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CHAPTER 5

Norms for vectors and matrices

5.0 Introduction

If one has several vectors in $\mathbf{C}^n$ or several matrices in M_n, what might it mean to say that some are "small" or that others are "large"? Under what circumstances might we say that two vectors are "close together" or "far apart"?

Questions of "size" and "proximity" in a two- or three-dimensional real vector space usually refer to Euclidean distance. The Euclidean length of a vector $z \in \mathbf{R}^n$ is $(z^T z)^{1/2} = (\sum z_i^2)^{1/2}$, and z is said to be "small" (with respect to this measure) if this nonnegative real number is small. The vectors x and y, furthermore, are "close" if the Euclidean length of the difference $z = x - y$ is a small number.

What may be said about the "size" of matrices, which may be thought of as vectors in a higher-dimensional space? What about vectors in infinite-dimensional spaces? What about complex vectors? Are there useful ways to measure the "size" of real vectors other than by Euclidean length?

One way to answer these questions is to study *norms,* or measures of size, of matrices and vectors. Norms may be thought of as generalizations of Euclidean length, but the study of norms is more than an exercise in mathematical generalization. It is necessary for a proper formulation of notions such as power series of matrices, and it is essential in the analysis and assessment of algorithms for numerical computations. Furthermore, different acceptable norms may be more or less convenient in various situations, so that it is appropriate to study properties common to all norms, rather than to restrict attention to any single norm.

The following examples indicate a few ways in which the need for norms arises.

5.0.1 **Example** (convergence). If x is a complex number such that $|x|<1$, we know that

$$(1-x)^{-1}=1+x+x^2+x^3+\cdots$$

This suggests the formula

$$(I-A)^{-1}=I+A+A^2+A^3+\cdots$$

for calculating the inverse of the square matrix $I-A$, but when is it valid? It turns out that it is sufficient that a matrix norm of A be less than 1, and any such norm will do! Similarly, many other power series which can be used to define matrix-valued functions of a matrix, such as

$$e^A \equiv \sum_{k=0}^{\infty} \frac{1}{k!}A^k$$

can be shown to be convergent and well-defined using norms. Norms may also be useful in determining the number of terms required in a power series in order to calculate a particular function value to a desired degree of accuracy. Similar remarks may be made about the analysis of convergence of iterative schemes to solve systems of equations.

5.0.2 **Example** (accuracy). If f is a real scalar-valued differentiable function of a real variable, we know that if the value of $f(x)$ is known for $x=x_0$, then its value at nearby points $x=x_0+h$ can be estimated in terms of the first derivative

$$\frac{f(x_0+h)-f(x_0)}{h}=\frac{\Delta f}{\Delta x}\cong f'(x_0)$$

Thus, we have a way of estimating the relative error in computing the value of f at x_0 if we actually compute the value of f at a nearby point x_0+h instead.

The same issue arises for matrix calculations. Suppose we wish to compute A^{-1} (or some other function of A), but the entries of A are obtained by experiment, by analysis of other data, or from prior calculation, and they are not known exactly. We may think of A as being composed of the "true" A_0 plus an error E, and we would like to assess the potential "relative error" (in terms of the "size" of E) in computing $A^{-1}=(A_0+E)^{-1}$ instead of the true A_0^{-1}. Bounds for the disparity between A^{-1} and A_0^{-1} may be as important to know as the exact value of A^{-1}, and norms provide a systematic way of dealing with such questions.

5.0.3 **Example** (bounds). Bounds for important quantities associated with a matrix, such as eigenvalues, often involve norms, as do bounds for possible changes in these quantities when a matrix is perturbed.

5.1 Defining properties of vector norms and inner products

We first consider norms on a vector space. Since M_n is a vector space, everything we do will also apply to norms of matrices.

In order to specify properties to be required of a function if it is to be a norm, we abstract from the familiar notion of absolute value of (real or complex) scalars. Of course, a significant difference is that, while the absolute value function is a real-valued function of one real or complex variable, we require a norm to be a real-valued function of the several variables that describe a vector. One such function on $\mathbf{C}^n$ is Euclidean length $(z^*z)^{1/2}$, but there are other functions that share some fundamental properties of Euclidean length and may be more relevant measures in some instances, may impart additional information, or may be more convenient to use in certain contexts.

Throughout this chapter we shall consider real or complex vector spaces only. All of the major results hold for both fields, but within each result one must be consistent as to which field is used. Thus, we shall often state results in terms of a field $\mathbf{F}$ (with $\mathbf{F}=\mathbf{R}$ or $\mathbf{C}$ at the outset) and then refer to the same field $\mathbf{F}$ in the rest of the argument.

5.1.1 **Definition.** Let V be a vector space over a field $\mathbf{F}$ ($\mathbf{R}$ or $\mathbf{C}$). A function $\|\cdot\|: V \to \mathbf{R}$ is a *vector norm* if for all $x, y \in V$,

(1)	$\|x\| \geq 0$	Nonnegative
(1a)	$\|x\| = 0$ if and only if $x = 0$	Positive
(2)	$\|cx\| = \|c\|\,\|x\|$ for all scalars $c \in \mathbf{F}$	Homogeneous
(3)	$\|x+y\| \leq \|x\| + \|y\|$	Triangle inequality

These four axioms are familiar properties of Euclidean length in the plane. Euclidean length possesses other properties that are independent of these four axioms [e.g., the parallelogram identity (5.1.8)], which we do not adopt as axioms because they are not essential to the general theory.

A function that satisfies axioms (1), (2), and (3), but not necessarily (1a) is called a *vector seminorm*. A seminorm generalizes the notion of a norm in that some vectors other than the zero vector are allowed to have zero length.

5.1.2 **Lemma.** If $\|\cdot\|$ is a vector seminorm on V, then

$$|\,\|x\| - \|y\|\,| \leq \|x - y\|$$

for all $x, y \in V$.

Proof: Since $y = x + (y - x)$, we have

$$\|y\| \leq \|x\| + \|y - x\| = \|x\| + \|x - y\|$$

from the triangle inequality (3) and the homogeneity axiom (2). From this it follows that

$$\|y\| - \|x\| \leq \|x - y\|$$

But $x = y + (x - y)$ as well, so we have

$$\|x\| \leq \|y\| + \|x - y\|$$

from the triangle inequality (3) again, and hence

$$\|x\| - \|y\| \leq \|x - y\|$$

Thus, we have shown that $\pm(\|x\| - \|y\|) \leq \|x - y\|$, which is equivalent to the assertion of the lemma. □

Associated with Euclidean length on $\mathbf{C}^n$ is the usual Euclidean inner product y^*x (sometimes called the "dot product"), which has something to do with the "angle" between two vectors: x and y are orthogonal if $y^*x = 0$. Just as for vector norms, one can abstract a few essential characteristics of the Euclidean inner product and use them as axioms for a general theory of inner products.

5.1.3 **Definition.** Let V be a vector space over the field $\mathbf{F}$ ($\mathbf{R}$ or $\mathbf{C}$). A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbf{F}$ is an *inner product* if for all $x, y, z \in V$,

(1)	$\langle x, x \rangle \geq 0$	Nonnegative
(1a)	$\langle x, x \rangle = 0$ if and only if $x = 0$	Positive
(2)	$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$	Additive
(3)	$\langle cx, y \rangle = c\langle x, y \rangle$ for all scalars $c \in \mathbf{F}$	Homogeneous
(4)	$\langle x, y \rangle = \overline{\langle y, x \rangle}$	Hermitian property

Exercise. Show that the Euclidean inner product $\langle x, y \rangle = y^*x$ satisfies all four of the above axioms for an inner product.

Exercise. Let $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ and consider the function $(x, y) \equiv y^*Dx$. Which of the axioms for an inner product does $(\cdot, \cdot)$ satisfy? Under what conditions on D is $(\cdot, \cdot)$ an inner product?

Exercise. Deduce the following properties of an inner product from the four axioms in Definition (5.1.3):

(a) $\langle x, cy\rangle = \bar{c}\langle x, y\rangle$
(b) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$
(c) $\langle ax+by, cw+dz\rangle = a\bar{c}\langle x, w\rangle + b\bar{c}\langle y, w\rangle + a\bar{d}\langle x, z\rangle + b\bar{d}\langle y, z\rangle$
(d) $\langle x, y\rangle = 0$ for all $y \in V$ if and only if $x = 0$
(e) $\langle x, \langle x, y\rangle y\rangle = |\langle x, y\rangle|^2$

An important property shared by all inner products is the Cauchy–Schwarz inequality.

5.1.4 **Theorem** (Cauchy–Schwarz inequality). If $\langle\cdot, \cdot\rangle$ is an inner product on a vector space V over the field $\mathbf{F}$ ($\mathbf{R}$ or $\mathbf{C}$), then

$$|\langle x, y\rangle|^2 \le \langle x, x\rangle\langle y, y\rangle \quad \text{for all} \quad x, y \in V$$

Equality occurs if and only if x and y are linearly dependent, that is, $x = \alpha y$ or $y = \alpha x$ for some $\alpha \in \mathbf{F}$.

Proof: Let $x, y \in V$ be given. If $y = 0$, the assertion is trivial, so we may assume that $y \ne 0$. Let $t \in \mathbf{R}$ and consider $p(t) \equiv \langle x+ty, x+ty\rangle = \langle x, x\rangle + t\langle y, x\rangle + t\langle x, y\rangle + t^2\langle y, y\rangle = \langle x, x\rangle + 2t\,\mathrm{Re}\langle x, y\rangle + t^2\langle y, y\rangle$, which is a real quadratic polynomial with real coefficients. Because of axiom (5.1.3(1)), we know that $p(t) \ge 0$ for all real t, and hence $p(t)$ can have no real simple roots. The discriminant of $p(t)$ must therefore be nonpositive

$$(2\,\mathrm{Re}\langle x, y\rangle)^2 - 4\langle y, y\rangle\langle x, x\rangle \le 0$$

and hence

$$(\mathrm{Re}\langle x, y\rangle)^2 \le \langle x, x\rangle\langle y, y\rangle \tag{5.1.5}$$

Since this inequality must hold for any pair of vectors, it must hold if y is replaced by $\langle x, y\rangle y$, so we also have the inequality

$$(\mathrm{Re}\langle x, \langle x, y\rangle y\rangle)^2 \le \langle x, x\rangle\langle y, y\rangle|\langle x, y\rangle|^2$$

But $\mathrm{Re}\langle x, \langle x, y\rangle y\rangle = \mathrm{Re}\,\overline{\langle x, y\rangle}\langle x, y\rangle = \mathrm{Re}\,|\langle x, y\rangle|^2 = |\langle x, y\rangle|^2$, so

$$|\langle x, y\rangle|^4 \le \langle x, x\rangle\langle y, y\rangle|\langle x, y\rangle|^2 \tag{5.1.6}$$

If $\langle x, y\rangle = 0$, then the statement of the theorem is trivial; if not, then we may divide (5.1.6) by the quantity $|\langle x, y\rangle|^2$ to obtain the desired inequality. Because of axiom (1a), $p(t)$ can have a real (double) root only if $x+ty=0$ for some t. Thus, equality can occur in the discriminant condition (5.1.5) if and only if x and y are linearly dependent. □

5.1.7 **Corollary.** If $\langle\cdot,\cdot\rangle$ is a vector inner product on V, then $\|x\| \equiv (\langle x,x\rangle)^{1/2}$ is a vector norm on V.

Exercise. Prove (5.1.7). *Hint:* The only nontrivial axiom to verify is the triangle inequality. Compute $\|x+y\|^2$ and use the Cauchy–Schwarz inequality.

If $\|\cdot\|$ is a vector norm such that $\|x\| = \langle x,x\rangle^{1/2}$ for some inner product $\langle\cdot,\cdot\rangle$, then we say that the vector norm $\|\cdot\|$ is *derived from an inner product* (namely, from $\langle\cdot,\cdot\rangle$).

