Real Analysis

The underlying space for the real analysis is the set of real numbers

1. Axioms of real numbers

- The set $\mathbb{N}$ of Natural Numbers
  - Peano Axioms
    (a) 1 belongs to $\mathbb{N}$
    (b) If n belongs to $\mathbb{N}$, then its successor $n+1$ belongs to $\mathbb{N}$
    (c) 1 is not successor of any elements in $\mathbb{N}$
    (d) If n and m have the same successor, then $n = m$
    (e) A subset of $\mathbb{N}$ which contains 1, and which contains $n+1$ whenever it contains n, must equal $\mathbb{N}$[This is the basis of mathematical induction]
- The set $\mathbb{Q}$ of Rational Numbers
  - Algebraic Number: A number satisfies a polynomial equation

$$c_n x^n + c_{n-1}x^{n-1} + ... + c_1 x + c_0 = 0$$

where the coefficients $c_0, c_1, ..., c_n \in \mathbb{Z}$, $c_n \neq 0$ and $n \geq 1$
Rational number are always algebraic number

  - **Rational Zeros Theorem**
Suppose $c_0, ..., c_n \in \mathbb{Z}$, and $r \in \mathbb{Q}$ satisfying the polynomial equation

$$c_n x^n + ...c_1 x + c_0 = 0 \tag{1}$$

where $n \geq 1$,$c_n \neq 0$ and $c_0 \neq 0$. Let $r = \frac{c}{d}$ where c, d are integers having no common factors and $d \neq 0$.
Then c divides c_0 and d divides c_n
Remark: The only rational candidates for solutions of (1) have the form $\frac{c}{d}$ where c divides c_0 and d divides c_n
Proof

$$c_n\left(\frac{c}{d}\right)^n + ... + c_1\left(\frac{c}{d}\right) + c_0 = 0 \tag{2a}$$

$$c_n c^n + ... + c_1 c d^{n-1} + c_0 d^n = 0 \tag{2b}$$

(a) Solve (2b) for $c_0 d^n$

$$c_0 d^n = -c\left[c_n c^{n-1} - ... - c_1 d^{n-1}\right]$$

(b) Solve (2b) for $c_n c^n$

$$c_n c^n = -d\left[c_{n-1}c^{n-1} + ... + c_1 c d^{n-2} + c_0 d^{n-1}\right]$$

□

  - Corollary
Consider the polynomial equation

$$x^n + c_{n-1}x^{n-1} + ... + c_1 x + c_0 = 0$$

where the coefficients $c_0, ..., c_{n-1} \in \mathbb{Z}$ and $c_0 \neq 0$. Any rational solution of this equation must be an integer that divides c_0

– Properties of $\mathbb{Q}$

(a) **A1.** *associative laws* $a + (b + c) = (a + b) + c$, $\forall a, b, c$
(b) **A2.** *commutative laws* $a + b = b + a$, $\forall a, b$
(c) **A3.** $a + 0 = a$, $\forall a$
(d) **A4.** $\forall a$, $\exists -a$ such that $a + (-a) = 0$
(e) **M1.** *associative laws* $a(bc) = (ab)c$, $\forall a, b, c$
(f) **M2.** *commutative laws* $ab = ba$, $\forall a, b$
(g) **M3.** $a \cdot 1 = a$ $\forall a$
(h) **M4.** $\forall a \neq 0$, $\exists a^{-1}$ such that $aa^{-1} = 1$
(i) **DL** *distributive law* $a(b + c) = ab + ac$, $\forall a, b, c$
Remark: a system that has more than one elements satisfies these nine properties is called a **filed**

– Order structure of $\mathbb{Q}$

(a) **O1.** Give a and b, either $a \leq b$ or $b \leq a$
(b) **O2.** If $a \leq b$ and $b \leq a$, then $a = b$
(c) **O3.** *transitive law* If $a \leq b$ and $b \leq c$, then $a \leq c$
(d) **O4.** If $a \leq b$ then $a + c \leq b + c$
(e) **O5.** if $a \leq b$ and $0 \leq c$, then $ac \leq bc$
Remark: A filed with an ordering satisfying properties $O1$ through $O5$ is called an **Ordering Filed**

- The set $\mathbb{R}$ of Real Numbers

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- The following are consequences of the field properties:

(a) $a + c = b + c$ $a = b$
(b) $a \cdot 0 = 0$, $\forall a$
(c) $(-a)b = -ab$, $\forall a, b$
(d) $(-a)(-b) = ab$, $\forall a, b$
(e) $ac = bc$, $c \neq 0$ implies $a = b$
(f) $ab = 0$ implies $a = 0$ or $b = 0$
$\forall a, b, c \in \mathbb{R}$

Proof. (a)
(b)
(c)
(d)
(e)
(f)
(g)

□

- The following are consequences of the properties of an ordered field:

(a) If $a \leq b$ then $-b \leq -a$
(b) If $a \leq b$ and $c \leq 0$, then $bc \leq ac$

(c) If $0 \leq a$ and $0 \leq b$ then $0 \leq ab$

(d) $\forall a,\ 0 \leq a^2$

(e) $0 < 1$

(f) If $0 < a$, then $0 < a^{-1}$

(g) If $0 < a < b$, then $0 < b^{-1} < a^{-1}$
$\forall a, b, c \in \mathbb{R}$

Proof. (a)

□

- **distance between** a **and** b: $dist(a, b) = |a - b|$
- Theorem

(a) $|a| \geq 0,\ \forall a \in \mathbb{R}$

(b) $|ab| = |a| \cdot |b|,\ \forall a, b \in \mathbb{R}$

(c) $|a + b| \leq |a| + |b|,\ \forall a, b \in \mathbf{R}$

Proof. (a)

(b)

(c)

□