Numerical solution for eigenvalue of Helmoltz eqution in a annulus domain (outer radius 1 and inner radius 0.5) with Neumann boundary condition on [0,10]

Zhengjie Xiong

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Problem

According to the textbook, the solution we are looking for is

$$f = c_1 J_m(\sqrt{\lambda} r) + c_2 Y_m(\sqrt{\lambda} r) \tag{1}$$

where J_m,Y_m are Bessel functions of the first and the second kind with order m, c_1, c_2 are constants. Plug in

$$f'(\frac{1}{2}) = f'(1) = 0 \tag{2}$$

and we have

$$c_1\sqrt{\lambda}J'_m(\frac{1}{2}\sqrt{\lambda}) + c_2\sqrt{\lambda}Y'_m(\frac{1}{2}\sqrt{\lambda}) = 0 \tag{3}$$

$$c_1\sqrt{\lambda}J'_m(\sqrt{\lambda}) + c_2\sqrt{\lambda}Y'_m(\sqrt{\lambda}) = 0 \tag{4}$$

Cancel coefficients c_1 and c_2 we have

$$J'_m(\sqrt{\lambda})Y'_m(\frac{1}{2}\sqrt{\lambda}) = J'_m(\frac{1}{2}\sqrt{\lambda})Y'_m(\sqrt{\lambda}) \tag{5}$$

It is difficult to find an analytic solution for equation (4), therefore we want to numerically find $\sqrt{\lambda}$ and corresponding m on the interval $[0, 10]$ such that

$$g = \left| J'_m(\sqrt{\lambda})Y'_m(\frac{1}{2}\sqrt{\lambda}) - J'_m(\frac{1}{2}\sqrt{\lambda})Y'_m(\sqrt{\lambda}) \right| \to 0 \tag{6}$$

Solution

First to apply monitor function ONLY and there are six rough solutions (local minimum):

```
sol=
    4.81
    5.21
    6.41
    7.11
    8.81
    9.61
```

Next to apply monitor function, Chebyshev polynomial and companion matrix to find accurate solutions and check how they satisfy equation (6):

```
sol1=
    4.656596861260596
g(sol1)=
    0.028233260617185 (m=0)

sol2=
    5.157587746541136
g(sol2)=
    8.130546683048079e-06 (m=4)

sol3=
    6.449975656112474
g(sol3)=
    0.008373345005586 (m=1)

sol4=
    7.062879796584355
g(sol4)=
    2.026122698505634e-05 (m=2)

sol5=
    8.835673257127437
g(sol5)=
    4.686173672266425e-05 (m=4)

sol6=
    9.639056147052665
g(sol6)=
    3.726689410363795e-04 (m=8)
```

Notice $g(sol1)$ only has accuracy 10^-2, $g(sol3)$ has has accuracy 10^-3 while other solutions have accuracy at least 10^-4.