IE 529 - Homework 2: Estimators and Matrix analysis

Due Friday, September 29th

I. Estimators:

(i) Recall the Bayes' estimator for the mean of a random sample taken from a Normal distribution; provide a sketch of a proof, i.e.

Suppose $\overline{x}$ is the sample mean for a random sample of size n taken from a normal distribution with unknown mean (denoted μ) and known variance σ^2 (i.e., $x_1, x_2, \ldots, x_n \in \mathcal{N}(\mu, \sigma^2)$). Also assume that the (*prior*) distribution for the mean is also normal, i.e., $\mu \in \mathcal{N}(\nu, \rho^2)$.

Show that the *posterior* distribution for the population mean μ is normal, with mean $\mu*$ and standard deviation σ^* given by

$$\mu^* = \left(\frac{\rho^2}{\rho^2 + \frac{\sigma^2}{n}}\right)\overline{x} + \left(\frac{\frac{\sigma^2}{n}}{\rho^2 + \frac{\sigma^2}{n}}\right)\nu; \text{ and } \sigma^* = \sqrt{\frac{\rho^2\sigma^2}{n\rho^2 + \sigma^2}}.$$

Hints: recall we know the following density functions apply:

$$f(x_1, x_2, \ldots, x_n|\mu) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \left(\frac{1}{\sigma}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right\};$$

$$f(\mu) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \left(\frac{1}{\rho}\right) \exp\left\{-\frac{1}{2\rho^2}(\mu - \nu)^2\right\}.$$

The posterior distribution is $f(\mu|x_1, x_2, \ldots, x_n)$; show that this posterior has a normal form.

(ii) In this problem, we want to determine the maximum likelihood estimator of p, the probability of success, when $X_1, X_2, \ldots, X_n$ are the outcomes of n repeated Bernoulli trials. Consider the following questions:

1. What distribution will the number of successes have? Write an expression for this in terms of a probability mass function or cumulative distribution function.
2. Write an expression for the likelihood function for p, and for the log-likelihood function for p.
3. Find the argmax of this function, that is, find the maximum likelihood estimator for p. If it helps to assume n is some fixed constant, say 10 that is acceptable.

II. Linear Algebra:

a. Sketch the unit ball in $\mathbf{R}^2$ for the l_1-, l_2- and l_∞- norms. That is, provide a sketch of the set of points that satisfy $||x||_a = 1$ for $a = 1, 2, \infty$.

b. A vector norm $|| \cdot ||$ is said to be *unitarily invariant* if for any *unitary* matrix U[1], the norm satisfies

$$||Ux|| = ||x||,$$

for all real or complex-valued vectors x, and all unitary matrices U. Show that the vector 2-norm is unitarily invariant.

c. Show that the following statements are true:
For $x,\ y \in \mathbf{R}^n$,

$$\text{(i) } \|y\|_\infty = \max_{\|x\|_1 \neq 0} \left(\frac{|y^* x|}{\|x\|_1} \right), \qquad \text{(ii) } \|y\|_1 = \max_{\|x\|_\infty \neq 0} \left(\frac{|y^* x|}{\|x\|_\infty} \right)$$

d. Suppose $a_1, a_2, \ldots, a_n$ are fixed positive real numbers. Determine which, if any, of the following are proper vector norms on $\mathbf{R}^n$ (i.e., which of the following satisfy the four conditions required of functions to be vector norms).

1. $\|x\| := \max_i \{a_i |x_i|\}$
2. $\|x\| := \sum_{i=1}^n a_i |x_i|$

e. For this problem we will prove that the induced matrix 2-norm for a matrix $A \in \mathbf{R}^{m \times n}$ is given by the maximum singular value, $\sigma_1(A)$.

In particular, show that

$$\begin{aligned} \max f(x) &= \|Ax\|_2^2 = x^T A^T A x \\ \text{subj. to } x^T x &= 1 \end{aligned},$$

is given by σ_1^2. Hint: consider using the SVD of the matrix A.

[1] Recall: a matrix U is said to be unitary if it satisfies $U^* U = I$.