Topic 1: Variational Inequality Theory

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Background

Equilibrium is a central concept in numerous disciplines including economics, management science/operations research, and engineering.

Methodologies that have been applied to the formulation, qualitative analysis, and computation of equilibria have included:

- systems of equations,
- optimization theory,
- complementarity theory, and
- fixed point theory.

Variational inequality theory is a powerful unifying methodology for the study of equilibrium problems.

Background

Variational inequality theory was introduced by Hartman and Stampacchia (1966) as a tool for the study of partial differential equations with applications principally drawn from mechanics. Such variational inequalities were *infinite-dimensional* rather than *finite-dimensional* as we will be studying here.

The breakthrough in finite-dimensional theory occurred in 1980 when Dafermos recognized that the traffic network equilibrium conditions as stated by Smith (1979) had a structure of a variational inequality.

This unveiled this methodology for the study of problems in economics, management science/operations research, and also in engineering, with a focus on transportation.

Background

To-date problems which have been formulated and studied as variational inequality problems include:

- traffic network equilibrium problems
- spatial price equilibrium problems
- oligopolistic market equilibrium problems
- financial equilibrium problems
- migration equilibrium problems, as well as
- environmental network and ecology problems,
- knowledge network problems,
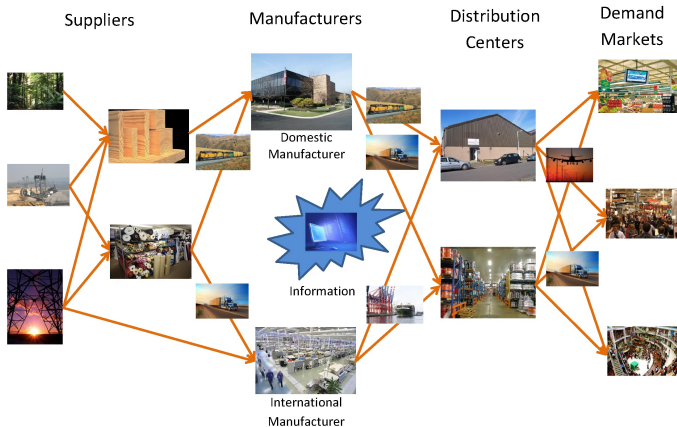- supply chain network equilibrium problems, and even
- the Internet!

Observe that many of the applications explored to-date that have been formulated, studied, and solved as variational inequality problems are, in fact, network problems.

In addition, as we shall see, many of the advances in variational inequality theory have been spurred by needs in practice!

Multimodal Transportation

Complex Logistical Networks

Suppliers   Manufacturers   Distribution Centers   Demand Markets

Domestic Manufacturer

Information
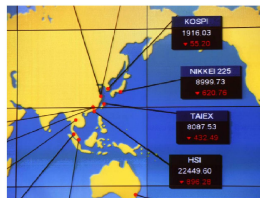
International Manufacturer

Electric Power Generation and Distribution Networks

Financial Networks

Social Networks

Variational Inequality Theory

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Variational Inequality Theory

Variational inequality theory provides us with a tool for:

• formulating a variety of equilibrium problems;

• qualitatively analyzing the problems in terms of existence and uniqueness of solutions, stability and sensitivity analysis, and

• providing us with algorithms with accompanying convergence analysis for computational purposes.

It contains, as special cases, such well-known problems in mathematical programming as: systems of nonlinear equations, optimization problems, complementarity problems, and is also related to fixed point problems.

Variational Inequality Theory

Also, as shown by Dupuis and Nagurney (1993), there is associated with a variational inequality problem, a *projected dynamical system*, which provides a natural underlying dynamics until an equilibrium state is achieved, under appropriate conditions.

This result further enriches the scope and reach of variational inequalities in terms of theory and especially applications!

Definition 1 (Variational Inequality Problem)
The finite - dimensional variational inequality problem,
$\mathrm{VI}(F, K)$, *is to determine a vector* $x^* \in K \subset R^n$, *such that*

$$F(x^*)^T \cdot (x - x^*) \geq 0, \quad \forall x \in K,$$

or, equivalently,

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K \qquad (1)$$

where F is a given continuous function *from K to R^n, K is a given* closed convex set, *and $\langle \cdot, \cdot \rangle$ denotes the inner product in n-dimensional Euclidean space, as does "$\cdot$".* Closed set does not guarantee the existence of solution, but it is enough to define a problem.

