

NUFFT and Compressed Sensing

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Outline

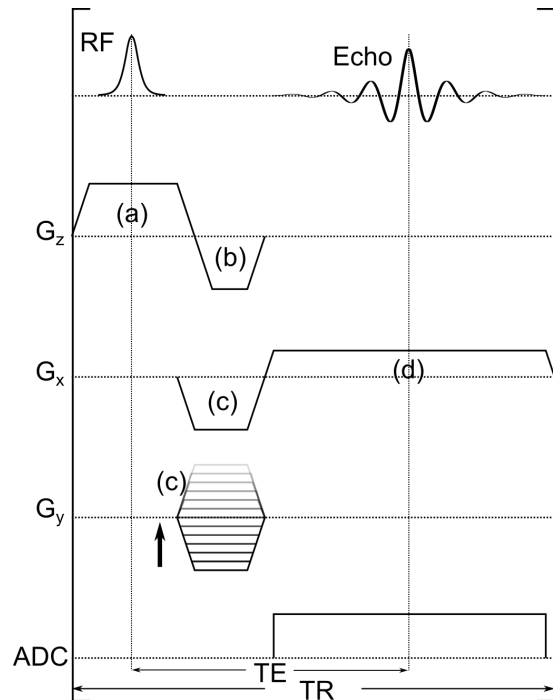
Non-Uniform FFT (NUFFT)

Compressed Sensing (CS)

1 Non-Uniform FFT (NUFFT)

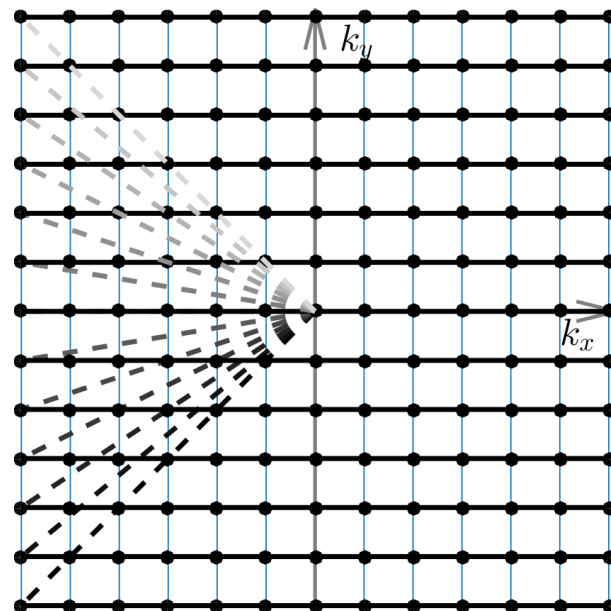


Introduction: Cartesian Sampling (e.g. FLASH) ¹



Gradients:

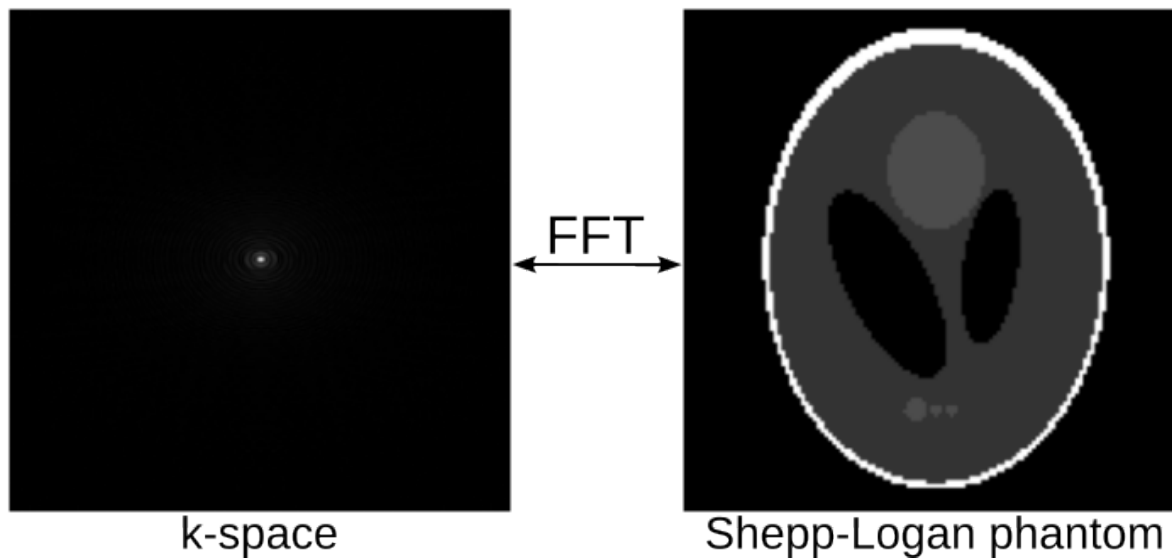
- (a) Slice selection
- (b) Rewinder
- (c) Prephasing
- (d) Readout



