Curvature and Vector Fields

Lecture Notes of Loring W. Tu [1]

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Last updated: October 4, 2025

Contents

※ 1. Riemannian manifold

A Riemannian manifold is a manifold endowed with a Riemannian metric. The Riemannian metric is a smoothly varying inner product on tangent space at each point. This section first recall the definition of inner product, then we prove the existence of a Riemannian metric on any smooth manifolds.

1.1 Inner Products on a Vector Space

The **Euclidean inner product** on $\mathbb{R}^n$ is defined by

$$\langle u, v\rangle = \sum_1^n u^i v^i, \tag{1}$$

and the length of a vector is

$$\|u\| = \sqrt{\langle u, u\rangle}, \tag{2}$$

the **angle θ** between two vectors is

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|}, \tag{3}$$

the **arc length** of a curve $c(t) \in \mathbb{R}^n, a \leq t \leq b$ is

$$s = \int_a^b \|c'(t)\|\,dt \tag{4}$$

Definition 1.1. An inner product in a real vector space V is a postive-definite, bilinear and symmetric map: $\langle\cdot,\cdot\rangle : V \times V \longrightarrow \mathbb{R}$ so that for $u, v, w \in V$ and $a, b \in \mathbb{R}$, satisfies

(i) **Postive-definiteness** $\langle v, v\rangle = 0$ iff. $v = 0$

(ii) **Symmetry** $\langle u, v\rangle = \langle v, u\rangle$

(iii) **Bilinear** $\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$

Proposition 1.2. If W is a subspace of V, then the restriction

$$\langle,\rangle_W := \langle,\rangle|_{W\times W} : W \times W \longrightarrow \mathbb{R}, \tag{5}$$

of an inner product $\langle,\rangle$ on V is also an innver prodcut.

Proof. The subspace construction preserves the properites in Definition 1.1. □

Proposition 1.3. The **nonnegative linear combinition** of inner products $\langle,\rangle_i$ on V: $\langle,\rangle := \sum_{i=1}^r a_i\langle,\rangle_i, a_i \geq 0$ is again an inner product on V.

Proof. The **nonnegativity** of a_i preserves condition (i) in Definition 1.1, the linearity makes condition $(ii), (iii)$ hold. □

1.2 Representations of Inner Products by Symmetric Matrices

Let $e_1, \ldots, e_n$ be the basis of vector space V, each vector $x \in V$ can be represented as a column vector

$$x = \sum_{i=1}^{n} x^i e_i \leftrightarrow \mathbf{x} = \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix}. \tag{6}$$

Let $\boldsymbol{A}$ be an $n \times n$ matrix whose entries $a_{ij} = \langle e_i, e_j \rangle$, the matrix form of an inner product on V is

$$\langle x, y \rangle = \sum_{ij} x^i y^j \langle e_i, e_j \rangle = \mathbf{x}^\top \boldsymbol{A}\mathbf{y}. \tag{7}$$

We find that, once a basis of V is chosen, the inner product on V determines a postive-definite symmetric matrix. Conversely, an $n \times n$ postive-definite symmetric matrix with a basis of V determines an inner product on V

It follows that there is an one-to-one correspondence

$$\left\{ \begin{array}{c} \text{inner product on a } n\text{-dimensional} \\ \text{vector space} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{An } n \times n \text{ postive-definite} \\ \text{symmetric matrix} \end{array} \right\}. \tag{8}$$

Let a basis of dual space $V^\vee := \mathrm{Hom}(V, \mathbb{R})$ be $\alpha^1, \ldots, \alpha^n$ w.r.t. the basis $e_1, \ldots, e_n$ of V, an inner product $\langle , \rangle$ of $x, y \in V$ is

$$\begin{aligned} \langle x, y \rangle &= \sum_{i,j} a_{ij} x^i y^j = \sum_{i,j} a_{ij} \alpha^i(x) \alpha^j(y) \\ &= \sum_{i,j} a_{ij} \alpha^i \otimes \alpha^j (x, y) \end{aligned}$$

In terms of tensor product, an inner product on V may be written as

$$\langle , \rangle = \sum_{ij} a_{ij} \alpha^i \otimes \alpha^j \tag{9}$$

1.3 Riemannian Metrics

Definition 1.4. A **Riemannian metric** is an inner product **assignment** to each $p \in M$ of an inner product $\langle , \rangle_p$ on the tangent space T_pM. This assignment should be C^∞ in the following sense: if $p \mapsto \langle X_p, Y_p \rangle_p$ is a C^∞ function for any C^∞ vector fields X, Y. A **Riemannian manifold** is a pair $(M, \langle , \rangle)$, which consists of a C^∞ manifold M together with a Riemannian metric on M.

