MIT open Course : Linear Algebra

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Lec 1: The geometry of linear equations.

n linear equations, n unknowns.

Row pictures : merge of lines

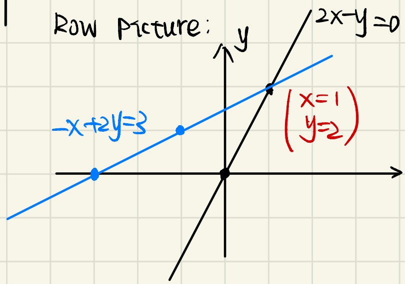
* column pictures : combination of vectors

matrix form

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned} \Rightarrow \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$A \quad x = b$$

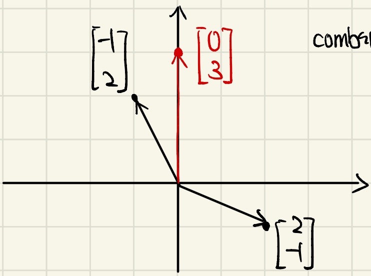
Row Picture:



column picture

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Linear combination of columns



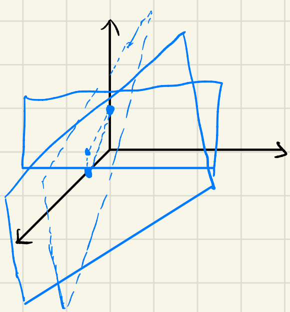
combination: $\begin{bmatrix} x=1 \\ y=2 \end{bmatrix}$

Q: What are all the combination?

A 3x3 example:

$$\begin{aligned} 2x - y \quad &= 0 \\ -x + 2y - z &= -1 \\ -3y + 4z &= 4 \end{aligned} \qquad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

Row picture.



3 planes meet at point, the solution.

Hard to see and imagine the geometry.

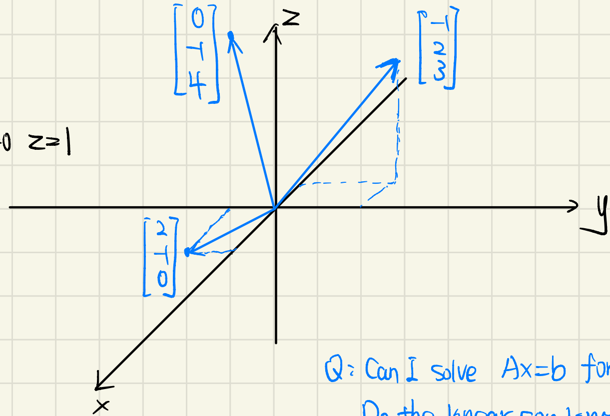
column picture:

$$x\begin{bmatrix}2\\-1\\0\end{bmatrix}+y\begin{bmatrix}-1\\2\\3\end{bmatrix}+z\begin{bmatrix}0\\-1\\4\end{bmatrix}=\begin{bmatrix}0\\-1\\4\end{bmatrix}$$

A is nonsingular here.

solution:

$x=0 \quad y=0 \quad z=1$



Q: Can I solve $Ax=b$ for every b?
Do the linear combination of the columns fill 3-D space?

If A is singular, for example, $\text{rank}(A)=2$, then I can only solve $Ax=b$ for some b. The ones in the same plane of A's columns.

Matrix × vector

$$\begin{bmatrix}2&5\\1&3\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=1\begin{bmatrix}2\\1\end{bmatrix}+2\begin{bmatrix}5\\3\end{bmatrix}=\begin{bmatrix}12\\7\end{bmatrix}$$

column way.

Ax is a combinations of columns in A!

$$\begin{bmatrix}2&5\\1&3\end{bmatrix}\begin{bmatrix}1&2\\2&3\end{bmatrix}\Rightarrow 1\begin{bmatrix}2\\1\end{bmatrix}+2\begin{bmatrix}5\\3\end{bmatrix}=\begin{bmatrix}12\\7\end{bmatrix}\Rightarrow\begin{bmatrix}12&19\\7&11\end{bmatrix}$$

$$\searrow 2\begin{bmatrix}2\\1\end{bmatrix}+3\begin{bmatrix}5\\3\end{bmatrix}=\begin{bmatrix}19\\11\end{bmatrix}\nearrow$$

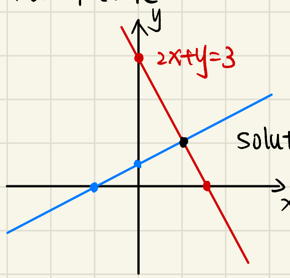
Exercise

solve $\begin{cases} 2x+y=3 \\ x-2y=-1 \end{cases}$, and find out its "row picture" and "column picture".

$x = 2y-1 \Rightarrow 2(2y-1)+y=3 \Rightarrow 4y-2+y=3 \Rightarrow 5y=5$

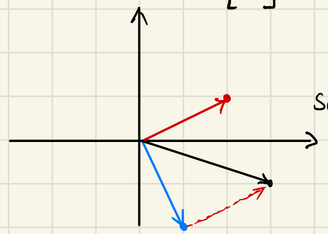
$\therefore y=1, \Rightarrow x-2=-1 \Rightarrow x=1$, solution $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Row picture



solution $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

column picture $\underset{v_1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} x + \underset{v_2}{\begin{bmatrix} 1 \\ -2 \end{bmatrix}} y = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$



solution $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Matrix form : $A = [v_1, v_2] = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

Linear system : $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

scalar system : $ax=b \Rightarrow x = \frac{b}{a} = a^{-1}b \Rightarrow a^{-1}a = 1$

same idea : $A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\therefore$ solution is $\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$A^{-1}A = I$

Mathematical methods for engineers

Vectors / matrices / subspaces

Vectors u, v, w. linear combination: $x_1 u + x_2 v + x_3 w = b$ (scalers: x_1, x_2, x_3)

$$\begin{bmatrix}1\\-1\\0\end{bmatrix}\begin{bmatrix}0\\1\\-1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

3 vectors are independent.

All the combinations of u, v form a plane in 3-D space.

$$Ax = \begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1\\x_2-x_1\\x_3-x_2\end{bmatrix} = b = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

A: difference matrix

combination of columns

$$A\begin{bmatrix}1\\4\\9\end{bmatrix} = \begin{bmatrix}1\\3\\5\end{bmatrix}$$

Given x, solve b

Reverse: given b, find x. solution: $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}b_1\\b_1+b_2\\b_1+b_2+b_3\end{bmatrix} = \begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$

$Ax=b$

$x=A^{-1}b$

A matrix multiplies b.

An different perspective:

Matrix A transforms vector x to b.

Inverse matrix: $Ax=b$, $x=A^{-1}b$. perfectly only one solution.

Example 2:

$$C = \begin{bmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{bmatrix} \qquad Cx = \begin{bmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1-x_3\\x_2-x_1\\x_3-x_2\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

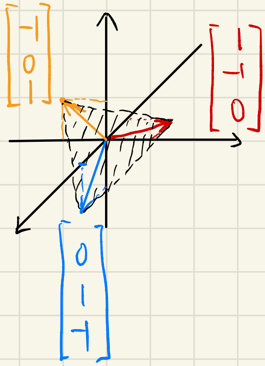
OK if $0 = b_1+b_2+b_3$

If $Cx = \begin{bmatrix}0\\0\\0\end{bmatrix}$ and $x \neq 0$, it means, these 3 vectors are at the same plane, C is singular.

solution: $x = \begin{bmatrix}a\\a\\a\end{bmatrix}$ a is any number.

No C^{-1} such that $x = C^{-1}0$! An big problem.

column picture of C



These 3 vectors are in one plane.

3 vectors are dependent.

Do we know an equation for this plane? $\Rightarrow$ all combinations of $u, v, \overset{\text{dependent}}{\overset{\downarrow}{w^*}}$

= all vectors Cx.

what $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ do we get?

$b_1 + b_2 + b_3 = 0$, this is the equation of this plane.

