STOR 614 - Linear Programming, Spring 2019

Homework No. 6

Zhenghan Fang

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Problem 1.

(a)

Convert to standard form

$$\begin{array}{rrrrrrrr}
\max & z = & -x_1 & & -x_3 & & & \\
s.t & & -2x_1 & -x_2 & +x_3 & +s_1 & & = -5 \\
& & -x_1 & +2x_2 & -2x_3 & & +s_2 & = -2 \\
& & x_1, & x_2, & x_3, & s_1, & s_2 & \geqslant 0
\end{array}$$

The initial tableau:

z	x_1	x_2	x_3	s_1	s_2	rhs	Basic var
1	1	0	1	0	0	0	$z = 0$
0	-2	-1	1	1	0	-5	$s_1 = -5$
0	-1	2	-2	0	1	-2	$s_2 = -2$

Iteration 1:

z	x_1	x_2	x_3	s_1	s_2	rhs	Basic var
1	1	0	1	0	0	0	$z = 0$
0	-2	$\boxed{-1}$	1	1	0	-5	$s_1 = -5$
0	-1	2	-2	0	1	-2	$s_2 = -2$
ratio	$1/2$	0	NA	NA	NA		

Dual BFS: (0,0)

s_1 leaves and x_2 enters.

Iteration 2:

z	x_1	x_2	x_3	s_1	s_2	rhs	Basic var
1	1	0	1	0	0	0	$z = 0$
0	2	1	-1	-1	0	5	$x_2 = 5$
0	$\boxed{-5}$	0	0	2	1	-12	$s_2 = -12$
ratio	1/5	NA	NA	NA	NA		

Dual BFS: (0,0)

s_2 leaves and x_1 enters.

Iteration 3:

z	x_1	x_2	x_3	s_1	s_2	rhs	Basic var
1	0	0	1	2/5	1/5	$-12/5$	$z = -12/5$
0	0	1	-1	$-1/5$	2/5	1/5	$x_2 = 1/5$
0	1	0	0	$-2/5$	$-1/5$	12/5	$x_1 = 12/5$
ratio	1/5	NA	NA	NA	NA		

Dual BFS: (2/5,1/5)

No leaving variable to choose.

Terminate. $x = (12/5, 1/5, 0)$ is optimal, and the optimal value is -12/5.

(b)

Convert to standard form.

$$\begin{array}{llrrrrrrrrl}
\max & z = & -3x_1 & -4x_2 & -2x_3 & -x_4 & -5x_5 & & & & \\
s.t & & x_1 & -2x_2 & -x_3 & +x_4 & +x_5 & +s_1 & & & = -3 \\
& & -x_1 & -x_2 & -x_3 & +x_4 & +x_5 & & +s_2 & & = -2 \\
& & x_1 & +x_2 & -2x_3 & +2x_4 & -3x_5 & & & +s_3 & = 4 \\
& & x_1, & x_2, & x_3, & x_4, & x_5, & s_1, & s_2 & s_3 & \geqslant 0
\end{array}$$

The initial tableau:

z	x_1	x_2	x_3	x_4	x_5	s_1	s_2	s_3	rhs	Basic var
1	3	4	2	1	5	0	0	0	0	$z=0$
0	1	-2	-1	1	1	1	0	0	-3	$s_1=-3$
0	-1	-1	-1	1	1	0	1	0	-2	$s_2=-2$
0	1	1	-2	2	-3	0	0	1	4	$s_3=4$

Iteration 1:

z	x_1	x_2	x_3	x_4	x_5	s_1	s_2	s_3	rhs	Basic var
1	3	4	2	1	5	0	0	0	0	$z=0$
0	1	-2	$\boxed{-1}$	1	1	1	0	0	-3	$s_1=-3$
0	-1	-1	-1	1	1	0	1	0	-2	$s_2=-2$
0	1	1	-2	2	-3	0	0	1	4	$s_3=4$
ratio	NA	2	2	NA	NA	NA	NA	NA		

Dual BFS: (0,0,0)

s_1 leaves and x_3 enters.