Example 1.5. Since the tangent space at a point in Euclidean space $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$, the Euclidean inner product induces a Riemannian metric on $\mathbb{R}^n$ called the **Euclidean metric**.

Example 1.6. A surface M in $\mathbb{R}^3$ is a 2-dimensional regular submanifold of $\mathbb{R}^3$, the tangent space at p is a subspace of $T_p\mathbb{R}^3$, so the surface M inherits a Riemannian metric from the Euclidean metric by restriction $\langle , \rangle_M$.

Definition 1.7. A C^∞ map $F : (N, \langle , \rangle') \to (M, \langle , \rangle)$ of Riemannian manifolds is said to be **metric-preserving** if

$$\langle u, v \rangle'_p = \langle F_* u, F_* v \rangle_{F(p)} \tag{10}$$

for all point $p \in N$ and tangent vectors $u, v \in T_p N$. An **isometry** is a metric-preserving diffeomorphism.

For a Riemannian manifold $(M, \langle , \rangle)$, if there is a diffeomorphism that maps some manifolds N to M, the induced metric $\langle , \rangle'$ on N can be defined by (10).

Example 1.8 (Metric-preserving but not an isometry)**.** Let N and M be the unit circle in $\mathbb{C}$. Define $F : N \to M$ a **2-sheeted covering space map** (for any $w \in M$, $F^{-1}(w)$ contains 2 points in N), by $F(z) = z^2$. Given M any Riemannian metric $\langle , \rangle$, and define the induced metric on N is (10), The map F is metric-preserving but not an isometry because F is not a diffeomorphism (not inject).

Example 1.9 (Topological equivalant Riemannian manifolds may not isometric)**.**

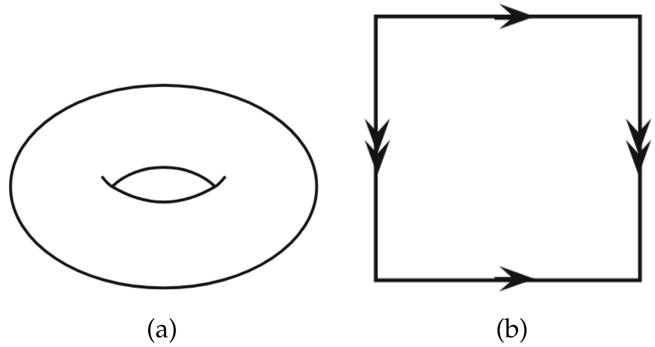

Figure 1.1: Two Riemannian metrics on torus [1, p. 6].

1.4 Existence of a Riemannian Metric

The local diffeomorphism ϕ defines a Riemannian metric on a coordinate chart $(U, x^1, \ldots, x^n)$ of M that $x^i = r^i \circ \phi$, as

$$\langle X, Y \rangle = \sum_{ij} a^i b^i \langle \partial_i, \partial_j \rangle = \sum_{ij} a^i b^i, \tag{11}$$

since $\phi_* \partial_j = \frac{\partial}{\partial r^j}$, the induced metric is the same as the Euclidean ones.

To obtain a Riemannian metric on M, we need to piece together the Riemannian metrics on all charts of an atlas of M. Here, we use the **partition of the unity** as the standard tools.

Theorem 1.10 (Existence of a Riemannian metric)**.** There exists a Riemannian metric on every manifold.

Proof. Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ an atlas of M. We have a partition of unity $\{\rho_\alpha\}$ that subcoordinates to open sets $\{U_\alpha\}$. Let $\langle , \rangle_\alpha$ the Riemannian metric on U_α as in (11), from Proposition 1.3, we define a metric on $T_p M$ at p is

$$\langle , \rangle = \sum_{\alpha \in A} \rho_\alpha \langle , \rangle_\alpha. \tag{12}$$

Since U_p intersects finite number of U_α, (12) is a finite sum. Since ρ_α and $\langle , \rangle_\alpha$ are both smooth, for any C^∞ vector fields X, Y, $\sum_{\alpha \in A} \rho_\alpha \langle X, Y \rangle_\alpha$ is a finite sum of smooth functions at arbitary p (By Definition 1.4). So $\sum_{\alpha \in A} \rho_\alpha \langle , \rangle_\alpha$ is a Riemannian metric on M. □

Problems

1.1 Suppose $(M, \langle , \rangle)$ is a Riemannian manifold. Show that two C^∞ vector fields $X, Y \in \mathfrak{X}(M)$ are equal if and only if $\langle X, Z \rangle = \langle Y, Z \rangle$ for all C^∞ vector fields $Z \in \mathfrak{X}(M)$.