Vector space: the place that are able to take all the combinations.

subspace: e.g. a 2-D subspace in 3-D space.

Within $\mathbb{R}^3$, subspace could be a plane (2D), a line (1D), and $\underbrace{\text{a point}}_{\text{origin}}$ (0-D)

3D is also a subspace.

$0 \cdot x = 0$ if it is not $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, e.g. $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, then multiply something

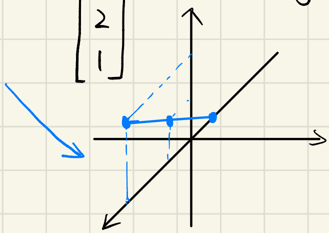
will be away from $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

Rectangular matrix A. 7x3 A^TA: 3x3. always square and symmetric.

Exercise

Suppose A is a matrix, s.t. the complete solution to $Ax=\begin{bmatrix}1\\4\\1\\1\end{bmatrix}$

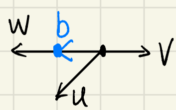
is $x=\begin{bmatrix}0\\1\\1\end{bmatrix}+\underset{\text{any number}}{c}\cdot\begin{bmatrix}0\\2\\1\end{bmatrix}$, what can you say about the columns of A?



my solution:

$$b=4\times 1,\quad x=3\times 1 \quad \therefore A=4\times 3 \Rightarrow A=\begin{array}{c} \begin{array}{ccc} u & v & w \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \end{array}$$

- A does not have inverse.
- The column vectors of A are not independent, otherwise, there will be no solution or only one solution.
- $v+w+c(2v+w)=b \Rightarrow 2v+w=0$



Answer

$A=[c_1,c_2,c_3]$ $\quad c_1,c_2,c_3$ are in $\mathbb{R}^4$

$$x=\underbrace{\begin{bmatrix}0\\1\\1\end{bmatrix}}_{x_p}+c\underbrace{\begin{bmatrix}0\\2\\1\end{bmatrix}}_{x_s} \qquad A(x_p+cx_s)=b \text{ for } c\in\mathbb{R}$$

$$\left.\begin{array}{l} c=0:\ Ax_p=b \\ c=1:\ Ax_p+Ax_s=b \end{array}\right\} Ax_s=0$$

$$[c_1,c_2,c_3]\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}1\\4\\1\\1\end{bmatrix}=[c_2+c_3]=b$$

$$[c_1, c_2, c_3]\begin{bmatrix}0\\2\\1\end{bmatrix} = 0 \Rightarrow 2c_2 + c_3 = 0 \Rightarrow \begin{matrix} c_2 = -b \\ c_3 = 2b \end{matrix}$$

What about the first column? $A = \begin{bmatrix} & -1 & 2 \\ & -4 & 8 \\ c_1 & -1 & 2 \\ & -1 & 2 \end{bmatrix}$

$$\left.\begin{matrix} A(x_p + c x_s) = b \\ A[c x_s] = 0 \end{matrix}\right\} \quad A \cdot c \begin{bmatrix}0\\2\\1\end{bmatrix} = 0 \Rightarrow Null(A) = \left\{ c \cdot \begin{bmatrix}0\\2\\1\end{bmatrix} \,\middle|\, \forall c \in \mathbb{R} \right\}$$

A line.

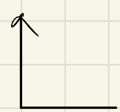
$\Downarrow$

$\dim(Null(A)) = 1$

$\therefore rank(A) = 3 - 1 = 2$

$\therefore c_1$ is not a multiple of b.

$$\boxed{Null\underset{\substack{\Downarrow \\ \mathbb{R}^{m\times n}}}{(A)} = \{ x \mid x \text{ is in } \mathbb{R}^n \text{ and } Ax = 0 \}}$$



Lec 2: Elimination with matrices. (purpose: solve the system)

$$x + 2y + z = 2$$
$$3x + 8y + z = 12$$
$$4y + z = 2$$
$$Ax = b$$

1. Elimination — success / Failure: singular case.
2. Back substitution
3. Elimination matrices
4. Matrix multiplication

1st pivot, 2nd pivot, 3rd pivot

$$A = \begin{bmatrix} \boxed{1} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{bmatrix} \rightarrow \overset{U}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & \boxed{5} \end{bmatrix}}$$

pivot can't be 0.

$\det|A| = 1 \times 2 \times 5 = 10$

U: upper triangular

2. Back substitution

Augmented matrix

$$\underset{A \quad\quad b}{\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right]} \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \rightarrow \underset{U \quad\quad C}{\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right]}$$

$A \rightarrow U$

$b \rightarrow C$

$$\Downarrow$$

$$\left.\begin{aligned} x + 2y + z &= 2 \\ 2y - 2z &= 6 \\ 5z &= -10 \end{aligned}\right\} \quad \begin{aligned} x &= 2 \\ y &= 1 \\ z &= -2 \end{aligned} \uparrow \text{back}$$

Columns: geometric interpretation.

Rows: solving this system.

3. Elimination matrices

step 1:

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

step 2: subtract 2x row 2

$$\underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

elimination matrix

The whole elimination process:

$$E_{32}(E_{21}A) = U \qquad \text{Let } E = E_{32}E_{21}, \quad EA = U$$
$$(E_{32}E_{21})A = U$$

Inverses.

$$\underset{E^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Permutation matrices: Exchange rows 1 and 2.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

P: permutation matrix

How about change columns?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Exercise:

solve, using the method of elimination:

$$\begin{aligned} x - y - z + u &= 0 \\ 2x \quad + 2z \quad &= 8 \\ -y - 2z \quad &= -8 \\ 3x - 3y - 2z + 4u &= 7 \end{aligned}$$

Augmented matrix

$$\left[\begin{array}{cccc|c} \boxed{1} & -1 & -1 & 1 & 0 \\ 2 & 0 & 2 & 0 & 8 \\ 0 & -1 & -2 & 0 & -8 \\ 3 & -3 & -2 & 4 & 7 \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{cccc|c} \boxed{1} & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -2 & 8 \\ 0 & -1 & -2 & 0 & -8 \\ 3 & -3 & -2 & 4 & 7 \end{array}\right] \xrightarrow{r_4 - 3r_1} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -2 & 8 \\ 0 & -1 & -2 & 0 & -8 \\ 0 & 0 & 1 & 1 & 7 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & \boxed{2} & 4 & -2 & 8 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & -1 & -2 & 0 & -8 \end{array}\right] \xrightarrow{r_4 + \frac{1}{2}r_2} \left[\begin{array}{cccc|c} \boxed{1} & -1 & -1 & 1 & 0 \\ 0 & \boxed{2} & 4 & -2 & 8 \\ 0 & 0 & \boxed{1} & 1 & 7 \\ 0 & 0 & 0 & \boxed{-1} & -4 \end{array}\right] \Rightarrow \begin{aligned} x - y - z + u &= 0 \\ 2y + 4z - 2u &= 8 \\ z + u &= 7 \\ -u &= -4 \end{aligned}$$

$\therefore$ $u = 4$

$z = 3$

$y = 2$

$x = 1$

back substitution

Lec 3: Multiplication and inverse matrices

- Matrix multiplication (5 ways!)
- Inverse of A $\quad AB \quad A^T$
- Gauss-Jordan / find A^{-1}

- Matrix multiplication

① $\underset{A^{m\times n}}{\begin{bmatrix} & & \\ & & \\ r_3 & \text{— — —} & \end{bmatrix}} \underset{B^{n\times p}}{\begin{bmatrix} & & c_4 \\ & & \vdots \\ & & \vdots \end{bmatrix}} \underset{C^{m\times p}=AB}{\begin{bmatrix} c_{34} & & \\ & & \\ & & \end{bmatrix}}$

$$c_{34} = r_3(A)\cdot c_4(B)$$
$$= a_{31}\cdot b_{14} + a_{32}\cdot b_{24} + \cdots$$
$$= \sum_{k=1}^{n} a_{3k} b_{k4}$$

② $\underset{A^{m\times n}}{\begin{bmatrix} & & \\ & & \end{bmatrix}} \underset{B^{n\times p}}{\begin{bmatrix} \vdots & & \\ \vdots & & \end{bmatrix}} = \underset{C^{m\times p}=AB}{\begin{bmatrix} & & \\ & & \end{bmatrix}} = \begin{bmatrix} A\times c_1(B), & \underset{\text{column}}{A\times c_2(B)}, & \cdots \end{bmatrix}$

Each column of C is a combination of columns in A.

