Outline

Contents

Key Questions.

What are the conditions under which we can recover the original (i.e., continuous, analog, ..) signal from its samples (i.e., discrete time or space)? Why should we even care?

1 Sampling

In this set of notes/slides we will first review the basics of signal sampling. While much of the data that you see (or hear), whether they are from cameras or on your CD, are already in the digital form, they are sensed and sampled from a continuous space. For audio, the continuous variable is time. For images, it is the two dimensional space (think of your old printed photographs that you want to convert to digital JPEG files). For video, it is space and time! And, in many scientific imaging cases, these could be the 3D space over time, leading to 4D signals. You may even have a 5th dimension–a spectral dimension–where the source has a wavelength associated with it to illuminate a given (biological) specimen;

Note that there is sampling (of the dimension, such as time or space) and then quantization of the signal (the amplitude of the signal is a continuous number as well, and this amplitude needs to be quantized too with a finite number of bits.) The following discussion only focuses on sampling.

We will begin with the familiar and simpler case of 1-D signals and then generalize those to the 2D case. We begin by asking the simple question: suppose we sample a continuous signal; **Under what conditions can we recover the original signal from its sampled version?**

Note: Please review your basics of Fourier series representation of periodic signals as we will be using some of those in the following discussion. I am going to add an appendix to contain some review material on this topic.

Sampling 1-D functions

- Consider a one-dimensional continuous function $f(t)$. We want to represent it by its regularly spaced samples,

  $$f(kT), \quad -\infty \leq k \leq \infty$$

  where T is the *sampling period*.

- How do you reconstruct the original signal $f(t)$ from the $f(kT)$? Can you?
  - yes! by suitably interpolating between the samples.

    $$f(t) = \sum_{k=-\infty}^{\infty} f(kT)\ g(t-kT) \tag{1}$$

  - The value at a specific t is calculated as a weighted combination of the samples $f(kT)$. Note that the interpolating function is centered at t.

- What are these interpolating functions $g(t)$??

Condition for perfect reconstruction
Noting

$$f(kT)\ g(t-kT) = \int_{\infty}^{\infty} f(\tau)g(t-\tau)\delta(\tau-kT)\ d\tau$$

we can write (1)

$$f(t) = \int_{\infty}^{\infty} f(\tau)g(t-\tau)\left(\sum_{k=-\infty}^{\infty}\delta(\tau-kT)\right)\ d\tau \tag{2}$$

The summation in parenthesis above is nothing but an impulse train – a sequence of impulse functions at regular intervals of T. This being a periodic pulse train, it can be expanded using Fourier series as:

$$\sum_{k=-\infty}^{\infty}\delta(\tau-kT) = \sum_{n=-\infty}^{\infty} a_n \exp\left(j\frac{2\pi n}{T}\tau\right) \tag{3}$$

$$\begin{aligned} a_n &= \frac{1}{T}\int_{-T/2}^{T/2}\left(\sum_{k=-\infty}^{\infty}\delta(\tau-kT)\right)\exp\left(-j\tfrac{2\pi n}{T}\tau\right)\ d\tau \\ &= \frac{1}{T}\int_{-T/2}^{T/2}\delta(\tau)\exp\left(-j\tfrac{2\pi n}{T}\tau\right)\ d\tau \\ &= \frac{1}{T}, \quad \forall n \end{aligned} \tag{4}$$

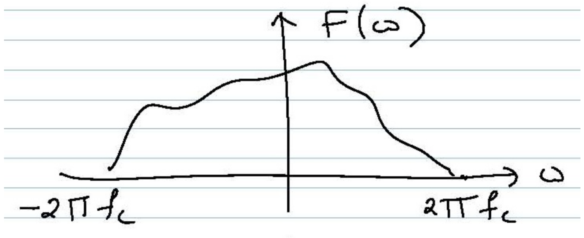


Figure 1: Bandlimited funtion

Substituting (3), (4) in (2),

$$f(t) = \sum_{n=-\infty}^{\infty} \underbrace{\int_{\infty}^{\infty} \left(f(\tau) \exp\left(j\frac{2\pi n}{T}\tau \right) \right) \left(\frac{g(t-\tau)}{T} \right) \; d\tau}_{\text{convolution integral}}$$

$$f(t) = \sum_{n=-\infty}^{\infty} \left\{ f(t) \exp\left(j\frac{2\pi nt}{T} \right) * \frac{g(t)}{T} \right\}$$