Problems

1. Let e_i denote the ith unit coordinate vector in $\mathbf{C}^n$ and suppose that $\|\cdot\|$ is a vector seminorm on $\mathbf{C}^n$. Show that

$$\|x\| \le \sum_{i=1}^{n} |x_i|\,\|e_i\|$$

2. If $\|\cdot\|$ is a vector seminorm on V, show that $V_0 = \{v \in V\colon \|v\| = 0\}$ is a subspace of V (called the *null space* of $\|\cdot\|$). (a) If V_1 is any subspace of V such that $V_0 \cap V_1 = \{0\}$, show that $\|\cdot\|$ is a vector norm on V_1. (b) Consider the relation $x \sim y$ defined by

$$x \sim y \quad \text{if and only if} \quad \|x-y\| = 0$$

Show that $\sim$ is an equivalence relation on V, that the cosets of this equivalence relation are of the form $\hat{x} = \{x+y \in V\colon y \in V_0\}$, and that the set of these cosets forms a vector space in a natural way. Show that the function $\|\hat{x}\| \equiv \{\|x\|\colon x \in \hat{x}\}$ is well defined and is a vector norm on the vector space of cosets. (c) Explain why there is a natural norm associated with every vector seminorm. (d) Is $\|x\| \equiv 0$ a seminorm? (e) Give an example of a nontrivial seminorm that is not a norm.

3. Show that if we define the "angle" between the nonzero vectors x and y to be the value of

$$\cos^{-1}\left(\frac{|\langle x,y\rangle|}{(\langle x,x\rangle\langle y,y\rangle)^{1/2}}\right)$$

that lies between 0 and $\pi/2$, then this notion of angle is well defined for any inner product.

4. Show that any vector norm derived from an inner product as in (5.1.7) must satisfy the *parallelogram identity*

$$\tfrac{1}{2}(\|x+y\|^2+\|x-y\|^2)=\|x\|^2+\|y\|^2 \tag{5.1.8}$$

Why is this identity so named? The equation (5.1.8) is, in fact, necessary and sufficient that a given norm $\|\cdot\|$ be derived from an inner product. See Problem 10.

5. Consider the function $\|x\|_\infty \equiv \max_{1\le i\le n}|x_i|$ defined on $\mathbf{C}^n$. Show that $\|\cdot\|_\infty$ is a vector norm that cannot be derived from an inner product.

6. If $\|\cdot\|$ is a vector norm derived from an inner product $\langle\cdot,\cdot\rangle$, show that

$$\mathrm{Re}\langle x,y\rangle=\tfrac{1}{4}(\|x+y\|^2-\|x-y\|^2) \tag{5.1.9}$$

This is known as the *polarization identity*. Show also that

$$\mathrm{Re}\langle x,y\rangle=\tfrac{1}{2}(\|x+y\|^2-\|x\|^2-\|y\|^2)$$

7. Show that the l_1 norm $\|x\|\equiv|x_1|+\cdots+|x_n|$ on $\mathbf{C}^n$ satisfies the axioms (5.1.1) but does not obey the polarization identity (5.1.9). It is not, therefore, derived from any inner product.

8. If $\|\cdot\|$ is a vector norm on V derived from an inner product, then

$$\|x+y\|\,\|x-y\|\le\|x\|^2+\|y\|^2$$

for all $x, y\in V$. When does equality hold? Does this inequality hold for all vector norms? Give a geometric interpretation of this inequality.

9. Let x and y be given vectors in V, which has a norm $\|\cdot\|$ derived from an inner product $\langle\cdot,\cdot\rangle$, and suppose that y is nonzero. Show that the scalar α_0 that minimizes the value of $\|x-\alpha y\|$ is $\alpha_0=\langle x,y\rangle/\|y\|^2$, and that $x-\alpha_0 y$ and y are othogonal.

10. It is not difficult to show that the parallelogram identity (5.1.8) is a sufficient condition for a given norm to be derived from an inner product, but some ingenuity is required. First consider the case of a vector space V over $\mathbf{R}$. Let $\|\cdot\|$ be a given norm on V. (a) Define

$$\langle x,y\rangle=\frac{\|x+y\|^2-\|x\|^2-\|y\|^2}{2} \tag{5.1.10}$$

Show that $\langle\cdot,\cdot\rangle$ defined in this way satisfies axioms (1), (1a), and (4) in (5.1.3) and that $\langle x,x\rangle=\|x\|^2$. (b) Use (5.1.8) to show that

$$\begin{aligned}4\langle x,y\rangle+4\langle z,y\rangle&=2\|x+y\|^2+2\|z+y\|^2-2\|x\|^2-2\|z\|^2-4\|y\|^2\\&=\|x+2y+z\|^2-\|x+z\|^2-4\|y\|^2=4\langle x+z,y\rangle\end{aligned}$$

and conclude that the additivity axiom (2) in (5.1.3) is satisfied. (c) Use the additivity axiom to show that $\langle nx,y\rangle=n\langle x,y\rangle$ and $m\langle m^{-1}nx,y\rangle=$

$\langle nx, y\rangle = n\langle x, y\rangle$ whenever m and n are nonnegative integers. Use (5.1.8) and (5.1.10) to show that $\langle -x, y\rangle = -\langle x, y\rangle$ and conclude that $\langle ax, y\rangle = a\langle x, y\rangle$ whenever $a \in \mathbf{R}$ is rational. (d) Let $p(t) = t^2\|x\|^2 + 2t\langle x, y\rangle + \|y\|^2$, $t \in \mathbf{R}$, and show that $p(t) = \|tx+y\|^2$ if t is rational. Conclude from the continuity of $p(t)$ that $p(t) \geq 0$ for all $t \in \mathbf{R}$. Deduce the Cauchy–Schwarz inequality $|\langle x, y\rangle|^2 \leq \|x\|^2\|y\|^2$ from the fact that the discriminant of $p(t)$ must be nonpositive. (e) Now let $a \in \mathbf{R}$ be given. Show that

$$\begin{aligned}|\langle ax, y\rangle - a\langle x, y\rangle| &= |\langle (a-b)x, y\rangle + (b-a)\langle x, y\rangle|\\ &\leq |\langle (a-b)x, y\rangle| + |(b-a)\langle x, y\rangle| \leq 2|a-b|\,\|x\|\,\|y\|\end{aligned}$$

for any rational b, and observe that the upper bound can be made arbitrarily small. Conclude that the homogeneity axiom (3) in (5.1.3) is satisfied. This shows that $\langle \cdot, \cdot \rangle$ is an inner product on V.

A careful reader will observe that the triangle inequality for the norm $\|\cdot\|$ [axiom (3) in (5.1.1)] is not used in this argument. Thus, the axioms (1), (1a), and (2) in (5.1.1) together with (5.1.8) imply that the function $\|\cdot\|$ is derived from an inner product, is therefore a norm, and hence must satisfy the triangle inequality. (f) If V is a complex vector space, define

$$\langle x, y\rangle = \frac{\|x+y\|^2 - \|x\|^2 - \|y\|^2}{2} + \frac{i(\|x+iy\|^2 - \|x\|^2 - \|y\|^2)}{2}$$

The real part of $\langle x, y\rangle$ is an inner product of V considered as a vector space over $\mathbf{R}$. Use this fact and (5.1.8) to show that $\langle \cdot, \cdot \rangle$ is an inner product for V as a vector space over $\mathbf{C}$.

Further Reading. The first proof that that parallelogram identity is both necessary and sufficient for a given vector norm to be derived from an inner product seems to be due to P. Jordan and J. Von Neumann, "On Inner Products in Linear Metric Spaces," *Ann. Math.* 36(2) (1935), 719–723. The outline of a proof of this result given in Problem 10 follows D. Fearnley-Sander and J. S. V. Symons, "Apollonius and Inner Products," *Amer. Math. Monthly* 81 (1974), 990–993.

5.2 Examples of vector norms

The following are some examples of frequently encountered vector norms.

5.2.1 The *Euclidean norm* (or l_2 norm) on $\mathbf{C}^n$ is

$$\|x\|_2 \equiv (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$$

This is perhaps the best known vector norm since $\|x-y\|_2$ measures the standard Euclidean distance between two points $x, y \in \mathbf{C}^n$. This norm is also derived from the usual Euclidean inner product; that is, $\|x\|_2^2 = \langle x, x\rangle = x^*x$.

Exercise. Verify that $\|\bullet\|_2$ is a vector norm on $\mathbf{C}^n$.

Exercise. A norm $\|\bullet\|$ is said to be *unitarily invariant* if $\|Ux\| = \|x\|$ for all $x \in \mathbf{C}^n$ and all unitary matrices $U \in M_n$. Show that the Euclidean norm $\|\bullet\|_2$ is unitarily invariant.

5.2.2 The *sum norm* (or l_1 norm) on $\mathbf{C}^n$ is

$$\|x\|_1 \equiv |x_1| + \cdots + |x_n|$$

This norm is also called the one-norm or, more picturesquely, the *Manhattan norm* because of the rectilinear measurement of length in coordinate directions only.

Exercise. Verify that the sum norm is a vector norm on $\mathbf{C}^n$, but that it is not derived from an inner product. *Hint:* Use (5.1.8).

5.2.3 The *max norm* (or l_∞ norm) on $\mathbf{C}^n$ is

$$\|x\|_\infty \equiv \max\{|x_1|, \ldots, |x_n|\}$$

Exercise. Verify that $\|\bullet\|_\infty$ is a vector norm on $\mathbf{C}^n$.

Exercise. Is $\|\bullet\|_\infty$ derived from an inner product?

5.2.4 The l_p *norm* on $\mathbf{C}^n$ is

$$\|x\|_p \equiv \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

for $p \geq 1$.

Exercise. Verify that each l_p norm for $p \geq 1$ is a vector norm on $\mathbf{C}^n$ and that $\|x\|_\infty = \lim_{p\to\infty} \|x\|_p$ for each $x \in \mathbf{C}^n$. *Hint:* The triangle inequality is the only nontrivial axiom to verify. The triangle inequality for the l_p norms is a classical inequality known as *Minkowski's inequality*.

Exercise. Give an example of a vector norm that is not an l_p norm.

The foregoing examples of vector norms have all been norms on $\mathbf{C}^n$, but they can be used to create vector norms on any finite-dimensional real or complex vector space V. If $\mathcal{B} = \{b^{(1)}, \ldots, b^{(n)}\}$ is a basis for V, then recall that

$$x \to [x]_{\mathcal{B}} \equiv \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbf{C}^n, \qquad x = \sum_{i=1}^{n} x_i b^{(i)}$$

is an isomorphism of V onto $\mathbf{C}^n$. If $\|\cdot\|$ is any vector norm on $\mathbf{C}^n$, then

$$\|x\|_{\mathcal{B}} \equiv \|[x]_{\mathcal{B}}\| = \|[x_1, \ldots, x_n]^T\|, \qquad x = \sum_{i=1}^{n} x_i b^{(i)}$$

is easily shown to be a vector norm on V.

Exercise. Verify the last assertion.

A matrix $B \in M_n$ is said to be an *isometry for the vector norm* $\|\cdot\|$ on $\mathbf{C}^n$ if

$$\|Bx\| = \|x\| \quad \text{for all} \quad x \in \mathbf{C}^n$$

Exercise. Show that an isometry for any vector norm must be a nonsingular matrix.

Exercise. Show that the set of isometries for a given norm forms a group (known as the *isometry group* of the norm). Are there any isometries for $\|\cdot\|_2$ besides the unitary matrices?

Exercise. Show that the isometry group of the sum norm is the set (group) of all matrices that look like permutation matrices except that the "+1" entries are replaced by arbitrary complex numbers with absolute value 1.

Exercise. What is the isometry group of the max norm?

