STOR 614 - Linear Programming, Spring 2019
Homework No. 1

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May 22, 2019

Problem 1.

Let N_{ijg} be the number of students in grade g in neighborhood i and assigned to school j.

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{g=1}^{G} d_{ij}N_{ijg} \\
\text{subject to} \quad & \sum_{i=1}^{I} N_{ijg} \le C_{jg}, \quad j = 1, ..., J, \quad g = 1, ..., G \\
& \sum_{j=1}^{J} N_{ijg} = S_{ig}, \quad i = 1, ..., I, \quad g = 1, ..., G \\
& N_{ijg} \ge 0, \quad i = 1, ..., I, \quad j = 1, ..., J, \quad g = 1, ..., G
\end{aligned}
$$

Problem 2.

Problem 2(a)

Let N_1 and N_2 be the produced units of the first and second products, respectively.

$$
\begin{aligned}
\text{maximize} \quad & 6N_1 + 5.4N_2 - 3N_1 - 2N_2 \\
\text{subject to} \quad & 3N_1 + 4N_2 \le 20,000 \\
& 3N_1 + 2N_2 \le 4000 + (0.45)(6)N_1 + (0.30)(5.40)N_2 \\
& N_1 \ge 0 \\
& N_2 \ge 0
\end{aligned}
$$

Problem 2(b)

Simplify the equations in 2(a) to

$$
\begin{aligned}
\text{maximize} \quad & 3N_1 + 3.4N_2 \\
\text{subject to} \quad & 3N_1 + 4N_2 \le 20,000 \\
& 3N_1 + 3.8N_2 \le 40,000 \\
& N_1 \ge 0 \\
& N_2 \ge 0
\end{aligned}
$$

Plot the feasible region and isoprofit lines in Fig. 1. As shown in Fig. 1, the optimal solution is $N_2 = 0$, $N_1 = 20,000/3$, and the object function value there is 20,000.

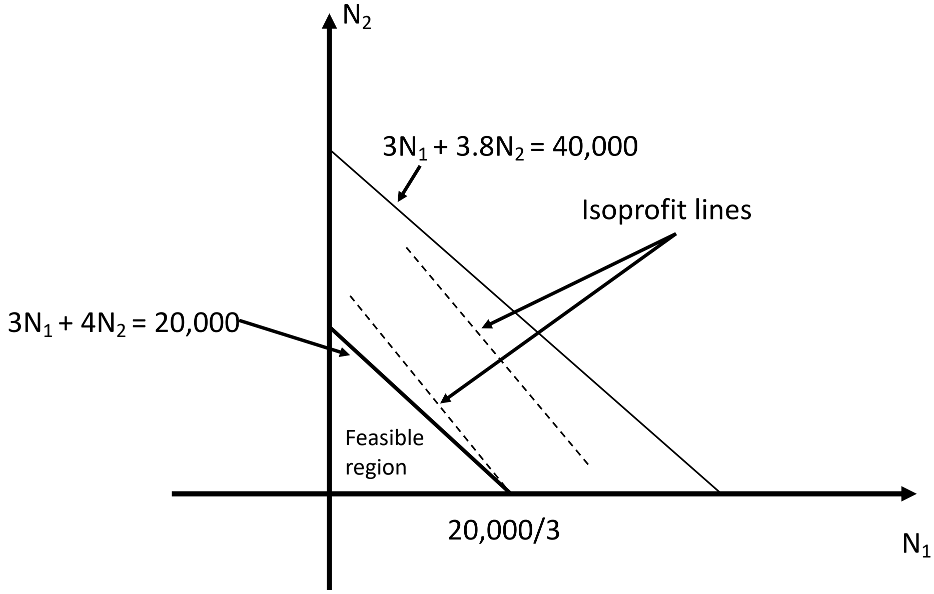

Figure 1: Graphic solution to Problem 2(b).

