Randomized Algorithms

Spezialvorlesung

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Randomized Quicksort

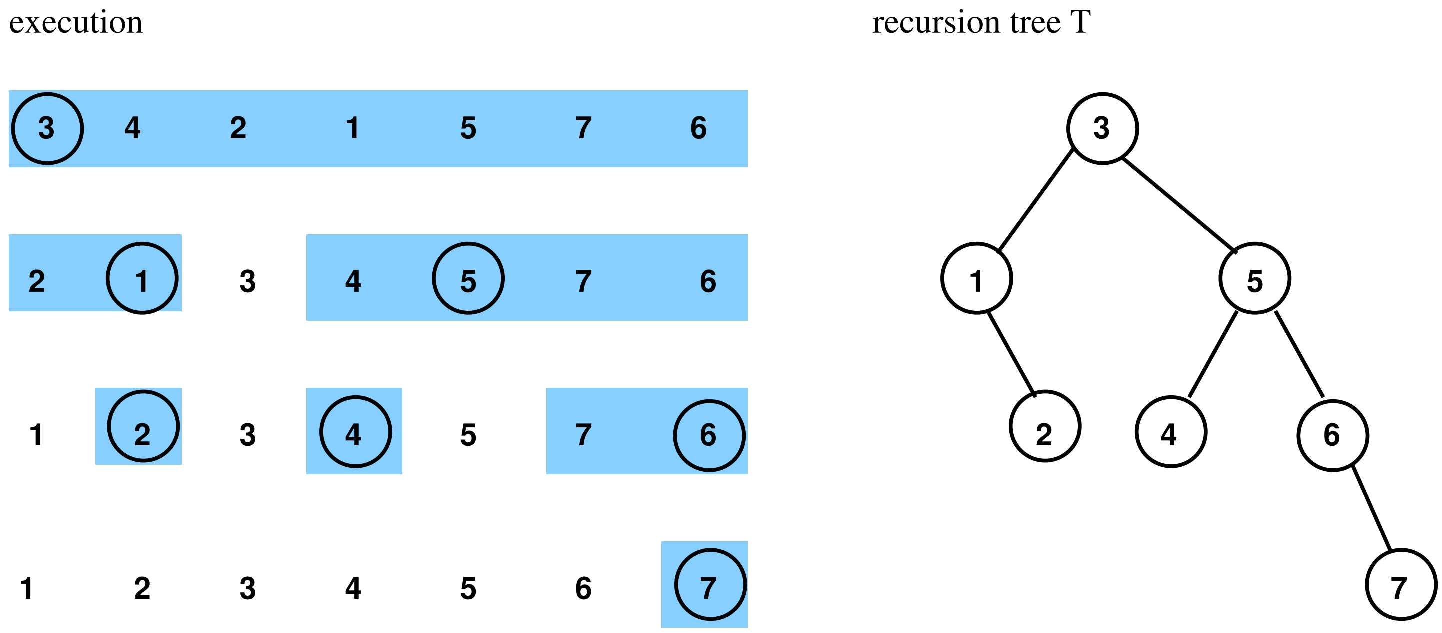
Input: a set S of n different numbers

Output: the numbers of S in increasing order

Algorithm: randQS(S)

1. Choose y form S uniformly at random
2. $S_1 := \{x \in S \mid x < y\}; S_2 := \{x \in S \mid x > y\}$
3. If $S_1 \neq \emptyset$ then randQS(S_1)
4. output y
5. If $S_2 \neq \emptyset$ then randQS(S_2)

Example



execution
3 4 2 1 5 7 6
2 1 3 4 5 7 6
1 2 3 4 5 7 6
1 2 3 4 5 6 7
recursion tree T
3
1
5
2
4
6
7

The worst case

execution	#comp.	probability
1 2 3 4 5 6 7	6	1/7
2 3 4 5 6 7	5	1/6
3 4 5 6 7	4	1/5
6 7	1	1/2
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#comparisons: $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} = \Theta(n^2)$

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prob: $\prod_{i=1}^{n} \frac{1}{i} = \frac{1}{n!}$