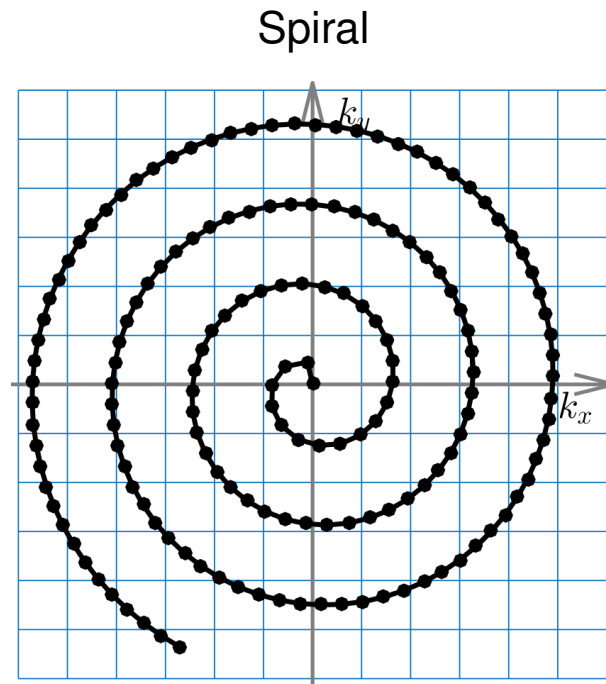
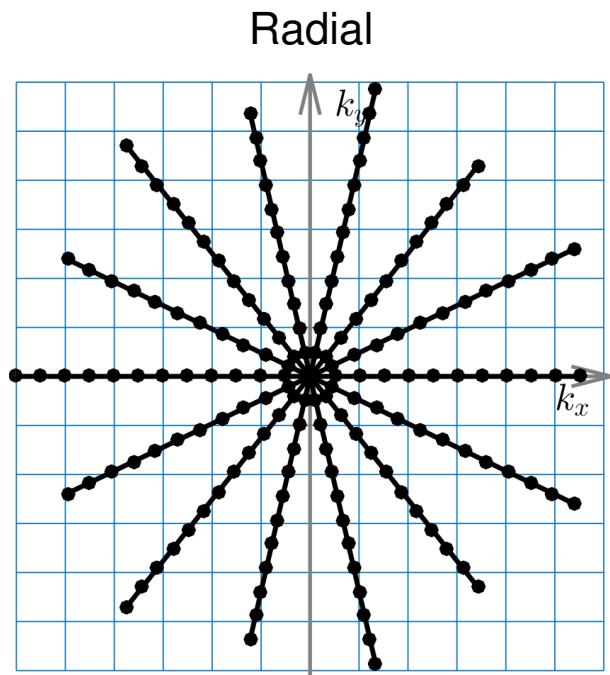
¹Haase A, Frahm J, Matthaei D, Hancike W, Merboldt KD. FLASH imaging. Rapid NMR imaging using low flip-angle pulses. *J Magn Reson* (1986).

Introduction: Fast Fourier Transform

$$s(k_x, k_y) = \int \rho(x, y) \cdot e^{-i(k_x \cdot x + k_y \cdot y)} dx dy \quad (1)$$



Non-Cartesian Sampling (e.g. Radial² and Spiral³)



²Lauterbur PC. Image formation by induced local interactions: Examples employing nuclear magnetic resonance. *Nature* (1973).

³Nishimura DG, Irarrazabal P, Meyer C. A velocity k-space analysis of flow effects in echo-planar and spiral imaging. *Magn Reson Med* (1995).

Image Reconstruction of Non-Cartesian Data Requires NUFFT

NUFFT: Non-Uniform FFT ⁴ , ⁵ , ⁶

1. Density compensation
2. Gridding: Convolution with the Kaiser-Bessel window
3. Deapodization & Inverse FFT

⁴OSullivan J. A fast sinc function gridding algorithm for Fourier inversion in computed tomography. *IEEE Trans Med Imaging* (1985).

⁵Jackson J, Meyer CH, Nishimura DG, Macovski A. Selection of a convolution function for Fourier inversion using gridding. *IEEE Trans Med Imaging* (1991).

⁶Fessler JA, Sutton BP. Nonuniform fast Fourier transforms using min-max interpolation. *IEEE Trans Med Imaging* (2003).

