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Stochastic Equilibrium Traffic Assignment with Value-of-time Distributed Among Users

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In this paper, existing fixed-point models for stochastic equilibrium traffic assignment to transportation networks are extended to deal with the value of time distributed among users. This problem is relevant when transporation demand management measures affecting travelling monetary costs are to be evaluated. Analysis of existence and uniqueness of solutions is carried out, leading to mild assumptions about link cost-flow functions, path choice model, and value-of-time distribution. Some quite simple solution algorithms are also specified and their convergence analysed. Results of some applications are also reported.

Key words: Stochastic equilibrium, traffic assignment, value-of-time.

1. INTRODUCTION

Models for traffic assignment to transportation networks simulate how Origin–Destination demand flows affect link flows according to user path choice behaviour and network performances. Most existing assignment models follow a *user equilibrium* approach, searching for a link flow pattern in which no user can reduce the cost of his/her choice by unilaterally changing it (reviews and references in Sheffi, 1985, Cascetta, 1990, Patriksson, 1994).

Deterministic user equilibrium (DUE) assignment is defined assuming that all the users exactly perceive path costs, which can be exactly estimated through a network model embedded within the assignment model. DUE can be formulated through variational inequalities (Smith, 1979; Dafermos, 1980).

On the other hand, *stochastic user equilibrium* (SUE) assignment is defined assuming that each user may perceive different path costs, modelled as random variables due to different sources of randomness, such as user perception errors and/or behaviour dispersion, modelling errors, daily fluctuations (caused for instance by weather conditions). SUE can be effectively formulated through fixed-point models (Daganzo, 1983; Cantarella, 1997).*

These models have been applied to deal with assignment to road networks as well as to transit networks with overlapping and competitive lines modelling user route choice strategies through hyperpaths and, more generally, to multi-user equilibrium assignment with elastic demand. They provide useful tools for both theoretical analysis and algorithm specifications.†

All the above quoted models have been developed assuming that a generalised transportation cost may be associated to each link of the network. These costs are generally made up by a lin-

*Fixed-point models can be also used to analyse DUE.

†If link cost-flow functions have a symmetric Jacobian matrix, DUE and SUE can also be formulated through mathematical programs which allow developing efficient solution algorithms.

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ear combination of different attributes, such as travel time, monetary costs, etc. and allow defining path costs which affect user path choice behaviour.

The evaluation of transporation demand management measures affecting travelling monetary costs requires explicitly taking into account the value of time (VoT) distribution among users*. Recently, several papers have addressed the *deterministic equilibrium assignment with VoT distributed among users* extending variational inequality models for DUE† (Dial, 1996; Leurent, 1993, 1995, 1996; Marcotte and Zhu, 1996; Marcotte *et al.*, 1996). The analysis of these models requires some mathematical complications, but they allow specifying solution algorithms, effective for real-size systems.

To the present authors knowledge no paper has addressed the similarly relevant topic of *stochastic equilibrium assignment with VoT distributed among user*. In this paper, fixed-point models and algorithms developed for SUE have been extended to analyse and solve equilibrium assignment with VoT distributed among user. For simplicity's sake, reference will be made to (multiuser) single-mode equilibrium assignment with rigid demand and fully pre-trip path choice behaviour. Extension to multi-user multi-mode equilibrium assignment with elastic demand and/or pre-trip/en-route path choice behaviour is quite straightforward as indicated in Cantarella (1997).

Section 2 briefly presents commonly adopted notations and introduces the stochastic network loading function, whilst Section 3 describes its extension to deal with VoT distributed among users. Section 4 presents fixed-point models for user equilibrium with VoT distributed among users, as well as solution algorithms. Section 5 reports the results of an application to a medium-size network. Finally, some conclusions and indications for further research work are presented in Section 6.

2. NOTATIONS AND ASSUMPTIONS

In this section, basic variables and relations are introduced to specify path choice behaviour models and network loading maps. Extensions to deal with VoT distributed among users is discussed in the next section.

A transportation system is generally analysed by dividing the whole study area into zones. Origins and destinations of journeys are concentrated into centroids, one for each zone. Users travelling between the same origin-destination pair with common behavioural parameters (e.g., due to common trip purpose, income class, etc.) are grouped into a user class. As noted in the introduction user choice behaviour will be restricted to path choice only, assuming that other choice dimensions like mode and destination are not affected by network performances, or more properly by congestion.

2.1. Path choice behaviour models

Path choice behaviour can be modelled through random utility theory (Ben Akiva and Lerman, 1987) assuming that each user of class i

- examines all paths in a (non empty) set K_i,
- associates to each path k within set K_i a value of *perceived utility* $u_{i,k}$, modelled by a random variable,
- chooses the maximum utility path.

From these hypotheses the probability $p_{i,k}$ that a user of class i choose path k is given by the probability that path k is the highest utility path:

*The traffic assignment with VoT distributed among users, addressed in this paper, is also referred to in the literature as bicriterion assignment or multiclass assignment. However, these definitions may be misleading since monetary and nonmonetary costs are actually combined into one criterion through the VoT, and users can be distinguished into classes with other parameters than VoT.

