Outline

Contents

1 Discrete Images

2-D digital images are represented using 2-D arrays. In the following, we will explore basic image processing as matrix operations. We can represent images either as 2D matrices or as very long one dimensional vectors, by either scanning along the rows or along the columns.

As we will see, any arbitrary two-dimensional (linear) transformation of an image array can be written as a matrix operation. All of the invertible transformations that you are familiar with, the DFTs, the DCTs, wavelets, etc, can be re-written in this form using unitary or othogonal matrix operations. Why is this interesting/important? This representation helps us understand the underlying concepts in a better/simpler way than writing them out as long equations. It will also help us distinguish clearly between the continuous domain operations and their discrete equivalents, their similarities and where some of the approximations we make are not valid.

1.1 Matrix operations

Representations

Image Matrix $[f]$

$f(m,n) =$ an $M \times N$ array

$$[f] = \begin{pmatrix} f(0,0) & f(0,1) & \cdots & f(0,N-1) \\ f(1,0) & f(1,1) & \cdots & f(1,N-1) \\ \vdots & \cdots & \vdots & \\ f(M-1,0) & f(M-1,1) & \cdots & f(M-1,N-1) \end{pmatrix}$$

This can be written as a linear sum of displaced delta functions,

$$f(m,n) = \sum_{m',n'} f(m',n')\delta(m-m', n-n')$$

and for a *LSI system with psf* $h(m,n) = \mathcal{O}(\delta(m,n))$, the output $g(m,n)$ can be written as:

$$g(m,n) = \sum_{m',n'} f(m',n')h(m-m', n-n')$$

A **row-ordered** vector is constructed by stacking each row to the right of the previous row of $[f]$. One can similarly construct a **column-ordered** vector.

Row Ordered Vector

$$\vec{f} = \begin{bmatrix} f(0,0) \\ f(0,1) \\ \vdots \\ f(0,N-1) \\ f(1,0) \\ \vdots \\ f(1,N-1) \\ \vdots \\ f(M-1,0) \\ \vdots \\ f(M-1,N-1) \end{bmatrix}$$

Some Basic Matrix Properties

Properties: A and B are complex matrices.

- $(A^*)^T = (A^T)^*$
- $(AB)^T = (B^T)(A^T)$
- $(A^{-1})^T = (A^T)^{-1}$
- $(AB)^* = A^*B^*$
- Rank $[A]$: Number of linearly independent rows or columns
- Eigenvalues, λ_k: All roots $|A - \lambda_k I| = 0$
- Eigenvectors, ϕ_k, All solutions $A\phi_k = \lambda_k\phi_k,\ \phi_k \neq 0$

1.2 Toeplitz and Circulant Matrices

Toeplitz Matrix

A *Toeplitz* matrix $\mathbf{T}$ is a matrix that has constant diagonal elements along the main diagonal and subdiagonals.

Toeplitz Matrix: An $N \times N$ matrix is Toeplitz if
$T_N = [t_{k,j} : k, j = 0, 1, 2, \cdots, N-1]$ where $t_{k,j} = t_{k-j}$

$$T_N = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-(N-1)} \\ t_1 & t_0 & t_{-1} & \cdots & \vdots \\ t_2 & t_1 & t_0 & \cdots & \vdots \\ \vdots & \cdots & \cdots & t_0 & t_{-1} \\ t_{N-1} & \cdots & \cdots & t_1 & t_0 \end{bmatrix}$$

Toeplitz matrices describe the input-output transformations of 1D LSI systems.

Linear Convolution as Toeplitz Matrix operation

Suppose $\vec{x} = (x_0 \ x_1 \ \cdots \ x_{n-1})^T$ and let $t_k = 0$ for $k < 0$.

$$y = [T_n][\vec{x}] = \begin{bmatrix} t_0 & & 0 & \\ t_1 & t_0 & & \\ \vdots & \ddots & \ddots & \\ t_{n-1} & \cdots & t_1 & t_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

$$y = \begin{bmatrix} x_0 t_0 \\ t_1 x_0 + t_0 x_1 \\ \sum_{i=0}^{2} t_{2-i} x_i \\ \vdots \\ \sum_{i=0}^{n-1} t_{n-1} x_i \end{bmatrix} \implies y_k = \sum_{i=0}^{k} t_{k-i} x_i$$

This corresponds to output of a LSI (causal) system with psf $h(.)$ given by $\{t_k\}$ for a given input $\vec{x}$.