Very Basic Probability Theory

Probability Space: A (finite) sample space Ω and a probability measure Pr

Sample Space Ω: A set of elementary events

Probability measure Pr: A function $Pr : \mathcal{P}(\Omega) \to \mathbb{R}$ with

- For each $\mathcal{E} \subseteq \Omega, 0 \leq Pr[\mathcal{E}] \leq 1$
- $Pr[\Omega] = 1$
- For each $\mathcal{E} \subseteq \Omega, Pr[\mathcal{E}] = \sum_{\omega \in \mathcal{E}} Pr[\{\omega\}]$

Random Variable X: a function $X : \Omega \to \mathbb{R}$.

Expectation of a random variable: $E[X] = \sum_{x \in \mathbb{N}} x \cdot Pr[X = x]$

Fact: $E[X+Y] = E[X] + E[Y]$.

Analyzing Quick-sort

- We want to count the number of comparisons we perform.
- Let s_i be the ith smallest element of S.
- We define random variables X_{ij} as

$$X_{ij} = \begin{cases} 1 & \text{if we compare } s_i \text{ and } s_j \\ 0 & \text{otherwise} \end{cases}$$

- The expected running time (#comparisons) is

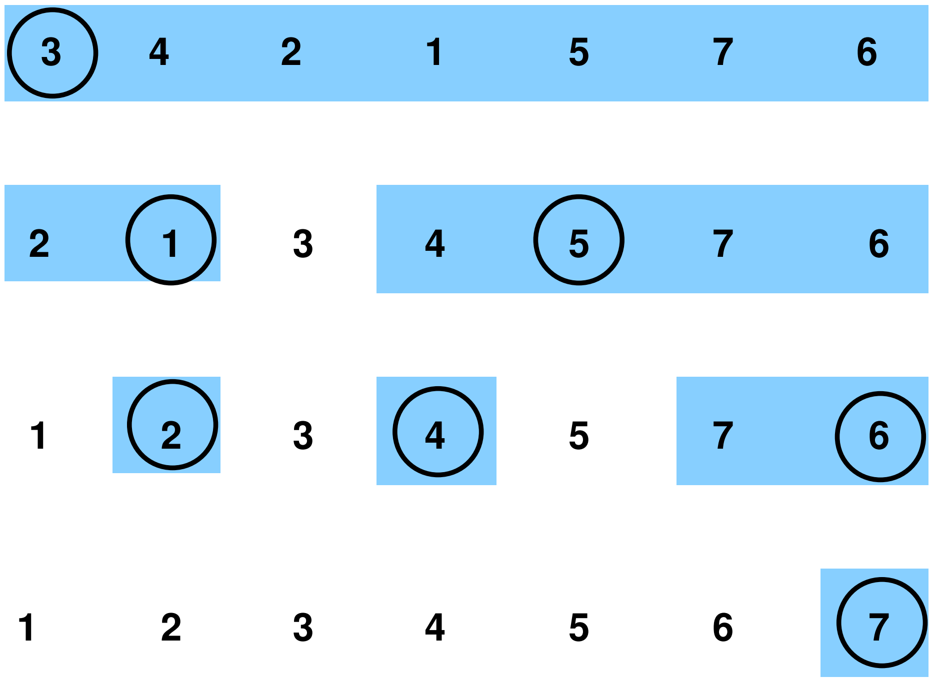
$$E[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$$

- Thus we have to compute $E[X_{ij}]$.

