\mathcal{D}}^j = \{d \in [D] \,:\, \tilde{\boldsymbol{r}}_d \in \mathcal{B}(\boldsymbol{r}^j, \frac{\Delta}{6})\}$, it is clear that $\tilde{\mathcal{E}}^j \subseteq \tilde{\mathcal{D}}^j$ for every $j \in [K]$. By the construction of $\tilde{S}^k$, $\tilde{\mathcal{P}}^k$ has a radius of at most $\frac{\Delta}{3}$. We first discuss some properties of these sets that will be useful in our subsequent proofs.

Observation 1.

(i) For every $k, j \in [K]$, $\tilde{S}^k \cap \tilde{S}^j = \emptyset$ *whenever* $k \neq j$.

(ii) For every $k, j \in [K]$, $\tilde{\mathcal{P}}^k \cap \mathcal{B}(\boldsymbol{r}^j, \frac{\Delta}{6}) \neq \emptyset$ *whenever* $\tilde{S}^k \cap \tilde{\mathcal{D}}^j \neq \emptyset$ *holds.*

(iii) For every $k, j \in [K]$, $\tilde{\mathcal{D}}^k \cap \tilde{\mathcal{D}}^j = \emptyset$ *whenever* $k \neq j$.

Proof of Observation 1: Without loss of generality, assume that $k < j$. Then, $\tilde{S}^k \cap \tilde{S}^j = \emptyset$ follows from the fact that $\tilde{S}^j$ is generated after removing the donors $d \in \tilde{S}^k$ from the consideration set $\mathcal{D}$ in Algorithm 2. As for the second statement, assume that $d \in \tilde{S}^k \cap \tilde{\mathcal{D}}^j$. Then, $\tilde{\boldsymbol{r}}_d \in \tilde{\mathcal{P}}^k$ by construction of $\tilde{\mathcal{P}}^k$ as well as $\tilde{\boldsymbol{r}}_d \in \mathcal{B}(\boldsymbol{r}^j, \frac{\Delta}{6})$ by construction of $\tilde{\mathcal{D}}^j$, respectively. We thus conclude that $\tilde{\mathcal{P}}^k \cap \mathcal{B}(\boldsymbol{r}^j, \frac{\Delta}{6}) \neq \emptyset$. The third statement follows from the definition of $\tilde{\mathcal{D}}^j, j \in [K]$ and the distance between any donor types is at least Δ. □

Observation 2. *Fix any ball* $\mathcal{B} \subseteq [0, \max \mathcal{R}]^C$ *of radius* $\frac{\Delta}{3}$. $\mathcal{B}$ *intersects with at most one ball* $\mathcal{B}(\boldsymbol{r}^k, \frac{\Delta}{6}), k \in [K]$.

We note that this result, together with the second statement of Observation 1, is used to show that most donors in each of our identified clusters are from the same underneath cluster, i.e., the identified clusters are relatively clean.

Proof of Observation 2: Suppose $\mathcal{B}$ intersects with $\mathcal{B}(\boldsymbol{r}^j, \frac{\Delta}{6}), j \in [K]$. We conclude our result by showing that $d(\mathcal{B}, \mathcal{B}(\boldsymbol{r}^k, \frac{\Delta}{6})) > 0$ for any $k \in [K], k \neq j$. Let us bound the distance between the two sets as follows.

$$\begin{aligned}
d(\mathcal{B}, \mathcal{B}(\boldsymbol{r}^k, \frac{\Delta}{6})) &= \inf_{\boldsymbol{r} \in \mathcal{B}, \boldsymbol{r}' \in \mathcal{B}(\boldsymbol{r}^k, \frac{\Delta}{6})} \|\boldsymbol{r}' - \boldsymbol{r}\|_2 \quad \geq \inf_{\boldsymbol{r} \in \mathcal{B}, \boldsymbol{r}' \in \mathcal{B}(\boldsymbol{r}^k, \frac{\Delta}{6})} \|\boldsymbol{r}' - \boldsymbol{r}^j\|_2 - \|\boldsymbol{r}^j - \boldsymbol{r}\|_2 \\
&= \inf_{\boldsymbol{r} \in \mathcal{B}, \boldsymbol{r}' \in \mathcal{B}(\boldsymbol{r}^k, \frac{\Delta}{6})} \|\boldsymbol{r}^j - \boldsymbol{r}'\|_2 - \|\boldsymbol{r}^j - \boldsymbol{r}\|_2 \quad \geq \inf_{\boldsymbol{r} \in \mathcal{B}, \boldsymbol{r}' \in \mathcal{B}(\boldsymbol{r}^k, \frac{\Delta}{6})} \|\boldsymbol{r}^j - \boldsymbol{r}^k\|_2 - \|\boldsymbol{r}^k - \boldsymbol{r}'\|_2 - \|\boldsymbol{r}^j - \boldsymbol{r}\|_2 \\
&\geq \inf_{\boldsymbol{r} \in \mathcal{B}, \boldsymbol{r}' \in \mathcal{B}(\boldsymbol{r}^k, \frac{\Delta}{6})} \|\boldsymbol{r}^j - \boldsymbol{r}^k\|_2 - \|\boldsymbol{r}^k - \boldsymbol{r}'\|_2 - \left(\|\boldsymbol{r}^j - \boldsymbol{r}''\|_2 + \|\boldsymbol{r}'' - \boldsymbol{r}\|_2\right) \text{ for any } \boldsymbol{r}'' \in \mathcal{B} \cap \mathcal{B}(\boldsymbol{r}^j, \frac{\Delta}{6}) \\
&\geq \Delta - \frac{\Delta}{6} - \left(\frac{\Delta}{6} + \frac{2\Delta}{3}\right) \geq 0.
\end{aligned}$$

In the above, the first two inequalities follow from the triangle inequality, which implies that $\|\boldsymbol{a} - \boldsymbol{b}\|_2 \geq \|\boldsymbol{a} - \boldsymbol{c}\|_2 - \|\boldsymbol{c} - \boldsymbol{b}\|_2$ for any $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in [0, \max \mathcal{R}]^C$. The third inequality applies the triangle inequality in the form $\|\boldsymbol{a} - \boldsymbol{b}\|_2 \leq \|\boldsymbol{a} - \boldsymbol{c}\|_2 + \|\boldsymbol{c} - \boldsymbol{b}\|_2$. The fourth inequality holds because $\|\boldsymbol{r}^j - \boldsymbol{r}^k\|_2 \geq \Delta$ by the definition of Δ, both $\|\boldsymbol{r}^k - \boldsymbol{r}'\|_2$ and $\|\boldsymbol{r}^j - \boldsymbol{r}''\|_2$ are bounded from above by $\frac{\Delta}{6}$ as $\mathcal{B}(\boldsymbol{r}^k, \frac{\Delta}{6})$ and $\mathcal{B}(\boldsymbol{r}^j, \frac{\Delta}{6})$ are open balls of radius $\frac{\Delta}{6}$, and $\|\boldsymbol{r}'' - \boldsymbol{r}\|_2 \leq \frac{2\Delta}{3}$ for any $\boldsymbol{r}''$ and $\boldsymbol{r} \in \mathcal{B}$ as $\mathcal{B}$ is an open ball of radius $\frac{\Delta}{3}$. The no-intersection conclusion follows from that $\mathcal{B}(\boldsymbol{r}^k, \frac{\Delta}{6})$ is an open ball. □

Observation 3. *If event* ***(E)*** *is true,* $|\tilde{\mathcal{D}}^j| \geq (1 - \eth - \eta) N_j$ *for every* $j \in [K]$

Proof of Observation 3: The result holds since

$$|\tilde{\mathcal{D}}^j| = \tilde{N}^j + \tilde{N}^{-j} \geq \tilde{N}^j \geq (1 - \eth - \eta)\, N_j.$$

The equality follows from decomposing $|\tilde{\mathcal{D}}^j| = |d \in [D] : \|\tilde{\boldsymbol{r}}_d - \boldsymbol{r}^j\|_2 < \frac{\Delta}{6}|$ into $\tilde{N}^j = |d \in \mathrm{K}^{-1}(j) : \|\tilde{\boldsymbol{r}}_d - \boldsymbol{r}^j\|_2 < \frac{\Delta}{6}|$ and $\tilde{N}^{-j} = |d \in [D] \setminus \mathrm{K}^{-1}(j) : \|\tilde{\boldsymbol{r}}_d - \boldsymbol{r}^j\|_2 < \frac{\Delta}{6}|$. The penultimate inequality holds since $\tilde{N}^{-j} \geq 0$, and the ultimate inequality uses the definition of the event **(E)**. □

Proof of Lemma 3: We prove the result in three steps. Step 1 shows that each $\tilde{S}^k$, $k \in [K]$, intersects with precisely one $\tilde{D}^j$, $j \in [K]$, that is, the first direction of the first claim in the statement of the lemma. Step 2 proves the second claim in the statement of the lemma, and Step 3 shows that each $\tilde{D}^j$, $j \in [K]$, intersects with precisely one $\tilde{S}^k$, $k \in [K]$, that is, the reverse direction of the first claim in the statement of the lemma.

In view of Step 1, we proceed in two Sub-steps. Sub-step 1 *(i)* shows that each $\tilde{S}^k$, $k \in [K]$, intersects with *at most* one $\tilde{D}^j$, $j \in [K]$, and Sub-step 1 *(ii)* subsequently argues that each $\tilde{S}^k$, $k \in [K]$, also has to intersect with *at least* one $\tilde{D}^j$, $j \in [K]$.

As for Sub-step 1 *(i)*, suppose that $\tilde{S}^k$ intersects with $\tilde{\mathcal{D}}^j$ and fix a donor $d \in \tilde{S}^k \cap \tilde{\mathcal{D}}^j$. We then have $\tilde{\boldsymbol{r}}_d \in \tilde{\mathcal{P}}^k \cap \mathcal{B}(\boldsymbol{r}^j, \frac{\Delta}{6})$ by definition of $\tilde{\mathcal{P}}^k$ and $\tilde{\mathcal{D}}^j$, and hence $\tilde{\mathcal{P}}^k \cap \mathcal{B}(\boldsymbol{r}^j, \frac{\Delta}{6}) \neq \emptyset$. As $\tilde{\mathcal{P}}^k$ has a radius of at most $\frac{\Delta}{3}$, Observation 2 establishes that $\tilde{\mathcal{P}}^k$ cannot intersect with any other set $\mathcal{B}(\boldsymbol{r}^l, \frac{\Delta}{6}), l \in [K] \setminus \{j\}$. This implies that $\tilde{S}^k \cap \tilde{\mathcal{D}}^l = \emptyset$ for all $l \in [K] \setminus \{j\}$ by Observation 1 (ii). Thus, $\tilde{S}^k$ indeed intersects with at most one set $\tilde{\mathcal{D}}^j, j \in [K]$.

We prove Sub-step 1 *(ii)* by contradiction: We first show that if $\tilde{S}^k$ does not intersect with any set $\tilde{\mathcal{D}}^j, j \in [K]$, then $|\tilde{S}^k|$ is smaller than all $|\tilde{\mathcal{D}}^j|, j \in [K]$. We then show that this would contradict the construction of $\tilde{S}^k$ in Algorithm 2, which selects $\tilde{S}^k$ as the largest donor set among the remaining donors.

To see that $\tilde{S}^k$ not intersecting with any $\tilde{\mathcal{D}}^j, j \in [K]$, implies that $|\tilde{S}^k| < |\tilde{\mathcal{D}}^j|, j \in [K]$, note that $\tilde{S}^k \cap \tilde{\mathcal{D}}^j = \emptyset$ for all $j \in [K]$ implies that $\tilde{S}^k \subseteq [D] \setminus F$, where $F = \bigcup_{j \in [K]} \tilde{\mathcal{D}}^j$, and thus $|\tilde{S}^k| \leq D - |F|$. The cardinality of F is lower bounded by

$$|F| = \left| \bigcup_{j \in [K]} \tilde{\mathcal{D}}^j \right| = \sum_{j \in [K]} |\tilde{\mathcal{D}}^j| \geq \sum_{j \in [K]} (1 - \eth - \eta) N_j = (1 - \eth - \eta) D$$

The second equality holds since the sets $\tilde{\mathcal{D}}^j, j \in [K]$ are pairwise disjoint by Observation 1 (iii). The inequality holds due to event **(E)**, which implies that $|\tilde{\mathcal{D}}^j| \geq (1 - \eth - \eta) N_j$ for every $j \in [K]$ (*cf.* Observation 3). The last equality holds since $\sum_{j \in [K]} N_j = D$. We thus conclude that $|\tilde{S}^k| \leq D - (1 - \eth - \eta) D = (\eth + \eta) D$.

We next show that our upper bound on $|\tilde{S}^k|$ is strictly less than all $|\tilde{\mathcal{D}}^j|, j \in [K]$. Observation 3 implies that every set $\tilde{\mathcal{D}}^j, j \in [K]$, has a cardinality of at least

$$(1 - \eth - \eta) N_K > N_K - \frac{N_K}{2 + \beta (K-1)} = \frac{1 + \beta (K-1)}{2 + \beta (K-1)} N_K \geq \frac{D}{2 + \beta (K-1)} > (\eth + \eta) D.$$

The first expression lower bounds the cardinality of all $\tilde{\mathcal{D}}^j, j \in [K]$, since $N_1 \geq N_2 \geq \cdots \geq N_K$ by construction. The two strict inequalities use Condition **(C)**, which states that $\eth + \eta < \frac{1}{2 + \beta(K-1)}$. The equality rearranges terms. The weak inequality is due to $N_K \geq \frac{1}{1 + \beta(K-1)} D$, which is derived from the definition of β. Indeed, we have $N_j \leq \beta N_K$ for every $j \in [K], j \neq K$, and hence $D = \sum_{j \in [K]} N_j \leq (K-1) \beta N_K + N_K = (1 + \beta(K-1)) N_K$. As a result, we conclude that $|\tilde{\mathcal{D}}^j|$ is larger than $|\tilde{S}^k|$ for every $j \in [K]$.

To conclude Sub-step 1 *(ii)*, we show that $\tilde{S}^k$ not intersecting any $\tilde{\mathcal{D}}^j$, $j \in [K]$, and $|\tilde{S}^k| < |\tilde{\mathcal{D}}^j|$, $j \in [K]$, and hence contradicts the construction of $\tilde{S}^k$ in Algorithm 2. To this end, fix the index $k \in [K]$ of the first set $\tilde{S}^k$ generated by Algorithm 2 that does not intersect any $\tilde{\mathcal{D}}^j$, $j \in [K]$. Sub-step 1 *(i)* then implies that every set $\tilde{S}^l$, $l < k$, generated by Algorithm 2 must intersect exactly one set $\tilde{\mathcal{D}}^j$, and thus there are $K - k + 1$ sets $\tilde{\mathcal{D}}^j$ that do not intersect any set $\tilde{S}^l$, $l < k$. Since, for each of these sets $\tilde{\mathcal{D}}^j$, we can construct $\tilde{\mathcal{D}}^j_L$ which constitutes a valid choice for $\tilde{S}^k$ with $|\tilde{\mathcal{D}}^j_L| \geq |\tilde{\mathcal{D}}^j| > |\tilde{S}^k|$, Algorithm 2 would have chosen one of those sets instead of $\tilde{S}^k$. This leads to the desired contradiction, and we thus conclude that each $\tilde{S}^k$ must intersect at least one $\tilde{\mathcal{D}}^j$, $j \in [K]$. The construction of $\tilde{\mathcal{D}}^j_L$ involves taking the empirical donation probability vector $\tilde{\boldsymbol{r}}_d$ of a donor $d \in \tilde{\mathcal{D}}^j$ as the center and including all remaining donors who are within a distance of $\frac{\Delta}{3}$ from $\tilde{\boldsymbol{r}}_d$. By Observation 3 and Condition **(C)**, it follows that $(1-\eth-\eta)N_j > 0$. Consequently, the existence of a donor $d \in \tilde{\mathcal{D}}^j$ is established. Additionally, since $\tilde{\mathcal{D}}^j$ has a radius of $\frac{\Delta}{6}$, all donors in $\tilde{\mathcal{D}}^j$ differ from each other by at most $\frac{\Delta}{3}$, which implies that $\tilde{\mathcal{D}}^j_L$ encompasses $\tilde{\mathcal{D}}^j$. This is because $\tilde{\mathcal{D}}^j_L$ is defined as a ball with a radius of $\frac{\Delta}{3}$ and is centered within $\tilde{\mathcal{D}}^j$.

As for Step 2, assume to the contrary that $\tilde{S}^k, k \in [K]$, is the first set generated by Algorithm 2 whose cardinality is less than $(1-\eth-\eta)N_k$. A similar reasoning as in the previous paragraph shows that there exist $K-k+1$ sets in $\tilde{\mathcal{D}}^j, j \in [K]$, that do not intersect any set $\tilde{S}^l, l < k$. Moreover, Observation 3 implies that there are at least k sets $\tilde{\mathcal{D}}^j, j \in [K]$, that have a cardinality of at least $(1-\eth-\eta)N_k$. Combining both findings, we conclude that there is at least one set $\tilde{\mathcal{D}}^j, j \in [K]$, that does not intersect any set $\tilde{S}^l, l < k$, and that has a cardinality of at least $(1-\eth-\eta)N_k$. Again, we can construct a set $\tilde{\mathcal{D}}^j_L$ from $\tilde{\mathcal{D}}^j$ following exactly the same step as we described in the past paragraph, which is a valid choice for $\tilde{S}^k$ with $|\tilde{\mathcal{D}}^j_L| \geq |\tilde{\mathcal{D}}^j| > |\tilde{S}^k|$, Algorithm 2 would have chosen $\tilde{\mathcal{D}}^j_L$ instead of $\tilde{S}^k$. This leads to the desired contradiction, and we thus conclude that each $\tilde{S}^k$ has a cardinality of at least $(1-\eth-\eta)N_k$.

In view of Step 3, it is sufficient to show that each set $\tilde{\mathcal{D}}^j$, $j \in [K]$, intersects *at most* one set $\tilde{S}^k, k \in [K]$. Indeed, in this case each set $\tilde{\mathcal{D}}^j, j \in [K]$ also has to intersect *at least* one set $\tilde{S}^k, k \in [K]$, for otherwise, there would be sets $\tilde{S}^k$ that do not intersect any $\tilde{\mathcal{D}}^j$, in contradiction to the findings of Step 1.