Example of Toeplitz matrix operation

Let $h(n) = n, \ -1 \leq n \leq 1$. Let $x(n) = 0$ outside of the interval $0 \leq n \leq 4$.

$$y(n) = h(n) * x(n) = \sum_{k=0}^{4} x(k) h(n-k)$$

Note: $y(n)$ is zero outside the interval $-1 \leq n \leq 5$.

$$\begin{bmatrix} y(-1) \\ y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \end{bmatrix}$$

$$\boxed{\vec{y} = H\ \vec{x}}$$

Circulant Matrix

Circulant Matrix: An $N \times N$ matrix is Circulant if
$C(m, n) = C((m - n) \mod N)$

$$C_N = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{N-1} \\ c_{N-1} & c_0 & c_1 & \cdots & c_{N-2} \\ \vdots & \ddots & \cdots & & \vdots \\ c_1 & c_2 & \cdots & c_{N-1} & c_0 \end{bmatrix}$$

Convolution of periodic sequences

Consider two periodic sequences, $h(n)$ and $x(n)$, both with period N. Convolving such sequences results in a periodic output, also with period N.

$$y(n) = h(n) *_c x(n)$$

The subscript c denotes circular convolution.

Example 1. Consider $N = 4$, and $h(n) = (n + 3) \mod 4 = \begin{pmatrix} 3 & 0 & 1 & 2 \end{pmatrix}$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 3 & 2 & 1 \\ 1 & 0 & 3 & 2 \\ 2 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\boxed{\vec{y} = H\ \vec{x}}$$

H is circulant.

1.3 Orthogonal and Unitary Matrices

Orthogonal and Unitray Matrices

Orthogonal if

$$A^{-1} = A^T$$

or

$$A^T A = AA^T = I$$

(I is the Identity Matrix)

Unitary if the matrix inverse is equal to its conjugate transpose,

$$A^{-1} = A^{*T}$$

or

$$AA^{*T} = A^{*T} A = I$$

A real orthogonal matrix is also unitary (converse not true).

The columns (or rows) of an $N \times N$ unitary matrix are orthogonal and form a complete set of basis vectors in an N-dimensional vector space.

1.4 Block Matrices

Block Matrices

A matrix $\mathcal{A}$ whose elements are matrices themeslves is called a *bock matrix.*

$$\mathcal{A} = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & & & \\ A_{m,1} & \cdots & & A_{m,n} \end{bmatrix}_{mp \times nq}$$

where $\{A_{i,j}\}$ are $p \times q$ matrices, and $\mathcal{A}$ *is an* $m \times n$ *block matrix of basic dimension* $p \times q$.

- If the block structure is Toeplitz, i.e., $A_{i,j} = A_{i-j}$, then $\mathcal{A}$ is *Block Toeplitz.*
- Similarly, if the block structure is circulant, i.e., $A_{i,j} = A_{((i-j) \mod n)}$, then then $\mathcal{A}$ is *Block Circulant*

Block Matrices

- *Doubly block Toeplitz:* In addition to block Toeplitz, each block itself is Toeplitz. These Doubly Toeplitz arise in 2D linear convolution operations. The output of a 2D LSI system can be written as a Doubly Block Toeplitz operation.
- *Doubly block Circulant:* In addition to block Circulant, each block is Circulant. 2D Circular convolution can be written as a Doubly Circulant matrix operation.

- *Toeplitz Block:* If $\{A_{i,j}\}$ are Toeplitz but $A_{i,j} \neq A_{i-j}$, then $\mathcal{A}$ is *Toeplitz Block.*

HW#2 will explore constructing these block matrices for 2D convolutions.

1.4.1 2D Linear Convolution: Example

Consider the 2D linear convolution

$$y(m,n) = \sum_{k=0}^{2}\sum_{l=0}^{1} h(m-k, n-l)x(k,l), \quad 0 \leq m \leq 3, \quad 0 \leq n \leq 2$$

where

$$x(m,n) = \begin{bmatrix} 2 & 1 \\ 5 & 4 \\ 3 & 1 \end{bmatrix}, \quad h(m,n) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

with the array index increasing from left to right and top to bottom, with the origin located at the top left-hand corner of the array. Let the column-ordered vectors for $x(m,n)$ be

$$\mathbf{x_0} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}, \quad \mathbf{x_1} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

Then we can write the column-ordered output vector $\mathbf{y}_n$ as:

$$\mathbf{y}_n = \sum_{n'=0}^{1} \mathbf{H}_{n-n'}\mathbf{x}_{n'}, \quad \mathbf{H}_n = \{h(m-m', n), \quad 0 \leq m \leq 3, \quad 0 \leq m' \leq 2\}$$

With this notation, note that we get

$$\mathbf{H}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{H}_{-1} = \mathbf{0}, \quad \mathbf{H}_2 = \mathbf{0}$$

Finally, defining $\mathcal{Y}$ and $\mathcal{X}$ as column ordered vectors, we get

$$\mathcal{Y} = \begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{H}_0 & \mathbf{0} \\ \mathbf{H}_1 & \mathbf{H}_0 \\ \mathbf{0} & \mathbf{H}_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{bmatrix} \triangleq \mathcal{H}\mathcal{X}$$

Additional notes.

