Outline

Contents

The primary goal of this lecture notes is to introduce you to projective transformations in the 2D space, and in particular, to the homography transformation. Homography transformations are quite useful in understanding image to image mappings. While he first few lectures focused on image intentities and transforming those values, here we will consider the coordinate transformations. Images can be geometrically transformed in many ways, and these include 3D space to 2D image projections that you all are quite familiar with. The first topic is on a coordinate representation that makes these computations simple to manipulate.

1 Homogeneous Coordinates

Homogeneous representation of vectors in 2D and 3D space is convenient in describing *geometric transformations* of points, particularly in the context of projective transformations of the 3D space. For the purposes of this course, we will be concerned with a class of transformations known as **Homography**. These are transformations of a 2D plane when viewed through a pin-hole camera, i.e., a projection of an 2D plane onto another 2D plane. Such transformations arise, for example, in **image registration** where we want to map two or more images onto a common reference coordinate system.

The main take-home message in homogeneous representations is that the 2D image coordinates are represented as 3D homogeneous vectors. And that points and lines in 2D have a similar 3D representation! and that only the directions matter and not the absolute scale/magnitude.

For example, a 2D point (x, y) will be represented as a column vector $(\lambda x, \lambda y, \lambda)^T$. Similarly, a 2D line $ax + by + c = 0$ will have a homogeneous representation $(a, b, c)^T \sim k(a, b, c)^T, k \neq 0$. The only point that is ill-defined in the 3D vector space of homogeneous vectors is the origin $(0, 0, 0)^T$. This 3D Euclidean space excluding origin forms the **two dimensional projective space $\mathbf{P}^2$**.

1.1 Line Representation

Homogeneous Representation of Lines

Homogeneous Representation of Lines
Consider $ax + by + c = 0$. This represents a line in 2D. We can think of this being parametrized by a *3D vector* $(a,\ b,\ c)^T$. Equivalently,

$$(ka)x + (kb)y + kc = 0, \forall k \neq 0$$

$$(a, b, c)^T \sim k(a, b, c)^T$$

Equivalence class of vectors

$$(a, b, c)^T \sim k(a, b, c)^T$$

forms an equivalence class of vectors, any vector is representative of the line.

Set of equivalence classes in $\mathbf{R}^3 - (0, 0, 0)^T$ (3D Euclidean space but excluding the origin) forms the two dimensional projective space $\mathbf{P}^2$

1.2 Point Representation

Homogeneous Representation of Points
A 2D point $(x, y)^T$ lies on a line $l = (a, b, c)^T$ if and only if

$$ax + by + c = 0$$

$$(x, y, 1)(a, b, c)^T = (x, y, 1)\ l = 0$$

Note that $(x, y, 1) \sim k(x, y, 1)^T, \forall k \neq 0$.
Thus, the homogeneous point $\mathtt{x} = (x, y, 1)^T$ lies on the line l if and only if

$$\mathtt{x}^T l = l^T \mathtt{x} = 0$$

Homogeneous coordinates
Homogeneous coordinates $(x_1, x_2, x_3)^T$ have only two degrees of freedom, representing the inhomogeneous coordinates $x = x_1/x_3, y = x_2/x_3$.

Points from Lines and vice-versa

- Intersection of lines: The intersection of two lines l and l' is $x = l \times l'$
- Line joining two points: The line through two points x and x' is $l = x \times x'$

Recap: Vector Cross Product

Given two vectors, $\mathbf{a} = (a_x, a_y, a_z),\ \mathbf{b} = (b_x, b_y, b_z)$, the vector product $\mathbf{a} \times \mathbf{b}$ is given by,

$$\mathbf{a} \times \mathbf{b} = \hat{\boldsymbol{x}}(a_y b_z - a_z b_y) - \hat{\boldsymbol{y}}(a_x b_z - a_z b_x) + \hat{\boldsymbol{z}}(a_x b_y - a_y b_x)$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Ideal Points and Line at infinity

Intersection of parallel lines:

$$l = (a, b, c)^T \text{ and } l' = (a, b, c')^T$$

$$l \times l' = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ a & b & c \\ a & b & c' \end{vmatrix} = (bc' - bc, -(ac' - ac), 0)^T$$

$$l \times l' = (b, -a, 0)^T$$

Ideal Points

Ideal Points are those points whose third coordinate is zero. These are the set of points $(x_1, x_2, 0)^T$.

Ideal Points and Line at Infinity

Ideal Points

Ideal Points are those points whose third coordinate is zero. These are the set of points $(x_1, x_2, 0)^T$. Note that all these points lie on a line $l = (0, 0, 1)^T$ (verify: $\mathbf{x}^T l = 0$).