Suppose to the contrary that there are $k, j, l \in [K], k \neq l$, such that $\tilde{S}^k$ and $\tilde{S}^l$ are the first two sets generated by Algorithm 2 that have non-empty intersections with $\tilde{\mathcal{D}}^j$. Recall that $F = \bigcup_{j\in[K]} \tilde{\mathcal{D}}^j$ and define $F^c = [D] \setminus F$. We have $\tilde{S}^k \cup \tilde{S}^l \subseteq \tilde{\mathcal{D}}^j \cup F^c$ because $\tilde{S}^k$ and $\tilde{S}^l$ cannot intersect any other set $\tilde{\mathcal{D}}^i, i \in [K] \setminus \{j\}$ by Step 1. Additionally, since $\tilde{S}^k \cap \tilde{S}^l = \emptyset$ (*cf.* Observation 1 (i)), we have $|\tilde{S}^k| + |\tilde{S}^l| = |\tilde{S}^k \cup \tilde{S}^l| \leq |\tilde{\mathcal{D}}^j \cup F^c| = |\tilde{\mathcal{D}}^j| + |F^c|$. Recall from before that $|\tilde{S}^k|, |\tilde{S}^l| \geq (1-\eth-\eta)N_K$ and $|F^c| \leq (\eth+\eta)D < (1-\eth-\eta)N_K$. Thus, both $|\tilde{S}^k|$ and $|\tilde{S}^l|$ must be strictly smaller than $|\tilde{\mathcal{D}}^j|$. Without loss of generality, assume that $\tilde{S}^k$ is generated before $\tilde{S}^l$. We can construct $\tilde{\mathcal{D}}^j_L$ from $\tilde{\mathcal{D}}^j$ as above, which is a valid choice for $\tilde{S}^k$ with $|\tilde{\mathcal{D}}^j_L| > |\tilde{S}^k|$, Algorithm 2 would have chosen $\tilde{\mathcal{D}}^j_L$ instead of $\tilde{S}^k$. This leads to the desired contradiction, and we thus conclude that each set $\tilde{\mathcal{D}}^j, j \in [K]$, intersects at most one set $\tilde{S}^k, k \in [K]$. □

Before proving other results, we first modify the cardinality bound of $\tilde{S}^k$ in Lemma 3 (ii).

LEMMA 7. *Assume that Condition **(C)** holds and the event **(E)** is true. For every $k \in [K]$, $|\tilde{S}^k| \geq (1-\eth-\eta)N_{\tau(k)}$ where $\tau(k)$ is the index of $\tilde{\mathcal{D}}^j, j \in [K]$ that intersects with $\tilde{S}^k$.*

Proof of Lemma 7: Under Condition **(C)** and event **(E)**, Lemma 3 holds. The statement follows directly from Lemma 3 (ii) if $\tau(k) = k$ for any $k \in [K]$. Assume the reverse that $\tau(k) \neq k$ for some $k \in [K]$. By Lemma 3 (i), the set $\tilde{\mathcal{D}}^{\tau(k)}$ does not intersect any set $\tilde{S}^j, j < k$, and thus $\tilde{\mathcal{D}}^{\tau(k)}_L$, which is constructed from $\tilde{\mathcal{D}}^{\tau(k)}$ as discussed in the proof of Lemma 3, is an admissible choice for $\tilde{S}^k$ in the k-th iteration of Algorithm 2. The statement now follows from

$$|\tilde{S}^k| \geq |\tilde{\mathcal{D}}^{\tau(k)}_L| \geq |\tilde{\mathcal{D}}^{\tau(k)}| \geq (1-\eth-\eta)N_{\tau(k)}.$$

The first inequality follows from the construction of Algorithm 2 which selects the largest admissible set in each iteration and $\tilde{\mathcal{D}}_L^{\tau(k)}$ is a valid candidate. We note that the second inequality follows from that $\tilde{\mathcal{D}}^{\tau(k)}$ is contained in $\tilde{\mathcal{D}}_L^{\tau(k)}$ by construction, and the last inequality follows from Observation 3. $\square$

Proof of Lemma 4: Recall that by construction $\tilde{\mathcal{P}}^k$ is the smallest open ball that covers the empirical probability vector $\tilde{\boldsymbol{r}}_d$ of all donors $d \in \tilde{S}^k$, and that $\tilde{\mathcal{P}}^k$ has a radius of at most $\frac{\Delta}{3}$. Assume the center of $\tilde{\mathcal{P}}^k$ is $\tilde{\boldsymbol{r}}'$. The number of contaminating donor in $\tilde{S}^k$ is therefore upper bounded by $|d \in \mathrm{K}^{-1}(j), j \in [K] \setminus \{k\} : \tilde{\boldsymbol{r}}_d \in \tilde{\mathcal{P}}^k|$. The remainder of the proof proceeds in two steps. We first lower bound the distance of $\tilde{\mathcal{P}}^k$ to any cluster center $\boldsymbol{r}^j, j \in [K] \setminus \{k\}$. We then employ event **(E)** to conclude that the number of donors from cluster j, whose empirical probability vectors falls in $\tilde{\mathcal{P}}^k$, must be small.

In view of the first step, we observe that for any $j \in [K] \setminus \{k\}$, we have

$$\begin{aligned} d(\boldsymbol{r}^j, \tilde{\mathcal{P}}^k) &= \min_{\boldsymbol{r} \in \tilde{\mathcal{P}}^k} \|\boldsymbol{r}^j - \boldsymbol{r}\|_2 \quad \geq \min_{\boldsymbol{r} \in \tilde{\mathcal{P}}^k} \|\boldsymbol{r}^j - \boldsymbol{r}^k\|_2 - \|\boldsymbol{r}^k - \boldsymbol{r}\|_2 \\ &\geq \min_{\boldsymbol{r} \in \tilde{\mathcal{P}}^k} \|\boldsymbol{r}^j - \boldsymbol{r}^k\|_2 - \big(\|\boldsymbol{r}^k - \tilde{\boldsymbol{r}}'\|_2 + \|\tilde{\boldsymbol{r}}' - \boldsymbol{r}\|_2\big) \\ &\geq \Delta - \left(\frac{\Delta}{2} + \frac{\Delta}{3}\right) \quad = \frac{\Delta}{6}. \end{aligned} \tag{10}$$

The first two inequalities apply the triangle inequality. As for the third inequality, we have $\|\boldsymbol{r}^j - \boldsymbol{r}^k\|_2 \geq \Delta$ by the definition of Δ, $\|\tilde{\boldsymbol{r}}' - \boldsymbol{r}\|_2 \leq \frac{\Delta}{3}$ holds since $\tilde{\mathcal{P}}^k$ has a radius no larger than $\frac{\Delta}{3}$, and we observe that

$$\|\boldsymbol{r}^k - \tilde{\boldsymbol{r}}'\|_2 \leq \|\boldsymbol{r}^k - \tilde{\boldsymbol{r}}_d\|_2 + \|\tilde{\boldsymbol{r}}_d - \tilde{\boldsymbol{r}}'\|_2 \leq \frac{\Delta}{6} + \frac{\Delta}{3} = \frac{\Delta}{2}.$$

Here, the first inequality applies the triangle inequality using any $d \in \tilde{S}^k \cap \tilde{\mathcal{D}}^k$, which exists due to the first result in Lemma 3. The definition of d and the constructions of $\tilde{\mathcal{P}}^k$ and $\tilde{\mathcal{D}}^k$ implies that $\tilde{\boldsymbol{r}}_d \in \tilde{\mathcal{P}}^k \cap \mathcal{B}(\boldsymbol{r}^k, \frac{\Delta}{6})$. Since $\tilde{\mathcal{P}}^k$ is a ball of radius at most $\frac{\Delta}{3}$, we have $\|\tilde{\boldsymbol{r}}_d - \tilde{\boldsymbol{r}}'\|_2 \leq \frac{\Delta}{3}$. Moreover, $\|\boldsymbol{r}^k - \tilde{\boldsymbol{r}}_d\|_2 \leq \frac{\Delta}{6}$ as $\mathcal{B}(\boldsymbol{r}^k, \frac{\Delta}{6})$ is a ball of radius $\frac{\Delta}{6}$. Combining the previous bounds leads us to the penultimate inequality in (10). The last inequality in (10) follows with simple algebraic computation.

As for the second step, we bound $|d \in \mathrm{K}^{-1}(j), j \in [K] \setminus \{k\} : \tilde{\boldsymbol{r}}_d \in \tilde{\mathcal{P}}^k|$ as follows:

$$\begin{aligned} &|d \in \mathrm{K}^{-1}(j), j \in [K] \setminus \{k\} : \tilde{\boldsymbol{r}}_d \in \tilde{\mathcal{P}}^k| = \sum_{j \in [K] \setminus \{k\}} |d \in \mathrm{K}^{-1}(j) : \tilde{\boldsymbol{r}}_d \in \tilde{\mathcal{P}}^k| \\ &\leq \sum_{j \in [K] \setminus \{k\}} \left| d \in \mathrm{K}^{-1}(j) : \|\tilde{\boldsymbol{r}}_d - \boldsymbol{r}^j\|_2 \geq \frac{\Delta}{6} \right| \leq \sum_{j \in [K] \setminus \{k\}} (\eth + \eta) N_j. \end{aligned}$$

Here, the first equality exploits the disjointness of the clusters. The first inequality follows from Step 1, which implies that $\|\tilde{\boldsymbol{r}}_d - \boldsymbol{r}^j\|_2 \geq d(\boldsymbol{r}^j, \tilde{\mathcal{P}}^k) \geq \frac{\Delta}{6}$ whenever $\tilde{\boldsymbol{r}}_d$ in $\tilde{\mathcal{P}}^k$. The definition of event **(E)** implies the last inequality. $\square$

For the favor of notation simplicity, we denote $\psi = \kappa \cdot (\eth + \eta) = (\beta \cdot (K-1) + 1)(\eth + \eta)$ henceforward.

Proof of Lemma 5: For each $\tilde{S}^k$ identified at line 7, $|\tilde{S}^k| \geq (1 - \eth - \eta) N_k$ by Lemmas 7. The number of contaminating donors does not exceed $\sum_{j \in [K], j \neq k} N_j$ according to Lemma 4. Consequently, at least $(1 - \eth - \eta) N_k - \sum_{j \in [K], j \neq k} N_j = N_k - (\eth + \eta) \sum_{j \in [K]} N_j = N_k - (\eth + \eta) D$ donors are correctly clustered. Since the assignment at line 11 does not alter any previous assignments in the loop, we ensure that at least $N_k - (\eth + \eta) D$ donors are correctly clustered for each donor type $k \in [K]$. Therefore, the number of mis-clustered donors is upper bounded by $(\eth + \eta) D$ for each type, concluding the first part of the result.

To demonstrate the second part of the result, define $\tilde{F} = [D] \setminus \cup_{j\in[K]} \tilde{E}^j$, the set of donors whose deviation from their true expected donation vector exceeds $\frac{\Delta}{6}$. At line 7, by Lemma 3 (i), we know that $\tilde{S}^k = \tilde{E}^k \cup \tilde{F}$. Only donors from $\tilde{F}$, who are not of type k, can be the contaminating donors in $\tilde{S}^k$. Thus, in this step, the number of mis-clustered donors can be upper bounded by the cardinality of $\tilde{F}$, which is bounded by $|F| \le (\eth+\eta)D$ due to the event **(E)**. At line 11, the number of unclustered donors remains upper bounded by $(\eth+\eta)D$, as $|\tilde{S}^k| \ge (1-\eth-\eta)N_k$ for each $k \in [K]$ by Lemma 7. Assuming all of these donors are mis-clustered, the total number of mis-clustered donors is then upper bounded by $2(\eth+\eta)D$.

The third part of the result requires careful analysis. Define $\tilde{L}$ to be the set of donors that have not been clustered before line 11. Let $\tilde{n}^k$ be the number of contaminating donors in $\tilde{S}^k$ identified at line 7. The number of donors in $\tilde{L}$ whose empirical donation vector deviates from their true value by $\frac{\Delta}{6}$ is upper bounded by $|F| - \sum_{k\in[K]} \tilde{n}^k$, and the number of donors of type k in $\tilde{L}$ whose empirical donation vector is within $\frac{\Delta}{6}$ to their true value is upper bounded by $\tilde{n}^k$. Hence, for each identified cluster, the number of contaminating donors are $\tilde{n}^k + |F| - \sum_{k\in[K]} \tilde{n}^k + \sum_{k\in[K]} \tilde{n}^k - \tilde{n}^k = |F| \le (\eth+\eta)D$. □

Proof of Lemma 6: In this proof, let us denote $\zeta = \max \mathcal{R}$. For any $k \in [K]$, define $\tilde{\boldsymbol{q}}^k = N_j^{-1} \sum_{d\in \mathrm{K}^{-1}(k)} \tilde{\boldsymbol{r}}_d$ as the estimate of $\boldsymbol{r}^k$ based on the donors from cluster k. We prove the claim in two steps. Step 1 proves the bound

$$\mathbb{P}\left[\|\boldsymbol{r}^k - \tilde{\boldsymbol{q}}^k\|_\infty \le \nu \;\; \forall k \in [K]\right] \ge 1 - \sum_{c\in[C]} \sum_{j\in[K]} 2\exp\left(-2N_j M \nu^2\right).$$

Recall the definition of ψ, Step 2 shows that $\|\tilde{\boldsymbol{q}}^k - \tilde{\boldsymbol{r}}^k\|_\infty \le \psi$ for every $k \in [K]$ under the conditions of the lemma. Combining the results from Steps 1 and 2 via the triangle inequality of the infinity norm yields the statement of the lemma.

As for Step 1, we note that

$$\begin{aligned}
&\mathbb{P}\left[\|\boldsymbol{r}^k - \tilde{\boldsymbol{q}}^k\|_\infty \le \nu \;\; \forall k \in [K]\right] = \mathbb{P}\left[|r_c^k - \tilde{q}_c^k| \le \nu \;\; \forall c \in [C], k \in [K]\right] \\
=&1 - \mathbb{P}\left[\exists c \in [C], k \in [K] \,:\, |r_c^k - \tilde{q}_c^k| > \nu\right] \ge 1 - \sum_{c\in[C]} \sum_{k\in[K]} \mathbb{P}\left[|r_c^k - \tilde{q}_c^k| \ge \nu\right] \\
\ge&1 - \sum_{c\in[C]} \sum_{j\in[K]} 2\exp\left(-2N_j M \nu^2/\zeta^2\right).
\end{aligned}$$

Here, the first equality applies the definition of the infinity norm, the second equality considers the complementary event, and the first inequality uses the union bound and replaces the strict inequality with a weak one. In view of the final inequality, we note that $\tilde{\boldsymbol{r}}_d, d \in \mathrm{K}^{-1}(j)$, is the average of M independent random vectors with mean $\boldsymbol{r}^j$, and hence $\tilde{q}_c^j = N_j^{-1} \sum_{d\in\mathrm{K}^{-1}(j)} \tilde{r}_{dc}, c \in [C]$, is the average of $N_j M$ independent random variables with mean r_c^j supported on $[0, \zeta]$. Lemma 12 thus implies that $\mathbb{P}[|r_c^j - \tilde{q}_c^j| \ge \nu] \le 2\exp(-2N_j M \nu^2)$.

Step 2 is split into three sub-steps. Sub-step 2 *(i)* expresses $\tilde{\boldsymbol{q}}^k$ in terms of $\tilde{\boldsymbol{r}}^k$ and $\tilde{\boldsymbol{r}}_d$. Subsequently, Sub-steps 2 *(ii)* and 2 *(iii)* exploit the expression of Sub-step 2 *(i)* to show that $\tilde{\boldsymbol{q}}^k - \tilde{\boldsymbol{r}}^k \le \psi\zeta\mathbf{1}$ and $\tilde{\boldsymbol{q}}^k - \tilde{\boldsymbol{r}}^k \ge -\psi\zeta\mathbf{1}$ component-wise, respectively. Taken together, both inequalities imply that $\|\tilde{\boldsymbol{q}}^k - \tilde{\boldsymbol{r}}^k\|_\infty \le \psi\zeta$ as desired.

In view of Sub-step 2 *(i)*, fix any $k \in [K]$ and consider the decomposition of $\tilde{S}^k$ into $\tilde{T} \cup \tilde{W}$, where $\tilde{T} = \tilde{S}^k \cap \mathrm{K}^{-1}(k)$, that is, $\tilde{T}$ contains the donors in $\tilde{S}^k$ that stem from cluster k, and $\tilde{W} = \tilde{S}^k \setminus \mathrm{K}^{-1}(k)$, that is, $\tilde{W}$ contains the donors in $\tilde{S}^k$ that do not stem from cluster k. We then have

$$\tilde{\boldsymbol{q}}^k = \frac{\sum_{d\in\tilde{T}} \tilde{\boldsymbol{r}}_d + \sum_{d\in\tilde{T}^c} \tilde{\boldsymbol{r}}_d}{N_k} = \frac{(|\tilde{T}| + |\tilde{W}|)\tilde{\boldsymbol{r}}^k - \sum_{d\in\tilde{W}} \tilde{\boldsymbol{r}}_d + \sum_{d\in\tilde{T}^c} \tilde{\boldsymbol{r}}_d}{N_k}. \tag{11}$$

where $\tilde{T}^{\mathrm{c}} = \mathrm{K}^{-1}(k) \setminus \tilde{S}^k$ is the set of donors from cluster k that are not contained in $\tilde{S}^k$. The first equality follows from the definition of $\tilde{\boldsymbol{q}}^k$ and the fact that $\tilde{T} \cup \tilde{T}^{\mathrm{c}} = \mathrm{K}^{-1}(k)$. The second equality holds since $\tilde{\boldsymbol{r}}^k = |\tilde{S}^k|^{-1} \sum_{d \in \tilde{S}^k} \tilde{\boldsymbol{r}}_d = \frac{\sum_{d\in\tilde{T}} \tilde{\boldsymbol{r}}_d + \sum_{d\in\tilde{W}} \tilde{\boldsymbol{r}}_d}{|\tilde{T}|+|\tilde{W}|}$, which implies that $\sum_{d\in\tilde{T}} \tilde{\boldsymbol{r}}_d = (|\tilde{T}| + |\tilde{W}|)\tilde{\boldsymbol{r}}^k - \sum_{d\in\tilde{W}} \tilde{\boldsymbol{r}}_d$.

As for Sub-step 2 *(ii)*, we upper bound the last expression in (11) as follows:

$$\begin{aligned}
\tilde{\boldsymbol{q}}^k &= \frac{(|\tilde{T}| + |\tilde{W}|)\tilde{\boldsymbol{r}}^k - \sum_{d\in\tilde{W}} \tilde{\boldsymbol{r}}_d + \sum_{d\in\tilde{T}^{\mathrm{c}}} \tilde{\boldsymbol{r}}_d}{N_k} \le \frac{(|\tilde{T}| + |\tilde{W}|)\tilde{\boldsymbol{r}}^k + \sum_{d\in\tilde{T}^{\mathrm{c}}} \tilde{\boldsymbol{r}}_d}{N_k} \\
&\le \frac{|\tilde{T}|\tilde{\boldsymbol{r}}^k + |\tilde{W}|\tilde{\boldsymbol{r}}^k + |\tilde{T}^{\mathrm{c}}|\zeta\mathbf{1}}{N_k} \le \frac{|\tilde{T}|\tilde{\boldsymbol{r}}^k + (\eth+\eta)D\tilde{\boldsymbol{r}}^k + |\tilde{T}^{\mathrm{c}}|\zeta\mathbf{1}}{N_k} \\
&\le \frac{N_k\tilde{\boldsymbol{r}}^k + (\eth+\eta)D\zeta\mathbf{1}}{N_k} \le \tilde{\boldsymbol{r}}^k + \psi\zeta\mathbf{1}.
\end{aligned} \tag{12}$$

Here, the first inequality removes a non-positive term, while the second inequality overestimates each $\tilde{\boldsymbol{r}}_d, d \in \tilde{T}^{\mathrm{c}}$, by $\zeta\mathbf{1}$. The third inequality is due to the third result of Lemma 5, which shows that $|\tilde{W}| \le (\eth+\eta)D$. In the proof of Lemma 5, we know that $|\tilde{T}| \ge N_k - (\eth+\eta)D$. Then, the penultimate inequality in (12) follows from

$$|\tilde{T}|\tilde{\boldsymbol{r}}^k + |\tilde{T}^{\mathrm{c}}|\mathbf{1} = |\tilde{T}|\tilde{\boldsymbol{r}}^k + \left(N_k - |\tilde{T}|\right)\zeta\mathbf{1} = N_k\zeta\mathbf{1} + |\tilde{T}|\left(\tilde{\boldsymbol{r}}^k - \zeta\mathbf{1}\right) \le (N_k - (\eth+\eta)D)\tilde{\boldsymbol{r}}^k + (\eth+\eta)D\zeta\mathbf{1},$$

where the first equality holds because $|\tilde{T}| + |\tilde{T}^{\mathrm{c}}| = N_k$ and the second equality rearranges terms. Because $\tilde{\boldsymbol{r}}^k - \zeta\mathbf{1} \le \mathbf{0}$, all components of $|\tilde{T}|\left(\tilde{\boldsymbol{r}}^k - \zeta\mathbf{1}\right)$ are monotonically non-increasing in $|\tilde{T}|$. The inequality then follows from substituting of the lower bound of $|\tilde{T}|$ and rearranging terms. The final inequality in (12) follows from $\frac{D}{N_k} \le (\beta(K-1)+1)$ and the definition of ψ. Moving $\tilde{\boldsymbol{r}}^k$ to the left-hand-side of (12) concludes Sub-step 2 *(ii)*.