Here we assume that all vectors are column vectors, except where noted.
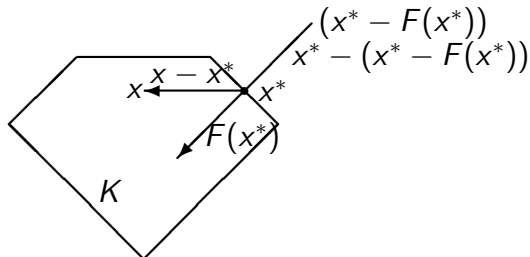
The Variational Inequality Problem

Figure: Geometric Depiction of the Variational Inequality Problem

In geometric terms, the variational inequality (1) states that $F(x^*)^T$ is "orthogonal" to the feasible set K at the point x^*.

Systems of Equations

Indeed, many mathematical problems can be formulated as variational inequality problems, and several examples applicable to equilibrium analysis follow.

Systems of Equations
Many classical economic equilibrium problems have been formulated as systems of equations, since market clearing conditions necessarily equate the total supply with the total demand. In terms of a variational inequality problem, the formulation of a system of equations is as follows.

Systems of Equations

Proposition 1
Let $K = R^n$ and let $F : R^n \mapsto R^n$ be a given function. A vector $x^* \in R^n$ solves $\mathrm{VI}(F, R^n)$ if and only if $F(x^*) = 0$.

Proof: If $F(x^*) = 0$, then inequality (1) holds with equality. Conversely, if x^* satisfies (1), let $x = x^* - F(x^*)$, which implies that

$$F(x^*)^T \cdot (-F(x^*)) \geq 0, \quad \text{or} \quad -\|F(x^*)\|^2 \geq 0 \qquad (2)$$

and, therefore, $F(x^*) = 0$.

Note that systems of equations, however, preclude the introduction of inequalities, which may be needed, for example, in the case of nonnegativity assumptions on certain variables such as prices.

An Example (Market Equilibrium with Equalities Only)

As an illustration, we now present an example of a system of equations. Consider m consumers, with a typical consumer denoted by j, and n commodities, with a typical commodity denoted by i. Let p denote the n-dimensional vector of the commodity prices with components: $\{p_1, \ldots, p_n\}$.

Assume that the demand for a commodity i, d_i, may, in general, depend upon the prices of all the commodities, that is,

$$d_i(p) = \sum_{j=1}^{m} d_i^j(p),$$

where $d_i^j(p)$ denotes the demand for commodity i by consumer j at the price vector p.

An Example (Market Equilibrium with Equalities Only)

Similarly, the supply of a commodity i, s_i, may, in general, depend upon the prices of all the commodities, that is,

$$s_i(p) = \sum_{j=1}^{m} s_i^j(p),$$

where $s_i^j(p)$ denotes the supply of commodity i of consumer j at the price vector p.

An Example (Market Equilibrium with Equalities Only)

We group the aggregate demands for the commodities into the n-dimensional column vector d with components: $\{d_1, \ldots, d_n\}$ and the aggregate supplies of the commodities into the n-dimensional column vector s with components: $\{s_1, \ldots, s_n\}$.

The market equilibrium conditions that require that the supply of each commodity must be equal to the demand for each commodity at the equilibrium price vector p^*, are equivalent to the following system of equations:

$$s(p^*) - d(p^*) = 0.$$

An Example (Market Equilibrium with Equalities Only)

Clearly, this expression can be put into the standard nonlinear equation form, if we define the vectors $x \equiv p$ and $F(x) \equiv s(p) - d(p)$.

Note, however, that the problem class of nonlinear equations is not sufficiently general to guarantee, for example, that $x^* \geq 0$, which may be desirable in this example in which the vector x refers to prices.

Optimization Problems

An optimization problem is characterized by its specific objective function that is to be maximized or minimized, depending upon the problem and, in the case of a constrained problem, a given set of constraints.

Possible objective functions include expressions representing profits, costs, market share, portfolio risk, etc. Possible constraints include those that represent limited budgets or resources, nonnegativity constraints on the variables, conservation equations, etc. Typically, an optimization problem consists of a single objective function.

Optimization Problems

Both **unconstrained** and **constrained** optimization problems can be formulated as variational inequality problems. The subsequent two propositions and theorem identify the relationship between an optimization problem and a variational inequality problem.