Necessary and Sufficient Condition for Reconstruction

$$f(t) = \sum_{n=-\infty}^{\infty} \left\{ f(t) \exp\left(j\frac{2\pi nt}{T} \right) * \frac{g(t)}{T} \right\}$$

and hence,

$$\boxed{F(\omega) = \frac{G(\omega)}{T} \sum_{n=-\infty}^{\infty} F(\omega - \frac{2\pi n}{T})} \tag{5}$$

This is the necessary and sufficient condition for the exact reconstruction of $f(t)$ from its samples $f(kT)$. Note that all the steps above are reversible.

Transform of the sampled sequence

$$\boxed{F(\omega) = \frac{G(\omega)}{T} \sum_{n=-\infty}^{\infty} F(\omega - \frac{2\pi n}{T})}$$

Suppose $f(t)$ is bandlimited, i.e., $F(w) = 0$ for $|w| \geq 2\pi f_c$, for some cut-off frequency f_c.

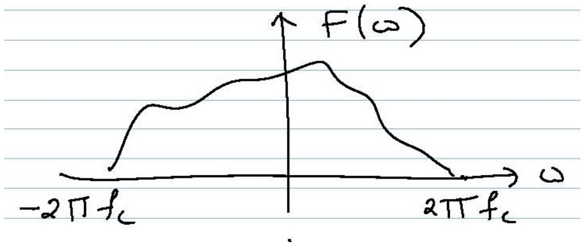


Figure 2: Bandlimited funtion $F(w)$

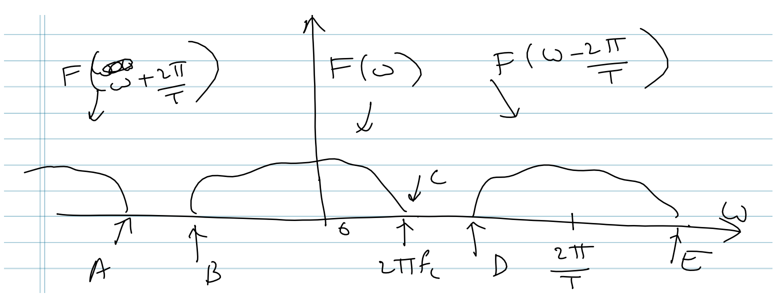
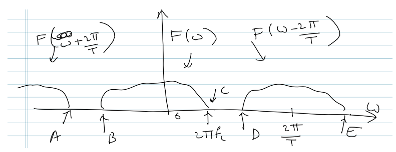


Figure 3: Sampled signal $\sum_k f(t)\delta(t-kT) \rightarrow \sum_{n=-\infty}^{\infty} F(\omega - \frac{2\pi n}{T})$

8. Image Sampling

- Sampled signal is the product of the signal $f(t)$ with the impulse train $\sum \delta(t - kT)$.
- Multiplication in time is convolution in frequency. So, the spectrum of the sampled signal is the convolution of the transform of the impuse train with the transform of the signal, $\mathcal{F}\{f(t)\} = F(\omega)$.



If $T \leq \frac{1}{2f_c}$, there is no overlap of the neighboring spectra.

- For such a bandlimited $f(t)$, we can find a $G(w)$ that picks up one full period of the sampled spectrum, thus fully recovering $F(w)$.

$$G(\omega) = \left\{ \begin{array}{ll} T & : |\omega| \leq 2\pi f_c) \\ 0 & : otherwise \end{array} \right. \tag{6}$$

- Note that (6) would satisfy the condition (5) for perfect reconstruction.

Interpolation function

$$G(\omega) = \left\{ \begin{array}{ll} T & : |\omega| \leq 1/(2\pi f_c) \\ 0 & : otherwise \end{array} \right. \tag{7}$$

and

$$g(t) = \frac{1}{2\pi} \int_{-2\pi f_c}^{2\pi f_c} T e^{j\omega t} d\omega = \frac{\sin 2\pi f_c t}{(\pi t/T)} \tag{8}$$

If $T < \frac{1}{2f_c}$, then in the intervals for which $F(\omega - 2\pi n/T) = 0$, $G(w)$ can be chose arbitrarily.

