Curvature and Vector Fields

Lecture Notes of Loring W. Tu [1]

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Contents

※ 1. Riemannian manifold

A Riemannian manifold is a manifold endowed with a Riemannian metric. The Riemannian metric is a smoothly varying inner product on tangent space at each point. This section first recall the definition of inner product, then we prove the existence of a Riemannian metric on any smooth manifolds.

1.1 Inner Products on a Vector Space

The **Euclidean inner product** on $\mathbb{R}^n$ is defined by

$$\langle u, v\rangle = \sum_1^n u^i v^i, \tag{1}$$

and the length of a vector is

$$\|u\| = \sqrt{\langle u, u\rangle}, \tag{2}$$

the **angle** θ between two vectors is

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|}, \tag{3}$$

the **arc length** of a curve $c(t) \in \mathbb{R}^n, a \leq t \leq b$ is

$$s = \int_a^b \|c'(t)\|\, dt \tag{4}$$

Definition 1.1. An inner product in a real vector space V is a postive-definite, bilinear and symmetric map: $\langle\cdot,\cdot\rangle : V \times V \longrightarrow \mathbb{R}$ so that for $u, v, w \in V$ and $a, b \in \mathbb{R}$, satisfies

(i) **Postive-definiteness** $\langle v, v\rangle = 0$ iff. $v = 0$

(ii) **Symmetry** $\langle u, v\rangle = \langle v, u\rangle$

(iii) **Bilinear** $\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$

Proposition 1.2. If W is a subspace of V, then the restriction

$$\langle,\rangle_W := \langle,\rangle|_{W\times W} : W \times W \longrightarrow \mathbb{R}, \tag{5}$$

of an inner product $\langle,\rangle$ on V is also an innver prodcut.

Proof. The subspace construction preserves the properites in Definition 1.1. □

Proposition 1.3. The **nonnegative linear combinition** of inner products $\langle,\rangle_i$ on V: $\langle,\rangle := \sum_{i=1}^r a_i\langle,\rangle_i, a_i \geq 0$ is again an inner product on V.

Proof. The **nonnegativity** of a_i preserves condition (i) in Definition 1.1, the linearity makes condition $(ii), (iii)$ hold. □

1.2 Representations of Inner Products by Symmetric Matrices

Let $e_1, \dots, e_n$ be the basis of vector space V, each vector $x \in V$ can be represented as a column vector

$$x = \sum_{i=1}^{n} x^i e_i \leftrightarrow \mathbf{x} = \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix}. \tag{6}$$

Let $\boldsymbol{A}$ be an $n \times n$ matrix whose entries $a_{ij} = \langle e_i, e_j \rangle$, the matrix form of an inner product on V is

$$\langle x, y \rangle = \sum_{ij} x^i y^j \langle e_i, e_j \rangle = \mathbf{x}^\top \boldsymbol{A} \mathbf{y}. \tag{7}$$

We find that, once a basis of V is chosen, the inner product on V determines a postive-definite symmetric matrix. Conversely, an $n \times n$ postive-definite symmetric matrix with a basis of V determines an inner product on V

It follows that there is an one-to-one correspondence

$$\left\{ \begin{array}{c} \text{inner product on a } n\text{-dimensional} \\ \text{vector space} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{An } n \times n \text{ postive-definite} \\ \text{symmetric matrix} \end{array} \right\}. \tag{8}$$

Let a basis of dual space $V^\vee := \mathrm{Hom}(V, \mathbb{R})$ be $\alpha^1, \dots, \alpha^n$ w.r.t. the basis $e_1, \dots, e_n$ of V, an inner product $\langle , \rangle$ of $x, y \in V$ is

$$\begin{aligned} \langle x, y \rangle &= \sum_{i,j} a_{ij} x^i y^j = \sum_{i,j} a_{ij} \alpha^i(x) \alpha^j(y) \\ &= \sum_{i,j} a_{ij} \alpha^i \otimes \alpha^j (x, y) \end{aligned}$$

In terms of tensor product, an inner product on V may be written as

$$\langle , \rangle = \sum_{ij} a_{ij} \alpha^i \otimes \alpha^j \tag{9}$$

1.3 Riemannian Metrics

Definition 1.4. A **Riemannian metric** is an inner product **assignment** to each $p \in M$ of an inner product $\langle , \rangle_p$ on the tangent space T_pM. This assignment should be C^∞ in the following sense: if $p \mapsto \langle X_p, Y_p \rangle_p$ is a C^∞ function for any C^∞ vector fields X, Y. A **Riemannian manifold** is a pair $(M, \langle , \rangle)$, which consists of a C^∞ manifold M together with a Riemannian metric on M.

