Outline

Contents

1 Linear, Shift Invariant Systems

Many image processing operations involve *convolution* computations. Hence a good understanding the basics of linear systems and transforms is critical. Most of the theory and application is a straightforward extension of what you might have seen in 1-D. The addition of the second dimension to a signal brings some interesting new possibilities, including the rotation operator on a 2D signal. In the following slides we will review some basic 2D signals, 2D linear convolution and conclude with the 2D Fourier transform.

1.1 Linear Systems

We will start with the definition of a linear system and 2D point sources. Then we will consider two 2D functions that are important in image processing (2D impulse function, 2D rectangular function). It is also useful to distinguish between the continuous and discrete functions in 2D, and the *sifting* property.

Linear Systems

Definition
Let Θ be an operator that takes pictures into pictures. Θ is *linear* if $\Theta(af+bg) = a\Theta(f) + b\Theta(g)$ for *all pictures* f and g, and for all constants a and b.

Point Sources
Any arbitrary picture f could be considred to be a sum of point sources. The output of Θ for a point source inpout is called the *point spread function (psf)* of Θ.

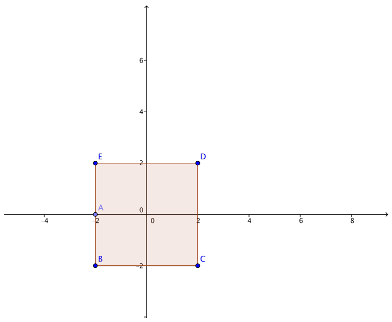


Figure 1: Shows a rectangle region over which the function is non-zero. This function can be described as $f(x, y) = rect(x/4, y/4)$.

Example: Rectangle function

$$rect(x, y) = \begin{cases} 1 & \text{if } |x| \leq 1/2 \text{ and } |y| \leq 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

2D Dirac Delta

Example 1. Let $\delta_n(x, y) = n^2 rect(nx, ny),\ n = 1, 2, 3, ...$Thus δ_n is zero outside the $\frac{1}{n} \times \frac{1}{n}$ square defined by $|x| \leq 1/2n,\ |y| \leq 1/2n$, and has a constant value n^2 inside the square.

dirac delta as the limit of the rect function

$$\int \int \delta_n(x, y)\ dx\ dy = 1, \quad \forall n$$

$$\delta(x, y) = \lim_{n \to \infty} \delta_n(x, y)$$

$\delta(x, y)$ is called the dirac-delta function. Note that

$$\int \int \delta(x, y)\ dx\ dy = 1,$$

1.2 Continuous vs Discrete

Delta functions and Sifting Property

- Continuous case: *dirac* delta $\delta(x, y)$ is defined by $\int \delta(x, y)\ dx\ dy = 1$
- Discrete case: *Kronecker delta* $\delta(m, n) = 1$ if $m = 0$ and $n = 0$. Note $\sum \sum \delta(m, n) = 1$.

- Note that both these functions are *separable functions*.

$$\text{Dirac: } \delta(x, y) = \delta(x)\ \delta(y)$$

$$\text{Kronecker: } \delta(m, n) = \delta(m)\ \delta(n)$$

- *Sifting property* of the delta functions:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y')\delta(x' - x, y' - y)\ dx'\ dy' = f(x, y)$$

$$\sum_{m'=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} g(m', n')\delta(m' - m, n' - n) = g(m, n)$$

Averaging

Example 2. $\int\int g(x, y)\delta_n(x, y)\ dx\ dy$ This integral gives the average value of $g(x, y)$ over the $\frac{1}{n} \times \frac{1}{n}$ rectangular region centered at the origin. Thus, in the limit:

$$\int\int g(x, y)\delta(x, y)\ dx\ dy = g(0, 0)$$

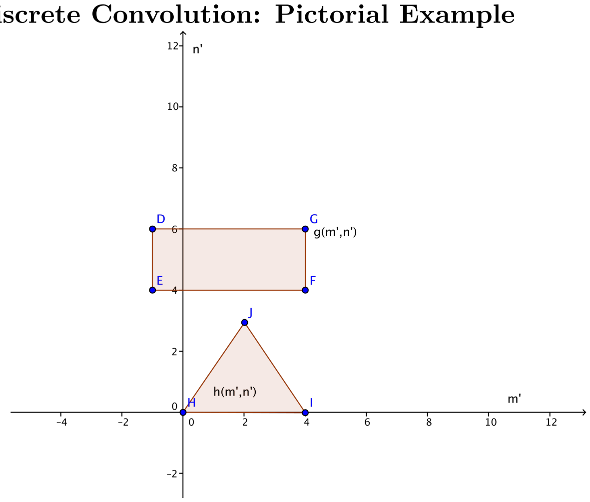
Generalization of the above concept leads us to the *sifting* property discussed before.

Prove this!

