Effect of noise on front propagation in reaction-diffusion equations

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Based on ongoing work with **Clayton Barnes** and **Leonid Mytnik**

Webinar
November, 2020

KFPP equations

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$$\partial_t u_{t,x} = \frac{1}{2}\partial_x^2 u_{t,x} + u_{t,x}(1 - u_{t,x}), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

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- $u_{t,x}$ – the proportion of population at time t at site x with the advantageous gene;
- $\frac{1}{2}\partial_x^2 u$ – the microscopic movements of the members of the population are Brownian motions.
- More fit members replace less fit members. Frequency of this interaction is proportional to $u(1-u)$.

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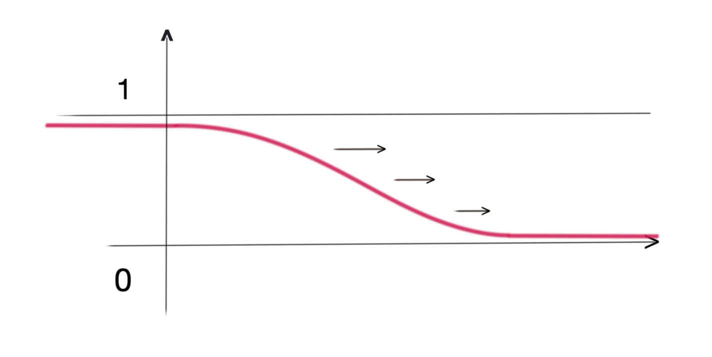

Traveling wave solution

Kolmogorov-Petrovsky-Piskunov (1937) and Bramson (1983):

- For any $\mathrm{v} \geq \mathrm{v}_0 := \sqrt{2}$, there exists a function U_{v} on $\mathbb{R}$ such that $(x,t) \mapsto U_{\mathrm{v}}(x - \mathrm{v}t)$ is a solution to the FKPP equation which satisfies $U_{\mathrm{v}}(-\infty) = 1$, and $U_{\mathrm{v}}(\infty) = 0$.

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- If $-\infty < \inf\{x : u_{0,x} < 1\} \leq \sup\{x : u_{0,x} > 0\} < \infty$, then

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u_{t,x} - U_{\mathrm{v}_0}(x - m(t))| = 0$$

where $m(t)$ is some suitable centering satisfying $m(t)/t \to \mathrm{v}_0$.

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- If $-\infty < \inf\{x : u_{0,x} < 1\} \leq \sup\{x : u_{0,x} > 0\} < \infty$, then

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  where $m(t)$ is some suitable centering satisfying $m(t)/t \to \mathrm{v}_0$.
- Bramson's correction:

$$m(t) = O(1) + \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t.$$

Mckean's duality

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- Then $\mathrm{P}(R_t > x) = u_{t,x}$ solves the FKPP equation.

Therefore $R_t/t \to \mathrm{v}_0 = \sqrt{2}$ and

$$R_t - (\sqrt{2}t - \frac{3}{2\sqrt{2}} \log t) \xrightarrow[t\to\infty]{d} \text{some random variable.}$$

Stochastic FKPP equation

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$$\partial_t u = \frac{1}{2}\partial_x^2 u + u(1-u) + \epsilon\sqrt{u(1-u)}\dot{W} \tag{1}$$

where $\dot{W}$ is a Gaussian space-time white noise.

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where $\dot{W}$ is a Gaussian space-time white noise.

We say u is a (mild) solution to (1) if for each $t > 0$ and $x \in \mathbb{R}$,

$$u_{t,x} = \iint_0^t G_{s,y;t,x}\Big(u_{0,y}\delta_0(\mathrm{d}s)\mathrm{d}y + u_{s,y}(1-u_{s,y})\mathrm{d}s\mathrm{d}y \\ + \epsilon\sqrt{u_{s,y}(1-u_{s,y})}\dot{W}_{s,y}ds\mathrm{d}y\Big)$$

almost surely where $G_{s,y;t,x}$ is the transition density for Brownian motion from time-space (s, y) to (t, x).

Motivation

The outcome of interaction between fit and unfit individuals may be random. The "variance" of this interaction is proportion to $\epsilon^2 u(1-u)$. Interactions are independent at different space-time points (space-time white noise).