Example 1.5. Since the tangent space at a point in Euclidean space $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$, the Euclidean inner product induces a Riemannian metric on $\mathbb{R}^n$ called the **Euclidean metric**.

Example 1.6. A surface M in $\mathbb{R}^3$ is a 2-dimensional regular submanifold of $\mathbb{R}^3$, the tangent space at p is a subspace of $T_p\mathbb{R}^3$, so the surface M inherits a Riemannian metric from the Euclidean metric by restriction $\langle , \rangle_M$.

Definition 1.7. A C^∞ map $F : (N, \langle , \rangle') \to (M, \langle , \rangle)$ of Riemannian manifolds is said to be **metric-preserving** if

$$\langle u, v \rangle'_p = \langle F_* u, F_* v \rangle_{F(p)} \tag{10}$$

for all point $p \in N$ and tangent vectors $u, v \in T_pN$. An **isometry** is a metric-preserving diffeomorphism.

For a Riemannian manifold $(M, \langle , \rangle)$, if there is a diffeomorphism that maps some manifolds N to M, the induced metric $\langle , \rangle'$ on N can be defined by (10).

Example 1.8 (Metric-preserving but not an isometry)**.** Let N and M be the unit circle in $\mathbb{C}$. Define $F : N \to M$ a **2-sheeted covering space map** (for any $w \in M$, $F^{-1}(w)$ contains 2 points in N), by $F(z) = z^2$. Given M any Riemannian metric $\langle , \rangle$, and define the induced metric on N is (10), The map F is metric-preserving but not an isometry because F is not a diffeomorphism (not inject).

Example 1.9 (Topological equivalant Riemannian manifolds may not isometric)**.**

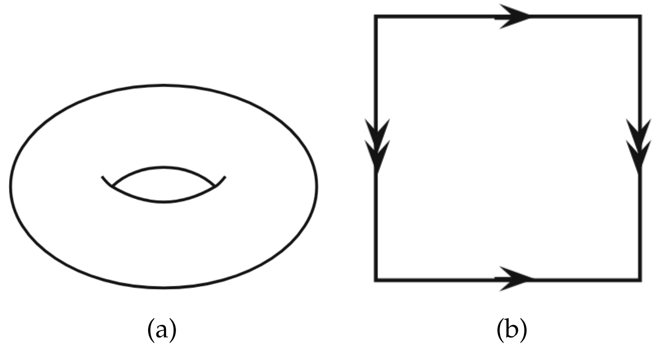

Figure 1.1: Two Riemannian metrics on torus [1, p. 6].

1.4 Existence of a Riemannian Metric

The local diffeomorphism ϕ defines a Riemannian metric on a coordinate chart $(U, x^1, \ldots, x^n)$ of M that $x^i = r^i \circ \phi$, as

$$\langle X, Y\rangle = \sum_{ij} a^i b^i \langle \partial_i, \partial_j\rangle = \sum_{ij} a^i b^i, \tag{11}$$

since $\phi_* \partial_j = \frac{\partial}{\partial r^j}$, the induced metric is the same as the Euclidean ones.

To obtain a Riemannian metric on M, we need to piece together the Riemannian metrics on all charts of an atlas of M. Here, we use the **partition of the unity** as the standard tools.

Theorem 1.10 (Existence of a Riemannian metric)**.** There exists a Riemannian metric on every manifold.