Proposition 2

Let x^* be a solution to the optimization problem:

$$\text{Minimize} \quad f(x) \tag{3}$$

$$\text{subject to:} \quad x \in K,$$

where f is continuously differentiable and K is closed and convex. Then x^* is a solution of the variational inequality problem:

$$\nabla f(x^*)^T \cdot (x - x^*) \geq 0, \quad \forall x \in K. \tag{4}$$

Proof: Let $\phi(t) = f(x^* + t(x - x^*))$, for $t \in [0, 1]$. Since $\phi(t)$ achieves its minimum at $t = 0$, $0 \leq \phi'(0) = \nabla f(x^*)^T \cdot (x - x^*)$, that is, x^* is a solution of (4).

Optimization Problems

Proposition 3

If $f(x)$ is a convex function and x^* is a solution to $\mathrm{VI}(\nabla f, K)$, then x^* is a solution to the optimization problem (3).

Proof: Since $f(x)$ is convex,

$$f(x) \geq f(x^*) + \nabla f(x^*)^T \cdot (x - x^*), \quad \forall x \in K. \qquad (5)$$

But $\nabla f(x^*)^T \cdot (x - x^*) \geq 0$, since x^* is a solution to $\mathrm{VI}(\nabla f, K)$. Therefore, from (5) one concludes that

$$f(x) \geq f(x^*), \quad \forall x \in K,$$

that is, x^* is a minimum point of the mathematical programming problem (3).

Optimization Problems

If the feasible set $K = R^n$, then the unconstrained optimization problem is also a variational inequality problem.

Relationship Between Optimization Problems and Variational Inequalities

On the other hand, in the case where a certain symmetry condition holds, the variational inequality problem can be reformulated as an optimization problem.

In other words, in the case that the variational inequality formulation of the equilibrium conditions underlying a specific problem is characterized by a function with a symmetric Jacobian, then the solution of the equilibrium conditions and the solution of a particular optimization problem are one and the same. We first introduce the following definition and then fix this relationship in a theorem.

Definition 2

An $n \times n$ matrix $M(x)$, whose elements $m_{ij}(x)$; $i = 1, \ldots, n$; $j = 1, \ldots, n$, are functions defined on the set $S \subset R^n$, is said to be positive-semidefinite on S if

$$v^T M(x) v \geq 0, \quad \forall v \in R^n, x \in S.$$

It is said to be positive-definite on S if

$$v^T M(x) v > 0, \quad \forall v \neq 0, v \in R^n, x \in S.$$

It is said to be strongly positive-definite on S if

$$v^T M(x) v \geq \alpha \|v\|^2, \quad \text{for some} \quad \alpha > 0, \quad \forall v \in R^n, x \in S.$$

Note that if $\gamma(x)$ is the smallest eigenvalue, which is necessarily real, of the symmetric part of $M(x)$, that is, $\frac{1}{2}\left[M(x) + M(x)^T\right]$, then it follows that (i). $M(x)$ is positive-semidefinite on S if and only if $\gamma(x) \geq 0$, for all $x \in S$; (ii). $M(x)$ is positive-definite on S if and only if $\gamma(x) > 0$, for all $x \in S$; and (iii). $M(x)$ is strongly positive-definite on S if and only if $\gamma(x) \geq \alpha > 0$, for all $x \in S$.

Relationship Between Optimization Problems and Variational Inequalities

Theorem 1

Assume that $F(x)$ is continuously differentiable on K and that the Jacobian matrix

$$\nabla F(x) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}$$

is symmetric and positive-semidefinite.

Then there is a real-valued convex function $f : K \mapsto R^1$ satisfying

$$\nabla f(x) = F(x)$$

with x^* the solution of $VI(F, K)$ also being the solution of the mathematical programming problem:

$$\text{Minimize} \quad f(x) \tag{6}$$

$$\text{subject to:} \quad x \in K.$$

Proof: Under the symmetry assumption it follows from Green's Theorem that

$$f(x) = \int F(x)^T dx, \tag{7}$$

where $\int$ is a line integral. The conclusion follows from Proposition 3.

Hence, although the variational inequality problem encompasses the optimization problem, a variational inequality problem can be reformulated as a convex optimization problem, only when the symmetry condition and the positive-semidefiniteness condition hold.

The variational inequality, therefore, is the more general problem in that it can also handle a function $F(x)$ with an asymmetric Jacobian.

Complementarity Problems

The variational inequality problem also contains the complementarity problem as a special case. Complementarity problems are defined on the nonnegative orthant.