- $\mathcal{H}$ is a doubly Toeplitz 3×2 block matrix of basic dimensions 4×3.
- $\mathcal{H}$ is **NOT** Toeplitz, $[\mathcal{H}]_{m,n} \neq [\mathcal{H}]_{m-n}$.
- The one-dimensional system $\mathcal{Y} = \mathcal{H}\mathcal{X}$ is linear but not shift-invariant (since $\mathcal{H}$ is not Toeplitz, but the original 2-D system is.

1.4.2 2D Circular Convolution: Example

Consider the 2D circular convolution defined by

$$y(m,n) = \sum_{k=0}^{2}\sum_{l=0}^{3} h(m-k,n-l)x(k,l), \quad 0 \le m \le 2, \quad 0 \le n \le 2$$

where $h(m,n)$ is doubly periodic with periods (3, 4), i.e., $h(m,\ n) = h(m+3,\ n+4), \forall m,n$, and $h(m,\ n)$ over one period is given by:

$$h(m,n) = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 8 & 5 & 2 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix}$$

Similar to the previous example, using column-ordered vectors, we can write

$$\mathbf{y}_n = \sum_{n'=0}^{3} \mathbf{H}_{n-n'}\mathbf{x}_{n'}, \quad 0 \le m \le 3$$

where $\mathbf{H}_n$ is a periodic sequence of 3×3 circulant matrices with period 4, given by

$$\mathbf{H}_0 = \begin{bmatrix} 4 & 3 & 8 \\ 8 & 4 & 3 \\ 3 & 8 & 4 \end{bmatrix}, \mathbf{H}_1 = \begin{bmatrix} 3 & 1 & 5 \\ 5 & 3 & 1 \\ 1 & 5 & 3 \end{bmatrix}, \mathbf{H}_2 = \begin{bmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & 2 & 2 \end{bmatrix}, \mathbf{H}_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and the output can be written as (column-ordered vectors):

$$\mathcal{Y} = \begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{H}_0 & \mathbf{H}_3 & \mathbf{H}_2 & \mathbf{H}_1 \\ \mathbf{H}_1 & \mathbf{H}_0 & \mathbf{H}_3 & \mathbf{H}_2 \\ \mathbf{H}_2 & \mathbf{H}_1 & \mathbf{H}_0 & \mathbf{H}_3 \\ \mathbf{H}_3 & \mathbf{H}_2 & \mathbf{H}_1 & \mathbf{H}_0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \triangleq \mathcal{H}\mathcal{X}$$

- $\mathbf{H}_{-n} = \mathbf{H}_{4-n}$.
- $\mathcal{H}$ is a doubly circulant, 4×4 block matrix with basic dimensions 3×3

1.5 Kronecker Product

Kronecker Products

$A : M_1 \times M_2$; $B = N_1 \times N_2$. Then the *Kronecker product* of A and B, represented as $A \otimes B \triangleq \{a_{m,n}B\}$, is an $M_1 \times M_2$ block matrix with basic dimensions $N_1 \times N_2$.

$$A \otimes B = \begin{bmatrix} a_{1,1}B & \cdots a_{1,M_2}B \\ a_{2,1}B & \cdots a_{2,M_2}B \\ \vdots & \\ a_{M_1,1}B & \cdots a_{M_1,M_2}B \end{bmatrix}$$

- Note: $A \otimes B \neq B \otimes A$.

2 Discrete Transforms

Fourier Transform Revisited

1D Continuous

$$F(u) = \int_{-\infty}^{\infty} f(x) \exp\left(-j2\pi ux\right) \, dx$$

1D Discrete: N-point sequence

$$F(k) = \sum_{m=0}^{N-1} f(m) \exp\left(-j\frac{2\pi}{N}km\right), k = 0, 1, \cdots, N-1.$$

Note that $F(k)$ is periodic, with period N. This implicitly imposes the periodicity on $f(m)$ as well!!

