Outline

Contents

Key Questions.
Is there an optimal sampling strategy? Suppose you are given a sample budget, say the number of 'samples' you can keep, then what is the best one can do so as to minimize the error between the original data and the reconstructed data from the 'samples'?

Note: The reason I have put the 'samples' under quotes is that we can consider these 'samples' beyond the traditional samples obtained directly in the signal space. For examples, these could be samples in the transform domain. A good example is the standard JPEG compression where our 'samples' are coming from Cosine Transform coefficients.

Terminology used: Random fields, Homogeneous random fields, wide-sense stationary random fields vs strict-sense stationarity, Autocorrelation functions, Spectral density function.

1 Sampling beyond 'samples'

This lecture focuses on sampling beyond spatial domain samples, leading to basic theory/principles of image compression. We will start with generalizing the sampling requirements (from previous lecture) to collections of pictures. Assuming that these pictures are drawn from a "distribution", we could think of each pictures as an instance of a random field. Further assuming that these random fields are homogeneous, we then derive the necessary and sufficient conditions for "optimal sampling". This leads to an interesting result for random fields, that if you want to sample a random field and reconstruct the original (continuous) field without errors, all you need is to ensure that the autocorrelation function is reproducible without error.

Since we are discussing sampling, focus is on continuous fields (and the derivations are simpler as well). As we move towards compressing discrete signals (next week), we will come back to discrete transformations and discrete versions of what you will learn today. Conceptually, they are very similar.

The take-home message is that if you want to compress, whether it is a continuous signal or a discrete signal, your samples in the compressed domain should be less correlated than the original signal samples. Further, uncorrelated samples lead to optimal compression.

1.1 Generalization to Random fields

Generalization to random fields

- So far: single, band-limited image.
- Lesson learnt: do not let aliasing happen. Mixed components can not be recovered.
  - Example: fast moving "wheels" tend to rotate in the opposite direction at a much lower speed on film. The sampling rate of 24 frames/second is not fast enough.
  - If you have a signal that can not be sampled at the right speed, then filter (smooth) the signal first, and then sample.
- Q: What would be the sampling strategy if you are not given a single image, but a whole collection of them? E.g., a collection of face pictures? What is the appropriate interpolation function?
- A: Use the average properties of the picture collection to determine the right strategy.

Pictures and random fields

Basics

Let $\Omega = \{\omega_1, \omega_2, \cdots, \omega_n\}$ be a set of outcomes of an experiment. Let the probability of the i-th outcome be $p_i = P(\omega_i), p_i \geq 0$ and $\sum p_i = 1$.

Let $\omega_i \to f(\omega_i)$ be a mapping from the deterministic outcome to a real number. Note that ω_i is a random variable (outcome of the experiment) and f(.) is a real number.

Example: mapping the tossing of the coin: Heads = 1 and Tails =0.

Picture as a random field
Let us consider the above experiment of tossing the coin again; but now, when "head" happens, I choose a picture that contains the sand/ocean as viewed from the Santa Barbara pier. If the "Tail" happens, I will chose the picture of the east gate of the campus.

Each of these outcomes results in a two dimensional "field", whose pixel intensities depend on the outcome of the coin-tossing experiment. I can complicate this a bit further by having a collection of pictures, for each of the two scenes, and picking from one of the two sets. One can now index this as $f_s(\omega_i)$.

Picture $f_s(\omega_i)$ as a random field
$f_s(\omega_i)$ is a one parameter family of functions over the set of all outcomes of an experiment and $s \in I$ is an interval on the real axis, or a region of multidimensional Euclidean space.

1. $f_s(\omega_i)$ is a family of random variables, each member of the family being generated by a value of "s".
2. $f_s(\omega_i)$ can also be considered as a family of functions of "s", each member of the family corresponding to an outcome ω_i.
3. If I is 1-D, $f_s(\omega_i)$ is usually referred to as the random process.
4. For $I \geq 2$, $f_s(\omega_i)$ is referred to as a random field.