Problem 2(c)

If the investment is made, cash will decrease to \$3600, and machine hours will increase to 22,000. Thus, the new linear programming problem is

$$\begin{aligned} \text{maximize} \quad & 3N_1 + 3.4N_2 \\ \text{subject to} \quad & 3N_1 + 4N_2 \leq 22,000 \\ & 3N_1 + 3.8N_2 \leq 36,000 \\ & N_1 \geq 0 \\ & N_2 \geq 0 \end{aligned}$$

The optimal solution for the new problem is $N_1 = 22,000/3$, $N_2 = 0$, and the object function value there is 22,000. Because the maximal net income after investment (= \$22,000) is greater than that before investment (= \$20,000), the investment should be made.

Problem 3.

Lemma 1. If $A \in \mathbb{R}^{m\times n}$ and $m < n$, $Ax = 0$ has infinite many solutions.

Proof. Using the Gauss-Jordan method, we can convert $Ax = 0$ to its "reduced row-echelon form", where at least $(n - m) \geq 1$ unknowns can have arbitrary values and any arbitrary value of those unknowns corresponds to a solution of $Ax = 0$. Therefore, $Ax = 0$ has infinite many solutions.

Suppose that $Ax = b$ has at least one solution x_0. Let S be the solution set of $Ax = b$. Then

$$\{x_0 + v \mid Av = 0\} \subseteq S$$

where $\{x_0 + v \mid Av = 0\}$ has infinite many elements, because $Ax = 0$ has infinite many solutions (lemma 1). Therefore, $Ax = b$ has infinite many solutions.

Therefore, $Ax = b$ either has no solution at all, or has infinite many solutions.

Problem 4.

Prove that after each of the three types of row operations, the rows of the new matrix are still linearly independent. Let $r_1, r_2, r_3, ..., r_m$ be the rows of A.

Type 1. Multiplying a row by a nonzero number.

Suppose that after one type 1 row operation, the ith row of the new matrix is kr_i ($k \neq 0$).

Suppose that the rows of the new matrix are linearly dependent. Then, $\exists a_1, a_2, a_3, ..., a_m \in \mathbb{R}$, not all 0, such that

$$a_1 r_1 + a_2 r_2 + ... + a_i(kr_i) + ... + a_m r_m = 0$$

Let

$$b_p = \begin{cases} a_p & p = 1, 2, ..., m \text{ and } p \neq i \\ ka_p & p = i \end{cases}$$

Then

$$b_1 r_1 + b_2 r_2 + ... + b_m r_m = 0$$

where $b_1, ..., b_m$ are not all 0 (because $a_1, ..., a_m$ are not all 0 and $k \neq 0$). This contradicts with that the rows of A are linearly independent.

Therefore, the rows of the new matrix after a finite number of type 1 row operations are still linearly independent.

Type 2. Adding a multiple of one row to another row.

Suppose that after one type 2 row operation, the ith row of the new matrix is $r_i + kr_j$ ($i \neq j, k \in \mathbb{R}$).

Suppose that the rows of the new matrix are linearly dependent. Then, $\exists a_1, a_2, a_3, ..., a_m \in \mathbb{R}$, not all 0, such that

$$a_1 r_1 + a_2 r_2 + ... + a_i(r_i + kr_j) + ... + a_m r_m = 0 \tag{1}$$

Let

$$b_p = \begin{cases} a_p & p = 1, 2, ..., m \text{ and } p \neq j \\ a_j + a_i k & p = j \end{cases}$$

Then

$$b_1 r_1 + b_2 r_2 + ... + b_m r_m = 0 \tag{2}$$

If $b_1, b_2, ..., b_m$ are all 0, then

$$a_p = \begin{cases} b_p = 0 & p = 1, 2, ..., m \text{ and } p \neq j \\ b_j - a_i k = -a_i k = 0 & p = j \end{cases} \tag{3}$$

which contradicts with that $a_1, ..., a_m$ are not all 0. Therefore, $b_1, b_2, ..., b_m$ are not all 0. Then, Equ. 2 contradicts with that the rows of A are linearly independent.

Therefore, the rows of the new matrix after a finite number of type 2 row operations are still linearly independent.

Type 3. Switching two rows.