The definition of a vector norm does not require that the vector space V be finite-dimensional. The space V might, for example, be the vector space $C[a, b]$ of all continuous real- or complex-valued functions on the real interval $[a, b]$.

5.2.5 **Example.** Some examples of norms on $C[a, b]$ are similar to norms already defined for C^n. For example,

$$\|f\|_2 \equiv \left[\int_a^b |f(t)|^2 \, dt\right]^{1/2} \qquad L_2 \text{ norm}$$

$$\|f\|_1 \equiv \int_a^b |f(t)| \, dt \qquad L_1 \text{ norm}$$

$$\|f\|_p \equiv \left[\int_a^b |f(t)|^p\,dt\right]^{1/p}, \qquad p \geq 1 \qquad L_p \text{ norm}$$

$$\|f\|_\infty \equiv \max\{|f(x)| : x \in [a, b]\} \qquad L_\infty \text{ norm}$$

are all norms on $C[a, b]$.

Ignore Problems

Problems

1. Show that if $0 < p < 1$, then (5.2.4) defines a function on $\mathbf{C}^n$ that satisfies all but one of the axioms for a vector norm. Which one fails? Give an example.

2. Let $f \in C[0, 1]$. Show that $\|f\|_\infty = \lim_{p \to \infty} \|f\|_p$.

3. What does the triangle inequality look like for $\|\cdot\|_p$ on $C[0, 1]$? How could you prove it starting from Minkowski's inequality (Appendix B) for $\mathbf{C}^n$?

4. Let $p_1, p_2, \ldots, p_n$ be given positive real numbers. Which of the following is a vector norm on $\mathbf{C}^n$?

(a) $\|x\| = \sum_{i=1}^{n} p_i |x_i|$

(b) $\|x\| = \left(\sum_{i=1}^{n} p_i |x_i|^2\right)^{1/2}$

(c) $\|x\| = \max\{p_1|x_1|, \ldots, p_n|x_n|\}$

5. Let $x_0 \in [a, b]$ be a given point. Show that the function $\|f\|_{x_0} \equiv |f(x_0)|$ on $C[a, b]$ is a seminorm that is not a norm if $a < b$.

6. If $\|\cdot\|$ is an unitarily invariant vector norm on $\mathbf{C}^n$, show that $\|\cdot\| = \alpha\|\cdot\|_2$ for some $\alpha > 0$ and that $\|\cdot\|_2$ is the only unitarily invariant vector norm for which $\|e_1\| = 1$.

7. Show that $\|y\|_\infty = \max_{\|x\|_1 = 1} |y^*x|$ and that $\|x\|_1 = \max_{\|y\|_\infty = 1} |x^*y|$.

8. Use the preceding exercise to show that if A^* is in the isometry group of the sum norm, then A is in the isometry group of the max norm, and vice versa.

9. What is the intersection of all the isometry groups of all the l_p norms?

Further Readings. For a detailed discussion of the classical inequalities of Minkowski and Hölder, see [BB].

two points, each of which is in the metric convex hull of some pair of points of S. Show that this agrees with the above definition when $k=2$ and describe the l_1 convex hull of the set of unit orthonormal basis vectors $\{e_1, e_2, \ldots, e_n\}$ in $\mathbf{R}^n$. What is the l_2 convex hull of this set? What is the ordinary linear algebraic convex hull of this set?

Further Readings. See [Hou 64] for more discussion of geometrical aspects of vector norms. The key idea for the proof of the duality theorem (the identification of the unit ball of the second dual of a norm or pre-norm with the intersection of all the half-spaces containing the unit ball of the norm or pre-norm) is used by Von Neumann in the paper cited at the end of Section (5.4). See [Val] for a detailed discussion of convex sets, convex hulls, half-spaces, and so forth.

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5.6 Matrix norms

Since M_n is itself a vector space of dimension n^2, one can measure the "size" of a matrix by using any vector norm on $\mathbf{C}^{n^2}$. However, M_n is not just a high-dimensional vector space; it has a natural multiplication operation, and it is often useful in making estimates to relate the "size" of AB to the "sizes" of A and B.

We call a function $|||\cdot|||: M_n \to \mathbf{R}$ a *matrix norm* if for all $A, B \in M_n$ it satisfies the following five axioms:

(1)	$|||A||| \geq 0$	Nonnegative
(1a)	$|||A||| = 0$ if and only if $A=0$	Positive
(2)	$|||cA||| = |c|\, |||A|||$ for all complex scalars c	Homogeneous
(3)	$|||A+B||| \leq |||A||| + |||B|||$	Triangle inequality
(4)	$|||AB||| \leq |||A|||\, |||B|||$	Submultiplicative

Notice that properties (1)–(3) are identical to the axioms for a vector norm (5.1.1). A vector norm on matrices, that is, a function that satisfies (1)–(3) and not necessarily (4), is often called a *generalized matrix norm.* The notions of a matrix seminorm and a generalized matrix seminorm may also be defined via omission of axiom (1a).

Since $|||A^2||| = |||AA||| \leq |||A|||\, |||A||| = |||A|||^2$ for any matrix norm, it must be that $|||A||| \geq 1$ for any nonzero matrix A for which $A^2 = A$. In particular, $|||I||| \geq 1$ for any matrix norm. If A is invertible, then $I = AA^{-1}$, so $|||I||| = |||AA^{-1}||| \leq |||A|||\, |||A^{-1}|||$, and we have the lower bound

$$|||A^{-1}||| \geq \frac{|||I|||}{|||A|||}$$

for any matrix norm $|||\cdot|||$.

Exercise. Show that if $\|\cdot\|$ is a matrix norm, then $\|A^k\| \geq \|A\|^k$ for every $k=1,2,\ldots$, and all $A \in M_n$. Give an example of a vector norm on matrices for which this inequality is not true.

Some of the vector norms introduced in (5.2) are matrix norms when applied to the vector space M_n and some are not. The most familiar examples are the l_p norms for $p=1,2,\infty$. They are already known to be vector norms, so one needs to verify only axiom (4).

Example. The l_1 *norm* defined for $A \in M_n$ by

$$\|A\|_1 \equiv \sum_{i,j=1}^{n} |a_{ij}|$$

is a matrix norm because

$$\begin{aligned}\|AB\|_1 &= \sum_{i,j=1}^{n} \left| \sum_{k=1}^{n} a_{ik} b_{kj} \right| \leq \sum_{i,j,k=1}^{n} |a_{ik} b_{kj}| \\ &\leq \sum_{i,j,k,m=1}^{n} |a_{ik} b_{mj}| = \left(\sum_{i,k=1}^{n} |a_{ik}| \right) \left(\sum_{j,m=1}^{n} |b_{mj}| \right) \\ &= \|A\|_1 \|B\|_1\end{aligned}$$

The first inequality comes from the triangle inequality, while the second comes from adding additional terms to the sum.

Example. The *Euclidean norm* or l_2 *norm* defined for $A \in M_n$ by

$$\|A\|_2 \equiv \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

is a matrix norm because

$$\begin{aligned}\|AB\|_2^2 &= \sum_{i,j=1}^{n} \left| \sum_{k=1}^{n} a_{ik} b_{kj} \right|^2 \leq \sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} |a_{ik}|^2 \right) \left(\sum_{m=1}^{n} |b_{mj}|^2 \right) \\ &= \left(\sum_{i,k=1}^{n} |a_{ik}|^2 \right) \left(\sum_{m,j=1}^{n} |b_{mj}|^2 \right) = \|A\|_2^2 \|B\|_2^2\end{aligned}$$

This inequality is just the Cauchy-Schwarz inequality. When applied to matrices, this norm is sometimes called the *Frobenius norm,* the *Schur norm,* or the *Hilbert-Schmidt norm.* Notice that if $A = [a_1\, a_2 \cdots a_n] \in M_n$ is written in terms of its column vectors $a_i \in \mathbf{C}^n$, then

$$\|A\|_2^2 = \|a_1\|_2^2 + \cdots + \|a_n\|_2^2$$

Since the l_2 norm on $\mathbf{C}^n$ is unitarily invariant, we have the important fact that

$$\|UA\|_2^2 = \|Ua_1\|_2^2 + \cdots + \|Ua_n\|_2^2 = \|a_1\|_2^2 + \cdots + \|a_n\|_2^2 = \|A\|_2^2$$

whenever $U \in M_n$ is unitary. Since $\|B^*\|_2 = \|B\|_2$ for all $B \in M_n$, this implies that

$$\|UAV\|_2 = \|AV\|_2 = \|V^*A^*\|_2 = \|A^*\|_2 = \|A\|_2$$

whenever $U, V \in M_n$ are unitary. Thus, the l_2 norm on M_n is a unitarily invariant matrix norm.

Example. The l_∞ *norm* defined for $A \in M_n$ by

$$\|A\|_\infty \equiv \max_{1 \le i, j \le n} |a_{ij}|$$

is a norm on the vector space M_n but is not a matrix norm. Consider the matrix $J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in M_2$ and compute $J^2 = 2J$, $\|J\|_\infty = 1$, $\|J^2\|_\infty = \|2J\|_\infty = 2\|J\|_\infty = 2$. It is not the case that $\|J^2\|_\infty \le \|J\|_\infty^2$, and hence $\|\cdot\|_\infty$ is not a submultiplicative norm. However, if we define

$$|||A||| \equiv n\|A\|_\infty, \qquad A \in M_n$$

then we have

$$\begin{aligned} |||AB||| &= n \max_{1 \le i, j \le n} \left| \sum_{k=1}^{n} a_{ik} b_{kj} \right| \le n \max_{1 \le i, j \le n} \sum_{k=1}^{n} |a_{ik} b_{kj}| \\ &\le n \max_{1 \le i, j \le n} \sum_{k=1}^{n} \|A\|_\infty \|B\|_\infty = n\|A\|_\infty n\|B\|_\infty \\ &= |||A|||\ |||B||| \end{aligned}$$

Thus, only a minor modification of the vector norm $\|\cdot\|_\infty$ is required to make it a matrix norm.

Associated with each vector norm $\|\cdot\|$ on $\mathbf{C}^n$ is a natural matrix norm $|||\cdot|||$ that is "induced" by $\|\cdot\|$ on M_n. The norm $|||\cdot|||$ is constructed from $\|\cdot\|$, and this construction adds to the list of methods for producing one norm from another.

5.6.1 **Definition.** Let $\|\cdot\|$ be a vector norm on $\mathbf{C}^n$. Define $|||\cdot|||$ on M_n by

$$|||A||| \equiv \max_{\|x\| = 1} \|Ax\|$$

The "max" in the above definition (rather than "sup") is justified since $\|Ax\|$ is a continuous function of x and the unit ball $B_{\|\cdot\|}$ is a compact set (see Appendix E).

Exercise. Show that the norm (5.6.1) may also be computed in the following equivalent ways:

$$\begin{aligned}\|A\| &= \max_{\|x\|=1} \|Ax\| \\ &= \max_{\|x\|\le 1} \|Ax\| \\ &= \max_{x\ne 0} \frac{\|Ax\|}{\|x\|} \\ &= \max_{\|x\|_\alpha=1} \frac{\|Ax\|}{\|x\|}, \qquad \text{where } \|\bullet\|_\alpha \text{ is any vector norm}\end{aligned}$$

5.6.2 **Theorem.** The function $\|\bullet\|$ defined in (5.6.1) is a matrix norm on M_n, $\|Ax\| \le \|A\|\|x\|$ for all $A \in M_n$ and all $x \in \mathbf{C}^n$, and $\|I\| = 1$.