†An early paper (not available to authors) on this topic was an unpublished manuscript by Dafermos (1981). See also Dial (1979).

$$p_{i,k} = \Pr[u_{i,k} \geq u_{i,j} \forall j \in K_i] \;\forall i,k$$

The perceived utility $u_{i,k}$ is generally expressed as the sum of two terms:

$$u_{i,k} = v_{i,k} + \xi_{i,k} \;\forall i,k$$

or

$$\mathbf{u}_i = \mathbf{v}_i + \xi_i \;\forall i$$

where

$\mathbf{u}_i$ is the path perceived utility vector for user class i, with entries $u_{i,k}$;

$\mathbf{v}_i$ is the path systematic utility vector for user class i, with entries given by *systematic utility* values $v_{i,k} = E[u_{i,k}]$, and $E[v_{i,k}] = v_{i,k}$, $Var[v_{i,k}] = 0$;

ξ_i is the path random residual vector for user class i, with entries given by *random residuals* (r.r.) $\xi_{i,k}$, and $E[\xi_{i,k}] = 0$, $Var[\xi_{i,k}] = Var[u_{i,k}] = \sigma^2_{i,k}$,

$\phi_{\xi i}(\xi_i)$ be the joint probability density function of random residuals,

$\mathbf{\Sigma}_i$ be the variance-covariance matrix of random residuals, with entries given by the covariance between random residuals of path k and j, $\sigma_{i,kj}$, and $\sigma_{i,kk} = \sigma^2_{i,k}$.

The above relations yields that the path choice probabilities depend on path systematic utilities, as expressed by the *path choice map*, with functional expression depending on the random residual joint distribution:

$$p_{i,k} = \Pr[\xi_{i,k} - \xi_{i,j} \geq v_{i,j} - v_{i,k} \;\forall j \in K_i] = p_{i,k}(\mathbf{v}_i) \;\forall i,k$$

or

$$\mathbf{p}_i = \boldsymbol{p}_i(\mathbf{v}_i) \;\forall i$$

where

$\mathbf{p}_i$ is the path choice fraction vector for user class i ($\mathbf{p}_i \geq \mathbf{0}$ and $\mathbf{1}^T \mathbf{p}_i = 1$), with entries given by $p_{i,k}$, the fraction of users of class i choosing path k.

Probabilistic models are obtained for non singular variance-covariance matrix of random residuals, $|\mathbf{\Sigma}| \neq 0$, that is when r.r. variance is non zero, $Var[\xi_{i,k}] \neq 0$, and each pair of random residuals is also assumed non perfectly linearly correlated each other, $(\sigma_{i,kj})^2 < \sigma^2_{i,k}\ \sigma^2_{i,j}$ for $k \neq j$. *(continued on next page)*

Probabilistic models usually adopted lead to continuous and differentiable *probabilistic path choice functions*, $\boldsymbol{p}_i(\mathbf{v}_i)$. If the random residual distribution does not depend on the path systematic utilities $\mathbf{v}_i$, the path choice function is non-decreasing monotone with respect to systematic utilities $\mathbf{v}_i$ (Cantarella, 1997):

$$\phi_{\xi i}(\xi_i | \boldsymbol{v}_i) = \phi_{\xi i}(\xi_i) \Longrightarrow [\boldsymbol{p}_i(biv_i{}') - \boldsymbol{p}_i(bi\mathbf{v}_i{}'')]^T (biv_i{}'' - biv_i{}'') \geq 0 \;\forall biv_i{}', biv_i{}'' \;\forall i$$

A path choice function with all the above features will be referred to in this paper as a *regular probabilistic path choice function.*

The *Logit* model is obtained assuming that the random residuals $\xi_{i,k}$ are identically and independently distributed (i.i.d.) Gumbel random variables with parameter $\pi/(\sqrt{6}\sigma_i)$, and $\sigma^2_{i,k} = \sigma^2_i$, leading to a simple closed form. In spite of this advantage the Logit model does not allow to properly simulate overlapping between paths for the hypothesis of independence, which implies null correlation, $\sigma_{i,kj} = 0$ for $k \neq j$.

The recently proposed *C-Logit* model (Cascetta *et al.*, 1996) is obtained from a Logit model by including in the systematic utility of a path a commonality factor, increasing with the number of links shared with other paths. This model still has the closed form of a Logit model, but allows choice between overlapping paths to be more realistically simulated.

The *Probit* model is obtained assuming that random residuals are jointly distributed as a multivariate Normal with zero mean and variance-covariance matrix $\mathbf{\Sigma}_i$. This model allows a realistic simulation of the choice among overlapping paths, but Probit path choice probabilities cannot be expressed in a closed form, and their computation requires MonteCarlo techniques.

Assuming no random residual, $\mathrm{Var}[\xi_{i,k}] = 0$, the *deterministic* model is obtained. The corresponding *deterministic path choice map* is a superior semi-continuous multi-valued map non-increasing monotone with respect to path costs (Daganzo, 1983; Cantarella, 1997). With respect to the deterministic model, probabilistic models assure more realism and wider modelling flexibility, even if they may present some computational disadvantages.

2.2. Network variables and relations

Transportation supply is usually modelled through a network, with cost attributes associated to links and/or to paths. In particular, a transportation cost c_l is associated to each link l, and a corresponding cost g_k to each path k. Let

$\mathbf{A}_i$ be the link-path incidence matrix for user class i, with entries $a_{i,lk} = 1$ if link l belongs to path k, $a_{i,lk} = 0$ otherwise;

$\mathbf{c}$ be the link transportation cost vector, with entries c_l;

$\mathbf{g}_i$ be the path cost vector for user class i, with entries $g_{i,k}$.

