Outline

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1 Introduction

This lecture introduces you to the fundamentals of multiresolution image analysis. In the 1980s Multiresolution analysis evolves somewhat independently in mathematics and in signal processing. Perhaps the first well known paper on this topic in the image processing literature is by Burt and Adelson on the Laplacian Pyramids. You implemented the approximation pyramid in your homework earlier. In signal processing, a good reference is the book by PP Vaidyanathan (who got his PhD working with Professor Mitra here at UCSB) and is currently at Caltech. For a more detailed exposure to this topic you should take ECE258a/b.

My plan is to provide a basic understanding of the multiresolution analysis, mostly from an image compression point of view. We will start with the Laplacian pyramid as it provides an intuitive intro to the topic. Note, however, that the Gaussian functions used in constructing the Laplacian pyramid do not form basis functions. Instead, it has found use in computer vision, recall the SIFT descriptors and initial pre-processing to compute these descriptors. The term **wavelets** is used to denote an orthogonal basis set that form the core of the multiresolution decomposition. These wavelets are often carefully constructed, mostly numerically as analytical derivations are often hard. There exist a number of different wavelet families (obviously named after those who came up with those functions first), and JPEG 2000 standard specifies certain wavelet functions as part of the standard.

For this lecture, we will discuss the general concepts underlying the wvelet analysis, and much of it is taken from Chapter 8 of Gonzalez and Woods. We will conclude with the simple example of a Haar Wavelet basis. You have a homework problem relating to this Haar Wavelet.

Take home points

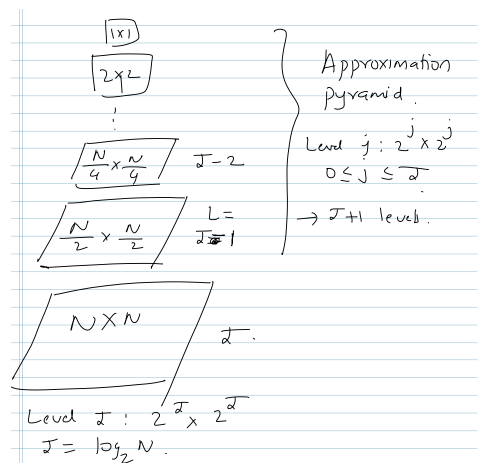
- Wavelets form an orthogonal family of basis functions
- Laplacian pyramid representation is one of the first "multiresolution" decomposition introduced in image processing

- The Gaussian functions in the Laplacian decomposition are not orthogonal and do not form a basis in the strict sense of the terminology
- However, the wavelets and (the difference of) Gaussians share a common property: they form a self-similar basis set for the successive approximations of the signal that are derived from the original
- The basis functions in a wavelet decomposition are all obtained from what is known as the "mother wavelet". This mother wavelet function is dilated (stretched) and translated to create the orthogonal basis set. This is unlike the Unitary Transformations we have discussed so far (e.g., Fourier, Cosine, Sine, ..)

2 Image Pyramids

Image Pyramids

Image Information is at different spatial resolutions



Reference: Image pyramids, Burt and Adelson, 1983. usually stop after p decompositions.