※ 2. Curves

A curve in manifold means either a parameterized curve, i.e., a smooth map $c : [a, b] \to M$, or a set of points in M that is the image of this map. This section focus on the plane curve, first introduces the regular curves whose velocity never zero so that can be reparameterized by arc length. We can define the signed curvature by the second derivative of this parameterization.

2.1 Regular Curves

Definition 2.1 (Regular curve)**.** A parameterized curve $c : [a, b] \longrightarrow M$ is **regular** if its velocity $c'(t) \neq 0$ for all t in $[a, b]$, which means an immersion from $[a, b]$ to M.

Example 2.2. The curve $c(t) = (t^3, t^2), t \in [-1, 1]$ in $\mathbb{R}^2$ is not regular since $c'(t)$ is zero at $t = 0$. Although c is smooth, but the image of c is not smooth as shown in Figure 2.1.

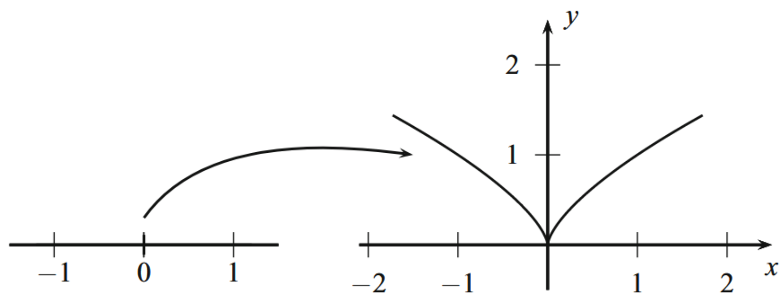

Figure 2.1: A nonregular curve [1, p. 9].

2.2 Arc Length Parameterization

The most important **reparameterization** ($\beta(u) := c(t(u))$ if $t = t(u)$ is a diffeomorphism from one to another closed interval) is the **arc length reparameterization**. We define the **speed** of a curve $c : [a, b] \longrightarrow M$ is $\|c'(t)\|$, and the arc length is

$$\ell = \int_a^b \|c'(t)\| \, dt.$$

Then, the **arc length function** $s : [a, b] \longrightarrow [0, \ell]$ of c is

$$s(t) = \int_a^t \|c'(u)\| \, du.$$

Proposition 2.3. The arc length function s of a regular curve has a C^∞ inverse.

Proof. The regular property gaurantees $s'(t) = \|c'(t)\| > 0$, which means $s(t)$ is monotonically increasing, so $t(s)$ is a C^∞ function. □

Thus, we can write the **arc length reparameterization** of a regular curve by $\gamma(s) = c(t(s))$.

Proposition 2.4. A curve $\gamma(s)$ is reparameterized by arc length if and only if it has **unit speed** and its parameter starts at 0.

Proof. ($\Rightarrow$): as $\gamma(s) = c(t(s))$, the speed is

$$\left\|\frac{d\gamma}{ds}\right\| = \left\|\frac{dc}{dt}\right\| \cdot \left|\frac{dt}{ds}\right| = \frac{ds}{dt}\left|\frac{dt}{ds}\right| = 1. \tag{13}$$

($\Leftarrow$): If $c(t) : [a, b] \to M$ has unit speed that $\|c'(t)\| = 1$, the arc length fucntion $s(t) = \int_a^t dt = t - a$. Since $a = 0$, we have $s = t$. Thus, a unit speed curve starts at $t = 0$ is reparameterized by arc length. $\square$

Here, we do not emphasize that the curve need to be regular since **"reparameterized by arc length" implies regularity**. The parameter is s or t depends on the way of reparameterization.

Example 2.5. The regular curve $c : [0, 2\pi] \to \mathbb{R}^2$,

$$c(t) = (a\cos t, a\sin t), \quad a > 0,$$

is a circle of radius a centered at the origin. The arc length function is

$$s(t) = \int_0^t \|c'(t)\| = at.$$

So the reparameterization is

$$\gamma(s) = (a\cos\frac{s}{a}, a\sin\frac{s}{a}).$$

2.3 Signed Curvature of a Plane Curve

The signed curvature measures how and what direction a curve bends. In this section, we quantify the signed curvature of a plane curve $\gamma : [0, \ell] \to \mathbb{R}^2$ parameterized by arc length s in $\mathbb{R}^2$.