The *link path cost consistency* yields:

$$big_i = biA_i^T bic \ \forall i \tag{1}$$

The user path choice behaviour can be modelled as described in Section 2.1. Assumed link and path costs homogeneous with utility, the systematic utility of a path is generally expressed as a decreasing linear function of path cost attributes, called the *utility function*:

$$biv_i = -big_i \ \forall i \tag{2}$$

The path choice probability vector depends on the systematic utility vector, through the *path choice map* (Section 2.1):

$$bip_i = \boldsymbol{p}_i(biv_i) \ \forall i \tag{3}$$

The *demand path flow* and *link path flow consistency* yields:

$$bih_i = d_i bip_i \ \forall i \tag{4}$$

$$bif = \Sigma_i biA_i bih_i \tag{5}$$

where

$d_i \geq 0$ is the demand flow for users belonging to class i, measured in a common units through duly defined equivalence coefficients;

$\mathbf{h}_i$ is the path flow vector for user class i, with entries $h_{i,k}$;

$\mathbf{f}$ is the link flow vector, with entries f_l.

A link flow vector **f** is called *feasible* if there exists a probability vector $\mathbf{p}_i$ for each user class i, such that **f** is given by equation (4) and equation (5). Let n be the number of links; $F = \{\mathbf{f} = \Sigma_i \, d_i \, \mathbf{A}_i \, \mathbf{p}_i : \mathbf{p}_i \geq \mathbf{0}, \, \mathbf{1}^T \mathbf{p}_i = 1 \ \forall i\} \subseteq R^n$ be the *feasible link flow set*; it is *non empty* (for connected O–D pairs), *compact* and *convex* ($\mathbf{f} \in F \Rightarrow \mathbf{f} \geq \mathbf{0}$).

2.3. Network loading map

A relation between link flows and costs, called the *network-loading map* (NL), can be defined by combining equations (1)–(5):

$$\boldsymbol{f}(bic) = \sum_i d_i biA_i \boldsymbol{p}_i(-biA_i^T bic) = \Sigma_i \boldsymbol{f}_i(bic) \in F \tag{6}$$

where $\boldsymbol{f}_i(\mathbf{c}) = d_i \, \mathbf{A}_i \, \boldsymbol{p}_i(-biA_i^T bic)$ is the NL map for user class i.

For probabilistic path choice functions the NL map is a function called *stochastic network loading (SNL) function*. For regular probabilistic path choice functions the SNL function, as well as each term $\boldsymbol{f}_i(\mathbf{c})$, is continuous and differentiable. If the random residual distribution does not depend on the path costs $\mathbf{g}_i$, the path choice functions are non-increasing monotone with respect to path costs $\mathbf{g}_i$. The resulting NL function, as well as each term $\boldsymbol{f}_i(\mathbf{c})$, is a non-increasing monotone with respect to link costs, with a symmetric negative semi-definite Jacobian matrix (details in Cantarella, 1997):

$$(\boldsymbol{f}(bic') - \boldsymbol{f}(bic''))^T (bic' - bic'') \leq 0 \ \forall bic', bic''$$

In the following NL map with all the above features* will be refereed to as a *regular stochastic network loading (RSNL) function.*

The NL map can be easily computed if paths available for each user class are explicitly enumerated. There are also efficient algorithms for computing it without explicit path enumeration for Logit, C-Logit, Probit (based on MonteCarlo simulation) or deterministic choice models, under mild assumptions on relevant paths† (reviews and references in Sheffi, 1985; Cascetta, 1990; Patriksson, 1994).

3. EXTENSION OF NL MAP TO VoT DISTRIBUTED AMONG USERS

This section shows how the modelling approach described above can be extended to take into account VoT and monetary cost explicitly. The same approach can be followed for combining any two cost attributes (such as length and time). With this aim, a *monetary cost* m_l is associated to each link l in addition to the *non monetary* cost c_l, thus let

$\mathbf{m}^i$ be the link monetary cost vector for user class i, with entries m_l^i;

$\alpha_i \geq 0$ be the Value-of-Time for users of class i;

$\boldsymbol{\alpha}$ be the VoT vector with entries α_i.

The link path cost consistency equation (1) becomes:

$$big_i = biA_i^T bic + (1/\alpha_i) biA_i^T bim^i \ \forall i \tag{7}$$

whilst relations equations (2)–(5) remain unchanged.

*The multi-valued deterministic NL map has similar features (Daganzo, 1983; Cantarella, 1997).

†Some problems may arise for particular specifications of Logit model if the set of available paths depends on congested costs, introduced in Section 4.

If the VoT for each class is a deterministic variable, results in the previous section can easily be applied. With reference to probabilistic path choice models, the SNL function defined by equation (6) becomes:

$$\boldsymbol{f}_{\mathrm{VoT}}(bic,bim;bi\alpha) = \sum_{\mathrm{i}} \mathrm{d}_{\mathrm{i}} biA_{\mathrm{i}} \boldsymbol{p}_i(-biA_{\mathrm{i}}^{\mathrm{T}} bic - (1/\alpha_{\mathrm{i}}) biA_{\mathrm{i}}^{\mathrm{T}} bim^{\mathrm{i}}) = \sum_{\mathrm{i}} \boldsymbol{f}_{\mathrm{VoT,i}}(bic,bim^{\mathrm{i}};\alpha_{\mathrm{i}}) \in F \quad (8)$$

where $\boldsymbol{f}_{\mathrm{VoT,i}}(\mathbf{c},\mathbf{m}^{\mathrm{i}};\ \alpha_{\mathrm{i}}) = \mathrm{d}_{\mathrm{i}}\ \mathbf{A}_{\mathrm{i}}\ \boldsymbol{p}_{\mathrm{i}}(-biA_{\mathrm{i}}^{\mathrm{T}}\mathbf{c} - (1/\alpha_{\mathrm{i}})biA_{\mathrm{i}}^{\mathrm{T}}\mathbf{m}^{\mathrm{i}})$ is the NL map with explicit monetary costs and VoT for user class i.

