

Curvature and Vector Fields

Lecture Notes of Loring W. Tu [1]

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❖ Lecture 1 (1/9)

1.1 Inner Products on a Vector Space

The **Euclidean inner product** on \mathbb{R}^n is defined by

$$\langle u, v \rangle = \sum_1^n u^i v^i, \quad (1)$$

and the length of a vector is

$$\|u\| = \sqrt{\langle u, u \rangle}, \quad (2)$$

the **angle** θ between two vectors is

$$\cos \theta = \frac{\langle u, v \rangle}{\|u\| \|v\|}, \quad (3)$$

the **arc length** of a curve $c(t) \in \mathbb{R}^n, a \leq t \leq b$ is

$$s = \int_a^b \|c'(t)\| dt \quad (4)$$

Definition 1.1. An inner product in a real vector space V is a positive-definite, bilinear and symmetric map: $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ so that for $u, v, w \in V$ and $a, b \in \mathbb{R}$, satisfies

- (i) **Postive-definiteness** $\langle v, v \rangle = 0$ iff. $v = 0$
- (ii) **Symmetry** $\langle u, v \rangle = \langle v, u \rangle$
- (iii) **Bilinear** $\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$

Proposition 1.2. If W is a subspace of V , then the restriction

$$\langle \cdot, \cdot \rangle_W := \langle \cdot, \cdot \rangle|_{W \times W} : W \times W \rightarrow \mathbb{R}, \quad (5)$$

of an inner product $\langle \cdot, \cdot \rangle$ on V is also an inner product.

Proof. The subspace construction preserves the properties in Definition 1.1. \square

Proposition 1.3. The **nonnegative linear combination** of inner products $\langle \cdot, \cdot \rangle_i$ on V : $\langle \cdot, \cdot \rangle := \sum_{i=1}^r a_i \langle \cdot, \cdot \rangle_i, a_i \geq 0$ is again an inner product on V .

Proof. The **nonnegativity** of a_i preserves condition (i) in Definition 1.1, the linearity makes condition (ii), (iii) hold. \square

1.2 Representations of Inner Products by Symmetric Matrices

1.3 Riemannian Metrics

1.4 Existence of a Riemannian Metric

1.5 Problems

References

- [1] L.W. Tu. *An Introduction to Manifolds*. Universitext. Springer New York.