Curvature and Vector Fields

Lecture Notes of Loring W. Tu [1]

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※ 1. Riemannian manifold

1.1 Inner Products on a Vector Space

The **Euclidean inner product** on $\mathbb{R}^n$ is defined by

$$\langle u, v\rangle = \sum_1^n u^i v^i, \tag{1}$$

and the length of a vector is

$$\|u\| = \sqrt{\langle u, u\rangle}, \tag{2}$$

the **angle θ** between two vectors is

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\|v\|}, \tag{3}$$

the **arc length** of a curve $c(t) \in \mathbb{R}^n, a \leq t \leq b$ is

$$s = \int_a^b \|c'(t)\| dt \tag{4}$$

Definition 1.1. An inner product in a real vector space V is a postive-definite, bilinear and symmetric map: $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ so that for $u, v, w \in V$ and $a, b \in \mathbb{R}$, satisfies

(i) **Postive-definiteness** $\langle v, v\rangle = 0$ iff. $v = 0$

(ii) **Symmetry** $\langle u, v\rangle = \langle v, u\rangle$

(iii) **Bilinear** $\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$

Proposition 1.2. If W is a subspace of V, then the restriction

$$\langle , \rangle_W := \langle , \rangle|_{W\times W} : W \times W \to \mathbb{R}, \tag{5}$$

of an inner product $\langle , \rangle$ on V is also an innver prodcut.

Proof. The subspace construction preserves the properites in Definition 1.1. □

Proposition 1.3. The **nonnegative linear combinition** of inner products $\langle , \rangle_i$ on V: $\langle , \rangle := \sum_{i=1}^r a_i \langle , \rangle_i, a_i \geq 0$ is again an inner product on V.

Proof. The **nonnegativity** of a_i preserves condition (i) in Definition 1.1, the linearity makes condition $(ii), (iii)$ hold. □

1.2 Representations of Inner Products by Symmetric Matrices

Let $e_1, \ldots, e_n$ be the basis of vector space V, each vector $x \in V$ can be represented as a column vector

$$x = \sum_{i=1}^n x^i e_i \leftrightarrow \mathbf{x} = \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix}. \tag{6}$$

Let $\boldsymbol{A}$ be an $n \times n$ matrix whose entries $a_{ij} = \langle e_i, e_j\rangle$, the matrix form of an inner product on V is

$$\langle x, y\rangle = \sum_{ij} x^i y^j \langle e_i, e_j\rangle = \mathbf{x}^\top \boldsymbol{A}\mathbf{y}. \tag{7}$$

We find that, once a basis of V is chosen, the inner product on V determines a postive-definite symmetric matrix. Conversely, an $n \times n$ postive-definite symmetric matrix with a basis of V determines an inner product on V

It follows that there is an one-to-one correspondence

$$\left\{ \begin{array}{c} \text{inner product on a } n\text{-dimensional} \\ \text{vector space} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{An } n \times n \text{ postive-definite} \\ \text{symmetric matrix} \end{array} \right\}. \tag{8}$$

Let a basis of dual space $V^\vee := \mathrm{Hom}(V, \mathbb{R})$ be $\alpha^1, \ldots, \alpha^n$ w.r.t. the basis $e_1, \ldots, e_n$ of V, an inner product $\langle , \rangle$ of $x, y \in V$ is

$$\begin{aligned} \langle x, y \rangle &= \sum_{i,j} a_{ij} x^i y^j = \sum_{i,j} a_{ij} \alpha^i(x) \alpha^j(y) \\ &= \sum_{i,j} a_{ij} \alpha^i \otimes \alpha^j(x, y) \end{aligned}$$

In terms of tensor product, an inner product on V may be written as

$$\langle , \rangle = \sum_{ij} a_{ij} \alpha^i \otimes \alpha^j \tag{9}$$

1.3 Riemannian Metrics

Definition 1.4. A **Riemannian metric** is an inner product **assignment** to each $p \in M$ of an inner product $\langle , \rangle_p$ on the tangent space T_pM. This assignment should be C^∞ in the following sense: if $p \mapsto \langle X_p, Y_p \rangle_p$ is a C^∞ function for any C^∞ vector fields X, Y. A **Riemannian manifold** is a pair $(M, \langle , \rangle)$, which consists of a C^∞ manifold M together with a Riemannian metric on M.