Each term $\boldsymbol{f}_{\mathrm{VoT,i}}(\mathbf{c},\mathbf{m}^{\mathrm{i}};\ \alpha_{\mathrm{i}})$ of the stochastic NL function equation (8) is regular, as defined in Section 2.3, under the same assumptions assuring this feature for the NL function equation (6). In this case, it has all the features already discussed in Section 2.3. Thus, it is continuous, differentiable, and a non-increasing monotone with respect to $\mathbf{c}$ for given $\mathbf{m}^{\mathrm{i}}$. Moreover, it can be computed with the same algorithms used for NL map equation (6).

3.1. VoT modelled through a continuous random variable

If the VoT dispersion among users is modelled through a continuous random variable, users can be grouped into classes with the same O–D pair and behavioural parameters as well as VoT probability density function, let

α_{i} be the Value-of-Time for users of class i, considered a random variable, assumed defined over $[0,\infty)$, with probability density function $\phi_{\mathrm{i}}(\alpha_{\mathrm{i}};\ \boldsymbol{\theta}_{\mathrm{i}})$ and parameters $\boldsymbol{\theta}_{\mathrm{i}}$.

With reference to probabilistic path choice models, the SNL function is obtained by integrating each term of relation equation (8) with respect to α_{i}:

$$\boldsymbol{f}_{\mathrm{CON}}(bic,bim;\boldsymbol{\theta}) = \sum_{\mathrm{i}} \int_0^{\infty} \boldsymbol{f}_{\mathrm{VoT,i}}(bic,bim^{\mathrm{i}};\alpha_{\mathrm{i}})\phi_{\mathrm{i}}(\alpha_{\mathrm{i}};\boldsymbol{\theta}_{\mathrm{i}})\mathrm{d}\alpha_{\mathrm{i}} \in F \quad (9)$$

where $\boldsymbol{\theta}$ is a vector formed by vectors $\boldsymbol{\theta}_{\mathrm{i}}$ which defines the VoT distribution.

Clearly, if each term $\boldsymbol{f}_{\mathrm{VoT,i}}(\mathbf{c},\ \mathbf{m}^{\mathrm{i}};\ \alpha_{\mathrm{i}})$ of the SNL function equation (8) is continuous and differentiable, the SNL function equation (9) is continuous and differentiable for continuous and differentiable probability density functions $\phi_{\mathrm{i}}(\alpha_{\mathrm{i}};\ \boldsymbol{\theta}_{\mathrm{i}})$. Moreover, it is a convex combination of non-increasing monotone functions, $\boldsymbol{f}_{\mathrm{VoT,i}}(\mathbf{c},\ \mathbf{m}^{\mathrm{i}};\ \alpha_{\mathrm{i}})$. Then, it is a non-increasing monotone with respect to $\mathbf{c}$ for given $\mathbf{m}$. Hence, the SNL function equation (9) is a regular SNL function under the same assumptions assuring this feature for the SNL function equation (6).

Theoretically, the SNL function equation (9) could be computed from SNL function equation (8), through direct integration. However, usually the VoT is assumed distributed according a log-normal or gamma variable over $[0,\infty)$, which may require MonteCarlo techniques.

3.2. VoT modelled through a discrete random variable

If the VoT is modelled through a discrete random variable distributed among users, each user class can be subdivided in subclasses containing users with the same O–D pair and behavioural parameters as well as VoT. All notations and relations introduced in Section 2 still apply, using a couple of indices i,j instead of index i for variables relative to paths. The link path incidence matrix $\mathbf{A}_{\mathrm{i}}$ and the path choice map $\boldsymbol{p}_{\mathrm{i}}(.)$ are common to all subclasses j in user class i. Let

α_{ij} be the Value-of-Time for user subclass j of class i;

η_{ij} be the probability that a user of class i belongs to subclass j, that is the probability that a user of class i has a VoT equal to α_{ij}, with $\eta_{ij} \geq 0$ and $\Sigma_j\ \eta_{ij} = 1$; thus $d_i\ \eta_{ij}$ is the demand flow for user subclass j of class i.

The SNL function is obtained by summing up terms of relation equation (8) with respect to subclasses and classes:

$$\boldsymbol{f}_{\mathrm{DIS}}(bic,bim;\boldsymbol{\alpha},\boldsymbol{\eta}) = \sum_{\mathrm{ij}} \mathrm{d_i}\eta_{\mathrm{ij}} biA_{\mathrm{i}}\boldsymbol{p}_{\mathrm{i}}(-biA_{\mathrm{i}}^{\mathrm{T}}bic - (1/\alpha_{\mathrm{ij}})biA_{\mathrm{i}}^{\mathrm{T}}bim^{\mathrm{i}}) =$$

$$= \sum_{\mathrm{ij}} \eta_{\mathrm{ij}}\boldsymbol{f}_{\mathrm{VoT,i}}(bic,bim^{\mathrm{i}};\alpha_{\mathrm{ij}}) \in F \tag{10}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\eta}$ are vectors with entries α_{ij} and η_{ij} respectively.

Equation (10) is similar to the SNL function equation (8) by considering each subclass j of each class i as a class with demand flow given by $d_i\ \eta_{ij}$. Therefore, conditions which assure that the SNL function equation (8) is regular also assure that the SNL function equation (10) is a regular. In this case, it has all the features already discussed for the SNL function equation (8). The SNL function equation (10) can easily be computed through methods available for the SNL function equation (8).