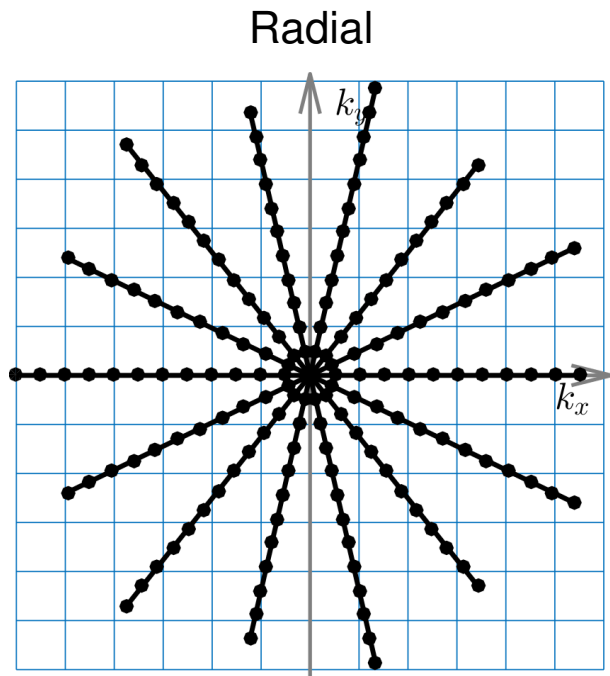
NUFFT Step 1: Density Compensation

- ▶ Non-Cartesian samples are usually not uniformly acquired in k -space.
- ▶ e.g. In radial sampling, the central k -space points are more densely acquired than peripheral k -space points.
- ▶ Analytical density compensation function (DCF) in radial sampling.

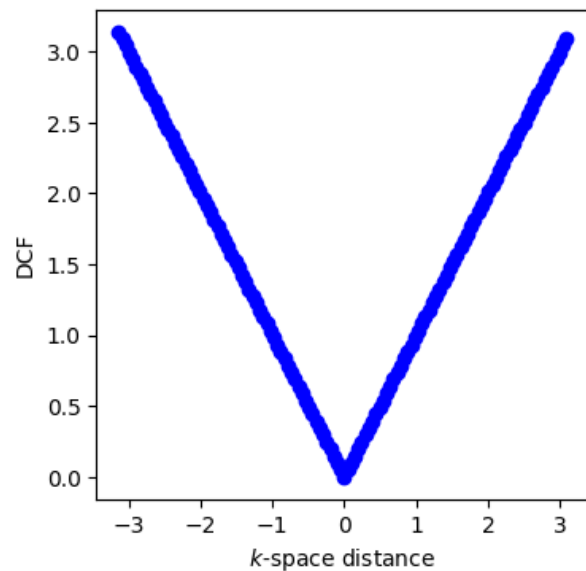
Given $k_{trj} = k_x + 1i * k_y$,

$$DCF = \sqrt{k_x^2 + k_y^2} \quad (2)$$

DCF in Radial Sampling

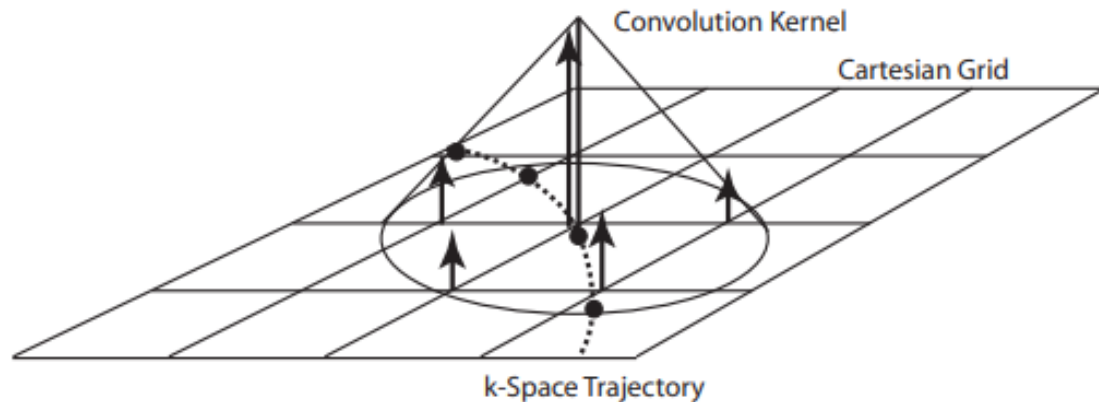


DCF



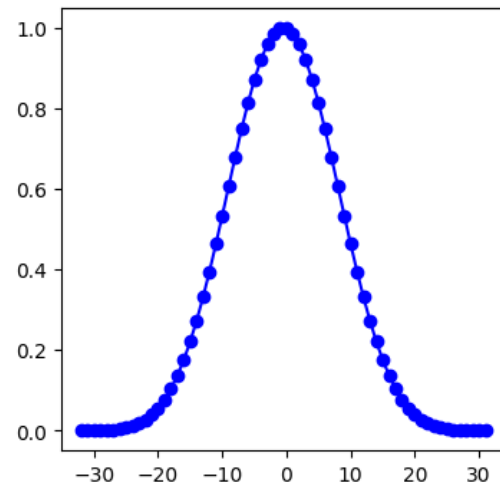
NUFFT Step 2: Gridding (Interpolation) with the Kaiser-Bessel Window