Proof. Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ an atlas of M. We have a partition of unity $\{\rho_\alpha\}$ that subcoordinates to open sets $\{U_\alpha\}$. Let $\langle , \rangle_\alpha$ the Riemannian metric on U_α as in (11), from Proposition 1.3, we define a metric on T_pM at p is

$$\langle , \rangle = \sum_{\alpha \in A} \rho_\alpha \langle , \rangle_\alpha. \tag{12}$$

Since U_p intersects finite number of U_α, (12) is a finite sum. Since ρ_α and $\langle , \rangle_\alpha$ are both smooth, for any C^∞ vector fields X, Y, $\sum_{\alpha \in A} \rho_\alpha \langle X, Y\rangle_\alpha$ is a finite sum of smooth functions at arbitary p (By Definition 1.4). So $\sum_{\alpha \in A} \rho_\alpha \langle , \rangle_\alpha$ is a Riemannian metric on M. □

Problems

1.1 Suppose $(M, \langle , \rangle)$ is a Riemannian manifold. Show that two C^∞ vector fields $X, Y \in \mathfrak{X}(M)$ are equal if and only if $\langle X, Z\rangle = \langle Y, Z\rangle$ for all C^∞ vector fields $Z \in \mathfrak{X}(M)$.

※ 2. Curves

A curve in manifold means either a parameterized curve, i.e., a smooth map $c : [a, b] \to M$, or a set of points in M that is the image of this map. This section focus on the plane curve, first introduces the regular curves whose velocity never zero so that can be reparameterized by arc length. We can define the signed curvature by the second derivative of this parameterization.

2.1 Regular Curves

Definition 2.1 (Regular curve)**.** A parameterized curve $c : [a, b] \to M$ is **regular** if its velocity $c'(t) \neq 0$ for all t in $[a, b]$, which means an immersion from $[a, b]$ to M.

Example 2.2. The curve $c(t) = (t^3, t^2)$, $t \in [-1, 1]$ in $\mathbb{R}^2$ is not regular since $c'(t)$ is zero at $t = 0$. Although c is smooth, but the image of c is not smooth as shown in Figure 2.1.

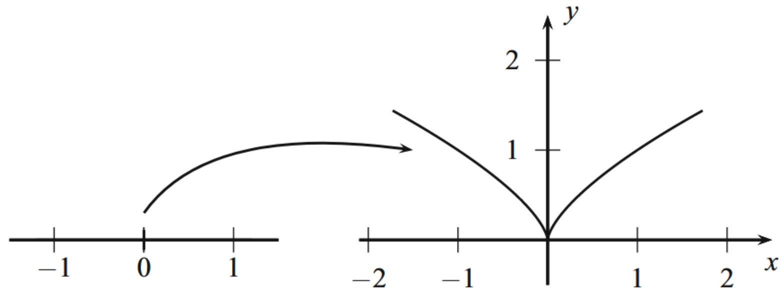

Figure 2.1: A nonregular curve [1, p. 9].

2.2 Arc Length Parameterization

The most important **reparameterization** ($\beta(u) := c(t(u))$ if $t = t(u)$ is a diffeomorphism from one to another closed interval) is the **arc length reparameterization**. We define the **speed** of a curve $c : [a, b] \to M$ is $\|c'(t)\|$, and the arc length is

$$\ell = \int_a^b \|c'(t)\| \, dt.$$

Then, the **arc length function** $s : [a, b] \to [0, \ell]$ of c is

$$s(t) = \int_a^t \|c'(u)\| \, du.$$

Proposition 2.3. The arc length function s of a regular curve has a C^∞ inverse.

Proof. The regular property gaurantees $s'(t) = \|c'(t)\| > 0$, which means $s(t)$ is monotonically increasing, so $t(s)$ is a C^∞ function. $\square$

Thus, we can write the **arc length reparameterization** of a regular curve by $\gamma(s) = c(t(s))$.

Proposition 2.4. A curve $\gamma(s)$ is reparameterized by arc length if and only if it has **unit speed** and its parameter starts at 0.