☆ What about the rank? $\quad \text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$

③ $\underset{A^{m\times n}}{\begin{bmatrix} & & \\ & & \end{bmatrix}} \underset{B^{n\times p}}{\begin{bmatrix} & & \\ & & \end{bmatrix}} = \underset{C^{m\times p}}{\begin{bmatrix} & & \\ & & \end{bmatrix}} = \begin{bmatrix} r_1(A)\times B \\ r_2(A)\times B \\ \vdots \end{bmatrix}$ (row)

Each row of C is a combination of rows in B.

④ column of A $\times$ row of B $\qquad AB = \text{sum}(\text{cols of } A \times \text{rows of } B)$

$\quad m\times 1 \qquad\qquad 1\times p$

e.g. $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$

$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0, & 0 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$

⑤ Block multiplication

$A_1B_1 + A_2B_3$

$$\underset{A}{\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]} \underset{B}{\left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]} = \underset{C}{\left[\begin{array}{c|c} \downarrow & \\ \hline & \end{array}\right]}$$

- Inverse (square matrices)

— Invertible = nonsingular

$A^{-1}A = I = AA^{-1}$

↑

If this matrix exists.

A left inverse is also a right inverse.

Analogy: $\frac{1}{3} \times 3 = 3 \times \frac{1}{3}$

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$A \times$ column j of A^{-1} = column j in I

$\therefore$ get inverse = solve a linear system.

Gauss-Jordan (solve 2 equations at once)

Eq.1 $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Eq.2 $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Augmented matrix

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 7 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

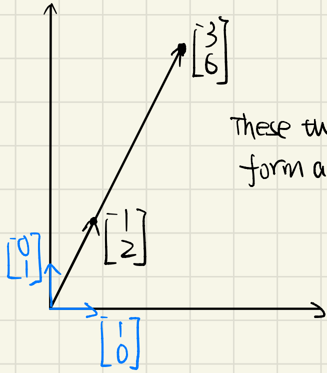
$$\quad A \quad\ I \qquad\qquad\qquad\qquad\qquad\qquad I \quad A^{-1}$$

$[A \mid I] \Rightarrow [A^{-1}A \mid A^{-1}I] = [I \mid A^{-1}]$

$E[A\ I] = [I\ A^{-1}]$

— Singular case, no inverse

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \Rightarrow$$



These two columns can never form an identity matrix.

I can find a non-zero vector x that $Ax=0$, $x=\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

prove by contradiction:

If A is invetible and there is a non-zero vector x such that $Ax=0$.

$\therefore A^{-1}Ax = A^{-1}0$

$\therefore x = 0$ contradict x is non-zero.

Exercize

Find the condition on a and b that make the matrix A invertible, and find A^{-1} when it exists.

$$A=\begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$$

Solution:

The columns should be independent, i.e. $Ax=0$ iff $x=0$.

$$A \Rightarrow \begin{bmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & a-b & a-b \end{bmatrix} \Rightarrow \begin{bmatrix} a & b & b \\ 0 & a-b & 0 \\ 0 & 0 & a-b \end{bmatrix}$$

$$\det = a\,(a-b)^2 \neq 0$$

$$\therefore\ a \neq 0,\ a \neq b$$

Inverse of this matrix, Gauss-Jordan method:

$$\left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ a & a & b & 0 & 1 & 0 \\ a & a & a & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & a-b & a-b & -1 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & 0 & a-b & 0 & -1 & 1 \end{array}\right]$$

$$\frac{b}{a-b}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} a & 0 & b & \frac{a}{a-b} & -\frac{b}{a-b} & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & 0 & a-b & 0 & -1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} a & 0 & 0 & \frac{a}{a-b} & 0 & -\frac{b}{a-b} \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & 0 & a-b & 0 & -1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} \frac{1}{a-b} & 0 & \frac{-b}{a(a-b)} \\ \frac{-1}{a-b} & \frac{1}{a-b} & 0 \\ 0 & \frac{-1}{a-b} & \frac{1}{a-b} \end{matrix}}_{\text{inverse matrix}} \\ 0 & 0 & 1 & \end{array}\right]$$

$$A^{-1} = \frac{1}{a-b}\begin{bmatrix} 1 & 0 & -b \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Lec 4: Factorization into $A=LU$

- Inverse of AB, A^T
- Product of elimination matrices $A=LU$ (no row exchanges)

① Inverse.

$$A^{-1}A = I = AA^{-1}$$

$$ABB^{-1}A^{-1} = I \qquad \therefore \text{ For } AB. \ B^{-1}A^{-1} \text{ is its inverse matrix.}$$

$$B^{-1}A^{-1}AB = I$$

transpose $\quad (AB)^T = B^TA^T$

$$AA^{-1} = I \qquad (AA^{-1})^T = (A^{-1})^TA^T = I$$

② $A=LU$

elementary matrix

$$\begin{array}{ccc} E_{21} & A & U \\ \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} & = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \end{array}$$

$$\begin{array}{ccc} A & L & U \\ \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = & \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \end{array}$$

$$\begin{array}{ccc} \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} & \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \\ L & D & U \end{array}$$

Eliminate

$E_{32}E_{31}E_{21}A = U$ (assume no row exchange)

$$\Downarrow$$

$$A = \underbrace{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}_{L}U$$

suppose

$$\begin{array}{cccc} E_{32} & E_{31} & E_{21} & \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \boxed{10} & -5 & 1 \end{bmatrix} = E \end{array} \qquad EA=U$$

product of interfering

Inverses, reverse order

$$\begin{array}{ccc} E_{21}^{-1} & E_{31}^{-1} & E_{32}^{-1} \\ \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = L \qquad A = LU$$

right order, no interfere

$A = LU$. If no row exchanges, the multipliers go directly into L.

interfere

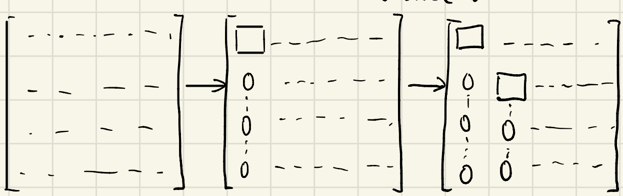
$$EA = \begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} \Rightarrow \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & x & x \end{bmatrix} \Rightarrow \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}$$

$$LU = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix} \Rightarrow \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & x & x \end{bmatrix} \Rightarrow \begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}$$ no interfere

How many operations on $n \times n$ matrix A? Time complexity. n? n^2? n^3? $n!$?

$n = 100$ about n^2 about $(n-1)^2$

full of non-zeros



count: $n^2 + (n-1)^2 + \cdots + 2^2 + 1^2 \approx \frac{1}{3}n^3 \iff \int_1^n x^2 dx$ $\qquad \int_1^n 2x^2 dx = \frac{2}{3}x^3$

cost of $b \approx n^2$

Permutation natrix row exchanges 6 P's A_3^3 $P^{-1} = P^T$

3x3
$$\underset{123}{}\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad \underset{213}{P_{12}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \underset{321}{P_{13}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \underset{132}{P_{23}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$231 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad 312 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

A group of 6 matrices. multiplication and inverse are still in the group.
multi-permutation undo permutation

4x4, how many P's? $A_4^4 = 4 \times 3 \times 2 \times 1 = 24$

Exercise

Find the LU-decomposition of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{pmatrix}$$

when it exists. For which real numbers a and b does it exist?

