Multi-level Monte Carlo Method and its Numerical Analysis in Black-Scholes Option Model

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Background

Standard Monte Carlo Method

- Flexible
- Independent of dimensions
- Estimation Variance
- Convergence Speed

Problem

How many samples needed?

I want accuracy and don't want too much cost. How?

Objectives: Estimated variance, Convergence Speed, Definition of Cost…

Table of Contents

Methodology

1. Mathematical Derivation of MLMC

Brownian Motion, Ito Integral, SDE→

→Standard MC, Two-level MC, MLMC

2. Numerical Analysis(EM method)

Standard MC, MLMC
on European, Asian and lookback Options

1.Brownian Motion

A Randomly Moving Suspended Particle:

$$W_{t+\Delta t} = W_t + \alpha * N(0, \Delta t)$$

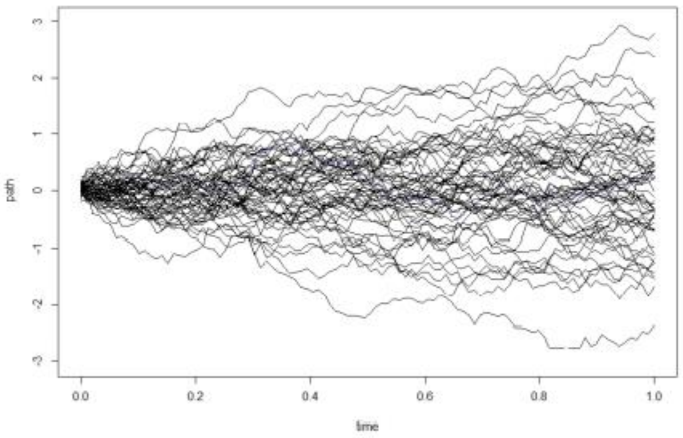


Figure 2.2: 60 paths of standard Brownian Motion(dt=0.01)

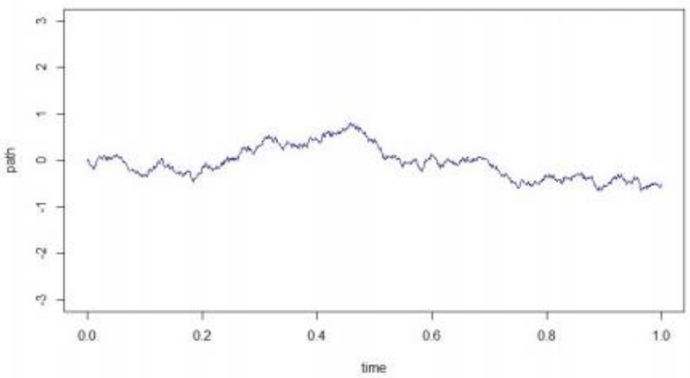


Figure 2.3: Average of 60 paths of standard Brownian Motion

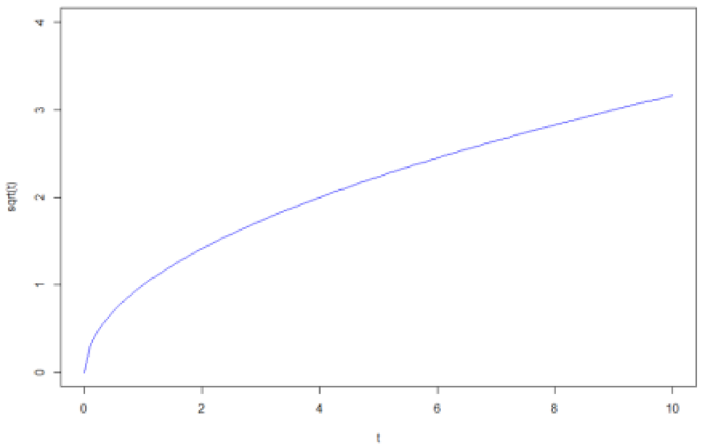


Figure 2.4: Expected Variance of standard Brownian Motion

Definition

Definition 1 [2]

A scalar standard Brownian motion, or standard Wiener process, over [0, T] is a random variable W(t) that depends continuously on $t \in [0, T]$ *and satisfies the following three conditions.*

1. W(0) = 0 (with probability 1).

2. For $0 \leqslant s < t \leqslant 1$ *the random variable given by the increment W(t)−W(s) is normally distributed with mean zero and variance t − s; equivalently, W(t) −W(s)*$\sqrt{t}$ *− s N(0, 1), where N(0, 1) denotes a normally distributed random variable with zero mean and unit variance.*