1.1 Sampling Theorem

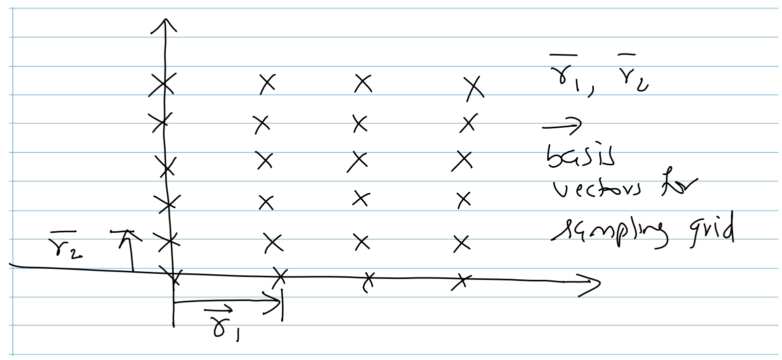
Sampling Theorem

1-D case

If the Fourier transform of a function $f(t)$ vanishes for $|\omega| \geq 2\pi f_c$, then $f(t)$ can be exactly reconstructed from samples of its values taken $\frac{1}{2f_c}$ apart or closer.

2 2D Sampling

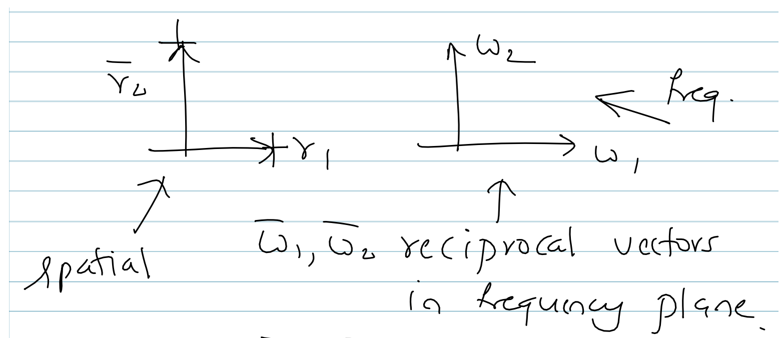
2D Sampling Lattices



Sampling Lattices

$$\vec{r}_{mn} = m\vec{r_1} + n\vec{r_2},\ m, n : 0, \pm 1, \pm 2, \cdots \tag{9}$$

Reciprocal Vectors

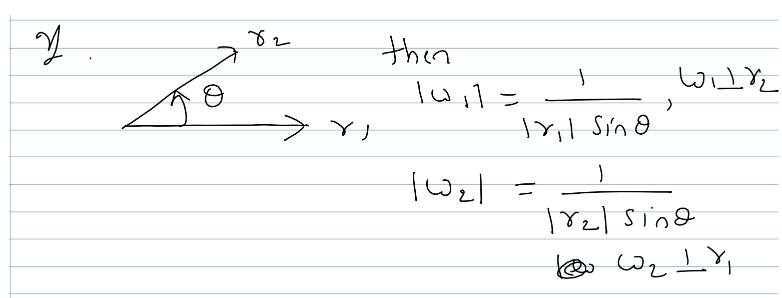


Given $\vec{r_1}$ and $\vec{r_2}$, $\vec{w_1}$ and $\vec{w_2}$ can be uniquely derived.