Solution:

$\because\ EA = U \qquad A = LU$

$\therefore\ E^{-1} = L$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{bmatrix} \xrightarrow{E_{21}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ b & b & a \end{bmatrix} \xrightarrow{E_{31}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & b & a-b \end{bmatrix} \xrightarrow{E_{32}} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a-b \end{bmatrix}}_{U} = U$$

$-b$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{b}{a} & 1 \end{bmatrix}}_{E_{32}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{bmatrix}}_{E_{31}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & -\frac{b}{a} & 1 \end{bmatrix} = E$$

$$L = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix}}_{E_{31}^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{b}{a} & 1 \end{bmatrix}}_{E_{32}^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & \frac{b}{a} & 1 \end{bmatrix}$$

requirement $a \neq 0$

comment: if $a=b$, U is singular.

Singular matrices can have LU decompositions.

LU decomposition comes with Gauss elimination naturally.

Lec 5: Transposes, Permutations, Spaces R^n

- $PA = LU$
- vector spaces and subspaces

Permutations P: excute row exchanges. (shift away 0 on pivot position)

$$A = LU = \begin{bmatrix} 1 & & 0 \\ - & 1 & \\ - & - & 1 \end{bmatrix}\begin{bmatrix} \diagdown & - & - \\ & \diagdown & - \\ 0 & & \diagdown \end{bmatrix}$$ becomes row exchange $PA = LU$ for any invertible A
$\uparrow$ permutation matrix

P = identity matrix with reordered rows.
counts reorderings : all $n \times n$ permutations. $n! = n(n-1)\cdots(3)(2)(1)$

$P^{-1} = P^T \qquad P^T P = I$

- Transpose matrix

$$(A^T)_{ij} = A_{ji}$$

E.g. $\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$ ← R^T ← R

- symmetric matrix

$$A^T = A$$

$R^T R$ is always symmetric.

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & 13 & 11 \\ 7 & 11 & 17 \end{bmatrix}$$

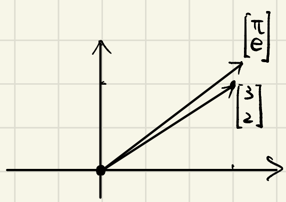
why ?

$$(R^T R)^T = R^T R^{TT} = R^T R$$

- vector spaces

Examples : R^2 = all 2-dimensional real vectors. $\begin{bmatrix} 3 \\ 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}\begin{bmatrix} \pi \\ e \end{bmatrix}$
= "x-y" plane

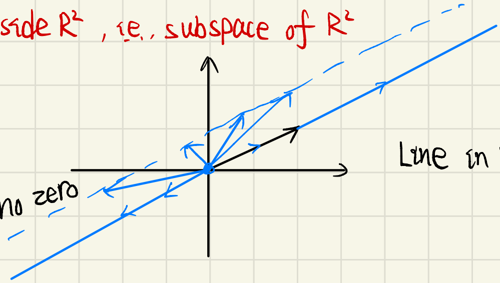
R^3 = all 3-dimensional real vectors.
R^n = all column vectors with real numbers.



not a vector space

addition ✓
multiplication with real number ✗
(may go out)

A vector space inside R^2, i.e., subspace of R^2



Line in R^2 through zero vector.

☆

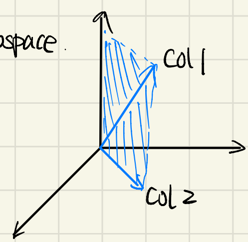
subspaces of R^2:
① all of R^2
② any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
③ zero vector only.

Pick any vector in these subspaces, do some simple operations, the product is still in that subspace.

subspaces of R^3:
① all of R^3
② planes through origin
③ lines through origin
④ zero vector only

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

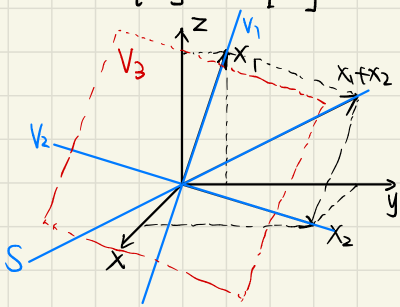
columns are in R^3
How to create a subspace?
All their linear combinations form a subspace.
called column space. $C(A)$



give a plane throgh origin

Exercise:

$x_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ $x_2 = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$



1. Find V_1 = subspace generated by x_1,
   V_2 = subspace generated by x_2
   Describe $V_1 \cap V_2$.
2. Find V_3 = subspace generated by $\{x_1, x_2\}$
   Is V_3 equal to $V_1 \cup V_2$?
   Find a subspace S of V_3, s.t. $x_1 \notin S$, $x_2 \notin S$.
3. what is $V_3 \cap \{x\text{-}y \text{ plane}\}$?

two conditions: preserve addition
preserve multiplication
In short: linear combination

Solution:

1. $V_1 = \{a x_1, \forall a \in R\}$ line
   $V_2 = \{a x_2, \forall a \in R\}$ line
   $V_1 \cap V_2 = \{(0,0,0)^T\}$ zero vector

2. $V_3 = \{a x_1 + b x_2, \forall a, b \in R\}$ plane
   $V_3 \neq V_1 \cup V_2$ not a subspace $x_1 + x_2 \notin V_1 \cup V_2$
   plane two lines

   A subspace of V_3: $S = \{a x_1 + a x_2, \forall a \in R\}$ a line
   or zero vector

3. $V_3 \cap \{x\text{-}y \text{ plane}\} = V_2$

Lec 6: column space and nullspace

- vector spaces and subspaces
- column space of A: solving $Ax=b$
- Null space of A

vector space requirements:

$\underbrace{v+w}_{\text{sum}}$ and $\underbrace{cv}_{\text{multiplication}}$ are in the space or all combinations $cv+dw$ are in the space

2 subspaces: P and L (Plane and line in R^3)

$P \cup L$ = all the vectors in P and L. This is not a subspace.

$P \cap L$ = all the vectors in both P and L. This is a subspace.

Subspaces S and T. intersection $S \cap T$ is a subspace.

- Column space of A is a subspace of R^4. $C(A)$ For A

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

$C(A)$ = all linear combinations of columns.

Does $Ax=b$ have a solution for every b? No.

4 equations 3 unknows

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

which b's allow this system can be solved?

Answer: $b \in C(A)$

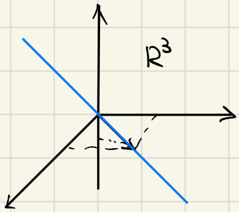
$C(A)$ is a 2-dimensional subspace in R^4.

- Null space of A = all solutions $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $Ax=0$. For x.

in R^3

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$N(A)$ contains $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} c \\ c \\ -c \end{bmatrix}$



$\Rightarrow c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ line in R^3

check that solutions to $Ax=0$ always give a subspace.

If $Av=0$ and $Aw=0$ then $A(v+w)=0$

- consider another condition of solution space.

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

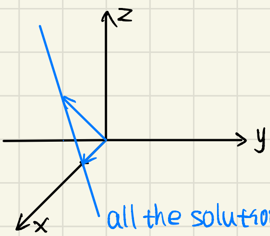
Does this have a solution? ✓

Do the solutions form a vector space? ✗

consider zero vector.

$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$



is not a vector space.

Exercise

which are subspaces of $R^3 = \left\{ \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right\}$?

(1) $b_1 + b_2 - b_3 = 0$

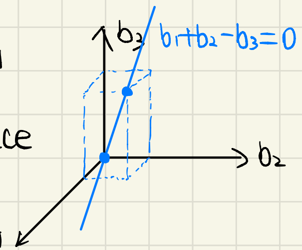
(2) $b_1 b_2 - b_3 = 0$

(3) $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(4) $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

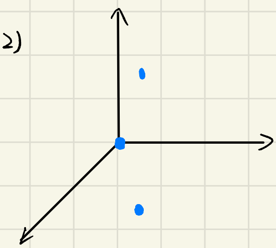
solution: (1)

a subspace



$\begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0$ null space of $\begin{pmatrix} 1 & -1 & -1 \end{pmatrix}$

(2)



not a subspace.