Interpolation ^a



^ahttps://users.fmrib.ox.ac.uk/~karla/reading_group/lecture_notes/AdvRecon_Paulyn_read.pdf

Kaiser-Bessel Window



NUFFT Step 3: Deapodization

- ▶ The gridded signal on Cartesian grids can be written as the convolution between the non-Cartesian signal and the Kaiser-Bessel window,

$$s_{\text{Cart}} = s_{\text{non-Cart}} \circledast W_{\text{kb}} \quad (3)$$

- ▶ with FFT, it becomes

$$\mathcal{F}\{s_{\text{Cart}}\} = \mathcal{F}\{s_{\text{non-Cart}}\} \cdot \mathcal{F}\{W_{\text{kb}}\} \quad (4)$$

- ▶ Then deapodization is

$$I = \mathcal{F}\{s_{\text{Cart}}\} ./ \mathcal{F}\{W_{\text{kb}}\} \quad (5)$$

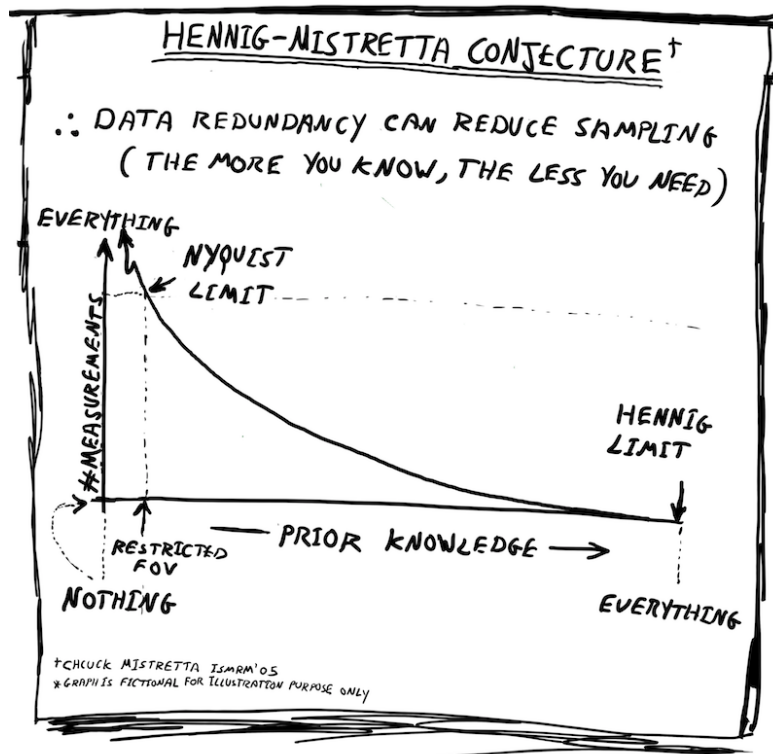
NUFFT Recap

- ▶ Any questions with regard to NUFFT?
- ▶ Any question with regard to NUFFT exercises?
- ▶ What if the density compensation function is unknown?

2 Compressed Sensing (CS)



Compressed Sensing



- ▶ Candes EJ, Romberg JK, Tao T. Stable signal recovery from incomplete and inaccurate measurements. *Commun Pure Appl Math* (2006).
- ▶ Lustig M, Dohono D, Pauly JM. Sparse MRI: The application of compressed sensing for rapid MR imaging. *Magn Reson Med* (2007).
- ▶ Block KT, Uecker M, Frahm J. Undersampled radial MRI with multiple coils. Iterative image reconstruction using a total variation constraint. *Magn Reson Med* (2007).
- ▶ Comics from <http://people.eecs.berkeley.edu/~mlustig/comics1.html>

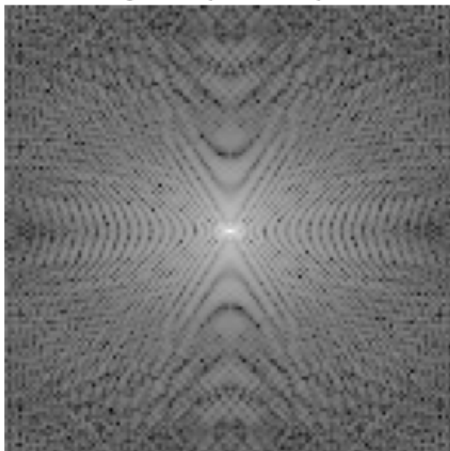
What is Nyquist Limit?