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- Let the system start with one particle at 0.
- Let $R^{bc}(t)$ be the position of the rightmost particle at time t, then $\mathrm{P}(R^{bc}(t) > x) = \mathrm{E}_{\mathbf{1}_{(-\infty,0]}}[u(t,x)]$ where u satisfies the stochastic FKPP equation.

Propagation of compact support interface

For any $u : x \to [0, 1]$, define

$$L(u) = \inf\{x : u_x < 1\}, \quad R(u) = \sup\{x : u_x > 0\}.$$

Define $\mathcal{B}_I := \{u \in \mathcal{B}(\mathbb{R}, [0, 1]) : -\infty < L(u) \leq R(u) < \infty\}$ and $\mathcal{C}_I := \mathcal{B}_I \cap \mathcal{C}(\mathbb{R}, [0, 1])$.

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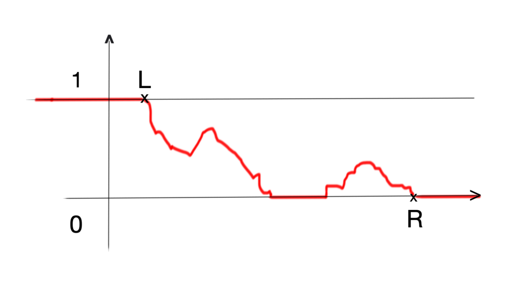

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Muller-Sowers (1995):

If $u_0 \in \mathcal{B}_I$ then almost surely for any $t > 0$, $u_t \in \mathcal{C}_I$. (This doesn't hold for the deterministic FKPP equation.)

Effect of small noise on the speed

Mueller-Sowers (1995):

For small enough $\epsilon > 0$, there exists a deterministic speed $V(\epsilon) \in \mathbb{R}$ such that almost surely

$$\lim_{t\to\infty} \frac{R(u_t)}{t} = V(\epsilon).$$

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Brunet-Derrida conjecture (1997) proved by Mueller-Mytnik-Quastel (2011):

When $\epsilon \to 0$,

$$V(\epsilon) = \sqrt{2} - \frac{\pi^2}{2|\log \epsilon|^2} + O\Big(\frac{\log|\log \epsilon|}{|\log \epsilon|^3}\Big)$$

Unexpected slow down

From the property of space-time white noise, one has $V(-\epsilon) = V(\epsilon)$. So $V(\epsilon)$ takes its local maximum at $\epsilon = 0$. A naive Taylor's expansion argument expects that $V^{\text{ex}}(\epsilon) \approx \sqrt{2} - c\epsilon^2$.

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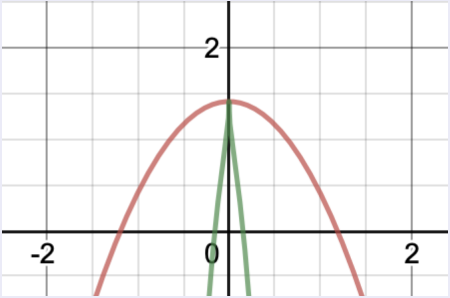

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Conlon-Doering (2005):

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Conlon-Doering conjecture (2005) proved by Mueller-Mytnik-Ryzhik (2019):

$$\epsilon^2 V(\epsilon) \xrightarrow[\epsilon\to\infty]{} 1$$

Dramatic slow down

Let's define $v_{t,x} = u_{\epsilon^{-4}t,\epsilon^{-2}x}$. Then v satisfies the equation

$$\partial_t v_{t,x} = \partial_x^2 v_{t,x} + \epsilon^{-4} v_{t,x}(1 - v_{t,x}) + \sqrt{v_{t,x}(1 - v_{t,x})}\dot{W}_{t,x}.$$

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The deterministic equation

$$\partial_t \bar{v}_{t,x} = \partial_x^2 \bar{v}_{t,x} + \epsilon^{-4} \bar{v}_{t,x}(1 - \bar{v}_{t,x}).$$

has speed of order $\epsilon^{-2} \gg \epsilon^{-4}$ when ϵ is large.

