STOR 614 - Linear Programming, Spring 2019

Homework No. 4

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Problem 1.

(a)

Phase I.

Add artificial variables y_1, y_2.

$$
\begin{array}{llrrrrrl}
\max & t = & & & & -y_1 & -y_2 & \\
s.t & & 2x_1 & +x_2 & +x_3 & +y_1 & & = 4 \\
& & x_1 & +x_2 & +2x_3 & & +y_2 & = 2 \\
& & x_1, & x_2, & x_3, & y_1, & y_2 & \geqslant 0
\end{array}
$$

Transform LP to canonical form.

$$
\begin{array}{llrrrrrl}
\max & t = & 3x_1 & +2x_2 & +3x_3 & -6 & & \\
s.t & & 2x_1 & +x_2 & +x_3 & +y_1 & & = 4 \\
& & x_1 & +x_2 & +2x_3 & & +y_2 & = 2 \\
& & x_1, & x_2, & x_3, & y_1, & y_2 & \geqslant 0
\end{array}
$$

The initial tableau is as follows

t	x_1	x_2	x_3	y_1	y_2	rhs	Basic var
1	-3	-2	-3	0	0	-6	$t = -6$
0	2	1	1	1	0	4	$y_1 = 4$
0	1	1	2	0	1	2	$y_2 = 2$

First iteration: x_1 enters and y_1 leaves.

t	x_1	x_2	x_3	y_1	y_2	rhs	Basic var
1	0	$-1/2$	$-3/2$	$3/2$	0	0	$t = 0$
0	1	$1/2$	$1/2$	$1/2$	0	2	$x_1 = 2$
0	0	$1/2$	$3/2$	$-1/2$	1	0	$y_2 = 0$

Second iteration: x_2 enters and y_2 leaves.

t	x_1	x_2	x_3	y_1	y_2	rhs	Basic var
1	0	0	0	1	1	0	$t = 0$
0	1	0	-1	1	-1	2	$x_1 = 2$
0	0	1	3	-1	2	0	$x_2 = 0$

The optimal $t = 0$. Case 2.1. Obtain a simplex tableau for the original LP.

z	x_1	x_2	x_3	rhs	Basic var
1	0	0	2	2	$z = 2$
0	1	0	-1	2	$x_1 = 2$
0	0	1	3	0	$x_2 = 0$

First iteration: x_3 enters and x_2 leaves.

z	x_1	x_2	x_3	rhs	Basic var
1	0	$-2/3$	0	2	$z = 2$
0	1	$1/3$	0	2	$x_1 = 2$
0	0	$1/3$	1	0	$x_3 = 0$

The original LP has an optimal solution $(x_1, x_2, x_3) = (2, 0, 0)$ and the optimal value is $z = 2$.

(b)

Convert to standard form.

$$
\begin{array}{llrrrrl}
\min & z = & 2x_1 & +3x_2 & & & \\
s.t & & 1/2x_1 & +1/4x_2 & +s_1 & & = 4 \\
 & & x_1 & +3x_2 & & -s_2 & = 20 \\
 & & x_1 & +x_2 & & & = 10 \\
 & & x_1, & x_2, & s_1, & s_2 & \geqslant 0
\end{array}
$$

Add artificial vriables y_1, y_2. The phase I LP is as follows.

$$\begin{array}{llllllllll}
\max & t = & & & & & -y_1 & -y_2 & \\
s.t & & 1/2x_1 & +1/4x_2 & +s_1 & & & & = 4 \\
& & x_1 & +3x_2 & & -s_2 & +y_1 & & = 20 \\
& & x_1 & +x_2 & & & & +y_2 & = 10 \\
& & x_1, & x_2, & s_1, & s_2, & y_1, & y_2 & \geqslant 0
\end{array}$$

Convert phase I LP to canonical form.

$$\begin{array}{llllllllll}
\max & t = & 2x_1 & +4x_2 & & -s_2 & -30 & & \\
s.t & & 1/2x_1 & +1/4x_2 & +s_1 & & & & = 4 \\
& & x_1 & +3x_2 & & -s_2 & +y_1 & & = 20 \\
& & x_1 & +x_2 & & & & +y_2 & = 10 \\
& & x_1, & x_2, & s_1, & s_2, & y_1, & y_2 & \geqslant 0
\end{array}$$

The initial tableau of phase I LP is as follows

t	x_1	x_2	s_1	s_2	y_1	y_2	rhs	Basic var
1	-2	-4	0	1	0	0	-30	$t = -30$
0	$1/2$	$1/4$	1	0	0	0	4	$s_1 = 4$
0	1	3	0	-1	1	0	20	$y_1 = 20$
0	1	1	0	0	0	1	10	$y_2 = 10$

First iteration: x_1 enters, s_1 leaves.

t	x_1	x_2	s_1	s_2	y_1	y_2	rhs	Basic var
1	0	-3	4	1	0	0	-14	$t = -14$
0	1	$1/2$	2	0	0	0	8	$x_1 = 8$
0	0	$5/2$	-2	-1	1	0	12	$y_1 = 12$
0	0	$1/2$	-2	0	0	1	2	$y_2 = 2$

