Outline

Contents

Key Questions. *Among all reversible transformations, is there one that minimizes the reconstruction error for a given number of retained coefficients? What are the conditions under which this minimum error is achieved?*

In this lecture notes we will further explore the relationship between the transform coefficients and signal (image) statistics. In particular, it is shown that the **optimal transformation results in decorrelated coefficients**. Take home points from this discussion are:

- The Karhunen-Loeve transform is optimal in the sense of resulting in decorrelated coefficients for a given random field (collection of picture samples).
- The KLT diagonalizes the covariance matrix representing the zero-mean random field (corresponding to the collection of pictures).
- KLT is a Unitary Transform, and so are the Fourier, Cosine, and Sine transforms.
- KLT minimizes the *basis restriction error.* That is, given any fixed number of M coefficients to keep, the basis images of the KLT transform corresponding to the largest M eigenvalues of the covariance matrix of the random field, pack the most energy among all Unitary Transforms.
- the 1-D DFT transform matrix is the KLT for a circulant matrix, i.e., it diagonalizes the circulangt matrix. This helps explain why you can do the circular convolution in the Fourier domain by multiplying the corresponding frequency coefficients and taking the inverse transform.

- Similarly, the 2D DFT matrix (see the detailed notes) diagonalizes the doubly circulant matrix.
- Finally, the reason the Cosine transform works better than Fourier transform for compression is that it approximates the characteristics of KLT for a wide range of image signals.

1 KL Transform

1.1 Introduction

Notations/Recap

$f(x, y)$: continuous image $f(m, n)$: sampled image, discrete picture, an array of numbers. $[f]_{N\times N}$: An $N \times N$ array. $[P]$ and $[Q]$ are deterministic and nonsingular. A general 2D transform can then be written as:

$$[F] = [P][f][Q]$$

$$[f] = [P'][F][Q'],\ [P'] = [P]^{-1}, [Q'] = \ [Q]^{-1}$$

$$f(m, n) = \sum_{u=0}^{N-1}\sum_{v=0}^{N-1} F(u, v)P'(m, u)Q'(v, n),\ m, n : 0, 1, \ldots , N-1$$

Rewriting,

$$[f] = \sum_{u,v}[\Phi^{(u,v)}]F(u, v) \tag{1}$$

where $[\Phi^{(u,v)}]$ is an $N \times N$ matrix, and

$$[\Phi^{(u,v)}]_{m,n} = \phi^{(u,v)}(m, n) = P'(m, u)Q'(v, n)$$

It would be convenient to consider $N \times N$ matrices as vectors in an N^2-dimensional space.

e.g., $[Q] = [Q(0, 0), \cdots , Q(0, N-1), \cdots , Q(N-1, N-1)]^T$.

Define the *complex dot product* of two vectors as,

$$[Q] \cdot [\Gamma] = \sum_{m,n} Q(m, n)\Gamma^*(m, n)$$

Consider again

$$[f] = \sum_{u,v}[\Phi^{(u,v)}]F(u, v)$$

The picture matrix $[f]$ can be considered as a linear combination of N^2 matrices $[\Phi^{(u,v)}]$, for $u = 0, 1, \ldots , N-1; v = 0, 1, \ldots , N-1$. These N^2 matrices form an orthonormal set provided

$$[\Phi^{(u,v)}] \cdot [\Phi^{(r,s)}] = \begin{cases} 0 & : u \neq r, v \neq s \\ 1 & : u = r \ \&\ v = s \end{cases} \tag{2}$$