In view of Sub-step 2 *(iii)*, we lower bound the last expression in (11) as follows:

$$\begin{aligned}
\tilde{\boldsymbol{q}}^k &= \frac{(|\tilde{T}| + |\tilde{W}|)\tilde{\boldsymbol{r}}^k - \sum_{d\in\tilde{W}} \tilde{\boldsymbol{r}}_d + \sum_{d\in\tilde{T}^{\mathrm{c}}} \tilde{\boldsymbol{r}}_d}{N_k} \\
&\ge \frac{(|\tilde{T}| + |\tilde{W}|)\tilde{\boldsymbol{r}}^k - \sum_{d\in\tilde{W}} \tilde{\boldsymbol{r}}_d}{N_k} \ge \frac{|\tilde{T}|\tilde{\boldsymbol{r}}^k - \sum_{d\in\tilde{W}}(\zeta\mathbf{1} - \tilde{\boldsymbol{r}}^k)}{N_k} \\
&\ge \frac{(N_k - (\eth+\eta)D)\tilde{\boldsymbol{r}}^k - \sum_{d\in\tilde{W}}(\zeta\mathbf{1} - \tilde{\boldsymbol{r}}^k)}{N_k} \\
&\ge \frac{(N_k - (\eth+\eta)D)\tilde{\boldsymbol{r}}^k - (\eth+\eta)D(\zeta\mathbf{1} - \tilde{\boldsymbol{r}}^k)}{N_k} \\
&\ge \tilde{\boldsymbol{r}}^k - \psi\zeta\mathbf{1}.
\end{aligned} \tag{13}$$

The first inequality removes a non-negative term, while the second inequality overestimates $\tilde{\boldsymbol{r}}_d, d \in \tilde{T}^{\mathrm{c}}$, by $\zeta\mathbf{1}$ and rearrange terms. The third inequality use the lower bound of $|\tilde{T}|$ while the penultimate inequality applies an upper bound of $|\tilde{W}|$ which we have demonstrated above. The last inequality uses again that $\frac{D}{n_k} \le (\beta(K-1)+1)$ and the definition of ψ. Moving $\tilde{\boldsymbol{r}}^k$ to the left-hand-side of (13) then concludes Sub-step 2 *(iii)*. □

C.3. Proof of the Result in Section 5.2

Proof of Theorem 2: In this proof, we define $\Delta_d^{\mathrm{c}} = |r_c^{\mathrm{K}}(d) - e_c|$, and $\Delta_{\max}^{\mathrm{c}} = \max_{d\in[D]} \Delta_d^{\mathrm{c}}$. The proof is divided into six steps. Steps 1 to 5 determine the regret based on Condition **(C)**. Then, Step 6 determines the condition on T (which is regulated in the statement of the theorem) such that the conditions in Condition **(C)** holds.

Step 1 decomposes the regret incurred into two parts: the regret incurred during the exploration periods and during the exploitation periods. For the regret during the exploitation periods, we further decompose it into two cases. The first case occurs under the complement of event **(E)**, and the second case under the event **(E)**. The result of this decomposition is provided in (14).

Step 2 upper bounds the first two terms in (14). Step 3 is dedicated to upper bounding the last conditional expectation in (14). It is divided into two sub-steps: Sub-step 3 (i) further decomposes the last conditional expectation in (14) into two scenarios depending on whether $\|\boldsymbol{r}^k - \tilde{\boldsymbol{r}}^k\|_\infty \le \psi\zeta + \nu$ for all $k \in [K]$. Sub-step 3 (ii) combines the bounds from Sub-step 3 (i) and obtains a tighter bound by minimizing over the hyper-parameter ν (the significance of which will be clear in context).

Step 4 reformulates and relaxes the bounds obtained in Steps 2 and 3 for better representation and further operations, divided into: Sub-step 4 (i) combines the bounds obtained in Steps 2 and 3. Sub-step 4 (ii) optimizes over another hyper-parameter η, eliminates it, and then reformulates and relaxes the bounds to derive the final regret formulation in terms of M and other problem parameters. This results in the formula given in (24).

Step 5 aims to minimize the expression in (24) by selecting M, the number of exploration periods, and deriving the upper bound. This optimization process is inherently challenging; therefore, we focus on minimizing the simplified term $\frac{M}{2} + 1.2\kappa\xi T \exp(-M\phi)$, denoted by the optimal solution $M^\star$. This approach exploits the assumption that $N_K \ge N_K$, which ensures that the last two terms within the square brackets of (24) are small at the optimal $M^\star$. This underscores the philosophy of our algorithm where clustering benefits from high dimensionality and the estimation based on clustering achieves higher precision due to the substantial size of the clusters. Step 6 establishes the condition on T such that the choice of M makes the prerequisites outlined in Condition **(C)** satisfied.

Let us begin with Step 1 of the proof. According to Algorithm 2, we initially conduct all campaigns M times for all donors. This implementation ensures that $\tilde{a}_{dct} = 1$ for all $d \in [D], c \in [C]$, and $t \in [M]$. Consequently, the indicator function $\mathbb{1}\{\tilde{a}_{dct} \neq \mathbb{1}\{r_c^{\mathrm{K}(d)} \ge e_c\}\}$ simplifies to $\mathbb{1}\{r_c^{\mathrm{K}(d)} < e_c\}$. Thus, the regret $R(T)$ expressed in (1) can be decomposed accordingly into the first term as follows:

$$
\begin{aligned}
R(T) &\le \frac{1}{D}\mathbb{E}\left[\sum_{d\in[D]}\sum_{c\in[C]}\sum_{t\in[M]} \Delta_d^c \cdot \mathbb{1}\{r_c^{\mathrm{K}(d)} < e_c\} + \sum_{d\in[D]}\sum_{c\in[C]}\sum_{t>M} \Delta_d^c \cdot \mathbb{1}\{\tilde{a}_{dct} \neq \mathbb{1}\{r_c^{\mathrm{K}(d)} \ge e_c\}\}\right] \\
&= \frac{1}{D}\left(\sum_{d\in[D]}\sum_{c\in[C]}\sum_{t\in[M]} \Delta_d^c \cdot \mathbb{1}\{r_c^{\mathrm{K}(d)} < e_c\} + \mathbb{E}\left[\sum_{d\in[D]}\sum_{c\in[C]}\sum_{t>M} \Delta_d^c \cdot \mathbb{1}\{\tilde{a}_{dct} \neq \mathbb{1}\{r_c^{\mathrm{K}(d)} \ge e_c\}\}\right]\right) \\
&= \frac{1}{D}\left(\sum_{d\in[D]}\sum_{c\in[C]}\sum_{t\in[M]} \Delta_d^c \cdot \mathbb{1}\{r_c^{\mathrm{K}(d)} < e_c\} + \right. \\
&\qquad \mathbb{E}\left[\sum_{d\in[D]}\sum_{c\in[C]}\sum_{t>M} \Delta_d^c \cdot \mathbb{1}\{\tilde{a}_{dct} \neq \mathbb{1}\{r_c^{\mathrm{K}(d)} \ge e_c\}\} | \neg\mathcal{E}\right] \mathbb{P}[\neg\mathcal{E}] + \\
&\qquad \left. \mathbb{E}\left[\sum_{d\in[D]}\sum_{c\in[C]}\sum_{t>M} \Delta_d^c \cdot \mathbb{1}\{\tilde{a}_{dct} \neq \mathbb{1}\{r_c^{\mathrm{K}(d)} \ge e_c\}\} | \mathcal{E}\right] \mathbb{P}[\mathcal{E}]\right).
\end{aligned} \tag{14}
$$

The initial equality in the decomposition of $R(T)$ follows from the linearity of expectation. The expectation operator is not necessary in the first term, as it does not involve any random quantities. The second equality arises from the property of conditional expectation, stated as: $\mathbb{E}[\tilde{Y}] = \mathbb{E}[\tilde{Y}|\tilde{X}]\mathbb{P}[\tilde{X}] + \mathbb{E}[\tilde{Y}|\neg\tilde{X}]\mathbb{P}[\neg\tilde{X}]$, for any random event $\tilde{X}$.

In terms of Step 2, we upper bound the first term in (14) by assuming $\mathbb{1}\{r_c^{\mathrm{K}(d)} < e_c\} = 1$. This assumption simplifies the terms and leads to the following first inequality:

$$\sum_{d\in[D]}\sum_{c\in[C]}\sum_{t\in[M]} \Delta_d^c \cdot \mathbb{1}\{r_c^{\mathrm{K}(d)} < e_c\} \le \sum_{d\in[D]}\sum_{c\in[C]}\sum_{t\in[M]} \Delta_d^c \le DM \sum_{c\in[C]} \Delta_{\max}^c.$$

The second inequality is derived by the definition of $\Delta_{\max}^c = \max_{d\in[D]} \Delta_d^c$ and rearranges terms.

We upper bound $\mathbb{1}\{\tilde{a}_{dct} \ne \mathbb{1}\{r_c^{\mathrm{K}(d)} \ge e_c\}\}$ by 1, which corresponds to the scenario where each campaign decision does not align with the optimal condition. This assumption leads to the first inequality in our upper bound for the first conditional expectation in (14) as follows:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{d\in[D]}\sum_{c\in[C]}\sum_{t>M} \Delta_d^c \cdot \mathbb{1}\{\tilde{a}_{dct} \ne \mathbb{1}\{r_c^{\mathrm{K}(d)} \ge e_c\}\}|\neg\mathcal{E}\right] \le \mathbb{E}\left[\sum_{d\in[D]}\sum_{c\in[C]}\sum_{t>M} \Delta_d^c|\neg\mathcal{E}\right] \\
=& \sum_{d\in[D]}\sum_{c\in[C]}\sum_{t>M} \Delta_d^c \le D\cdot(T-M)\sum_{c\in[C]} \Delta_{\max}^c.
\end{aligned}$$

The first equality arises from the condition $\neg\mathcal{E}$ and the expectation operator are not needed as no random quantity is involved. The second inequality results from the definition of $\Delta_{\max}^c$ and through algebraic summation.

In terms of Sub-step 3 *(i)*, we upper bound the last conditional expectation in (14) as below:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{d\in[D]}\sum_{c\in[C]}\sum_{t>M} \Delta_d^c \cdot \mathbb{1}\{\tilde{a}_{dct} \ne \mathbb{1}\{r_c^{\mathrm{K}(d)} \ge e_c\}\}|\mathcal{E}\right] \\
=&\mathbb{E}\left[\sum_{d\in\mathcal{D}_w\cup\mathcal{D}_c}\sum_{c\in[C]}\sum_{t>M} \Delta_d^c \cdot \mathbb{1}\{\tilde{a}_{dct} \ne \mathbb{1}\{r_c^{\mathrm{K}(d)} \ge e_c\}\}|\mathcal{E}\right] \\
\le&\mathbb{E}\left[\sum_{d\in\mathcal{D}_w}\sum_{c\in[C]}\sum_{t>M} \Delta_d^c + \sum_{d\in\mathcal{D}_c}\sum_{c\in[C]}\sum_{t>M} \left[(\psi\zeta+\nu)(1-f(\nu)) + \Delta_d^c \cdot f(\nu)\right]|\mathcal{E}\right] \\
\le&\mathbb{E}\left[\sum_{d\in\mathcal{D}_w}\sum_{c\in[C]}\sum_{t>M} \Delta_{\max}^c + \sum_{d\in\mathcal{D}_c}\sum_{c\in[C]}\sum_{t>M} \left[(\psi\zeta+\nu)(1-f(\nu)) + \Delta_{\max}^c \cdot f(\nu)\right]|\mathcal{E}\right] \\
\le&\psi D\cdot(T-M)\sum_{c\in[C]} \Delta_{\max}^c + D\cdot(T-M)\sum_{c\in[C]} \left[(\psi\zeta+\nu)(1-f(\nu)) + \Delta_{\max}^c \cdot f(\nu)\right] \\
\le&\psi D\cdot(T-M)\sum_{c\in[C]} \Delta_{\max}^c + D\cdot(T-M)\sum_{c\in[C]} \left[(\psi\zeta+\nu) + \Delta_{\max}^c \cdot f(\nu)\right].
\end{aligned} \tag{15}$$

Conditional on the "good" event **(E)**, donors are categorized into those who are mis-clustered, denoted by $\mathcal{D}_w$, and those who are correctly clustered, denoted by $\mathcal{D}_c$. This categorization leads to the first equality. We note that the first inequality will take a long page to prove, we defer it to the end (next paragraph) of this sub-step. The second inequality in (15) is deduced from $\Delta_d^c \le \Delta_{\max}^c$. Conditioned on the event **(E)**, Lemma 5 establishes that $|\mathcal{D}_w| \le \psi D$. This finding, combined with the fact that $|\mathcal{D}_c| \le D$, results in the third inequality in (15). By noting that $(\psi\zeta+\nu)(1-f(\nu)) \le \psi\zeta+\nu$, we achieve the final inequality in (15).

Now, let us derive the first inequality in (15) step by step. Firstly, for $d \in \mathcal{D}_w$, we establish that:

$$\mathbb{E}\left[\sum_{d\in\mathcal{D}_w}\sum_{c\in[C]}\sum_{t>M}\Delta_d^c \cdot \mathbb{1}\{\tilde{a}_{dct} \neq \mathbb{1}\{r_c^{\mathrm{K}(d)} \geq e_c\}\}|\mathcal{E}\right] \leq \mathbb{E}\left[\sum_{d\in\mathcal{D}_w}\sum_{c\in[C]}\sum_{t>M}\Delta_d^c|\mathcal{E}\right], \tag{16}$$

as $\mathbb{1}\{\tilde{a}_{dct} \neq \mathbb{1}\{r_c^{\mathrm{K}(d)} \geq e_c\}\} \leq 1$ for all $c \in [C]$ and $t > M$. Then, for every donor who is correctly clustered, denoted as $d \in \mathcal{D}_c$, we bound the regret incurred based on whether $\|\boldsymbol{r}^k - \tilde{\boldsymbol{r}}^k\|_\infty \leq \psi\zeta + \nu$ for all $k \in [K]$. Here, $\tilde{\boldsymbol{r}}^k$ represents the estimated donation probability vector of cluster k. Referencing Lemma 6,

$$\mathbb{P}\left[\|\boldsymbol{r}^k - \tilde{\boldsymbol{r}}^k\|_\infty \leq \psi\zeta + \nu \quad \forall k \in [K]\right] \geq 1 - 2CK\exp\left(-2N_K M\nu^2/\zeta^2\right) = 1 - f(\nu), \tag{17}$$

where $f(\nu) = 2CK\exp\left(-2N_K M\nu^2/\zeta^2\right)$ and the inequality is due to $N_K \leq N_k$ for all $k \in [K]$.

When $\|\boldsymbol{r}^k - \tilde{\boldsymbol{r}}^k\|_\infty \leq \psi\zeta + \nu$ for every $k \in [K]$, that is,

$$p_c^k - (\psi\zeta + \nu) \leq \tilde{p}_c^k \leq p_c^k + \psi\zeta + \nu \quad \forall c \in [C], k \in [K], \tag{18}$$

we show

$$\Delta_d^c \cdot \mathbb{1}\{\tilde{a}_{dct} \neq \mathbb{1}\{r_c^{\mathrm{K}(d)} \geq e_c\}\} \leq \psi\zeta + \nu. \tag{19}$$

To demonstrate this bound, it is sufficient to show that $\tilde{a}_{dct} = \mathbb{1}\{r_c^{\mathrm{K}(d)} \geq e_c\}$ whenever $\Delta_d^c = |r_c^{\mathrm{K}(d)} - e_c| > \psi\zeta + \nu$. We observe that $|r_c^{\mathrm{K}(d)} - e_c| > \psi\zeta + \nu$ encompasses the following two cases:

$$\text{case } 1: r_c^{\mathrm{K}(d)} - e_c > \psi\zeta + \nu \text{ and case } 2: r_c^{\mathrm{K}(d)} - e_c < -(\psi\zeta + \nu).$$

In case 1, $\tilde{p}_c^{\mathrm{K}(d)} \geq r_c^{\mathrm{K}(d)} - (\psi\zeta + \nu) > e_c$, where the first inequality follows from the left inequality in (18) and the second from the condition of case 1. This implies that our action $\tilde{a}_{dct} = \mathbb{1}\{\tilde{p}_c^{\mathrm{K}(d)} \geq e_c\} = \mathbb{1}\{r_c^{\mathrm{K}(d)} \geq e_c\} = 1$, with the last equality reflecting the condition of case 1. In case 2, $\tilde{p}_c^{\mathrm{K}(d)} \leq r_c^{\mathrm{K}(d)} + (\psi\zeta + \nu) < e_c$, where the first inequality follows from the right inequality in (18) and the second follows from the stipulation in case 2. Hence, $\tilde{a}_{dct} = \mathbb{1}\{\tilde{p}_c^{\mathrm{K}(d)} \geq e_c\} = \mathbb{1}\{r_c^{\mathrm{K}(d)} \geq e_c\} = 0$.

When $\|\boldsymbol{r}^k - \tilde{\boldsymbol{r}}^k\| > \psi\zeta + \nu$ for some $k \in [K]$, we apply the bound:

$$\Delta_d^c \cdot \mathbb{1}\{\tilde{a}_{dct} \neq \mathbb{1}\{r_c^{\mathrm{K}(d)} \geq e_c\}\} \leq \Delta_d^c. \tag{20}$$

Given the probability bound in (17) and the regret bounds in (19) and (20), we have

$$\begin{aligned}
&\mathbb{E}\left[\sum_{d\in\mathcal{D}_c}\sum_{c\in[C]}\sum_{t>M}\Delta_d^c \cdot \mathbb{1}\{\tilde{a}_{dct} \neq \mathbb{1}\{r_c^{\mathrm{K}(d)} \geq e_c\}\}|\mathcal{E}\right] \\
\leq &\mathbb{E}\left[\sum_{d\in\mathcal{D}_c}\sum_{c\in[C]}\sum_{t>M}\left[(\psi\zeta + \nu)(1 - f(\nu)) + \Delta_d^c \cdot f(\nu)\right]|\mathcal{E}\right].
\end{aligned}$$

Together with the bound in (16), we derive the first inequality, and hence the whole bound in (15).

