STOR 614
Homework Assignment No. 5

1. Consider the following problem as the primal LP.

$$
\begin{array}{llrrrcr}
\max & z = & 2x_1 & +3x_2 & +x_3 & & \\
\text{s.t} & & x_1 & +x_2 & +x_3 & \leq & 5, \\
& & -x_1 & -x_2 & +3x_3 & = & -2, \\
& & x_1 & & -x_3 & \geq & 1, \\
& & \multicolumn{5}{l}{x_1 \geq 0, x_2 \geq 0, x_3 \text{ free.}}
\end{array}
$$

   (a) Write down its dual LP.

   (b) Convert the primal LP into standard form, and write down the dual LP of the LP in standard form.

   (c) Show that the two dual problems obtained in (a) and (b) are equivalent.

2. Let the following problem be the primal LP.

$$
\begin{array}{llrrrcr}
\max & z = & 5x_1 & +3x_2 & +x_3 & & \\
\text{s.t.} & & 2x_1 & +x_2 & +x_3 & \leq & 6 \\
& & x_1 & +2x_2 & +x_3 & \leq & 7 \\
& & \multicolumn{5}{l}{x_1, x_2, x_3 \geq 0}
\end{array}
$$

   Graphically solve the dual LP. Then use complementary slackness conditions to find all optimal solutions to the primal LP.

3. Let A be a symmetric $n \times n$ matrix (i.e., $A = A^T$), and $c \in \mathbb{R}^n$. Consider the LP below

$$
\begin{array}{ll}
\min & c^T x \\
\text{s.t.} & Ax \geq c \\
& x \geq 0.
\end{array}
$$

   Prove that if $x^* \in \mathbb{R}^n$ satisfies $Ax^* = c$ and $x^* \geq 0$, then x^* is an optimal solution.

4. Consider the following pair of primal and dual LPs.

$$
\text{(P)} \quad
\begin{array}{llrrrrrrcr}
\min & 35x_1 & +30x_2 & +60x_3 & +50x_4 & +27x_5 & +22x_6 & & \\
\text{s.t} & x_1 & & +2x_3 & +2x_4 & +x_5 & +2x_6 & \geq & 9, \\
 & & x_2 & +3x_3 & +x_4 & +3x_5 & +2x_6 & \geq & 19, \\
 & & & & & x & & \geq & 0.
\end{array}
$$

$$
\text{(D)} \quad
\begin{array}{llrcr}
\max & 9\pi_1 & +19\pi_2 & & \\
\text{s.t} & \pi_1 & & \leq & 35, \\
 & & \pi_2 & \leq & 30, \\
 & 2\pi_1 & +3\pi_2 & \leq & 60, \\
 & 2\pi_1 & +\pi_2 & \leq & 50, \\
 & \pi_1 & +3\pi_2 & \leq & 27, \\
 & 2\pi_1 & +2\pi_2 & \leq & 22, \\
 & & \pi & \geq & 0.
\end{array}
$$

Check if $x^* = (0, 0, 0, 0, 5, 2)$ is an optimal solution to the primal LP. If it is a primal optimal solution, write down the set of all optimal solutions to the dual LP, and decide whether x^* is the unique optimal solution or one of multiple optimal solutions to the primal LP.