Let R_+^n denote the nonnegative orthant in R^n, and let $F : R^n \mapsto R^n$. The nonlinear complementarity problem over R_+^n is a system of equations and inequalities stated as:

Find $x^* \geq 0$ such that

$$F(x^*) \geq 0 \quad \text{and} \quad F(x^*)^T \cdot x^* = 0. \tag{8}$$

Complementarity Problems

Whenever the mapping F is affine, that is, whenever $F(x) = Mx + b$, where M is an $n \times n$ matrix and b an $n \times 1$ vector, problem (8) is then known as the linear complementarity problem.

Relationship Between Complementaroty Problems and Variational Inequalities

The relationship between the complementarity problem and the variational inequality problem is as follows.

Proposition 4

$\mathrm{VI}(F, R_+^n)$ *and (8) have precisely the same solutions, if any.*

Relationship Between Complementaroty Problems and Variational Inequalities

Proof: First, it is established that if x^* satisfies $\mathrm{VI}(F, R^n_+)$, then it also satisfies the complementarity problem (8). Substituting $x = x^* + e_i$ into $\mathrm{VI}(F, R^n_+)$, where e_i denotes the n-dimensional vector with 1 in the i-th location and 0, elsewhere, one concludes that $F_i(x^*) \geq 0$, and $F(x^*) \geq 0$.

Substituting now $x = 2x^*$ into the variational inequality, one obtains

$$F(x^*)^T \cdot (x^*) \geq 0. \tag{9}$$

Relationship Between Complementaroty Problems and Variational Inequalities

Substituting then $x = 0$ into the variational inequality, one obtains

$$F(x^*)^T \cdot (-x^*) \geq 0. \qquad (10)$$

(9) and (10) together imply that $F(x^*)^T \cdot x^* = 0$.

Conversely, if x^* satisfies the complementarity problem, then

$$F(x^*)^T \cdot (x - x^*) \geq 0$$

since $x \in R_+^n$ and $F(x^*) \geq 0$.

An Example (Market Equilibrium with Equalities and Inequalities)

A nonlinear complementarity formulation of market equilibrium is now presented.

Assume that the prices must now be nonnegative in the market equilibrium example presented earlier. Hence, we consider the following situation, in which the demand functions are given as previously as are the supply functions, but now, instead of the market equilibrium conditions, which are represented by a system of equations, we have the following equilibrium conditions. For each commodity i; $i = 1, \ldots, n$:

$$s_i(p^*) - d_i(p^*) \begin{cases} = 0, & \text{if } p_i^* > 0 \\ \geq 0, & \text{if } p_i^* = 0. \end{cases}$$

An Example (Market Equilibrium with Equalities and Inequalities)

These equilibrium conditions state that if the price of a commodity is positive in equilibrium then the supply of that commodity must be equal to the demand for that commodity.

On the other hand, if the price of a commodity at equilibrium is zero, then there may be an excess supply of that commodity at equilibrium, that is, $s_i(p^*) - d_i(p^*) > 0$, or the market clears.

Furthermore, this system of equalities and inequalities guarantees that the prices of the instruments do not take on negative values, which may occur in the system of equations expressing the market clearing conditions.

An Example (Market Equilibrium with Equalities and Inequalities)

The nonlinear complementarity formulation of this problem is as follows.

Determine $p^* \in R_+^n$, satisfying:

$$s(p^*) - d(p^*) \geq 0 \quad \text{and} \quad \langle (s(p^*) - d(p^*)), p^* \rangle = 0.$$

Moreover, since a nonlinear complementarity problem is a special case of a variational inequality problem, we may rewrite the nonlinear complementarity formulation of the market equilibrium problem above as the following variational inequality problem:

Determine $p^* \in R_+^n$, such that

$$\langle (s(p^*) - d(p^*)), p - p^* \rangle \geq 0, \quad \forall p \in R_+^n.$$

Note, first, that in the *special* case of demand functions and supply functions that are separable, the Jacobians of these functions are symmetric since they are diagonal and given, respectively, by

$$\nabla s(p) = \begin{pmatrix} \frac{\partial s_1}{\partial p_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{\partial s_2}{\partial p_2} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\partial s_n}{\partial p_n} \end{pmatrix},$$

$$\nabla d(p) = \begin{pmatrix} \frac{\partial d_1}{\partial p_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{\partial d_2}{\partial p_2} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\partial d_n}{\partial p_n} \end{pmatrix}.$$

Indeed, in this **special case model**, the supply of a commodity depends only upon the price of that commodity and, similarly, the demand for a commodity depends only upon the price of that commodity.