In terms of Sub-step 3 *(ii)*, given that the final bound in (15) is valid for every $\nu \geq 0$ and any fixed M, a tighter bound for (15) can be obtained by optimizing it with respect to ν. Furthermore, minimizing the regret expressed in (15) over ν corresponds to minimizing

$$\sum_{c\in[C]}\nu + \sum_{c\in[C]}\Delta_{\max}^c \cdot f(\nu) = C\nu + \sum_{c\in[C]}\Delta_{\max}^c 2CK\exp\left(-2N_K M\nu^2/\zeta^2\right).$$

Taking the derivative with respect to ν and setting it to zero, we obtain

$$C+\sum_{c\in[C]}\Delta^c_{\max}2CK\exp(-2N_KM\nu^2/\zeta^2)\cdot(-4N_KM\nu/\zeta^2)=0$$

$$\Longleftrightarrow\exp(-2N_KM\nu^2/\zeta^2)=\frac{C\zeta^2}{8CKN_KM\nu(\sum_{c\in[C]}\Delta^c_{\max})}\Longleftrightarrow\nu^2=\frac{\zeta^2\ln(8KN_KM\nu(\sum_{c\in[C]}\Delta^c_{\max})/\zeta^2)}{2N_KM}.$$

The last expression includes ν, which is undesired as our goal is to eliminate it from the final expression. Hence, we approximate it by setting

$$\nu^2=\frac{\zeta^2\ln(8KN_KMC)}{2N_KM}\Longleftrightarrow\nu=\sqrt{\frac{\zeta^2\ln(8KN_KMC)}{2N_KM}}. \tag{21}$$

We note that we take the form in (21) to simplify the expression and to reduce the complexity of the mathematical formulation. We note that this choice of ν does not affect it to be an upper bound but only tightness of the bound. Under this value of ν,

$$f(\nu)=2CK\exp\left(-2N_KM\nu^2/\zeta^2\right)=\frac{2CK}{8KN_KMC}=\frac{1}{4N_KM}$$

and

$$\sum_{c\in[C]}[\nu+\Delta^c_{\max}f(\nu)]\le\sum_{c\in[C]}\left(\sqrt{\frac{\zeta^2\ln(8KN_KMC)}{2N_KM}}+\frac{1}{4N_KM}\right),$$

where the inequality applies $\Delta^c_{\max}\le 1$ for all $c\in[C]$. Substituting this result back into the last term in (15), we obtain the first inequality below:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{d\in[D]}\sum_{c\in[C]}\sum_{t>M}\Delta^c_d\cdot\mathbb{1}\{\tilde{a}_{dct}\neq\mathbb{1}\{r^{\mathrm{K}(d)}_c\ge e_c\}\}|\mathcal{E}\right]\\
\le&\psi D\cdot(T-M)\sum_{c\in[C]}\Delta^c_{\max}+D\cdot(T-M)\sum_{c\in[C]}\left(\psi\zeta+\sqrt{\frac{\zeta^2\ln(8KN_KMC)}{2N_KM}}+\frac{1}{4N_KM}\right)\\
=&\psi D\cdot(T-M)\sum_{c\in[C]}(\Delta^c_{\max}+\zeta)+D\cdot(T-M)C\left(\sqrt{\frac{\zeta^2\ln(8KN_KMC)}{2N_KM}}+\frac{1}{4N_KM}\right)\\
\le&\psi D\cdot(T-M)\sum_{c\in[C]}(\Delta^c_{\max}+\zeta)+\frac{3}{2}D\cdot(T-M)C\sqrt{\frac{\zeta^2\ln(8KN_KMC)}{2N_KM}}\\
=&\psi D\cdot(T-M)\sum_{c\in[C]}(\Delta^c_{\max}+\zeta)+D\cdot(T-M)Cg(M),
\end{aligned}$$

where $g(M):=\frac{3}{2}\sqrt{\frac{\zeta^2\ln(8KN_KTC)}{2N_KM}}$. The first equality rearranges terms. The second inequality follows from $\frac{1}{4N_KM}=\frac{1}{2\times 2N_KM}\le\frac{1}{2\times\sqrt{2N_KM}}\le\frac{1}{2}\sqrt{\frac{\ln(8KN_KMC)}{2N_KM}}$ where the last inequality applies the condition $\ln(8KN_KMC)\ge 1$ as K,N_K,M, and $C\ge 1$.

As for Sub-step 4 *(i)*, we derive an upper bound of the regret that can be worked around easier. To achieve this, we put together all the upper bounds in Steps 2 and 3, which gives the first inequality in (22).

$$R(T)\le\frac{1}{D}\left(DM\sum_{c\in[C]}\Delta^c_{\max}+\mathbb{P}[\neg\mathcal{E}]D\cdot(T-M)\sum_{c\in[C]}\Delta^c_{\max}+\right.$$

$$
\begin{aligned}
&\mathbb{P}[\mathcal{E}]\left(\psi D\cdot(T-M)\sum_{c\in[C]}(\Delta^c_{\max}+\zeta)+D\cdot(T-M)Cg(M)\right)\Bigg)\\
\leq &M\sum_{c\in[C]}\Delta^c_{\max}+(T-M)\left(\psi\sum_{c\in[C]}(\Delta^c_{\max}+\zeta)+Cg(M)+\mathbb{P}[\neg\mathcal{E}]\sum_{c\in[C]}\Delta^c_{\max}\right)\\
\leq &M\sum_{c\in[C]}\Delta^c_{\max}+(T-M)\left(\sum_{c\in[C]}(\Delta^c_{\max}+\zeta)\cdot\left(\psi+\frac{\mathbb{P}[\neg\mathcal{E}]}{2}\right)+Cg(M)\right)\\
\leq &\frac{M}{2}\sum_{c\in[C]}(\Delta^c_{\max}+\zeta)+T\left(\sum_{c\in[C]}(\Delta^c_{\max}+\zeta)\cdot\left(\psi+\frac{\mathbb{P}[\neg\mathcal{E}]}{2}\right)+Cg(M)\right)\\
\leq &\sum_{c\in[C]}(\Delta^c_{\max}+\zeta)\cdot\left(\frac{M}{2}+T\left(\psi+\frac{\mathbb{P}[\neg\mathcal{E}]}{2}+g(M)\right)\right)\\
\leq &\sum_{c\in[C]}(\Delta^c_{\max}+\zeta)\cdot\left(\frac{M}{2}+T\left(\kappa\cdot(\eth+\eta)+\frac{K}{2}\exp(\frac{-N_K\eta^2}{2\eth})+g(M)\right)\right).
\end{aligned}
\tag{22}
$$

The second inequality follows by firstly applying $\mathbb{P}[\mathcal{E}]\cdot a\leq a$ for any $a\geq 0$, then moving $\frac{1}{D}$ into the bracket and canceling out with D in each term, and finally rearranging terms. Because $\Delta^c_{\max}\leq\zeta$, $\mathbb{P}[\neg\mathcal{E}]\sum_{c\in[C]}\Delta^c_{\max}\leq\mathbb{P}[\neg\mathcal{E}]\sum_{c\in[C]}\zeta$, and hence $\mathbb{P}[\neg\mathcal{E}]\sum_{c\in[C]}\Delta^c_{\max}\leq\frac{\mathbb{P}[\neg\mathcal{E}]}{2}\sum_{c\in[C]}(\Delta^c_{\max}+\zeta)$. Substituting this inequality and rearranging terms give the third inequality. Using the same trick gives $M\sum_{c\in[C]}\Delta^c_{\max}\leq\frac{M}{2}\sum_{c\in[C]}(\Delta^c_{\max}+\zeta)$, and inflating $T-M$ to T, we obtain the fourth inequality. The penultimate inequality applies $Cg(M)=g(M)\sum_{c\in[C]}1\leq g(M)\sum_{c\in[C]}(\Delta^c_{\max}+1)$ and rearranges terms. On top of this bound, the final inequality holds by observing further that $\mathbb{P}[\neg\mathcal{E}]=1-\mathbb{P}[\mathcal{E}]\leq\sum_{k\in[K]}\exp(\frac{-N_k\eta^2}{2\eth\cdot(1-\eth)})\leq K\exp(\frac{-N_K\eta^2}{2\eth})$, where the first inequality uses Lemma 1 and the second inequality uses $1-\eth\in(0,1)$ and $N_1\geq N_2\geq\cdots\geq N_K$ and that $\psi=\kappa\cdot(\eth+\eta)$.

In terms of Sub-step 4 *(ii)*, we note that for any fixed M, the bound in (22) is valid for any $\eta\geq 0$. Hence, we can minimize the last term in (22) over η to get a tighter upper bound, which is the same as minimizing

$$\kappa\eta+\frac{K}{2}\exp\left(\frac{-N_K\eta^2}{2\eth}\right).$$

Taking the derivative of this function with respect to η and setting it to 0, we obtain

$$\kappa+\frac{K}{2}\exp\left(\frac{-N_K\eta^2}{2\eth}\right)\frac{-N_K\eta}{\eth}=0\Longleftrightarrow\exp\left(\frac{-N_K\eta^2}{2\eth}\right)=\frac{2\kappa\eth}{KN_K\eta}\Longleftrightarrow\eta^2=\frac{2\eth}{N_K}\ln\left(\frac{KN_K\eta}{2\kappa\eth}\right).$$

Similar to the determination of the value of ν in (21), we set

$$\eta^2=\frac{2\eth}{N_K}\ln\left(\frac{2KN_K}{2\kappa\eth}\right)\Longleftrightarrow\eta=\sqrt{\frac{2\eth}{N_K}\ln\left(\frac{KN_K}{\kappa\eth}\right)}.$$

This will not affect the outcome value to be an upper bound but just tightness of the bound. We replace $\frac{KN_K\eta}{2\kappa\eth}$ by $\frac{2KN_K}{2\kappa\eth}$ for simplicity of expressions. With this value of η, $\frac{K}{2}\exp(\frac{-N_K\eta^2}{2\eth})=\frac{\kappa\eth}{2N_K}$. Substituting this value of η and $\frac{K}{2}\exp(\frac{-N_K\eta^2}{2\eth})$ back to the last term of (22), we obtain an upper bound of the regret as shown in the first inequality.

$$R(T)\leq\sum_{c\in[C]}(\Delta^c_{\max}+1)\left(\frac{M}{2}+T\left[\kappa\eth+\kappa\sqrt{\frac{2\eth}{N_K}\ln\left(\frac{KN_K}{\kappa\eth}\right)}+\frac{\kappa\eth}{2N_K}+g(M)\right]\right)$$

$$
\begin{aligned}
&= A \cdot \left(\frac{M}{2} + T \left[\kappa\eth + \kappa\sqrt{\frac{2\eth}{N_K}\ln\left(\frac{2KN_K}{\kappa\eth}\right)} + \frac{\kappa\eth}{2N_K} + g(M) \right] \right) \\
&= A. \left(\frac{M}{2} + T \left[\left(1 + \frac{1}{2N_K}\right)\kappa\xi\exp(-M\phi) + \kappa\sqrt{\frac{2\xi\exp(-M\phi)(\ln(KN_K) + M\phi - \ln(\kappa\xi))}{N_K}} + \right.\right. \\
&\qquad \left.\left. \frac{3}{2}\sqrt{\frac{\zeta^2\ln(8KN_KTC)}{2N_KM}} \right] \right) \\
&\le A \cdot \left(\frac{M}{2} + T \left[1.2\kappa\xi\exp(-M\phi) + \kappa\sqrt{\frac{2\xi(\ln(KN_K) + M\phi)}{N_K\exp(M\phi)}} + \frac{3}{2}\sqrt{\frac{\zeta^2\ln(8KN_KTC)}{2N_KM}} \right] \right). \qquad (23)
\end{aligned}
$$

The first equality is by defining $A = \sum_{c\in[C]}(\Delta^c_{\max} + 1)$, the second equality substitutes the expression for $\eth = \xi\exp(-M\phi)$ and that of $g(M)$ and rearranges terms, and the last inequality applies $1 + \frac{1}{2N_K} \le 1.2$ as $N_K \ge 3$ and $\ln(\kappa\xi) \ge 0$, and rearranges terms. Because the factor $A > 0$ would not play a role in our minimization of (23) with respect to M, we omit it from now on.

In terms of Step 5, we define and minimize

$$
R'(T) = \frac{M}{2} + T\left[1.2\kappa\xi\exp(-M\phi) + \kappa\sqrt{\frac{2\xi(\ln(KN_K) + M\phi)}{N_K\exp(M\phi)}} + \frac{3}{2}\sqrt{\frac{\zeta^2\ln(8KN_KTC)}{2N_KM}} \right], \qquad (24)
$$

with respect to M. However, acknowledging the formidable challenge of deriving the optimal M analytically from Equation (24), if indeed feasible, we refrain from attempting to optimize the regret to its utmost. However, because $N_K \ge N_K$ as assumed, we can show later that $\kappa\sqrt{\frac{2\xi(\ln(KN_K)+M\phi)}{N_K\exp(M\phi)}}$ and $\frac{3}{2}\sqrt{\frac{\zeta^2\ln(8KN_KTC)}{2N_KM}}$ remain relatively small if M is obtained by minimizing

$$
\frac{M}{2} + 1.2\kappa\xi T\exp(-M\phi),
$$

i.e., try to find a M that balance exploration and clustering error. We acknowledge that the M obtained by such method is not guaranteed to be the optimal one but is sufficient to generate clustering with reasonable error (a balance between too much exploration regret and too much regret from erroneous clustering). By computing the derivative with respect to M and equating it to zero, we obtain

$$
\frac{1}{2} - 1.2\kappa\xi T\exp(-M^\star\phi)\cdot\phi = 0 \Longleftrightarrow \exp(-M^\star\phi) = \frac{1}{2.4\kappa\xi T\phi} \Longleftrightarrow M^\star = \frac{1}{\phi}\ln(2.4\kappa\xi T\phi). \qquad (25)
$$

Let us now assume that $M^\star \ge 0$, i.e., ϕ is no smaller than $\frac{1}{2.4\kappa\xi T}$. As M must be an integer, we choose $M = \lceil M^\star \rceil \ge M^\star$, which implies that

$$
1.2\kappa\xi\exp(-M\phi) \le 1.2\kappa\xi\exp(-M^\star\phi) = 1.2\kappa\xi\frac{1}{2.4\kappa\xi T\phi} = \frac{1}{2T\phi}. \qquad (26)
$$

Accordingly, the regret in (24) can be upper bounded by:

$$
\begin{aligned}
R'(T) &\le \frac{M}{2} + T\left(\frac{1}{2T\phi} + \kappa\sqrt{\frac{2\xi(\ln(KN_K) + M\phi)}{2.4\kappa\xi TN_K\phi}} + \frac{3}{2}\sqrt{\frac{\zeta^2\ln(8KN_KTC)}{2N_KM}} \right) \\
&\le \frac{M}{2} + T\left(\frac{1}{2T\phi} + \sqrt{\frac{2\kappa(\ln(KN_K) + \ln(2.4\kappa\xi T\phi) + \phi)}{2.4TN_K\phi}} + \frac{3}{2}\sqrt{\frac{\zeta^2\ln(8KN_KTC)\phi}{2N_K\ln(2.4\kappa\xi T\phi)}} \right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{M}{2}+T\left(\frac{1}{2T\phi}+\sqrt{\frac{2\kappa(\ln(KN_K)+\ln(2.4\kappa\xi T\phi)+\phi)}{2.4N_KT\phi}}+\frac{3}{2}\sqrt{\frac{\zeta^2\ln(8KN_KTC)\phi}{2N_K\ln(2.4\kappa\xi T\phi)}}\right) \qquad (27)\\
&\leq \frac{\ln(2.4\kappa\xi T\phi)}{2\phi}+\frac{1}{2}+\frac{1}{2\phi}+\sqrt{\frac{\kappa T(\ln(KN_K)+\ln(2.4\kappa\xi T\phi)+\phi)}{1.2N_K\phi}}+\frac{3}{2}\sqrt{\frac{\zeta^2\ln(8KN_KTC)\phi}{2N_K\ln(2.4\kappa\xi T\phi)}}.
\end{aligned}
$$

Here, the inequality in (26) states that $1.2\kappa\xi\exp(-M\phi)\leq\frac{1}{2T\phi}$, also implying $\exp(M\phi)\geq 2.4\kappa\xi T\phi$. Substituting these two inequalities into the first term and the second term in the square bracket of (24), respectively, yields the first inequality. Additionally, as

$$
\kappa\sqrt{\frac{2\xi(\ln(KN_K)+M\phi)}{2.4\kappa\xi TN_K\phi}}\leq\sqrt{\frac{2\kappa(\ln(KN_K)+(M^\star+1)\,\phi)}{2.4TN_K\phi}}=\sqrt{\frac{2\kappa(\ln(KN_K)+\ln(2.4\kappa\xi T\phi)+\phi)}{2.4TN_K\phi}}
$$

where the inequality follows from $M\leq M^\star+1$ and rearranging terms and the equality arises from $M^\star=\frac{1}{\phi}\ln(2.4\kappa\xi T\phi)$, and as

$$
\sqrt{\frac{\ln(8KN_KTC)}{2N_KM}}\leq\sqrt{\frac{\ln(8KN_KTC)}{2N_KM^\star}}=\frac{3}{2}\sqrt{\frac{\ln(8KN_KTC)\,\phi}{2N_K\ln(2.4\kappa\xi T\phi)}},
$$

where the first inequality utilizes $M^\star\leq M$ and the equality employs $M^\star=\frac{1}{\phi}\ln(2.4\kappa\xi T\phi)$, we derive the second inequality of (27). The third inequality ensues from both $\sqrt{\frac{2\kappa(\ln(KN_K)+\ln(2.4\kappa\xi T\phi)+\phi)}{2.4TN_K\phi}}$ and $\frac{3}{2}\sqrt{\frac{\ln(8KN_KTC)\phi}{2N_K\ln(2.4\kappa\xi T\phi)}}$ being decreasing in $N_K\geq 3$ as indicated by Lemma 11, thus, both terms are upper bounded by substituting the lower bound of N_K. We note that we substitute x with $\ln(N_K)$ in both terms and $a=\ln K+\ln(2.4\kappa\xi T\phi)+\phi\geq\ln K+M\phi>0$ and $a=\ln(8KTC)>1$, respectively, in the application of Lemma 11. The last inequality of (27) applies $M\leq M^\star+1$, substitutes the value of $M^\star$, multiplies T inside the bracket, and rearranges terms.

