**Outline**

# Contents



# 1 2D DCT

**The Cosine Transform**

The $N \times N$ cosine transform matrix, $[C] = \{c(k,n)\}$, is defined as:

$$c(k,n) = \begin{cases} \frac{1}{\sqrt{N}}, & k = 0, 0 \leq n \leq N-1 \\ \sqrt{\frac{2}{N}} \cos \frac{\pi(2n+1)k}{2N}, & 1 \leq k \leq N-1, 0 \leq n \leq N-1 \end{cases} \tag{1}$$

Alternatively, the forward and inverse transforms are given by

$$C(k) = \alpha(k) \sum_{n=0}^{N-1} f(n) \cos\left(\frac{\pi(2n+1)k}{2N}\right), \ 0 \leq k \leq N-1 \tag{2}$$

$$f(n) = \sum_{k=0}^{N-1} \alpha(k) C(k) \cos\left(\frac{\pi(2n+1)k}{2N}\right), \ 0 \leq n \leq N-1 \tag{3}$$

where $\alpha(0) \triangleq \sqrt{\frac{1}{N}}$, $\alpha(k) \triangleq \sqrt{\frac{2}{N}}$ for $1 \leq k \leq N-1$.

**Notes:** Note that DCT of a sequence is NOT the real part of the DFT of the same sequence. However, the DCT and DFT are related as explained in the next slide. This derivation relates the 2N-point DFT of the extended sequence with mirror symmetry to the N-point DCT.

Note the frequency of the cosine basis functions: for an N-point DCT, the DCT has 2N-periodicity. This implicitly helps in reducing blocking artifact in the standard JPEG compression.

**DCT and DFT**

We will consider the relationship between DCR and DCT here. Consider an $N$-point sequence $x(n), 0 \leq n \leq N-1$.

Let

$$y(n) = \begin{cases} x(n), & x(n), 0 \leq n \leq N-1 \\ x(2N-1-n), & N \leq n \leq 2N-1 \end{cases}$$

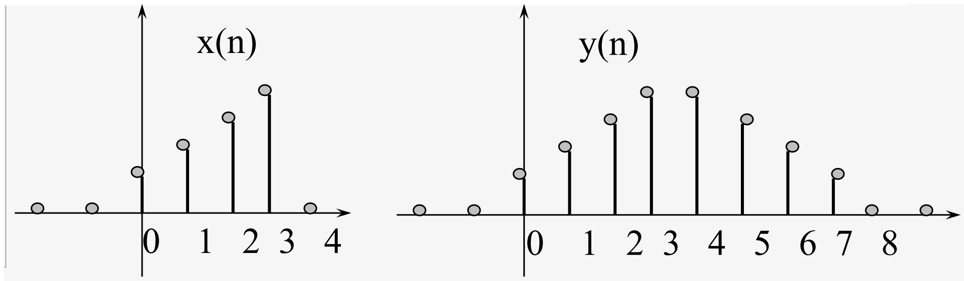


Figure 1: Extended sequence with mirror symmetry

Taking the 2D-point DFT of $y(n)$, we get:

$$Y(u) = \sum_{n=0}^{2N-1} y(n) e^{-j\frac{2\pi}{2N}un} = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{2N}un} + \sum_{n=N}^{2N-1} x(2N-1-n) e^{-j\frac{2\pi}{2N}un}$$
$$= \sum_{n=0}^{N-1} \left( x(n) e^{-j\frac{2\pi}{2N}un} + x(n) e^{-j\frac{2\pi}{2N}u(2N-1-n)} \right)$$

$$Y(u) = e^{i\frac{\pi}{2N}u} \sum_{n=0}^{N-1} 2x(n) \cos(\frac{\pi}{2N} u(2n+1))$$

**DCT Properties**

1. The cosine transform is real and orthogonal, i.e.,
$$[C] = [C^*] \Rightarrow [C]^{-1} = [C]^T \tag{4}$$

2. Note that the cosine transform is *not* the real part of unitary DFT. It is, however, related to the DFT of its symmetric extension.

