Outline

Contents

1 The DWT

November 21, 2017: These are draft notes, will update them tomorrow with additional details based on how the lecture goes today. This para will be deleted when the update is posted.

First, we will go over (again) the discrete transform basics, with the transform functions as samples derived from their continuous counterparts. For the case of the Haar transform, it helps to keep in mind that at the finest level you are considering either taking the average of the two adjacent values (i.e., [1, 1] as the filter) or taking their difference (i.e., [1, -1] as the filter).

Next, recall the concept of nested spaces, with the coarsest resolution subspace (V_0) at the center. As more detailed information (we called them error signals in the Laplacian pyramid representation) are added, we get increasingly better resolution of the signals. The corresponding detail spaces are represented by W_js in our notation below.

Finally, the approximation spaces can be fully represented by what we refer to as the scaling basis and the detail spaces are characterized by the corresponding wavelet basis. These basis are at the core of multiresolution signal analysis. The resulting transformation is referred to as the Wavelet transformation. The beauty of this overall scheme is that we could do this very efficiently: the approximation at any given level ("coarser images") are the signal input for the next level of analysis in such a framework. This is due to the fact that the wavelet basis are self-similar, so the coputations can be recursively performed at each level of approximation. The rest of the story is in the details! (pun intended).

The DWT

We can write

$$f(x) = \sum_k c_{j_0}(k)\,\varphi_{j_0,k}(x)$$

$$+ \sum_{j=j_0}^{\infty} \sum_k d_j(k)\,\psi_{j,k}(x).$$

$j_0 \rightarrow$ arbitrary starting scale.

$c_{j_0}(k) \rightarrow$ approximation coefficients.

$d_j(k) \rightarrow$ detail coefficients.

DWT (contd.)

$$f(x) = \frac{1}{\sqrt{M}} \sum_k W_\varphi(j_0,k)\,\varphi_{j_0,k}(x)$$

$$+ \frac{1}{\sqrt{M}} \sum_{j=j_0}^{\infty} \sum_k W_\psi(j,k)\,\psi_{j,k}(x).$$

$$\boxed{\begin{aligned} W_\varphi(j_0,k) &= \frac{1}{\sqrt{M}} \sum_x f(x)\,\varphi_{j_0,k}(x) \\ W_\psi(j,k) &= \frac{1}{\sqrt{M}} \sum_x f(x)\,\psi_{j,k}(x). \end{aligned}}$$

for $j \geq j_0$ DWT pair.

where Note that x here is a discrete variable, $x = 0, 1, \ldots, M-1$

Example: Haar wavelet

Let $f(n) = [1, 4, -3, 0]$; $M = 4$, $J = 2$ and $j_0 = 0$. The corresponding indices $j = 0, 1$ and $k = 0$ for $j = 0$ and $k = 0, 1$ for $j = 1$.

$$W_\phi(0,0) = \frac{1}{2}\sum_{n=0}^{3} f(n)\phi_{0,0}(n)$$
$$= \frac{1}{2}(1.\ (1) + 4.\ (1) - 3.\ (1) + 0.\ (1)) = 1.$$
$$W_\psi(0,0) = \frac{1}{2}(1.\ (1) + 4.\ (1) - 3.\ (-1) + 0.\ (-1)) = 4$$
$$W_\psi(1,0) = \frac{1}{2}\left(1\sqrt{2} + 4(-\sqrt{2}) - 3.\ (0) + 0.\ (-0)\right) = -1.5\sqrt{2}$$
$$W_\psi(1,1) = \frac{1}{2}\left(1.\ (0) + 4.\ (0) - 3\sqrt{2} + 0(-\sqrt{2})\right) = -1.5\sqrt{2}$$

Example: contd.