Proof: Axiom (1) at the beginning of this section follows from the fact that $\|A\|$ is the maximum of a nonnegative valued function, and (1a) follows from the fact that $Ax = 0$ for all x precisely when $A = 0$. Axiom (2) follows from the calculation

$$\|cA\| = \max\|cAx\| = \max|c|\,\|Ax\| = |c| \max\|Ax\| = |c|\,\|A\|$$

Similarly, the triangle inequality (3) is inherited, since

$$\begin{aligned}\|A+B\| &= \max\|(A+B)x\| = \max\|Ax+Bx\| \le \max(\|Ax\|+\|Bx\|) \\ &\le \max\|Ax\| + \max\|Bx\| = \|A\| + \|B\|\end{aligned}$$

The submultiplicative axiom (4) follows from the fact that

$$\begin{aligned}\|AB\| &= \max\frac{\|ABx\|}{\|x\|} = \max\frac{\|ABx\|}{\|Bx\|}\frac{\|Bx\|}{\|x\|} \\ &\le \max\frac{\|Ay\|}{\|y\|}\max\frac{\|Bx\|}{\|x\|} = \|A\|\,\|B\|\end{aligned}$$

where we assume, without loss of generality, that the maximum is taken over only those x that are not in the null space of B. For the next assertion, we observe that if $x \ne 0$, then $\|Ax/\|x\|\| \le \|A\|$ because of the definition of this norm as a maximum. By homogeneity of the vector norm we obtain $\|Ax\| \le \|A\|\,\|x\|$, which also holds when $x = 0$. Finally,

$$\|I\| = \max_{\|x\|=1}\|Ix\| = \max_{\|x\|=1}\|x\| = 1 \qquad \square$$

5.6.3 **Definition.** We say that the matrix norm $|||\cdot|||$ defined in (5.6.1) is the matrix norm *induced* by the vector norm $\|\cdot\|$. It is sometimes called the *operator norm* or *lub* (least upper bound) norm associated with the vector norm $\|\cdot\|$.

Notice that the operator norm is a matrix norm as a consequence of general properties of all vector norms. Therefore, one way to prove that a certain function on M_n is a matrix norm is to show that it is induced by some vector norm. We shall adopt this strategy when we discuss an important matrix norm called the spectral norm.

The inequality in the statement of Theorem (5.6.2) says that the vector norm $\|\cdot\|$ is *compatible* with the induced matrix norm $|||\cdot|||$, and this theorem shows that associated with any vector norm on $\mathbf{C}^n$ there is a compatible matrix norm on M_n. The theorem also gives the necessary condition $|||I|||=1$ for a matrix norm $|||\cdot|||$ to be induced by some vector norm; unfortunately, this necessary condition is not also sufficient.

We next note several important examples of matrix norms that are induced by familiar l_p norms but can also be calculated independent of the definition (5.6.1). In each case, we take $A=[a_{ij}]\in M_n$.

5.6.4 The *maximum column sum matrix norm* $|||\cdot|||_1$ is defined on M_n by

$$|||A|||_1 \equiv \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$$

The norm $|||\cdot|||_1$ is induced by the l_1 vector norm and hence must be a matrix norm. One can show this as follows. Write $A\in M_n$ in terms of its columns as $A=[a_1 \cdots a_n]$. Then $|||A|||_1=\max_{1\leq i\leq n}\|a_i\|_1$. If $x=[x_i]$, then

$$\begin{aligned}\|Ax\|_1 &= \|x_1a_1+\cdots+x_na_n\|_1 \leq \sum_{i=1}^{n}\|x_ia_i\|_1 = \sum_{i=1}^{n}|x_i|\,\|a_i\|_1\\ &\leq \sum_{i=1}^{n}|x_i|\left(\max_{1\leq k\leq n}\|a_k\|_1\right) = \sum_{i=1}^{n}|x_i|\,|||A|||_1\\ &= \|x\|_1\,|||A|||_1\end{aligned}$$

Thus, $\max_{\|x\|_1=1}\|Ax\|_1\leq |||A|||_1$. If we now choose $x=e_k$ (the kth unit basis vector), then for any $k=1,2,\ldots,n$ we have

$$\max_{\|x\|_1=1}\|Ax\|_1 \geq \|1\,a_k\|_1=\|a_k\|_1$$

and hence

$$\max_{\|x\|_1=1}\|Ax\|_1 \geq \max_{1\leq k\leq n}\|a_k\|_1 = |||A|||_1$$

Since we have now proved that the matrix norm induced by the l_1 vector norm is both an upper bound and a lower bound on $\|A\|_1$, we are done.

Exercise. Prove directly from the definition that $\|\cdot\|_1$ is a matrix norm.

5.6.5 The *maximum row sum matrix norm* $\|\cdot\|_\infty$ is defined on M_n by

$$\|A\|_\infty \equiv \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

The norm $\|\cdot\|_\infty$ is induced by the l_∞ vector norm and hence must be a matrix norm. The argument is similar to the proof for the maximum column sum norm. We compute

$$\|Ax\|_\infty = \max_{1\le i\le n} \left|\sum_{j=1}^{n} a_{ij}x_j\right| \le \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}x_j| \le \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|\,\|x\|_\infty$$
$$= \|A\|_\infty \|x\|_\infty$$

and hence $\max_{\|x\|_\infty=1}\|Ax\|_\infty \le \|A\|_\infty$. If $A=0$ there is nothing to prove, so we may assume that $A\neq 0$. Suppose the kth row of A is nonzero and define the vector $z=[z_i]\in \mathbf{C}^n$ by

$$z_i = \frac{\bar{a}_{ki}}{|a_{ki}|} \quad \text{if} \quad a_{ki}\neq 0$$
$$z_i = 1 \quad \text{if} \quad a_{ki}=0$$

Then $\|z\|_\infty=1$, $a_{kj}z_j=|a_{kj}|$ for all $j=1,2,\ldots,n$, and

$$\max_{\|x\|_\infty=1} \|Ax\|_\infty \ge \|Az\|_\infty = \max_{1\le i\le n} \left|\sum_{j=1}^{n} a_{ij}z_j\right| \ge \left|\sum_{j=1}^{n} a_{kj}z_j\right| = \sum_{j=1}^{n} |a_{kj}|$$

Thus,

$$\max_{\|x\|_\infty=1} \|Ax\|_\infty \ge \max_{1\le k\le n} \sum_{j=1}^{n} |a_{kj}| = \|A\|_\infty$$

and we are done.

Exercise. Verify directly from the definition that $\|\cdot\|_\infty$ is a matrix norm on M_n.

5.6.6 The *spectral norm* $\|\cdot\|_2$ is defined on M_n by

$$\|A\|_2 \equiv \max\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } A^*A\}$$

Notice that if $A^*Ax=\lambda x$ and $x\neq 0$, then $x^*A^*Ax=\|Ax\|_2^2=\lambda\|x\|_2^2$, so $\lambda\ge 0$ and $\sqrt{\lambda}$ is real and nonnegative.

Exercise. If B is a normal matrix and $B = U^*\Lambda U$ with U unitary and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, show that

$$|x^*Bx| \leq \max\{|\lambda| : \lambda \text{ is an eigenvalue of } B\} \|x\|_2^2$$

Exercise. Show that $\|Ax\|_2^2 = x^*A^*Ax$ for all $x \in \mathbf{C}^n$ and use the previous exercise to show that $|||\cdot|||_2$ is the matrix norm induced by the Euclidean vector norm $\|\cdot\|_2$. Conclude from this that the spectral norm is in fact a matrix norm.

Exercise. Show that $|||UAV|||_2 = |||A|||_2$ for any $A \in M_n$ and any unitary matrices $U, V \in M_n$. Thus, the spectral norm is a unitarily invariant matrix norm.

We next show that one matrix norm may be transformed into another by a fixed similarity.

5.6.7 **Theorem.** If $|||\cdot|||$ is a matrix norm on M_n and if $S \in M_n$ is nonsingular, then

$$|||A|||_S \equiv |||S^{-1}AS||| \quad \text{for all} \quad A \in M_n$$

is a matrix norm.

Proof: The axioms (1), (1a), (2), and (3) are verified in a straightforward manner for $|||\cdot|||_S$. The submultiplicativity of $|||\cdot|||_S$ follows from the calculation

$$\begin{aligned}|||AB|||_S &= |||S^{-1}ABS||| = |||(S^{-1}AS)(S^{-1}BS)||| \leq |||S^{-1}AS|||\ |||S^{-1}BS||| \\ &= |||A|||_S |||B|||_S\end{aligned}$$ □

Theorem (5.6.7) can be of great use in tailoring a matrix norm for a specific purpose. Some applications of this type are developed here and in the following section.

One important area of application of matrix norms is in giving bounds for the spectrum of a matrix.

5.6.8 **Definition.** The *spectral radius* $\rho(A)$ of a matrix $A \in M_n$ is

$$\rho(A) \equiv \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$$

Observe that if λ is any eigenvalue of A, then $|\lambda| \leq \rho(A)$; moreover, there is at least one eigenvalue λ for which $|\lambda| = \rho(A)$. If $Ax = \lambda x$, $x \neq 0$, and if $|\lambda| = \rho(A)$, consider the matrix $X \in M_n$ all the columns of which are equal to the eigenvector x, and observe that $AX = \lambda X$. If $|||\cdot|||$ is any matrix norm,

$$|\lambda|\ \|X\| = \|\lambda X\| = \|AX\| \leq \|A\|\ \|X\|$$

and therefore $|\lambda| = \rho(A) \leq \|A\|$. This is a proof of the following theorem.

5.6.9 **Theorem.** If $\|\cdot\|$ is any matrix norm and if $A \in M_n$, then $\rho(A) \leq \|A\|$.

Exercise. Give an example of a vector norm $\|\cdot\|$ on matrices and a matrix $A \in M_n$ such that $\|A\| < \rho(A)$.

Exercise. Let $\|\cdot\|$ be a matrix norm on M_n, and consider the mapping $F: \mathbf{C}^n \to M_n$ defined by $F(x) = [x\ x \ldots x] =$ the matrix in M_n all of whose columns are just x. Show that the function $\|\cdot\|$ defined on $\mathbf{C}^n$ by $\|x\| \equiv \|F(x)\|$ is a norm on $\mathbf{C}^n$ and that $\|Ax\| \leq \|A\|\ \|x\|$ for all $x \in \mathbf{C}^n$ and all $A \in M_n$. This inequality says that the vector norm $\|\cdot\|$ is *compatible* with the matrix norm $\|\cdot\|$, and this exercise shows that any matrix norm on M_n has a compatible vector norm on $\mathbf{C}^n$.

Although the spectral radius function is not itself a matrix or vector norm on M_n (see Problem 19), for each fixed $A \in M_n$, it is the greatest lower bound for the values of all matrix norms of A.

5.6.10 **Lemma.** Let $A \in M_n$ and $\epsilon > 0$ be given. There is a matrix norm $\|\cdot\|$ such that $\rho(A) \leq \|A\| \leq \rho(A) + \epsilon$.

Proof: By the Schur triangularization theorem (2.3.1), there is a unitary matrix U and an upper triangular matrix Δ such that $A = U^*\Delta U$. Set $D_t \equiv \operatorname{diag}(t, t^2, t^3, \ldots, t^n)$ and compute

$$D_t \Delta D_t^{-1} = \begin{bmatrix} \lambda_1 & t^{-1}d_{12} & t^{-2}d_{13} & \ldots & t^{-n+1}d_{1n} \\ 0 & \lambda_2 & t^{-1}d_{23} & \ldots & t^{-n+2}d_{2n} \\ 0 & 0 & \lambda_3 & \ldots & t^{-n+3}d_{3n} \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & 0 & \ldots & t^{-1}d_{n-1,n} \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

Thus, for $t > 0$ large enough, we can be certain that the sum of all the absolute values of the off-diagonal entries of $D_t \Delta D_t^{-1}$ is less than ϵ. In particular, we can be sure that $\|D_t \Delta D_t^{-1}\|_1 \leq \rho(A) + \epsilon$ for large enough t. Thus, if we define the matrix norm $\|\cdot\|$ by

$$\|B\| \equiv \|D_t U^* B U D_t^{-1}\|_1 = \|(U D_t^{-1})^{-1} B (U D_t^{-1})\|_1$$

for any $B \in M_n$, and if we choose t large enough, then we will have constructed a matrix norm such that $\|A\| \le \rho(A)+\epsilon$. Since $\|A\| \ge \rho(A)$ for any matrix norm, we are done. □

Exercise. Explain why the preceding results show that $\rho(A) = \inf\{\|A\| : \|\cdot\| \text{ is a matrix norm}\}$.