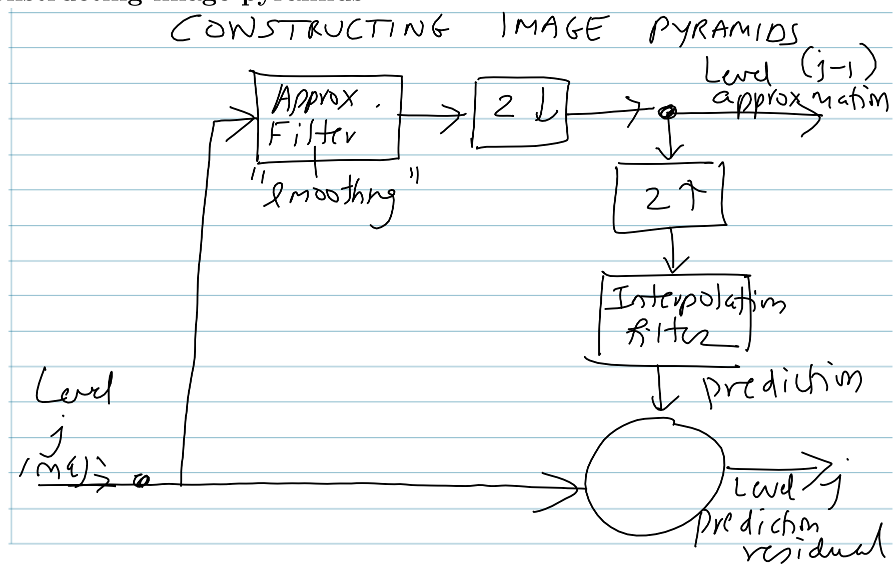
$$j = J - p, J - p + 1, \ldots, J - 1, J$$

$\rightarrow$ p+1 levels, $1 \le p \le J$ Total number of elements in a $p+1$ level pyramid is:

$$\begin{aligned} & N^2\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^p}\right) \\ & \le N^2\left(\frac{4}{3}\right) \end{aligned} \tag{1}$$

2.1 Gaussian and Laplacian pyramids

Constructing image pyramids



Constructing image pyramids

- If the approximation filter is Gaussian → Gaussian pyramid.
- For Gaussian approximation, the residuals → residual pryramid → *Laplacian Pyramid*, as the *difference of Gaussians* approximate the Laplacian function.

Recall that an image with a "uniform" histogram has a higher entropy == requires more bits to code compared to one with a "peaky" histogram. As you can see from the following example of the Laplacian pyramid, the error histograms should require fewer bits to code than the original counterparts at the same pixel resolution.

Laplacian pyramid

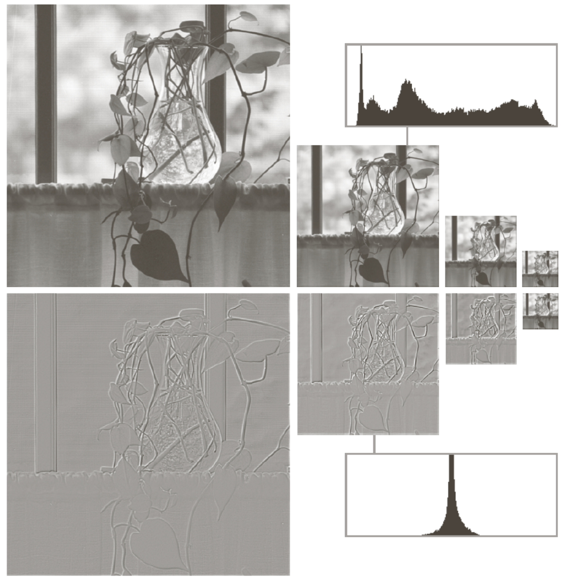
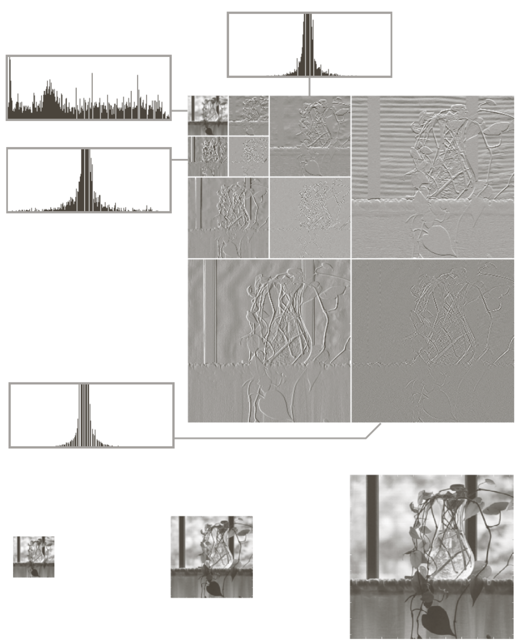


Figure 7.10 from Gonzalez & Woods, 3E Top row: Gaussian approximation bottom row: Laplacian pyramid of the residuals (error)

The following one decomposed and synthesized back using Haar wavelets. Histograms of coeeficients at different resolutions are shown together with reconstructed images at 64×64, 128×128 and 256×256.