$\begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} \Rightarrow b_1 b_2 - b_3 = 0$

$\begin{pmatrix} 4 \\ 4 \\ 8 \end{pmatrix} \Rightarrow b_1 b_2 - b_3 \neq 0$

(3) is a subspace $-1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ can get $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

a plane through origin in R^3

(4) not a subspace. can not return to zero vector.

a plane does not through origin in R^3

Lec 7: solving $Ax=0$: Pivot variables, special solutions

- computing the nullspace ($Ax=0$)
- pivot variables - free variables
- special solutions - $rref(A)=R$

Echelon matrix: step matrix

- computing the nullspace ($Ax=0$) $\Rightarrow Ux=0 \Rightarrow Rx=0$

$\boxed{\ }$: pivot

2 pivots

$$A=\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \xrightarrow{\text{elimination}} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

2 pivot column

free columns

Rank of A = # of pivots in U.

free varibles = # of varibles − # of pivot variables

Compared to invertible square matrix, this require:
① fix the free columns
② backward substitution.

$$x_1+2x_2+2x_3+2x_4=0$$
$$2x_3+4x_4=0$$

$$x = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

fix fix fix fix

2 special solutions

An algorithm for finding null space.

R = reduced row echelon form: zeros above and below pivots

Matlab operation: $R=rref(A)$

$$U=\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Pivot rows, pivots columns

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

free columns

$$\begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}$$

$$1x_1 + 2x_2 \quad -2x_4 = 0$$
$$x_3 + 2x_4 = 0$$

group the pivot columns
and free columns.

rref form

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow r \text{ pivot rows} \\ \leftarrow m-r \text{ free columns} \end{matrix}$$

$\uparrow$ r pivot columns

Pivot columns, free columns

$$= \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 \end{bmatrix} \\ \begin{matrix} 1 & 3 & 2 & 4 \end{matrix} \end{array}$$

I F : free part

$$RX=0 \Rightarrow RN=0$$

N is null space matrix (columns = special solutions)

$$\therefore N = \begin{bmatrix} -F \\ I \end{bmatrix} \qquad RN = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore N = \begin{array}{cc} \begin{bmatrix} -2 & 2 \\ 0 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{matrix} 1 \\ 3 \\ 2 \\ 4 \end{matrix} \end{array} \Rightarrow x = c \begin{array}{cc} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \end{array} + d \begin{array}{cc} \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \end{array}$$

match the $UX=0$ solution.

null space matrix, columns are null basis

Another example:

U $\text{rank}(A)=2$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot columns   free column. (one special solution)

$$x = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \text{fix} \qquad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_2 + 2x_3 &= 0 \end{aligned}$$

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{c} \begin{matrix} I & F \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} = R \qquad RX=0 \qquad x = c \begin{bmatrix} -F \\ I \end{bmatrix} = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$\uparrow$ null space matrix

Exercise:

The set S of points $P(x,y,z)$ s.t. $x-5y+2z=9$ is a ___①___ in $\mathbb{R}^3$.
It is ___②___ to the ___③___ S_0 of $P(x,y,z)$ s.t. $x-5y+2z=0$.
All points of S have the form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \; \\ 0 \\ 0 \end{bmatrix} + c_1 \cdot \begin{bmatrix} \; \\ 1 \\ 0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} \; \\ 0 \\ 1 \end{bmatrix}$$

Solution:

① A plane without origin point.

② parallel ③ plane

$x-5y+2z=0$

$A = [1 \ -5 \ 2]$ $\quad A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \therefore \ R = [\boxed{1} \ -5 \ 2]$

Pivot varible

free columns.

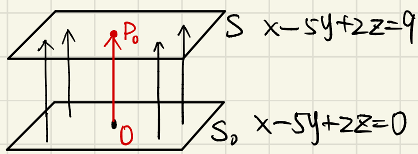
$$\Downarrow$$

$$[I \ F]\begin{bmatrix} -F \\ I \end{bmatrix} = 0$$

$\therefore$ the null space matrix is $\begin{bmatrix} 5 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \overset{9}{?} \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

null basis $\qquad P_0$

$? = 9$ if we take $c_1 = 0$ and $c_2 = 0$.



Any point in $S = P_0 +$ any point in S_0

Lec 8: solving $Ax=b$: Row reduced form R

- complete solution of $Ax=b$
- Rank r $\qquad x = x_p + x_n$ $\quad$ particular solution + null space
- $r=m$: solution exists. $r=n$: solution is unique

$$x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$$
$$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$$
$$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$$

pivots

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & \underline{b_3 - b_2 - b_1} \end{bmatrix} \begin{array}{l} b_1 = 1 \\ b_2 - 2b_1 = 3 \\ b_3 - b_2 - b_1 = 0 \end{array}$$

Augmented matrix $= [A \ b]$ $\qquad 0 = b_3 - b_2 - b_1$

If $b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$, OK, it allows a solution

Solvability: condition on b

$Ax=b$ is solvable when b is in $C(A)$, column space of A.

If a combination of rows of A gives zero row, then the same combination of entries of b must give 0.

To find the complete solution to $Ax=b$

① $X_{particular}$: set all free variables to zero, and solve $Ax=b$ for pivot variables.

$+$

$$\begin{array}{l} x_1 + 2x_3 = 1 \\ \quad\;\; 2x_3 = 3 \end{array} \Rightarrow \begin{array}{l} x_3 = \frac{3}{2} \\ x_1 = -2 \end{array} \Rightarrow x_p = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}$$

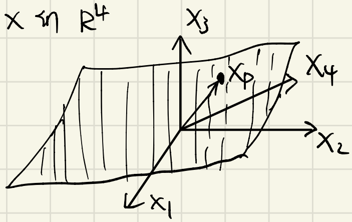
$$\left.\begin{array}{l} Ax_p = b \\ Ax_n = 0 \\ A(x_p + x_n) = b \end{array}\right\} ☆$$

② $X_{null\ space}$

$X = x_p + x_n$

$$X_{complete} = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Plot all solutions x in $\mathbb{R}^4$



$m \times n$ matrix A of rank r (know $r \le m$, $r \le n$)

- Full column rank means $r = n$: no free variables

  $N(A) = $ [zero vector] Solution to $Ax = b$: $x = x_p$. unique solution if it exists.

  An example: $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}$ $\underset{\text{rref}}{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ If $b = \begin{bmatrix} 4 \\ 3 \\ 7 \\ 6 \end{bmatrix} \Rightarrow x = x_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- Full row rank means $r = m < n$

  Can solve $Ax = b$ for every b. Exists.

  $n - r$ free variables.

  $A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$ $R = \left[\begin{array}{cc|cc} 1 & 0 & - & - \\ 0 & 1 & - & - \end{array}\right]$ with the dashed block labelled F

- $r = m = n$, full rank, invertible

  $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ $\underset{\text{rref}}{R} = I$

$r = m = n$	$r = n < m$	$r = m < n$	$r < m,\ r < n$
$R = I$	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$	$R = \begin{bmatrix} I & F \end{bmatrix}$	$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
1 solution	0 or 1 solution	∞ solutions	0 or ∞ solutions

Exercise:

Find all solutions, depending on b_1, b_2, b_3:

$$\begin{aligned} x-2y-2z &= b_1 \\ 2x-5y-4z &= b_2 \\ 4x-9y-8z &= b_3 \end{aligned}$$

Solution:

$$[A\ x] = \begin{bmatrix} 1 & -2 & -2 & b_1 \\ 2 & -5 & -4 & b_2 \\ 4 & -9 & -8 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -2 & b_1 \\ 0 & -1 & 0 & b_2-2b_1 \\ 0 & -1 & 0 & b_3-4b_1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -2 & b_1 \\ 0 & 1 & 0 & 2b_1-b_2 \\ 0 & 0 & 0 & b_3-b_2-2b_1 \end{bmatrix}$$