3. For $0 \leqslant s < t < u < v \leqslant T$ *the increments W(t) − W(s) and W(v) − W(u) are independent.*

2.SDE and Ito Integral

Difference from ODE

$$\frac{dX_t}{dt} = f\left(X_t, t\right)$$

$$\frac{dX_t}{dt} = f\left(X_t, t\right) + g\left(X_t, t\right) * \text{ noise}$$

$$\frac{dX_t}{dt} = f\left(X_t, t\right) + g\left(X_t, t\right) * \Delta W_t$$

2.SDE and Ito Integral

Ito Integral

$$\frac{dX_t}{dt} = f\left(X_t, t\right) + g\left(X_t, t\right) * \Delta W_t$$

$$X_t = X_0 + \int_0^t f\left(X_s, s\right) ds + \int_0^t g\left(X_s, s\right) dW_s$$

$$I = \int_0^t g\left(X_s, s\right) * N(0, dt)$$

$$I = \lim_{N \to \infty} \sum_{n=1}^{N} g\left(X_{t_n}, t_n\right) * N\left(0, \frac{t}{N}\right) \quad t_n = \frac{tn}{N}$$

3. Standard MC

Step1: Find a distribution density function $f_X(x)$, where $\int_a^b f_X(x)dx = 1$, in order to facilitate calculations, Uniform Distribution is generally used;

Step2: Assign the function g according to the probability density to obtain a new function

$g^*(x) = \frac{g(x)}{f_X(x)} \Rightarrow I = \int_a^b g^*(x) f_X(x)dx;$

Step3: According to Step1, let X_i obey Uniform Distribution, that is$X_i (i = 1, 2, ..., N) \Rightarrow f_X(x) = \frac{1}{b-a}$, where X_i are MC sample points;

Step4: Find the new function g*'s average of the values at each sample point(i.e.$I \approx \widehat{I} = \frac{1}{N}\sum_{i=1}^{N} g^*(X_i)$).