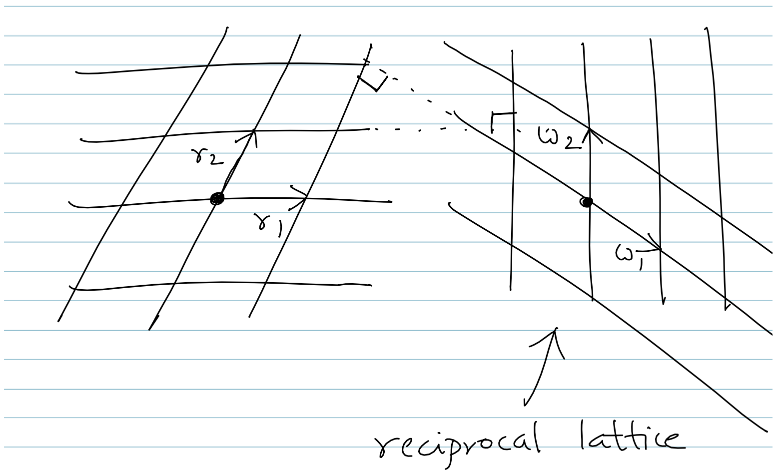
$$\vec{r_i} \cdot \vec{w_j} = \begin{cases} 1 & : i = j \\ 0 & : otherwise \end{cases} \tag{10}$$

$$\vec{w_1} \textit{ is perpendicular to } \vec{r_2}$$

Reciprocal Vectors



Reciprocal Lattice



$$\bar{\omega}_{mn} = m\bar{\omega}_1 + n\bar{\omega}_2, \quad m,n: 0, \pm 1, \ldots$$

Sampling of Pictures using a Sampling Lattice

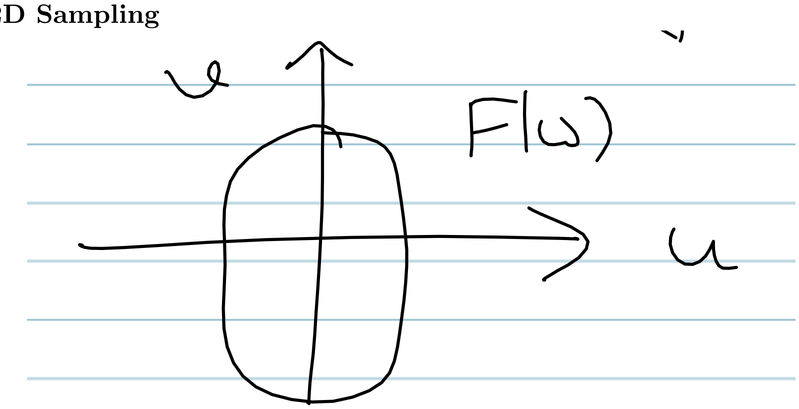
- $f(x, y) \leftrightarrow F(u, v)$
- $$F(u, v) = \int\int f(x, y) \exp(-j2\pi(ux + vy))\; dx\; dy$$
- $$F(\vec{w}) = \int f(\vec{r}) \exp(-j2\pi(\vec{w} \cdot \vec{r}))\; d\vec{r}$$
- Assume $f(x, y)$ is bandlimited. $f(r)$ represented using samples $f(r_{mn})$.

$$f(r) = \sum_m \sum_n f(r_{mn}) g(r - r_{mn})$$

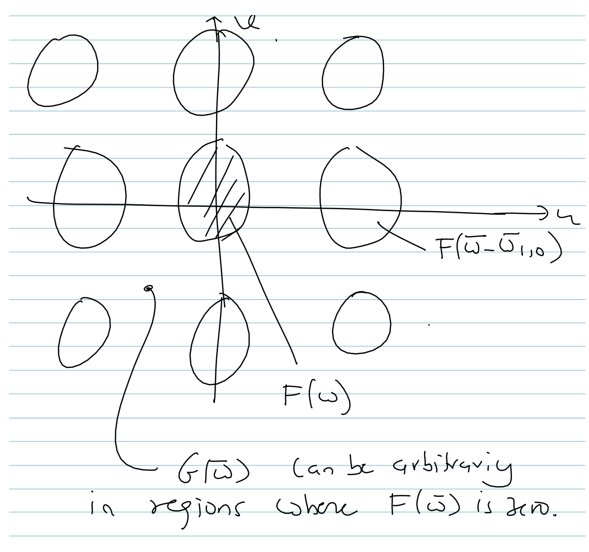
- Similar to 1D case, with $\{w_{pq}\}$ a reciprocal points, and Q area of the parallelogram formed by the vectors,