Iteration 2:

z	x_1	x_2	x_3	x_4	x_5	s_1	s_2	s_3	rhs	Basic var
1	5	0	0	3	7	2	0	0	-6	$z=-6$
0	-1	2	1	-1	-1	-1	0	0	3	$x_3=3$
0	-2	1	0	0	0	-1	1	0	1	$s_2=1$
0	-1	5	0	0	-5	-2	0	1	10	$s_3=10$
ratio	NA	2	2	NA	NA	NA	NA	NA		

Dual BFS: (2,0,0)

No leaving variable to choose.

Terminate. $x=(0,0,3,0,0)$ is optimal. The optimal value is -6.

Problem 2.

Proof. If (a) holds, then (b) cannot hold, because

$$p^T A > 0, x \geqslant 0, x \neq 0 \implies p^T A x > 0,$$

which contradicts with $Ax = 0$.

Suppose that (a) does not hold. Let $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$. Consider the pair

$$\begin{array}{llll} \text{Primal:} & \max & \mathbf{1}^T x \\ & s.t & Ax = 0 \\ & & x \geqslant 0 \end{array} \qquad \begin{array}{lll} \text{Dual:} & \min & 0 \\ & s.t & p^T A \geqslant \mathbf{1} \end{array}$$

The primal has a unique optimal solution $x = 0$ and the optimal value is 0. Then the dual is feasible, so there exists p such that $p^T A \geqslant \mathbf{1} > 0$. $\square$

Problem 3.

Proof. If (b) holds, then

$$a^T x \leqslant \sum_{i=1}^{m} \lambda_i a_i^T x \leqslant \sum_{i=1}^{m} \lambda_i (\max_{i=1,\dots,m} a_i^T x) = \max_{i=1,\dots,m} a_i^T x$$

for any $x \geqslant 0$.

Suppose that (a) holds. Consider the pair

$$\begin{array}{lll} \text{Primal:} & \max & a^T x \\ & s.t & a_i^T x \leqslant 1, \quad i = 1, \dots, m \\ & & x \geqslant 0 \end{array} \qquad \begin{array}{lll} \text{Dual:} & \min & \sum_{i=1}^m p_i \\ & s.t & \sum_{i=1}^m p_i a_i \geqslant a \\ & & p_i \geqslant 0, \quad i = 1, \dots, m \end{array}$$

The primal is not infeasible ($x = 0$ is a trivial feasible solution). Thus, the primal has an optimal solution and the optimal value is bounded by 1. Thus, the optimal value of the dual is $\leqslant 1$. Thus, there exist coefficients p_i's, such that

$$\begin{aligned} \sum_{i=1}^{m} p_i &\leqslant 1, \\ \sum_{i=1}^{m} p_i a_i &\geqslant a, \\ p_i \geqslant 0, &\quad i = 1, \dots, m. \end{aligned} \tag{3.1}$$

Furthermore, consider the pair

$$
\begin{array}{llll}
\text{Primal:} & \max & a^T x & \\
 & s.t & a_i^T x \leqslant -1, & i = 1, ..., m \\
 & & x \geqslant 0 &
\end{array}
\qquad
\begin{array}{llll}
\text{Dual:} & \min & \sum_{i=1}^m -q_i & \\
 & s.t & \sum_{i=1}^m q_i a_i \geqslant a & \\
 & & q_i \geqslant 0, & i = 1, ..., m
\end{array}
$$

If the primal is infeasible, then the dual is unbounded (dual cannot be infeasible because of (3.1)). If the primal is feasible, then the primal optimal value is $\leqslant -1$. Therefore, either way, there exist coefficients q_i's, such that

$$
\begin{gathered}
\sum_{i=1}^m -q_i \leqslant -1 \implies \sum_{i=1}^m q_i \geqslant 1, \\
\sum_{i=1}^m q_i a_i \geqslant a, \\
q_i \geqslant 0, \quad i = 1, ..., m.
\end{gathered} \tag{3.2}
$$

From (3.1) and (3.2), and that the feasible set of dual problem is convex, we get statement (b). □

Problem 4.