Assuming orthonormality,

$$[f] \cdot [\Phi^{(r,s)}] = F(r,s) \tag{3}$$

1.2 Basis Images

Matrix expansion using orthonormal matrices

Summary
Given a set of N^2 orthonormal matrices $[\Phi^{(u,v)}]$, we can expand any arbitrary $N \times N$ matrix (picture) $[f]$ as,

$$[f] = \sum_{u,v} F(u,v)[\Phi^{(u,v)}]$$

where the coefficients of expansion are given by,

$$F(u,v) = [f] \cdot [\Phi^{(u,v)}]$$

Example

Consider $[F] = [P][f][Q]$ with $[P] = [Q] = [A]$,

$$[f] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad [A] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Then,

$$\begin{aligned} [F] &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -1 \\ -2 & 0 \end{bmatrix} \end{aligned}$$

$$[\Phi^{(u,v)}]_{m,n} = P'(m,u)Q'(v,n)$$

$$[\Phi^{(u,v)}]_{N \times N} = (\text{u-th column of P'})_{N \times 1} \times (\text{v-th row of Q'})_{1 \times N}$$

$[A]$ is orthogonal, $\quad A' = A^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = A$

$$\begin{aligned} [\Phi^{(0,0)}] &= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} & &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ [\Phi^{(0,1)}] &= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} & &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \\ [\Phi^{(1,0)}] &= \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} & &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \\ [\Phi^{(1,1)}] &= \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} & &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned} \tag{4}$$

$[\Phi^{(u,v)}]$ are also called the *basis images*.

Verify that $F(u,v) = [f] \cdot [\Phi^{(u,v)}]$ $\rightarrow$ projection of $[f]$ onto the space spanned by the basis images.

1.3 The KLT

K-L Transform for Discrete Pictures

Let $R(m,n,p,q) = \mathcal{E}\{f(m,n)f(p,q)\}$: autocorrelation function of $[f]$. For zero-mean random fields, the orthonormal matrices $[\Phi^{(u,v)}]$ that result in uncorrelated coefficients $F(u,v)$ satisfy the equation:

$$\sum_{p,q} R(m,n,p,q)\phi^{(u,v)}(p,q) = \gamma_{u,v}\phi^{(u,v)}(m,n) \tag{5}$$

where

$$\gamma_{u,v} = \mathcal{E}\{|F(u,v)|^2\} \tag{6}$$

The matrices $[\Phi^{(u,v)}]$ are called the *Eigenmatrices* or the *basis matrices* of $R(m,n,p,q)$.

Proof:

$$\begin{aligned} F(u,v) &= [f] \cdot [\Phi^{(u,v)}] \\ &= \sum_{m,n} f(m,n)\phi^{*(u,v)}(m,n) \end{aligned} \tag{7}$$

$$\mathcal{E}\{F(u,v)\} = \sum_{m,n} \mathcal{E}\{f(m,n)\}\phi^{*(u,v)}(m,n)$$

but $\mathcal{E}\{f(m,n)\} = 0 \Rightarrow \mathcal{E}\{F(u,v)\} = 0,\ \forall u,v$. Suppose $F(u,v)$ are uncorrelated.

$$\mathcal{E}\{F(u,v)F^*(u',v')\} = \mathcal{E}\{F(u,v)\}\mathcal{E}\{F^*(u',v')\}, u \neq u', v \neq v'.$$

$$\Rightarrow \mathcal{E}\{F(u,v)F^*(u',v')\} = 0$$

$$[f] = \sum_{u,v} F(u,v)[\Phi^{(u,v)}]$$

$$f(m,n) = \sum_{u,v} \phi^{(u,v)}(m,n), \quad m,n : 0,1,\ldots,N-1 \tag{8}$$

$$\begin{aligned} \mathcal{E}\{f(m,n)F^*(u',v')\} &= \sum_{u,v} \mathcal{E}\{F(u,v)F^*(u',v')\}\phi^{(u,v)}(m,n) \\ &= \boxed{\mathcal{E}\{|F(u',v')|^2\}\phi^{(u',v')}(m,n)} \end{aligned} \tag{9}$$

from (7),

$$F^*(u',v') = \sum_{m,n} f(m,n)\phi^{(u',v')}(m,n) \tag{10}$$

$$\begin{aligned}\mathcal{E}\{f(m,n)F^*(u',v')\} &= \sum_{p,q} \mathcal{E}\{f(m,n)f(p,q)\}\phi^{(u',v')}(p,q) \\ &= \boxed{\sum_{p,q} R(p,q,m,n)\phi^{(u',v')}(p,q)}\end{aligned} \tag{11}$$

Comparing equations (9) and (11), we get

$$\boxed{\sum_{p,q} R(m,n,p,q)\phi^{(u',v')}(p,q) = \mathcal{E}\{|F(u',v')|^2\}\phi^{(u',v')}(m,n)}$$

This completes the proof.