Thus the DWT of $f(n) = [1,\ 4,\ \ -3,\ 0]$ is

$$DWT f(n) = \{1,\ 4, -1.5\sqrt{2}, -1.5\sqrt{2}\}$$

$$f(n) = \frac{1}{2}\Big[W_\phi(0,0)\phi_{0,0}(n) + W_\psi(0,0)\psi_{0,0}(n) + W_\psi(1,0)\psi_{1,0}(n) + W_\psi(1,1)\psi_{1,1}(n)\Big]$$

for $n = 0, 1, 2, 3$. For example,

$$f(0) = \frac{1}{2}[1.1 + 4.\ 1 - 1.5\sqrt{2}(\sqrt{2}) - 1.5\sqrt{2}\ (0)] = 1$$

2 Fast Wavelet Transforms

Fast Wavelet Transforms

Computationally efficient implementation of DWT resembles sub-band coding. Recall,

$$\phi(x) = \sum_n h_\phi(n)\sqrt{2}\phi(2x - n) \tag{1}$$
$$\phi(2^j x - k) = \sum_n h_\phi(n)\sqrt{2}\phi(2(2^j x - k) - n) \tag{2}$$

substituting $m = 2k + n$, we get

$$\phi(2^j x - k) = \sum_m h_\phi(m - 2k)\sqrt{2}\phi(2^{j+1}x - m) \tag{3}$$

Similarly,

$$\psi(2^j x - k) = \sum_m h_\psi(m - 2k)\sqrt{2}\phi(2^{j+1}x - m) \tag{4}$$

From the DWT pair, we have

$$\begin{aligned}
W_\psi(j,k) &= \frac{1}{\sqrt{M}}\sum_x f(x)\psi_{j,k}(x) &(5)\\
&= \frac{1}{\sqrt{M}}\sum_x f(x)2^{j/2}\psi(2^j x-k) &(6)\\
&= \frac{1}{\sqrt{M}}\sum_x f(x)2^{j/2}\left[\sum_m h_\psi(m-2k)\sqrt{2}\phi(2^{j+1}x-m)\right] &(7)\\
&= \sum_m h_\psi(m-2k)\left[\frac{1}{\sqrt{M}}\sum_x f(x)2^{(j+1)/2}\phi(2^{j+1}x-m)\right] &(8)
\end{aligned}$$

Thus,

$$W_\psi(j,k) = \sum_m h_\psi(m-2k)W_\phi(j+1,m) \tag{9}$$

i.e., *Detail coefficients at scale j are a function of the DWT approximation coefficients at scale $(j+1)$.*

Similarly,

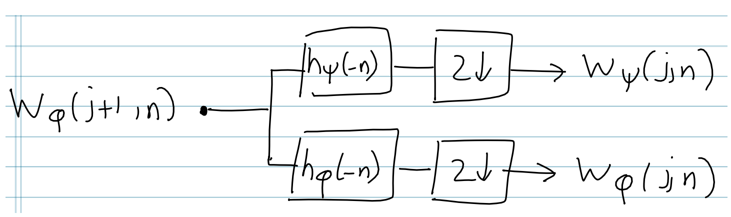
$$W_\phi(j,k) = \sum_m h_\phi(m-2k)W_\phi(j+1,m) \tag{10}$$

These two equations relate the DWT coefficients at adjacent scales.