It is obvious that the rows of the new matrix after a finite number of type 3 row operations are still linearly independent.

In summary, if the rows of a matrix are linearly independent, then after a finite number of elementary row operations, the rows of the new matrix are still linearly independent.

Problem 5.

Prove that after each of the three types of row operations, the columns of the new matrix are still linearly independent. Let $r_1, r_2, r_3, ..., r_m$ be the rows of A and $c_1, c_2, c_3, ..., c_n$ be the columns of A.

Type 1. Multiplying a row by a nonzero number.

Suppose that after one type 1 row operation, the ith row of the new matrix A' is kr_i ($k \neq 0$). Suppose that the columns of the new matrix are linearly dependent. Then $\exists x \in \mathbb{R}^n$, $x \neq 0$, such that

$$\begin{aligned}
& A'x = 0 \\
\Rightarrow & \begin{cases} r_p x = 0 & p = 1, 2, ..., m \text{ and } p \neq i \\ kr_p x = 0 & p = i \end{cases} \\
\Rightarrow & \ r_p x = 0, \quad p = 1, 2, ..., m \quad (k \neq 0) \\
\Rightarrow & \ Ax = 0
\end{aligned}$$

which contradicts with that the columns of A are linearly independent.

Therefore, the columns of the new matrix after a finite number of type 1 row operations are still linearly independent.

Type 2. Adding a multiple of one row to another row.

Suppose that after one type 2 row operation, the ith row of the new matrix A' is $r_i + kr_j$ ($i \neq j, k \in \mathbb{R}$).

Suppose that the columns of the new matrix are linearly dependent. Then $\exists x \in \mathbb{R}^n$,

$x \neq 0$, such that

$$
\begin{aligned}
& A'x = 0 \\
\Rightarrow & \begin{cases} r_p x = 0 & p = 1, 2, ..., m \text{ and } p \neq i \\ (r_p + k r_j) x = 0 & p = i \end{cases} \\
\Rightarrow & \begin{cases} r_p x = 0 & p = 1, 2, ..., m \text{ and } p \neq i \\ r_p x = -k r_j x & p = i \end{cases} \\
\Rightarrow & \, r_p x = 0, \quad p = 1, 2, ..., m \quad (i \neq j) \\
\Rightarrow & \, Ax = 0
\end{aligned}
$$

which contradicts with that the columns of A are linearly independent.

Therefore, the columns of the new matrix after a finite number of type 2 row operations are still linearly independent.

Type 3. Switching two rows.

It is obvious that the columns of the new matrix after a finite number of type 3 row operations are still linearly independent.

In summary, if the columns of a matrix are linearly independent, then after a finite number of elementary row operations, the columns of the new matrix are still linearly independent.

Problem 6.

Problem 6(a)

For any two elements x and y of S_0 and two real numbers α and β,

$$
\begin{aligned}
2\alpha x + x_0 &= 2\alpha(x + x_0) + (1 - 2\alpha)x_0 \in S \quad (x + x_0 \in S, x_0 \in S) \\
2\beta y + x_0 &= 2\beta(y + x_0) + (1 - 2\beta)x_0 \in S \quad (y + x_0 \in S, x_0 \in S)
\end{aligned}
$$

Therefore,

$$\alpha x + \beta y + x_0 = \frac{1}{2}(2\alpha x + x_0) + \frac{1}{2}(2\beta y + x_0) \in S$$

Therefore, $\alpha x + \beta y \in S_0$. Therefore, S_0 is a subspace.

Problem 6(b)

The dimension of S is 3.

$$S = \begin{bmatrix} x_1 \\ x_1 - 2 \\ 2x_1 - 3 \\ x_4 \\ x_5 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ -3 \\ 0 \\ 0 \end{bmatrix}, \quad x_1, x_4, x_5 \in \mathbb{R}$$

Let $x_0 = \begin{bmatrix} 0 & -2 & -3 & 0 & 0 \end{bmatrix}^T$, then

$$S_0 = S - x_0 = \text{span}(\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix})$$

and the dimension of S_0 is 3. Thus, the dimension of S is 3.