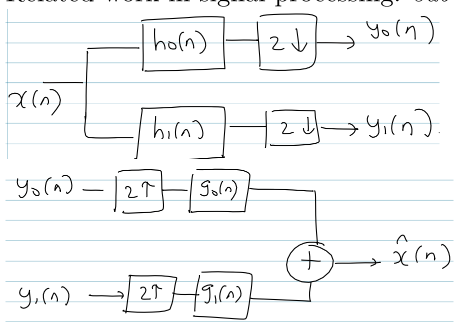
Example: 2D Haar Wavelets (G& W Figure 7.10)

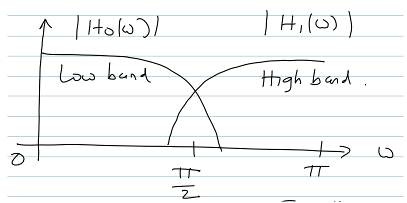


3 in Signal Processing terminology: Sub-band coding

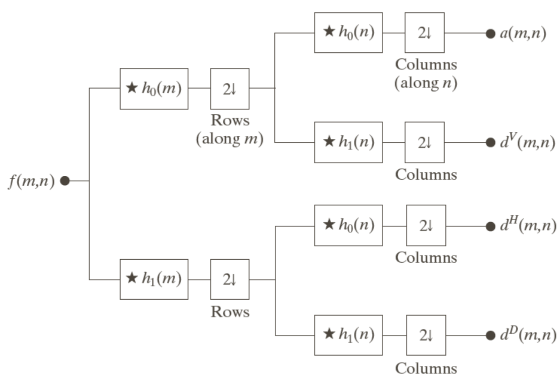
Sub-band coding

Related work in signal processing: *sub-band coding*





2D sub-band coding (G&W Figure 7.7)



$a(m,n)$: approximation; $d^V(m,n)$: vertical details; $d^H(m,n)$: horizontal details; $d^D(m,n)$: diagnonal details.

In order for this analysis/synthesis to work, the filters $\{h_i(.)\}$ must satisfy certain constraints. Further reading: see P. P. Vaidyanathan, Multirate systems and filter banks, Prentice Hall, 1993.

Basis Functions and Series Expansion

Let us go back to signal analysis using a set of basis functions,

$$f(x) = \sum_k \alpha_k \phi_k(x)$$

where α_k are the expansion coefficients and $\phi_k(x)$ are the *basis functions*. The coefficients of expansion are given by $\alpha_k =< f|\tilde{\phi}_k >$ where $< f|g >$ is the standard inner product of two (complex) functions f and g,

$$< f|g >= \int f(x)g^*(x)dx$$

Dual functions $\tilde{\phi}_k(x)$

Two possible scenarios of interest:

- *Case 1: Orthogonal.* $< \phi_j|\phi_k >= \delta_{jk}$. In this case, the basis functions $\{\phi_k(x)\}$ form an orthonormal basis, the basis and its *dual* are equivalent, i.e., $\phi_k(x) = \tilde{\phi}_k(x)$.
- *Case 2. biorthogonal.* $< \phi_j|\tilde{\phi}_k >= \delta_{jk}$. In this case the basis functions are not orthogonal but form a biorthogonal pair with their dual functions $\tilde{\phi}_k$. In filter design, biorthogonality offers more flexibility/freedom.