Hence, in this special case, the price vector p^* that satisfies the equilibrium conditions can be obtained by solving the following optimization problem:

$$\text{Minimize } \sum_{i=1}^{n} \int_0^{p_i} s_i(x)dx - \sum_{i=1}^{n} \int_0^{p_i} d_i(y)dy$$

subject to:

$$p_i \geq 0, \quad i = 1, \ldots, n.$$

One also obtains an optimization reformulation of the equilibrium conditions, provided that the following symmetry condition holds: $\frac{\partial s_i}{\partial p_k} = \frac{\partial s_k}{\partial p_i}$ and $\frac{\partial d_i}{\partial p_k} = \frac{\partial d_k}{\partial p_i}$ for all commodities i, k. In other words, the price of a commodity k affects the supply of a commodity i in the same way that the price of a commodity i affects the supply of a commodity k. A similar situation must hold for the demands for the commodities.

However, such symmetry conditions are limiting from the application standpoint and, hence, the appeal of variational inequality problem that enables the formulation and, ultimately, the computation of equilibria where such restrictive symmetry assumptions on the underlying functions need no longer hold. Indeed, such symmetry assumptions were not imposed in the variational inequality problem.

An Example (Market Equilibrium with Equalities and Inequalities and Policy Interventions)

We now provide a generalization of the preceding market equilibrium model to allow for price policy interventions in the form of price floors and ceilings. Let p_i^C denote the imposed price ceiling on the price of commodity i, and we let p_i^F denote the imposed price floor on the price of commodity i.

Then we have the following equilibrium conditions. For each commodity i; $i = 1, \ldots, n$:

$$s_i(p^*) - d_i(p^*) \begin{cases} \leq 0, & \text{if } p_i^* = p_i^C \\ = 0, & \text{if } p_i^F < p_i^* < p_i^C \\ \geq 0, & \text{if } p_i^* = p_i^F. \end{cases}$$

An Example (Market Equilibrium with Equalities and Inequalities and Policy Interventions)

These equilibrium conditions state that if the price of a commodity in equilibrium lies between the imposed price floor and ceiling, then the supply of that commodity must be equal to the demand for that commodity.

On the other hand, if the price of a commodity at equilibrium is at the imposed floor, then there may be an excess supply of that commodity at equilibrium, that is, $s_i(p^*) - d_i(p^*) > 0$, or the market clears. In contrast, if the price of a commodity in equilibrium is at the imposed ceiling, then there may be an excess demand of the commodity in equilibrium.

An Example (Market Equilibrium with Equalities and Inequalities and Policy Interventions)

The variational inequality formulation of the governing equilibrium conditions is:

Determine $p^* \in \mathcal{K}$, such that

$$\langle (s(p^*) - d(p^*)), p - p^* \rangle \geq 0, \quad \forall p \in \mathcal{K},$$

where the feasible set $\mathcal{K} \equiv \{p | p^F \leq p \leq p^C\}$, where p^F and p^C denote, respectively, the n-dimensional column vectors of imposed price floors and ceilings.

Fixed Point Problems

Fixed point theory has been used to formulate, analyze, and compute solutions to economic equilibrium problems. The relationship between the variational inequality problem and a fixed point problem can be made through the use of a projection operator. First, the projection operator is defined.

Lemma 1

Let K be a closed convex set in R^n. Then for each $x \in R^n$, there is a unique point $y \in K$, such that

$$\|x - y\| \leq \|x - z\|, \quad \forall z \in K, \qquad (11)$$

and y is known as the orthogonal projection of x on the set K with respect to the Euclidean norm, that is,

$$y = P_K x = \arg \min_{z \in K} \|x - z\|.$$

Fixed Point Problems

Proof: Let x be fixed and let $w \in K$. Minimizing $\|x - z\|$ over all $z \in K$ is equivalent to minimizing the same function over all $z \in K$ such that $\|x - z\| \leq \|x - w\|$, which is a compact set. The function g defined by $g(z) = \|x - z\|^2$ is continuous. Existence of a minimizing y follows because a continuous function on a compact set always attains its minimum. To prove that y is unique, observe that the square of the Euclidean norm is a strictly convex function. Hence, g is strictly convex and its minimum is unique.

Theorem 2

Let K be a closed convex set. Then $y = P_K x$ if and only if

$$y^T \cdot (z - y) \geq x^T \cdot (z - y), \quad \forall z \in K$$

or

$$(y - x)^T \cdot (z - y) \geq 0, \quad \forall z \in K. \tag{12}$$

Proof: Note that $y = P_K x$ is the minimizer of $g(z)$ over all $z \in K$. Since $\nabla g(z) = 2(z - x)$, the result follows from the optimality conditions for constrained optimization problems.

