STOR 614
Homework Assignment No. 7

1. Consider the LP
$$\begin{array}{lrcl} \max & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0, \end{array}$$
where $A \in \mathbb{R}^{m\times n}$. Suppose that there exists some $b^* \in \mathbb{R}^m$, such that the LP above has an optimal solution when $b = b^*$. Prove that the LP has an optimal solution whenever it is feasible. In other words, prove that the LP has an optimal solution for each $b \in B$, where
$$B = \{b \in \mathbb{R}^m \mid \text{there exists some } x \in \mathbb{R}^n_+ \text{ such that } Ax = b\}.$$

2. Consider the LP
$$\begin{array}{lrcl} \min & c^T x & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0 \end{array}$$
and its dual
$$\begin{array}{lrcl} \max & b^T y & & \\ \text{s.t.} & A^T y & \leq & c \end{array}$$
where $A \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}^n$ are fixed. Suppose that the dual feasible set is nonempty. Let $f(b)$ denote the optimal value of the primal LP for each $b \in \mathbb{R}^m$ (if the LP is infeasible at some b then $f(b) = \infty$). Let $b^* \in \mathbb{R}^m$ be fixed.

   (a) Prove that any dual optimal solution y^* at $b = b^*$ satisfies
   $$f(b) - f(b^*) \geq (y^*)^T(b - b^*) \text{ for all } b \in \mathbb{R}^m. \tag{1}$$

   (b) Suppose a vector $y^* \in \mathbb{R}^m$ satisfies (1). Prove that y^* is a dual optimal solution at $b = b^*$. Hint: for each $x \geq 0$, consider the choice $b = Ax$ and prove that $(y^*)^T Ax \leq c^T x - f(b^*) + (y^*)^T b^*$, and use this to show that y^* is a dual feasible solution.

   Note: any y^* that satisfies (1) is called a *subgradient* of f at b^*. The goal of this exercise is to show that dual optimal solutions are exactly subgradients of the optimal value function. See Theorem 5.2 of the textbook.

3. Write down all extreme points (if any) and a complete set of extreme rays (if any) for each of the following polyhedra.

   (a) $P = \{x \in \mathbb{R}^2 \mid 4x_1 + 2x_2 \geq 8, \;\; x_1 \geq 0\}$.

   (b) $P = \{x \in \mathbb{R}^2 \mid 4x_1 + 2x_2 \geq 8, \;\; 2x_1 + x_2 \leq 8\}$.