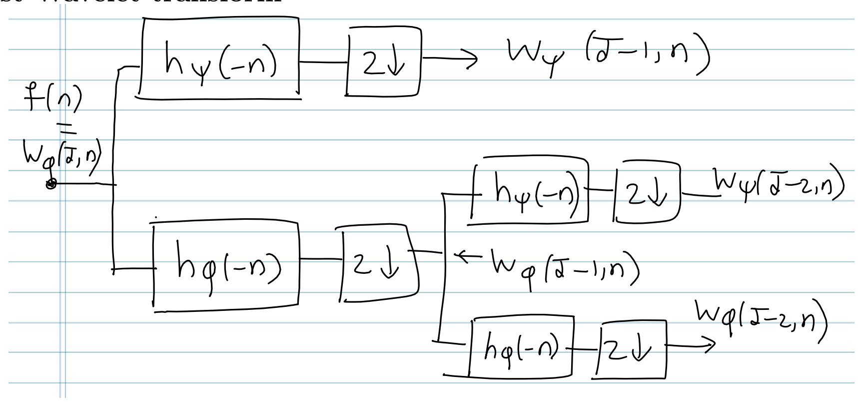
We can rewrite these as,

$$W_\psi(j,k) = h_\psi(-n) * W_\phi(j+1,n)\Big|_{n=2k,k\geq 0}$$

$$W_\phi(j,k) = h_\phi(-n) * W_\phi(j+1,n)\Big|_{n=2k,\ k\geq 0}$$



Fast Wavelet transform



Example Revisited (1)

Revisiting the earlier example

$f[n] = \{1, 4, -3, 0\}$ based on Haar scaling and wavelet coefficients,

recall $h_\varphi(n) = \begin{cases} 1/\sqrt{2} & n = 0,1. \\ 0 & \text{else} \end{cases} \rightarrow \boxed{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}}$.

and $h_\psi(n) = \begin{cases} 1/\sqrt{2} & n = 0 \\ -1/\sqrt{2} & n = 1 \\ 0 & \text{else} \end{cases} \quad \boxed{\left\{\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right\}}$

Example Revisited (2)

$f(n) = W_\varphi(2,n) = \{1, 4, -3, 0\}$.

$W_\psi(1,n) = \left[h_\psi(-n) * W_\varphi(2,n)\right] \rightarrow \boxed{2\downarrow}$

$= \left\{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\} * \{1, 4, -3, 0\} \rightarrow \boxed{2\downarrow}$

$= \left\{-\frac{1}{\sqrt{2}}, \frac{-3}{\sqrt{2}}, \frac{7}{\sqrt{2}}, \frac{-3}{\sqrt{2}}, 0\right\} \rightarrow \boxed{2\downarrow}$

$= \left\{-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right\} \equiv W_\psi(1,n)$.

Example Revisited (3)

$$W_\varphi(1,n) = \left\{\frac{1}{\sqrt{2}}, \frac{1}{2}\right\} * \{1, 4, -3, 0\} \rightarrow \boxed{2\downarrow}$$

$$= \left\{\frac{1}{\sqrt{2}}, \frac{5}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{-3}{\sqrt{2}}, 0\right\} \rightarrow \boxed{2\downarrow}$$

$$= \left\{\frac{5}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right\}.$$

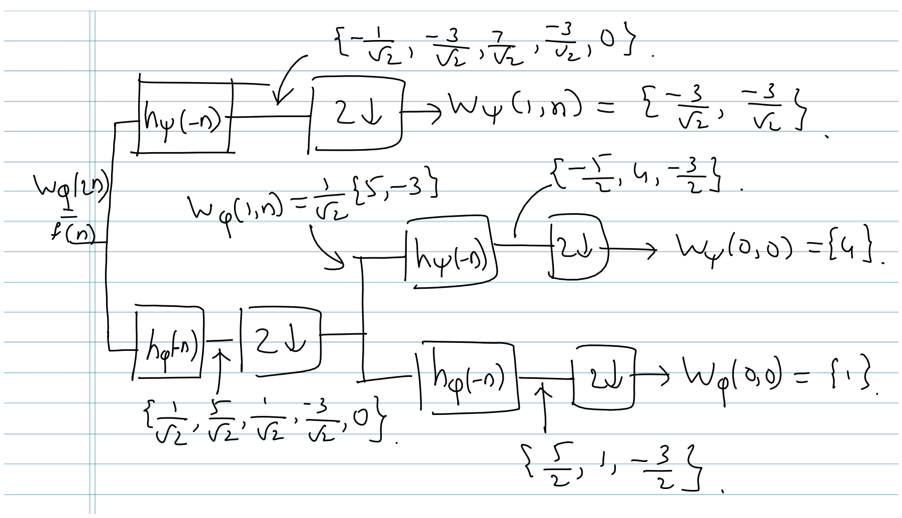
$$W_\psi(0,0) = [h_\psi(-n) * W_\varphi(1,n)] \downarrow 2$$