4. EQUILIBRIUM ASSIGNMENT WITH VoT DISTRIBUTED AMONG USERS

Traffic congestion is generally simulated assuming that the time needed to traverse a link depends on the flow of users on the link, and possibly on other links. Thus non monetary cost depends on link flows through *link cost-flow functions*:

$$bic = \boldsymbol{c}(bif) \tag{11}$$

If monetary costs simulate tolls and fees due to Traffic Demand Management measures, such park and/or cordon pricing and/or limitations, they can be considered flow-independent. On the other hand, other monetary costs, such as fuel costs, generally depend on link flows:

$$bim = \boldsymbol{m}(bif) \tag{12}$$

It is, however, common practice, at least in urban transportation planning, to assume that these latter costs do not affect user path choice behaviour.

The *equilibrium* approach, usually adopted to deal with traffic assignment to congested networks, searches for mutually consistent link flows and costs, by combining cost functions with the SNL functions (or deterministic NL maps) described above. This approach can be generally expressed by a system of non-linear equations:

$$bic^* = \boldsymbol{c}(bif^*) \tag{13}$$

$$bim^* = \boldsymbol{m}(bif^*) \tag{14}$$

$$bif^* = \boldsymbol{f}(bic^*,bim^*) \tag{15}$$

The system of non-linear equations equations (13)–(15) can be analysed through equivalent fixed-point models, which generalises models usually adopted for stochastic user equilibrium (SUE)* (Daganzo, 1983; Cantarella, 1997), with respect to link flows:

$$bif^* = \boldsymbol{f}(\boldsymbol{c}(bif^*),\boldsymbol{m}(bif^*)) \text{ with } bif^* \in F \tag{16}$$

Other models can be defined with respect to link costs (Cantarella, 1997).

Existence of solutions of the fixed-point problem equation (16) can be analysed through Brouwer's theorem, since the link flow feasible set, F, is non empty (for connected O–D pairs),

*The same approach can be also adopted to deal with deterministic user equilibrium (DUE).

compact and convex, and the link flows resulting from the SNL function are always feasible, $\mathbf{f}(\boldsymbol{c}(\mathbf{f})) \in F$. Thus, if the SNL function is continuous, it is sufficient that cost functions are continuous. This condition implies that capacity constraints cannot be explicitly considered. Conditions to state uniqueness of solutions of fixed-point problem equation (16) are still an open issue.

If the monetary costs do not depend on link flows, problem equation (16) becomes:

$$bif^* = \boldsymbol{f}(\boldsymbol{c}(bif^*), bim) \text{ with } bif^* \in F \tag{17}$$

In this case, if the SNL function is a non-increasing monotone with respect to link costs for given monetary costs (as it occurs for regular SNL functions) uniqueness is assured if link cost-flow functions are a strictly increasing monotone (Daganzo, 1983; Cantarella, 1997). This condition can be relaxed to non decreasing cost functions for usually adopted models like Logit and Probit, which assign a strictly positive choice probability to each path, whichever the path costs (Cantarella, 1997).

In spite of its "direct" formulation and flexibility for theoretical analysis, only few algorithms have been proposed to solve fixed-point models as such. They are based on the Method of Successive Average (MSA) in which link flows, or costs, are recursively averaged until convergence.

The *flow averaging* (FA) algorithm was originally proposed by Daganzo (1983) (and further analysed by Cantarella, 1997) to solve fixed-point models for SUE without explicit monetary costs and VoT. It can be applied also to deal with VoT distributed among user, as specified below:

GIVEN t: = 0; $\mathbf{f}^0 \in F$
REPEAT
 t: = t + 1; $\mathbf{c}^t := \boldsymbol{c}(\mathbf{f}^{t-1})$; $\mathbf{m}^t := \boldsymbol{m}(\mathbf{f}^{t-1})$
 $\mathbf{f}^t_{SNL} := \boldsymbol{f}(\mathbf{c}^t, \mathbf{m}^t)$
 $\mathbf{f}^t := (\mathbf{f}^t_{SNL} + (t-1)\, \mathbf{f}^{t-1})/t$
UNTIL $\mathbf{f}^{t-1} \cong \mathbf{f}^t_{SNL}$

Convergence analysis for SUE without explicit monetary costs and VoT (Daganzo, 1983; Cantarella, 1997) can be only extended when monetary costs do not depend on link flows, with reference to a fixed-point model equation (17). In this case the above algorithm becomes:

GIVEN t: = 0; $\mathbf{f}^0 \in F$; $\mathbf{m}$
REPEAT
 t: = t + 1; $\mathbf{c}^t := \boldsymbol{c}(\mathbf{f}^{t-1})$;
 $\mathbf{f}^t_{SNL} := \boldsymbol{f}(\mathbf{c}^t, \mathbf{m})$
 $\mathbf{f}^t := (\mathbf{f}^t_{SNL} + (t-1)\, \mathbf{f}^{t-1})/t$
UNTIL $\mathbf{f}^{t-1} \cong \mathbf{f}^t_{SNL}$

If existence and uniqueness conditions hold, and the Jacobian matrix of link cost-flow functions is symmetric, the above algorithm can be proved converging to the solution of model equation (17), say $\lim_{t\to\infty} \mathbf{f}^t = \mathbf{f}^*$.