In view of Step 6, we check the validity of Condition **(C)**. Recalling that $\eth=\xi\exp(-M\phi)$ and $\eta=\sqrt{\frac{2\eth}{N_K}\ln\left(\frac{KN_K}{\kappa\eth}\right)}$, $\eth+\eta\leq\frac{1}{2+\beta(K-1)}=\frac{1}{\kappa+1}$ is equivalent to the first condition below.

$$
\begin{aligned}
&\xi\exp(-M\phi)+\sqrt{\frac{2\eth}{N_K}\ln\left(\frac{KN_K}{\kappa\eth}\right)}\leq\frac{1}{2+\beta(K-1)}\\
\Longleftarrow\ &\xi\exp(-M\phi)+\sqrt{2\xi\frac{\ln(3K)+M\phi-\ln(\kappa\xi)}{3\exp(M\phi)}}\leq\frac{1}{2+\beta(K-1)}\\
\Longleftarrow\ &\xi\exp(-M\phi)+\sqrt{2\xi\frac{\ln(3K)+(M^\star+1)\phi-\ln(\kappa\xi)}{3\exp(M\phi)}}\leq\frac{1}{2+\beta(K-1)} \qquad (28)\\
\Longleftarrow\ &\frac{1}{2.4\kappa T\phi}+\sqrt{2\frac{\ln(3K\exp(\phi))+\ln(2.4\kappa T\phi)}{3\times 2.4\kappa T\phi}}\leq\frac{1}{2+\beta(K-1)}.
\end{aligned}
$$

As $\kappa\eth\leq\kappa\frac{1}{\kappa+1}\leq 1$, where the first inequality comes from $\eth+\eta\leq\frac{1}{\kappa+1}$, we have $2\eth\ln(\frac{K}{\kappa\eth})\geq 0$. Applying Lemma 11 with $x=\ln(N_K)\geq\ln(3)\geq 1$ and $a=0$, we know $\frac{\ln(N_K)}{N_K}$ decreases in $N_K\geq 3$. This implies the first inequality of the following equations:

$$
\frac{\ln(N_K)}{N_K}2\eth\ln(\frac{K}{\kappa\eth})\geq\frac{\ln(3)}{3}2\eth\ln(\frac{K}{\kappa\eth})=\frac{\ln(3)}{3}2\xi\exp(-M\phi)\ln(\frac{K}{\kappa\xi\exp(-M\phi)})=2\xi\frac{\ln(3K)+M\phi-\ln(\kappa\xi)}{3\exp(M\phi)}.
$$

The first equality substitutes the expression of $\eth=\xi\exp(-M\phi)$ and the last equality rearranges terms. All of these give us the first sufficient condition in (28). The second sufficiency condition in (28) is derived from

$M \leq M^\star + 1$, while the last sufficient condition follows from the inequalities below where the first inequality utilizes $M^\star\phi = \ln(2.4\kappa\xi T\phi)$, $\exp(M\phi) \geq 2.4\kappa\xi T\phi$ from inequality (26).

$$\begin{aligned} 2\xi\frac{\ln(3K) + (M^\star+1)\phi - \ln(\kappa\xi)}{3\exp(M\phi)} &\leq 2\xi\frac{\ln(3K) + \ln(2.4\kappa\xi T\phi) + \phi - \ln(\kappa\xi)}{3\times 2.4\kappa\xi T\phi} \\ &= 2\frac{\ln(3K\exp(\phi)) + \ln(2.4\kappa T\phi) - \ln(\kappa)}{3\times 2.4\kappa T\phi} \leq 2\frac{\ln(3K\exp(\phi)) + \ln(2.4\kappa T\phi)}{3\times 2.4\kappa T\phi}. \end{aligned}$$

The equality rearranges terms and the last inequality uses $\ln(\kappa) \geq 0$.

Now, let us establish a suitable lower bound for T to satisfy the condition specified in (28). Let $y = 2 + \beta(K-1) \geq 3$ considering $\beta \geq 1$ and $K \geq 2$. We demonstrate below that $2.4\kappa T\phi = 3K\exp(\phi)y^3$ fulfills the condition in (28) by showing this value will make the condition satisfied.

$$\begin{aligned} &\frac{1}{3K\exp(\phi)y^3} + \sqrt{2\frac{2\ln(3K\exp(\phi)) + 3\ln(y)}{3\times 3K\exp(\phi)y^3}} \\ \leq &\frac{1}{3K\exp(\phi)y^3} + \sqrt{2\left(\frac{2}{3y^3} + \frac{4}{9K\exp(\phi)y^2}\right)} \leq \frac{1}{54y} + \sqrt{2\left(\frac{2}{9} + \frac{2}{9}\right)\frac{1}{y^2}} \leq \frac{1}{y}. \end{aligned}$$

Here, the first inequality applies $\ln(3K\exp(\phi)) \leq 3K\exp(\phi)$ and $\ln(y) \leq y$ since both $3K\exp(\phi)$ and y are greater than 1, and then cancels out terms. The second inequality applies $3K\exp(\phi)y^3 = 3K\exp(\phi)y^2 \cdot y \geq 3\times 2\times 1\times 3^2 y = 54y$, $\frac{2}{3y^3} \leq \frac{2}{3\times 3y^2} = \frac{2}{9y^2}$, and $\frac{4}{9K\exp(\phi)y^2} \leq \frac{4}{9\times 2y^2} = \frac{2}{9y^2}$, all following from $K \geq 2$, $\exp(\phi) \geq 1$, and $y \geq 3$. The final inequality is an algebraic computation.

Now we have shown that $2.4\kappa T\phi = 3K\exp(\phi)y^3$ will make the last condition in (28) hold. We further notice that $\frac{1}{2.4\kappa T\phi} + \sqrt{2\frac{\ln(3K\exp(\phi)) + \ln(2.4\kappa T\phi)}{3\times 2.4\kappa T\phi}}$ is decreasing in $2.4\kappa T\phi$ where the decreasing property of the first term is easy to observe while the decreasing property of $\sqrt{2\frac{\ln(3K\exp(\phi)) + \ln(2.4\kappa T\phi)}{3\times 2.4\kappa T\phi}}$ follows from Lemma 11 with $x = \ln(2.4\kappa T\phi)$ and $a = \ln(3K\exp(\phi))$. Hence, $3K\exp(\phi)y^3$ is an upper bound of $2.4\kappa T\phi$ that satisfies the condition in (28). Hence, the condition in (28) is satisfied if the following condition holds:

$$\frac{1}{2.4\kappa T\phi} + \sqrt{2\frac{\ln(3K\exp(\phi)) + \ln(3K\exp(\phi)y^3)}{3\times 2.4\kappa T\phi}} \leq \frac{1}{y} \tag{29}$$

$$\iff \frac{1}{2.4\kappa T\phi} + \sqrt{2\frac{2\ln(3K\exp(\phi)) + 3\ln(y)}{3\times 2.4\kappa T\phi}} \leq \frac{1}{y} \tag{30}$$

Now, let us define $\sqrt{2\frac{2\ln(3K\exp(\phi)) + 3\ln(y)}{3\times 2.4\kappa T\phi}} = \frac{1}{ay}$ for some $a > 1$, which leads to that

$$\frac{1}{2.4\kappa T\phi} = \frac{3}{2(2\ln(3K\exp(\phi)) + 3\ln(y))a^2y^2}, \tag{31}$$

by taking square operation on both sides and rearranging terms. Finally, we transform the condition in (30) to the first term below.

$$\begin{aligned} &\frac{3}{2(2\ln(3K\exp(\phi)) + 3\ln(y))a^2y^2} + \frac{1}{ay} \leq \frac{1}{y} \iff \frac{3}{2(2\ln(3K\exp(\phi)) + 3\ln(y))a^2y^2} \leq \frac{a-1}{ay} \\ \iff &\frac{3}{2(2\ln(3K\exp(\phi)) + 3\ln(y))y} \leq a^2 - a \Longleftarrow \frac{3}{2(2\ln(3K\exp(\phi)) + 3\ln(y))y} \leq \frac{a^2}{2} - \frac{1}{2} \\ \iff &a^2 \geq 1 + \frac{3}{(2\ln(3K\exp(\phi)) + 3\ln(y))y}. \end{aligned}$$

Here, the first equivalence rearranges terms, moving $\frac{1}{ay}$ to the right-hand side and merging it with $\frac{1}{y}$. The second equivalence multiplies both sides by a^2y and rearranges terms. The first sufficient condition applies the Cauchy-Schwarz inequality in the format $a \le \frac{1+a^2}{2}$. The last equivalence follows by multiplying both sides by 2 and then adding 1 to both sides.

By substituting this value of a^2 back into (31), we obtain $2.4\kappa T\phi \ge \frac{2(2\ln(3K\exp(\phi))+3\ln(y))y^2+6y}{3}$, which implies that

$$\begin{aligned} T &\ge \frac{2(2\ln(3K\exp(\phi))+3\ln(y))y^2+6y}{7.2\kappa\phi} \\ &= \frac{[2\ln(3K\exp(\phi))+3\ln(\kappa+1)](\kappa+1)^2+3(\kappa+1)}{3.6\kappa\phi}, \end{aligned}$$

where the last equality substitute the definition of $y = 2+\beta\cdot(K-1) = \kappa+1$.

Last, if ϕ is very small, i.e., $\phi \le \frac{1}{2.4\kappa\xi T}$, we set $M^\star = 0$. The regret is upper bounded by

$$R(T) \le \frac{1}{D}\sum_{d\in[D]}\sum_{c\in[C]}\sum_{t\in[T]}\Delta_d^c\cdot\mathbb{1}\{\tilde{a}_{dct}\neq\mathbb{1}\{r_c^{\mathrm{K}(d)}\ge e_c\}\} \le \frac{1}{D}\sum_{d\in[D]}\sum_{c\in[C]}\sum_{t\in[T]}\Delta_{\max}^c = \sum_{c\in[C]}T\Delta_{\max}^c.$$

The second inequality applies that $\Delta_{\max}^c = \max_{d\in[D]}\Delta_d^c$ and the equality rearranges terms. Additionally, we this bound is valid without any conditions. □

C.4. Proof of the Result in Section 5.5

Proof of Theorem 3 The proof closely follows that of Theorem 2, with the initial step providing the decomposition of the regret as follows:

$$\begin{aligned} R(T) = \frac{1}{D}\Bigg(&\sum_{d\in[D]}\sum_{c\in[C]}\sum_{t\in[M]}\Delta_d^c\cdot\mathbb{1}\{r_c^{\mathrm{K}(d)}<e_c\}+ \\ &\mathbb{E}\left[\sum_{d\in[D]}\sum_{c\in[C]}\sum_{t>M}\Delta_d^c\cdot\mathbb{1}\{\tilde{a}_{dct}\neq\mathbb{1}\{r_c^{\mathrm{K}(d)}\ge e_c\}\}|\mathcal{E}\right]\Bigg). \end{aligned} \tag{32}$$

We simplify three terms in (14) into two because we assume that each donor type is large enough for event (**E**) to always hold, making clear the concept for this analysis. A precise and detailed analysis can be developed by following the steps adopted in the proof of Theorem 2.

As for the regret bound, the first term can be bounded by the same value as Theorem 2. In the following, we bound the conditional expectation as below:

$$\begin{aligned} &\mathbb{E}\left[\sum_{d\in[D]}\sum_{c\in[C]}\sum_{t>M}\Delta_d^c\cdot\mathbb{1}\{\tilde{a}_{dct}\neq\mathbb{1}\{r_c^{\mathrm{K}(d)}\ge e_c\}\}|\mathcal{E}\right] \\ =&\mathbb{E}\left[\sum_{d\in\mathcal{G}_1\cup\mathcal{G}_2}\sum_{c\in[C]}\sum_{t>M}\Delta_d^c\cdot\mathbb{1}\{\tilde{a}_{dct}\neq\mathbb{1}\{r_c^{\mathrm{K}(d)}\ge e_c\}\}|\mathcal{E}\right] \\ \le&\mathbb{E}\left[\sum_{d\in\mathcal{G}_1}\sum_{c\in[C]}\sum_{t>M}\eth\Delta_d^c+\sum_{d\in\mathcal{G}_2}\sum_{t>M}\left(\eth\sum_{c\in[C]}\Delta_d^c+(1-\eth)\sqrt{C\Delta^2}\right)\right] \end{aligned} \tag{33}$$

In the above, the first equality follows directly from the decomposition of donors into two different groups. Group 1 ($\mathcal{G}_1$) includes all donors whose expected donation vectors significantly deviate from all other types by more than Δ, whereas Group 2 ($\mathcal{G}_2$) comprises the remaining donors. For the first group of donors, following the reasoning in Lemmas 3 to 7 we demonstrate that at most $\eth+\eta$ portion of donors are mis-clustered.

Under the simplifying assumption in Section 5.5 that N_K is sufficiently large so that we consider η to be zero and that campaign actions based on empirical estimates of correctly clustered types are accurate. Thus, only mis-clustered donors incur regret, as detailed in the first term. The second term is derived directly from Lemma 8, and is also the source of difference in our analysis between Theorems 2 and 3.

In the following, we consolidate all the upper bounds previously discussed and rearrange the terms, resulting in the first inequality presented in (22).

$$\begin{aligned} R(T) \leq & \frac{1}{D}\left(DM\sum_{c\in[C]}\Delta^c_{\max} + (T-M)\left(\sum_{d\in[D]}\sum_{c\in[C]}\eth\Delta^c_d + \sum_{d\in\mathcal{G}_2}\sqrt{C\Delta^2}\right)\right) \\ \leq & \left[\sum_{c\in[C]}\Delta^c_{\max}(M+(T-M)\xi\exp(-M\phi))\right] + (T-M)\sqrt{C\Delta^2} \\ \leq & \left[\sum_{c\in[C]}\Delta^c_{\max}(M+T\xi\exp(-M\phi))\right] + T\sqrt{C\Delta^2} \end{aligned} \tag{34}$$

The second inequality follows from $\Delta^c_d \leq \Delta^c_{\max}$ for all $d\in[D]$, $|\mathcal{G}_2| \leq D$ with $\theta_\Delta \in [0,1]$, substituting $\eth = \xi\exp(-M\phi)$, and rearranging terms. The last inequality results from elevating $T-M$ to T, and θ_Δ to 1.

Next, we minimize the term within the square brackets by selecting an M that optimizes this term. Following the methodology employed for choosing M in Theorem 2 and reminding that $\phi = \min_{k\in[K]}\frac{\Delta^2}{36\left(\sigma_k^2+R_k\Delta/18\right)}$, we determine M as $\lceil\frac{\ln(T\xi\phi)}{\phi}\rceil$ to appropriately bound the regret, as follows:

$$\begin{aligned} R(T) \leq & \left[\sum_{c\in[C]}\Delta^c_{\max}\left(\frac{\ln(T\xi\phi)+1}{\phi}+1\right)\right] + T\sqrt{C\Delta^2} \\ \leq & \left[\sum_{c\in[C]}\Delta^c_{\max}\left(\frac{(\ln(T\xi\phi)+1)11C(\max\mathcal{R})^2}{\Delta^2}+1\right)\right] + T\sqrt{C\Delta^2} \\ \leq & \left[\sum_{c\in[C]}\Delta^c_{\max}\left(\frac{(\ln(T\xi)+1)11C(\max\mathcal{R})^2}{\Delta^2}+1\right)\right] + T\sqrt{C\Delta^2} \end{aligned}$$

The second inequality comes from $\phi \geq \frac{\Delta^2}{36(\frac{C(\max\mathcal{R})^2}{4}+\sqrt{C(\max\mathcal{R})^2}\Delta/18)} \geq \frac{\Delta^2}{11C(\max\mathcal{R})^2}$ as $\sigma_k^2 \leq \frac{C(\max\mathcal{R})^2}{4}$ and $R_k \leq \sqrt{C(\max\mathcal{R})^2}$ for each $k\in[K]$, as well as the fact that $\Delta \leq \sqrt{C(\max\mathcal{R})^2}$. The last inequality in the above follows from that $\phi \leq \frac{\Delta^2}{2R_k\Delta} \leq \frac{\Delta}{2\sqrt{C(\max\mathcal{R})^2/4}} \leq 1$. In this series of upper bounds, the first inequality comes from $\sigma_k^2 \geq 0$, the second inequality comes from $R_k \geq \sqrt{C(\max\mathcal{R})^2/4}$ by its definition, and the last inequality comes from again $\Delta \leq \sqrt{C(\max\mathcal{R})^2}$.

Next, we transition to the final part of our analysis, i.e., selecting an optimal value of Δ^2 to minimize the regret, a step not required in the analysis of Theorem 2. For notation simplification, define $\iota = \sum_{c\in[C]}\Delta^c_{\max}\left(\ln(T\xi)+1\right)11C\left(\max\mathcal{Z}\right)^2$. The regret is then expressed as

$$\frac{\iota}{\Delta^2} + T\sqrt{C\Delta^2} + \sum_{c\in[C]}\Delta^c_{\max}.$$

This expression can be minimized by setting $\Delta^2 = \left(\frac{2\iota}{T\sqrt{C}}\right)^{2/3}$, yielding the minimal regret as

$$\left(T\sqrt{C}/2\right)^{2/3}(\iota)^{1/3}\left(2^{-2/3}+2^{1/3}\right) + \sum_{c\in[C]}\Delta^c_{\max}.$$

We note that we can adopt the same approach as in Theorem 2 to establish the conditions on T ensuring that condition **(C)** is satisfied, though this is omitted here for simplicity. $\square$

LEMMA 8. *Let Δ be the input value to Algorithm 2. For any donor from a type whose distance to at least one other type is smaller than Δ, the expected regret incurred to this donor is upper bounded by $\eth \sum_{c\in[C]} \Delta_d^c + (1-\eth)\sqrt{C\Delta^2}$ during exploitation periods.*

Proof of Lemma 8: Without loss of generality, assume the donor is of type 1 with an expected donation vector $\boldsymbol{r}^1$. By Lemma 2, the probability that this donor's empirical donation vector deviates from its expected vector by more than $\frac{\Delta}{6}$ is at most $\eth$, and the probability of deviation within $\frac{\Delta}{6}$ is at least $1-\eth$. When the deviation exceeds $\frac{\Delta}{6}$, we assume all campaign actions are incorrect, resulting in a regret of $\sum_{c\in[C]} \Delta_d^c$ per period. If the deviation is within $\frac{\Delta}{6}$, the regret is bounded by $\sqrt{C\Delta^2}$. Combining these two cases completes the proof.

We obtain the bound $\sqrt{C\Delta^2}$ by considering two cases regarding the identification of type 1: correct identification and its complement. In the correct identification case, as described in Line 11, the donor will be allocated to type 1. Given our assumptions in Section 5.5, the regret for donors correctly clustered is zero. In the incorrect identification case, the donor is allocated to a mixed cluster of types each within a Δ radius of each other. As this is simply a convex combination of types, it means our empirical estimate $\hat{\boldsymbol{r}}$ will not deviate from $\boldsymbol{r}^1$ by more than Δ, thus $\|\boldsymbol{r}^1 - \hat{\boldsymbol{r}}\|_2 \le \Delta$.

For any campaign $c \in [C]$, if the decision derived from $\hat{r}_c$ coincides with the optimal decision for type 1, no regret is incurred. Otherwise, recall e_c is the cost of sending a letter of campaign c, there must be $\min\{r_c^1, \hat{r}_c\} < e_c \le \max\{r_c^1, \hat{r}_c\}$, the incurred regret for campaign c is $|e_c - r_c^1| \le |\hat{r}_c - r_c^1|$. Therefore, the total regret to the donor is $\sum_{c\in[C]} |\hat{r}_c - r_c^1|$. Since the maximum distance between $\hat{\boldsymbol{r}}$ and $\boldsymbol{r}^1$ is Δ, i.e., $\|\hat{\boldsymbol{r}} - \boldsymbol{r}^1\|_2^2 \le \Delta^2$, we have

$$\sum_{c\in[C]} |\hat{r}_c - r_c^1| = \sqrt{(\sum_{c\in[C]} |\hat{r}_c - r_c^1|)^2} \le \sqrt{C(\sum_{c\in[C]} (\hat{r}_c - r_c^1)^2)} = \sqrt{C\|\hat{\boldsymbol{r}} - \boldsymbol{r}^1\|_2^2} \le \sqrt{C\Delta^2},$$

where the first inequality applies the Cauchy-Schwarz inequality. $\square$

Appendix D: Some Auxiliary Results

LEMMA 9. *Let S_n be the sum of n independent Bernoulli random variables, each with success probability p. The additive Chernoff bounds can be expressed as: for $\lambda > 0$:*

$$\mathbb{P}\left[\frac{1}{n}S_n \geq p+\lambda\right] \leq \exp\left(-\frac{n\cdot\lambda^2}{2p(1-p)}\right)$$

Proof of Lemma 9: This result is taken from classic textbook (I will find out the reference). We omit it here for simplicity.