(a) The optimal solution is $(0, 25, 25, 0, 0)$. The optimal objective function value is 300.

(b) The dual LP:

$$
\begin{array}{lcccl}
\min & 50y_1 & +100y_2 & & \\
s.t & y_1 & +2y_2 & \geqslant 3 \\
 & y_1 & +3y_2 & \geqslant 7 \\
 & y_1 & +y_2 & \geqslant 5 \\
 & y_1 & & \geqslant 0 \\
 & & y_2 & \geqslant 0
\end{array}
$$

A dual optimal solution is $(y_1, y_2) = (4, 1)$. The dual has only one optimal solution, because the primal LP has a nondegenerate optimal BFS.

(c) The reduced cost for x_1 will be $3 - \Delta$. For $\Delta \leqslant 3$, $\{x_3, x_2\}$ continues to be an optimal basis. For $\Delta \leqslant 3$, an optimal solution is $(0, 25, 25)$, and the optimal value is 300. For $\Delta = 4$, the tableau is

z	x_1	x_2	x_3	s_1	s_2	rhs	Basic var	ratio
1	-1	0	0	4	1	300	$z = 300$	
0	0.5	0	1	1.5	-0.5	25	$x_3 = 25$	50
0	0.5	1	0	-0.5	0.5	25	$x_2 = 25$	50

x_1 enters and x_3 leaves.

z	x_1	x_2	x_3	s_1	s_2	rhs	Basic var
1	0	0	2	7	0	350	$z = 350$
0	1	0	2	3	-1	50	$x_1 = 50$
0	0	1	-1	-2	1	0	$x_2 = 0$

An optimal solution is $(50, 0, 0, 0, 0)$, and the optimal value is 350.

(d)

z	x_1	x_2	x_3	s_1	s_2	rhs	Basic var
1	3	0	$-\Delta$	$4-\Theta$	1	300	$z = 300$
0	0.5	0	1	1.5	-0.5	25	$x_3 = 25$
0	0.5	1	0	-0.5	0.5	25	$x_2 = 25$

z	x_1	x_2	x_3	s_1	s_2	rhs	Basic var
1	$3+0.5\Delta$	0	0	$4-\Theta+1.5\Delta$	$1-0.5\Delta$	$300+25\Delta$	$z = 300+25\Delta$
0	0.5	0	1	1.5	-0.5	25	$x_3 = 25$
0	0.5	1	0	-0.5	0.5	25	$x_2 = 25$

For the current basis to remain optimal, we need

$$3 + 0.5\Delta \geqslant 0$$

$$4 - \Theta + 1.5\Delta \geqslant 0$$

$$1 - 0.5\Delta \geqslant 0$$

Thus, for $-6 \leqslant \Delta \leqslant 2$ and $\Theta \leqslant 4 + 1.5\Delta$, this tableau shows an optimal solution. Optimal solution: $(0, 25, 25, 0, 0)$, optimal value: $300 + 25\Delta$. $(\Delta, \Theta) = (2, 2)$ belongs to that range.

(e)

z	x_1	x_2	x_3	s_1	s_2	rhs	Basic var
1	3	0	0	4	1	$300 + 4\Delta$	$z = 300 + 4\Delta$
0	0.5	0	1	1.5	-0.5	$25 + 1.5\Delta$	$x_3 = 25 + 1.5\Delta$
0	0.5	1	0	-0.5	0.5	$25 - 0.5\Delta$	$x_2 = 25 - 0.5\Delta$

For the current basis to remain optimal, we need

$$25 + 1.5\Delta \geqslant 0$$

$$25 - 0.5\Delta \geqslant 0$$

Thus, for $-50/3 \leqslant \Delta \leqslant 50$, the current basis continue to be optimal. The optimal value is $300 + 4\Delta$. When $\Delta = 100$,

z	x_1	x_2	x_3	s_1	s_2	rhs	Basic var
1	3	0	0	4	1	700	$z = 700$
0	0.5	0	1	1.5	-0.5	175	$x_3 = 175$
0	0.5	1	0	-0.5	0.5	-25	$x_2 = -25$

x_2 leaves and s_1 enters.

z	x_1	x_2	x_3	s_1	s_2	rhs	Basic var
1	7	8	0	0	5	500	$z = 500$
0	2	3	1	0	1	100	$x_3 = 100$
0	-1	-2	0	1	-1	50	$s_1 = 50$

An optimal solution is $(0, 0, 100, 50, 0)$, and the optimal value is 500.