We are interested in characterizing matrices A such that $A^k \to 0$ as $k \to \infty$. The following result is the last tool we need to attack this problem.

5.6.11 **Lemma.** Let $A \in M_n$ be a given matrix. If there is a matrix norm $\|\cdot\|$ such that $\|A\| < 1$, then $\lim_{k\to\infty} A^k = 0$; that is, all the entries of A^k tend to zero as $k \to \infty$.

Proof: If $\|A\| < 1$, then $\|A^k\| \le \|A\|^k \to 0$ as $k \to \infty$. This says that $A^k \to 0$ with respect to the norm $\|\cdot\|$, but since all vector norms on the n^2 dimensional space M_n are equivalent, it must also be the case that $A^k \to 0$ with respect to the vector norm $\|\cdot\|_\infty$. □

Exercise. Give an example of a matrix A and two matrix norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ such that $\|A\|_\alpha < 1$ and $\|A\|_\beta > 1$. Conclusion?

Matrices $A \in M_n$ such that $\lim_{k\to\infty} A^k = 0$ are called *convergent* and are important in many applications, for example, in the analysis of iterative processes. In is therefore important to be able to characterize convergent matrices.

5.6.12 **Theorem.** Let $A \in M_n$. Then $\lim_{k\to\infty} A^k = 0$ if and only if $\rho(A) < 1$.

Proof: If $A^k \to 0$ and if $x \ne 0$ is a vector such that $Ax = \lambda x$, then $A^k x = \lambda^k x \to 0$ only if $|\lambda| < 1$. Since this inequality must hold for every eigenvalue of A, we conclude that $\rho(A) < 1$. Conversely, if $\rho(A) < 1$, then by Lemma (5.6.10) there is some matrix norm $\|\cdot\|$ such that $\|A\| < 1$. Thus, $A^k \to 0$ as $k \to \infty$ by Lemma (5.6.11). □

Exercise. Consider the matrix $A = \begin{bmatrix} 1/2 & 1 \\ 0 & 1/2 \end{bmatrix} \in M_2$. Compute A^k and $\rho(A^k)$ explicitly for $k = 2, 3, \ldots$. Show that $\rho(A^k) = [\rho(A)]^k$. How do the following behave as $k \to \infty$? The entries of A^k; $\|A^k\|_1$; $\|A^k\|_\infty$; $\|A^k\|_2$.

Exercise. Let $A=\begin{bmatrix} 1/2 & 1 \\ -1/8 & 1/2 \end{bmatrix}$, and define a sequence of vectors $\{x^{(k)}\}\in \mathbf{C}^2$ by the recursion $x^{(k+1)}=Ax^{(k)}$, $k=0,1,\dots$. Show that, regardless of the initial vector $x^{(0)}$ chosen, $x^{(k)}\to 0$ as $k\to\infty$.

Sometimes one needs bounds on the size of the entries of A^k as $k\to\infty$. One useful bound is an immediate consequence of the previous theorem.

5.6.13 **Corollary.** Let $A\in M_n$ be a given matrix, and let $\epsilon>0$ be given. There is a constant $C=C(A,\epsilon)$ such that

$$|(A^k)_{ij}|\le C(\rho(A)+\epsilon)^k$$

for all $k=1,2,3,\dots$ and all $i,j=1,2,3,\dots,n$.

Proof: Since the matrix $\tilde{A}\equiv[\rho(A)+\epsilon]^{-1}A$ has spectral radius strictly less than 1, it is convergent and hence $\tilde{A}^k\to 0$ as $k\to\infty$. In particular, the elements of the sequence $\{\tilde{A}^k\}$ are bounded, so there is some finite $C>0$ such that $|(\tilde{A}^k)_{ij}|\le C$ for all $k=1,2,3,\dots$ and all $i,j=1,2,\dots,n$. This is the asserted bound. $\square$

Exercise. Let $A=\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$, compute A^k explicitly, and show that one may not always take $\epsilon=0$ in (5.6.13).

Even though it is not accurate to say that individual entries of A^k behave like $\rho(A)^k$ as $k\to\infty$, the sequence $\{|||A^k|||\}$ does have this asymptotic behavior for any matrix norm $|||\cdot|||$.

5.6.14 **Corollary.** Let $|||\cdot|||$ be a matrix norm on M_n. Then

$$\rho(A)=\lim_{k\to\infty}|||A^k|||^{1/k}$$

for all $A\in M_n$.

Proof: Since $\rho(A)^k=\rho(A^k)\le|||A^k|||$, we have that $\rho(A)\le|||A^k|||^{1/k}$ for all $k=1,2,\dots$. If $\epsilon>0$ is given, the matrix $\tilde{A}\equiv[\rho(A)+\epsilon]^{-1}A$ has spectral radius strictly less than 1 and hence it is convergent. Thus, $|||\tilde{A}^k|||\to 0$ as $k\to\infty$ and there is some $N=N(\epsilon,A)$ such that $|||\tilde{A}^k|||<1$ for all $k\ge N$. This is just the statement that $|||A^k|||\le[\rho(A)+\epsilon]^k$ for all $k\ge N$, or that $|||A^k|||^{1/k}\le\rho(A)+\epsilon$ for all $k\ge N$. Since $\rho(A)\le|||A^k|||^{1/k}$ for all k and since $\epsilon>0$ is arbitrary, we conclude that $\lim_{k\to\infty}|||A^k|||^{1/k}$ exists and equals $\rho(A)$. $\square$

Questions about the convergence of infinite sequences or series of matrices can be treated with vector norms just as one treats infinite sequences or series of vectors.

Exercise. Let $\{A_k\} \subset M_n$ be a given infinite sequence of matrices. Show that the series $\sum_{k=0}^{\infty} A_k$ converges to some matrix in M_n if there is a vector norm $\|\cdot\|$ on M_n such that the numerical series $\sum_{k=0}^{\infty} \|A_k\|$ is convergent (or even if its partial sums are bounded). *Hint:* Show that the partial sums form a Cauchy sequence.

One special case for matrices that does not arise in the study of infinite series of vectors is the case of power series of matrices. But because of the submultiplicative property of matrix norms, it is easy to give a simple sufficient condition for convergence of matrix power series.

5.6.15 **Theorem.** If $A \in M_n$, then the series $\sum_{k=0}^{\infty} a_k A^k$ converges if there is a matrix norm $|||\cdot|||$ on M_n such that the numerical series $\sum_{k=0}^{\infty} |a_k|\, |||A|||^k$ converges, or even if the partial sums of this series are bounded.

Exercise. Prove (5.6.15).

Exercise. Show by example that it is possible that the series $\sum_{k=0}^{\infty} a_k A^k$ converges and the series $\sum_{k=0}^{\infty} |a_k|\, |||A|||^k$ diverges. This is analogous to conditional convergence (convergence but not absolute convergence) for numerical series.

Exercise. Let the function $f(z)$ be defined by the power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$, which has radius of convergence $R > 0$, and let $|||\cdot|||$ be a matrix norm on M_n. Show that $f(A) \equiv \sum_{k=0}^{\infty} a_k A^k$ is well defined for all $A \in M_n$ such that $|||A||| < R$. More generally, show that $f(A)$ is well defined for all $A \in M_n$ such that $\rho(A) < R$.

Exercise. If A is diagonalizable and $A = S^{-1}\Lambda S$, one sometimes defines $f(A) \equiv S^{-1} f(\Lambda) S$, where $f(\Lambda) \equiv \mathrm{diag}(f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n))$. Show that this definition of $f(A)$ agrees with the power series definition in the preceding exercise if A is diagonalizable. Is one of the two definitions more general than the other?

Exercise. Show that the matrix exponential given by the power series

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

is well defined for every $A \in M_n$.

Exercise. How would you define $\cos(A)$? For what A is this defined?

5.6.16 **Corollary.** A matrix $A \in M_n$ is invertible if there is a matrix norm $\|\cdot\|$ such that $\|I-A\|<1$. If this condition is satisfied,

$$A^{-1}=\sum_{k=0}^{\infty}(I-A)^k$$

Proof: If $\|I-A\|<1$, then the series

$$\sum_{k=0}^{\infty}(I-A)^k$$

converges to some matrix C because the radius of convergence of the series $\sum z^k$ is 1. But since

$$A\sum_{k=0}^{N}(I-A)^k=[I-(I-A)]\sum_{k=0}^{N}(I-A)^k=I-(I-A)^{N+1}\to I$$

as $N\to\infty$, we conclude that $C=A^{-1}$. □

Exercise. Show that the preceding result is equivalent to the following statement: If $\|\cdot\|$ is a matrix norm, and if $\|A\|<1$, then $I-A$ is invertible and

$$(I-A)^{-1}=\sum_{k=0}^{\infty}A^k$$

Exercise. Let $\|\cdot\|$ be a matrix norm on M_n, and suppose a given matrix $A\in M_n$ has an "approximate inverse" $B\in M_n$ with the property that $\|BA-I\|<1$. Show that A and B are both invertible.

Exercise. If the matrix norm $\|\cdot\|$ has the property that $\|I\|=1$ (which would be the case if it were an induced norm), and if $A\in M_n$ is such that $\|A\|<1$, show that

$$\frac{1}{1+\|A\|}\le\|(I-A)^{-1}\|\le\frac{1}{1-\|A\|}$$

Hint: Use the inequality $\|(I-A)^{-1}\|\le\sum_{k=0}^{\infty}\|A\|^k$ to get the upper bound. Use the general inequality $\|B^{-1}\|\ge 1/\|B\|$ and the triangle inequality for the lower bound.

Exercise. If $\|\cdot\|$ is a general matrix norm, all we know is that $\|I\|\ge 1$. In this case, show that

$$\frac{\|I\|}{\|I\|+\|A\|}\leq\|(I-A)^{-1}\|\leq\frac{\|I\|-(\|I\|-1)\|A\|}{1-\|A\|}$$

whenever $\|A\|<1$.

Exercise. If $A,B\in M_n$, if A is invertible, and if $A+B$ is singular, show that $\|B\|\geq 1/\|A^{-1}\|$ for any matrix norm $\|\cdot\|$. Thus, there is an intrinsic limit to how well a nonsingular matrix can be approximated by a singular one. *Hint:* $A+B=A(I+A^{-1}B)$. If $\|A^{-1}B\|<1$, then $I+A^{-1}B$ would be invertible, so it must be that $\|A^{-1}B\|\geq 1$.

One useful and easily computed criterion for invertibility follows easily from the last corollary.

5.6.17 Corollary. Let $A=[a_{ij}]\in M_n$, and suppose that

$$|a_{ii}|>\sum_{\substack{j=1\\ j\neq i}}^{n}|a_{ij}| \quad \text{for all} \quad i=1,2,\ldots,n$$

Then A is invertible.

Proof: The hypothesis ensures that all main diagonal entries a_{ii} are nonzero. Set $D=\operatorname{diag}(a_{11},\ldots,a_{nn})$, so that D is an invertible diagonal matrix, $D^{-1}A$ has all 1's on the main diagonal, the matrix $B=[b_{ij}]=I-D^{-1}A$ has all 0's on the main diagonal, and $b_{ij}=-a_{ij}/a_{ii}$ if $i\neq j$. Consider the maximum row sum norm $\|\cdot\|_\infty$. The hypothesis guarantees that $\|B\|_\infty<1$, so $I-B=D^{-1}A$ is invertible by (5.6.16), and hence A is invertible. □

A matrix that satisfies the hypothesis of (5.6.17) is said to be *strictly diagonally dominant*. This sufficient condition for invertibility is known as the Levy–Desplanques theorem, and it can be improved somewhat. See Sections (6.1), (6.2), and (6.4).