Then we define the velocity vector $T(s) = \gamma'(s)$, which has unit length and tangent at $\gamma(s)$. We can measure the curvature by how fast the velocity changes:

$$T'(s) = \frac{dT}{ds}(s) = \gamma''(s),$$

Here, we have already a tangent vector $T(s)$ at $\gamma(s)$, there is a unit normal vector $\mathbf{n}$ that perpendicular to $T(s)$ at $\gamma(s)$. We usually choose $(T(s), \mathbf{n})$ is counterclockwise, i.e., rotate from $T(s)$ to $\mathbf{n}$ counterclockwisely.

Since T has unit speed that $\langle T, T\rangle = 1$, we have $\langle T', T\rangle = 0$, which means T' is perpendicular to T so that we can write $T' = \kappa \mathbf{n}$. The scalar κ is the **signed curvature**, or simply **curvature**. We can write

$$\kappa = \langle T', \mathbf{n}\rangle = \langle \gamma'', \mathbf{n}\rangle. \tag{14}$$

The sign of κ means whether the curve is bending towards or away from $\mathbf{n}$.

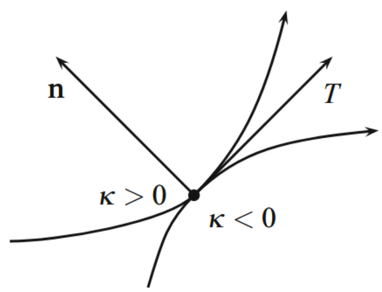

Figure 2.2: The sign of the curvature [1, p. 12].

Example 2.6. Recall Example 2.5, we can easily compute

$$T' = [-\frac{1}{a}\cos\frac{s}{a}, -\frac{1}{a}\sin\frac{s}{a}]^\top,$$

the normal vector $\mathbf{n} = [-\cos\frac{s}{a}, -\sin\frac{s}{a}]^\top$. So the curvature $\kappa = \frac{1}{a}$.

2.4 Orientation and Curvature

For a arc length parameterized curve that the two endpoints are fixed, will have two parameterization that inverse the orientation. Let the arc length be ℓ, then, we have another parameterization by

$$\tilde{\gamma}(s) = \gamma(\ell - s).$$

Then, the velocity and its derivative give

$$\tilde{T}(s) = -T(\ell - s), \quad \tilde{T}'(s) = T'(\ell - s).$$

The unit norm vector is given by rotate $\tilde{T}(s)$ by $\frac{\pi}{2}$

$$\tilde{\mathbf{n}}(s) = \mathrm{rot}\left(\frac{\pi}{2}\right)\tilde{T}(s) = -\mathrm{rot}\left(\frac{\pi}{2}\right)T(\ell - s) = -\mathbf{n}(\ell - s).$$

The sign of curvature will be reversed by

$$\tilde{\kappa}(s) = \langle \tilde{\mathbf{n}}(s), \tilde{T}'(s)\rangle = \langle -\mathbf{n}(\ell - s), T'(\ell - s)\rangle = -\kappa(\ell - s).$$

Example 2.7. From Example 2.5, the clockwise circle has the signed curvature $-1/a$.

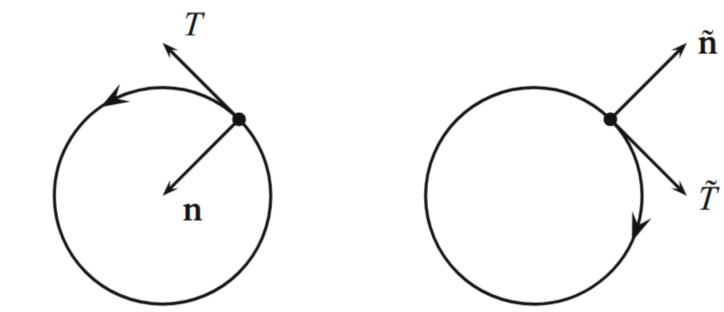

Figure 2.3: Reverse of a curve and its curvature [1, p. 13].

Problems

2.1 Let $T(s)$ be the unit tangent vector field on a plane curve $\gamma(s)$ parametrized by arc length. Write

$$T(s) = \begin{bmatrix} \cos\theta(s) \\ \sin\theta(s) \end{bmatrix},$$

where $\theta(s)$ is the angle of $T(s)$ with respect to the positive horizontal axis. Show that the **signed curvature** κ is the derivative $d\theta/ds$.