Vector Notation

$$[f] = \begin{pmatrix} f(0,0) & \cdots & f(0,N-1) \\ \vdots & \ddots & \\ f(N-1,0) & \cdots & f(N-1,N-1) \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N^2-1} \end{pmatrix}$$

$$\begin{pmatrix} \phi^{(uv)}(0,0) & \cdots & \phi^{(uv)}(0,N-1) \\ \vdots & \ddots & \\ \phi^{(uv)}(N-1,0) & \cdots & \phi^{(uv)}(N-1,N-1) \end{pmatrix} = \begin{pmatrix} \phi^s_0 \\ \vdots \\ \phi^s_{N-1} \\ \vdots \\ \phi^s_{N^2-1} \end{pmatrix}$$

$$(u,v) \to s$$

$$R(m,n,p,q) \to N^2 \times N^2 \text{ matrix } \{K(i,j)\} \text{ where}$$

$$K(i,j) = \mathcal{E}\{f_i f_j\},\ i,j : 0,1,\ldots,N^2-1.$$

We can re-write

$$\sum_{p,q} R(m,n,p,q)\phi^{(u,v)}(p,q) = \gamma_{u,v}\phi^{(u,v)}(m,n)$$

$$\sum_{j=0}^{N^2-1} K(i,j)\phi^s_j = \gamma_s\phi^s_i, \quad i = 0,1,\ldots,N^2-1$$

$$\begin{pmatrix} K(0,0) & \cdots & K(0,N^2-1) \\ K(1,0) & \cdots & K(1,N^2-1) \\ \vdots & \ddots & \\ K(N^2-1,0) & \cdots & K(N^2-1,N^2-1) \end{pmatrix} \begin{pmatrix} \phi_0^s \\ \phi_1^s \\ \vdots \\ \phi_{N^2-1}^s \end{pmatrix} = \gamma_s \begin{pmatrix} \phi_0^s \\ \phi_1^s \\ \vdots \\ \phi_{N^2-1}^s \end{pmatrix}$$

$$[K]\overrightarrow{\phi^s} = \gamma_s \overrightarrow{\phi^s}$$

The K-L Transform

$$\boxed{[K]\overrightarrow{\phi^s} = \gamma_s \overrightarrow{\phi^s}} \tag{12}$$

$\vec{\phi}^s \rightarrow$ eigenvectors of $[K]$, and $\gamma_s \rightarrow$ Eigenvalues of $[K]$.

The KL transform of $[f]$ is defined as:

$$\vec{F} = [\Phi^{*T}]\vec{f} = \begin{pmatrix} \vec{\phi}^{0^{*T}} \\ \vec{\phi}^{1^{*T}} \\ \vdots \\ \vec{\phi}^{N^2-1^{*T}} \end{pmatrix} \vec{f} \tag{13}$$

$$\vec{f} = [\Phi]\vec{F} = \sum_{j=0}^{N^2-1} F(j)\vec{\phi}^j \tag{14}$$

where

$$[\Phi] = \begin{pmatrix} \vec{\phi}^1 & \vec{\phi}^2 & \ldots & \vec{\phi}^{N^2-1} \end{pmatrix} \tag{15}$$

Notes:

- KLT Diagonalizes the covariance matrix.

$$[\Phi^{*T}]\ [K]\ [\Phi] = \Lambda = \text{Diagonal } \{\lambda^s\}$$

- KLT is a *Unitary Transform*:

$$[\Phi]^{-1} = [\Phi^{*T}]$$

1.4 Unitary transforms and properties

Unitary Transforms: Properties

Energy conservation and rotation

$$\vec{F} = [A]\vec{f} \Rightarrow ||\vec{F}||^2 = ||f||^2$$

Proof:

$$||F||^2 = F^{*T}F = f^{*T}A^{*T}Af = ||f||^2$$

$\rightarrow$ rotation of the basis coordinates in some N-dimensional space. The new components are the projections of f onto the new basis.