Another example of a random field

- Consider $I \rightarrow (x, y)$ plane, and s is a point in the image (x, y) plane. Then $f_s(\omega_i) = f(s, \omega_i)$ is a random variable for a given s.
- For a given outcome ω_i, $f(s, \omega_i)$ is a function over the (x, y) plane.
- Let ω_1=water, ω_2=buildings, ω_3=sand, and ω_4=grass. A given image may consist of any of the four regions with probability $P(\omega_i)$. Given ω_i, $f(s, \omega_i)$ is specified as a function over the 2D space.

2D Sampling

Auto-correlation and Auto-covariance functions

Autocorrelation function

$$R_{ff}(\vec{r_1}, \vec{r_2}) = \mathcal{E}\{f(\vec{r_1}), f(\vec{r_2})\} \tag{1}$$

Autocovariance function

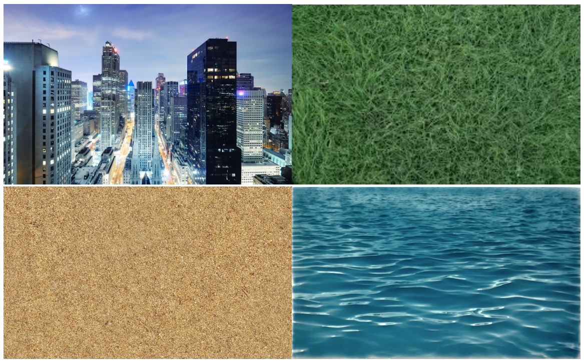


Figure 1: Four different "textures", examples of a random field.

$$\begin{aligned} C_{ff}(\vec{r_1}, \vec{r_2}) &= \mathcal{E}\left\{(f(\vec{r_1}) - \mu_f(\vec{r_1})) \cdot (f(\vec{r_2}) - \mu_f(\vec{r_2}))\right\} \\ &= R_{ff}(\vec{r_1}, \vec{r_2}) - \mu_f(\vec{r_1})\mu_f(\vec{r_2}) \end{aligned} \tag{2}$$

Wide-sense stationary (wss)

A random field is called *wide sense stationary (w.s.s)* if $\mu_f(\vec{r}) = \mu =$ constant and its autocorrelation function is shift-nvariant, i.e.,

$$R_{ff}(\vec{r_1}, \vec{r_2}) = R_{ff}(\vec{r_1} + \vec{r_0}, \vec{r_2} + \vec{r_0}) = R_{ff}(\vec{r_1} - \vec{r_2}) = R_{ff}(\vec{r_2} - \vec{r_1}) \tag{3}$$

Sampling random fields

Consider a random field $f(r)$, represented by its samples $f(r_{mn})$, generated by basis vectors (r_1, r_2). We would like to reconstruct $f(r)$,

$$f(r) = \sum_{m,n} f(r_{mn}) g(r - r_{mn}) \tag{4}$$

- Note that the above is a family of equations, one for each picture in the collection.
- We seek $g(r)$ such that the ensemble error is minimized,

$$e = \mathcal{E}\left\{(f(r) - f_e(r))^2\right\} \tag{5}$$

where

$$f_e(r) = \sum_{m,n} f(r_{mn}) g(r - r_{mn}) \tag{6}$$

Sampling random fields

With a bit of calculus, it is easy to show that the necessary and sufficient condition on $g(r)$ for minimizing the error given in Equation (5) is given by:

$$\boxed{R_{ff}(r) = \sum_{m,n} R_{ff}(r_{mn}) g(r - r_{mn})} \tag{7}$$

The optimum $g(r)$ for the random field $f(r)$ is the one that exactly reproduces the non-random $R_{ff}(r)$ from its samples.