Example 1.5. Since the tangent space at a point in Euclidean space $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$, the Euclidean inner product induces a Riemannian metric on $\mathbb{R}^n$ called the **Euclidean metric**.

Example 1.6. A surface M in $\mathbb{R}^3$ is a 2-dimensional regular submanifold of $\mathbb{R}^3$, the tangent space at p is a subspace of $T_p\mathbb{R}^3$, so the surface M inherits a Riemannian metric from the Euclidean metric by restriction $\langle , \rangle_M$.

Definition 1.7. A C^∞ map $F : (N, \langle , \rangle') \to (M, \langle , \rangle)$ of Riemannian manifolds is said to be **metric-preserving** if

$$\langle u, v \rangle'_p = \langle F_* u, F_* v \rangle_{F(p)} \tag{10}$$

for all point $p \in N$ and tangent vectors $u, v \in T_pN$. An **isometry** is a metric-preserving diffeomorphism.

For a Riemannian manifold $(M, \langle , \rangle)$, if there is a diffeomorphism that maps some manifolds N to M, the induced metric $\langle , \rangle'$ on N can be defined by (10).

Example 1.8 (Metric-preserving but not an isometry)**.** Let N and M be the unit circle in $\mathbb{C}$. Define $F : N \to M$ a **2-sheeted covering space map** (for any $w \in M$, $F^{-1}(w)$ contains 2 points in N), by $F(z) = z^2$. Given M any Riemannian metric $\langle , \rangle$, and define the induced metric on N is (10), The map F is metric-preserving but not an isometry because F is not a diffeomorphism (not inject).

Example 1.9 (Topological equivalant Riemannian manifolds may not isometric)**.**

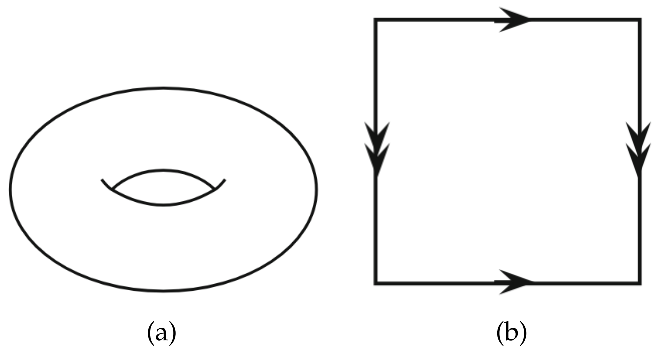

Figure 1: Two Riemannian metrics on torus.

1.4 Existence of a Riemannian Metric

The local diffeomorphism ϕ defines a Riemannian metric on a coordinate chart $(U, x^1, \ldots, x^n)$ of M that $x^i = r^i \circ \phi$, as

$$\langle X, Y \rangle = \sum_{ij} a^i b^i \langle \partial_i, \partial_j \rangle = \sum_{ij} a^i b^i, \tag{11}$$

since $\phi_* \partial_j = \frac{\partial}{\partial r^j}$, the induced metric is the same as the Euclidean ones.

To obtain a Riemannian metric on M, we need to piece together the Riemannian metrics on all charts of an atlas of M. Here, we use the **partition of the unity** as the standard tools.

Theorem 1.10 (Existence of a Riemannian metric)**.** There exists a Riemannian metric on every manifold.

Proof. Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ an atlas of M. We have a partition of unity $\{\rho_\alpha\}$ that subcoordinates to open sets $\{U_\alpha\}$. Let $\langle,\rangle_\alpha$ the Riemannian metric on U_α as in (11), from Proposition 1.3, we define a metric on T_pM at p is

$$\langle,\rangle = \sum_{\alpha\in A} \rho_\alpha \langle,\rangle_\alpha. \tag{12}$$

Since U_p intersects finite number of U_α, (12) is a finite sum. Since ρ_α and $\langle,\rangle_\alpha$ are both smooth, for any C^∞ vector fields X, Y, $\sum_{\alpha\in A} \rho_\alpha\langle X, Y\rangle_\alpha$ is a finite sum of smooth functions at arbitary p (By Definition 1.4). So $\sum_{\alpha\in A} \rho_\alpha\langle,\rangle_\alpha$ is a Riemannian metric on M. □

Problems

1.1 Suppose $(M, \langle,\rangle)$ is a Riemannian manifold. Show that two C^∞ vector fields $X, Y \in \mathfrak{X}(M)$ are equal if and only if $\langle X, Z\rangle = \langle Y, Z\rangle$ for all C^∞ vector fields $Z \in \mathfrak{X}(M)$.