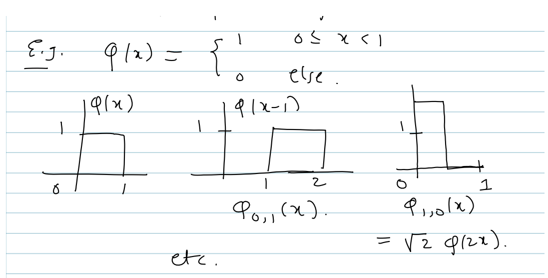
Multiresolution Analysis: Scaling functions

Consider again

$$f(x) = \sum_{j,k} \alpha_{jk}\phi_{jk}(x),$$

$$\phi_{jk}(x) = 2^{\frac{j}{2}}\ \phi(2^{\ j}x - k)$$

k : position of the function and j : scale or width of the function. By choosing $\phi(x)$ appropriately, we can cover the set of all measurable, square integrable functions (the set represented by $L^2(R)$.)



Multiresolution Analysis: Requirements

1. Scaling functions $\phi(.)$ are orthogonal to their integer translations, i.e., $< \phi_{j,k}|\phi_{j,k'} >= 0,\ k \neq k'$.

2. Nested subspaces: The subspaces spanned by the scaling functions at "coarse" (lower) scales are nested within those spanned at "finer" (higher)scales.

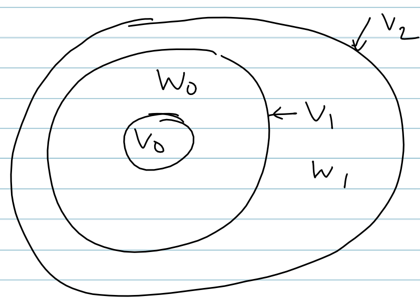
$$V_{-1} \in V_0 \in V_1 \in V_2 \ldots$$

V_j : subspace spanned by the scaling function ϕ_j.

$$V_\infty = L^2(R)$$

$$\{\phi_{j,k}(x)\} \rightarrow \text{basis for } V_j$$

Nested Subspaces (1)



$V_\infty = \{L^2(R)\}$.

$V_1 = V_0 \oplus W_0$

$V_2 = V_1 \oplus W_1$

etc.

$\{\varphi_{j,k}(x)\} \rightarrow$ basis for V_j.

$\{\psi_{j,k}(x)\} \rightarrow$ basis for W_j.

$$\psi_{j,k}(x) = 2^{j/2}\,\psi(2^j x - k).$$

$\psi \rightarrow$ wavelets fns.

Nested Subspaces (2)

The orthogonal complement of V_j in V_{j+1} is W_j. All members of V_j are othogonal to the members of W_j.

$$\Rightarrow \langle \varphi_{j,k}(x), \psi_{j,\ell}(x) \rangle = 0 \quad \forall\, j,k,\ell \in Z.$$

Nested Subspaces (3)

Consider $\varphi_{j,k}(x) \in V_j \subset V_{j+1}$

$\downarrow$

Can be expanded using $\varphi_{j+1,k}$

$$\varphi_{j,k}(x) = \sum_n \alpha_n \varphi_{j+1,n}(x)$$

$$= \sum_n h_\varphi(n)\, 2^{(j+1)/2} \varphi(2^{j+1}x - n)$$

where

$$\varphi_{0,0}(x) = \varphi(x) = \sum_n h_\varphi(n) \sqrt{2}\, \varphi(2x-n).$$

Scaling fn coefficients. scale translation

Nested Subspaces: Haar scaling functions

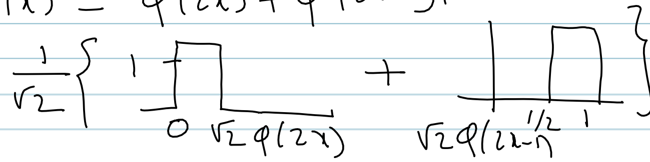
Going back to the Haar fn example

$$\varphi(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & \text{else} \end{cases}$$