Let $S_{ff}(\omega) = \mathcal{F}\{R_{ff}(r)\}$. Then

$$S_{ff}(\omega) = \frac{G(\omega)}{Q} \sum_{p,q} S_{ff}(\omega - \omega_{pq}) \tag{8}$$

Sampling randm fields: Summary

- Note that $R_{ff}(r)$ is real (for real random fields) and is symmetric with respect to 180 deg rotation.
- For this case, $S_{ff}(\omega)$ is also real and has the same symmetry.
- Finally, *if $S_{ff}(\omega)$ is bandlimited, then the error goes to zero.*

2 Orthonormal Sampling

2.1 Orthonormal functions

Sampling using orthonormal functions

- Picture samples need not correspond to gray levels on a sampling lattice.
  - Example: use coefficients of an orthonormal expansion as samples! (think of Fourier coefficients or DCT coefficients or wavelet coefficients, etc etc...)
- *Orthonormal functions:* Let $f(x, y)$ be a real, square integrable ($\int f(x, y)dxdy < \infty$) function defined over a region $\mathcal{R}$ in the (x, y) plane. Consider $\phi_{mn}(x, y), m, n : 0, 1, 2, \cdots$, where

$$\int\int_{\mathcal{R}} \phi_{mn}(x, y)\phi^*_{pq}(x, y)dxdy = 0, m \neq p \text{ or } n \neq q$$

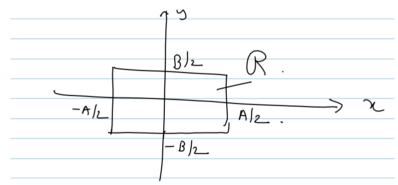
$$\int\int_{\mathcal{R}} |\phi_{mn}(x, y)|^2 \, dx \, dy = 1, \ m = p \text{ and } n = q \tag{9}$$

2.2 Sampling using orthonormal basis

Orthonormal functions: an example

$$\phi_{mn}(x,y) = \frac{1}{\sqrt{AB}} \exp\left(j2\pi\left(\frac{mx}{A} + \frac{ny}{B}\right)\right), \quad m,n: 0,1,2,\cdots \tag{10}$$

form an orthonormal set of function in the region outlined below:



We wish to approximate $f(x,y)$ at all points in $\mathcal{R}$ by

$$\tilde{f}(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} a_{mn}\phi_{mn}(x,y)$$

such that the error

$$e_{MN}^2 = \int\int_{\mathcal{R}} |f(x,y) - \tilde{f}(x,y)|^2 dxdy \tag{11}$$

is minimized.

Orthonormal Sampling

Theorem 1. *The constants a_{mn} that minimize e_{MN}^2 are given by*

$$a_{mn} = \int\int_{\mathcal{R}} f(x,y)\phi_{mn}^*(x,y)dxdy \tag{12}$$

Proof: Prove by showing

$$\int\int_{\mathcal{R}} \left(f(x,y) - \sum_{m,n} a_{mn}\phi_{mn}(x,y)\right)^2 dxdy$$

$$\leq \int\int_{\mathcal{R}} \left(f(x,y) - \sum_{m,n} b_{mn}\phi_{mn}(x,y)\right)^2 dxdy \tag{13}$$

where a_{mn} are given by (12) and b_{mn} arbitrary.

Orthonormal basis

The orthonormal set of functions $\{\phi_{mn}\}$ are called *complete* if

$$\lim_{M,N\to\infty} e_{MN}^2 = 0 \tag{14}$$

A complete orthonormal set of functions is also called an *orthonormal basis*

Orthonormal sampling

Given an orthonormal basis $\{\phi_{mn}(x,y)\}$, $m,n : 0,1,2,\cdots$; defined over a region $\mathcal{R}$ of the $xy-$plane, any function $f(x,y)$, square integrable over $\mathcal{R}$, can be expanded as

$$f(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}\phi_{mn}(x,y)$$

where

$$a_{mn} = \int\int_{\mathcal{R}} f(x,y)\phi^*_{mn}(x,y)dxdy$$

a_{mn} can be viewed as *picture samples* using which $f(x,y)$ can be reconstructed.