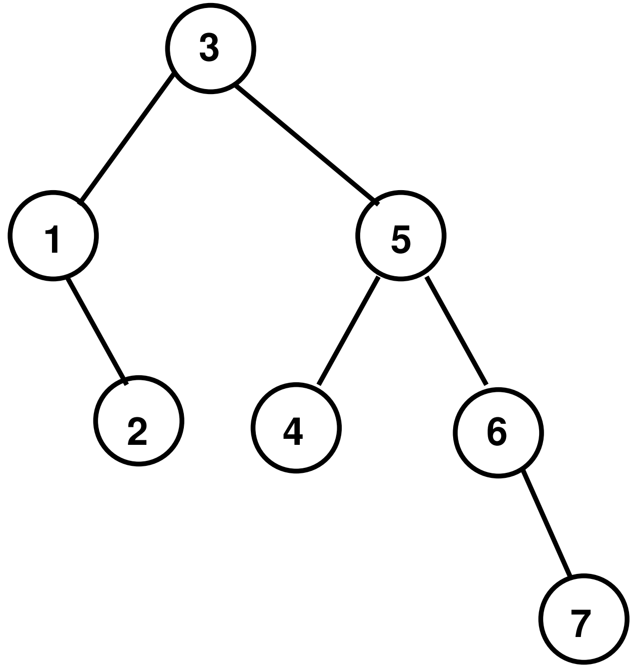
$$E[X_{ij}] = 0 \cdot Pr[X_{ij} = 0] + 1 \cdot Pr[X_{ij} = 1] = Pr[X_{ij} = 1]$$

Recursion Tree

execution



recursion tree T



$\pi = (3, 1, 5, 2, 4, 6, 7)$: level order traversal

Lemma

Lemma: There is a comparison between s_i and s_j $(i < j)$, iff s_i or s_j occurs earlier in π than any other element in $S_{ij} = \{s_i, s_{i+1}, \ldots, s_j\}$.

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s^*: element from S_{ij} that occurs first in π.

Observe: No pivot preceding s^* in π splits $S_{i,j}$
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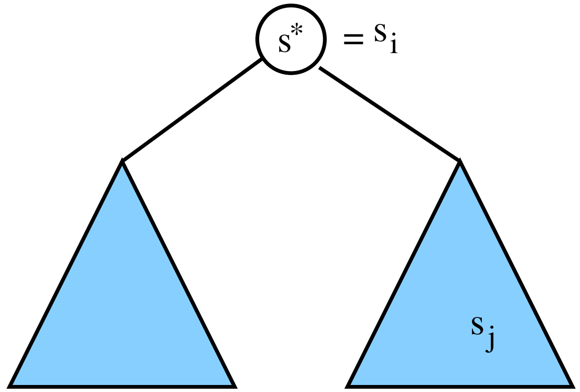
Case I: $s^* \in \{s_i, s_j\}$:



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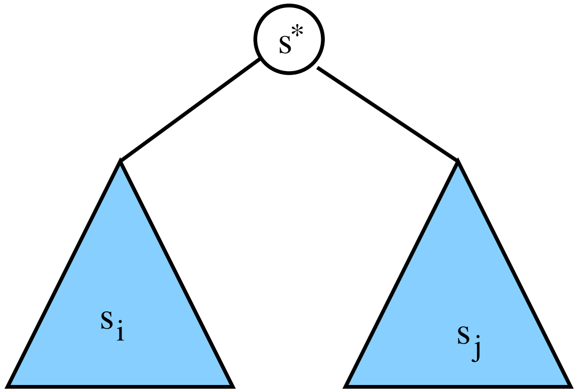
Case II: $s^* \notin \{s_i, s_j\}$:



Lemma

Lemma:

$$\mathbf{Pr}[s^* \in \{s_i, s_j\}] = \frac{2}{j-i+1}$$

Each element in $S_{i,j}$ equally likely to be the first element in π.

Finalizing the Analysis

Lemma: $\mathbf{Pr}[X_{ij} = 1] = 2/(j - i + 1)$

$$\Rightarrow \mathbf{E}[X_{ij}] = 0 \cdot \mathbf{Pr}[X_{ij} = 0] + 1 \cdot \mathbf{Pr}[X_{ij} = 1] = 2/(j - i + 1)$$

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Theorem: The expected number of comparisons performed is

$$\begin{aligned}
\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathbf{E}[X_{ij}] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=2}^{n} \sum_{k=2}^{i} \frac{2}{k} \\
&= 2 \sum_{i=2}^{n} (H_i - 1) \\
&\leq 2nH_n \approx 2n \log n
\end{aligned}$$

The problem

Verify $A \cdot B = C$ for given matrices A, B and C.