SUE with non separable cost functions (that is with asymmetric Jacobian) can be solved through the algorithm proposed by Daganzo (1983) based on the inverse cost functions. Cantarella (1997) has proposed a simpler *cost averaging* (CA) algorithm, which can be applied also to deal with VoT distributed among users, as specified below:

GIVEN t: = 0; $\mathbf{f}^0 \in F$; $\mathbf{c}^0 := \boldsymbol{c}(\mathbf{f}^0)$; $\mathbf{m}^0 := \boldsymbol{m}(\mathbf{f}^0)$
REPEAT
 t: = t + 1; $\mathbf{f}^t_{SNL} := \boldsymbol{f}(\mathbf{c}^{t-1}, \mathbf{m}^{t-1})$
 $\mathbf{c}^t_{SNL} := \boldsymbol{c}(\mathbf{f}^t_{SNL})$; $\mathbf{m}^t_{SNL} := \boldsymbol{m}(\mathbf{f}^t_{SNL})$
 $\mathbf{c}^t := (\mathbf{c}^t_{SNL} + (t-1)\, \mathbf{c}^{t-1})/t$
 $\mathbf{m}^t := (\mathbf{m}^t_{SNL} + (t-1)\, \mathbf{m}^{t-1})/t$
UNTIL $\boldsymbol{f}(\mathbf{c}^t, \mathbf{m}^t) \cong \mathbf{f}^t_{SNL}$

Convergence analysis (Cantarella, 1997) can be only extended when monetary costs do not depend on link flows, and the above algorithm becomes:

GIVEN t: = 0; $\mathbf{f}^0 \in F$; $\mathbf{c}^0 := c(\mathbf{f}^0)$; $\mathbf{m}$

REPEAT

$t := t + 1$; $\mathbf{f}^t_{SNL} := f(\mathbf{c}^{t-1}, \mathbf{m})$

$\mathbf{c}^t_{SNL} := c(\mathbf{f}^t_{SNL})$

$\mathbf{c}^t := (\mathbf{c}^t_{SNL} + (t-1)\,\mathbf{c}^{t-1})/t$

UNTIL $f(\mathbf{c}^t, \mathbf{m}) \cong \mathbf{f}^t_{SNL}$

If existence and uniqueness conditions hold, and the Jacobian matrix of SNL functions is symmetric (as it occurs for regular SNL functions), the above algorithm converges to link costs from which the equilibrium link flows can be obtained, say $\lim_{t \to \infty} \mathbf{c}^t = \mathbf{c}^\infty$ with $\mathbf{f}^* = f(\mathbf{c}^\infty)$. It is worth noting that the algorithm provides a sequence of feasible link flows $\mathbf{f}^t = \mathbf{f}^t_{SNL}$. Generally, for separable link cost functions, the cost-averaging algorithm is less efficient than the flow-averaging one. Enhancements of a cost-averaging algorithm speed of convergence is still an open issue.

All the above algorithms can be defined as *simple*, since they can be implemented using only cost-flow $\mathbf{c}(.)$ and SNL functions $\mathbf{f}(.)$, and *feasible*, since they provide a sequence of feasible link flow vectors $\mathbf{f}^t$.

If the choice probabilities cannot be evaluated exactly, as it occurs for Probit model, it is assumed that an unbiased estimation of the SNL function can be obtained through MonteCarlo techniques. In this case, only *almost sure* convergence can be assured.

5. NUMERICAL EXAMPLES

This section presents the results of an application to a medium-size urban network relative to the town of Crotone (population 59 000) in the South of Italy. The whole area has been subdivided into 22 zones, each represented by a centroid, 326 O–D pairs over 462 have a non-zero demand flow in the morning peak hour. The road network (length 65 km) has been modelled through a graph with 145 nodes and 390 links (Fig. 1).

Congestion for non-monetary costs has been simulated through separable travel time functions. A flow-independent monetary cost of 0.75 EURO has been associated to links 290 and 367, highly saturated without any monetary costs.

A Probit model has been used to realistically simulate the choice among overlapping paths. The entries of the variance-covariance matrix between pairs of random residuals have been assumed proportional, through a parameter θ, to reference path costs obtained from uncongested travel times. This assumption guarantees that random residual distribution does not depend on the path systematic utilities and the SNL function is regular. Three variance-covariance matrices have been considered, $\theta = 1, 5, 10$ sec. For an average uncongested journey time equal to 480 sec, values of the average standard deviation of path random residuals are $\bar{\sigma}_\xi = 22$, 49, 69 sec, to be compared with an average congested journey time equal to 720 sec (without monetary costs).

Two demand scenarios have been analysed: *low demand* (total demand flow 6251 vehicles/hour) with average saturation ratio 40% (without monetary costs), and *high demand* (total demand flow 9401 vehicles/hour) with average saturation ratio 50% (without monetary costs).

A Lognormal distribution has been assumed for VoT. Two scenarios for VoT have been analysed, *low VoT*, $\mu_{VoT} = 5$EURO, and *high VoT*, $\mu_{VoT} = 10$EURO, and six values of standard deviation to mean ratio have been considered, $\sigma_{VoT}/\mu_{VoT} = 0.00$, 0.10, 0.25, 0.50, 0.75, 1.00.

The flow averaging algorithm has been applied to solve the problem. The values of the SNL function have been estimated (without explicit path enumeration) by averaging over 50 link flows, assigned to the shortest paths obtained by MonteCarlo sampling of link costs and VoT (not taking into account sampled negative link costs). A starting solution has been obtained through the flow averaging, estimating SNL with only one MonteCarlo sampling, and iterated until $\mathbf{f}^{t-1} \cong \mathbf{f}^t$.