The result below, Lemma 10 gives a variance bound on any measurable random variable supported on $[0,1]$. Lemma 11 gives monotonicity property of a class of function. These two results will be useful for our proof of the main results in the text.

LEMMA 10. *Let $\tilde{X}$ be any measurable random variable supported on $[0,1]$, the variance of $\tilde{X}$ is upper bounded by $\frac{1}{4}$.*

Proof of Lemma 10: Suppose $\mathbb{E}[\tilde{X}] = p \in [0,1]$ as $\tilde{X}$ is supported on $[0,1]$. Then, $\mathrm{var}(\tilde{X}) = \mathbb{E}[\tilde{X}^2] - p^2 \leq \mathbb{E}[\tilde{X}] - p^2 = p - p^2 \leq \frac{1}{4}$, where the first equality is the identity of variance, the first inequality is because $\tilde{X}^2 \leq \tilde{X}$ as $\tilde{X} \in [0,1]$, and the last equality is by the definition of p. The last inequality follows from $\max_{p\in[0,1]} p(1-p) = \frac{1}{4}$, which is obtained at $p = \frac{1}{2}$. □

LEMMA 11. *Let $f(x) = \frac{a+x}{\exp(x)}$ with $x \in \mathbb{R}_+$ then, $f(x)$ is decreasing in x as long as $x \geq 1-a$. Additionally, $\sqrt{af(x)}, a > 0$ has the same monotonicity as $f(x)$ suppose $f(x) \geq 0$.*

Proof of Lemma 11: Note that $f(x) = (a+x)\exp(-x)$. The result follows from $f'(x) = \exp(-x) - (a+x)\exp(-x) = \exp(-x)(1-a-x) \leq 0$ as $x \geq 1-a$. The second claim follows directly from basic convex analysis results. □

Lemma 12 gives an expression of the Hoeffiding inequality, the proof of which can be found in most text book or even wikipedia, hence we skip its proof here.

LEMMA 12. *Given a sequence of n independent random variables $X_1, X_2, \ldots, X_n$, where each X_i is bounded such that $a_i \leq X_i \leq b_i$, the Hoeffding's inequality is given by:*

$$P\left(\sum_{i=1}^n X_i - \mathbb{E}\left[\sum_{i=1}^n X_i\right] \geq \lambda\right) \leq \exp\left(-\frac{2\lambda^2}{\sum_{i=1}^n (b_i-a_i)^2}\right)$$

And for the lower bound:

$$P\left(\mathbb{E}\left[\sum_{i=1}^n X_i\right] - \sum_{i=1}^n X_i \geq \lambda\right) \leq \exp\left(-\frac{2\lambda^2}{\sum_{i=1}^n (b_i-a_i)^2}\right)$$

Proof of Lemma 12: Proof of this result can be found at sources such as Wainwright (2019) and Lattimore and Szepesvári (2020)

LEMMA 13. *Let $\{\tilde{\boldsymbol{A}}_i\}_{i=1}^m$ be independent zero-mean matrices of $\mathbb{R}^{d_1\times d_2}$ that are almost surely bounded as $\|\tilde{\boldsymbol{A}}_i\|_2 \leq R$ and $\sigma^2 = \max\{\|\frac{1}{m}\sum_{i=1}^m \mathbb{E}[\tilde{\boldsymbol{A}}_i\tilde{\boldsymbol{A}}_i^\top]\|_2, \|\frac{1}{m}\sum_{i=1}^m \mathbb{E}[\tilde{\boldsymbol{A}}_i^\top\tilde{\boldsymbol{A}}_i]\|_2\}$. We have*

$$\mathbb{P}\left[\|\frac{1}{m}\sum_{i=1}^m \boldsymbol{A}_i\|_2 \geq \delta\right] \leq (d_1+d_2)\exp\left(\frac{-m\delta^2}{\sigma^2 + R\delta/3}\right).$$

Proof of Lemma 13: the proof of this result follows exactly that of Theorem 1.6 in Tropp (2012). □

Appendix E: Regrets of Other Campaign Strategies

In this section, we present the regret of other campaign strategies that are of direct application of literature. However, as there are only limited periods as we have motivated already, popular algorithms like UCB type would not perform well as they require time to explore. Algorithms, like Thompson sampling that requires prior assumption on the donation distribution, would not be a meaningful comparison with the algorithm we use which does not assume any prior knowledge. Carrying out this type of algorithms would also require certain advanced knowledge of the domain, which makes the application harder. Therefore, we focus on the explore-then-exploit algorithm for the comparability with the algorithm that we employ.

E.1. Alternative Campaign Algorithms with Unknown Types

In this section, we present an alternative algorithm (*cf.* Algorithm 3) that can be applied to our campaign problem when donor types are unknown. This algorithm is the same as our proposed Algorithm 1 except that it does not require clustering of donors.

Algorithm 3 Explore-then-exploit algorithm for campaign c.

1: Set exploration periods M_c

 # Exploration

2: Send the campaign the the donor for M_c periods. Compute the empirical donation vector $\hat{r}_c$, average of donation over M_c periods.

 # Exploitation

3: For each period $t \in \{M_c+1, \cdots, T\}$, send the campaign to the donor if $\hat{r}_c \geq e_c$.

There are two approaches to analyze the performance of Algorithm 3 depends on the availability of the knowledge of Δ_c that is the smallest distance between the expected donation to campaign c and the cost of campaign c, both of which are presented in Lemma 14. We note that the definition of Δ_c is different from that of Δ which is the minimal distance between the expected donations vector of different donor types.

LEMMA 14. *Suppose there are C campaigns and T periods and donors are separated into K types.*

1. *If the minimum distance between the expected donation to campaign c and the decision threshold among donor types is Δ_c, with $M_c = \max\{1, \lceil \frac{4}{\Delta_c^2} \ln(\frac{T\Delta_c^2}{4}) \rceil\}$, the regret incurred by Algorithm 3 is upper bounded by*
$$\sum_{c\in[C]} \min\{T\Delta_c, \Delta_c + \frac{4}{\Delta_c}\left(1+\max\{0, \log(\frac{T\Delta_c^2}{4})\}\right)\}.$$
2. *If the minimum distance between the expected donation to campaign c and the decision threshold among donor types is unknown, with $M_c = T^{2/3}(2\ln(T))^{1/3}$, the regret incurred by Algorithm 3 is upper bounded by*
$$\sum_{c\in[C]} 3(\max \mathcal{R})T^{2/3}(2\ln(T))^{1/3}.$$

Proof of Lemma 14: The proof of the first result adheres to the approach outlined in equation (6.6) of Lattimore and Szepesvári (2020), including a summation over all campaigns. The proof of the second result primarily follows the discussion in Section 1.2 of Slivkins (2019), substituting the big $\mathcal{O}$ notation with a more appropriate constant. We note that the value $\max \mathcal{R}$ in the regret upper bound arises because the random outcome of sending brochure of campaign c lies within $[0, \max \mathcal{R}]$, rather than $[0, 1]$, which is the usual assumption. By scaling our random outcome by $\max \mathcal{R}$, we transform it into a random variable within $[0, 1]$, allowing the argument from Section 1.2 of Slivkins (2019) to be applied successfully. □

We note that in the above analysis, the value of Δ differs from the one utilized in comparison with Algorithm 2. However, if the distances between donor types are unknown, we cannot ascertain whether a donor is correctly clustered; we can only assert, with high probability, that their empirical donation vectors do not significantly deviate from their true values using the Hoeffding inequality. Consequently, we cannot utilize all donors, whose empirical donation vectors are concentrated within a certain radius, to decrease the estimation error of the donation vector of each type. This implies the absence of the benefits provided by our proposed clustering procedure, thus necessitating independent campaign decisions among donors, a scenario that aligns with Algorithm 3.

		ϵ										
		0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
Γ	1	-5.06	-5.06	-5.05	-5.01	-4.98	-4.90	-4.49	-3.84	-2.28	-0.60	0.50
		0.20	0.20	0.20	0.25	0.34	0.43	0.81	1.38	2.26	5.65	8.28
	2	-5.06	-5.06	-5.06	-5.00	-5.02	-4.93	-4.40	-3.24	-0.81	1.73	3.74
		0.20	0.20	0.21	0.30	0.37	0.47	0.88	1.80	3.08	9.00	13.28
	3	-5.07	-5.06	-5.05	-4.98	-4.98	-4.82	-4.03	-2.48	0.35	3.45	5.68
		0.20	0.20	0.20	0.34	0.39	0.57	1.21	2.05	3.11	11.11	14.90
	4	-5.07	-5.06	-5.06	-4.96	-4.94	-4.69	-3.65	-1.53	1.43	5.02	7.71
		0.20	0.20	0.20	0.35	0.43	0.51	1.46	2.79	4.21	14.34	16.15
	5	-5.07	-5.06	-5.06	-5.01	-5.00	-4.60	-3.28	-1.02	1.93	5.87	8.93
		0.20	0.20	0.21	0.33	0.43	0.68	1.61	2.77	4.92	13.55	14.04
	6	-5.08	-5.06	-5.03	-4.96	-4.98	-4.54	-2.89	-0.36	3.00	6.69	9.62
		0.20	0.20	0.27	0.42	0.45	0.79	2.00	2.51	7.35	14.32	12.54
	7	-5.08	-5.06	-5.05	-4.94	-4.96	-4.33	-2.55	0.05	4.04	7.63	10.31
		0.20	0.20	0.24	0.44	0.48	0.86	1.92	2.13	9.76	15.27	11.71
	8	-5.08	-5.06	-5.04	-4.97	-4.99	-4.05	-1.91	0.40	4.71	8.58	10.79
		0.21	0.20	0.27	0.46	0.50	1.28	2.16	2.30	12.41	13.37	9.54
	9	-5.08	-5.06	-5.05	-5.01	-4.99	-3.92	-1.72	0.58	4.93	8.92	10.99
		0.20	0.20	0.25	0.41	0.45	1.36	1.88	1.97	11.85	14.14	8.56
	10	-5.09	-5.07	-5.04	-4.95	-4.89	-3.85	-1.31	0.98	5.80	9.97	11.62
		0.20	0.20	0.27	0.47	0.52	1.46	2.29	2.25	15.06	12.58	6.29

Table 12 **Relative performance of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 2. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.**

Appendix F: Additional Numerical Results on the Robustness Check of Algorithm 1

		ϵ										
		0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
Γ	1	-5.06	-5.06	-5.05	-5.01	-4.98	-4.90	-4.49	-3.84	-2.28	-0.60	0.50
		0.20	0.20	0.20	0.25	0.34	0.43	0.81	1.38	2.26	5.65	8.28
	2	-5.06	-5.06	-5.06	-5.00	-5.02	-4.93	-4.40	-3.24	-0.81	1.73	3.74
		0.20	0.20	0.21	0.30	0.37	0.47	0.88	1.80	3.08	9.00	13.28
	3	-5.07	-5.06	-5.05	-4.98	-4.98	-4.82	-4.03	-2.48	0.35	3.45	5.68
		0.20	0.20	0.20	0.34	0.39	0.57	1.21	2.05	3.11	11.11	14.90
	4	-5.07	-5.06	-5.06	-4.96	-4.94	-4.69	-3.65	-1.53	1.43	5.02	7.71
		0.20	0.20	0.20	0.35	0.43	0.51	1.46	2.79	4.21	14.34	16.15
	5	-5.07	-5.06	-5.06	-5.01	-5.00	-4.60	-3.28	-1.02	1.93	5.87	8.93
		0.20	0.20	0.21	0.33	0.43	0.68	1.61	2.77	4.92	13.55	14.04
	6	-5.08	-5.06	-5.03	-4.96	-4.98	-4.54	-2.89	-0.36	3.00	6.69	9.62
		0.20	0.20	0.27	0.42	0.45	0.79	2.00	2.51	7.35	14.32	12.54
	7	-5.08	-5.06	-5.05	-4.94	-4.96	-4.33	-2.55	0.05	4.04	7.63	10.31
		0.20	0.20	0.24	0.44	0.48	0.86	1.92	2.13	9.76	15.27	11.71
	8	-5.08	-5.06	-5.04	-4.97	-4.99	-4.05	-1.91	0.40	4.71	8.58	10.79
		0.21	0.20	0.27	0.46	0.50	1.28	2.16	2.30	12.41	13.37	9.54
	9	-5.08	-5.06	-5.05	-5.01	-4.99	-3.92	-1.72	0.58	4.93	8.92	10.99
		0.20	0.20	0.25	0.41	0.45	1.36	1.88	1.97	11.85	14.14	8.56
	10	-5.09	-5.07	-5.04	-4.95	-4.89	-3.85	-1.31	0.98	5.80	9.97	11.62
		0.20	0.20	0.27	0.47	0.52	1.46	2.29	2.25	15.06	12.58	6.29

Table 13 Relative performance of dynamic version of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 2. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.

		ϵ										
		0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
Γ	1	-6.65	-6.65	-6.64	-6.59	-6.54	-6.42	-5.88	-5.01	-2.93	-0.66	0.79
		0.15	0.14	0.15	0.21	0.34	0.54	1.17	2.11	3.69	10.06	14.36
	2	-6.65	-6.65	-6.65	-6.57	-6.60	-6.47	-5.77	-4.21	-0.98	2.40	5.12
		0.15	0.15	0.15	0.29	0.40	0.56	1.46	3.04	5.10	15.35	22.84
	3	-6.67	-6.66	-6.65	-6.54	-6.54	-6.30	-5.24	-3.14	0.61	4.74	7.69
		0.15	0.15	0.16	0.41	0.53	0.73	1.90	3.41	5.32	19.36	25.79
	4	-6.67	-6.66	-6.66	-6.51	-6.47	-6.13	-4.72	-1.89	2.02	6.81	10.39
		0.15	0.14	0.14	0.42	0.54	0.63	2.28	4.77	7.10	24.84	28.23
	5	-6.67	-6.65	-6.65	-6.59	-6.58	-6.02	-4.28	-1.23	2.71	7.96	12.04
		0.15	0.14	0.16	0.40	0.53	0.98	2.69	4.51	8.55	23.81	24.17
	6	-6.67	-6.66	-6.61	-6.53	-6.55	-5.93	-3.72	-0.34	4.15	9.08	12.97
		0.15	0.15	0.24	0.46	0.55	1.22	3.15	4.25	12.89	24.91	21.97
	7	-6.68	-6.66	-6.65	-6.48	-6.51	-5.67	-3.27	0.21	5.54	10.32	13.87
		0.15	0.15	0.23	0.57	0.66	1.37	3.18	3.57	17.39	27.06	20.35
	8	-6.68	-6.65	-6.63	-6.55	-6.56	-5.31	-2.43	0.66	6.41	11.58	14.52
		0.15	0.15	0.23	0.46	0.58	1.91	3.77	3.91	21.67	23.50	16.50
	9	-6.69	-6.66	-6.64	-6.58	-6.55	-5.09	-2.14	0.90	6.71	12.03	14.80
		0.15	0.15	0.21	0.45	0.56	2.35	3.37	3.21	20.46	25.11	15.50
	10	-6.69	-6.66	-6.62	-6.48	-6.42	-5.02	-1.60	1.46	7.87	13.43	15.63
		0.15	0.15	0.30	0.54	0.64	2.51	3.86	3.73	26.71	22.12	11.13

Table 14 Relative performance of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 3. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.

		ϵ										
		0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
Γ	1	-4.96	-4.95	-4.95	-4.88	-4.83	-4.71	-4.18	-3.31	-1.14	1.13	2.82
		0.13	0.13	0.13	0.24	0.44	0.73	1.37	2.20	3.50	9.16	12.67
	2	-4.96	-4.96	-4.95	-4.85	-4.89	-4.80	-4.12	-2.42	1.07	4.66	7.55
		0.13	0.13	0.13	0.34	0.50	0.73	1.49	2.91	4.73	12.45	17.83
	3	-4.97	-4.96	-4.94	-4.79	-4.81	-4.61	-3.54	-1.20	2.89	7.30	10.52
		0.13	0.13	0.14	0.52	0.69	0.88	1.94	3.31	4.81	15.22	20.26
	4	-4.97	-4.96	-4.96	-4.75	-4.73	-4.44	-2.94	0.19	4.58	9.44	13.07
		0.13	0.13	0.13	0.54	0.72	0.74	2.19	4.22	5.98	19.44	22.04
	5	-4.97	-4.95	-4.94	-4.86	-4.87	-4.35	-2.40	1.02	5.58	10.84	14.90
		0.13	0.13	0.17	0.51	0.70	1.07	2.59	4.09	6.95	17.54	18.16
	6	-4.98	-4.96	-4.89	-4.77	-4.85	-4.23	-1.77	2.08	6.99	12.01	15.67
		0.13	0.14	0.27	0.58	0.71	1.30	3.00	3.69	10.24	18.65	17.28
	7	-4.98	-4.96	-4.93	-4.71	-4.79	-3.96	-1.19	2.79	8.40	13.09	16.59
		0.13	0.13	0.25	0.75	0.84	1.41	2.97	3.13	13.76	20.98	14.99
	8	-4.98	-4.95	-4.90	-4.80	-4.88	-3.58	-0.27	3.34	9.45	14.45	17.20
		0.13	0.13	0.27	0.65	0.77	1.91	3.40	3.36	16.80	17.48	12.79
	9	-4.98	-4.95	-4.91	-4.84	-4.88	-3.34	0.13	3.77	9.92	15.01	17.54
		0.13	0.13	0.24	0.60	0.70	2.31	2.91	2.84	16.00	18.56	11.38
	10	-4.99	-4.95	-4.88	-4.73	-4.72	-3.23	0.80	4.40	10.99	16.23	18.32
		0.13	0.13	0.36	0.73	0.79	2.43	3.42	3.18	21.19	16.49	7.88

Table 15 Relative performance of dynamic version of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 3. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.

