STOR 614
Homework Assignment No. 6

1. Use the dual simplex method to solve the following linear programs. Write down the dual BFS associated with each tableau.

   (a)
$$
\begin{array}{llrrrcr}
\max & z = & -x_1 & & -x_3 & & \\
\text{s.t.} & & -2x_1 & -x_2 & +x_3 & \leq & -5 \\
 & & -x_1 & +2x_2 & -2x_3 & \leq & -2 \\
 & & \multicolumn{3}{l}{x_1, x_2, x_3 \geq 0} & &
\end{array}
$$

   (b)
$$
\begin{array}{lllcr}
\max & z = & -3x_1 - 4x_2 - 2x_3 - x_4 - 5x_5 & & \\
\text{s.t} & & x_1 - 2x_2 - x_3 + x_4 + x_5 & \leq & -3, \\
 & & -x_1 - x_2 - x_3 + x_4 + x_5 & \leq & -2, \\
 & & x_1 + x_2 - 2x_3 + 2x_4 - 3x_5 & \leq & 4, \\
 & & x & \geq & 0
\end{array}
$$

2. Prove that exactly one of the following two statements holds for any given matrix $A \in \mathbb{R}^{m\times n}$. (A vector y satisfies $y > 0$ if all of its components are strictly positive, and it satisfies $y \neq 0$ if some of its components are nonzero.)

   (a) There exists $x \in \mathbb{R}^n$ such that $Ax = 0,\ x \geq 0,\ x \neq 0$.

   (b) There exists $p \in \mathbb{R}^m$ such that $p^T A > 0$.

3. Let a and $a_1, \cdots, a_m$ be fixed vectors in $\mathbb{R}^n$. Prove that the following two statements are equivalent.

   (a) $a^T x \leq \max_{i=1,\cdots,m} a_i^T x$ for all $x \geq 0$.

   (b) There exist nonnegative coefficients $\lambda_i, i = 1, \cdots, m$, such that

$$\sum_{i=1}^{m} \lambda_i = 1 \quad \text{and} \quad a \leq \sum_{i=1}^{m} \lambda_i a_i.$$

4. Consider the following LP.

$$
\begin{array}{lrrrrrrcr}
\max & z = & 3x_1 & +7x_2 & +5x_3 & & & & \\
\text{s.t} & & x_1 & +x_2 & +x_3 & +s_1 & & = & 50, \\
& & 2x_1 & +3x_2 & +x_3 & & +s_2 & = & 100, \\
& & x_1, & x_2, & x_3, & s_1, & s_2 & \geq & 0.
\end{array}
$$

At the end of simplex method we have the following optimal tableau.

z	x_1	x_2	x_3	s_1	s_2	rhs	Basic var
1	3	0	0	4	1	300	$z = 300$
0	0.5	0	1	1.5	-0.5	25	$x_3 = 25$
0	0.5	1	0	-0.5	0.5	25	$x_2 = 25$

$\max z; x, s \geq 0$

(a) What is the optimal solution for the LP, and what is the optimal objective function value?

(b) Write down the dual LP, and read off a dual optimal solution from the tableau. Does the dual have multiple optimal solutions?

(c) Suppose that the coefficient for x_1 in the objective function changes from 3 to $3+\Delta$. How does this change affect the simplex tableau with $\{x_3, x_2\}$ as the basis? For what range of Δ does $\{x_3, x_2\}$ continue to be an optimal basis? Write down an optimal solution and the optimal value for Δ in that range. Find an optimal solution and the optimal value for the case $\Delta = 4$.

(d) Suppose that the objective function changes to $\max z = 3x_1 + 7x_2 + (5+\Delta)x_3 + \Theta s_1$. Write down the new simplex tableau in which x_3 and x_2 are basic variables. For which Δ and Θ does that tableau show an optimal solution? Write down an optimal solution and a formula for the optimal value for Δ belonging to that range. Does $(\Delta, \Theta) = (2, 2)$ belong to that range?

(e) Now suppose that the right hand side for the first equation changes from 50 to $50+\Delta$ in the original LP. For what range of Δ does the basis $\{x_3, x_2\}$ continue to be optimal? What is the optimal value when Δ belongs to this range? Find an optimal solution and the optimal value for the case $\Delta = 100$.

5. The following linear program involves the making of four products — represented by x_1, x_2, x_3 and x_4 — maximizing profits in dollars, and subject to three resource constraints:

$$\begin{array}{rrcrcrcrcr}
\max z = & 3x_1 & + & x_2 & + & 4x_3 & + & x_4 & & \\
 & 6x_1 & + & 3x_2 & + & 5x_3 & + & 4x_4 & \leq & 25 \\
 & 3x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & \leq & 15 \\
 & 3x_1 & + & 4x_2 & + & 5x_3 & + & 2x_4 & \leq & 20 \\
 & x_1 \geq 0 & & x_2 \geq 0 & & x_3 \geq 0 & & x_4 \geq 0 & &
\end{array}$$

This LP was solved, resulting in the following simplex tableau:

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	**rhs**
1	0	2	0	1	1/5	0	3/5	?
0	1	−1/3	0	2/3	1/3	0	−1/3	?
0	0	0	0	−1	−2/5	1	−1/5	?
0	0	1	1	0	−1/5	0	2/5	?

(a) Give the basis, basis matrix A_B and its inverse A_B^{-1}, the basic cost vector c_B, and the basic feasible solution and its objective function value associated with this tableau.

(b) What is the largest value the objective coefficient of the second item (currently 1) can take for the current optimal basis to remain optimal? For that largest value, what is the associated optimal value and set of optimal solutions?

(c) What is the smallest value the objective coefficient of the third item (currently 4) can take for the current optimal basis to remain optimal? For that smallest value, what is the associated optimal value and set of optimal solutions?

(d) What is the largest value of the first resource (currently 25) for which the current basis is optimal? For values slightly larger than this value, what will the new optimal basis be?

(e) Suppose we can purchase a “package” of resources: 3 units of resource 1, 7 units of resource 2, and 4 units of resource 3, at a total cost of \$5. Should you purchase any of this package? Why or why not?