

Curvature and Vector Fields

Lecture Notes of Loring W. Tu [?]

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❖ Lecture 1 (1/9)

1.1 Inner Products on a Vector Space

The **Euclidean inner product** on \mathbb{R}^n is defined by

$$\langle u, v \rangle = \sum_1^n u^i v^i, \quad (1)$$

and the length of a vector is

$$\|u\| = \sqrt{\langle u, u \rangle}, \quad (2)$$

the **angle** θ between two vectors is

$$\cos \theta = \frac{\langle u, v \rangle}{\|u\| \|v\|}, \quad (3)$$

the **arc length** of a curve $c(t) \in \mathbb{R}^n, a \leq t \leq b$ is

$$s = \int_a^b \|c'(t)\| dt \quad (4)$$

Definition 1.1. An inner product in a real vector space V is a postive-definite, bilinear and symmetric map: $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ so that for $u, v, w \in V$ and $a, b \in \mathbb{R}$, satisfies

- (i) **Postive-definiteness** $\langle v, v \rangle = 0$ iff. $v = 0$
- (ii) **Symmetry** $\langle u, v \rangle = \langle v, u \rangle$
- (iii) **Bilinear** $\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$

Proposition 1.2. If W is a subspace of V , then the restriction

$$\langle \cdot, \cdot \rangle_W := \langle \cdot, \cdot \rangle|_{W \times W} : W \times W \rightarrow \mathbb{R}, \quad (5)$$

of an inner product $\langle \cdot, \cdot \rangle$ on V is also an innver prodcut.

Proof. The subspace construction preserves the properites in Definition 1.1. \square

Proposition 1.3. The **nonnegative linear combination** of inner products $\langle \cdot, \cdot \rangle_i$ on V : $\langle \cdot, \cdot \rangle := \sum_{i=1}^r a_i \langle \cdot, \cdot \rangle_i, a_i \geq 0$ is again an inner product on V .

Proof. The **nonnegativity** of a_i preserves condition (i) in Definition 1.1, the linearity makes condition (ii), (iii) hold. \square

1.2 Representations of Inner Products by Symmetric Matrices

Let e_1, \dots, e_n be the basis of vector space V , each vector $x \in V$ can be represented as a column vector

$$x = \sum_{i=1}^n x^i e_i \leftrightarrow \mathbf{x} = \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix}. \quad (6)$$

Let A be an $n \times n$ matrix whose entries $a_{ij} = \langle e_i, e_j \rangle$, the matrix form of an inner product on V is

$$\langle x, y \rangle = \sum_{ij} x^i y^j \langle e_i, e_j \rangle = \mathbf{x}^\top A \mathbf{y}. \quad (7)$$

We find that, once a basis of V is chosen, the inner product on V determines a postive-definite symmetric matrix. Conversely, an $n \times n$ postive-definite symmetric matrix with a basis of V determines an inner product on V

It follows that there is an one-to-one correspondence

$$\left\{ \begin{array}{c} \text{inner product on a } n\text{-dimensional} \\ \text{vector space} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{An } n \times n \text{ postive-definite} \\ \text{symmetric matrix} \end{array} \right\}. \quad (8)$$

Let a basis of dual space $V^\vee := \text{Hom}(V, \mathbb{R})$ be $\alpha^1, \dots, \alpha^n$ w.r.t. the basis e_1, \dots, e_n of V , an inner product \langle, \rangle of $x, y \in V$ is

$$\begin{aligned}\langle x, y \rangle &= \sum_{i,j} a_{ij} x^i y^j = \sum_{i,j} a_{ij} \alpha^i(x) \alpha^j(y) \\ &= \sum_{i,j} a_{ij} \alpha^i \otimes \alpha^j(x, y)\end{aligned}$$

In terms of tensor product, an inner product on V may be written as

$$\langle, \rangle = \sum_{ij} a_{ij} \alpha^i \otimes \alpha^j \quad (9)$$

1.3 Riemannian Metrics

Definition 1.4. A **Riemannian metric** is an inner product **assignment** to each $p \in M$ of an inner product \langle, \rangle_p on the tangent space $T_p M$. This assignment should be C^∞ in the following sense: if $p \mapsto \langle X_p, Y_p \rangle_p$ is a C^∞ function for any C^∞ vector fields X, Y . A **Riemannian manifold** is a pair (M, \langle, \rangle) , which consists of a C^∞ manifold M together with a Riemannian metric on M .

Example 1.5. Since the tangent space at a point in Euclidean space \mathbb{R}^n is isomorphic to \mathbb{R}^n , the Euclidean inner product induces a Riemannian metric on \mathbb{R}^n called the **Euclidean metric**.

Example 1.6. A surface M in \mathbb{R}^3 is a 2-dimensional regular submanifold of \mathbb{R}^3 , the tangent space at p is a subspace of $T_p \mathbb{R}^3$, so the surface M inherits a Riemannian metric from the Euclidean metric by restriction \langle, \rangle_M .

Definition 1.7. A C^∞ map $F : (N, \langle, \rangle') \rightarrow (M, \langle, \rangle)$ of Riemannian manifolds is said to be **metric-preserving** if

$$\langle u, v \rangle'_p = \langle F_* u, F_* v \rangle_{F(p)} \quad (10)$$

for all point $p \in N$ and tangent vectors $u, v \in T_p N$. An **isometry** is a metric-preserving diffeomorphism.

For a Riemannian manifold (M, \langle, \rangle) , if there is a diffeomorphism that maps some manifolds N to M , the induced metric \langle, \rangle' on N can be defined by (10).

Example 1.8 (Metric-preserving but not an isometry). Let N and M be the unit circle in \mathbb{C} . Define $F : N \rightarrow M$ a **2-sheeted covering space map** (for any $w \in M$, $F^{-1}(w)$ contains 2 points in N), by $F(z) = z^2$. Given M any Riemannian metric \langle, \rangle , and define the induced metric on N is (10), The map F is metric-preserving but not an isometry because F is not a diffeomorphism (not inject).

Example 1.9 (Topological equivalent Riemannian manifolds may not be isometric).

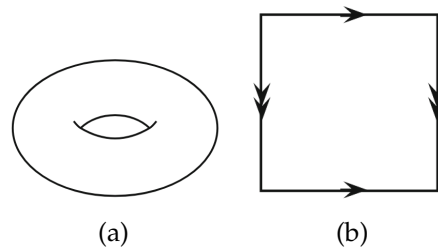


Figure 1: Two Riemannian metrics on torus.

1.4 Existence of a Riemannian Metric

1.5 Problems