18.06 Quiz 1 March 1, 2010 Professor Strang

Your PRINTED name is: ______________________________ **1.**

Your recitation number or instructor is ______________________ **2.**

3.

4.

1. Forward elimination changes $\overset{3\times3}{A}\mathbf{x} = \mathbf{b}$ to a row reduced $\overset{3\times3}{R}\mathbf{x} = \mathbf{d}$: the complete solution is

$$\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \mathbf{c_1} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \mathbf{c_2} \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

$\dim N(A) = 2$

$\text{rank}(A) = 1$

(a) **(14 points)** What is the 3 by 3 reduced row echelon matrix R and what is $\mathbf{d}$?

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \Rightarrow$ null matrix $= \begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\therefore R = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$d = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$

(b) **(10 points)** If the process of elimination subtracted 3 times row 1 from row 2 and then 5 times row 1 from row 3, what matrix connects R and $\mathbf{d}$ to the original A and $\mathbf{b}$? Use this matrix to find A and $\mathbf{b}$.

$E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$

$EAx = Eb \Rightarrow Rx = d$

$\therefore EA = R \quad Eb = d$

$\therefore A = E^{-1}R \quad b = E^{-1}d$

$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -5 \\ 3 & -6 & -15 \\ 5 & -10 & -25 \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 20 \end{bmatrix}$

2. Suppose A is the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_1 \end{bmatrix}$$

(a) **(16 points)** Find all special solutions to $Ax = 0$ and describe in words the whole nullspace of A.

$$A = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 3 & 8 & 7 & 0 \\ 0 & 4 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 4 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = R$$

$\therefore$ special solutions $\begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_1 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $N(A)$ is a 2-D plane through $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ in $\mathbb{R}^4$.

linear combination of all special solutions.

(b) **(10 points)** Describe the column space of this particular matrix A. "All combinations of the four columns" is not a sufficient answer.

$$R = \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} V_2 & V_3 & V_4 & V_1 \end{matrix} \end{array}$$

$\therefore$ V_2, V_3 is pivot columns.

$\therefore$ The 2nd, 3rd column in A is the basis for $C(A)$

$\therefore$ $C(A) = a\begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix} + b\begin{bmatrix} 2 \\ 7 \\ 2 \end{bmatrix}$, a 2-d plane in 3-d space.

(c) **(10 points)** What is the reduced row echelon form $R^* = \text{rref}(B)$ when B is the 6 by 8 block matrix

$$B = \begin{bmatrix} A & A \\ A & A \end{bmatrix} \text{ using the same } A?$$

$$R^* = \begin{bmatrix} R & R \\ 0 & 0 \end{bmatrix} = \begin{array}{cc} I & F \\ \multicolumn{2}{c}{\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \end{array}$$

$$R^* = \begin{bmatrix} \text{rref}(A) & \text{rref}(A) \\ 0 & 0 \end{bmatrix}, \quad \text{rref}(A) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3. **(16 points) Circle the words** that correctly complete the following sentence:

(a) Suppose a 3 by 5 matrix A has rank $r = 3$. Then the equation $Ax = b$

(always / sometimes but not always)

has (a unique solution / many solutions / no solution).

X

3x5

dim $N(A) = 2$

(b) What is the column space of A? Describe the nullspace of A.

$C(A) = \mathbb{R}^3$. $N(A)$ is a 2-d plane in $\mathbb{R}^5$.

4. Suppose that A is the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 5 \\ 2 & 4 \end{bmatrix}.$$

(a) **(10 points)** Explain in words how knowing all solutions to $A\mathbf{x} = \mathbf{b}$ decides if a given vector $\mathbf{b}$ is in the column space of A.

If the solution exist, $b \in C(A)$. In this case, the solution is unique.

$Ax=b$ has a solution if and only if $b \in C(A)$.

(b) **(14 points)** Is the vector $\mathbf{b} = \begin{bmatrix} 8 \\ 28 \\ 14 \end{bmatrix}$ in the column space of A?

Augmented matrix

$$\left[\begin{array}{cc|c} 2 & 1 & 8 \\ 6 & 5 & 28 \\ 2 & 4 & 14 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 4 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 4 \\ 0 & 1 & 2 \\ \hline 0 & 0 & 0 \end{array}\right]$$

有解 e.g. $x = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Yes.