2.2 In physics literature, it is customary to denote the derivative with respect to time by a dot, e.g., $\dot{x} = dx/dt$, and the derivative with respect to distance by a prime, e.g., $x' = dx/ds$. We will sometimes follow this convention. Let $c(t) = (x(t), y(t))$ be a regular plane curve. Using Problem 2.1 and the chain rule $d\theta/ds = (d\theta/dt)/(ds/dt)$, show that the signed curvature of the curve $c(t)$ is

$$\kappa = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}},$$

where $\dot{x} = dx/dt$ and $\ddot{x} = d^2x/dt^2$.

2.3 The *graph* of a C^∞ function $y = f(x)$ is the set

$$\{(x, f(x)) \mid x \in \mathbb{R}\}$$

in the plane. Show that the signed curvature of this graph at $(x, f(x))$ is

$$\kappa = \frac{f''(x)}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}}.$$

2.4 The ellipse with equation $x^2/a^2 + y^2/b^2 = 1$ in the (x, y)-plane (Figure 2.4) can be parametrized by

$$x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi.$$

Find the curvature of the ellipse at an arbitrary point (x, y).

2.5 Let I be a closed interval in $\mathbb{R}$. If $\gamma\colon I \to \mathbb{R}^3$ is a regular space curve parametrized by arc length, its *curvature* κ at $\gamma(s)$ is defined to be $\|\gamma''(s)\|$.[1] Consider the helix $c(t) = (a\cos t, a\sin t, bt)$ in space.

(a) Reparametrize c by arc length: $\gamma(s) = c(t(s))$.

(b) Compute the curvature of the helix at $\gamma(s)$.

※ 3. Surfaces in Space

There are two ways to study the curvature of a surface: (1)following Euler (1760), consider the curvatures of all normal sections. (2) the derivative of unit normal vector fields. In this section, we along the first way to define the curvature at a point on a surface. Then we introdce Gauss's Theorema Egregium and Gauss-Bonnet theorem that are through out the whole differential geometry. Here, a surface in $\mathbb{R}^3$ is treated as a **regular 2-dimensional submanifold**. We emphsize the regularity to gaurantee the surface is smooth and has no singular points.

3.1 Principal, Mean, and Gaussian Curvatures

A **normal vector** to M at p is a vector $N_p \in T_p\mathbb{R}^3$ that orthogonal to the tangent plane T_pM. A normal vector field N is a function that assigns each p a normal vector N_p

$$N_p = \sum_{i=1}^{3} a^i(p) \frac{\partial}{\partial x^i}\bigg|_p .$$

The C^∞ of all $a^i(p)$ implies N is a C^∞ vector field. Let N be a C^∞ unit normal

[1]The curvature of a space curve is always nonnegative, while the signed curvature of a plane curve could be negative. To distinguish the two for a space curve that happens to be a plane curve, one can use κ_2 for the signed curvature. We will use the same notation κ for both, as it is clear from the context what is meant.

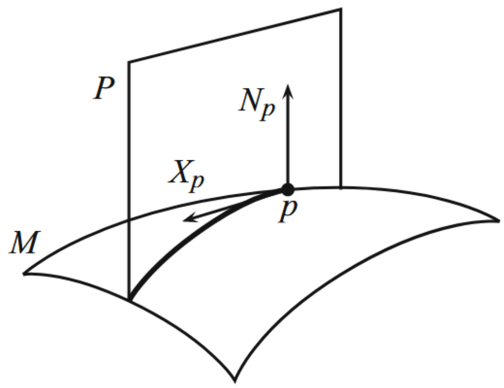

Figure 3.1: Normal Section at p [1, p. 17].

vector field, the plane P that contains N_p slice the surface M along a plane curve $P \cap M$ through p, as shown in Figure 3.1.

By the **transversality theorem** [2], $P \cap M$ is transversal, then being smooth. Such a plane curve is called **section** of the surface through p. Suppose the normal sections has C^∞ parameterizations, we can describe the how the surface curves at p by the collection of all the curvatures at p of all sections w.r.t. normal vector N_p.

Let $\gamma(s)$ be the normal section parameterized by arc length, which satisfies $\gamma(0) = p$ and $\gamma'(0) = X_p$, where X_p is a unit tangent vector, that determines the normal vector N_p and the orientation of $\gamma(s)$. Then, we can define the curvature of normal section γ at p w.r.t. N_p by

$$\kappa(X_p) = \langle \gamma''(0), N_p \rangle. \tag{15}$$

Since the set of all unit vectors X_p is a unit circle, we have a function $\kappa : S^1 \to \mathbb{R}$. Since reverse the sign of X_p does not change the sign of second derivative of γ, $\kappa(-X_p) = \kappa(X_p)$.

