STOR 614
Homework Assignment No. 8

1. Consider the LP

$$\begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Suppose that A has linearly independent rows, and that the primal simplex method terminates with the conclusion that the LP is unbounded. Explain how to obtain an extreme ray d of the feasible set of the LP with $c^T d > 0$.

Hint: Denote the basic variables in the last simplex tableau by $x_{B(1)}, \cdots, x_{B(m)}$, and let A_B be the basis matrix. Suppose that x_j is a nonbasic variable with negative reduced cost and that the jth column of the tableau, $A_B^{-1}A_j$, has no positive elements. Let $d^* \in \mathbb{R}^n$ be defined as $d_B^* = -A_B^{-1}A_j$ and $d_j^* = 1$ and $d_i^* = 0$ for every nonbasic index i other than j. Show that d^* satisfies the desired properties: (1) show that d^* satisfies $Ad^* = 0$ and $d^* \geq 0$, so that it belongs to the recession cone of the feasible set; (2) show that $c^T d^* > 0$, by using the fact that the reduced cost for x_j is negative; (3) show that d^* has $n-1$ linearly independent active constraints among all constraints defining the recession cone of the feasible set.

2. Let f be a convex function from $\mathbb{R}^n$ to $\mathbb{R}$, and let g be a convex and nondecreasing function from $\mathbb{R}$ to $\mathbb{R}$. (A nondecreasing function g from $\mathbb{R}$ to $\mathbb{R}$ satisfies $g(y) \leq g(z)$ whenever $y \leq z$.) Prove the function F defined by $F(x) = g(f(x))$ is a convex function on $\mathbb{R}^n$.

3. For each part below, use Karush-Kuhn-Tucker conditions to verify the given vector x^* is a global solution to the given quadratic program. Do those quadratic programs have multiple global solutions?

   (a) $x^* = (3, 2)$.

$$\begin{array}{ll} \min & z = x_1^2 + 4x_2^2 - 8x_1 - 16x_2 \\ \text{s.t.} & x_1 + x_2 \leq 5 \\ & x_1 \leq 3 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{array}$$

(b) $x^* = (2/3, 4/3)$.

$$\begin{array}{ll} \min & z = \frac{1}{2}x_1^2 - x_1x_2 + x_2^2 - 2x_1 - 6x_2 \\ \text{s.t.} & x_1 + x_2 \leq 2 \\ & -x_1 + 2x_2 \leq 2 \\ & 2x_1 + x_2 \leq 3 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{array}$$

4. Consider the half space defined by $H = \{x \in \mathbb{R}^n \mid a^T x + \alpha \geq 0\}$ where $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ are given. Formulate the problem of finding the point x in H with the smallest Euclidean norm as a quadratic program, and solve it by using Karush-Kuhn-Tucker conditions.