Fourier Sampling

Consider the orthonormal basis set defined by

$$\phi_{mn}(x,y) = \frac{1}{AB}\exp\left(j2\pi\left(\frac{mx}{A}+\frac{ny}{B}\right)\right) \tag{15}$$

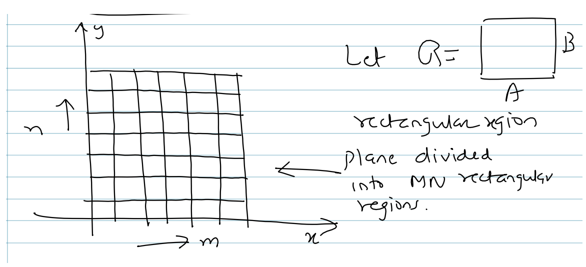
The Fourier samples are then given by

$$a_{mn} = \frac{1}{AB}\int_{-B/2}^{B/2}\int_{-A/2}^{A/2} f(x,y)\exp\left(-j2\pi\left(\frac{mx}{A}+\frac{ny}{B}\right)\right)\,dx\ dy \tag{16}$$

$$= \frac{1}{AB}F\left(\frac{m}{A},\frac{n}{B}\right) \tag{17}$$

$\rightarrow$ these are the values of the Fourier Transform $F(u,v)$ of $f(x,y)$ evaluated at $u = m/A$, $v = n/B$.

Standard Sampling



The pixel sample values in each of the subregions is then the average gray-level in that sub-region. *Verify* that this *standard sampling* can be written as sampling using an orthonormal basis set:

$$\phi_{mn}(x,y) = \begin{cases} \sqrt{\frac{MN}{AB}} & : \frac{mA}{M} \le x \le \frac{(m+1)A}{M},\ \frac{nB}{N} \le y \le \frac{(n+1)B}{N} \\ 0 & : \text{otherwise.} \end{cases} \tag{18}$$

for $m = 0,1,\ldots,M-1; n = 0,1,\ldots,N-1$.

2.3 Sampling random fields using orthonormal expansions

Sampling random fields using orthonormal expansions

- Let $f(x,y)$ be a real, homogeneous random field.
- R_{ff}: autocorrelation function of $f(.)$.
- $$f(x,y) = \sum_{m,n} a_{mn}\phi_{mn}(x,y)$$
  where
  $$a_{mn} = \int_{\mathcal{R}} f(x,y)\phi^*_{mn}(x,y)dxdy \tag{19}$$
- Note that a_{mn} is now a random variable.
- In fact, (19) is actually a family of equations, one for each picture in the random field. The value of a_{mn} will depend on the picture selected.
- If only $M \times N$ samples are retained for every picture in the rF, then one can compute the MSE as before, and then average this error over all the pictures in the rF. This gives us the samping error.

Random fields: Sampling Error

The sampling error is given by:

$$\begin{aligned} e^2_{MN} &= \mathcal{E}\left\{ \int\limits_{\mathcal{R}} \left| f(x,y) - \sum_{m,n} a_{mn}\phi_{mn}(x,y) \right|^2 dxdy \right\} \\ &= \mathcal{E}\left\{ \int\limits_{\mathcal{R}} [f(x,y)]^2 dxdy - \sum_{m,n} |a_{mn}|^2 \right\} \end{aligned} \tag{20}$$

(You can prove this using the orthonormality of the $\phi(.)$ functions in two steps. *verify.*)

Substituting $a_{mn} = \int\int f(x,y)\phi^*_{mn}(x,y)\ dx\ dy$,

$$\begin{aligned} e^2_{MN} = \mathcal{E}\{ & \int\limits_{\mathcal{R}} |f|^2 dxdy \\ & - \sum_{m,n} \int f(x,y)f(x',y')\phi^*_{mn}(x,y)\phi^*_{mn}(x',y')dx\ dy\ dx'dy' \} \\ = SR_{ff}(0,0) & - \sum_{m,n} \int R_{ff}(x-x', y-y')\phi^*_{mn}(x,y)\phi^*_{mn}(x',y')dx'dy'dxdy \end{aligned} \tag{21}$$

where S is the area of the region $\mathcal{R}$ in $xy-$ plane.