Nyquist Sampling Requirement

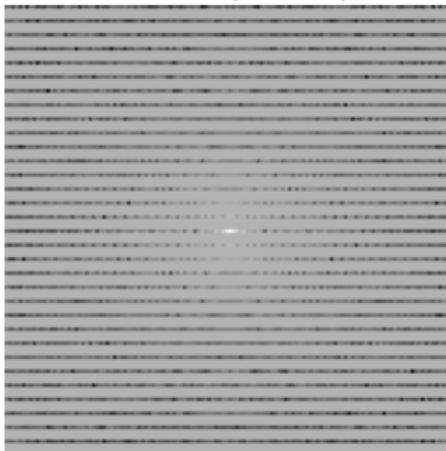
Given an imaging field-of-view (FOV), the sampling in k -space must satisfy

$$\Delta k \leq 1/\text{FOV} \quad (6)$$

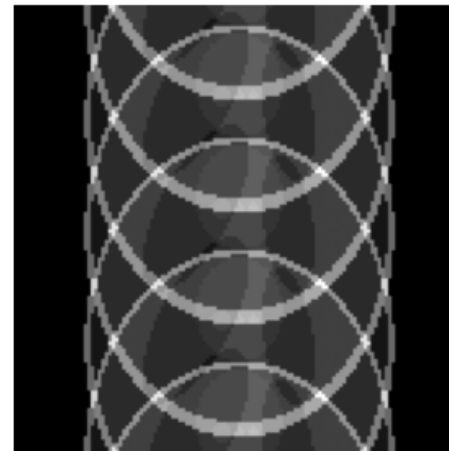
Fully sampled k -space



4X undersampled k -space



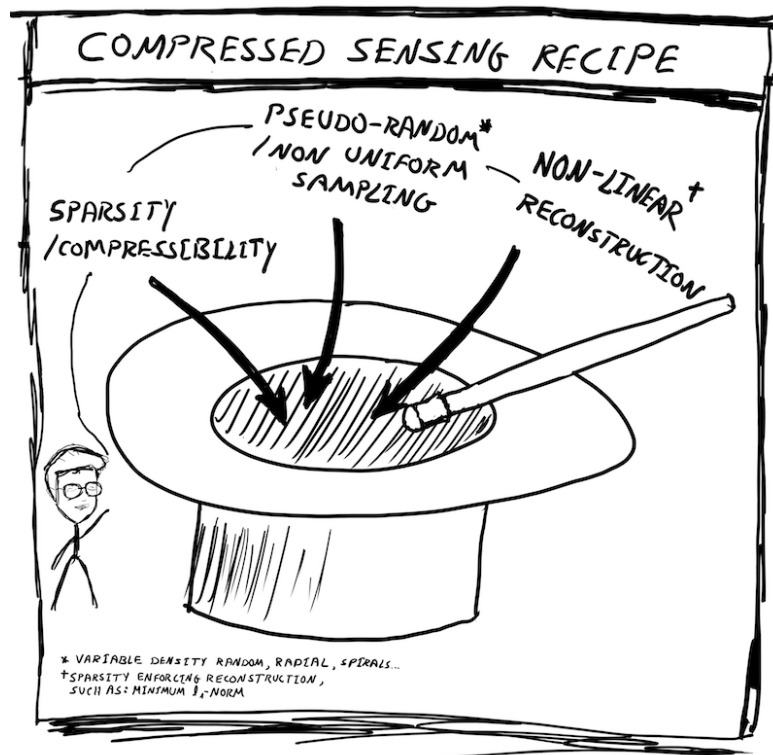
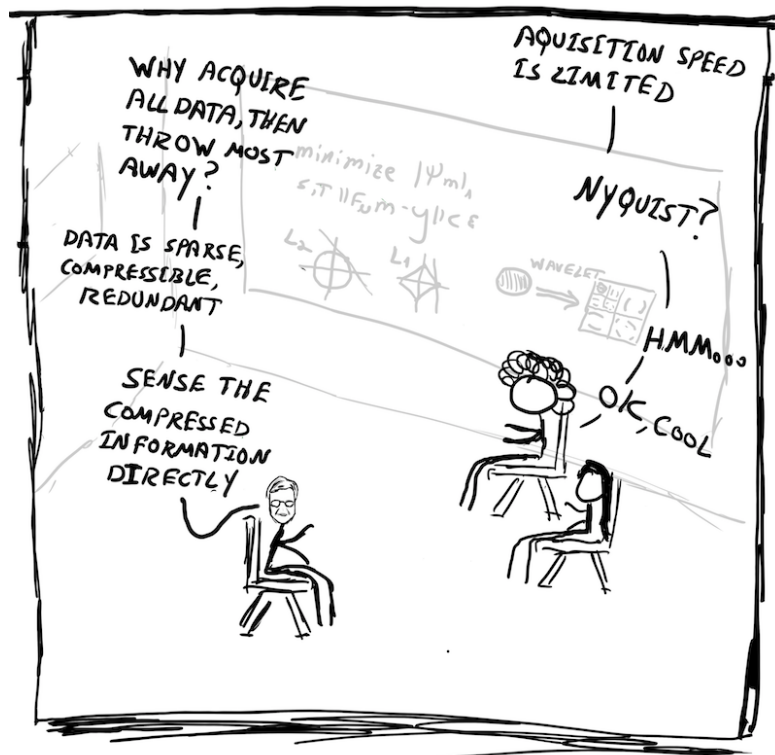
iFFT