General drift

One can consider a more general reaction term $f(u)$ instead of $u(1-u)$ in the reaction-diffusion equation

$$\partial_t u_{t,x} = \frac{1}{2}\partial_x^2 u_{t,x} + f(u_{t,x}), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R} \tag{1}$$

and the corresponding stochastic reaction-diffusion equation

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If f is a C^1 function on $[0,1]$ which satisfies the KPP condition:

$$f(0) = f(1) = 0; \quad 0 < f(u) \leq uf'(0), \quad u \in (0,1),$$

similar results mentioned above still hold. In particular, (1) has minimum speed $\sqrt{2f'(0)}$.

Non-Lipschitz drift with no noise

Question:

What if f is not Lipschitz?

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Aguirre and Escobedo (1986):

For the deterministic reaction-diffusion equation,

$$\partial_t u_{t,x} = \frac{1}{2}\partial_x^2 u_{t,x} + u_{t,x}^p(1-u_{t,x}), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}$$

with $p \in (0,1)$, if $u_0 \not\equiv 0$ taking values in $[0,1]$, then there exists a unique solution to (1). And the solution satisfies

$$\lim_{t\to\infty} \inf_{x\in\mathbb{R}} u_{t,x} = 1,$$

which says that there is no traveling wave solution.

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if $u_0 \in \mathcal{B}_I$ and f is continuous with $|f(u)| \leq K_f|u(1-u)|^{1/2}$, then:

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if $u_0 \in \mathcal{B}_I$ and f is continuous with $|f(u)| \leq K_f|u(1-u)|^{1/2}$, then:

- There exists a unique in law solution;
- Almost surely for any $t > 0$, $u_t \in \mathcal{C}_I$;
- The solution has finite deterministic speed

$$\lim_{t\to\infty} \frac{R(u_t)}{t} = V_{\epsilon,f} \in \mathbb{R};$$

Noise effect in the non-Lipschitz case

Question

What is the effect of the noise on the speed of stochastic reaction-diffusion equation with non-Lipschitz drift

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If $|f(u)| \leq K_f |u(1-u)|^\gamma$ for some $\gamma \in (1/2, 1]$, then $\lim_{\epsilon \to \infty} \epsilon^2 V_{\epsilon,f} = c_f$.

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Barnes-Mytnik-S. (ongoing work)

If $f(u) = u^p(1-u)$ with $p \in (1/2, 1)$, then there exists $0 < c < C < \infty$ and ϵ_0 such that

$$c\epsilon^{-2\frac{1-p}{1+p}} \leq V_{\epsilon,f} \leq C\epsilon^{-2\frac{1-p}{1+p}}, \quad \forall \epsilon \in (0, \epsilon_0).$$

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Consider a deterministic equation with free moving boundary: Find solution (F, v) so that $\varrho_{t,x} = F(x - \mathrm{v}t)$ satisfies

$$\begin{cases} \partial_t \varrho_{t,x} = \frac{1}{2}\partial_x^2 \varrho + \varrho^p(1-\varrho), & x < \mathrm{v}t, \\ \varrho_{t,x} = 0, & x \geq \mathrm{v}t. \end{cases}$$

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- For any ϵ small enough, there exists a unique solution (F, v) so that $F'(0-) = \epsilon^2$ and $F(-\infty) = 1$.

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- For any ϵ small enough, there exists a unique solution (F, v) so that $F'(0-) = \epsilon^2$ and $F(-\infty) = 1$.
- It can be verified that $\mathrm{v} = \Theta(\epsilon^{-2\frac{1-p}{1+p}})$.

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Now we use this (F, v) with $F'(0-) = \epsilon^2$ to construct a stochastic equation with moving boundary:

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Here, the SPDE with the moving boundary condition is understood as

$$v_{t,x} = \iint_0^t G^{\mathrm{v}}_{s,y;t,x}\Big(v_{0,y}\delta_0(\mathrm{d}s)\mathrm{d}y + v_{s,y}^p(1 - v_{s,y})\mathrm{d}s\mathrm{d}y + \epsilon\sqrt{v_{s,y}(1 - v_{s,y})}\dot{W}_{s,y}\Big)$$

where G^{v} is the transition kernel of a Brownian motion killed upon entering time-space region $\{(t, x) : x \geq \mathrm{v}t\}$.

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Comments on the proof

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Observe that v satisfies

$$\partial_t v_{t,x} = \frac{1}{2}\partial_x^2 v_{t,x} + v_{t,x}^p(1 - v_{t,x}) + \epsilon\sqrt{v_{t,x}(1 - v_{t,x})}\dot{W}_{t,x}^v - \delta_{\mathrm{vt}}(x)\dot{A}_t.$$

where A_t is the total mass loss at the boundary up to time t.