Problem 5.

(a) Basis: $\{x_1, x_6, x_3\}$. $A_B = \begin{bmatrix} 6 & 0 & 5 \\ 3 & 1 & 3 \\ 3 & 0 & 5 \end{bmatrix}$. $A_B^{-1} = \begin{bmatrix} 1/3 & 0 & -1/3 \\ -2/5 & 1 & -1/5 \\ -1/5 & 0 & 2/5 \end{bmatrix}$. $c_B = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$. BFS: $(5/3, 0, 3, 0, 0, 1, 0)$. Objective function value = 17.

(b) The largest value is 3. The optimal value is 17. If x_4, x_5, x_7 are zero, the optimal value is reached. Thus the set of optimal solutions is $\{\begin{bmatrix} 1/3\lambda + 5/3 \\ \lambda \\ 3-\lambda \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \mid 0 \leqslant \lambda \leqslant 3\}$.

(c) If the coefficient of x_3 is $4+\Delta$, the tableau will be

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs	Basic var
1	0	2	$-\Delta$	1	$1/5$	0	$3/5$	17	$z = 17$
0	1	$-1/3$	0	$2/3$	$1/3$	0	$-1/3$	$5/3$	$x_3 = 175$
0	0	0	0	-1	$-2/5$	1	$-1/5$	1	$x_2 = -25$
0	0	1	1	0	$-1/5$	0	$2/5$	3	$x_2 = -25$

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs	Basic var
1	0	$2+\Delta$	0	1	$1/5 - 1/5\Delta$	0	$3/5 + 2/5\Delta$	$17+\Delta$	$z = 17+\Delta$
0	1	$-1/3$	0	$2/3$	$1/3$	0	$-1/3$	$5/3$	$x_3 = 175$
0	0	0	0	-1	$-2/5$	1	$-1/5$	1	$x_2 = -25$
0	0	1	1	0	$-1/5$	0	$2/5$	3	$x_2 = -25$

Thus, the smallest value is $4 - 1.5 = 2.5$. The optimal value is 15.5. The optimal value is reached if $x_2, x_4, x_5 = 0$. Thus, the set of optimal solutions is $\{\begin{bmatrix} 1/3\lambda + 5/3 \\ 0 \\ -2/5\lambda + 3 \\ 0 \\ 0 \\ 1/5\lambda + 1 \\ \lambda \end{bmatrix} \mid 0 \leqslant \lambda \leqslant 5/2\}$.

(d) If the value is $25 + \Delta$, the tableau will be

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs	Basic var
1	0	2	0	1	1/5	0	3/5	$17 + 1/5\Delta$	$z = 17$
0	1	$-1/3$	0	2/3	1/3	0	$-1/3$	$5/3 + 1/3\Delta$	$x_3 = 175$
0	0	0	0	-1	$-2/5$	1	$-1/5$	$1 - 2/5\Delta$	$x_2 = -25$
0	0	1	1	0	$-1/5$	0	2/5	$3 - 1/5\Delta$	$x_2 = -25$

Thus, the largest value is 27.5. The new optimal basis will be $\{x_1, x_5, x_3\}$.

(e) Because the dual optimal solution is $(1/5, 0, 3/5)$, the rate of change of the primal optimal value in response to a change in the largest value of resource 1, 2, 3 are $1/5, 0, 3/5$, respectively. Thus, if we purchase n packages of resources, the primal optimal value increases by $3n \times 1/5 + 4n \times 3/5 = 3n$, and the cost increases by $5n$. The increase of cost is greater than the increase of earnings, so we should not purchase any of this package.