Unitary Transforms: Properties

Energy compaction
Variance of transform coefficients: Most energy in fewer coefficients.

$$\text{mean: } \mu_F = A\mu_f$$

$$\begin{aligned} \text{Covariance: } [C_F] &= \mathcal{E}\{(F-\mu_F)(F-\mu_G)^{*T}\} \\ &= A[C_f]A^{*T} \end{aligned} \tag{16}$$

Decorrelation of transform coefficients
Transform coefficients tend to be less correlated. KLT is optimum in the sense that it packs maximum energy in a given number of transform coefficients, and the coefficients are uncorrelated.

Unitary Transforms: Properties

Separable unitary transforms

$$F(u,v) = \sum_{u,v} f(m,n)a_{uv}(m,n)$$

Separable if $a_{uv}(m,n) = a_u(m)b_v(n) = a(u,m)b(v,n)$

$$a_u(m), u = 0,1,\cdots,N-1,\ b_v(n), v = 0,1,\cdots,N-1$$

$\rightarrow$ 1-D orthonormal basis vectors. $[A] = \{a(u,m)\},\ \ [B] = \{b(v,n)\} \rightarrow$ *Unitary*

Unitary transform: example

Consider zero-mean, first order, stationary Markov sequence of N elements.

$$R(n_1,n_2) = R(n_1-n_2) = R(n) = \rho^{|n|},\ \ |\rho| < 1\ \forall n$$

$$R = \begin{pmatrix} 1 & \rho & \rho^2 & \rho^3 & \cdots & \rho^{N-1} \\ \rho & & & & & \\ \vdots & & & & & \\ \vdots & \cdots & & & & \rho \\ \rho^{N-1} & \cdots & & & \rho & 1 \end{pmatrix}$$

$\rightarrow$ Toeplitz matrix.

Note: The covariance and autocorrelation matrices of any stationary sequence are Toeplitz. For the above R,

$$\lambda_k = \frac{1-\rho^2}{1-2\rho\cos\omega_k+\rho^2},$$

and

$$\phi^k(m) = \phi(m,k) = \left(\frac{2}{N+\lambda_k}\right)^{1/2} \sin\left(w_k(m+1-\frac{N+1}{2}) + \frac{k+1}{2}\pi\right)$$

$$0 \leq m,\ k \leq N-1$$

where ω_k are the positive roots of

$$\tan(n\omega) = \frac{(1-\rho^2)\sin\omega}{\cos\omega - 2\rho + \rho^2\cos\omega}, \quad N \text{ is even.}$$

(a similar result holds for N odd.) $\rightarrow$ similar to DCT coefficients!

1.5 Separable autocorrelation

Separable autocorrelation

$$R(m,n;m',n') = R_1(m,m')R_2(n,n')$$

$$\Rightarrow [K] = \mathcal{E}\{ff^T\} = [R_1] \otimes [R_2]$$

where $[R_1] \otimes [R_2] = \{R_1(m,m')[R_2]\}$ is the Kronecker product of $[R_1]$ with $[R_2]$.