2D Fourier Transform

2D Continuous

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \exp\left(-j2\pi(ux + vy)\right) \, dx \, dy$$

2D Discrete: $N \times N$-point sequence

$$F(k, l) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m, n) \exp\left(-j\frac{2\pi}{N}(km + ln)\right),$$
$$k = 0, 1, \cdots, N-1; \quad l = 0, 1, \cdots, N-1;$$

2.1 DFT Matrix Operations

Discrete Transform of $[f]$

A general 2D Discrete Transform of an $M \times N$ matrix $[f]$ is defined as:

$$\boxed{[F]_{M\times N} = [P]_{M\times M}[f]_{M\times N}[Q]_{N\times N}}$$

where $[P]$ and $[Q]$ are non-singular. Note that this can be written in the more familiar summation form as:

$$F(k, l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} P(k, m) f(m, n) Q(n, l)$$
$$k = 0, 1, \cdots, M-1; \quad l = 0, 1, \cdots, N-1;$$

Finally, $f(m, n)$ can be reconstructed from the samples $F(k, l)$ as

$$\boxed{[f] = [P]^{-1} \; [F] \; [Q]^{-1}}$$

Example: 2D DFT

Consider $[\Phi_{J\times J}]$ - a $J\times J$ matrix with elements $\{\phi_{m,n}\}$,

$$\phi_{m,n} = \frac{1}{J}\exp\left(-j\frac{2\pi}{J}mn\right), \quad m,n \ : \ 0,1,\cdots,J-1.$$

Then, with $[P]=[\Phi_{M\times M}]$, and $[Q]=[\Phi_{N\times N}]$, the 2D DFT of $[f]$ is

$$[F] = [P]\ [f]\ [Q]$$

where

$$F(k,l) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)e^{-j2\pi\left(\frac{mk}{M}+\frac{nl}{N}\right)}$$
$$k \ : \ 0,1,\cdots,M-1; \ l \ : \ 0,1,\cdots,N-1.$$

- The (m,n)-th element of the inverse matrix

$$[\Phi_{J\times J}]^{-1}_{m,n} = \exp\left(j\frac{2\pi}{J}mn\right)$$

- Orthogonality of the exponential family as basis functions:

$$\sum_{m=0}^{J-1}\exp\left(-j\frac{2\pi}{J}km\right)\exp\left(j\frac{2\pi}{J}mn\right) = \begin{cases} J & : k=n \\ 0 & : k\neq n \end{cases}$$

- Complete: any real signal(image) can be reconstructed as a linear combination of complex exponentials.

$$f(m,n) = \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} F(k,l)e^{+j2\pi\left(\frac{mk}{M}+\frac{nl}{N}\right)}$$

2D DFT Properties

Periodicity

The 2D DFT $F(k,l)$ (and hence the 2D signal $f(m,n)$) are periodic (assume $M\times N$ signals).

$$F(k,-l) = F(k,N-l)$$
$$F(-k,l) = F(M-k,l)$$
$$F(-k,-l) = F(M-k,N-l)$$
$$F(aM+k,bN+l) = F(k,l),\ k,\ l: 0,\pm1,\pm2,...$$
$$f(aM+m,bN+n) = f(m,n),\ m,\ n: 0,\pm1,\pm2,...$$

2D DFT Properties

Circular Convolution

Consider two periodic arrays, $[f]$ and $[d]$, both with period $M \times N$. Let the circular convolution of these two signal be given by:

$$g(m,n) = \sum_{m'=0}^{M-1} \sum_{n'=0}^{N-1} f(m',n')\ d(m-m', n-n')$$

$$m = 0, 1, 2, \cdots, M-1; \quad n = 0, 1, \cdots, N-1.$$

$$\begin{aligned} g(m,n) &= \sum_{m'} \sum_{n'} \left\{ \sum_{k} \sum_{l} F(k,l) e^{j2\pi\left(\frac{m'k}{M} + \frac{n'l}{N}\right)} \right\} \\ &\quad \times \left\{ \sum_{s} \sum_{t} D(s,t) e^{j2\pi\left(\frac{m-m'}{M}s + \frac{n-n'}{N}t\right)} \right\} \\ g(m,n) &= \sum_{k} \sum_{l} \sum_{s} \sum_{t} F(k,l) D(s,t) e^{j2\pi\left(\frac{ms}{M} + \frac{nt}{N}\right)} \\ &\quad \times \underbrace{\sum_{m'} \sum_{n'} e^{j2\pi\left(\frac{m'k}{M} + \frac{n'l}{N}\right)} e^{-j2\pi\left(\frac{m's}{M} + \frac{n't}{N}\right)}}_{\text{apply the orthogonality principle here}} \\ &\quad = \begin{cases} MN & : s = k \text{ and } t = l \\ 0 & : \text{otherwise} \end{cases} \\ &= \sum_{k} \sum_{l} F(k,l) D(k,l) e^{j2\pi\left(\frac{mk}{M} + \frac{nl}{N}\right)} \end{aligned}$$

Ignoring the constant scaling term associated with the forward/inverse transforms, you see that the above equation is nothing but the inverse Fourier Transform of the product of the transforms of f and d. Effectively,

$$G(k,l) = F(k,l)\ D(k,l)$$

Summary

- Images as matrices
- Convolution as matrix operations
- General 2D transforms on pictures/matrices
- 2D DFT and properties
- 2D circular convolution = multiplication in the 2D DFT space