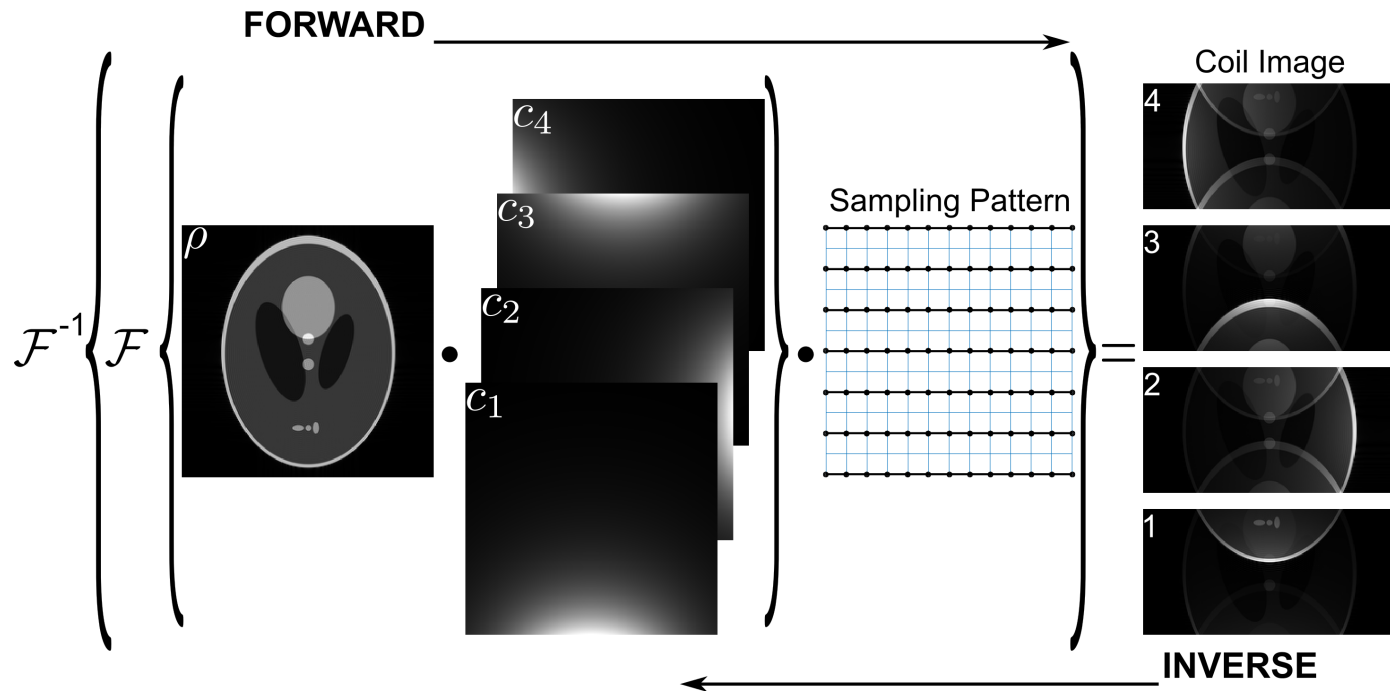
How Compressed Sensing Goes Beyond Nyquist?



How Compressed Sensing Goes Beyond Nyquist?



Before Compressed Sensing: Parallel Imaging as SENSE⁷



⁷Pruessmann KP, Weiger M, Scheidegger MB, Boesiger P. SENSE: sensitivity encoding for fast MRI. *Magn Reson Med* (1999).

Parallel Imaging as SENSE: Solving a Linear Inverse Problem

Objective function:

$$\operatorname{argmin}_x \|y - \mathcal{P}\mathcal{F}\mathcal{S}x\|_2^2 \quad (7)$$

- ▶ Solver: gradient descent, or conjugate gradient, or ADAM, ...
- ▶ Data update via gradient method, e.g. FISTA ⁸:

$$x^{(t+1)} = x^{(t)} + \alpha \cdot \mathcal{S}^H \mathcal{F}^{-1} \mathcal{P}^H (y - \mathcal{P}\mathcal{F}\mathcal{S}x^{(t)})$$

⁸Beck A, Teboulle M. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM J Imaging Sciences* (2009).

What is Linear Inverse Problem and What is Gradient?