$$F(w) = \frac{G(w)}{Q} \sum_p \sum_q F(w - w_{pq})$$

2D Sampling



2D Sampling



2.1 2D Sampling Theorem

2D Sampling Theorem

Theorem 1. *A function $f(\overrightarrow{r})$ whose Fourier transform $F(\overrightarrow{w})$ vanishes over all but a bounded region in the spatial frequency space cane be reproduced exactly from its values taken over a lattice of points $(m\overrightarrow{r_1} + n\overrightarrow{r_2}), m, n : 0, \pm 1, \pm 2, \cdots,$ provided that the vectors $\overrightarrow{r_1}$ abd $\overrightarrow{r_2}$ are small enough to ensure non-overlapping of the spectrum $F(\overrightarrow{w})$ with its images on a periodic lattice of points $(p\overrightarrow{w_1} + q\overrightarrow{w_2}), p, q : 0, \pm 1, \pm 2, \cdots,$ with $\overrightarrow{r_i} \cdot \overrightarrow{w_j} = 0, i \neq j,$ and $\overrightarrow{r_i} \cdot \overrightarrow{w_i} = 1, i, j : 1, 2.$*

Example of undersampling

An example of *aliasing*

$$f(x,y) = 2\cos(2\pi(3x+4y)), \text{ sampled at intervals of } \Delta x = 0.2 = \Delta y.$$

Note that the highest frequencies are $u_0 = 3$ units along u, and $v_0 = 4$ units along v (horizontal and vertical frequencies). So, for an error free reconstruction, the sampling frequencies should be at least $2u_0$ and $2v_0$ in the respective directions. However, given the sample spacings, we get the cut-off frequencies for the sampled spectrum to be $\frac{1}{\Delta x} = \frac{1}{\Delta y} = 5$, and this is less than the minimum needed along either of the axes.

$$\begin{aligned}
F(u,v) &= \delta(u-3, v-4) + \delta(u+3, v+4) && \text{(spectrum of signal)} \\
F_s(u,v) &= 25 \sum_m \sum_n \{\delta(u-3-5m, v-4-5n) \\
&\quad + \delta(u+3-5m, v+4-5n)\} && \text{(spectrum of sampled signal)}
\end{aligned}$$

$$\text{Let } G(u,v) = \begin{cases} \frac{1}{25} & : -2.5 \le u, v \le 2.5 \\ 0 & : else \end{cases}$$

Then the reconstructed signal is given by:

$$F_{recon}(u,v) = \delta(u-2, v-1) + \delta(u+2, v+1)$$

and

$$f_{recon}(x,y) = 2\cos(2\pi(2x+y))$$

Aliasing

- *Aliasing* occurs when you do not get the sampling right. If the signal is sampled at less than twice the highest frequency (*the Nyquist rate*) on a regular grid, mixing of the frequency components in the sampled spectrum makes reconstruction impossible. There is nothing one can do to undo the effect.
- One interesting consequence of this aliasing are the Moire patterns. When one mixes different high frequency patterns, you get the illusion of low frequency patterns that are not originally present. See the online demos.

2.2 Random fields

Generalization to random fields

- So far: single, band-limited image.

- Lesson learnt: do not let aliasing happen. Mixed components can not be recovered.
  - Example: fast moving "wheels" tend to rotate in the opposite direction at a much lower speed on film. The sampling rate of 24 frames/second is not fast enough.
  - If you have a signal that can not be sampled at the right speed, then filter (smooth) the signal first, and then sample.
- Q: What would be the sampling strategy if you are not given a single image, but a whole collection of them? E.g., a collection of face pictures? What is the appropriate interpolation function?
- A: Use the average properties of the picture collection to determine the right strategy.

Appendix

Some useful functions/definitions

Periodic Impulse or Impulse Train

$$\delta_N[n] = \sum_{m=-\infty}^{\infty} \delta(n - mN), \quad n \text{ an integer}$$

Discrete-time periodic impulse

$$\delta_T(t) = \sum_{-\infty}^{\infty} \delta(t - nT), \quad n \text{ an integer}$$

Sinc function

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

This definition of normalized SINC function is popular in the signal processing community, though another commonly used definition (un-normalized) is

$$\text{sinc}_2(t) = \frac{\sin(t)}{t}$$

As long as one is consistent with the use of either of the two, it really does not matter which one you use. I will make it clear in my notes the specific version that is used in any given context, with the default being the first definition with the normalization by π.