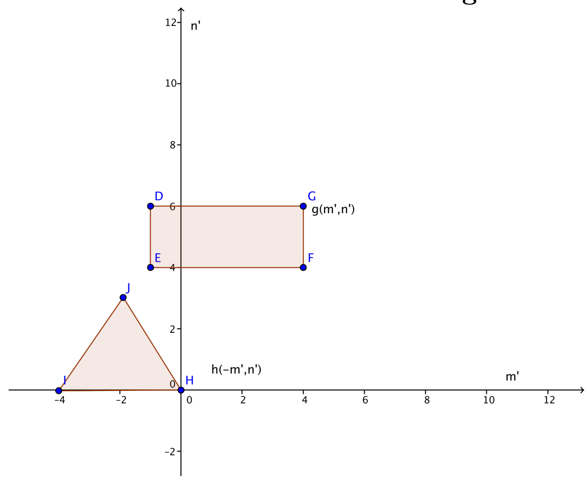
$$\int\int \exp(+j2\pi(ux + vy))\ du\ dv = \delta(x, y)$$

2 2D convolution

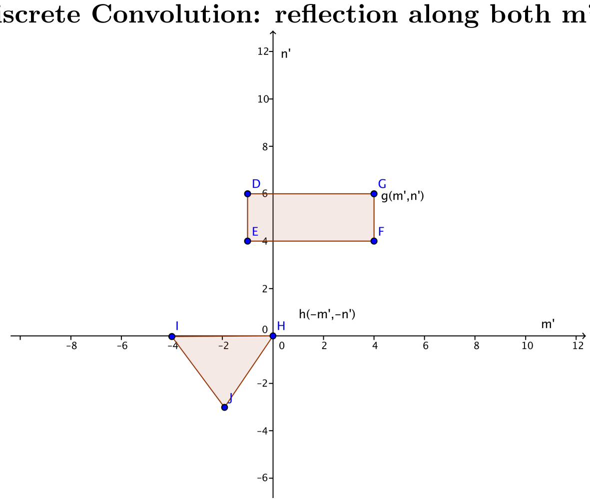
Discrete Convolution: Pictorial Example



Discrete Convolution: reflection along m'-axis

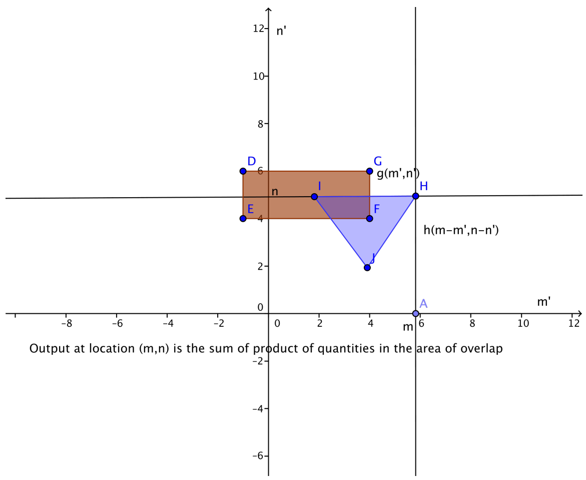


Discrete Convolution: reflection along both m' and n'



Note that this is essentially a rotation by 180 deg of the $h(m,n)$ function.

Discrete Convolution: rotation by 180 and shift by (m,n)



A numerical example

array index increases from left to right, and bottom to top
Red cell denotes the origin.

$$h(m,n) = \begin{array}{cc} \begin{array}{c}1\\0\end{array} & \begin{bmatrix}-1 & 1\\ \boxed{1} & 1\end{bmatrix} \\ n = & \begin{array}{cc}0 & \quad 1\end{array}\end{array} \quad , \quad g(m,n) = \begin{bmatrix}3 & 1\\ 5 & 4\\ \boxed{2} & 1\end{bmatrix}$$

Rotate $h(m,n)$ by 180 deg. Then shift by 1 unit along the rows

$$h(-m,-n) = \begin{array}{cc} \begin{array}{c}0\\-1\end{array} & \begin{bmatrix}1 & \boxed{1}\\ 1 & -1\end{bmatrix} \\ n = & \begin{array}{cc}-1 & \quad 0\end{array}\end{array} \quad , \quad h(1-m,-n) = \begin{array}{cc} \begin{array}{c}1\\0\end{array} & \begin{bmatrix}1 & 1\\ 1 & \boxed{-1}\end{bmatrix} \\ n = & \begin{array}{cc}-1 & \quad 0\end{array}\end{array} \quad ,$$

Now align red cells and compute the convolution
$f(1,0) = (h * g)$ evaluated at $m = 1, n = 0 : 2 * (-1) + 5 * 1 = 3$

Convolution: numerical example slide 2

Discrete convolution Definition

$$f(m,n) = g(m,n) * h(m,n) \triangleq \sum_{m'=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} h(m-m', n-n') g(m', n')$$

for our example from previous slide,

$$f(m,n) = \begin{bmatrix}-3 & 2 & 1\\ -2 & 5 & 5\\ 3 & 10 & 5\\ \boxed{2} & 3 & 1\end{bmatrix}_{4\times 3}$$

What about the size of $f(m,n)$?
Note that the size of $f(m,n)$ is 4×3. In general, if h is an $M \times N$ array and g is an $K \times L$ array, then f is $(M+K-1) \times (N+L-1)$