We now consider in more detail the induced matrix norms defined in (5.6.1). These are some of the most familiar matrix norms, and they have an important minimality property. Because one often wishes to establish that a given matrix A is convergent by using the test $\|A\|<1$, it is natural to prefer matrix norms that are uniformly as small as possible. As we shall show, the entire class of induced matrix norms has this desirable property, and this property characterizes the class of induced matrix norms.

Any two norms on a finite-dimensional space are equivalent, and so for each two matrix norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ there is a least finite positive constant $C_M(\alpha,\beta)$ such that $\|A\|_\alpha\leq C_M(\alpha,\beta)\|A\|_\beta$ for all $A\in M_n$. This constant can be computed as

$$C_M(\alpha,\beta) = \max_{A\neq 0} \frac{\|A\|_\alpha}{\|A\|_\beta}$$

If the roles of α and β are reversed, there must be a similarly defined least finite positive constant $C_M(\beta,\alpha)$ such that $\|A\|_\beta \leq C_M(\beta,\alpha)\|A\|_\alpha$ for all $A \in M_n$. In general, there is no obvious relation between the two constants $C_M(\alpha,\beta)$ and $C_M(\beta,\alpha)$, but if we examine the table in Problem 23 at the end of this section, we see that its upper left 3×3 corner is symmetric; that is, $C_M(\alpha,\beta) = C_M(\beta,\alpha)$ for any pair of the three matrix norms $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$. All three of these matrix norms are induced norms, and this symmetry is a property of all induced norms.

5.6.18 **Theorem.** Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be two given vector norms on $\mathbf{C}^n$, and let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ denote the respective induced matrix norms on M_n, that is,

$$\|A\|_\alpha \equiv \max_{x\neq 0} \frac{\|Ax\|_\alpha}{\|x\|_\alpha} \quad \text{and} \quad \|A\|_\beta \equiv \max_{x\neq 0} \frac{\|Ax\|_\beta}{\|x\|_\beta}$$

Define

$$R_{\alpha\beta} \equiv \max_{x\neq 0} \frac{\|x\|_\alpha}{\|x\|_\beta} \quad \text{and} \quad R_{\beta\alpha} \equiv \max_{x\neq 0} \frac{\|x\|_\beta}{\|x\|_\alpha} \tag{5.6.19}$$

Then

$$\max_{A\neq 0} \frac{\|A\|_\alpha}{\|A\|_\beta} = R_{\alpha\beta}R_{\beta\alpha} \tag{5.6.20}$$

In particular,

$$\max_{A\neq 0} \frac{\|A\|_\alpha}{\|A\|_\beta} = \max_{A\neq 0} \frac{\|A\|_\beta}{\|A\|_\alpha} = R_{\alpha\beta}R_{\beta\alpha} \tag{5.6.21}$$

Proof: Let $A \in M_n$ and $x \in \mathbf{C}^n$ be given, and suppose that $x \neq 0$ and $Ax \neq 0$. Then

$$\frac{\|Ax\|_\alpha}{\|x\|_\alpha} = \frac{\|Ax\|_\alpha}{\|Ax\|_\beta}\frac{\|Ax\|_\beta}{\|x\|_\beta}\frac{\|x\|_\beta}{\|x\|_\alpha} \leq R_{\alpha\beta}\frac{\|Ax\|_\beta}{\|x\|_\beta}R_{\beta\alpha}$$

an inequality that holds even if $Ax = 0$. Thus,

$$\|A\|_\alpha \equiv \max_{x\neq 0}\frac{\|Ax\|_\alpha}{\|x\|_\alpha} \leq R_{\alpha\beta}\max_{x\neq 0}\frac{\|Ax\|_\beta}{\|x\|_\beta}R_{\beta\alpha} \equiv R_{\alpha\beta}R_{\beta\alpha}\|A\|_\beta$$

and hence

$$\frac{\|\!|A|\!\|_\alpha}{\|\!|A|\!\|_\beta} \le R_{\alpha\beta} R_{\beta\alpha} \tag{5.6.22}$$

for all nonzero $A \in M_n$.

Each of the two extrema in (5.6.19) is achieved for some nonzero vector, so there are vectors $y, z \in \mathbf{C}^n$ such that $\|y\|_2 = \|z\|_2 = 1$, $\|y\|_\alpha = R_{\alpha\beta}\|y\|_\beta$, and $\|z\|_\beta = R_{\beta\alpha}\|z\|_\alpha$. By Corollary (5.5.15) there exists a vector $z_0 \in \mathbf{C}^n$ such that

(a) $|z_0^* x| \le \|x\|_\beta$ for all $x \in \mathbf{C}^n$; and
(b) $z_0^* z = \|z\|_\beta$.

Consider the matrix $A_0 \equiv y z_0^*$. Using (b), we have

$$\frac{\|A_0 z\|_\alpha}{\|z\|_\alpha} = \frac{\|y z_0^* z\|_\alpha}{\|z\|_\alpha} = \frac{\|y\|_\alpha |z_0^* z|}{\|z\|_\alpha} = \frac{\|y\|_\alpha \|z\|_\beta}{\|z\|_\alpha}$$

so we have the lower bound

$$\|\!|A_0|\!\|_\alpha \ge \frac{\|y\|_\alpha \|z\|_\beta}{\|z\|_\alpha} = R_{\alpha\beta} R_{\beta\alpha} \|y\|_\beta$$

On the other hand, we can use (a) to obtain

$$\frac{\|A_0 x\|_\beta}{\|x\|_\beta} = \frac{\|y z_0^* x\|_\beta}{\|x\|_\beta} = \frac{\|y\|_\beta |z_0^* x|}{\|x\|_\beta} \le \frac{\|y\|_\beta \|x\|_\beta}{\|x\|_\beta} = \|y\|_\beta$$

and hence we have the upper bound

$$\|\!|A_0|\!\|_\beta \le \|y\|_\beta$$

Combining these two bounds, we have

$$\frac{\|\!|A_0|\!\|_\alpha}{\|\!|A_0|\!\|_\beta} \ge \frac{R_{\alpha\beta} R_{\beta\alpha} \|y\|_\beta}{\|y\|_\beta} = R_{\alpha\beta} R_{\beta\alpha}$$

which shows that equality is possible in (5.6.22) and establishes (5.6.20). The assertion (5.6.21) follows because the right-hand side of the identity (5.6.20) is symmetric in α and β. □

Is it possible that two different vector norms on $\mathbf{C}^n$ could induce the same matrix norm on M_n? According to the following consequence of (5.6.18), this can happen if and only if one of the vector norms is a constant scalar multiple of the other.

5.6.23 Corollary. Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be vector norms on $\mathbf{C}^n$, and let $\|\!|\cdot|\!\|_\alpha$ and $\|\!|\cdot|\!\|_\beta$ denote the respective induced matrix norms on M_n. Then $\|\!|A|\!\|_\alpha = \|\!|A|\!\|_\beta$ for all $A \in M_n$ if and only if there is a positive constant c such that $\|x\|_\alpha = c\|x\|_\beta$ for all $x \in \mathbf{C}^n$.

Proof: Observe that

$$R_{\beta\alpha} = \max_{x \neq 0} \frac{\|x\|_\beta}{\|x\|_\alpha} = \left[\min_{x \neq 0} \frac{\|x\|_\alpha}{\|x\|_\beta} \right]^{-1} \geq \left[\max_{x \neq 0} \frac{\|x\|_\alpha}{\|x\|_\beta} \right]^{-1} = \frac{1}{R_{\alpha\beta}}$$

Thus, we have the general inequality

$$R_{\alpha\beta} R_{\beta\alpha} \geq 1 \tag{5.6.24}$$

with equality if and only if

$$\min_{x \neq 0} \frac{\|x\|_\alpha}{\|x\|_\beta} = \max_{x \neq 0} \frac{\|x\|_\alpha}{\|x\|_\beta}$$

which can occur if and only if the function $\|x\|_\alpha / \|x\|_\beta$ is constant for all $x \neq 0$. Thus, if $\|x\|_\alpha \equiv c\|x\|_\beta$, we certainly have $R_{\alpha\beta} R_{\beta\alpha} = 1$ and hence $\|A\|_\alpha \leq \|A\|_\beta$ and $\|A\|_\beta \leq \|A\|_\alpha$ for all $A \in M_n$ by (5.6.21); in this event, $\|A\|_\alpha = \|A\|_\beta$ for all $A \in M_n$. Conversely, if the two induced matrix norms are identical, then $R_{\alpha\beta} R_{\beta\alpha} = 1$ by (5.6.20) and hence equality holds in (5.6.24) and the ratio $\|x\|_\alpha / \|x\|_\beta$ is constant by the preceding argument. □

5.6.25 **Corollary.** Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be vector norms on $\mathbf{C}^n$, and let $|\!|\!|\cdot|\!|\!|_\alpha$ and $|\!|\!|\cdot|\!|\!|_\beta$ denote the respective induced matrix norms on M_n. Then $|\!|\!|A|\!|\!|_\alpha \leq |\!|\!|A|\!|\!|_\beta$ for all $A \in M_n$ if and only if $|\!|\!|A|\!|\!|_\alpha = |\!|\!|A|\!|\!|_\beta$ for all $A \in M_n$.

Proof: If $|\!|\!|A|\!|\!|_\alpha \leq |\!|\!|A|\!|\!|_\beta$ for all $A \in M_n$, then $R_{\alpha\beta} R_{\beta\alpha} \leq 1$, which [because of (5.6.24)] implies that $R_{\alpha\beta} R_{\beta\alpha} = 1$. Therefore, $|\!|\!|A|\!|\!|_\alpha \leq |\!|\!|A|\!|\!|_\beta$ and $|\!|\!|A|\!|\!|_\beta \leq |\!|\!|A|\!|\!|_\alpha$ for all $A \in M_n$ by (5.6.21). □

The last corollary says that no induced matrix norm can be uniformly dominated by another. What happens if we permit comparisons with other (not necessarily induced) matrix norms?

5.6.26 **Theorem.** Let $|\!|\!|\cdot|\!|\!|$ be a given matrix norm on M_n, and let $|\!|\!|\cdot|\!|\!|_\alpha$ be a given induced matrix norm on M_n. Then

(a) There is an induced matrix norm $N(\cdot)$ on M_n such that $N(A) \leq |\!|\!|A|\!|\!|$ for every $A \in M_n$; and
(b) $|\!|\!|A|\!|\!| \leq |\!|\!|A|\!|\!|_\alpha$ for every $A \in M_n$ if and only if $|\!|\!|A|\!|\!| = |\!|\!|A|\!|\!|_\alpha$ for every $A \in M_n$.

Proof: Define the vector norm $\|\cdot\|$ on $\mathbf{C}^n$ by

$$\|x\| \equiv |\!|\!|X|\!|\!|, \qquad X \equiv [x\ x \ldots x] \in M_n \tag{5.6.27}$$

and consider the matrix norm $N(\bullet)$ on M_n that is induced by $\|\bullet\|$. For any $A \in M_n$, we have

$$\begin{aligned} N(A) &\equiv \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{x \neq 0} \frac{|||[Ax\ Ax \dots Ax]|||}{|||[x\ x \dots x]|||} \\ &= \max_{x \neq 0} \frac{|||AX|||}{|||X|||} \\ &\leq \max_{x \neq 0} \frac{|||A|||\ |||X|||}{|||X|||} \qquad \text{(because } |||\bullet||| \text{ is a matrix norm)} \\ &= |||A||| \end{aligned} \tag{5.6.28}$$

which establishes (a). To prove (b), suppose that $|||A||| \leq |||A|||_\alpha$ for all $A \in M_n$. Then by (a) we have

$$N(A) \leq |||A||| \leq |||A|||_\alpha$$

for all $A \in M_n$. But $N(\bullet)$ and $|||\bullet|||_\alpha$ are both induced norms, so $N(A) \equiv |||A|||_\alpha$ by (5.6.25), and hence $|||A||| \equiv |||A|||_\alpha$ for all $A \in M_n$. □

The preceding result is the motivation for the following definition.