2.4 Optimal Sampling

Optimal Sampling

What are the basis functions $\phi_{mn}(x, y)$ *such that for any* M *and* N*, the resulting reconstruction yields the minimum value of the sampling error?*

$$\begin{aligned} e_{MN}^2 &= SR_{ff}(0,0) \\ &\quad - \sum_{m,n} \int R_{ff}(x-x', y-y')\phi_{mn}^*(x,y)\phi_{mn}^*(x',y')dx'dy'dxdy \end{aligned} \tag{22}$$

The problem is to find ϕ_{mn} for a given R_{ff} that maximizes the second term subject to the orthonormality constraints on ϕ_{mn}.

Karhunen Loeve Transform

The $\{\phi_{mn}(x, y)\}$ that maximise the second term in eq. (22) subject to the orthonormality constraints are the solutions of the following integral equation:

$$\int_{-A/2}^{A/2} \int_{-B/2}^{B/2} R_{ff}(x, y, x', y')\phi(x', y')\ dx'dy' = \gamma\phi(x, y) \tag{23}$$

For continuous R_f, such solutions do exis for certain values of γ_{mn} of γ, i.e.,

$$\int R_{ff}(x, y, x', y')\phi_{mn}(x', y')\ dx'dy' = \gamma_{mn}\phi_{mn}(x, y) \tag{24}$$

and, finally,

$$e_{MN}^2 = SR_{ff}(0,0) - \sum_{m,n} \gamma_{mn} \tag{25}$$

If the autocorrelation function is separable in x, y, that is,

$$R(x, y, x', y') = R_1(x, x')R_2(y, y')$$

then one can search for solutions that are also separable,

$$\phi(x, y) = \phi^1(x)\phi^2(y),$$

$$\int_{-A/2}^{A/2} R_1(x, x')\phi^{(1)}(x')dx' = \gamma^{(1)}\phi^{(1)}(x) \tag{26}$$

$$\int_{-B/2}^{B/2} R_2(y, y')\phi^{(2)}(y')dy' = \gamma^{(2)}\phi^{(2)}(y) \tag{27}$$

$$\phi_{mn}(x, y) = \phi_m^{(1)}(x)\phi_n^{(2)}(y) \tag{28}$$

$$\gamma_{mn} = \gamma_m^{(1)}\gamma_n^{(2)} \tag{29}$$

K-L functions and K-L sampling

These orthonomal basis functions – eq. (24), (28) are called the *Karhunen-Loeve functions* and the sampling procedure is called the *Karhunen-Loeve sampling*.

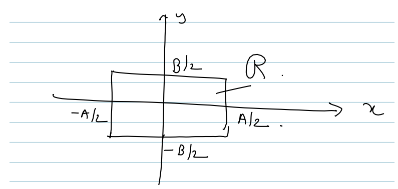
In some cases, one can obtain a closed form solutions to these equations.

Karhunen-Loeve Compression

- *Compression: you exploit the redundancy in the data; no redundancy- no compression.*
- example: Typical pixel intensity values are highly correlated.
- example: A sine function in time. In the time domain, you need to sample this signal at high enough frequncy. However, if you consider the frequency representation of the signal, then only the *amplitude*, *spatial frequency* and *phase shift* are sufficient to fully characterize the sinusoidal signal.
- message: transform the signal to another domain where the *samples* are less correlated.

2.5 Karhunen-Loeve Transform for continuous pictures

Theorem: K-L Transform for continuous pictures:



Let $\phi_{mn}(x, y)$ be a complete family of orthonormal functions defined over $\mathcal{R}$. Then a random field $f(x, y)$ can be expanded as

$$f(x, y) = \sum_{m,n} a_{mn}\phi_{mn}(x, y)$$

where

$$a_{mn} = \int_{\mathcal{R}} f(x, y)\phi^*_{mn}(x, y)dxdy$$

For zero-mean random fields, the functions $\phi_{mn}(x, y)$ that result in uncorrelated samples a_{mn} must satisfy

$$\boxed{\int R_{ff}(x, y, x', y')\phi_{mn}(x', y')\ dx'dy' = \gamma_{mn}\phi_{mn}(x, y)} \tag{30}$$

where $\gamma_{mn} = \mathcal{E}\{|a_{mn}|^2\}$

Proof

Uncorrelated coefficients a_{mn}:

$$\mathcal{E}\{a_{mn}a_{ij}^*\} = \mathcal{E}\{a_{mn}\}\mathcal{E}\{a_{ij}^*\}, \ m \neq i \ \ or \ \ n \neq j$$

$$\begin{aligned} \mathcal{E}\{a_{mn}\} &= \int_{\mathcal{R}} \mathcal{E}\{f(x,y)\}\phi_{mn}^*(x,y)dxdy \\ &= 0, \ \text{for} \ \ m,n = 0,1,2,\ldots \end{aligned} \tag{31}$$

