STOR 614 - Linear Programming, Spring 2019

Homework No. 3

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Problem 1.

Proof.

1. If.

   If a basis of x^* and a basis of y^* share $m-1$ elements, then the two bases also share $n-m-1$ nonbasic variables. Let $\{N(1),,N(n-m-1)\}$ be the indices of the shared nonbasic variables. Then x^* and y^* share the following $n-1$ linearly independent active constraints

$$\begin{cases} Ax = b \\ x_{N(1)} = 0 \\ \vdots \\ x_{N(n-m-1)} = 0 \end{cases}$$

   Therefore, x^* and y^* are adjacent.

2. Only if.

   Lemma 1. Let X be a set of vectors. If there are n linearly independent vectors in X, and $x_i, \ldots, x_m \in X$ are linearly independent, then there exists a set of n linearly independent vectors in X that contains $x_i, \ldots, x_m$.

Let S be the set of active constraints shared by x^* and y^*. Let T be the set of m linealy independent constraints imposed by $Ax = b$. Then, $T \subseteq S$ and S contains $n-1$ linearly independent elements. By Lemma 1, there exists $P \subseteq S$, such that P has $n-1$ elements, the elements of P are linearly independent, and $T \subseteq P$. Moreover, there exists a set of n linearly independent active constraints for x^*, B_x, and a set of n linearly independent active constraints for y^*, B_y, such that $P \subseteq B_x$, $P \subseteq B_y$. Then the bases corresponding to B_x and B_y share $m-1$ elements.

□

Problem 2.

(a)

Proof. If the reduced cost of every nonbasic variable is positive, then the object function reaches its maximum if and only if all nonbasic variables are zero. x^* is the only solution that satisfies that all nonbasic variables are zero. Therefore, x^* is the unique optimal solution. □

(b)

Proof. Let $x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix}$ be the vector of basic variables in x^*, and $x_N = \begin{bmatrix} x_{N(1)} \\ \vdots \\ x_{N(n-m)} \end{bmatrix}$ be the vector of nonbasic variables in x^*. The constraints of the linear programming problem can be written as

$$\begin{bmatrix} \mathbf{I}_m & \mathbf{P} \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b$$

where $\mathbf{I}_m \in \mathbb{R}^{m \times m}$ is identity matrix, $\mathbf{P} \in \mathbb{R}^{m \times (n-m)}$, and $b \in \mathbb{R}^m$.

Suppose that $x_{N(k)}$ has a nonpositive reduced cost, c. Because x^* is nondegenerate, all $b_i > 0$. Then there exists a feasible solution x^{**} which satisfies

$$\begin{cases} x^{**}_{N(k)} > 0 \\ x^{**}_{N(i)} = 0 \quad i \neq k \end{cases}$$

The objective function $z(x^{**}) = K - cx^{**}_{N(k)} \geqslant K - cx^{*}_{N(k)} = z(x^*)$, where $x^{*}_{N(k)} = 0$ and K is a constant. This contradicts with that x^* is the unique optimal solution. $\square$

Problem 3.

(a) Standard form.

$$\begin{aligned} \max \quad z = \;& 2x_1 + x_2 \\ \text{s.t} \quad & x_1 - x_2 + s_1 = 2, \\ & x_1 + x_2 + s_2 = 6, \\ & x_1, x_2, s_1, s_2 \geqslant 0 \end{aligned}$$

Simplex method. Boxed number indicate the pivot element.

z	x_1	x_2	s_1	s_2	rhs	Basic var	Ratio test
1	-2	-1	0	0	0	$z = 0$	
0	$\boxed{1}$	-1	1	0	2	$s_1 = 2$	$2/1 = 2$
0	1	1	0	1	6	$s_2 = 6$	$6/1 = 6$
1	0	-3	2	0	4	$z = 4$	
0	1	-1	1	0	2	$x_1 = 2$	N/A
0	0	$\boxed{2}$	-1	1	4	$s_2 = 4$	$4/2 = 2$
1	0	0	$1/2$	$3/2$	10	$z = 10$	
0	1	0	$1/2$	$1/2$	4	$x_1 = 4$	
0	0	1	$-1/2$	$1/2$	2	$x_2 = 2$	

The BFS

$$(x_1, x_2, s_1, s_2) = (4, 2, 0, 0)$$

is optimal, and the optimal objective function value is $z = 10$.

(b)

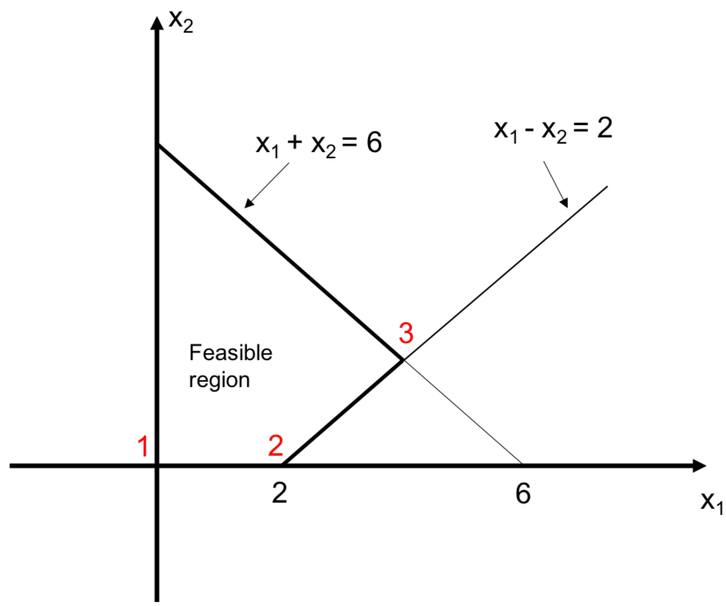

Figure 1: Optimization path of simplex method. Red numbers indicate the order of the BFS's reached in the simplex method.

Problem 4.

(a) False.

Proof. If the feasible solution is moved by a positive distance, then the right-hand-side of the pivot row is positive, then the z-value must be changed (increased). □

(b) True.

Proof. Let c be the reduced cost of the entering variable, p be the pivot element, and x^* be the leaving variable in one iteration. If the rules of simplex method are strictly followed, then $c < 0$, $p > 0$. Let d be the new reduced cost of x^* after that iteration, then $d = -c/p > 0$. Therefore, the new reduced cost of the leaving variable must be positive. Therefore, the leaving variable cannot reenter in the very next iteration. □

(c) False. Counter example:

z	x_1	x_2	s_1	s_2	rhs	Basic var	Ratio test
1	-2	-8	0	0	0	$z = 0$	
0	$\boxed{1}$	2	1	0	2	$s_1 = 2$	$2/1 = 2$
0	0	1	0	1	6	$s_2 = 6$	N/A
1	0	-4	2	0	4	$z = 4$	
0	1	$\boxed{2}$	1	0	2	$x_1 = 2$	$2/2 = 1$
0	0	1	0	1	6	$s_2 = 6$	$6/1 = 6$

Boxed number indicates pivot element. x_1 just entered in the first iteration but leaves in the next iteration.

(d) False. Counter example:

z	x_1	x_2	s_1	s_2	rhs	Basic var	Ratio test
1	2	0	0	0	4	$z = 4$	
0	1	2	1	0	2	$s_1 = 2$	
0	1	1	0	1	6	$s_2 = 6$	

This tableau shows a nondegenerate optimal solution

$$(x_1, x_2, s_1, s_2) = (0, 0, 2, 6),$$

but there exists another optimal solution

$$(x_1, x_2, s_1, s_2) = (0, 1, 0, 5).$$