		ϵ										
		0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
Γ	1	-7.93	-7.93	-7.92	-7.86	-7.81	-7.68	-7.07	-6.09	-3.73	-1.18	0.43
		0.11	0.11	0.11	0.24	0.42	0.59	1.38	2.63	4.54	12.42	18.18
	2	-7.94	-7.93	-7.93	-7.84	-7.86	-7.71	-6.94	-5.17	-1.54	2.28	5.31
		0.11	0.11	0.11	0.29	0.43	0.70	1.73	3.75	6.27	19.64	29.27
	3	-7.94	-7.94	-7.92	-7.82	-7.81	-7.54	-6.33	-3.99	0.23	4.88	8.20
		0.11	0.11	0.12	0.42	0.56	0.82	2.52	4.38	6.87	24.83	33.14
	4	-7.95	-7.94	-7.94	-7.77	-7.74	-7.36	-5.77	-2.59	1.81	7.19	11.25
		0.11	0.11	0.11	0.46	0.61	0.70	2.77	5.84	9.15	31.36	35.33
	5	-7.95	-7.93	-7.93	-7.85	-7.84	-7.24	-5.27	-1.84	2.59	8.49	13.08
		0.11	0.11	0.16	0.41	0.60	1.18	3.29	5.63	10.93	29.89	31.12
	6	-7.95	-7.93	-7.88	-7.78	-7.81	-7.10	-4.64	-0.84	4.19	9.73	14.13
		0.11	0.12	0.25	0.52	0.65	1.51	3.95	5.23	16.35	31.17	27.72
	7	-7.96	-7.95	-7.93	-7.74	-7.78	-6.84	-4.14	-0.23	5.77	11.14	15.13
		0.11	0.11	0.19	0.57	0.69	1.55	3.94	4.44	22.06	33.81	25.21
	8	-7.97	-7.94	-7.91	-7.82	-7.86	-6.42	-3.19	0.28	6.73	12.54	15.86
		0.11	0.11	0.23	0.54	0.68	2.35	4.64	4.91	27.18	29.77	21.03
	9	-7.97	-7.94	-7.91	-7.85	-7.82	-6.19	-2.88	0.53	7.06	13.06	16.18
		0.11	0.11	0.25	0.50	0.64	2.89	4.07	3.98	26.43	32.19	19.52
	10	-7.97	-7.94	-7.90	-7.75	-7.67	-6.12	-2.28	1.17	8.40	14.64	17.12
		0.11	0.11	0.31	0.68	0.74	2.99	4.66	4.54	33.60	28.05	14.26

Table 16 Relative performance of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 4. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.

Γ	ϵ = 0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
1	-5.03	-5.02	-5.02	-4.92	-4.85	-4.72	-4.12	-3.15	-0.63	1.96	3.95
	0.11	0.11	0.11	0.35	0.71	1.02	1.77	2.84	4.21	10.77	14.06
2	-5.03	-5.03	-5.03	-4.88	-4.92	-4.81	-4.05	-2.00	2.14	6.34	9.76
	0.10	0.10	0.10	0.45	0.72	1.10	1.85	3.55	5.63	14.12	20.19
3	-5.04	-5.03	-5.02	-4.82	-4.84	-4.62	-3.35	-0.52	4.41	9.43	13.06
	0.11	0.11	0.13	0.69	0.94	1.16	2.60	4.05	5.65	17.47	23.74
4	-5.04	-5.03	-5.03	-4.75	-4.74	-4.42	-2.64	1.19	6.50	11.88	16.01
	0.11	0.11	0.11	0.84	1.10	0.99	2.74	4.94	6.94	21.42	25.08
5	-5.04	-5.02	-5.01	-4.88	-4.91	-4.31	-1.95	2.26	7.71	13.59	18.05
	0.10	0.11	0.20	0.73	1.02	1.39	3.20	4.70	8.07	18.58	20.62
6	-5.05	-5.02	-4.94	-4.76	-4.88	-4.13	-1.11	3.59	9.35	14.91	18.95
	0.11	0.11	0.38	0.91	1.05	1.69	3.73	4.24	11.51	19.77	18.90
7	-5.05	-5.03	-4.99	-4.70	-4.84	-3.84	-0.39	4.55	10.98	16.14	19.98
	0.11	0.11	0.26	1.03	1.15	1.63	3.50	3.81	14.91	22.20	15.91
8	-5.06	-5.03	-4.95	-4.81	-4.95	-3.38	0.73	5.28	12.18	17.64	20.70
	0.10	0.10	0.37	0.98	1.10	2.38	3.82	3.83	17.60	18.48	13.61
9	-5.06	-5.03	-4.97	-4.86	-4.93	-3.06	1.29	5.86	12.85	18.37	21.06
	0.10	0.11	0.37	0.89	1.00	2.76	3.28	3.09	17.12	19.68	12.07
10	-5.06	-5.03	-4.92	-4.72	-4.74	-2.90	2.10	6.57	14.02	19.67	21.94
	0.11	0.11	0.49	1.16	1.17	2.86	3.81	3.28	22.47	17.35	8.27

Table 17 Relative performance of dynamic version of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 4. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.

Γ	ϵ = 0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
1	-6.77	-6.77	-6.75	-6.69	-6.64	-6.50	-5.84	-4.79	-2.28	0.44	2.19
	0.11	0.11	0.11	0.20	0.41	0.66	1.59	2.93	5.10	14.18	20.84
2	-6.77	-6.77	-6.77	-6.68	-6.70	-6.55	-5.71	-3.83	0.05	4.14	7.35
	0.11	0.11	0.11	0.32	0.46	0.71	1.92	4.09	7.07	22.06	33.34
3	-6.79	-6.78	-6.76	-6.64	-6.64	-6.35	-5.07	-2.58	1.95	6.89	10.44
	0.11	0.11	0.12	0.43	0.59	0.92	2.61	4.84	7.70	27.63	36.98
4	-6.79	-6.78	-6.78	-6.60	-6.56	-6.17	-4.47	-1.08	3.64	9.39	13.70
	0.11	0.11	0.11	0.53	0.69	0.84	3.24	6.68	10.34	35.52	39.88
5	-6.79	-6.77	-6.77	-6.68	-6.66	-6.00	-3.91	-0.25	4.47	10.77	15.67
	0.11	0.11	0.16	0.47	0.65	1.26	3.68	6.33	12.55	34.15	35.29
6	-6.80	-6.77	-6.71	-6.61	-6.65	-5.90	-3.26	0.81	6.19	12.10	16.77
	0.11	0.12	0.27	0.57	0.69	1.54	4.39	6.08	18.46	35.39	31.43
7	-6.80	-6.78	-6.76	-6.57	-6.60	-5.59	-2.72	1.47	7.85	13.59	17.86
	0.11	0.11	0.19	0.65	0.77	1.77	4.53	5.12	24.94	38.64	28.55
8	-6.80	-6.77	-6.73	-6.63	-6.65	-5.15	-1.72	2.01	8.89	15.09	18.64
	0.11	0.12	0.28	0.68	0.91	2.75	5.21	5.44	31.13	33.59	23.67
9	-6.81	-6.78	-6.76	-6.68	-6.64	-4.92	-1.41	2.27	9.26	15.66	18.98
	0.11	0.11	0.23	0.61	0.73	3.26	4.74	4.52	29.84	36.22	22.19
10	-6.82	-6.78	-6.73	-6.56	-6.48	-4.81	-0.72	2.96	10.67	17.34	19.97
	0.11	0.11	0.29	0.67	0.82	3.36	5.24	5.02	37.74	31.78	16.00

Table 18 Relative performance of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 5. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.

		ϵ										
		0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
Γ	1	-4.04	-4.03	-4.02	-3.92	-3.82	-3.65	-3.00	-1.94	0.79	3.63	5.95
		0.09	0.09	0.10	0.37	0.91	1.43	2.28	3.24	4.39	11.80	15.56
	2	-4.04	-4.04	-4.03	-3.85	-3.90	-3.79	-2.96	-0.66	3.97	8.57	12.25
		0.09	0.09	0.09	0.62	0.95	1.39	2.17	3.87	6.25	15.08	22.14
	3	-4.05	-4.04	-4.02	-3.77	-3.79	-3.56	-2.16	1.09	6.60	11.97	15.92
		0.09	0.09	0.12	0.95	1.34	1.57	2.80	4.38	5.95	18.39	25.36
	4	-4.05	-4.04	-4.04	-3.70	-3.70	-3.37	-1.33	3.02	8.95	14.70	19.12
		0.09	0.09	0.10	1.16	1.54	1.29	3.12	5.12	7.35	22.28	26.36
	5	-4.05	-4.03	-4.00	-3.84	-3.88	-3.18	-0.48	4.33	10.34	16.55	21.28
		0.09	0.09	0.23	0.98	1.29	1.60	3.51	5.05	8.72	19.35	21.51
	6	-4.06	-4.04	-3.91	-3.71	-3.88	-3.02	0.49	5.89	12.18	18.01	22.22
		0.10	0.10	0.46	1.23	1.37	1.84	3.96	4.72	11.86	20.12	19.57
	7	-4.06	-4.04	-3.99	-3.63	-3.79	-2.68	1.36	6.99	13.88	19.31	23.36
		0.09	0.10	0.33	1.45	1.52	1.86	3.90	4.08	14.99	22.77	16.28
	8	-4.06	-4.03	-3.93	-3.74	-3.91	-2.13	2.61	7.82	15.19	20.90	24.12
		0.09	0.09	0.53	1.49	1.64	2.76	4.02	3.93	17.79	18.54	13.69
	9	-4.07	-4.04	-3.97	-3.82	-3.92	-1.79	3.31	8.52	16.00	21.75	24.54
		0.09	0.10	0.40	1.30	1.32	3.03	3.56	3.19	17.09	19.38	11.87
	10	-4.07	-4.04	-3.89	-3.65	-3.69	-1.54	4.27	9.28	17.18	23.06	25.41
		0.09	0.10	0.56	1.45	1.51	3.16	3.93	3.23	22.17	17.28	8.27

Table 19 Relative performance of dynamic version of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 5. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.

		ϵ										
		0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
Γ	1	-6.85	-6.85	-6.84	-6.79	-6.71	-6.44	-5.44	-3.70	-0.25	3.00	5.44
		0.06	0.06	0.06	0.10	0.16	0.23	0.87	1.92	3.26	10.05	15.07
	2	-6.85	-6.85	-6.83	-6.75	-6.68	-6.17	-4.49	-1.47	3.68	8.16	11.69
		0.06	0.06	0.06	0.12	0.17	0.33	1.14	2.92	4.80	26.38	39.35
	3	-6.85	-6.85	-6.82	-6.70	-6.57	-5.76	-3.27	0.69	5.80	11.21	15.07
		0.06	0.06	0.06	0.15	0.21	0.47	1.71	3.36	9.06	33.31	43.94
	4	-6.85	-6.85	-6.82	-6.68	-6.48	-5.37	-2.15	2.46	7.61	13.90	18.62
		0.06	0.06	0.06	0.19	0.25	0.53	2.08	7.92	12.23	42.72	48.38
	5	-6.85	-6.84	-6.80	-6.68	-6.47	-4.93	-0.88	3.37	8.52	15.42	20.77
		0.06	0.06	0.08	0.19	0.25	0.79	2.67	7.51	14.90	40.64	42.02
	6	-6.85	-6.84	-6.75	-6.62	-6.38	-4.51	0.09	4.52	10.39	16.86	21.98
		0.06	0.06	0.09	0.19	0.27	0.84	5.18	7.21	22.11	42.52	37.39
	7	-6.85	-6.84	-6.77	-6.58	-6.33	-3.97	0.70	5.26	12.26	18.51	23.18
		0.06	0.06	0.09	0.23	0.32	1.16	5.43	6.31	29.97	46.46	34.40
	8	-6.85	-6.84	-6.74	-6.60	-6.30	-3.28	1.80	5.85	13.38	20.15	24.03
		0.06	0.06	0.12	0.22	0.33	1.55	6.29	6.58	37.11	40.21	28.06
	9	-6.85	-6.84	-6.74	-6.60	-6.19	-2.63	2.16	6.16	13.78	20.76	24.39
		0.06	0.06	0.11	0.22	0.30	1.66	5.60	5.47	36.05	43.74	26.85
	10	-6.85	-6.84	-6.72	-6.51	-6.03	-2.13	2.87	6.88	15.31	22.60	25.47
		0.06	0.06	0.12	0.23	0.37	2.01	6.18	5.75	45.69	38.02	19.02

Table 20 Relative performance of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 8. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.

		ϵ										
		0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
Γ	1	-3.31	-3.31	-3.30	-3.23	-3.13	-2.78	-1.72	-0.43	2.72	5.97	8.69
		0.05	0.05	0.05	0.16	0.30	0.45	1.11	4.23	4.93	12.87	17.61
	2	-3.31	-3.31	-3.30	-3.17	-3.11	-2.59	-1.58	1.28	6.77	11.93	16.11
		0.05	0.05	0.05	0.21	0.31	2.27	2.73	4.49	6.93	16.44	24.79
	3	-3.31	-3.31	-3.28	-3.11	-2.99	-2.26	-0.55	3.57	10.05	15.89	20.32
		0.05	0.05	0.06	0.28	0.38	2.43	3.50	5.00	6.54	20.05	27.50
	4	-3.31	-3.30	-3.28	-3.06	-2.89	-2.03	0.53	5.97	12.84	19.03	23.92
		0.05	0.05	0.05	0.37	0.47	1.99	3.64	5.69	8.07	22.89	27.78
	5	-3.31	-3.30	-3.26	-3.11	-2.92	-1.83	1.71	7.72	14.52	21.20	26.31
		0.05	0.05	0.09	0.35	0.39	2.17	4.13	5.46	9.13	19.62	21.93
	6	-3.31	-3.30	-3.19	-3.02	-2.84	-1.58	2.99	9.64	16.67	22.89	27.42
		0.05	0.05	0.15	0.40	0.45	2.43	4.46	5.07	12.08	19.88	19.50
	7	-3.31	-3.30	-3.23	-2.96	-2.75	-1.14	4.19	10.98	18.47	24.33	28.66
		0.05	0.05	0.12	0.46	0.51	2.40	4.44	4.89	14.45	22.32	16.32
	8	-3.31	-3.30	-3.19	-3.02	-2.77	-0.51	5.68	11.97	19.90	26.02	29.51
		0.05	0.05	0.19	0.45	2.48	3.17	4.46	4.06	16.82	17.87	13.28
	9	-3.31	-3.30	-3.20	-3.04	-2.78	-0.02	6.69	12.87	21.00	27.06	29.98
		0.05	0.05	0.16	0.41	2.18	3.51	3.90	3.14	16.34	18.11	11.28
	10	-3.31	-3.29	-3.16	-2.92	-2.47	0.35	7.85	13.60	22.21	28.40	30.88
		0.05	0.05	0.20	0.45	2.49	3.76	4.11	2.97	21.11	16.25	7.71

Table 21 Relative performance of dynamic version of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 8. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.

		ϵ										
		0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
Γ	1	-8.15	-8.15	-8.14	-8.10	-7.98	-7.55	-6.24	-4.29	-0.46	3.18	5.89
		0.03	0.03	0.03	0.05	0.07	0.17	0.87	2.30	4.15	12.47	18.73
	2	-8.15	-8.15	-8.14	-8.05	-7.90	-7.04	-5.16	-1.81	3.95	9.21	13.32
		0.03	0.03	0.03	0.06	0.09	0.37	1.37	3.71	5.99	18.13	29.81
	3	-8.15	-8.15	-8.12	-8.00	-7.76	-6.57	-3.80	0.61	7.24	13.16	17.35
		0.03	0.03	0.03	0.07	0.11	0.51	2.07	4.03	6.28	22.38	48.26
	4	-8.15	-8.14	-8.12	-7.97	-7.64	-6.15	-2.55	3.07	9.54	16.14	21.08
		0.03	0.03	0.03	0.10	0.16	0.64	2.56	4.99	13.20	46.82	53.01
	5	-8.15	-8.14	-8.10	-7.94	-7.55	-5.64	-1.16	4.78	10.52	17.73	23.33
		0.03	0.03	0.04	0.09	0.16	0.88	3.17	5.17	16.37	44.59	46.01
	6	-8.15	-8.14	-8.07	-7.88	-7.41	-5.21	0.09	6.29	12.47	19.22	24.59
		0.03	0.03	0.06	0.11	0.18	1.08	3.58	7.91	24.21	46.69	41.09
	7	-8.14	-8.13	-8.09	-7.85	-7.31	-4.56	1.33	7.08	14.41	20.97	25.86
		0.03	0.03	0.04	0.12	0.24	1.37	3.63	6.70	32.78	51.23	37.91
	8	-8.14	-8.13	-8.06	-7.84	-7.20	-3.78	2.69	7.71	15.58	22.69	26.74
		0.03	0.03	0.05	0.11	0.25	1.92	3.50	7.07	40.78	44.24	31.13
	9	-8.14	-8.13	-8.05	-7.81	-7.06	-3.09	3.64	8.01	16.00	23.32	27.13
		0.03	0.03	0.05	0.11	0.32	2.06	3.04	5.97	39.80	48.00	29.26
	10	-8.14	-8.13	-8.04	-7.75	-6.88	-2.52	4.59	8.77	17.61	25.25	28.27
		0.03	0.03	0.07	0.14	0.41	2.47	6.78	6.29	50.38	41.73	20.87

Table 22 Relative performance of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 12. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.

	ϵ										
	0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
Γ = 1	-4.45	-4.45	-4.44	-4.34	-4.21	-3.78	-2.54	-0.34	3.90	7.47	10.44
	0.04	0.04	0.04	0.20	0.46	0.77	1.51	2.72	5.20	13.69	18.94
2	-4.45	-4.45	-4.44	-4.27	-4.20	-3.49	-1.32	2.47	8.65	14.12	18.62
	0.04	0.04	0.04	0.33	0.48	0.78	1.60	4.82	7.53	16.96	25.69
3	-4.45	-4.45	-4.42	-4.17	-4.04	-2.89	0.38	5.24	12.42	18.48	23.26
	0.04	0.04	0.06	0.50	0.65	0.86	2.36	5.32	6.87	19.98	27.75
4	-4.45	-4.45	-4.41	-4.10	-3.91	-2.39	1.68	8.02	15.54	21.98	27.15
	0.04	0.04	0.04	0.66	0.80	0.89	3.84	5.65	7.99	22.13	27.68
5	-4.45	-4.44	-4.38	-4.18	-3.96	-1.82	3.14	10.15	17.46	24.37	29.68
	0.04	0.04	0.10	0.57	0.62	1.03	4.41	5.64	8.92	19.36	21.90
6	-4.45	-4.44	-4.29	-4.06	-3.89	-1.25	4.72	12.36	19.84	26.24	30.91
	0.04	0.04	0.25	0.68	0.73	1.31	4.68	5.24	11.94	19.19	19.54
7	-4.45	-4.44	-4.34	-3.98	-3.76	-0.48	6.18	13.81	21.69	27.79	32.19
	0.04	0.04	0.18	0.82	0.80	1.52	4.76	4.79	13.36	21.34	16.40
8	-4.45	-4.43	-4.29	-4.06	-3.76	0.39	7.87	14.89	23.16	29.49	33.12
	0.04	0.04	0.26	0.76	0.76	3.54	4.60	3.87	16.01	17.28	13.35
9	-4.44	-4.43	-4.30	-4.09	-3.68	0.99	9.11	15.92	24.49	30.67	33.62
	0.04	0.04	0.22	0.64	0.61	3.61	3.96	3.02	16.00	16.82	10.81
10	-4.44	-4.43	-4.25	-3.93	-3.44	1.51	10.43	16.61	25.73	32.03	34.54
	0.04	0.04	0.33	0.78	0.76	3.99	4.16	2.74	19.82	15.33	7.45

Table 23 Relative performance of dynamic version of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 12. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.

