STOR 614 - Linear Programming, Spring 2019
Homework No. 2

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Problem 1.

Suppose that P contains a line $Q = \{x + \lambda d \mid \lambda \in \mathbb{R}\}$, where $x \in \mathbb{R}, d \in \mathbb{R}, d \neq 0$. Because the set $\{a_1, \ldots, a_m\}$ contains n linearly independent vectors, there exists a_k, such that $a_k^T d \neq 0$. Therefore,

$$\left\{a_k^T q \mid q \in Q\right\} = \left\{a_k^T x + \lambda a_k^T d \mid \lambda \in \mathbb{R}\right\} = \mathbb{R},$$

which contradicts with

$$a_k^T q \geq b_k, \text{ for all } q \in Q$$

Problem 2.

Basic feasible solutions:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Degenerate BFS:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

because it has 3 active constraints:

$$\begin{aligned} x_1 + x_2 &\leq 1 \\ 4x_1 + x_2 &\leq 2 \\ 2x_1 + x_2 &\leq 4/3. \end{aligned}$$

Problem 3.

$$\begin{array}{lllllll} \min & z = 3x_1 & +x_2 & & & & \\ \text{s.t} & x_1 & & -\beta_1 & & = 3 \\ & x_1 & +x_2 & & +\beta_2 & = 4, \\ & 2x_1 & -x_2 & & & = 3, \\ & x_1, & x_2, & \beta_1, & \beta_2 & \geq 0 \end{array}$$

Problem 4.
Problem 4(a)

True.

Let $A \in \mathbb{R}^{m\times n}$. Because A has full row rank, $m \leq n$. Let x_{B1} and x_{B2} be the bases of x. Let x_{N1} and x_{N2} be the corresponding collections of nonbasic variables, i.e.

$$x_{N1} = \{x_1, \dots, x_n\} \setminus x_{B1}$$
$$x_{N2} = \{x_1, \dots, x_n\} \setminus x_{B2}$$

Let n_0 be the number of zeros in x. Then,

$$\begin{aligned} & x_{B1} \neq x_{B2} \\ \Rightarrow \quad & x_{N1} \neq x_{N2} \\ \Rightarrow \quad & |x_{N1} \cup x_{N2}| > n - m \quad (|\cdot| \text{ denotes cardinality}) \\ \Rightarrow \quad & n_0 > n - m \end{aligned}$$

Therefore, x is degenerate.

Problem 4(b)

False.

Counter example: $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \end{bmatrix}$. $x = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ is a degenerate basic solution but has only one basis, $\begin{bmatrix} 1 \end{bmatrix}$.

Problem 5.
Problem 5(a)

Proposition 1. Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^n, b \neq 0$. If

1. $Ax = b$ has a solution x^*, $x_i^* > 0$, for $i = 1, \dots, n$
2. The columns of A are not linearly independent

then there exists a solution of $Ax = b$, x', such that

1. $x_i' \geq 0$, for $i = 1, \dots, n$,
2. The columns of A corresponding to nonzero entries of x' are linearly independent.

Proof. Assume the induction hypothesis that the proposition holds for all $n < N (N \geq 3)$. Consider $n = N$. Let d be a nontrivial solution of $Ax = 0$. Then, $\exists \lambda \in \mathbb{R}$, such that $y = (x^* + \lambda d)$ has at least one zero entry and no negative entries. Let $A' \in \mathbb{R}^{m\times N'}$ be the matrix containing the columns of A corresponding to nonzero entries of y $(N' < N)$. If the columns of A' are linearly independent, then $x' = y$. Otherwise, by induction hypothesis, there exists a solution of $A'x = b$, y', such that $y_i' \geq 0$, for $i = 1, \dots, N'$, and the columns of A' corresponding to nonzero entries of y' are linearly independent. Then, x' can be obtained by replacing the nonzero entries of y by corresponding entries of y'. $\square$

Proof for problem 5(a).

If $b = 0$, then 0 is a degenerate basic feasible solution of P, so $b \neq 0$.

Suppose that x is not a basic feasible solution, then the columns of A corresponding to nonzero entries of x are not linearly independent. By proposition 1, there exists $x' \in \mathbb{R}^n$, such that x' is a feasible solution, x' has more than $(n-m)$ zeros, and the columns of A corresponding to nonzero entries of x' are linearly independent. Therefore, x' is a degenerate basic feasible solution, which contradicts with that all basic feasible solutions are nondegenerate.

Problem 5(b) A counter example:

$$A = \begin{bmatrix} 2 & 0 \end{bmatrix}$$
$$b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

x has exactly 1 positive component but is not a basic feasible solution.