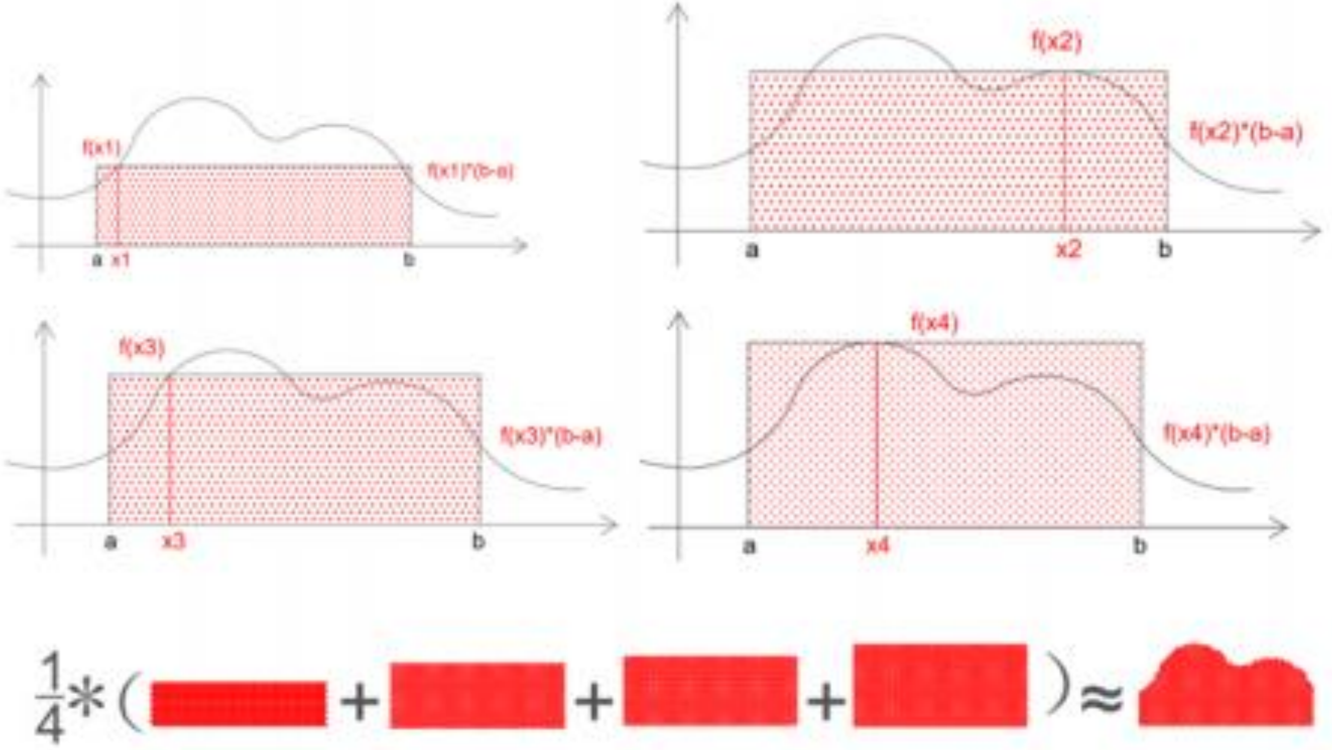


Figure 2.7: Process of MC for Integral

4. Two-level MC and MLMC

Two-level:

$$E[Q_1] = E[Q_0] + E[Q_1 - Q_0]$$

$$E[Q_1] \approx \frac{1}{N_0}\sum_{n=0}^{N_0} Q_0^{(n)} + \frac{1}{N_1}\sum_{n=0}^{N_1}(Q_1^{(n)} - Q_0^{(n)})$$

Simulation Cost

$$N_0C_0 + N_1C_1$$

Simulation Variance

$$\frac{N_1}{N_0} = \sqrt{\frac{V_1}{C_1}}\sqrt{\frac{V_0}{C_0}}$$

Multi-level:

$$E[Q_L] \approx \frac{1}{N_0}\sum_{i=1}^{N_0} Q_0(\omega_{i,0}) + \sum_{l=1}^{L}\{\frac{1}{N_l}\sum_{i=1}^{N_l}(Q_l(\omega_{i,l}) - Q_{l-1}(\omega_{i,l}))\}$$

Or $E[g(X_T)] = E[g_L(\hat{X}_T)]$

$$= (E[g_0(X_T)] + \sum_{l=1}^{L} E[(g_l(\hat{X}_T) - g_{l-1}(\hat{X}_T)])$$

$\sum_{l=0}^{L}(N_l^{-1}V_l + \lambda^2 N_l C_l)$ (Lagrange Multiplier) $\lambda = \epsilon^{-2}(\sum_{l=0}^{L}\sqrt{V_lC_l})$

Simulation Cost

$$C = \sum_{l=0}^{L} N_lC_l = \sum_{l=0}^{L}\lambda\sqrt{\frac{V_l}{C_l}}C_l = \epsilon^{-2}\sum_{l=0}^{L}\sqrt{V_lC_l}\sum_{l=0}^{L}\sqrt{V_lC_l} = \epsilon^{-2}(\sum_{l=0}^{L}\sqrt{V_lC_l})^2$$