$$= \left\{-\frac{5}{2}, \frac{8}{2}, \frac{-3}{2}\right\} \downarrow 2 = \{4\}.$$

$$W_\varphi(0,0) = [h_\varphi(-n) * W_\varphi(1,n)] \downarrow 2.$$

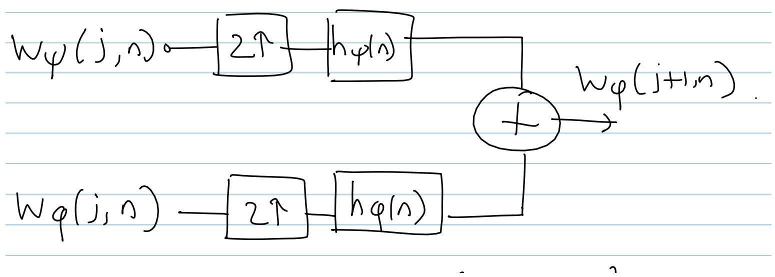
$$= \left\{\frac{5}{2}, \frac{2}{2}, \frac{-3}{2}\right\} \downarrow 2 = \{1\}.$$

Example Revisited (summary figure)

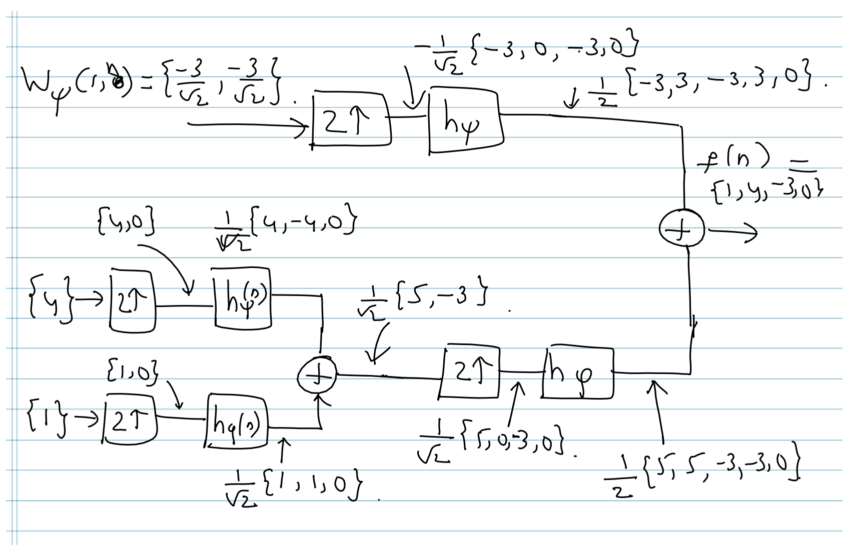


2.1 Inverse transformations

Inverse Transform



Inverse Transform: example



3 2D Separable Wavelets

Scaling function: $\phi(x,y)$ Wavelets: $\{\psi^H(x,y), \psi^V(x,y), \psi^D(x,y)\}$

These are all separable:

$$\begin{aligned}
\phi(x,y) &= \phi(x)\ \phi(y) \\
\psi^H(x,y) &= \psi(x)\ \phi(y), \quad \text{variations along columns} \\
\psi^V(x,y) &= \phi(x)\ \psi(y), \quad \text{variations along rows} \\
\psi^D(x,y) &= \psi(x)\ \psi(y), \quad \text{variations along diagonals}
\end{aligned}$$

Thus, extensions to 2D are straightforward.

$$\phi_{j,m,n}(x,y) = 2^{j/2}\phi(2^j x - m,\ 2^j y - n) \tag{11}$$

$$\psi^i_{j,m,n}(x,y) = 2^{j/2}\psi^i(2^j x - m,\ 2^j y - n), \quad i = \{H,\ V,\ D\} \tag{12}$$