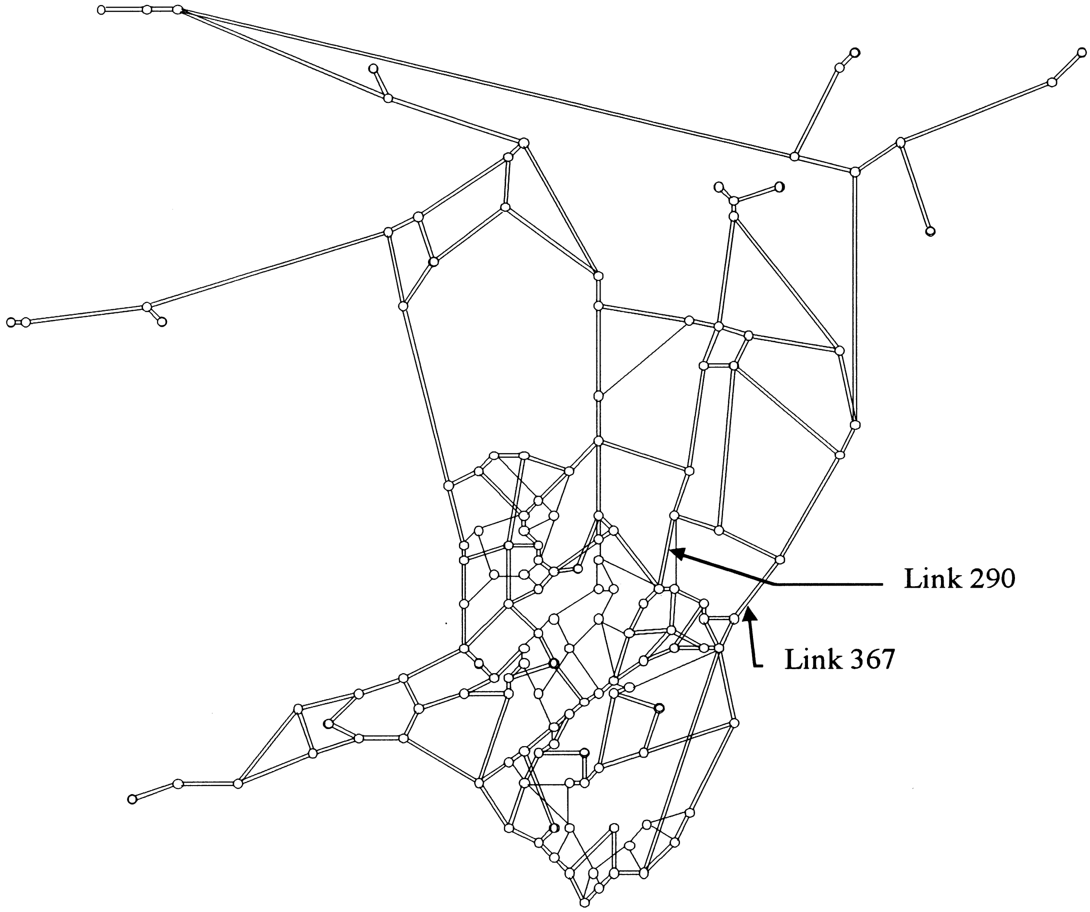

Fig. 1. The example network.

The degree of saturation (Gs), that is the flow to capacity ratio, for link 290, affected by the monetary cost, is plotted against different values of the VoT standard deviation to mean ratio (Cv), $=\sigma_{VoT}/\mu_{VoT}$, and of the Probit parameter (Teta), θ, in figures from Fig. 2–5 for the different demand and mean VoT scenarios. For comparison the degree of saturation for zero monetary cost is also shown.

The reported results show that the effects of VoT distribution is greater for low demand and low VoT, and that these effects generally depends on random residual dispersion. The impact of different mean VoT values in discouraging the use of the links 290 depends on both the non linearity of the path choice map and the asymmetry of the VoT distribution.* The reported results support the application of SUE with VoT distributed among the users when monetary costs are involved.

6. CONCLUSIONS

In this paper, models and algorithms for SUE with Value-of-Time distributed among users have been presented. Using the concept of regular stochastic network loading function existing fixed-point models and algorithms have been easily extended to deal with VoT distributed among

*The authors wish to thank an anonymous referee who raised this point.

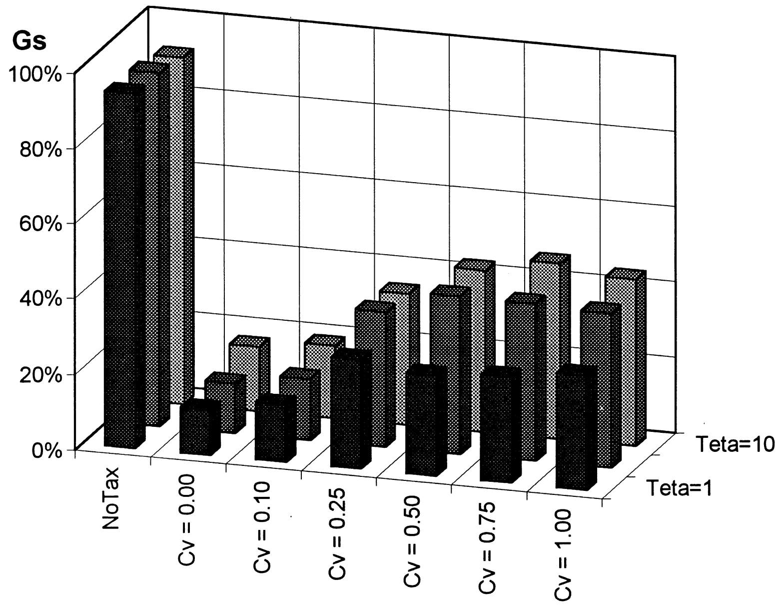

Fig. 2. Saturation degree for link 290, low VoT, low demand.

