STOR 614, Spring 2019
Homework Assignment No. 2

1. Suppose that the polyhedron $P = \{x \in \mathbb{R}^n \mid a_i^T x \geq b_i, i = 1, \cdots, m\}$ is nonempty, and that the set $\{a_1, \cdots, a_m\}$ contains n linearly independent vectors. Prove that P does not contain a straight line in it. (By definition, any straight line in $\mathbb{R}^n$ can be written as

$$\{x + \lambda d \mid \lambda \in \mathbb{R}\}$$

for some $x \in \mathbb{R}^n$, $d \in \mathbb{R}^n$, $d \neq 0$.)

2. Let

$$P = \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} x_1 + x_2 \leq 1 \\ 4x_1 + x_2 \leq 2 \\ 2x_1 + x_2 \leq \frac{4}{3} \\ x_1 \geq 0, \quad x_2 \geq 0 \end{array} \right\}.$$

List all the basic feasible solutions. Which of them are degenerate (a BFS for a polyhedron in $\mathbb{R}^n$ is degenerate, if it has more than n active constraints)? Why?

3. Convert the following LP to standard form.

$$\begin{array}{llrcl} \min & z = & 3x_1 + x_2 & & \\ \text{s.t} & & x_1 & \geq & 3, \\ & & x_1 + x_2 & \leq & 4, \\ & & 2x_1 - x_2 & = & 3, \\ & & x_1, x_2 & \geq & 0 \end{array}$$

4. Consider the polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ in standard form. Suppose that A has full row rank.

   (a) Suppose that two different bases lead to the same basic solution x. Does x have to be degenerate? Prove or give a counter example.

   (b) Suppose that x is a degenerate basic solution. Does it always correspond to two or more different bases? Prove or give a counter example.

5. Consider the standard form polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$. Suppose that the matrix A, of dimension $m \times n$, has linearly independent rows, and that all basic feasible solutions are nondegenerate. Let x be an element of P that has exactly m positive components.

   (a) Show that x is a basic feasible solution.

   (b) Show that the result of part (a) is false if the nondegeneracy assumption is removed.