Proof. ($\Rightarrow$): as $\gamma(s) = c(t(s))$, the speed is

$$\left\|\frac{d\gamma}{ds}\right\| = \left\|\frac{dc}{dt}\right\| \cdot \left|\frac{dt}{ds}\right| = \frac{ds}{dt}\left|\frac{dt}{ds}\right| = 1. \tag{13}$$

($\Leftarrow$): If $c(t) : [a, b] \longrightarrow M$ has unit speed that $\|c'(t)\| = 1$, the arc length fucntion $s(t) = \int_a^t dt = t - a$. Since $a = 0$, we have $s = t$. Thus, a unit speed curve starts at $t = 0$ is reparameterized by arc length. □

Here, we do not emphasize that the curve need to be regular since **"reparameterized by arc length" implies regularity**. The parameter is s or t depends on the way of reparameterization.

Example 2.5. The regular curve $c : [0, 2\pi] \longrightarrow \mathbb{R}^2$,

$$c(t) = (a\cos t, a\sin t), \quad a > 0,$$

is a circle of radius a centered at the origin. The arc length function is

$$s(t) = \int_0^t \|c'(t)\| = at.$$

So the reparameterization is

$$\gamma(s) = (a\cos\frac{s}{a}, a\sin\frac{s}{a}).$$

2.3 Signed Curvature of a Plane Curve

The signed curvature measures how and what direction a curve bends. In this section, we quantify the signed curvature of a plane curve $\gamma : [0, \ell] \longrightarrow \mathbb{R}^2$ parameterized by arc length s in $\mathbb{R}^2$.

Then we define the velocity vector $T(s) = \gamma'(s)$, which has unit length and tangent at $\gamma(s)$. We can measure the curvature by how fast the velocity changes:

$$T'(s) = \frac{dT}{ds}(s) = \gamma''(s),$$

Here, we have already a tangent vector $T(s)$ at $\gamma(s)$, there is a unit normal vector $\mathbf{n}$ that perpendicular to $T(s)$ at $\gamma(s)$. We usually choose $(T(s), \mathbf{n})$ is counterclockwise, i.e., rotate from $T(s)$ to $\mathbf{n}$ counterclockwisely.

Since T has unit speed that $\langle T, T\rangle = 1$, we have $\langle T', T\rangle = 0$, which means T' is perpendicular to T so that we can write $T' = \kappa\mathbf{n}$. The scalar κ is the **signed curvature**, or simply **curvature**. We can write

$$\kappa = \langle T', \mathbf{n}\rangle = \langle \gamma'', \mathbf{n}\rangle. \tag{14}$$

The sign of κ means whether the curve is bending towards or away from $\mathbf{n}$.

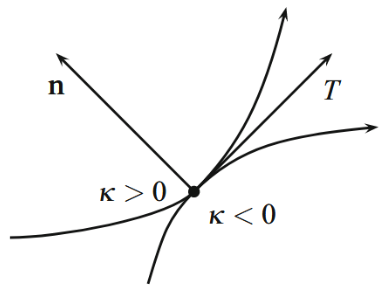

Figure 2.2: The sign of the curvature [1, p. 12].

Example 2.6. Recall Example 2.5, we can easily compute

$$T' = [-\frac{1}{a}\cos\frac{s}{a}, -\frac{1}{a}\sin\frac{s}{a}]^\top,$$

the normal vector $\mathbf{n} = [-\cos\frac{s}{a}, -\sin\frac{s}{a}]^\top$. So the curvature $\kappa = \frac{1}{a}$.

2.4 Orientation and Curvature

For a arc length parameterized curve that the two endpoints are fixed, will have two parameterization that inverse the orientation. Let the arc length be ℓ, then, we have another parameterization by

$$\tilde{\gamma}(s) = \gamma(\ell - s).$$

Then, the velocity and its derivative give

$$\tilde{T}(s) = -T(\ell - s), \quad \tilde{T}'(s) = T'(\ell - s).$$

The unit norm vector is given by rotate $\tilde{T}(s)$ by $\frac{\pi}{2}$

$$\tilde{\mathbf{n}}(s) = \mathrm{rot}\left(\frac{\pi}{2}\right)\tilde{T}(s) = -\mathrm{rot}\left(\frac{\pi}{2}\right)T(\ell - s) = -\mathbf{n}(\ell - s).$$

The sign of curvature will be reversed by

$$\tilde{\kappa}(s) = \langle\tilde{\mathbf{n}}(s), \tilde{T}'(s)\rangle = \langle -\mathbf{n}(\ell - s), T'(\ell - s)\rangle = -\kappa(\ell - s).$$

Example 2.7. From Example 2.5, the clockwise circle has the signed curvature $-1/a$.