(note: zero-mean random field). This implies then,

$$\mathcal{E}\{a_{mn}a_{ij}^*\} = 0, \ m \neq i \ \ or \ \ n \neq j$$

Now show that ϕ_{mn} that result in the above uncorrelated, zero-mean coefficients must also be the solutions of (24).

$$\begin{aligned} f(x,y) &= \sum_{m,n} a_{mn}\phi_{mn}(x,y) \\ f(x,y)a_{ij}^* &= \sum_{m,n} a_{mn}a_{ij}^*\phi_{mn}(x,y) \\ \mathcal{E}\{f(x,y)a_{ij}^*\} &= \mathcal{E}\{|a_{ij}^*|^2\}\phi_{ij}(x,y) \end{aligned} \tag{32}$$

Now consider:

$$\begin{aligned} a_{mn} &= \int_{\mathcal{R}} f(x,y)\phi_{mn}^*(x,y)dxdy \\ a_{ij}^* &= \int_{\mathcal{R}} f(x',y')\phi_{ij}(x',y')dx'dy' \end{aligned}$$

(random field $f(x,y)$ assumed to real valued.)

$$\begin{aligned} \mathcal{E}\{f(x,y)a_{ij}^*\} &= \int_{\mathcal{R}} \mathcal{E}\{f(x,y)f(x',y')\}\phi_{ij}(x',y')dx'dy' \\ &= \int_{\mathcal{R}} R(x,y,x',y')\phi_{ij}(x',y')dx'dy' \end{aligned} \tag{33}$$

Comparing equations (32) and (33), we get:

$$\boxed{\int_{\mathcal{R}} R(x,y,x',y')\phi_{ij}(x',y')dx'dy' = \mathcal{E}\{|a_{ij}^*|^2\}\phi_{ij}(x,y)} \tag{34}$$

This proves the theorem.

Final remark:

Converse is also true. Given a set of orthonormal functions that are the solutions of the integral equation (30), then if a zero-mean random field is expanded in terms of these orthonormal functions, the coefficients of the expansion are uncorrelated.

9. Image Sampling

2.6 Exponentials as Eigenfunctions of LSI systems

An **Eigenfunction** is a function that goes without a change in its shape through a system/operator. Its amplitude may be altered but not the shape of the function. That is,

$$\mathcal{T}[f] = \lambda f$$

where $\mathcal{T}$ is the system operator, f is the eigenfunction and λ is the eigenvalue associated with the eigenfunction, and is a constant.

For simplicity, let us consider the 1-D LSI system. The input-output relationship is given by the convolution operation,

$$\mathcal{T}[f] = \int h(t-\tau)f(\tau)d\tau$$

Considering a complex exponential as an input to such a system, we get

$$\int h(t-\tau)A\exp(j\omega\tau)d\tau = \int h(\tau)A\exp(j\omega(t-\tau))d\tau$$

$$= A\exp(j\omega t)\int h(\tau)\exp(-j\omega\tau)d\tau = A\exp(j\omega t)H(\omega)$$

Thus the output is the input scaled by $AH(\omega)$.

Shift-invariant Autocorrelation functions: Revisiting Equation (34), if the autocorrelation function is shif-invariant, we see that the LHS is a convolution function; Thus the complex exponentials of the Fourier transform basis form the Eigenfunctions, and the Eigenvalues are the corresponding Fourier Transform coefficients!

Thus, for homogeneous (wss) random fields, Fourier Transform is the KLT that results in uncorrelated coefficients.