Let

$$[A]\ [R_1]\ [A^{*T}] = \Lambda_1, \quad [B]\ [R_2]\ [B^{*T}] = \Lambda_2$$

Then

$$[\Phi] = [A] \otimes [B]$$

and the KLT of f is

$$\vec{F} = [\Phi^{*T}]\vec{f} = [A^{*T} \otimes B^{*T}]\vec{f}$$

and in matrix form,

$$[F] = [A^{*T}]\ [f]\ [B^*], \text{and}$$

$$[f] = [A]\ [F]\ [B^T]$$

Separable $[R]$ $\rightarrow$ one need to solve the eigenvalue problem for $N \times N$ matrices rather than $(N^2-1) \times (N^2-1)$. $N \times N$ requires $\mathcal{O}(N^3)$ operations.

1.6 KLT properties

KLT Properties

Uncorrelated coefficients

$$\mathcal{E}\{F(u,v)F(u',v')\} = \lambda_{uv}\delta(u-u', v-v')$$

Among all the unitary transforms $\vec{F} = \Phi\vec{f}$, the KLT packs the maximum average energy in $M \leq N^2 - 1$ samples of $[F]$.

KLT Properties

Basis restriction
Let $\vec{F} = [A]\vec{f}$. Keeping only M elements of F and reconstructing,

$$\overrightarrow{f_M} = [B]\overrightarrow{F_M}$$

where

$$F_M(j) = \begin{cases} F(j) & : 0 \leq j \leq M-1 \\ 0 & : \text{ otherwise} \end{cases}$$

Let

$$\begin{aligned} e_M &= \frac{1}{N}\mathcal{E}\{\sum_{n=0}^{N-1} |f(n) - f_M(n)|^2\} \\ &= \frac{1}{N}\text{Trace}(\mathcal{E}\{(f - f_M)(f - f_M)^{*}T\}) \end{aligned} \tag{17}$$

e_M is called the *basis restriction error*

KLT Properties

Theorem 1. *e_M is minimized when $[A] = [\Phi^{*T}]$, $[B] = [\Phi]$, and $[A]$ $[B] = [I]$ where the columns of $[\Phi]$ are arranged according to the decreasing order of the eignevalues of $[R] = \mathcal{E}\{ff^T\}$.*

1.7 Singular Value Decomposition

Q: Given just one image, what is the optimal transform?

Ans: Singular Value Decomposition (SVD) Ref: "SVD image coding" by Andrew & Patterson, IEEE Trans. Communications, April 1976, pp. 425-432. We can decompose, ($[f]$ is the image matrix.)

$$[f^T][f] = [U]\ [\Lambda]\ [U^T]$$

$$[f][f^T] = [V]\ [\Lambda]\ [V^T]$$

$$\begin{aligned} [f] &= [U]\ [\Lambda]^{1/2}[V^T] \\ &= \sum_{i=1}^{m} \lambda_i^{1/2}\vec{u_i}\vec{v_i}^T \end{aligned} \tag{18}$$

m is the rank of $[f]$. $\vec{u_i}$ and $\vec{v_i}$ are column vectors of the orthogonal matrices $[U]$ and $[v]$, respectively.

Properties of SVD

1. Once $\vec{v_i}$ are known, eigenvectors $\vec{u_i}$ can be determined as,

$$\vec{u_i} = \frac{1}{\sqrt{\lambda_i}}[f]\vec{v_i}$$

   and orthonormality of $\{\vec{v_i}\} \Rightarrow$ orthonormality of $\{\vec{u_i}\}$

2. SVD, in general, is *not* a unitary transform.

3. $$[f_n] = \sum_{i=1}^{n<m} \lambda_i^{1/2} \vec{u_i}\vec{v_i}^T$$

   is the best least-squares rank-n approximation of $[f]$ if λ_i are in decreasing order of magnitude. The LSE is given by,

$$e_n = \sum_{p,q} |f(p,q) - f_n(p,q)|^2 = \sum_{i=n+1}^{m=rank\ [f]} \lambda_i$$

2 2D Fourier transform and properties

2.1 Basis matrices of 1D and 2D DFT

Fourier transform revisited

The *basis matrices* of the Fourier transform can be written as:

$$\phi^{uv}(m,n) = \frac{1}{N}\exp(j\frac{2\pi}{N}(mu+nv))$$

$\{[\Phi^{uv}]\} \rightarrow$ basis matrices. (recall $[f] = \sum_{u,v} F(u,v)[\Phi^{uv}]$)