If $b_3-b_2-2b_1 \neq 0$, no solution; If $b_3-b_2-2b_1=0$, continue.

rref:

$$R = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

$\uparrow$ pivot $\nearrow$ $\quad$ $\uparrow$ free

$Ax_p = b$ ①

$Ax_n = 0$ ②

① get the particular solution by setting all free variables as zero.

$$\begin{aligned} x-2y &= b_1 \\ y &= 2b_1-b_2 \end{aligned} \Rightarrow x = 5b_1-2b_2 \Rightarrow x_p = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5b_1-2b_2 \\ 2b_1-b_2 \\ 0 \end{bmatrix}$$

② get the null space, i.e. get the special solution.

$$R\begin{bmatrix} -F \\ I \end{bmatrix} = 0 \Rightarrow \text{null matrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = x_s$$

$$\therefore \text{solution} = \begin{bmatrix} 5b_1-2b_2 \\ 2b_1-b_2 \\ 0 \end{bmatrix} + c\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = x_p + c\,x_s$$

This is the case $m \times n = 3\times 3$, $r=2$, $r<m$, $r<n$

cause no unique

may cause no solution

Lec 9: Independence, Basis, and Dimension

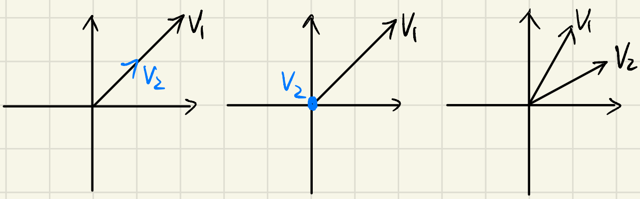
- Linear independence
- spanning a space
- Basis and dimension

suppose A is m by n with $m<n$, i.e., more unknowns than equations, then there are nonzero solutions to $Ax=0$.

Reason: There will be free variable!! Think about the rref.

Independence: Vectors $x_1, x_2, \dots, x_n$ are independent if no combination give zero vector, except the zero combination.

$c_1x_1 + c_2x_2 + \cdots + c_nx_n \neq 0$ except $c_1 = c_2 = \dots = c_n = 0$



dependent　　dependent　　independent

Repeat when $V_1, \dots, V_n$ are columns of A, they are independent if nullspace of A is $\{\text{zero vector}\}$. Rank$(A)=n$, no free variables.

They are dependent if $Ac=0$ for some nonzero c. Rank$(A)<n$. has free variable.

Vectors $V_1, \dots, V_l$ Span a space means: The space consists of all combinations of those vectors.

Basis for a vector space is a sequence of vectors $V_1, V_2, \dots, V_d$ with 2 properties:

① they are independent.

② they span the space.

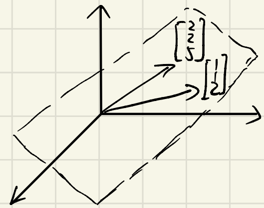
Example:

Space is $\mathbb{R}^3$

One basis is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ another basis $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix}$

$\mathbb{R}^n$: n vectors give basis if the $n \times n$ matrix with those columns is invertible.

Consider: $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$ is the basis of a plane in $\mathbb{R}^3$.



Given a space $\mathbb{R}^n$, every basis for the space has n vectors.

The dimension of the space.

Example:

space is $C(A)$.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

another basis

$$\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\text{a basis of null space}} \in N(A)$$

They span $C(A)$.
They are not independent.

A basis for $C(A)$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

Rank (A) = # of pivot columns = dimension of $C(A) = 2$

dim $[N(A)]$ = # of free variables = $n - \text{rank}(A)$

Exercise

Find the dimension of the vector space spanned by the vectors

$$\begin{bmatrix}1\\1\\-2\\0\\1\end{bmatrix}\begin{bmatrix}1\\2\\0\\-4\\1\end{bmatrix}\begin{bmatrix}0\\1\\3\\-3\\2\end{bmatrix}\begin{bmatrix}2\\3\\0\\-2\\0\end{bmatrix}$$

and find a basis for that space.

Solution:

Are they independent? Do elimination to find the number of pivots.

$$\begin{array}{c}\begin{bmatrix}1&1&0&2\\1&2&1&3\\-2&0&3&0\\0&-4&-3&-2\\1&1&2&0\end{bmatrix}\\ \uparrow\ \ \uparrow\ \ \uparrow\qquad\quad \\ \text{basis}\qquad\quad\end{array} \rightarrow \begin{bmatrix}1&1&0&2\\0&1&1&1\\0&2&3&4\\0&-4&-3&-2\\0&2&2&2\end{bmatrix} \rightarrow \begin{bmatrix}1&1&0&2\\0&1&1&1\\0&0&1&2\\0&0&1&2\\0&0&0&0\end{bmatrix} \rightarrow \begin{bmatrix}\boxed{1}&1&0&2\\0&\boxed{1}&1&1\\0&0&\boxed{1}&2\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

note: column space has changed.

These 4 vectors are dependent.

A basis is :

$$\begin{array}{ccc}\begin{bmatrix}1\\1\\-2\\0\\1\end{bmatrix}&\begin{bmatrix}1\\2\\0\\-4\\1\end{bmatrix}&\begin{bmatrix}0\\1\\3\\-3\\2\end{bmatrix}\\ v_1 & v_2 & v_3\end{array}$$

a basis of a 3-dimensional subspace in $\mathbb{R}^5$.

$$\dim[\text{span}(v_1, v_2, v_3)] = 4$$

Lec 10: The four foundamental subspaces. (for matrix A)

Example:

space is $\mathbb{R}^3$

one basis is $\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}$ Another is $\begin{bmatrix}1\\1\\2\end{bmatrix}\begin{bmatrix}2\\2\\5\end{bmatrix}\begin{bmatrix}3\\3\\8\end{bmatrix}$ X they are not independent

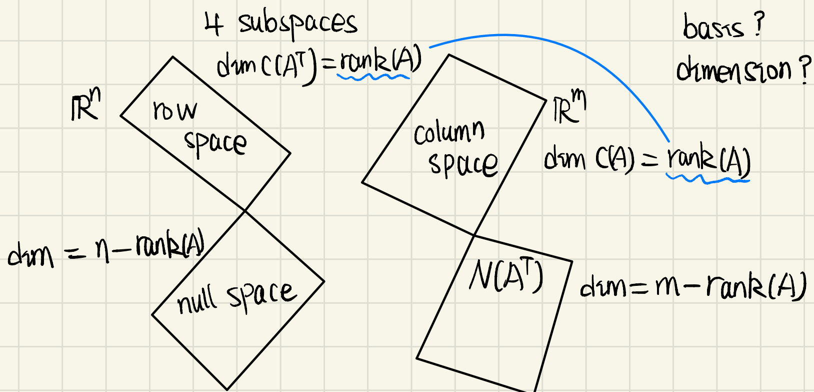
rows are dependent

4 subspaces

- column space $C(A)$
- null space $N(A)$
- row space = all combinations of rows = all combinations of columns of $A^T = C(A^T)$
- null space of $A^T = N(A^T)$ left null space of A

If $A = m \times n$, $C(A)$ is in $\mathbb{R}^m$
$N(A)$ is in $\mathbb{R}^n$
$C(A^T)$ is in $\mathbb{R}^n$
$N(A^T)$ is in $\mathbb{R}^m$

	C(A)	N(A)
basis?	pivot columns	special solution
dimension?	rank(A)	n − rank(A)



$$A = \begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix} \rightarrow \begin{array}{c} \quad I \qquad F \\ \begin{bmatrix}1&0&1&1\\0&1&1&0\\0&0&0&0\end{bmatrix} \\ 0 \end{array} = R$$

$C(R) \neq C(A)$ different column space

same row space

Basis for row space is first r rows in R.

