STOR 614 - Linear Programming, Spring 2019

Homework No. 8

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Problem 1.

(1)

$$Ad^* = -A_B A_B^{-1} A_j + A_j = 0.$$

(2)

$$c^T d^* = -c_B^T A_B^{-1} A_j + c_j = -(c_B^T A_B^{-1} A_j - c_j) > 0$$

because the reduced cost of x_j is $c_B^T A_B^{-1} A_j - c_j$ and is negative.

(3) The matrix

$$\begin{bmatrix} A_B & D \\ 0 & I_{n-m-1} \end{bmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}$$

is nonsingular because A_B is nonsingular. Thus, the matrix with the active constraints of d^* as row vectors has $n-1$ linearly independent columns. Thus, d^* has $n-1$ linearly independent active constraints.

Problem 2.

For any $x, y \in \mathbb{R}^n$ and $0 \leqslant t \leqslant 1$,

$$\begin{aligned} F[(1-t)x + ty] &= g\{f[(1-t)x + ty]\} \\ &\leqslant g[(1-t)f(x) + tf(y)] \quad (f \text{ is convex and } g \text{ is nondecreasing}) \\ &\leqslant (1-t)g(f(x)) + tg(f(y)) \quad (g \text{ is convex}) \\ &= (1-t)F(x) + tF(y) \end{aligned}$$

Thus, F is convex.

Problem 3.

(a)

First, we have

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}, c = \begin{bmatrix} -8 \\ -16 \end{bmatrix}, A = \begin{bmatrix} -1 & -1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, b = \begin{bmatrix} -5 \\ -3 \\ 0 \\ 0 \end{bmatrix}.$$

The matrix M is positive definite.

The point $(3,2)$ satisfies the first two constraints as equalities and the last two strictly. So the multipliers would need to satisfy $u_1 \geqslant 0, u_2 \geqslant 0, u_3 = 0, u_4 = 0$. The equation $Mx + c = A^T u$ becomes

$$\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{bmatrix}$$

which gives $u_1 = 0$ and $u_2 = 2$. Therefore there exists $u \in \mathbb{R}^4$ such that (x, u) satisfies all the KKT conditions. $x = (3,2)$ is a global solution.

M is positive definite, thus z is strictly convex, thus the QP has a unique global solution.

(b)

First, we have

$$M = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, c = \begin{bmatrix} -2 \\ -6 \end{bmatrix}, A = \begin{bmatrix} -1 & -1 \\ 1 & -2 \\ -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, b = \begin{bmatrix} -2 \\ -2 \\ -3 \\ 0 \\ 0 \end{bmatrix}.$$

The matrix M is positive definite. The point $(2/3, 4/3)$ satisfies the first two constraints as equalities and the last three strictly. So the multipliers would need to satisfy $u_1 \geqslant 0, u_2 \geqslant$

$0, u_3 = 0, u_4 = 0, u_5 = 0$. The equation $Mx + c = A^T u$ becomes

$$\begin{bmatrix} -8/3 \\ -4 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -2 & 1 & 0 \\ -1 & -2 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives $u_1 = 28/9, u_2 = 4/9$. Therefore there exists $u \in \mathbb{R}^5$ such that (x, u) satisfies all the KKT conditions. $x = (2/3, 4/3)$ is a global solution.

M is positive definite, thus z is strictly convex, thus the QP has a unique global solution.

Problem 4.

$$\begin{aligned} \min \quad & \frac{1}{2} x^T x \\ \text{s.t.} \quad & a^T x + \alpha \geqslant 0 \end{aligned}$$

We have

$$M = I_n, c = 0, A = a^T, b = -\alpha$$

. The KKT conditions state that

$$\begin{cases} I_n x = au \\ a^T x + \alpha \geqslant 0 \\ u \geqslant 0 \\ (a^T x + \alpha) u = 0 \end{cases}$$

If $\alpha \geqslant 0$, then $u = 0, x = 0$. The optimal solution is $x = 0$, and the optimal value is 0.

If $\alpha < 0$, then $a^T x + \alpha = a^T a u + \alpha = 0 \implies u = -\alpha/(a^T a)$ (assume $a \neq 0$). The optimal solution is

$$x = au = -\frac{\alpha}{a^T a} a,$$

and the optimal value is

$$\frac{1}{2} x^T x = \frac{1}{2} \alpha^2 / (a^T a).$$