2D Wavelets

Approximation at some arbitrary starting scale j_0

$$W_\phi(j_0, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)\phi_{j_0,m,n}(x, y)$$

Detail coefficients at $j \geq j_0$

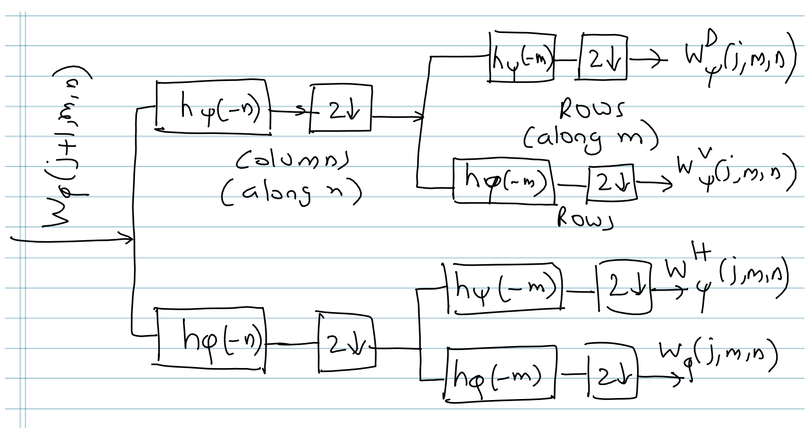
$$W_\psi(j, m, n)^i = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)\psi_{j_0,m,n}(x, y), \quad i = \{H,\ V,\ D\}$$

Reconstruction from wavelet transform coefficients

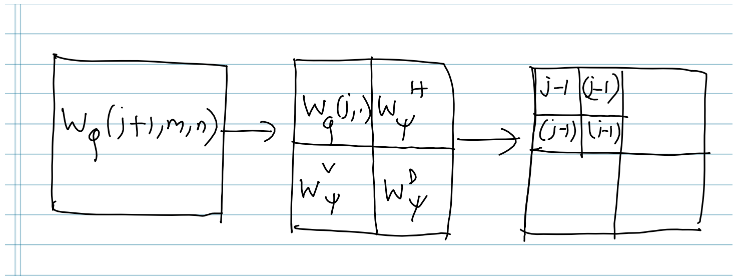
Given the approximation and detail coefficients

$$f(x, y) = \frac{1}{\sqrt{MN}} \sum_m \sum_n W_\phi(j_0, m, n)\phi_{j_0,m,n}(x, y) + \frac{1}{\sqrt{MN}} \sum_i \sum_{j \geq j_0}^{\infty} \sum_m \sum_n W_\psi^i(j, m, n)\psi_{j,m,n}(x, y)$$

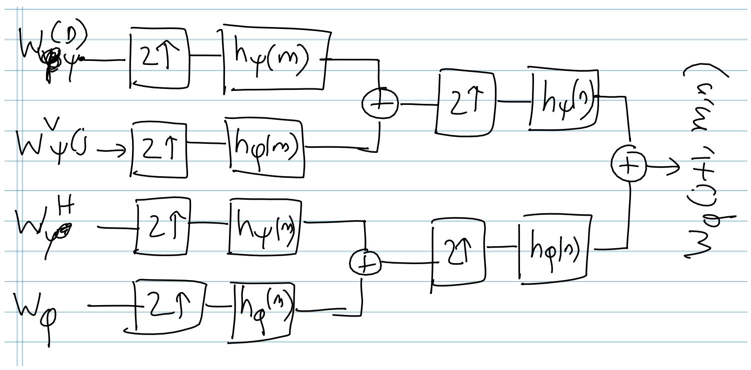
2D Wavelets



2D Wavelets


Wφ(j+1,m,n)
Wφ(j,·)
Wψ H
Wψ V
Wψ D
j-1
(j-1)

2D Wavelets


Wψ (D)
Wψ V
Wψ H
Wφ
2↑
hψ(m)
hφ(m)
hψ(n)
hφ(n)
Wφ(j+1, m,n)