Basis $[1\ 0\ 1\ 1]$
$[0\ 1\ 1\ 0]$

$$E = \begin{bmatrix}-1&2&0\\1&-1&0\\-1&0&1\end{bmatrix}$$ basis for $N(A^T)$

$N(A^T)$: $A^T y = 0 \xrightarrow{\text{Transpose}} y^T A = 0^T$ left null space

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix}\begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ 0 \\ \\ \end{bmatrix} \qquad \begin{bmatrix} & & \end{bmatrix}\begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} & 0 & \end{bmatrix}$$

$$\text{rref}\,[A_{m\times n} \;\; I_{m\times m}] \rightarrow [R_{m\times n} \;\; E_{m\times m}]$$

$$E\,[A_{m\times n} \;\; I_{m\times m}] = [R_{m\times n} \;\; E_{m\times m}]$$

new vector space M

All 3x3 matrices !! $A+B$, cA (not AB for now)

subspaces of M	upper triangular matrices	symmetric matrices	diagonal matrices

dim = 3

$\Downarrow$

basis: $\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}}$

They span the subspace of 3x3 diagonal matrices.

Exercise:

suppose

$$B=\begin{pmatrix}1&&\\2&1&\\-1&0&1\end{pmatrix}\begin{pmatrix}5&0&3\\0&1&1\\0&0&0\end{pmatrix}=\begin{pmatrix}5&0&3\\10&1&7\\-5&0&-3\end{pmatrix}$$

Find a basis for and compute the dimension of each of the 4 fundamental subspaces.

Solution:

- $C(B)$: a basis is $\begin{bmatrix}1\\2\\-1\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}$ dim=2.

- $N(B)$: $B=\begin{pmatrix}5&0&3\\10&1&7\\-5&0&-3\end{pmatrix}\to\begin{array}{c}\begin{array}{cc}I&F\end{array}\\\begin{pmatrix}1&0&\frac{3}{5}\\0&1&1\\0&0&0\end{pmatrix}\end{array}=R \Rightarrow$ a null basis is $\begin{bmatrix}-F\\I\end{bmatrix}=\begin{bmatrix}-\frac{3}{5}\\-1\\1\end{bmatrix}$ dim=1

- $C(B^T)$: a basis is $\begin{bmatrix}5\\0\\3\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix}$ dim = 2

- $N(B^T)$ $EB=R$. where $EB=\begin{bmatrix}\frac{1}{5}&0&0\\-2&1&0\\\boxed{1\quad 0\quad 1}\end{bmatrix}\begin{bmatrix}5&0&3\\10&1&7\\-5&0&-3\end{bmatrix}=\begin{bmatrix}1&0&\frac{3}{5}\\0&1&0\\\boxed{0\quad 0\quad 0}\end{bmatrix}$

  $\therefore$ a basis is $\begin{bmatrix}1\\0\\1\end{bmatrix}$ dim= 1

Vedio answer:

$$\begin{pmatrix}1&&\\-2&1&\\1&0&1\end{pmatrix}B=\begin{pmatrix}5&0&3\\0&1&1\\0&0&0\end{pmatrix}$$ a basis is $\begin{bmatrix}1\\0\\1\end{bmatrix}$

Lec 11 : Matrix space ; Rank 1 ; small world graphs

- Basis of new vector spaces $\longrightarrow$ M = all 3x3 matrices
  $\longrightarrow$ subspaces | symmetric S : 3x3 | upper triangular U: 3x3
- Rank one matrices
- Small world graphs

Basis for M = all 3x3 matrices   dim M = 9

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix} \cdots \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

Basis for symmetric matrices   dim S = 6

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix} \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix} \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

dim U = 6

$S \cap U$ = symmetric and upper triangular
= diagonal 3x3 matrices $\dim(S \cap U) = 3$

$S \cup U$ is not a subspace
$\cup \neq +$

$S + U$ = any element of S + any element of U = all 3x3 matrices
$\dim(S+U) = 9$

e.g. $$\underset{S}{\begin{bmatrix}1&1&0\\1&0&0\\0&0&0\end{bmatrix}} + \underset{U}{\begin{bmatrix}0&2&3\\0&3&1\\0&0&4\end{bmatrix}} = \begin{bmatrix}1&3&3\\1&3&1\\0&0&4\end{bmatrix}$$

$\boxed{\text{here } S+U \notin S \cup U \ ☆}$

$\dim S = 6 + \dim U = 6 = 12 = \dim(S \cap U) = 3 + \dim(S+U) = 9$

An example of differential equation:

$$\underbrace{\frac{d^2y}{dx^2} + y = 0}_{\text{find null space}} \qquad \text{sol: } y = \underbrace{\cos x, \sin x}_{\text{basis}} \qquad \dim\begin{pmatrix}\text{solution}\\ \text{space}\end{pmatrix} = 2$$

The complete solution $y = c_1 \cos x + c_2 \sin x$

- Rank 1 matrices

$$\underset{2\times 3}{A} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} = \underset{2\times 1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \underset{1\times 3}{\begin{bmatrix} 1 & 4 & 5 \end{bmatrix}}$$

$r = 1$

rank 1 matrix has the form

$A = UV^T$

a column $\times$ a row

M = all 5×17 matrices, subset of rank 1 matrices, is it a subspace?

$\text{rank}(A)=1$, $\text{rank}(B)=1$, $\text{rank}(A+B)$ may be 2, not a subspace

In $\mathbb{R}^4$, $V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$

S = all V in $\mathbb{R}^4$ with $V_1 + V_2 + V_3 + V_4 = 0$

= null space of $[1, 1, 1, 1]$

$AV = 0$

$m \times n = 1 \times 4$

S is a subspace

$\left.\begin{array}{l} \dim(S) = 3 \\ \text{rank}(A) = 1 \end{array}\right\} 3+1=4$

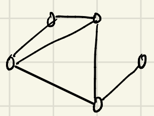
Basis for S:

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\left.\begin{array}{l} C(A) \text{ is } \mathbb{R}^1 \\ N(A^T) = \{0\} \end{array}\right\} 1+0=1$

- small world graph

Graph = {nodes, edges}



Exercise:

Show that the set of 2x3 matrices whose nulspace contains $\begin{bmatrix}2\\1\\1\end{bmatrix}$ is a vector subspace, and find a basis for it.

What about the set of those whose column space contains $\begin{bmatrix}2\\1\end{bmatrix}$?

Solution:

$$A = \begin{array}{c} 2\times3 \\ \begin{bmatrix} & & \\ & & \end{bmatrix} \\ v_1 \ v_2 \ v_3 \end{array}$$

Let M = set of 2x3 matrices whose nulspace contains $\begin{bmatrix}2\\1\\1\end{bmatrix}$.

$\therefore$ M is the set of matrices with rank ≤ 2 and $2v_1 + v_2 + v_3 = 0$.

hint: sum, multiply.

sum

$A + B \in M$ = set of 2x3 matrices whose nulspace contains $\begin{bmatrix}2\\1\\1\end{bmatrix}$.

$\begin{cases} A = [v_1, v_2, v_3] \\ 2v_1 + v_2 + v_3 = \begin{bmatrix}0\\0\end{bmatrix} \end{cases}$ $\begin{cases} B = [w_1, w_2, w_3] \\ 2w_1 + w_2 + w_3 = \begin{bmatrix}0\\0\end{bmatrix} \end{cases}$ $\Rightarrow$ $A+B = [v_1+w_1, v_2+w_2, v_3+w_3]$

$$[A+B]\begin{bmatrix}2\\1\\1\end{bmatrix} = A\begin{bmatrix}2\\1\\1\end{bmatrix} + B\begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

multiply:

$$c \cdot A\begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$\therefore$ M is a vector space, or matrix space more precisely.

$A\begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ Each row of A must be $[a\ b\ c]\begin{bmatrix}2\\1\\1\end{bmatrix} = 0$ $\quad$ $2a + b + c = 0$

$c = -2a - b$

$\therefore$ Must be $[a, b, -2a-b] = [a, 0, -2a] + [0, b, -b]$

= linear combination of $[1, 0, -2]$ and $[0, 1, -1]$

$\therefore$ basis is $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

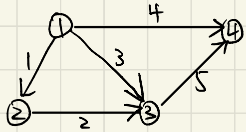
The set of those whose column space contains $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is not in the set. $\therefore$ not a vector space.