※ 2. Curves

2.1 Regular Curves

Definition 2.1 (Regular curve)**.** A parameterized curve $c : [a, b] \longrightarrow M$ is **regular** if its velocity $c'(t) \neq 0$ for all t in $[a, b]$, which means an immersion from $[a, b]$ to M.

Example 2.2. The curve $c(t) = (t^3, t^2), t \in [-1, 1]$ in $\mathbb{R}^2$ is not regular since $c'(t)$ is zero at $t = 0$. Although c is smooth, but the image of c is not smooth as shown in Figure 2.

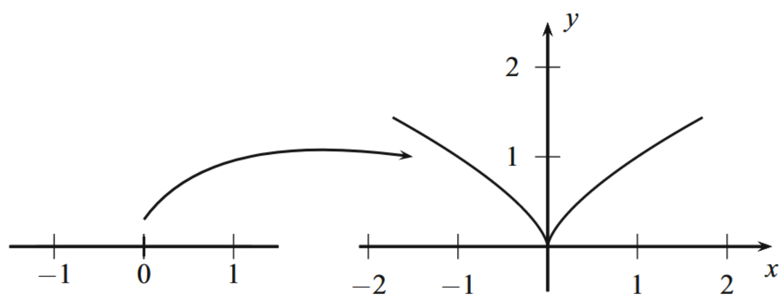

Figure 2: A nonregular curve.

2.2 Arc Length Parameterization

The most important **reparameterization** ($\beta(u) := c(t(u))$ if $t = t(u)$ is a diffeomorphism from one to another closed interval) is the **arc length reparameterization**. We define the **speed** of a curve $c : [a, b] \to M$ is $\|c'(t)\|$, and the arc length is

$$\ell = \int_a^b \|c'(t)\| dt.$$

Then, the **arc length function** $s : [a, b] \to [0, \ell]$ of c is

$$s(t) = \int_a^t \|c'(t)\| dt.$$

Proposition 2.3. The arc length function s of a regular curve has a C^∞ inverse.

Proof. The regular property gaurantees $s'(t) = \|c'(t)\| > 0$, which means $s(t)$ is monotonically increasing, so $t(s)$ is a C^∞ function. □

Thus, we can write the **arc length reparameterization** of a regular curve by $\gamma(s) = c(t(s))$.

Proposition 2.4. A curve $\gamma(s)$ is reparameterized by arc length if and only if it has **unit speed** and its parameter starts at 0.

Proof. ($\Rightarrow$): as $\gamma(s) = c(t(s))$, the speed is

$$\|\frac{d\gamma}{ds}\| = \|\frac{dc}{dt}\| |\frac{dt}{ds}| = \frac{ds}{dt} |\frac{dt}{ds}| = 1. \tag{13}$$

($\Leftarrow$): If $c(t) : [a, b] \to M$ has unit speed that $\|c'(t)\| = 1$, the arc length fucntion $s(t) = \int_a^t dt = t - a$. Since $a = 0$, we have $s = t$. Thus, a unit speed curve starts at $t = 0$ is reparameterized by arc length. □

Here, we do not emphasize that the curve need to be regular since **"reparameterized by arc length" implies regularity**. The parameter is s or t depends on the way of reparameterization.

Example 2.5. The regular curve $c : [0, 2\pi] \to \mathbb{R}^2$,

$$c(t) = (a \cos t, a \sin t), \quad a > 0,$$

is a circle of radius a centered at the origin. The arc length function is

$$s(t) = \int_0^t \|c'(t)\| = at.$$

So the reparameterization is

$$\gamma(s) = (a \cos \frac{s}{a}, a \sin \frac{s}{a}).$$

2.3 Signed Curvature of a Plane Curve

2.4 Orientation and Curvature

Problems

References

[1] L.W. Tu. *An Introduction to Manifolds*. Universitext. Springer New York.