5.6.29 **Definition.** A matrix norm $|||\bullet|||$ on M_n is a *minimal matrix norm* if the only matrix norm $N(\bullet)$ on M_n such that $N(A) \leq |||A|||$ for all $A \in M_n$ is $N(\bullet) = |||\bullet|||$.

Assertion (b) of Theorem (5.6.26) says that every induced norm on M_n is minimal. Assertion (a) implies immediately that every minimal norm is induced. Thus, if one wants to use a matrix norm that cannot be uniformly improved upon (in terms of small values on all matrices), one should use an induced norm, and any norm with this optimality property must be an induced norm.

The vector norm (5.6.27) is a special case of a whole family of vector norms that can be constructed from a given matrix norm. Let $|||\bullet|||$ be a given matrix norm on M_n, let $y \in \mathbf{C}^n$ be a given nonzero vector, and define the function $\|\bullet\|_y : \mathbf{C}^n \to \mathbf{R}$ by

$$\|x\|_y \equiv |||xy^*||| \qquad y \in \mathbf{C}^n, \quad y \neq 0 \tag{5.6.30}$$

Then $\|\bullet\|_y$ is a vector norm on $\mathbf{C}^n$ with the property that

$$\|Ax\|_y = |||A(xy^*)||| \leq |||A|||\ |||xy^*||| = |||A|||\ \|x\|_y$$

for all $A \in M_n$. If $y = [1 1 \dots 1]^T$, then (5.6.30) reduces to (5.6.27). If we denote by $N_y(\bullet)$ the matrix norm on M_n that is induced by $\|\bullet\|_y$, this inequality says that

$$N_y(A) \equiv \max_{x \neq 0} \frac{\|Ax\|_y}{\|x\|_y} \leq \max_{x \neq 0} \frac{\|A\| \, \|x\|_y}{\|x\|_y} = \|A\| \quad \text{for all} \quad A \in M_n \tag{5.6.31}$$

This is evidently a generalization of (5.6.26a).

If the given matrix norm $\|\cdot\|$ is a minimal norm, then (5.6.31) implies that $\|A\| = N_y(A)$ for all $A \in M_n$. Since the vector y used in this argument can be *any* nonzero vector, we would then have $N_y(\cdot) = \|\cdot\| = N_z(\cdot)$ for all nonzero $y, z \in M_n$.

5.6.32 **Theorem.** Let $\|\cdot\|$ be a matrix norm on M_n, and let $N_y(\cdot)$ be the induced norm defined by (5.6.31) and (5.6.30). The following are equivalent:

(a) $\|\cdot\|$ is an induced matrix norm.
(b) $\|\cdot\|$ is a minimal matrix norm.
(c) $\|\cdot\| = N_y(\cdot)$ for all nonzero $y \in \mathbf{C}^n$.

Proof: The assertion that (a) implies (b) is just (5.6.26b). We have just observed that if $\|\cdot\|$ is minimal, then $\|\cdot\| = N_y(\cdot)$, so (b) implies (c). If (c), then $\|\cdot\|$ is induced because $N_y(\cdot)$ is induced by definition. □

There is somewhat more to be gleaned from these observations. If $N_y(\cdot) = \|\cdot\|$ for all nonzero $y \in \mathbf{C}^n$, then $N_y(\cdot) = N_z(\cdot)$ for all nonzero $y, z \in \mathbf{C}^n$. But Corollary (5.6.23) says that the vector norm that induces a given matrix norm is unique up to a scale factor, so $\|\cdot\|_y = c_{yz}\|\cdot\|_z$ for some positive constant c_{yz}.

Exercise. If the matrix norm $\|\cdot\|$ on M_n is induced by the vector norm $\|\cdot\|$ on $\mathbf{C}^n$, show that $\|yz^*\| = \|y\| \, \|z\|^D$, $\|\cdot\|_z = \|\cdot\| \, \|z\|^D$, and $c_{yz} = \|y\|^D / \|z\|^D$ for all $y, z \in \mathbf{C}^n$. The vector norm $\|\cdot\|^D$ is the dual of the vector norm $\|\cdot\|$, as defined in (5.4.12).

5.6.33 **Theorem.** Let $\|\cdot\|$ be a given matrix norm on M_n and let $\|\cdot\|_y$ be the vector norm on $\mathbf{C}^n$ defined by (5.6.30). The following two assertions are equivalent:

(a) For each pair of nonzero vectors $y, z \in \mathbf{C}^n$ there is a positive constant c_{yz} such that

$$\|x\|_y = c_{yz}\|x\|_z \quad \text{for all} \quad x \in \mathbf{C}^n$$

(b) $$\|xy^*\| = \frac{\|xz^*\| \, \|zy^*\|}{\|zz^*\|} \quad \text{for all} \quad x, y, z \in \mathbf{C}^n \quad \text{with} \quad z \neq 0$$

If $|||\cdot|||$ is an induced matrix norm, then it satisfies the identity (b), and the vector norms constructed from it by (5.6.30) satisfy (a).

Proof: If (a), then

$$|||xz^*|||\,|||zy^*||| = \|x\|_z\|z\|_y = (1/c_{yz})\|x\|_y c_{yz}\|z\|_z = \|x\|_y\|z\|_z = |||xy^*|||\,|||zz^*|||$$

Conversely, if (b), then (a) follows with $c_{yz} = |||zy^*|||/|||zz^*|||$. We have already argued that if $N_y(\cdot) = |||\cdot|||$, then (a) [and hence (b) also] must follow, and this will be the case if $|||\cdot|||$ is an induced norm by (5.6.32). □

Exercise. Any positive scalar multiple of an induced norm satisfies the identity (5.6.33b). Show that the matrix norms $\|\cdot\|_1$ and $\|\cdot\|_2$ both satisfy this identity, but that neither norm is a scalar multiple of an induced norm.

We saw in (5.6.2) that if $|||\cdot|||$ is an induced matrix norm, then $|||I|||=1$. This property is unfortunately not sufficient for a matrix norm to be an induced norm. It is easy to show that the function

$$|||A||| \equiv \max\{|||A|||_1, |||A|||_\infty\} \tag{5.6.34}$$

defines a matrix norm on M_n, and that $|||I|||=1$. But since $|||A|||_1 \le |||A|||$ for all $A \in M_n$ and $|||A|||_1 < |||A|||$ for $A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$, $|||\cdot|||$ is not a minimal norm and hence cannot be an induced norm.

Exercise. Verify that (5.6.34) defines a matrix norm. More generally, show that if $|||\cdot|||_{(1)}, \ldots, |||\cdot|||_{(k)}$ are given matrix norms on M_n, then

$$|||A||| \equiv \max\{|||A|||_{(1)}, \ldots, |||A|||_{(k)}\}$$

defines a matrix norm on M_n.

The induced norms are minimal among all matrix norms, but suppose one considers only the important class of *unitarily invariant matrix norms*. These are the matrix norms $|||\cdot|||$ such that $|||A||| = |||UAV|||$ for all $A \in M_n$ and all unitary matrices $U, V \in M_n$. It turns out that in this class there is only one minimal matrix norm, and that is the spectral norm.

5.6.35 **Corollary.** If $|||\cdot|||$ is a unitarily invariant matrix norm, then $|||A|||_2 \le |||A|||$ for all $A \in M_n$. The spectral norm $|||\cdot|||_2$ is the only matrix norm on M_n that is both induced and unitarily invariant.

Proof: Suppose that $|||\cdot|||$ is a given unitarily invariant matrix norm. By part (a) of Theorem (5.6.26), we know that $N(A) \le |||A|||$ for all $A \in M_n$, where $N(A)$ is induced by the vector norm $\|\cdot\|$ defined by (5.6.27). If $U\in M_n$ is unitary, we have $\|Ux\|=\|UX\|=\|X\|=\|x\|$, and hence the vector norm $\|\cdot\|$ is unitarily invariant. If $x\in\mathbf{C}^n$ is a given nonzero vector, there exists a unitary matrix U such that $Ux=\|x\|_2e_1$. Thus, $\|x\|=\|\|x\|_2U^*e_1\|=\|x\|_2\|U^*e_1\|=\|x\|_2\|e_1\|$ for all $x\in\mathbf{C}^n$. The vector norm $\|\cdot\|$ is therefore a scalar multiple of the Euclidean norm and Corollary (5.6.23) says that $N(\cdot)$ (the matrix norm induced by $\|\cdot\|$) equals $\|\cdot\|_2$ (the matrix norm induced by $\|\cdot\|_2$). Therefore, $\|\cdot\|_2=N(A)\le\|A\|$ for all $A\in M_n$. If $\|\cdot\|$ is assumed to be induced, then it is minimal and hence $\|A\|_2=\|A\|$ for all $A\in M_n$. □

If $\|\cdot\|$ is a matrix norm on M_n, then the function $\|\cdot\|^*$ defined by

$$\|A\|^*\equiv\|A^*\|$$

is also a matrix norm on M_n. A direct calculation shows that $\|A\|_2^*=\|A^*\|_2=\|A\|_2$ and $\|A\|_1^*=\|A^*\|_1=\|A\|_1$ for all $A\in M_n$, but not every matrix norm has this property since $\|A\|_1^*=\|A\|_\infty\ne\|A\|_1$. A matrix norm such that $\|\cdot\|^*\equiv\|\cdot\|$ is said to be *self-adjoint*. The Frobenius and l_1 matrix norms are self-adjoint, and since

$$\|A^*\|_2^2=\rho(AA^*)=\rho(A^*A)=\|A\|_2^2$$

the spectral norm is self-adjoint, too. In fact, all unitarily invariant norms on M_n are self-adjoint [see (7.4), Problem 2]. The spectral norm is distinguished as the only induced matrix norm that is self-adjoint.

5.6.36 **Theorem.** Let $\|\cdot\|$ be a given matrix norm on M_n. Then

(a) $\|\cdot\|^*$ is an induced norm if and only if $\|\cdot\|$ is an induced norm.
(b) If the matrix norm $\|\cdot\|$ is induced by the vector norm $\|\cdot\|$, then $\|\cdot\|^*$ is induced by the dual norm $\|\cdot\|^D$.
(c) The spectral norm $\|\cdot\|_2$ is the only matrix norm on M_n that is both induced and self-adjoint.

Proof: If $N(\cdot)$ is a matrix norm, and if $N(A)\le\|A\|^*=\|A^*\|$ for all $A\in M_n$, then $N(A)^*=N(A^*)\le\|A\|$ for all $A\in M_n$. If $\|\cdot\|$ is a minimal matrix norm, then $N(\cdot)^*=\|\cdot\|$ and hence $N(\cdot)=\|\cdot\|^*$, so $\|\cdot\|^*$ is a minimal matrix norm. The assertion (a) follows from (5.6.32). Now suppose that $\|\cdot\|$ is induced by the vector norm $\|\cdot\|$. Using the duality theorem (5.5.14), we have

$$\|A\|^*=\|A^*\|=\max_{\|x\|=1}\|A^*x\|=\max_{\|x\|=1}(\|A^*x\|^D)^D$$

$$=\max_{\|x\|=1}\max_{\|z\|^D=1}|(A^*x)^*z|=\max_{\|z\|^D=1}\max_{\|x\|=1}|x^*Az|$$

$$= \max_{\|z\|^D=1} \|Az\|^D$$

and hence $|||\bullet|||^*$ is induced by $\|\bullet\|^D$. For the last assertion, we observe that if the matrix norm $|||\bullet|||$ is induced by the vector norm $\|\bullet\|$, and if $|||\bullet||| = |||\bullet|||^*$, then (b) says that $|||\bullet|||$ is also induced by $\|\bullet\|^D$. But Corollary (5.6.23) says that the vector norm that induces a given matrix norm is uniquely determined up to a positive scalar factor, and hence there exists some $c>0$ such that $\|\bullet\|^D = c\|\bullet\|$. By (5.4.16) we must then have $\|\bullet\| = \|\bullet\|_2/\sqrt{c}$. Since the given vector norm is a multiple of the Euclidean vector norm, they both induce the same matrix norm and we conclude that $|||\bullet||| = |||\bullet|||_2$. □

Exercise. Show that $|||\bullet|||^*$ is a matrix norm whenever $|||\bullet|||$ is a matrix norm.