Projection Operator

A property of the projection operator which is useful both in qualitative analysis of equilibria and their computation is now presented.

Corollary 1

Let K be a closed convex set. Then the projection operator P_K is nonexpansive, that is,

$$\|P_K x - P_K x'\| \leq \|x - x'\|, \quad \forall x, x' \in R^n. \qquad (13)$$

Projection Operator

Proof: Given $x, x' \in R^n$, let $y = P_K x$ and $y' = P_K x'$. Then from Theorem 2 note that

$$\text{for} \quad y \in K : y^T \cdot (z - y) \geq x^T \cdot (z - y), \quad \forall z \in K, \quad (14)$$

$$\text{for} \quad y' \in K : y'^T \cdot (z - y') \geq x'^T \cdot (z - y'), \quad \forall z \in K. \quad (15)$$

Setting $z = y'$ in (14) and $z = y$ in (15) and adding the resultant inequalities, one obtains:

$$\|y - y'\|^2 = (y - y')^T \cdot (y - y') \leq (x - x')^T \cdot (y - y')$$

$$\leq \|x - x'\| \cdot \|y - y'\|$$

by an application of the Schwarz inequality. Hence,

$$\|y - y'\| \leq \|x - x'\|.$$
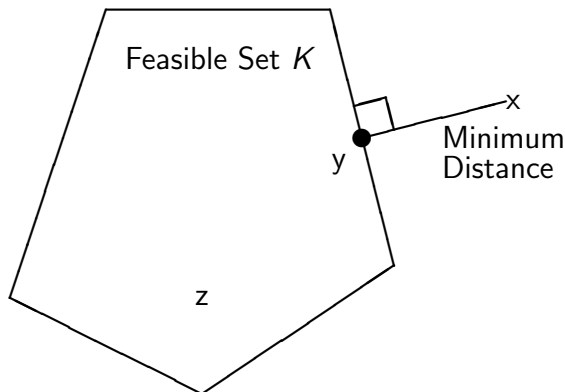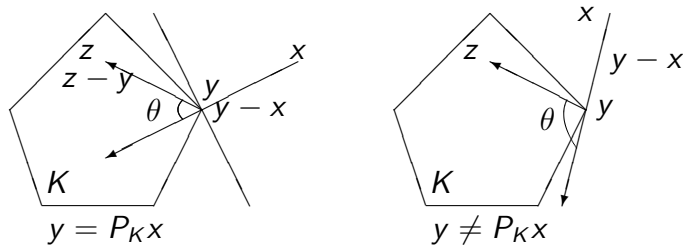
Geometric Interpretation of Projection

Figure: The projection y of x on the set K

Additional Geometric Interpretation

Figure: Geometric interpretation of $\langle (y - x), z - y \rangle \geq 0$, for $y = P_K x$ and $y \neq P_K x$

Relationship Between Fixed Point Problems and Variational Inequalities

The relationship between a variational inequality and a fixed point problem is as follows.

Theorem 3

Assume that K is closed and convex. Then $x^* \in K$ is a solution of the variational inequality problem $\mathrm{VI}(F, K)$ if and only if for any $\gamma > 0$, x^* is a fixed point of the map

$$P_K(I - \gamma F) : K \mapsto K,$$

that is,

$$x^* = P_K(x^* - \gamma F(x^*)). \tag{16}$$

Relationship Between Fixed Point Problems and Variational Inequalities

Proof: Suppose that x^* is a solution of the variational inequality, i.e.,

$$F(x^*)^T \cdot (x - x^*) \geq 0, \quad \forall x \in K.$$

Multiplying the above inequality by $-\gamma < 0$, and adding $x^{*T} \cdot (x - x^*)$ to both sides of the resulting inequality, one obtains

$$x^{*T} \cdot (x - x^*) \geq [x^* - \gamma F(x^*)]^T \cdot (x - x^*), \quad \forall x \in K. \quad (17)$$

From Theorem 2 one concludes that

$$x^* = P_K(x^* - \gamma F(x^*)).$$

Relationship Between Fixed Point Problems and Variational Inequalities

Conversely, if $x^* = P_K(x^* - \gamma F(x^*))$, for $\gamma > 0$, then

$$x^{*^T} \cdot (x - x^*) \geq (x^* - \gamma F(x^*))^T \cdot (x - x^*), \quad \forall x \in K,$$

and, therefore,

$$F(x^*)^T \cdot (y - x^*) \geq 0, \quad \forall y \in K.$$

Basic Existence and Uniqueness Results

Variational inequality theory is also a powerful tool in the qualitative analysis of equilibria. We now provide conditions for existence and uniqueness of solutions to $\mathrm{VI}(F, K)$ are provided.