3 LSI Systems and Linear Convolution (Continuous)

Linearity and Shift Invariance - revisited

- Consider a system Θ with the point spread function (psf) $h(x,y)$. i.e.,

$$\Theta[\delta(x,y)] = h(x,y)$$

- Θ is said to be linear if

$$\Theta[a\ f_1 + b\ f_2] = a\ F_1 + b\ F_2$$

  where $F_i = \Theta[f_i]$

- ... and shift-invariant if

$$\Theta[f(x-\alpha, y-\beta)] = F(x-\alpha, y-\beta)$$

LSI..revisited
From the sifting property of the delta functions, we can write:

$$f(x,y) = \int\int f(\alpha,\beta)\delta(x-\alpha, y-\beta)\ d\alpha\ d\beta$$

This can be interpreted as $f(x,y)$ consisting of a linear sum of point sources (though a very, very large - not finte) located at (α,β) in the $xy-$ plane, with α and β ranging from $-\infty$ to $+\infty$. In this sum, the point source at a particular value of (α,β) has a strength $f(\alpha,\beta)$.

2D Convolution - continuous case

- Consider the the output of a 2D LSI system to an input $f(x,y)$.
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$$\begin{aligned}
\Theta[f(x,y)] = &\quad \Theta\left[\int\int f(\alpha,\beta)\delta(x-\alpha, y-\beta)\ d\alpha\ d\beta\right] && (1)\\
= &\quad \int\int f(\alpha,\beta)\ \Theta\left[\delta(x-\alpha, y-\beta)\right]\ d\alpha\ d\beta && (2)\\
= &\quad \int\int f(\alpha,\beta)\ h(x-\alpha, y-\beta)\ d\alpha\ d\beta && (3)
\end{aligned}$$

- In deriving the above equation, we have used the *sifting* property of the delta function, and that the system is *linear and shift invariant.*
- If the output of the system for the input $f(x, y)$ is given by $g(x, y)$, then we can write

$$\boxed{g(x,y) = \int\int f(\alpha,\beta)\ h(x-\alpha, y-\beta)\ d\alpha\ d\beta}$$

Convolution Integral

- The *convolution integral.*

$$\boxed{g(x,y) = \int\int f(\alpha,\beta)\ h(x-\alpha, y-\beta)\ d\alpha\ d\beta}$$

- The output of a *linear, shift-invariant system can be written as a convolution of the input signal with the psf of the system.*
- This is often written as $g(x, y) = f(x, y) * h(x, y)$
- Convolution is commutative, i.e., $g = f * h = h * f$

Convolution integrat: steps in computing..

Note the steps involved in computing the convolution:

- signal $h(x', y')$ is first flipped along both $x-$ and $y-$ axes, to create $h(-x', -y')$.
- Then, to compute the output at a location (a, b), $g(a, b)$, this flipped signal $h(-x', -y')$ is shifted by a units along the x axis and by b units along y, to obtain $h(a - x', b - y')$.
- This signal is overlaid on the input signal $f(x', y')$, multiplied together point-by-point, and the results summed up to get the value $g(a, b)$.

4 2D Fourier Transform

2D Fourier Transform

- Continuous Case:

$$F(u,v) = \int\int f(x,y)\exp(-j2\pi(ux+vy)\ dx\ dy$$

In general, $F(u, v)$ is complex valued.

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Example 3. Rectangle function $f(x, y) = rect(x, y)$

$$F(u, v) = \frac{\sin(\pi u)}{\pi u}\frac{sin(\pi v)}{\pi v} = sinc(\pi u, \pi v)$$

- *Example* 4. Delta function: $f(x, y) = \delta(x, y)$

$$F(u, v) = \lim_{n\to\infty} sinc(u/n, v/n) = 1$$

Fourier Transform Properties

- *Linearity:*$\mathcal{F}\{af_1 + bf_2\} = aF_1(u, v) + bF_2(u, v)$
- *Scaling:* $\mathcal{F}\{f(\alpha x + \beta y)\} = \frac{1}{|\alpha\beta|}F(u/\alpha, v/\beta)$
- *Shift:* $\mathcal{F}\{f(x-\alpha, y-\beta)\} = F(u, v)\exp(-j2\pi(u\alpha+v\beta)$, and $\mathcal{F}\{f(x, y)exp(j2\pi(u_0\alpha+ v_0\beta)\} = F(u - u_0, v - v_0)$
- *Rotation:* $\mathcal{F}\{\mathcal{F}\{f(x, y)\}\} = f(-x, -y)$
- *Convolution:*

$$\mathcal{F}\left\{\int\int f_1(\alpha, \beta)\ f_2(x - \alpha, y - \beta)\ d\alpha\ d\beta\right\} = F_1(u, v)\ F_2(u, v)$$

$$\mathcal{F}\{f_1(x, y)\ f_2(x, y)\} = \int\int F_1(u - s, v - t)\ F_2(s, t) ds\ dt$$

Summary

Summary

- 2D Linear, Shift-Invariant systems
- point spread functions and convolution integral
- 2D discrete and continuous convolution examples
- 2D Fourier transform and properties