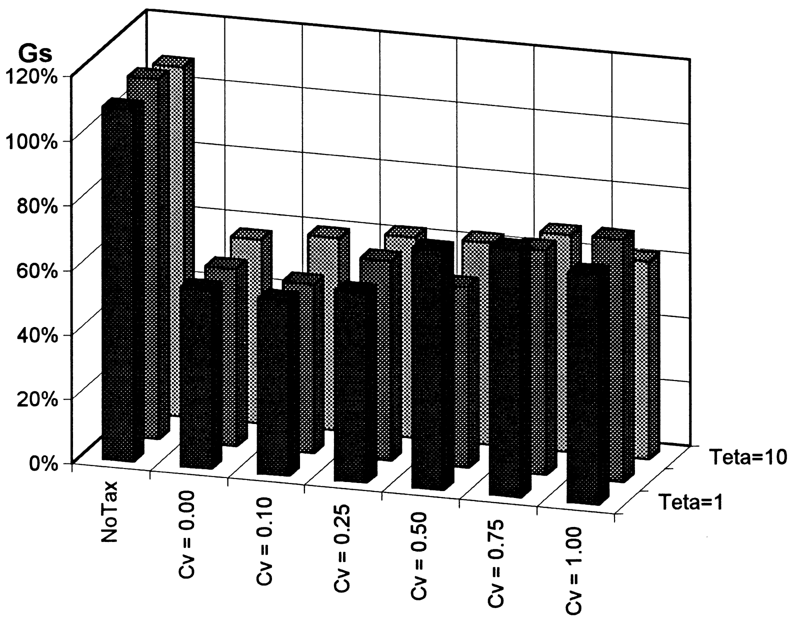

Fig. 3. Saturation degree for link 290, low VoT, high demand.

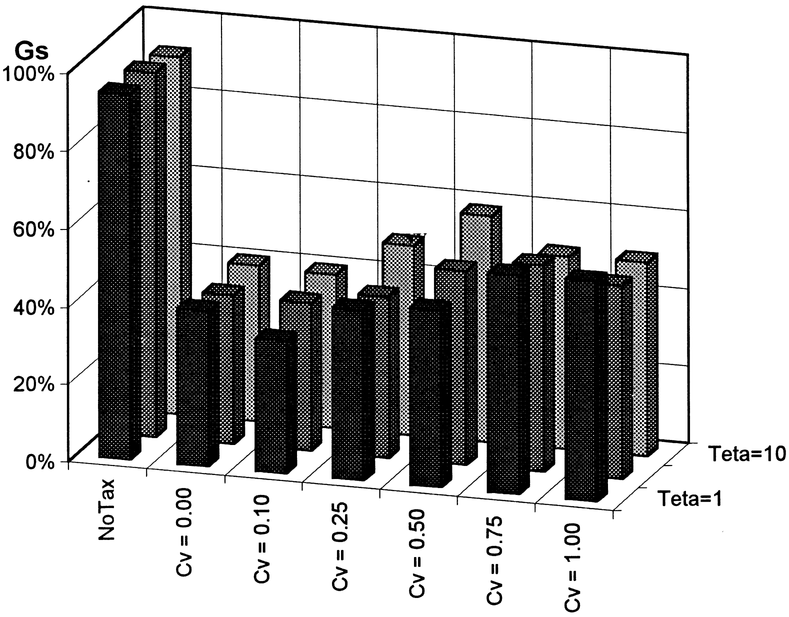

Fig. 4. Saturation degree for link 290, high VoT, low demand.

users, under mild assumption. The same approach can be followed for combining any two cost attributes (such as length and time).

Reported results of an application to a medium-size network indicate that the proposed approach is suitable for practical applications. They also show that the VoT distribution can greatly affect the simulation of transportation systems.

Conditions for uniqueness of solutions and algorithm convergence when monetary costs depend on link flows still remain an open issue worthy of further research work.

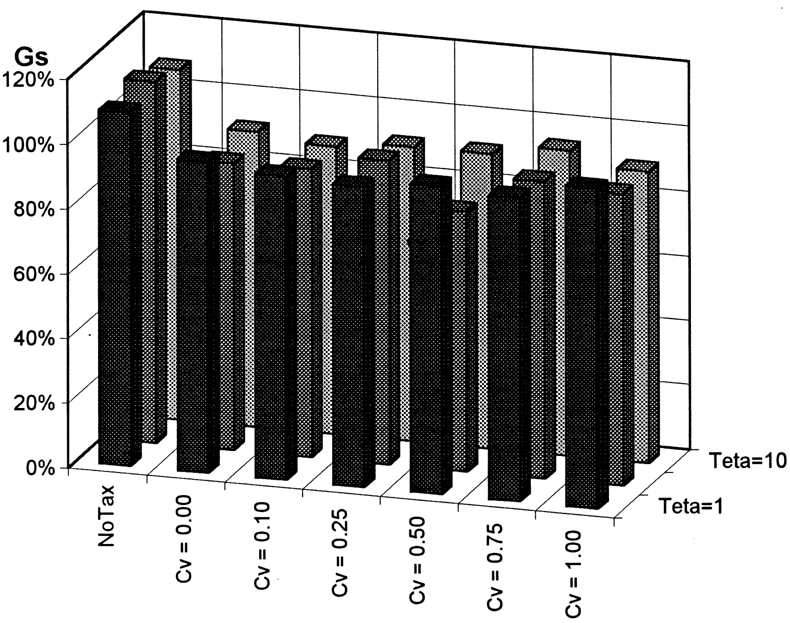

Fig. 5. Saturation degree for link 290, high VoT, high demand.

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