- compute $A \cdot C$ and compare to $C \rightarrow$ naive: $\Theta(n^3)$, best $\Theta(n^{2.37})$
- Faster randomized algorithm with small error probability:

1. Choose $r = (r_1, \ldots, r_n) \in \{0,1\}^n$ at random.
2. If $A(Br) = Cr$ output "TRUE" else output "FALSE"

One-sided error: false positives
Running time: $O(n^2)$

Theorem: If $AB \neq C$ then $\mathbf{Pr}[ABr = Cr] \leq 1/2$

Proof

Define $D = AB - C \neq 0$

$$ABr = Cr \Leftrightarrow (AB - C)r = 0 \Leftrightarrow Dr = 0$$

W.l.o.g, $d_{1,1} \neq 0$

$$\sum_{j=1}^{n} d_{1,j} r_j = 0$$

$$r_1 = \frac{\sum_{j=2}^{n} d_{1,j} r_j}{d_{1,1}}$$

- Fix all r_j but r_1.
- Equality holds for at most one of the two choices for r_1.

$$\Rightarrow \mathbf{Pr}[Dr = 0] \leq 1/2$$

Probability amplification

Can we decrease the error-probability?

Idea: Repeat the test k times with independent random choices for r

- If some test reports “FALSE” we know for sure $AB \neq C$
- If $AB \neq C$ then the probability that all test report “TRUE” is at most 2^{-k}
- running time: $O(kn^2)$

Las Vegas - Monte Carlo

Las Vegas Algorithm:

- always gives the correct/optimal solution.
- running time is a random variable.
- Example: Quick-sort.

Monte Carlo Algorithm:

- may produce incorrect solutions.
- has a fixed deterministic running time.
- Example: Verifying Matrix-Multiplication
- For decision problems there are two kinds:
  one-sided errors: either *yes* or *no* answers always correct.
  two-sided errors: both have non-zero probability to be wrong.

Markov inequality

Lemma: Let X be a random variable with non-negative domain.

$$\mathbf{Pr}\left[X \geq t\right] \leq \frac{\mathbf{E}\left[X\right]}{t}$$

or equivalently

$$\mathbf{Pr}\left[X \geq t\mathbf{E}\left[X\right]\right] \leq \frac{1}{t}$$

Proof: Assume for simplicity that the domain of X is $\mathbb{N}_0$

$$\begin{aligned}
\mathbf{E}\left[X\right] &= \sum_{i\geq 0} i\,\mathbf{Pr}\left[X = i\right] \\
&\geq \sum_{i\geq t} t\,\mathbf{Pr}\left[X = i\right] \\
&= t\,\mathbf{Pr}\left[X \geq t\right]
\end{aligned}$$

Transform LV -> MC

Given Las Vegas algorithm with expected running time at most $f(n)$.

Idea: Stop algorithm after $\alpha f(n)$ time.

- Only error source: Stopping algo before completion.
- Probability: $\mathbf{Pr}[T > \alpha f(n)] \leq 1/\alpha$

$\Rightarrow$ Monte Carlo algorithm with running time $\alpha f(n)$ and error rate $1/\alpha$

Transform LV -> MC

Given Monte Carlo algorithm MC with

- deterministic running time at most $f(n)$ and
- success probability at least $p(n)$.

Assume we can verify the correctness of the output in time $g(n)$.

Theorem: There exists Las Vegas algorithm with expected running time

$$\mathbf{E}[T] \leq \frac{f(n)+g(n)}{p(n)}$$

Proof

Proof: Repeat calling MC until verifier certifies correctness of output.

- Let i be the number of iterations of this scheme
- Running time $T \le i(f(n)+g(n))$
- $\mathbf{E}[T] \le (f(n)+g(n))\mathbf{E}[i]$

$$\mathbf{E}[i] = \sum_{k=1}^{\infty} k(1-p(n))^{k-1}p(n) = \frac{1}{p(n)}$$

as

$$\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}, \text{ for } x < 1$$