Objective function:

$$\operatorname{argmin}_x \|2 - x\|_2^2 \quad (8)$$

Given initial guess $x_{\text{prev}} = 0$ and learning rate $\alpha = 0.1$,

Iteration	x_{prev}	$\text{grad} = 2 * (2 - x)$	$x_{\text{curr}} = x_{\text{prev}} + \alpha * \text{grad}$
0	0	4	0.4
1	0.4	3.2	0.72
\vdots	\vdots	\vdots	\vdots
47	1.9999	0.0001	2

What is Linear Inverse Problem and What is Gradient?

```
x_prev = 0

for n in range(50):

    g = 2 * (2 - x_prev)

    x_curr = x_prev + 0.1 * g

    print(' iter %2d x_prev %6.4f grad %6.4f x_curr %6.4f'%(n, x_prev, g, x_curr))

    x_prev = x_curr
```

Inverse Problems can Become Under-determined / Ill-posed

Objective function:

$$\operatorname{argmin}_{x_1, x_2} \|2 - x_1 - x_2\|_2^2 \quad (9)$$

- ▶ You can't find an unique solution for this function.

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$$\text{s.t. } x_1 = x_2 \quad (10)$$

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$$\text{s.t. } x_1 = x_2 \quad (10)$$

- ▶ We can easily solve it, $x_1 = x_2 = 1$.

We can write this toy example in a matrix format

Objective function:

$$\operatorname{argmin}_{x_1, x_2} \|2 - x_1 - x_2\|_2^2 \quad (11)$$

It is equivalent to

$$\operatorname{argmin}_{\mathbf{x}} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 \quad (12)$$

$$\text{where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \text{ and } \mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

Does it look similar to the inverse problem in MRI?

$$\operatorname{argmin}_x \|y - \mathcal{PFS}x\|_2^2 \quad (13)$$

Prior is the Game Changer in Compressed Sensing

- ▶ Prof. Dr. Gary Golver from Stanford:
"We eventually only need to measure one point in k -space!!!"

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Prior is the Game Changer in Compressed Sensing

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- ▶ People like me:
"What is going on here??? Is there anyone who would like to share their codes???"
- ▶ Prof. Dr. Mike Lustig from Berkeley:
"Sparsifying transformation changes the game!!!"

$$\operatorname{argmin}_x \|y - \mathcal{PFS}x\|_2^2 + \lambda \|\Phi x\|_1 \quad (14)$$

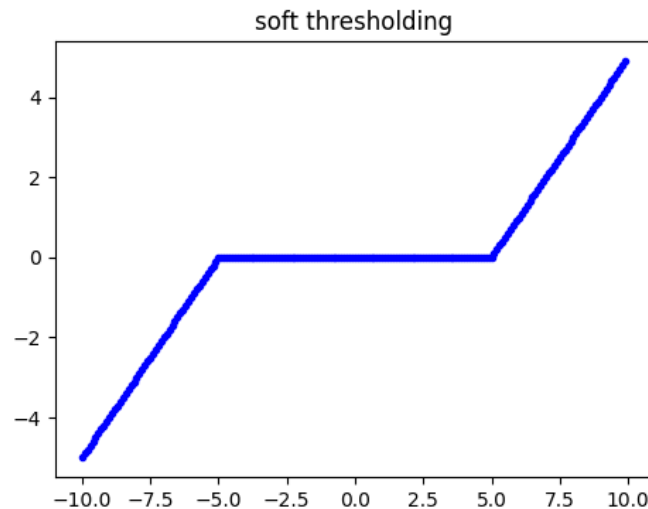
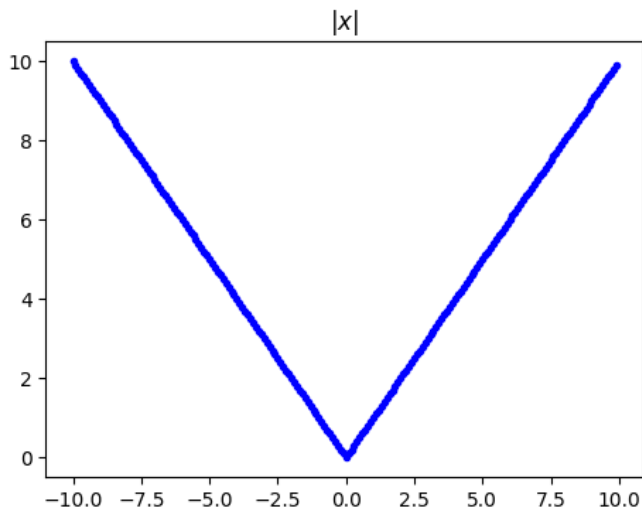
where Φ is wavelet transform, total variation, ...

How to Solve this Problem?