Line at Infinity l_∞

The set of ideal points lies on the *line at infinity* is given by $l_\infty = (0, 0, 1)^T$. A line $l = (a, b, c)^T$ intersects the line at infinity in the ideal point $l \times l_\infty = (b, -a, 0)^T$. A line $l' = (a, b, c')^T$ parallel to l also intersects l_∞ in the same ideal point, irrespective of the value of c'.

Summary

Ideal points and Line at infinity

In inhomogeneous notation, $(b, -a)^T$ is a vector tangent to the line. It is orthogonal to (a, b) - the line normal. Thus it represents the line direction. As the line's direction varies, the ideal point $(b, -a)^T$ varies over l_∞. The line at infinity can be thought of as the set of directions of lines in the plane.

4. Homogeneous Coordinates and Projective Transforms

2 Projective Transformations

Projective Transformations

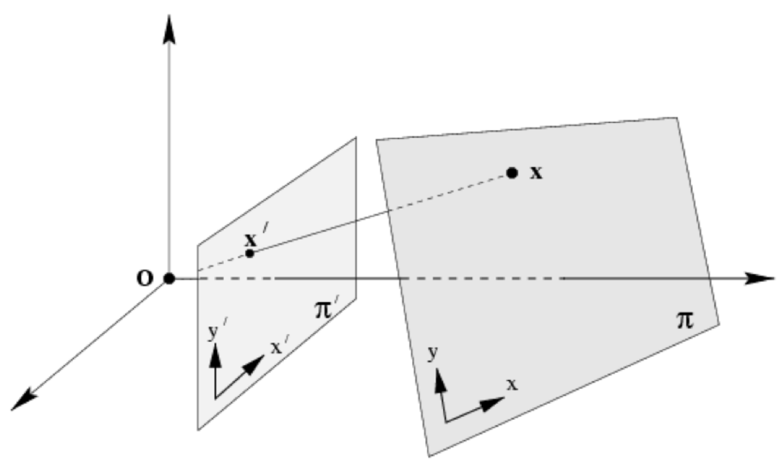
In this lecture notes, we consider projective transformations that map planes in 3D to planes in 3D. In homogeneous coordinates, this can be represented as a linear transformation via a 3×3 matrix H. The figure above shows some typical examples.

(left) Similarity: sqaures imaged as squares (middle) Affine: parallel lines remain parallel; circles become ellipses (right) Projective: parallel lines converge

2.1 Central Projection/projective transformation

Central Projection

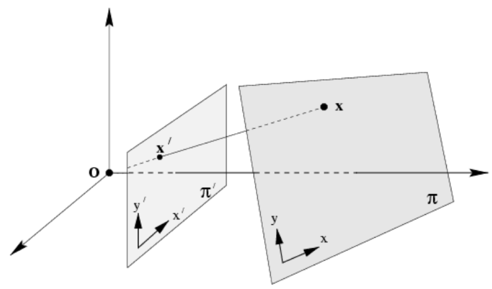
Central projection maps planes to planes



Projective transformations

Definition

A *projectivity* is an invertible mapping h from P^2 to itself such that three points x_1, x_2, x_3 lie on the same line if and only if $h(x_1), h(x_2), h(x_3)$ do.



Theorem
A mapping $h : P^2 \rightarrow P^2$ is a projectivity if and only if there exists a non-singular 3×3 matrix $\mathbf{H}$ such that any point in P^2 represented by a vector x it is true that $h(x) = \mathbf{H}x$

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T, \mathbf{x'} = \mathbf{Hx}$$

Note: all in homogeneous

projectivity = collineation = projective transformation = homography

3 Homography examples

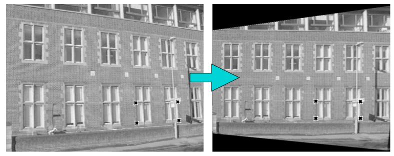
Homography

- Any point in P^2 is represented as a homogeneous 3-vector $\mathbf{x}$
- $\mathbf{x'} = \mathbf{Hx}$ is a *linear mapping* of *homogeneous coordinates*
- The theorem above asserts that any projectivity arises as such a linear transformation in the homogeneous coordinates
- conversely, *any such mapping is a projectivity*

bottom line: any invertible linear transformation of homogeneous coordinates is a projectivity

Example: mapping between planes

- Projection of rays through a common point - the *center of projection* - defines mapping from one plane to another
- This point-to-point mapping preserves lines: line in one plane is mapped to line in another
- If a coordinate system is defined in each plane and points are represented in homogeneous coordinates, then the central projection mapping may be expressed as $x' = Hx$