The maximum and minimum value κ_1, κ_2 of κ are the **principal curvatures** of the surface at p. Their average $(\kappa_1 + \kappa_2)/2$ is the **mean curvature** H. Their product $\kappa_1 \cdot \kappa_2$ is the **Gaussian Curvature** K. The unit tangent vector X_p that corresponds to principal curvature is called the **principal direction**.

Remark 3.1. If we reverse the sign of unit normal vector to $-N_p$, the sign of curvature will change, but the Gaussian curvature stays, which means that the Gaussian curvature is **independent** of the choice of the unit normal vector field.

Example 3.2 (Sphere with a radius a)**.** Recall the circle S^1 in Example 2.5, every normal section is circle, then we have the principal curvature $1/a$, the mean curvature $H = 1/2a$, the Gaussian curvature $K = 1/a^2$.

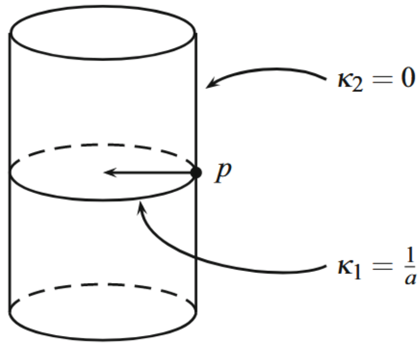

Figure 3.2: Principal curvatures of a cylinder [1, p. 19].

Example 3.3. For a cylinder of radius a with a unit inward normal vector field (Figure 3.2). The principal curvature $\kappa_1 = 1/a, \kappa_2 = 0$, the mean curvature $H = 1/2a$, the Gaussian curvature $K = 0$.

3.2 Gauss's Theorema Egregium

κ_1, κ_2 is not invariant under the isometry, but the Gaussian curvature K dose. The Gauss's Theorema Egregium tells us that the Gaussian curvature is independent of the embedding of the manifold, but the inherit geometry. This will be detailed in the following chapters.

3.3 The Gauss–Bonnet Theorem

The integral of Gaussian curvature gives a **topological invariant**, independent of Riemannian metric. For example, integral the Gaussian curvature $1/a^2$ of a sphere S^2 with radius a gives 4π.

For a compact oriented surface M in $\mathbb{R}^3$, the integral

$$\int_M K dS = 2\pi\chi(M),$$

where $\chi(M)$ denotes the Euler characteristic.

Problems

3.1 Suppose M is a smooth surface in $\mathbb{R}^3$, p a point in M, and N a smooth unit normal vector field on a neighborhood of p in M. Let P be the plane spanned by a unit tangent vector $X_p \in T_pM$ and the unit normal vector N_p. Denote by $C := P \cap M$ the normal section of the surface M at p cut out by the plane P.

(a) The plane P is the zero set of some linear function $f(x,y,z)$. Let $\tilde{f}: M \to \mathbb{R}$ be the restriction of f to M. Then the normal section C is precisely the level set $\tilde{f}^{-1}(0)$. Show that p is a regular point of $\tilde{f}$, i.e., that the differential $\tilde{f}_{*,p}: T_pM \to T_0\mathbb{R}$ is surjective. (Hint: Which map is $f_{*,p}: \mathbb{R}^3 = T_p\mathbb{R}^3 \to T_{f(p)}\mathbb{R} = \mathbb{R}$? What is its kernel?)

(b) Show that a normal section of M at p is a regular submanifold of dimension one in a neighborhood of p. (Hint: Apply [2, Proposition 11.4] and the regular level set theorem [2, Theorem 9.9].)

3.2 Show that if a curve C in a smooth surface is a regular submanifold of dimension one in a neighborhood of a point $p \in C$, then C has a C^∞ parametrization near p. (Hint: Relative to an adapted chart (U, x^1, x^2) centered at p, the curve C is the x^1-axis. A C^∞ parametrization of the x^1-axis in the (x^1, x^2)-plane is $(x^1, x^2) = (t, 0)$.)

References

[1] Loring W. Tu. *Differential Geometry*, volume 275 of *Graduate Texts in Mathematics*. Springer International Publishing.

[2] L.W. Tu. *An Introduction to Manifolds*. Universitext. Springer New York.