Note that $\text{sinc}(0) = 1$ (is defined in the limit) and that the $\text{sinc}(x)$ function is zero at all other integer values of x.

Also note that

$$\int \text{sinc}(x)dx = 1$$

$$\int \text{sinc}_2(x)dx = \pi$$

Periodic signal and its Fourier Series representation Consider now a periodic signal $x(t)$ with period T, i.e., $x(t+kT) = x(t)$. Then, this signal can be expressed as a sum of complex exponentials weighted by the *Fourier Coefficient* $X[k]$ as:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{+j2\pi kt/T}$$

$$X[k] = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-j2\pi kt/T}$$

Fourier Transform (1D)

$$X(f) = \mathcal{F}(x(t)) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt, \quad x(t) = \mathcal{F}^{-1}(X(f)) = \int_{-\infty}^{\infty} X(f) e^{+j2\pi ft} df$$

Fourier Transform of Periodic Signals

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi kt/T} \longleftrightarrow X(f) = \sum_{k=-\infty}^{\infty} X[k] \delta(f - k/T)$$

This follows immediately from the shifting property of the Fourier transform, where a phase shift translates to frequency shift in the transform domain. Note that the Fourier Transform of an impulse function is 1 for all frequencies and the transform of an impulse train (in time) is another impulse train (in frequency) with the spacings between impulses inversely proportional to each other.

3 Questions

1. Let $f(x, y)$ and $F(u, v)$ denote an analog signal and its analog Fourier transform. Let $f(m, n)$ denote a sequence of its (discrete space) Fourier transform. An ideal analog-to-digital converter converts $f(x, y)$ to $f(m, n)$ by

$$f(m,n) = \begin{cases} f(x,y)|_{x=mT_1, y=nT_2}, & \text{If both } m \text{ and } n \text{ are even, or both are odd} \\ 0 & \text{Otherwise} \end{cases} \tag{11}$$

The sampling periods T_1 and T_2 are related by $T_2 = \frac{\sqrt{3}}{3} T_1$. It is convenient to represent (1) by the system shown in Figure 4.

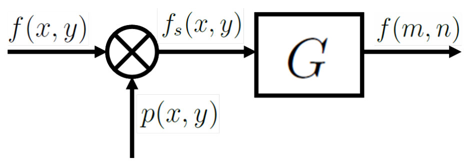


Figure 4: Analog to digital converter

Note that the function $p(x, y)$ is a periodic train of impulses. The system G converts an analog signal $f_s(x, y)$ to a sequence $f(m, n)$ by measuring the area under each impulse and using it as the amplitude of the sequence $f(m, n)$. You may want to use the following Fourier Transform relationship in answering the following.

$$\Im\left\{ \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - mT, y - nT) \right\} = \frac{1}{T^2} \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \delta\left(u - \frac{r}{T}, v - \frac{s}{T}\right)$$

where $\delta(x, y)$ is the Dirac-delta function.

(**a**) Determine $P(u, v)$, (Fourier transform of $p(x, y)$).

(**b**) Express $F_s(u, v)$, (Fourier transform of $f_s(x, y)$) in terms of $F(u, v)$.

(**c**) Sketch an example of $F(u, v)$ and the corresponding $F_s(u, v)$.

(**d**) Suppose $f(x, y)$ is band-limited to a circular region such that $F(u, v)$ has the following property: $F(u, v) = 0, \sqrt{u^2 + v^2} \geq b$. Determine the conditions on T_1 and T_2 such that $F(x, y)$ can be exactly recovered from $x(m, n)$.

(**e**) Comparing the result from (**d**) with a corresponding result based on a rectangular sampling grid, discuss which sampling grid is more efficient for exact reconstruction of circularly bandlimited signals.

(**f**) Show that the sampling grid used in this problem is efficient compared with sampling on a rectangular grid when $f(x, y)$ is bandlimited to the hexagonal region shown in Figure 5 below.

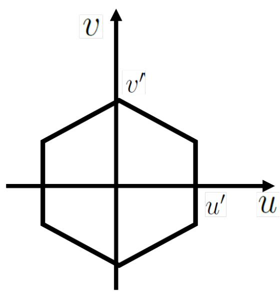


Figure 5: Hexagonal region