Example: removing projective distortion

- Left image: an image with *perspective distortion*: Shape is distorted under *perspective projection*. Notice that the windows are no longer rectangular. The lines of the windows converge to a single point (when extended)
- Synthesized, frontal, orthogonal view of the same wall. left image is related via a perspective transformation to the true geometry of the wall
- The *inverse transformation* is computed by mapping the four imaged window corners to corners of an appropriately sized rectangle
- The *four point correspondences* determine the transformation *(the H matrix)*. This transformation is then applied to the whole image

Example: removing perspective distortion

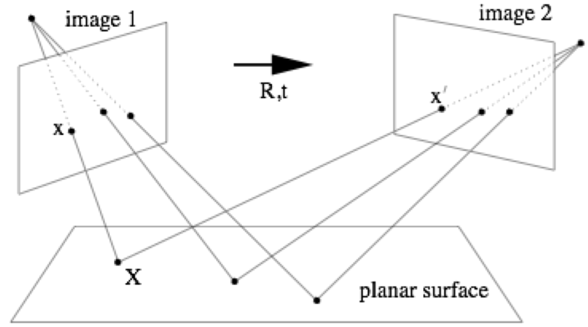
Note that the projective transformation is applied assuming *ALL points lie on the same plane*

- Not true for the ground plane! Ground and front wall are not on the same plane
- Projective transformation to rectify ground plane is not the same as the one for the wall

Your second homework explored this in greater detail

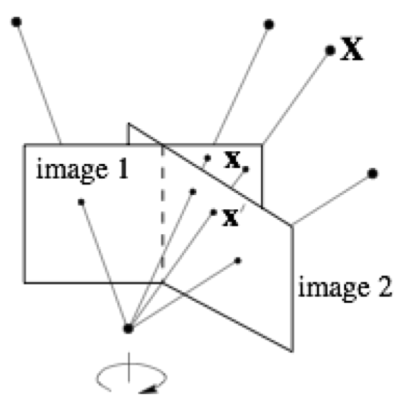
Homography: more examples

Transformation induced by a world plane



Homography: more examples

Projective transformation between two images with the same camera center



4 Hierarchy of transformations

Hierarchy of transformations: Isometry

Isometries

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \epsilon\cos\theta & -\sin\theta & t_x \\ \epsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \epsilon = \pm 1$$

$\epsilon = -1$ reverses the orientation. We can re-write this as

$$x' = \mathbf{H}_E x = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} x, \quad \mathbf{R}^T\mathbf{R} = \mathbf{I} \tag{1}$$

Here the homography matrix $\mathbf{H}_E$ has 3 degrees of freedom (dof): 1 for rotation and 2 for the translation vector t. Special cases include pure rotation and pure translation. Isometries preserve Euclidean distances.

Hierarchy of transformations: Similarity

Similarities = isometry + scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

We can re-write this as

$$x' = \mathbf{H}_S x = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} x, \quad \mathbf{R}^T\mathbf{R} = \mathbf{I} \tag{2}$$

Here the homography matrix $\mathbf{H}_S$has 4 degrees of freedom (dof): 1 for scale, 1 for rotation and 2 for the translation vector t. Similarities preserve shape.

Hierarchy of transformations: Affine

Affine

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = \mathbf{H}_A x = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} x \tag{3}$$

Here the homography matrix $\mathbf{H}_A$ has 6 degrees of freedom (dof): 2 for scale, 2 for rotation and 2 for the translation vector t. Affine preserves parallelism, ratios of parallel lengths, ratios of areas.

Hierarchy of transformations: full Projective

Affine

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ v_1 & v_2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix},$$

$$x' = \mathbf{H}_P x = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^T & 1 \end{bmatrix} x \tag{4}$$

Here the homography matrix $\mathbf{H}_P$ has 8 degrees of freedom (dof): 2 for scale, 2 for rotation, 2 for the translation vector t, and 2 for the line at infinity!.

Under a full projective transformation, line at infinity becomes finite. Allows one to observe vanishing points, horizon.

4.1 line at infinity under projective transformation

Action of affinities and projectivities on line at infinity

Affine transformation

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ 0 \end{bmatrix}$$

line at infinity stays at infinity, but points move along the line. (recall: line at infinity = set of ideal points = third coordinate is zero).

full projective transformation

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^T & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ v_1 x_1 + v_2 x_2 \end{bmatrix}$$

Note that the third entry in the homogeneous coordinates is no longer zero. Thus, under standard pin-hole camera geometry, parallel line map to the so called vanishing point. This is a finite point in the image plane.