1D DFT

$$\begin{aligned}[\Phi] &= \{\frac{1}{\sqrt{N}}\exp(j\frac{2\pi}{N}kn)\},\ 0 \le k,n \le N-1 \\ &= [\ \vec{\phi_0}\quad \vec{\phi_1}\quad \cdots\quad \vec{\phi}_{N-1}]\end{aligned} \tag{19}$$

where $\vec{\phi_k}$ =columns of $[\Phi] = \{\frac{1}{\sqrt{N}}\exp(j\frac{2\pi}{N}kn)\}^T$

1D DFT Poperties

1. The basis matrices of the 1D DFT are the orthonormal eigenvectors of any circulant matrix, i.e., if $[H]$ is circulant,

$$\boxed{[H]\vec{\phi_k} = \lambda_k\vec{\phi_k}}$$

where λ_k are the eigenvalues of $[H]$, and

$$\lambda_k = \sum_{l=0}^{N-1} h(l) \exp(-j\frac{2\pi}{N}kl), 0 \leq k \leq N-1$$

$\rightarrow$ DFT of the first column of $[H]$.

Proof:

$$[H]_{m,n} = h(m-n) = h((m-n) \text{ modulo } N), 0 \leq m, n \leq N-1$$

$$\overrightarrow{\phi_k} = \{\frac{1}{\sqrt{N}} \exp(j\frac{2\pi}{N}kn)\}^T, 0 \leq k \leq N-1$$

Consider the m-th element,

$$([H]\phi_k)_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} h(m-n) \exp(j\frac{2\pi}{N}kn)$$

substituting

$$l = m-n; n = 0 \Rightarrow l = m; n = N-1 \Rightarrow l = m-N+1 \rightarrow m+1;$$

$$\sum_{n=0}^{N-1} \rightarrow \sum_{l=m-N+1}^{-1} + \sum_{l=0}^{N-1} - \sum_{l=m+1}^{N-1}$$

$$\exp(j\frac{2\pi}{N}kn) \rightarrow \exp(j\frac{2\pi}{N}km) \exp(-j\frac{2\pi}{N}kl)$$

$$\begin{aligned} ([H]\phi_k)_m &= \tfrac{1}{\sqrt{N}} \exp(j\tfrac{2\pi}{N}km)\{ \sum_{l=0}^{N-1} h(l) \exp(-j\tfrac{2\pi}{N}kl) \\ &+ \sum_{l=-N+m+1}^{-1} h(l) \exp(-j\tfrac{2\pi}{N}kl) - \sum_{l=m+1}^{N-1} h(l) \exp(-j\tfrac{2\pi}{N}kl) \} \end{aligned}$$

The second and the third terms cancel (recall: periodic); and the first term under the summation is λ_k by definition. Hence,

$$([H]\phi_k)_m = \lambda_k \phi_k(m)$$

$$\boxed{[H]\overrightarrow{\phi_k} = \lambda_k \overrightarrow{\phi_k}}$$

where λ_k, the eigenvalues of $[H]$, are the DFT of the first column of $[H]$!

$$\boxed{\lambda_k \triangleq \sum_{l=0}^{N-1} h(l) \exp(-j\frac{2\pi}{N}kl), 0 \leq k \leq N-1}$$

2.2 Circulant matrices and 1D DFT

Circulant matrices and DFT

2. *Any circulant matrix can be diagonalized by the DFT.*

$$\boxed{[H]\overrightarrow{\phi_k} = \lambda_k \overrightarrow{\phi_k}}$$

$$[H][\Phi]_{N\times N} = \Lambda\ [\Phi]_{N\times N}$$
$$[\Phi]^*[H]\ [\Phi] = \Lambda$$

Circular convolution theorem

3. *Circular convolution theorem:* The DFT of the circular convolution of two sequences is equal to the product of their DFTs, i.e.,

$$g(n) = \sum_{k=0}^{N-1} h(n-k)f(k), \quad 0 \le n \le N-1$$

$$\Rightarrow \text{DFT}\{g(n)\}_N = \text{DFT}\{h(n)\}_N\ \text{DFT}\{f(n)\}_N$$

where $\text{DFT}\{f(n)\}_N$ is the N-point DFT of the sequence $f(n)$.