Objective function:

$$\operatorname{argmin}_x \|y - \mathcal{PFS}x\|_2^2 + \lambda \|\Phi x\|_1 \quad (15)$$

- ▶ the regularization term $\lambda \|\Phi x\|_1$ is not differentiable;
- ▶ soft thresholding $\mathcal{T}_\lambda(x) = (x - \lambda)_+ \operatorname{sign}(x)$



Compressed Sensing: Iterative Soft Thresholding

1. Gradient update:

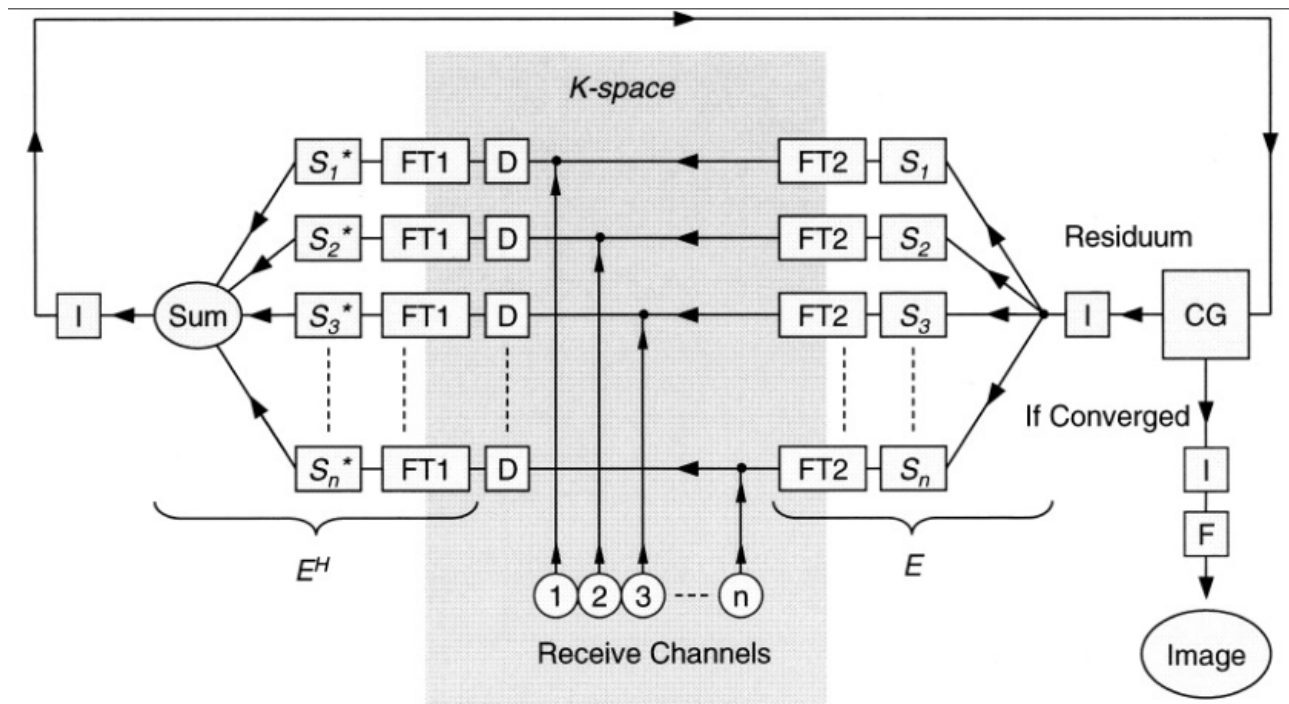
$$x^{(t+1)} = x^{(t)} + \alpha \cdot \mathcal{S}^H \mathcal{F}^{-1} \mathcal{P}^H (y - \mathcal{P} \mathcal{F} \mathcal{S} x^{(t)})$$

2. Soft thresholding:

$$x^{(t+1)} = \Phi^{-1} \mathcal{T}_\lambda (\Phi x^{(t+1)})$$

-
- ▶ This suffices when Φ is wavelet transform.
 - ▶ When you use total variation, you may need a more general solver such as ADAM.

What is Gradient Update in Parallel MRI? ⁹



⁹Pruessmann KP, Weiger M, Börnert P, Boesiger P. Advances in sensitivity encoding with arbitrary k -space trajectories. *Magn Reson Med* (2001).

Gradient Update in Parallel MRI: CG SENSE

- ▶ density correction D :

$D = 1/d\mathbf{k}$, \mathbf{k} is the relative density of the k -space sampling pattern at the position \vec{k} .

- ▶ this changes the system equation:

$$(E^H D E) \mathbf{v} = E^H D \mathbf{m}$$

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- ▶ Exercise 1: run the gradient step many times without soft thresholding, are you able to get a similar reconstruction as the NUFFT reconstruction with density compensation?

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- ▶ Exercise 1: run the gradient step many times without soft thresholding, are you able to get a similar reconstruction as the NUFFT reconstruction with density compensation?
- ▶ Exercise 2: try to tune the soft thresholding parameter, and show how it affects the reconstruction

It is not the end ...

- ▶ Prof. Dr. Florian Knoll ¹⁰:
"Instead of hand-crafted sparsifying transform, neural network learns it!!!"

¹⁰Hammernik K, Klatzer T, Kobler E, Recht MP, Sodickson DK, Pock T, Knoll F. Learning a variational network for reconstruction of accelerated MRI data. *Magn Reson Med* (2018).