Exercise. Give an example to show that a self-adjoint matrix norm need not be unitarily invariant.

Absolute and monotone vector norms were introduced in (5.5), and are the most commonly used vector norms. There is a simple and useful characterization of the matrix norms that are induced by monotone vector norms.

5.6.37 **Theorem.** Let $\|\bullet\|$ be a vector norm on $\mathbf{C}^n$ and let $|||\bullet|||$ be the matrix norm on M_n that it induces. The following are equivalent:

(a) $\|\bullet\|$ is an absolute norm; that is, $\| |x| \| = \|x\|$ for all $x \in \mathbf{C}^n$.
(b) $\|\bullet\|$ is a monotone norm; that is, $\|x\| \le \|y\|$ whenever $|x| \le |y|$.
(c) Whenever $D = \text{diag}(d_1, d_2, \ldots, d_n) \in M_n$, then

$$|||D||| = \max_{1\le i\le n} |d_i|$$

Proof: The equivalence of (a) and (b) is the content of (5.5.10). If $\|\bullet\|$ is monotone, and if we set

$$d \equiv \max_{1\le i\le n} |d_i|, \qquad d = |d_k|$$

then $|Dx| \le |dx|$ and hence $\|Dx\| \le d\|x\|$ with equality for $x = e_k$. Thus,

$$|||D||| = \max_{x\ne 0} \frac{\|Dx\|}{\|x\|} = d$$

and hence (b) implies (c). If we assume (c), let $x, y \in \mathbf{C}^n$ be given with $|x| \le |y|$ and note that there are complex numbers d_k such that $|x_k| =$

$d_k y_k$ and $|d_k| \le 1$, $k = 1, \ldots, n$. Thus, if $D \equiv \operatorname{diag}(d_1, \ldots, d_n)$, we have $Dy = |x|$ and $|||D||| \le 1$. Since

$$\| |x| \| = \|Dy\| \le |||D||| \, \|y\| \le \|y\|$$

the norm $\|\cdot\|$ must be monotone. □

Ignore Problems

Problems

1. Give an example of a vector norm for matrices for which $\|I\| < 1$.

2. A matrix A such that $A^2 = A$ is said to be *idempotent*. Give an example of a 2-by-2 idempotent matrix other than I and 0. Show that 0 and 1 are the only possible eigenvalues of an idempotent matrix. Show that an idempotent matrix A must always be diagonalizable and that $|||A||| \ge 1$ for any matrix norm $|||\cdot|||$ if $A \ne 0$.

3. If $|||\cdot|||$ is a matrix norm on M_n, show that $c|||\cdot|||$ is a matrix norm for all $c \ge 1$. Show, however, that neither $c|||\cdot|||_1$ nor $c\|\cdot\|_1$ is a matrix norm for any $c < 1$.

4. In Definition (5.6.1) the same vector norm is involved in two different ways. More generally, we might define $|||\cdot|||_{\alpha,\beta}$ by

$$|||A|||_{\alpha,\beta} \equiv \max_{\|x\|_\alpha = 1} \|Ax\|_\beta$$

where $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are two (possibly different) vector norms. Is such a function $|||\cdot|||_{\alpha,\beta}$ a matrix norm? Study $|||\cdot|||_{\alpha,\beta}$ to determine what interesting properties it might have; note that this notion might be used to define a norm on m-by-n matrices, since $\|\cdot\|_\alpha$ may be taken to be a vector norm on $\mathbf{C}^m$ and $\|\cdot\|_\beta$ may be taken to be a vector norm on $\mathbf{C}^n$. What properties like those of an induced matrix norm does $|||\cdot|||_{\alpha,\beta}$ have in this regard?

5. Show that both the Euclidean norm $\|\cdot\|_2$ and the spectral norm $|||\cdot|||_2$ are unitarily invariant norms on M_n; that is, A and UAV have the same norm whenever U and V are unitary. Compare the matrix norms $\|\cdot\|_2$ and $|||\cdot|||_2$ in as many respects as you can. Note that $\|A\|_2 = [\operatorname{tr} A^*A]^{1/2}$.

6. Verify that axioms (1)–(3) for $|||\cdot|||$ imply that the same axioms hold for $|||\cdot|||_S$ in (5.6.7). This verifies that (5.6.7) remains valid if "matrix norm" in the hypothesis and conclusion is replaced by "vector norm on matrices."

7. If $|||\cdot|||$ is an induced matrix norm on M_n and if $S \in M_n$ is nonsingular, show that $|||\cdot|||_S$ [as defined in (5.6.7)] is also an induced matrix norm. If

18. Let $A, B \in M_n$ be positive semidefinite and suppose A is positive definite. Use Problem 17 to show that

$$\|A^{1/2} - B^{1/2}\|_2 \leq \|A^{-1/2}\|_2 \, \|A - B\|_2 \tag{7.2.13}$$

and explain why this inequality implies that the function $f: C \to C^{1/2}$, defined on the set of positive semidefinite matrices in M_n, is continuous on the interior of this set, which is the open set of positive definite matrices. State and prove directly the inequality for the ordinary scalar square root function $f: t \to \sqrt{t}$ on $[0, \infty)$ that results from setting $n = 1$ in (7.2.13).

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7.3 The polar form and the singular value decomposition

We next develop two important related factorizations of complex matrices (not necessarily square) which depend heavily on positive definiteness.

7.3.1 **Lemma.** Let $A \in M_{m,n}$ with $m \leq n$ and rank $A = k \leq m$. There exists a unitary matrix $X \in M_m$, a diagonal matrix $\Lambda \in M_m$ with nonnegative diagonal entries $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > \lambda_{k+1} = \cdots = \lambda_m = 0$, and a matrix $Y \in M_{m,n}$ with orthonormal rows such that $A = X \Lambda Y$. The matrix $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$ is always uniquely determined and $\{\lambda_1^2, \ldots, \lambda_m^2\}$ are the eigenvalues of AA^*. The columns of the matrix X are eigenvectors of AA^*. If AA^* has distinct eigenvalues, then X is determined up to a right diagonal factor $D = \operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$ with all $\theta_i \in \mathbf{R}$; that is, if $A = X_1 \Lambda Y_1 = X_2 \Lambda Y_2$, then $X_2 = X_1 D$. Given X, the matrix Y is uniquely determined if rank $A = m$. If A is real, then X and Y may be taken to be real.

Proof: If $A = X \Lambda Y$ is a factorization of the asserted form, then $AA^* = X \Lambda YY^* \Lambda X^* = X \Lambda I \Lambda X^* = X \Lambda^2 X^*$, so $X \Lambda^2 X^*$ is a unitary diagonalization of the Hermitian matrix AA^*. If $X = [x_1 \, x_2 \ldots x_m]$ and if $\Lambda^2 = \operatorname{diag}(\lambda_1^2, \ldots, \lambda_m^2)$, then $AA^* x_j = \lambda_j^2 x_j$, $j = 1, 2, \ldots, m$, and the vectors $\{x_j\}$ are orthonormal. Because the diagonal entries of Λ are to be nonnegative and are to be arranged in nonincreasing order, Λ is uniquely determined by AA^*. If the numbers $\{\lambda_i^2\}$ are distinct, the corresponding normalized eigenvectors of AA^* are each determined up to a complex scalar factor of modulus 1, so if X_1 and X_2 are unitary matrices whose columns are eigenvectors of AA^*, we must have $X_2 = X_1 D$ with $D = \operatorname{diag}(d_1, \ldots, d_n)$ and all $|d_i| = 1$.

Eigenvectors of AA^* corresponding to a multiple eigenvalue are not uniquely determined, however, but once they are chosen and orthonormalized so that the unitary matrix X is fixed, then $Y = \Lambda^{-1} X^* A$ is uniquely determined if Λ is nonsingular, which is the case if $k = \operatorname{rank} A = m$. One checks easily that $YY^* = \Lambda^{-1}X^*(AA^*X)\Lambda^{-1} = \Lambda^{-1}X^*X\Lambda^2\Lambda^{-1} = \lambda^{-1}\Lambda^2\Lambda^{-1} = I$, so this matrix Y has orthonormal rows.

It remains only to handle the case in which $\operatorname{rank} A = k < m$. Since we want $Y = \Lambda^{-1}X^*A = \Lambda^{-1}(A^*X)^*$ when all $\lambda_i \neq 0$, we are led to define the jth row of Y to be the row vector y_j^*, where $y_j \equiv \lambda_j^{-1}(A^*x_j)$, $j = 1, \ldots, k$. Then

$$[\lambda_j^{-1}(A^*x_j)]^*[\lambda_k^{-1}(A^*x_k)] = x_j^*AA^*x_k/\lambda_j\lambda_k = x_j^*\lambda_k^2 x_k/\lambda_j\lambda_k = x_j^*x_k\lambda_k/\lambda_j$$

which is 0 if $j \neq k$ and is 1 if $j = k$ since the vectors $\{x_j\}$ are orthonormal. The vectors $\{y_1, \ldots, y_k\}$ are an orthonormal set in $\mathbf{C}^n$, and $n \geq m > k$, so there exist $m-k$ additional (but not uniquely determined) orthonormal vectors $y_{k+1}, \ldots, y_m$ such that the matrix $Y^* \equiv [y_1\, y_2 \ldots y_k\, y_{k+1} \ldots y_m] \in M_{n,m}$ has m orthonormal columns.

Now notice that $X^*A = \Lambda Y$. The first k rows of both sides of this identity are equal by construction of the vectors y_j. The last $m-k$ rows are all 0 on the right because the last $m-k$ diagonal entries of Λ are 0; the last $m-k$ rows are all 0 on the left because if $AA^*x_j = 0$, then $0 = x_j^*AA^*x_j = (A^*x_j)^*(A^*x_j) = 0$ and hence $A^*x_j = 0$.

Finally, if A is real, then AA^* is real and has real eigenvalues, and hence the eigenvectors X may be taken to be real. The first k rows of Y, which are determined by X, are real by construction, and the $m-k$ orthonormal vectors that are added may be taken to be real. Thus, all the factors may be taken to be real if A is real. $\square$

Every nonzero complex number z has a unique "polar representation" $z = pu$, where p is a positive real number and u is a complex number of modulus 1. Indeed, $p = |z|$ and $u = p^{-1}z = z/|z|$ if $z \neq 0$. If $z = 0$, then z can still be written in polar form with $p = 0$, but u is no longer uniquely determined. Indeed, u can be any complex number of modulus 1.

How does this generalize to a complex matrix $A \in M_n$? One answer is that $A = PU$ where P is positive (semi)definite and U is unitary. We can even generalize to the case in which A is not a square matrix.

7.3.2 Theorem. Let $A \in M_{m,n}$ with $m \leq n$. Then A may be written as

$$A = PU$$

where $P \in M_m$ is positive semidefinite, $\operatorname{rank} P = \operatorname{rank} A$, and $U \in M_{m,n}$ has orthonormal rows (that is, $UU^* = I$). The matrix P is always uniquely determined as $P = (AA^*)^{1/2}$, and U is uniquely determined when A has rank m. If A is real, then both P and U may be taken to be real.