$$C = O(TOL^{-2}(logTOL)^2)$$

Simulation Variance

$$min_{N_l} Var(A_{ML}) = \sum_{l=0}^{L}\frac{V_l}{N_l}$$

5. Experiment: Option Pricing

Estimation Error

$V_l = O(h_l)$

$E[Q_l - Q_{l-1}] = O(h_l)$

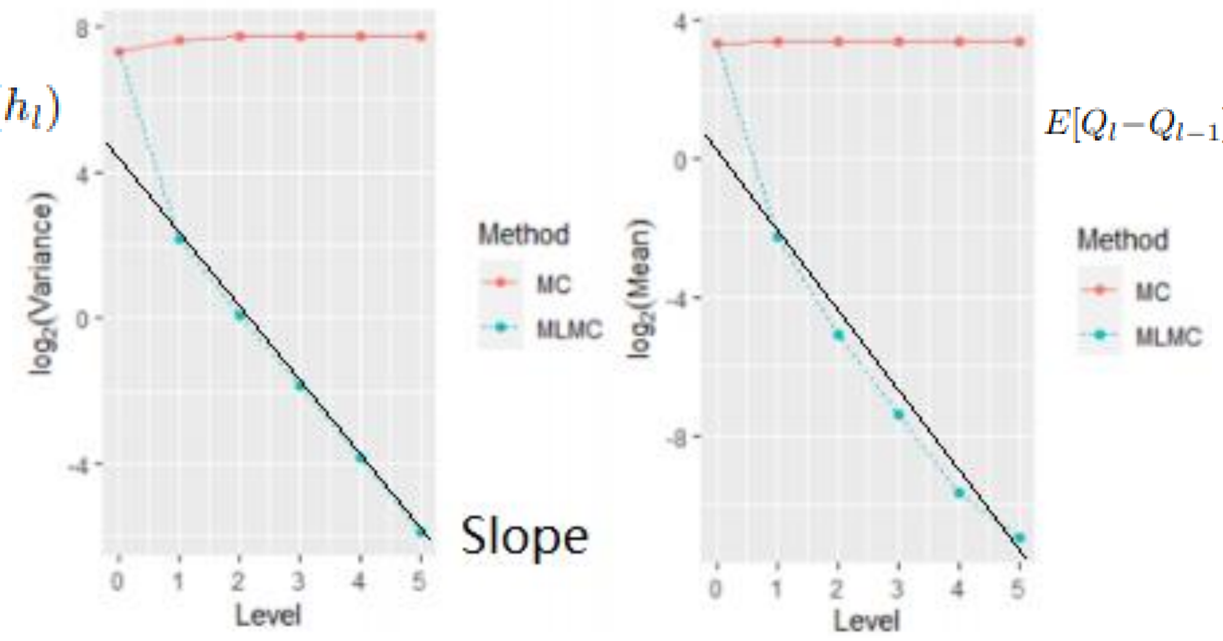


Figure 3.1: Variance/Mean to level l on European Options

5. Experiment: Option Pricing

Convergence Speed

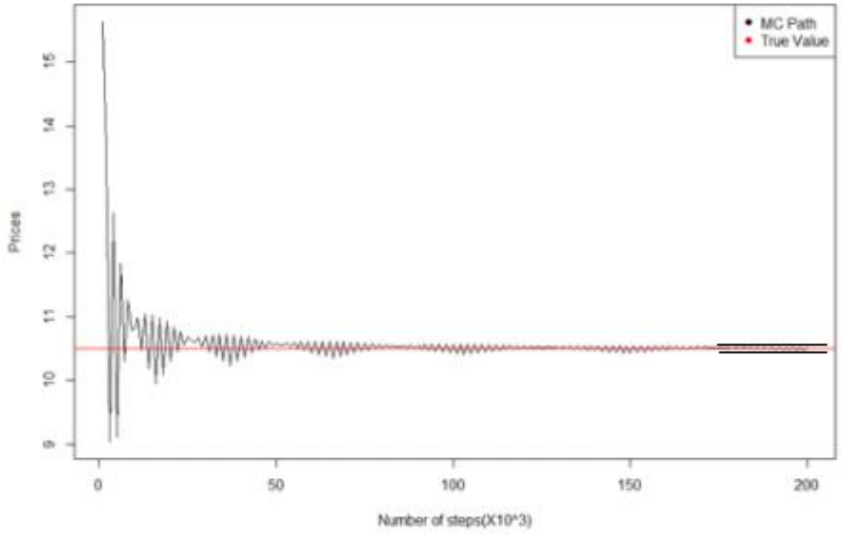
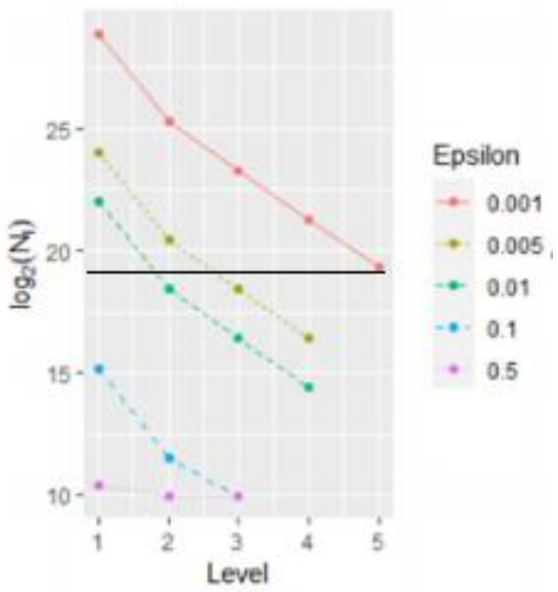


Figure 3.6: MC Convergence to Final Result



6. Conclusions

- Estimation Error
- Convergence Speed

MLMC is Better!

Future Direction

Uncertainty

MSE = $O(\varepsilon^2)$

Initial Samples N_0

Thank You