	ϵ										
	0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
Γ = 1	-8.92	-8.91	-8.90	-8.86	-8.74	-8.28	-6.88	-4.46	-0.06	3.78	6.68
	0.03	0.03	0.03	0.04	0.07	0.17	0.96	2.09	3.86	11.42	16.91
2	-8.92	-8.91	-8.90	-8.81	-8.64	-7.71	-5.23	-1.31	4.68	10.14	14.41
	0.03	0.03	0.03	0.06	0.10	0.31	1.35	3.78	6.26	19.36	32.05
3	-8.91	-8.91	-8.88	-8.76	-8.50	-7.04	-3.46	1.21	8.12	14.28	18.78
	0.03	0.03	0.03	0.08	0.13	0.48	1.88	4.39	6.90	24.34	34.80
4	-8.91	-8.91	-8.88	-8.72	-8.37	-6.44	-2.11	3.73	10.87	17.62	22.66
	0.03	0.03	0.03	0.09	0.17	0.60	2.74	5.38	9.19	30.52	37.04
5	-8.91	-8.90	-8.86	-8.69	-8.27	-5.71	-0.63	5.54	11.90	19.26	24.96
	0.03	0.03	0.03	0.10	0.17	0.91	3.42	5.51	17.03	46.52	47.81
6	-8.91	-8.90	-8.83	-8.63	-8.13	-5.05	0.68	7.41	13.92	20.79	26.26
	0.03	0.03	0.05	0.12	0.21	1.11	3.89	5.58	25.13	48.35	42.81
7	-8.91	-8.90	-8.84	-8.60	-8.03	-4.24	1.95	8.40	15.86	22.57	27.54
	0.03	0.03	0.05	0.13	0.26	1.36	3.89	7.09	34.17	52.92	39.14
8	-8.91	-8.89	-8.82	-8.58	-7.91	-3.39	3.35	9.05	17.09	24.31	28.44
	0.03	0.03	0.06	0.12	0.28	2.08	3.76	7.30	42.37	45.77	32.23
9	-8.91	-8.89	-8.81	-8.57	-7.74	-2.65	4.34	9.33	17.50	24.95	28.82
	0.03	0.03	0.05	0.13	0.27	2.19	3.30	6.10	41.44	49.71	30.36
10	-8.91	-8.89	-8.80	-8.49	-7.56	-2.06	5.56	10.15	19.14	26.91	29.99
	0.03	0.03	0.06	0.15	0.32	2.68	3.89	6.52	52.11	43.26	21.63

Table 24 Relative performance of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 15. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.

		ϵ										
		0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
Γ	1	-4.54	-4.54	-4.53	-4.42	-4.28	-3.80	-2.48	-0.15	4.48	8.42	11.52
		0.03	0.03	0.03	0.25	0.61	0.99	1.70	2.98	4.83	13.72	19.46
	2	-4.54	-4.54	-4.52	-4.32	-4.26	-3.52	-1.21	3.25	9.80	15.42	20.08
		0.03	0.03	0.03	0.42	0.65	0.95	1.73	4.05	7.55	16.90	26.01
	3	-4.54	-4.54	-4.51	-4.23	-4.09	-2.88	0.65	6.35	13.84	19.99	24.90
		0.03	0.03	0.05	0.63	0.86	1.08	2.59	5.36	7.06	20.06	28.10
	4	-4.54	-4.53	-4.51	-4.15	-3.95	-2.35	2.36	9.32	17.13	23.64	28.94
		0.03	0.03	0.03	0.84	1.01	1.03	2.97	5.83	8.09	21.65	27.82
	5	-4.54	-4.53	-4.47	-4.24	-4.02	-1.74	4.12	11.64	19.16	26.15	31.53
		0.03	0.03	0.14	0.75	0.81	1.21	4.67	5.65	8.98	19.42	22.19
	6	-4.54	-4.53	-4.38	-4.10	-3.94	-1.13	5.90	13.99	21.66	28.12	32.84
		0.03	0.03	0.30	0.91	0.93	1.47	4.83	5.25	11.91	18.99	19.82
	7	-4.54	-4.53	-4.43	-4.02	-3.80	-0.32	7.46	15.48	23.50	29.71	34.15
		0.03	0.03	0.21	1.05	1.00	1.58	4.92	4.67	13.07	20.99	16.59
	8	-4.54	-4.53	-4.37	-4.10	-3.81	0.64	9.26	16.58	24.98	31.42	35.10
		0.03	0.03	0.35	0.98	0.99	2.25	4.66	3.79	15.95	17.18	13.61
	9	-4.54	-4.52	-4.38	-4.13	-3.73	1.58	10.61	17.64	26.43	32.66	35.62
		0.03	0.03	0.29	0.88	0.78	2.53	4.05	2.92	16.29	16.61	10.81
	10	-4.54	-4.52	-4.31	-3.97	-3.47	2.33	12.00	18.29	27.67	34.01	36.56
		0.03	0.03	0.46	0.99	0.95	4.14	4.16	2.78	19.62	15.23	7.52

Table 25 Relative performance of dynamic version of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 15. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.

		ϵ										
		0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
Γ	1	-9.44	-9.44	-9.43	-9.39	-9.26	-8.77	-7.27	-4.70	-0.04	4.05	7.11
		0.03	0.03	0.03	0.05	0.08	0.20	1.10	2.35	4.36	12.86	19.37
	2	-9.44	-9.44	-9.43	-9.34	-9.17	-8.17	-5.53	-1.19	5.49	11.21	15.66
		0.03	0.03	0.03	0.06	0.11	0.36	1.51	3.88	6.45	17.31	28.97
	3	-9.44	-9.44	-9.41	-9.28	-9.00	-7.47	-3.68	1.97	9.21	15.59	20.29
		0.03	0.03	0.03	0.09	0.15	0.55	2.25	4.52	7.45	26.09	37.72
	4	-9.44	-9.43	-9.40	-9.24	-8.86	-6.81	-2.00	4.68	12.06	19.07	24.29
		0.03	0.03	0.03	0.11	0.19	0.68	2.62	5.77	9.70	32.85	40.03
	5	-9.44	-9.43	-9.39	-9.21	-8.76	-6.03	-0.11	6.55	13.56	21.03	26.84
		0.03	0.03	0.04	0.10	0.18	1.02	3.68	6.08	17.51	47.90	49.42
	6	-9.44	-9.43	-9.35	-9.14	-8.60	-5.31	1.49	8.48	15.60	22.60	28.16
		0.03	0.03	0.05	0.13	0.23	1.21	4.20	5.94	25.85	50.05	44.20
	7	-9.44	-9.43	-9.37	-9.10	-8.50	-4.48	2.81	9.82	17.58	24.39	29.46
		0.03	0.03	0.05	0.14	0.30	1.57	4.22	5.49	35.29	54.85	40.73
	8	-9.44	-9.42	-9.34	-9.09	-8.38	-3.52	4.25	10.64	18.82	26.16	30.37
		0.03	0.03	0.06	0.13	0.30	1.99	4.02	7.60	43.98	47.71	33.45
	9	-9.43	-9.42	-9.34	-9.07	-8.19	-2.50	5.32	10.98	19.24	26.82	30.77
		0.03	0.03	0.05	0.13	0.28	2.03	3.51	6.37	42.72	51.51	31.41
	10	-9.43	-9.42	-9.32	-8.99	-8.00	-1.71	6.56	11.77	20.91	28.83	31.96
		0.03	0.03	0.06	0.16	0.36	2.56	4.17	6.77	53.87	44.85	22.36

Table 26 Relative performance of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 20. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.

		ϵ										
		0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
Γ	1	-4.90	-4.90	-4.90	-4.82	-4.66	-4.05	-2.40	0.10	5.00	9.29	12.63
		0.02	0.02	0.02	0.11	0.26	0.46	1.29	3.28	5.35	14.25	19.95
	2	-4.90	-4.90	-4.88	-4.73	-4.57	-3.46	-0.97	3.80	11.09	16.87	21.65
		0.02	0.02	0.02	0.16	0.27	0.53	1.96	4.35	7.74	16.79	26.39
	3	-4.90	-4.89	-4.86	-4.64	-4.39	-2.78	1.01	7.60	15.44	21.68	26.73
		0.02	0.02	0.03	0.26	0.34	1.29	2.85	5.72	7.43	19.81	28.76
	4	-4.90	-4.89	-4.86	-4.58	-4.21	-2.21	2.92	10.87	18.95	25.54	30.96
		0.02	0.02	0.02	0.34	0.42	1.25	3.25	5.89	8.09	21.10	28.33
	5	-4.90	-4.88	-4.82	-4.60	-4.16	-1.57	5.16	13.39	21.09	28.18	33.65
		0.02	0.02	0.05	0.28	0.33	1.32	4.96	5.77	9.15	19.67	22.85
	6	-4.89	-4.88	-4.76	-4.48	-4.01	-0.88	7.25	15.85	23.72	30.27	35.02
		0.02	0.02	0.13	0.38	0.43	1.64	5.04	5.39	12.16	19.45	20.46
	7	-4.89	-4.88	-4.80	-4.42	-3.86	-0.03	8.99	17.39	25.58	31.89	36.37
		0.02	0.02	0.10	0.43	0.50	1.85	5.38	4.77	13.07	21.13	17.41
	8	-4.89	-4.87	-4.75	-4.45	-3.78	0.99	10.87	18.49	27.06	33.62	37.38
		0.02	0.02	0.14	0.39	1.22	2.44	4.84	3.91	16.29	17.55	14.17
	9	-4.89	-4.87	-4.75	-4.43	-3.70	2.02	12.37	19.63	28.66	34.93	37.92
		0.02	0.02	0.11	0.34	1.03	2.77	4.05	3.02	16.73	16.76	11.18
	10	-4.89	-4.87	-4.72	-4.31	-3.41	2.96	13.83	20.24	29.90	36.30	38.87
		0.02	0.02	0.18	0.43	1.27	3.52	4.22	2.78	19.57	15.43	7.65

Table 27 **Relative performance of dynamic version of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 20. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.**

		ϵ										
		0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
Γ	1	-9.40	-9.40	-9.38	-9.34	-9.18	-8.55	-6.83	-4.04	0.80	5.03	8.22
		0.02	0.02	0.02	0.02	0.05	0.17	1.12	2.48	4.66	13.67	20.55
	2	-9.40	-9.39	-9.38	-9.28	-9.04	-7.73	-4.89	-0.40	6.52	12.47	17.09
		0.02	0.02	0.02	0.04	0.10	0.33	1.59	4.09	6.78	18.69	31.27
	3	-9.40	-9.39	-9.36	-9.22	-8.85	-6.90	-2.99	2.87	10.66	17.20	22.01
		0.02	0.02	0.02	0.05	0.10	0.56	2.33	4.66	7.77	27.18	39.64
	4	-9.40	-9.39	-9.35	-9.18	-8.67	-6.22	-1.24	6.01	13.62	20.77	26.12
		0.02	0.02	0.02	0.07	0.15	0.73	2.86	5.48	10.28	34.02	41.59
	5	-9.39	-9.39	-9.34	-9.12	-8.51	-5.43	0.70	7.95	15.32	22.93	28.77
		0.02	0.02	0.02	0.08	0.18	1.06	3.90	6.19	12.21	49.12	34.21
	6	-9.40	-9.38	-9.31	-9.04	-8.29	-4.69	2.40	9.93	17.42	24.50	30.12
		0.02	0.02	0.03	0.08	0.18	1.31	4.03	6.16	26.77	51.45	45.43
	7	-9.39	-9.38	-9.32	-9.00	-8.15	-3.81	4.04	11.31	19.45	26.33	31.45
		0.02	0.02	0.03	0.10	0.27	1.64	4.02	5.81	36.00	55.95	41.45
	8	-9.39	-9.38	-9.29	-8.96	-7.97	-2.81	5.61	12.37	20.69	28.12	32.37
		0.02	0.02	0.04	0.10	0.27	2.16	4.31	5.18	44.63	48.48	34.03
	9	-9.39	-9.38	-9.29	-8.93	-7.75	-1.76	6.70	12.76	21.13	28.78	32.76
		0.02	0.02	0.03	0.10	0.29	2.18	3.68	6.46	43.62	52.50	31.98
	10	-9.39	-9.37	-9.26	-8.84	-7.53	-0.93	7.99	13.57	22.81	30.81	33.97
		0.02	0.02	0.05	0.13	0.38	2.79	4.34	6.86	54.91	45.72	22.82

Table 28 **Relative performance of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 25. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.**

		ϵ										
		0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
Γ	1	-5.34	-5.34	-5.33	-5.24	-5.08	-4.43	-2.72	0.18	5.27	9.75	13.20
		0.02	0.02	0.02	0.12	0.31	0.58	1.39	3.38	5.63	14.77	20.84
	2	-5.34	-5.33	-5.32	-5.15	-4.99	-3.84	-0.91	4.11	11.81	17.75	22.62
		0.02	0.02	0.02	0.21	0.35	0.58	2.02	4.68	8.01	17.06	27.10
	3	-5.34	-5.33	-5.30	-5.06	-4.80	-3.02	1.20	8.21	16.43	22.74	27.91
		0.02	0.02	0.03	0.33	0.42	0.72	3.02	5.91	7.70	19.84	29.27
	4	-5.33	-5.33	-5.30	-4.98	-4.60	-2.26	3.26	11.80	20.09	26.73	32.24
		0.02	0.02	0.02	0.44	0.54	0.85	3.54	6.10	8.43	21.26	28.96
	5	-5.33	-5.32	-5.26	-5.02	-4.57	-1.54	5.64	14.49	22.32	29.49	35.03
		0.02	0.02	0.06	0.37	0.44	1.53	5.42	5.82	9.44	20.22	23.59
	6	-5.33	-5.32	-5.19	-4.89	-4.41	-0.81	7.98	17.05	25.05	31.66	36.45
		0.02	0.02	0.16	0.45	0.49	1.84	5.36	5.62	12.62	20.19	21.29
	7	-5.33	-5.31	-5.23	-4.81	-4.24	0.08	9.96	18.61	26.92	33.33	37.84
		0.02	0.02	0.12	0.56	0.58	1.91	5.52	4.93	13.24	21.38	18.09
	8	-5.33	-5.31	-5.18	-4.85	-4.15	1.18	11.92	19.72	28.41	35.07	38.86
		0.02	0.02	0.17	0.51	0.53	2.65	5.12	4.05	16.75	17.96	14.71
	9	-5.33	-5.31	-5.18	-4.84	-3.99	2.26	13.49	20.89	30.09	36.42	39.43
		0.02	0.02	0.15	0.45	0.45	2.93	4.21	3.17	17.44	17.09	11.52
	10	-5.32	-5.30	-5.13	-4.70	-3.72	3.28	14.99	21.47	31.36	37.81	40.40
		0.02	0.02	0.23	0.55	0.62	3.77	4.25	2.88	19.88	15.75	7.88

Table 29 Relative performance of dynamic version of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 25. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.

		ϵ										
		0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
Γ	1	-9.53	-9.53	-9.52	-9.47	-9.31	-8.66	-6.88	-3.84	1.30	5.61	8.86
		0.02	0.02	0.02	0.03	0.05	0.17	1.26	2.62	4.88	14.35	21.47
	2	-9.53	-9.53	-9.51	-9.41	-9.16	-7.81	-4.62	0.06	7.14	13.21	17.93
		0.02	0.02	0.02	0.04	0.08	0.36	1.76	4.36	7.19	19.70	32.91
	3	-9.53	-9.52	-9.49	-9.35	-8.96	-6.93	-2.55	3.43	11.50	18.14	23.10
		0.02	0.02	0.02	0.06	0.12	0.60	2.41	4.88	7.46	23.47	40.67
	4	-9.53	-9.52	-9.48	-9.30	-8.78	-6.11	-0.78	6.63	14.59	21.86	27.28
		0.02	0.02	0.02	0.07	0.15	0.74	2.99	5.74	10.50	35.16	42.97
	5	-9.53	-9.52	-9.47	-9.24	-8.61	-5.14	1.21	8.85	16.33	24.15	29.97
		0.02	0.02	0.02	0.08	0.19	1.18	4.08	6.45	12.54	49.51	35.20
	6	-9.53	-9.52	-9.44	-9.16	-8.39	-4.31	2.93	10.85	18.58	25.72	31.38
		0.02	0.02	0.03	0.09	0.19	1.38	4.24	6.33	27.01	51.91	45.76
	7	-9.53	-9.52	-9.45	-9.12	-8.26	-3.41	4.62	12.25	20.63	27.57	32.71
		0.02	0.02	0.04	0.11	0.29	1.72	4.22	6.09	36.52	56.86	42.04
	8	-9.53	-9.51	-9.42	-9.08	-8.07	-2.36	6.27	13.33	21.88	29.36	33.64
		0.02	0.02	0.04	0.11	0.29	2.22	3.96	5.41	45.66	49.30	34.58
	9	-9.52	-9.51	-9.41	-9.04	-7.82	-1.31	7.57	13.88	22.32	30.05	34.05
		0.02	0.02	0.04	0.11	0.30	2.37	3.76	6.51	44.22	53.07	32.47
	10	-9.52	-9.51	-9.40	-8.95	-7.60	-0.45	8.88	14.70	24.01	32.07	35.26
		0.02	0.02	0.04	0.12	0.35	2.83	4.44	6.90	55.61	46.37	23.25

Table 30 Relative performance of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 30. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.

		ϵ										
		0.0001	0.0002	0.0005	0.001	0.002	0.005	0.01	0.02	0.05	0.1	0.2
Γ	1	-5.45	-5.45	-5.44	-5.34	-5.17	-4.51	-2.75	0.25	5.61	10.22	13.73
		0.01	0.01	0.01	0.15	0.37	0.62	1.48	2.90	5.91	15.42	21.46
	2	-5.45	-5.44	-5.43	-5.25	-5.08	-3.90	-0.70	4.47	12.40	18.49	23.41
		0.01	0.01	0.01	0.24	0.39	0.66	1.93	4.78	8.30	17.23	27.67
	3	-5.44	-5.44	-5.41	-5.15	-4.88	-3.07	1.51	8.82	17.26	23.64	28.86
		0.01	0.01	0.03	0.39	0.52	0.81	3.18	6.39	7.92	19.99	29.93
	4	-5.44	-5.43	-5.41	-5.06	-4.67	-2.29	3.68	12.61	21.06	27.75	33.34
		0.01	0.01	0.02	0.51	0.61	0.90	3.71	6.54	8.55	21.44	29.72
	5	-5.44	-5.43	-5.37	-5.10	-4.64	-1.43	6.21	15.43	23.34	30.59	36.18
		0.01	0.01	0.07	0.47	0.50	1.23	5.91	6.00	9.74	20.85	24.40
	6	-5.44	-5.43	-5.29	-4.97	-4.49	-0.59	8.64	18.06	26.17	32.84	37.64
		0.01	0.01	0.18	0.56	0.58	1.89	5.55	5.75	13.00	20.79	21.90
	7	-5.43	-5.42	-5.32	-4.87	-4.31	0.34	10.72	19.63	28.04	34.54	39.06
		0.01	0.01	0.14	0.66	0.67	2.05	5.98	5.02	13.56	22.02	18.89
	8	-5.44	-5.42	-5.28	-4.93	-4.22	1.50	12.73	20.75	29.54	36.29	40.11
		0.01	0.02	0.21	0.57	0.57	2.73	5.31	4.22	17.35	18.58	15.46
	9	-5.43	-5.41	-5.28	-4.91	-4.05	2.64	14.45	21.94	31.31	37.67	40.69
		0.01	0.01	0.18	0.53	0.51	3.16	4.34	3.21	18.10	17.54	12.19
	10	-5.43	-5.41	-5.22	-4.77	-3.78	3.71	15.98	22.49	32.58	39.07	41.68
		0.01	0.01	0.28	0.63	0.65	3.99	4.19	2.91	20.46	16.12	8.15

Table 31 Relative performance of dynamic version of Algorithm 1 comparing with that of Algorithm 2 under different mis-specification of the true donation distributions when the total period is 30. For each $\Gamma - \epsilon$ pair, the first entry represents the difference of mean of regret ratio while the second entry represents variance of the regret ratio.