Existence of a solution to a variational inequality problem follows from continuity of the function F entering the variational inequality, provided that the feasible set K is compact. Indeed, we have the following:

Theorem 4 (Existence Under Compactness and Continuity)
If K is a compact convex set and $F(x)$ is continuous on K, then the variational inequality problem admits at least one solution x^*.

Basic Existence and Uniqueness Results

Proof: According to Brouwer's Fixed Point Theorem, given a map $P : K \mapsto K$, with P continuous, there is at least one $x^* \in K$, such that $x^* = Px^*$. Observe that since P_K and $(I - \gamma F)$ are each continuous, $P_K(I - \gamma F)$ is also continuous. The conclusion follows from compactness of K and Theorem 3.

Basic Existence and Uniqueness Results

In the case of an unbounded feasible set K, Brouwer's Fixed Point Theorem is no longer applicable; the existence of a solution to a variational inequality problem can, nevertheless, be established under the subsequent condition.

Let $B_R(0)$ denote a closed ball with radius R centered at 0 and let $K_R = K \cap B_R(0)$. K_R is then bounded.
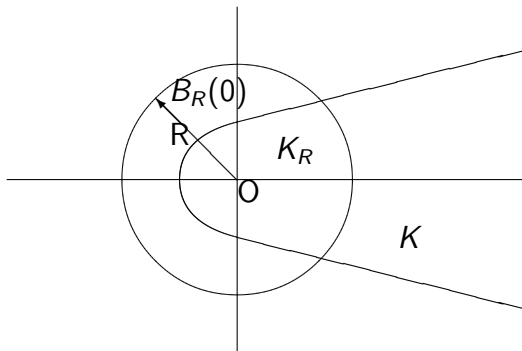
Figure: Depiction of bounded set K_R

Basic Existence and Uniqueness Results

Let VI_R denote the variational inequality problem:
Determine $x_R^* \in K_R$, such that

$$F(x_R^*)^T \cdot (y - x_R^*) \geq 0, \quad \forall y \in K_R. \tag{18}$$

Theorem 5
$VI(F, K)$ *admits a solution if and only if there exists an* $R > 0$
and a solution of VI_R, x_R^*, *such that* $\|x_R^*\| < R$.

Although $\|x_R^*\| < R$ may be difficult to check, one may be able to identify an appropriate R based on the particular application.

Basic Existence and Uniqueness Results

Existence of a solution to a variational inequality problem may also be established under the coercivity condition, as in the subsequent corollary.

Corollary 2 (Existence Under Coercivity)
Suppose that $F(x)$ satisfies the coercivity condition

$$\frac{(F(x) - F(x_0))^T \cdot (x - x_0)}{\|x - x_0\|} \to \infty \qquad (19)$$

as $\|x\| \to \infty$ for $x \in K$ and for some $x_0 \in K$. Then $\mathrm{VI}(F, K)$ always has a solution.

Corollary 3
Suppose that x^* is a solution of $\mathrm{VI}\ (F, K)$ and $x^* \in K^0$, the interior of K. Then $F(x^*) = 0$.

Basic Existence and Uniqueness Results

Qualitative properties of existence and uniqueness become easily obtainable under certain monotonicity conditions. First we outline the definitions and then present the results.

Definition 3 (Monotonicity)
$F(x)$ is monotone on K if

$$\left[F(x^1) - F(x^2)\right]^T \cdot (x^1 - x^2) \geq 0, \quad \forall x^1, x^2 \in K.$$

Definition 4 (Strict Monotonicity)
$F(x)$ is strictly monotone on K if

$$\left[F(x^1) - F(x^2)\right]^T \cdot (x^1 - x^2) > 0, \quad \forall x^1, x^2 \in K, \, x^1 \neq x^2.$$

Basic Existence and Uniqueness Results

Definition 5 (Strong Monotonicity)
F(x) is strongly monotone on K if for some $\alpha > 0$

$$\left[F(x^1) - F(x^2)\right]^T \cdot (x^1 - x^2) \geq \alpha \|x^1 - x^2\|^2, \quad \forall x^1, x^2 \in K.$$

Definition 6 (Lipschitz Continuity)
F(x) is Lipschitz continous on K if there exists an $L > 0$, such that

$$\|F(x^1) - F(x^2)\| \leq L\|x^1 - x^2\|, \quad \forall x^1, x^2 \in K.$$

Basic Existence and Uniqueness Results

A uniqueness result is presented in the subsequent theorem.