Second iteration: x_2 enters, y_2 leaves.

t	x_1	x_2	s_1	s_2	y_1	y_2	rhs	Basic var
1	0	0	-8	1	0	6	-2	$t=-2$
0	1	0	4	0	0	-1	6	$x_1=6$
0	0	0	8	-1	1	-5	2	$y_1=2$
0	0	1	-4	0	0	2	4	$x_2=4$

Third iteration: s_1 enters, y_1 leaves.

t	x_1	x_2	s_1	s_2	y_1	y_2	rhs	Basic var
1	0	0	0	0	1	1	0	$t=0$
0	1	0	0	$1/2$	$-1/2$	$3/2$	5	$x_1=5$
0	0	0	1	$-1/8$	$1/8$	$-5/8$	$1/4$	$s_1=1/4$
0	0	1	0	$-1/2$	$1/2$	$-1/2$	5	$x_2=5$

The optimal $t=0$. Case 2.1. Obtain a simplex tableau for the original LP.

z	x_1	x_2	s_1	s_2	rhs	Basic var
1	0	0	0	$-1/2$	25	$z=25$
0	1	0	0	$1/2$	5	$x_1=5$
0	0	0	1	$-1/8$	$1/4$	$s_1=1/4$
0	0	1	0	$-1/2$	5	$x_2=5$

The curretn BFS is optimal. The optimal solution of the original LP is $(x_1, x_2) = (5, 5)$ and the optimal value is $z = 25$.

Problem 2.

(a)

This problem is feasible if and only if there exists k such that $b/a_k \geqslant 0$.

Proof. If there exists k such that $b/a_k \geqslant 0$, then $\{x_i = 0 \ (i \neq k),\ x_k = b/a_k\}$ is a feasible solution.

If there doesn't exist k such that $b/a_k \geqslant 0$, then for all $a_i \neq 0$, $b/a_i < 0$. Suppose $b > 0$, then all $a_i \leqslant 0$. Then $\sum_{i=1}^n a_i x_i \leqslant 0$, thus this problem is not feasible. If $b < 0$, then $\sum_{i=1}^n a_i x_i \geqslant 0$, and this problem is not feasible. □

(b)

Let $K = \{i \mid b/a_i \geqslant 0\}$. The set of BFS's is $\{x_i = 0\ (i \neq k), x_k = b/a_k \mid k \in K\}$. Thus the set of objective function values at BFS's is $\{c_k b/a_k \mid k \in K\}$.

Find

$$p = \min_{i \in K} c_i b/a_i$$

The optimal solution is $x_i = 0\ (i \neq p), x_p = b/a_p$ and the optimal value is $c_p b/a_p$.

Problem 3.

Let z be the BFS at a certain step. In each step, replace a basis of z that is not a basis of y by a basis of y that is not a basis of z. In this way, the number of shared bases of y and z increases by 1 in each iteration. Thus we can go from x to y in a finite number of steps.

Problem 4.

Proposition 1. When formulating the phase I LP, if a non-artificial variable x satisfies:

x has coefficient 1 in an equation and zero coefficients in all the other equations. (condition 1)

then we do not need to add an artificial variable in the equation where x has coefficient 1.

When an artificial variable y becomes nonbasic, a non-artificial variable becomes basic and satisfies condition 1. By proposition 1, we can formulate a new phase I LP without y. The new phase I LP is equivalent to the LP obtained by eliminating the column of y from the old phase I LP.

Problem 5.

Use Matlab to solve the problems.

(a) The problem is unbounded.

(b) The optimal solution is $(0, 5, 0, 1, 0, 4)$ and the optimal value is -74.

Matlab code for revised simplex method:

```
%% LP (a)
A=[ 0 1 2 1 1 0 -5;
```

```
    0 2 1 0 -2 1 0;
    1 1 -2 0 1 0 3];
c = -[2;3;-4;3;1;-4;6];
b = [2;1;3];
B = [4 6 1];

%% LP (b)
A = [1 0 0 -1 1 1;
     1 1 0 -1 3 0;
     1 1 1 -3 0 1];
c = -[14;-19;0;21;52;0];
b = [3;4;6];
B = [1 2 6];

%% revised simplex method
k=0;
while(1)
    N = setdiff(1:length(c), B);
    k=k+1;
    k
    AB = A(:,B);
    AB_inv = inv(AB);

    reduced_cost = -c.' + c(B).'*AB_inv*A
    coef_x = AB_inv*A
    rhs_z=c(B).'*AB_inv*b
    rhs_x=AB_inv*b

    t = find(reduced_cost(N)<0);
    if length(t) == 0
        fprintf('found optimal value\n')
        break
    end
    pivot_column = N(t(1));
    u = AB_inv*A(:,pivot_column);
    t = find(u>0);
```

```
    if length(t) == 0
        fprintf('unbounded\n')
        break
    end
    [¬,min_i] = min(rhs_x(t) ./ u(t));
    pivot_row = t(min_i);
    B(pivot_row) = pivot_column;
end
z=c(B).'*AB_inv*b;
```