Lec 12: Graphs, Networks, Incidence Matrices

- Graphs and networks
- Incidence matrices
- Kirchhoff's laws

Graph: nodes, edges



$m=5$ edges $n=4$ nodes

Incidence matrix

$$A=\begin{array}{c} \begin{matrix}\text{node} & 1 & 2 & 3 & 4 & \text{edge}\end{matrix} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{array}$$

rows 1–3: loop → dependence $r_1+r_2=r_3$

Questions: 1. null space? $Ax=0$ $x=[x_1, x_2, x_3, x_4]$ potentials at nodes.

$$Ax=\begin{bmatrix} x_2-x_1 \\ x_3-x_2 \\ x_3-x_1 \\ x_4-x_1 \\ x_4-x_3 \end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix} \qquad x=C\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

Ax: the different of potentials on different edges.

$\downarrow$ C OHM's law

current $[y_1, y_2, y_3, y_4, y_5]$ on edges

$\downarrow$ Kirchoff's currency law

$A^T y=0$

$\dim N(A)=1$

2. rank of A? $\text{rank}(A)=3$

3. null space of A^T. $A^T y=0$ $\dim N(A^T)=m-r=5-3=2$

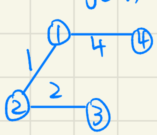
\# of independent loops $=$ \# edges $-$ (\#nodes-1)

\#nodes $-$ \#edges $+$ \#loops $=1$

Eular's formulation

Pivot columns

edge 1, 2, 4



$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\\y_5\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

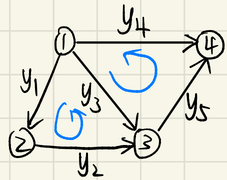
↑

A tree.

$$\begin{aligned} -y_1 - y_3 - y_4 &= 0 \\ y_1 - y_2 &= 0 \\ y_2 + y_3 - y_5 &= 0 \\ y_4 + y_5 &= 0 \end{aligned}$$

Basis for $N(A^T)$

$$\begin{array}{cc} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \\ \text{loop1} & \text{loop2} \end{array}$$

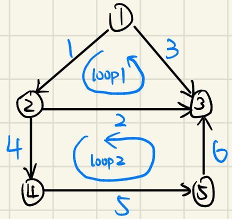


Ax	$y = Ce$	$A^T y = f$
potential difference	OHM's law	out sources

$$A^T C A x = f$$

Kirchhoff's circuit law: for any node (junction) in an electric circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node.

Exercise:



- Find incidence matrix A
- $N(A)$, $N(A^T)$ = ?
- Trace (A^TA) = ?

Solution:

Link-node incidence matrix $A = \begin{array}{c} \left[\begin{array}{ccccc} -1 & 1 & & & \\ & -1 & 1 & & \\ -1 & & 1 & & \\ & -1 & & 1 & \\ & & & -1 & 1 \\ & & 1 & & -1 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \\ \begin{array}{ccccc} ① & ② & ③ & ④ & ⑤ \end{array} \end{array}$ $\quad$ $\text{rank}(A) = 4$

$N(A)$: $\quad Ax = 0 \quad x \in \mathbb{R}^{5\times1}$

$$Ax = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_2 \\ x_5 - x_4 \\ x_3 - x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad x = c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The node potential difference is zero.
Then all nodes should have the same potential. $\dim N(A) = 1$

$N(A^T)$: $\quad A^Ty = 0 \quad y \in \mathbb{R}^{6\times1}$

y: current on edges.
A^Ty represents Kirchhoff's circuit laws, i.e., current conservation.

$$A^Ty = \begin{bmatrix} -y_1 - y_3 \\ y_1 - y_2 - y_4 \\ y_2 + y_3 + y_6 \\ y_4 - y_5 \\ y_5 - y_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad y = c_1 \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{loop1}} + c_2 \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\text{loop2}}$$

A^TA is a symmetric matrix. $tr(A)$ = sum of diagonal elements.

$$A^TA = \begin{bmatrix} -1 & & -1 & & & \\ 1 & -1 & & -1 & & \\ & 1 & 1 & & & 1 \\ & & & 1 & -1 & \\ & & & & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ -1 & & 1 & & \\ & -1 & & 1 & \\ & & & -1 & 1 \\ & & 1 & & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

$\therefore tr(A^TA) = 2+3+3+2+2 = 12$

$tr(A^TA) = \text{sum}(\text{col}1 \times \text{row}1) + \text{sum}(\text{col}2 \times \text{row}2) + \cdots$

$$= \sum_{i=1}^{\text{cols}} \|\text{col}_i\|_2^2$$

In this case, $(\text{column of } A^T)^2$ is node degree.

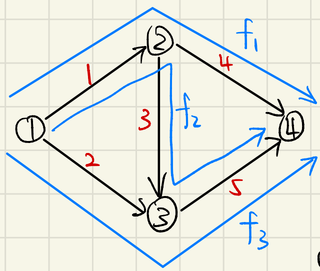
$tr(A^TA)$ = degree of node1 + degree of node2 + ...

$= 2+3+3+2+2 = 12$

Lec 13 : Review for Exam 1

Emphasizes Chapter 3

Some thoughts on network flow problem



node-link incidence matrix

$$A = \begin{array}{c|c} \begin{bmatrix} 1 & 1 & & & \\ -1 & & 1 & 1 & \\ & -1 & -1 & & 1 \\ & & & -1 & -1 \end{bmatrix} & \begin{matrix} ① \\ ② \\ ③ \\ ④ \end{matrix} \\ \begin{matrix} 1 & \;\;2 & \;\;3 & \;\;4 & \;\;5 \end{matrix} & \end{array}$$

S.P.:

Traffic: $Ax = b$

Network flow:

Q1: $N(A)$? find the combination that forms a loop.

$\therefore$

$$N(A) = c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \qquad \dim N(A) = 2$$

$\dim N(A^T) = 1$

$\dim C(A^T) = 3$ in $\mathbb{R}^5$

$\operatorname{rank}(A) = 3$

solution to $Ax = b \Rightarrow x = \begin{bmatrix} \\ \\ \\ \end{bmatrix} + N(A)$

$\boxed{\text{flow problem is to search the best solution for } c_1 \text{ and } c_2.}$

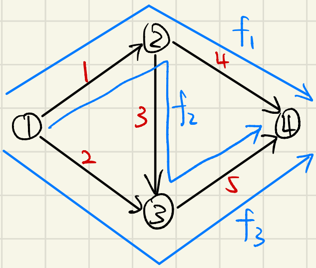
$$A^T y = \begin{bmatrix} y_1 - y_2 \\ y_1 - y_3 \\ y_2 - y_3 \\ y_2 - y_4 \\ y_3 - y_4 \end{bmatrix} \qquad N(A^T) = c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

link flow

$$\min c^T x \qquad \max b^T y \leftarrow \text{node potential}$$

$$Ax = b \qquad A^T y \le c$$

$$x \ge 0$$



$$\Gamma = [1 \quad 1 \quad 1]$$

$$x = \Delta f \qquad \Gamma f = d$$

link-path incidence matrix:

$$\Delta = \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \end{array}$$

Q1: rank$(A) = 3$

Q2: $N(A)$?

$$\Delta f = \begin{bmatrix} f_1 + f_2 \\ f_3 \\ f_2 \\ f_1 \\ f_2 + f_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow N(A) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} \Rightarrow x = \Delta f \text{ solution is unique.}$$

Q3: $N(A^T)$?

$$\Delta^T x = \begin{bmatrix} x_1 + x_4 \\ x_1 + x_3 + x_5 \\ x_2 + x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \dim N(A^T) = 2$$

这个表示什么?

Q: In CP, are the generated paths between an OD pair independent?