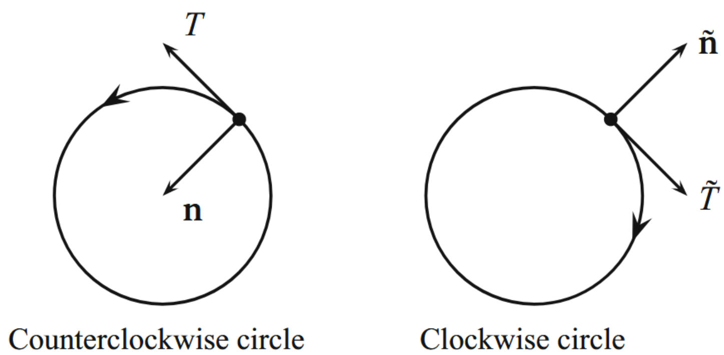

Figure 2.3: Reverse of a curve and its curvature [1, p. 13].

Problems

2.1 Let $T(s)$ be the unit tangent vector field on a plane curve $\gamma(s)$ parametrized by arc length. Write

$$T(s) = \begin{bmatrix} \cos\theta(s) \\ \sin\theta(s) \end{bmatrix},$$

where $\theta(s)$ is the angle of $T(s)$ with respect to the positive horizontal axis. Show that the **signed curvature** κ is the derivative $d\theta/ds$.

2.2 In physics literature, it is customary to denote the derivative with respect to time by a dot, e.g., $\dot{x} = dx/dt$, and the derivative with respect to distance by a prime, e.g., $x' = dx/ds$. We will sometimes follow this convention. Let $c(t) = (x(t), y(t))$ be a regular plane curve. Using Problem 2.1 and the chain rule $d\theta/ds = (d\theta/dt)/(ds/dt)$, show that the signed curvature of the curve $c(t)$ is

$$\kappa = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}},$$

where $\dot{x} = dx/dt$ and $\ddot{x} = d^2x/dt^2$.

2.3 The *graph* of a C^∞ function $y = f(x)$ is the set

$$\{(x, f(x)) \mid x \in \mathbb{R}\}$$

in the plane. Show that the signed curvature of this graph at $(x, f(x))$ is

$$\kappa = \frac{f''(x)}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}}.$$

2.4 The ellipse with equation $x^2/a^2 + y^2/b^2 = 1$ in the (x, y)-plane (Figure 2.4) can be parametrized by

$$x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi.$$

Find the curvature of the ellipse at an arbitrary point (x, y).

2.5 Let I be a closed interval in $\mathbb{R}$. If $\gamma\colon I \to \mathbb{R}^3$ is a regular space curve parametrized by arc length, its *curvature* κ at $\gamma(s)$ is defined to be $\|\gamma''(s)\|$.[1] Consider the helix $c(t) = (a\cos t, a\sin t, bt)$ in space.

(a) Reparametrize c by arc length: $\gamma(s) = c(t(s))$.

(b) Compute the curvature of the helix at $\gamma(s)$.

※ 2. Surfaces in Space

3.1 Principal, Mean, and Gaussian Curvatures

3.2 Gauss's Theorema Egregium

3.3 The Gauss–Bonnet Theorem

References

[1] L.W. Tu. *An Introduction to Manifolds*. Universitext. Springer New York.

[1] The curvature of a space curve is always nonnegative, while the signed curvature of a plane curve could be negative. To distinguish the two for a space curve that happens to be a plane curve, one can use κ_2 for the signed curvature. We will use the same notation κ for both, as it is clear from the context what is meant.