3. The basis vectors of the cosine transform (i.e., rows of the $[C]$matrix) are the eigen vectors of the symmetric tridiagonal matrix $Q_c$,
$$[Q_c] = \begin{bmatrix} 1-\alpha & -\alpha & & 0 & \\ -\alpha & 1 & & & \\ & \ddots & \ddots & \ddots & \\ 0 & & & 1 & -\alpha \\ & & & -\alpha & 1-\alpha \end{bmatrix} \tag{5}$$

4. The $N \times N$ cosine transform approximates the KL transform of a first order stationary Markov sequence of length $N$ when the correlation parameter $\rho$ is close to 1. Recall,

$$R = \begin{pmatrix} 1 & \rho & \rho^2 & \rho^3 & \cdots & \rho^{N-1} \\ \rho & & & & & \\ \vdots & & & & & \\ \vdots & \cdots & & & & \rho \\ \rho^{N-1} & \cdots & & & \rho & 1 \end{pmatrix}$$

for this $R$, $R^{-1}$ is a symmetric tridiagonal matrix. For scalars $\beta$ and $\alpha$ defined as below:

$$\beta^2 \triangleq \frac{(1-\rho^2)}{(1+\rho^2)}, \quad \alpha \triangleq \frac{\rho}{(1+\rho^2)}$$

we get

$$Q_c \approx \beta^2 \ R^{-1}$$

4. (contd.)

$$\beta^2 \ R^{-1} = \begin{bmatrix} 1-\rho\alpha & -\alpha & 0 & & \\ -\alpha & 1 & & & \\ & \ddots & \ddots & \ddots & \\ 0 & & & 1 & -\alpha \\ & & & -\alpha & 1-\rho\alpha \end{bmatrix},$$

$$Q_c \approx \beta^2 \ R^{-1} \quad \text{for } \rho \approx 1$$

Hence the eigenvectors of $R$ and the eigenvectors of $Q_c \to$ DCT, will be very close.

*For this reason, DCT has excellent compression properties for natural images whose statistics can be approximated well by a first order Markov process.*

# 2 The Hadamard transform

**Hadamard transform**

Consider

$$H_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{6}$$

and for $N \triangleq 2^n, n = 1, 2, 3, \cdots$, the Hadamard transform matrices are generated recursively as:

$$H_n = H_{n-1} \otimes H_1 = H_1 \otimes H_{n-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{pmatrix} \tag{7}$$

$$H_2 = H_1 \otimes H_1; \quad H_3 = H_1 \otimes H_2. \tag{8}$$

- Hadamard transform is a digital ($\pm 1$) transform, and has very good energy compaction properties.
- $H$ is real, symmetric and orthogonal: $H = H^* = H^T = H^{-1}$

# 3 The Haar transform

**The Haar Transfrom**

- The Haar functions are defined over the interval $x \in [0,1]$, and for $k = 0, 1, \ldots, N-1$, where $N = 2^n$.

  The integer $k$ can be uniquely decomposed as

$$k = 2^p + q - 1, \quad 0 \leq p \leq n-1; \tag{9}$$

$$q = \begin{cases} 0, 1 & : p = 0 \\ 1 \leq q \leq 2^p & : p \neq 0. \end{cases}$$

- Example: $N = 4$:

| $k$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p$ | 0 | 0 | 1 | 1 |
| $q$ | 0 | 1 | 1 | 2 |

## 3.1 Haar functions

- *Haar functions:* with $k \to (p, q)$,

$$h_0(x) = h_{0,0}(x) = \frac{1}{\sqrt{N}}, \quad x \in [0,1] \tag{10}$$

$$h_k(x) = h_{p,q}(x) = \frac{1}{\sqrt{N}} \begin{cases} 2^{p/2}, & \frac{q-1}{2^p} \leq x < \frac{q-0.5}{2^p} \\ -2^{p/2}, & \frac{q-0.5}{2^p} \leq x < \frac{q}{2^p} \\ 0, & \text{otherwise for } x \in [0,1] \end{cases} \tag{11}$$

- The Haar transform is computed by letting $x$ take discrete values at $m/N, \ m = 0, 1, \ldots, N-1$.
- Haar transform is real and orthogonal:

$$H = H^*; H^{-1} = H^T$$

**Haar transform $H_8$ for $N = 8$**

$$H_8 = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} \tag{12}$$

*We will revisit this transform after a discussion on multiresolution analysis*