2.3 2D DFT

2D DFT

4. Consider the 2D DFT, and working with rowscanning the images to column vectors, $\vec{F} \xleftrightarrow{2D-DFT} \vec{f}$. Using the notations from equations (13) - (15), the 2D DFT matrices can be written as:

$$\vec{F} = [\mathcal{F}_{2D}^{*T}]\ \vec{f} \text{ where } [\mathcal{F}_{2D}^{*T}] = \begin{pmatrix} \vec{\phi}_{0^*}^{\,T} \\ \vec{\phi}_{1^*}^{\,T} \\ \vdots \\ \vec{\phi}_{N^2-1^*}^{\,T} \end{pmatrix} \tag{20}$$

You can verify that

$$[\mathcal{F}_{2D}] = [\Phi]_{N\times N} \otimes [\Phi]_{N\times N} \tag{21}$$

where

$$[\Phi]_{N\times N} = \left\{ \frac{1}{\sqrt{N}} \exp\left(j\frac{2\pi}{N}kn \right) \right\}, \quad 0 \le k, n \le N-1 \tag{22}$$

is the 1-D DFT matrix defined earlier in (19).

5. Symmetric: $[\mathcal{F}_{2D}^T] = [\mathcal{F}_{2D}]$, and

$$[\mathcal{F}_{2D}^{-1}] = [\mathcal{F}_{2D}^*] = [\Phi^*] \otimes [\Phi^*] \tag{23}$$

6. *The columns of the 2D DFT matrix $[\mathcal{F}_{2D}]$ are the orthonormal eigenvectors of any doubly block circulant matrix (DBC).*

   If $\mathcal{D}$ is doubly block-circulant, then

$$\mathcal{D}\mathcal{F}_{2D} = \Lambda\mathcal{F}_{2D} \tag{24}$$

   The eignevalues of $\mathcal{D}$ are given by the 2D DFT of the first column of $\mathcal{D}$.

7. Periodicity:

$$\begin{aligned} F(k+N, l+N) &= F(k,l), \quad \forall k, l. \\ f(m+N, n+N) &= f(m,n), \quad \forall m, n \end{aligned} \tag{25}$$

8. Conjugate symmetry: DFT is conjugate symmetric for real images.

$$F(k,l) = F^*(N-k, N-l), \quad 0 \le k, l \le N-1 \tag{26}$$

   Thus, though complex valued, there are only N^2 independent real elements in the DFT for a given $N \times N$ image. The remaining elements an be computed using the conjugate symmetry property.

Appendix

For the KLT, keep in mind that you are decorrelating the coefficients, and this is data adaptive. So you first construct the covariance matrix and then diagonalize it.

To do this, you construct column vectors from your row-scanned images and put them in a large matrix, say A. To have a full rank matrix, you need as many (independent) images as the number of pixels–which is a tall order. Even for a 100x100 pixel image, you need 10K images to have a full rank matrix. If your number of images is less than the number of pixels, your covariance matrix X will have a rank no larger than the number of images M.

Now recall, SVD.Go over the notes earlier. if you have a lot fewer images than the number of pixels, say 40 images for a 100x100 image, your C rank is less than or equakl to 40. You can either diagnoalize a 10K x 10K C matrix or construct $A^T A$ and diagonalize a 40x40 matrix. Both will have the same number of non-zero eigen-values–they should. And you can construct the eigenvectors of C from that of $A^T A$!

hope this helps answer your programming assignment.