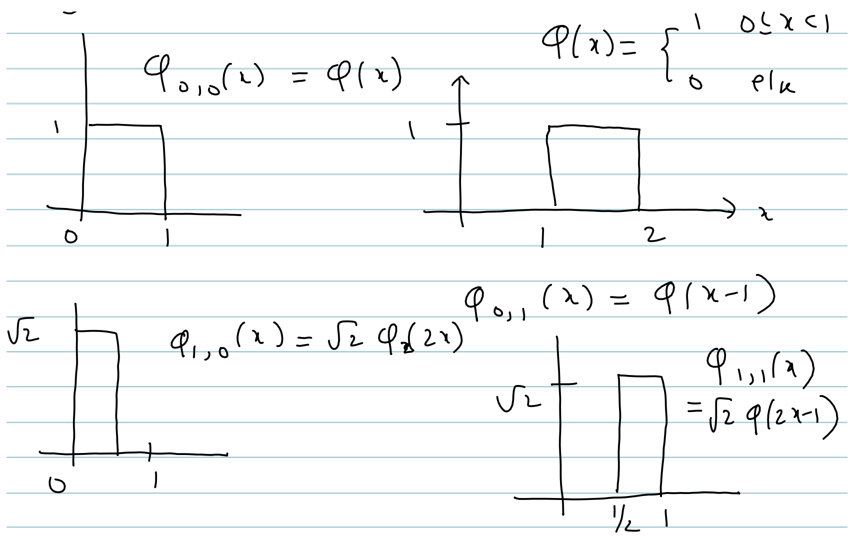
$$\varphi(x) = \frac{1}{\sqrt{2}}\left[\sqrt{2}\,\varphi(2x)\right] + \frac{1}{\sqrt{2}}\left[\sqrt{2}\,\varphi(2x-1)\right].$$

$$\left(h_\varphi(0) = h_\varphi(1) = \frac{1}{\sqrt{2}}\right).$$

$$\varphi(x) = \varphi(2x) + \varphi(2x-1).$$



Haar scaling functions: Recap



Wavelet Functions $\psi(x)$

These wavelet functions span the difference between two neighboring scaling sub-spaces.

$$\psi_{j,k}(x) = 2^{\,j/2}\psi(2^{\,j}x - k)$$

$$W_j = \text{span}_k\{\psi_{j,k}(x)\}$$

and

$$V_{j+1} = V_j \oplus W_j$$

where $\oplus$ denotes union of spaces.

Nested Subspaces: Haar Wavelets

For the wavelet counterpart, we can similarly write:

$$\psi(x) = \sum_n h_\psi(n)\sqrt{(}2)\phi(2x - n) \tag{2}$$

$h_\psi(n)$ is related to $h_\phi(n)$ as,

$$h_\psi(n) = (-1)^n h_\phi(1 - n)$$

Example 1. Haar

$$h_\phi(0) = h_\phi(1) = 1/\sqrt{2}, \text{ as before..}$$

$$h_\psi(0) = (-1)^0 h_\phi(1 - 0) = 1/\sqrt{2}$$

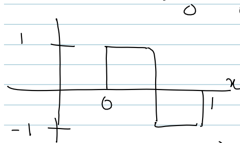
$$h_\psi(1) = (-1)^1 h_\phi(1 - 1) = -1/\sqrt{2}$$

Wavelets

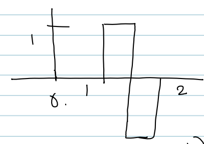
$$\psi(x) = \phi(2x) - \phi(2x-1)$$

Back to Haar

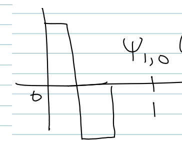
$$\psi(x) = \begin{cases} 1 & 0 \le x < 0.5 \\ -1 & 0.5 \le x < 1 \\ 0 & \text{else.} \end{cases}$$



$\psi(x) = \psi_{0,0}(x)$



$\psi_{0,1}(x) = \psi(x-1).$



$\psi_{1,0}(x) = \sqrt{2}\,\psi(2x).$

etc.