Theorem 6 (Uniqueness)

Suppose that $F(x)$ is strictly monotone on K. Then the solution is unique, if one exists.

Proof: Suppose that x^1 and x^* are both solutions and $x^1 \neq x^*$. Then since both x^1 and x^* are solutions, they must satisfy:

$$F(x^1)^T \cdot (x' - x^1) \geq 0, \quad \forall x' \in K \qquad (25)$$

$$F(x^*)^T \cdot (x' - x^*) \geq 0, \quad \forall x' \in K. \qquad (26)$$

Basic Existence and Uniqueness Results

After substituting x^* for x' in (25) and x^1 for x' in (26) and adding the resulting inequalities, one obtains:

$$(F(x^1) - F(x^*))^T \cdot (x^* - x^1) \geq 0. \qquad (27)$$

But inequality (27) is in contradiction to the definition of strict monotonicity. Hence, $x^1 = x^*$.

More on Monotonicity

Monotonicity is closely related to positive-definiteness.

Theorem 7
Suppose that $F(x)$ is continuously differentiable on K and the Jacobian matrix

$$\nabla F(x) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix},$$

which need not be symmetric, is positive-semidefinite (positive-definite). Then $F(x)$ is monotone (strictly monotone).

Proposition 5

Assume that $F(x)$ is continuously differentiable on K and that $\nabla F(x)$ is strongly positive-definite. Then $F(x)$ is strongly monotone.

One obtains a stronger result in the special case where $F(x)$ is linear.

Corollary 4
Suppose that $F(x) = Mx + b$, where M is an $n \times n$ matrix and b is a constant vector in R^n. The function F is monotone if and only if M is positive-semidefinite. F is strongly monotone if and only if M is positive-definite.

Proposition 6

Assume that $F : K \mapsto R^n$ is continuously differentiable at $\bar{x}$. Then $F(x)$ is locally strictly (strongly) monotone at $\bar{x}$ if $\nabla F(\bar{x})$ is positive-definite (strongly positive-definite), that is,

$$v^T F(\bar{x}) v > 0, \quad \forall v \in R^n, v \neq 0,$$

$$v^T \nabla F(\bar{x}) v \geq \alpha \|v\|^2, \quad \text{for some} \quad \alpha > 0, \quad \forall v \in R^n.$$

A Strong Result

The following theorem provides a condition under which both existence and uniqueness of the solution to the variational inequality problem are guaranteed. Here no assumption on the compactness of the feasible set K is made.

Theorem 8 (Existence and Uniqueness)
Assume that $F(x)$ is strongly monotone. Then there exists precisely one solution x^* to $\mathrm{VI}(F, K)$.

Proof: Existence follows from the fact that strong monotonicity implies coercivity, whereas uniqueness follows from the fact that strong monotonicity implies strict monotonicity.

Hence, in the case of an unbounded feasible set K, strong monotonicity of the function F guarantees both existence and uniqueness. If K is compact, then existence is guaranteed if F is continuous, and only the strict monotonicity condition needs to hold for uniqueness to be guaranteed.

A Contraction

Assume now that $F(x)$ is both strongly monotone and Lipschitz continuous. Then the projection $P_K[x - \gamma F(x)]$ is a contraction with respect to x, that is, we have the following:

Theorem 9

Fix $0 < \gamma \leq \frac{\alpha}{L^2}$ where α and L are the constants appearing, respectively, in the strong monotonicity and the Lipschitz continuity condition definitions. Then

$$\|P_K(x - \gamma F(x)) - P_K(y - \gamma F(y))\| \leq \beta\|x - y\| \qquad (31)$$

for all $x, y \in K$, where

$$(1 - \gamma\alpha)^{\frac{1}{2}} \leq \beta < 1.$$

An immediate consequence of Theorem 9 and the Banach
Fixed Point Theorem is:

Corollary 5
The operator $P_K(x - \gamma F(x))$ has a unique fixed point x^*.

Summary

In this lecture, the fundamental qualitative tools for the formulation and analysis of finite-dimensional variational inequalities have been provided.

In subsequent lectures, we will describe algorithms and a plethora of applications of variational inequalities.

Since many of the applications are network-based,, we will also cover such applications and special-purpose algorithms.

References

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