Comments on the proof

Given the equation for u and v, we can couple them together so that their difference $w = u - v$ satisfies the SPDE

$$\partial_t w_{t,x} = \frac{1}{2}\partial_x^2 w_{t,x} + f_{t,x}^w + \epsilon\sigma_{t,x}^w \dot{W}_{t,x}^w + \delta_{\mathrm{vt}}(x)\dot{A}_t$$

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When $w_{t,x}$ takes small values, one can verify that $f^w_{t,x} \sim w^p_{t,x}$ and $\sigma^w_{t,x} \sim \sqrt{w_{t,x}}$. So basically

$$\partial_t w_{t,x} \approx \frac{1}{2}\partial_x^2 w_{t,x} + w^p_{t,x} + \epsilon\sqrt{w_{t,x}}\dot{W}^v_{t,x} + \delta_{\mathrm{vt}}(x)\dot{A}_t.$$

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We want to get rid of the drift $f_{t,x}^w$ using Girsanov transformation.

Comments on the proof

Define an exponential martingale,

$$M_t^{(\epsilon)} = \exp\Big\{ -\iint_0^t \frac{f_{s,y}^w}{\epsilon\sigma_{s,y}} W(\mathrm{d}s\mathrm{d}y) + \frac{1}{2}\iint_0^t \Big(\frac{f_{s,y}^w}{\epsilon\sigma_{s,y}}\Big)^2 \mathrm{d}s\mathrm{d}y \Big\}$$

and a new probability Q with $\mathrm{d}Q|_{\mathscr{F}_t} = M_t^{(\epsilon)}\mathrm{d}P|_{\mathscr{F}_t}$.

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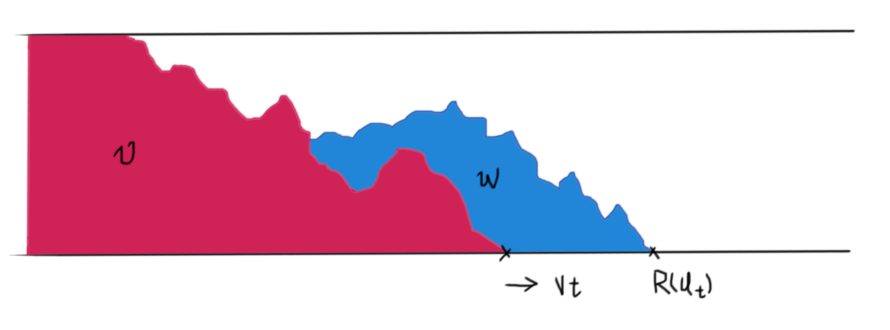
Now we can study w (under probability Q) since it is basically a critical super-Brownian motion with immigration.

Comments on the proof



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Under probability Q, we have

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(Similar result is known for critical branching random walk with immigration, Kesten (1994).)

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Therefore, under probability Q, centered at the immigration source, the propagation speed of w is

$$\frac{R(w_t) - \mathrm{v}t}{t} \xrightarrow[t\to\infty]{\text{in probability}} 0$$

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So we want to find a small time $t = T_\epsilon$, which allows us to compare $(w_s)_{0\leq s\leq t}$ under P with $(w_s)_{0\leq s\leq t}$ under Q in the sense that

$$Q[(M_t^{(\epsilon)})^{-2}] = O(1), \quad \epsilon \to 0. \quad (*)$$

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A important observation

We need to choose $T_\epsilon \lesssim \epsilon^{4\frac{1-p}{1+p}}$ so that $(*)$ holds.

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Now, given time $t = T_\epsilon \sim \epsilon^{4\frac{1-p}{1+p}}$, under both probability Q and P, the (average) propagation speed of $(w_s)_{0\le s\le T_\epsilon}$ centered by $\mathrm{v}t$ is

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Therefore, the order of the average speed of u in time interval $[0, T_\epsilon]$ is equal to v.

To get the propagation speed for $t \to \infty$, we use a standard updating procedure (repeating the argument for each time interval $[nT_\epsilon, (n+1)T_\epsilon]$, $n \in \mathbb{N}$.)

Thank you!