IE 529 - Homework 1: Random variables

Due Friday, Sept. 15

Part I: Review of basic probability and statistics concepts (these should be easy):

I. Probability density functions
The times taken by each of two students to complete his and her portions of a team project are denoted by continuous random variables, x and y. A joint probability density function, $f(x, y)$, is given to model these times, which is defined by

$$f(x,y) = \begin{cases} c(x+y), & 0 < x < 3 \text{ and } 0 < y < x \\ 0 & \text{elsewhere} \end{cases}$$

where c is a constant.

a. Calculate the value of c that makes $f(x, y)$ a proper bivariate density function.

b. Find the **marginal** probability distributions for x and y, denoted $f_X(x)$ and $f_Y(y)$.

c. Are x and y independent? Why or why not?

d. Suppose the proportion of "dead time" (i.e., the time when no assigned jobs are begin performed by either worker) is given by the new random variable Z, defined by

$$z = 1 - \frac{c(x+y)}{6}$$

where c is the constant determined in part **a.** Find the expected value of Z, i.e., $E(z)$.

II. Conditional probabilities
Shutdowns of the campus computing system have been categorized as being due to either a hardware failure (electronics), or a software failure, or a power failure (localized components). *These three categories are assumed to be the only causes of failure leading to a shutdown.*

When a hardware failure occurs, the system is forced to shut down 73% of the time; when a software failure occurs, the system is shut down 12% of the time; when a power failure occurs, the system is shut down 88% of the time.

In a one-month time frame, hardware failures occur with probability 0.12, software failures occur with probability 0.05 and power component failures occur with probability 0.20.

a. What is the probability that a shutdown occurs in a 1 month time-frame?

Suppose the computing system is currently undergoing a shutdown for repairs. What is the **probability** that the current shutdown is due to:

b. Hardware failure?

c. Software failure?

d. Suppose that, in fact, the probability that a combined hardware/software failure occurs is .006. Are these two types of failures independent? (Justify).

III. The very basics (this should be really easy):
Consider three events, A, B, and C satisfying $A \cup B \cup C = \mathcal{S}$, $P(A) = .4$, $P(B) = .2$ and $P(C) = .5$. Assume events B and C are mutually exclusive.

True or False:

a. $P(B \cup C) = .7$.

b. $P(B \cap C) = 0$.

c. B and C are independent events.

d. $P((A \cup B) \cap (A \cup C)) = 0$.

e. Events A, B and C are mutually exclusive.

f. $P(A \cup B \cup C) = 1$.

g. Given $P(A \cup B) = .52$, then $P(A \cap B) = .08$.

h. Given $P(A \cap B) = .08$ holds, then A and B are independent events.

i. Given $P(A \cap B) = .08$ holds, then A^C and B are independent events.

j. Given $P(A \cap B) = .08$ holds, then $P(A|B) = .08$.

Part II: Advanced topics (these should be harder):

IV. Let $\{X_i\},\ i = 1, 2, \ldots, n, \ldots$ be independent Poisson random variables with respective rates $\{\lambda_i\},\ i = 1, 2, \ldots, n, \ldots$. Show that if

$$\sum_{i=1}^{\infty} \lambda_i \text{ converges,}$$

then

$$\sum_{i=1}^{\infty} X_i \text{ converges a.s.}$$

V. Suppose $X_n \xrightarrow{d.} X$ and $Y_n \xrightarrow{p.} c$ where c is a constant.
Show:

$$X_n Y_n \xrightarrow{d.} cX.$$

VI. Suppose $X_1, X_2, \ldots, X_n, \ldots$ is a series of random variables, where $|X_i| \le Y,\ \forall i$, and where $E(Y) < \infty$.
Show: If

$$X_n \xrightarrow{p.} X,$$

then $E(|X_n - X|) \to 0$ as $n \to \infty$.
(*Hint: Consider Fatou's lemma.*)

VII. Let $X_1, X_2, \ldots, X_n$ be a set of **i.i.d.** random variables, with $X_i \in \mathcal{N}(\mu, \sigma)$, (i.e., all r.v.'s are normally distributed, with mean μ and variance σ^2) and μ, σ both finite. Suppose s^2 is the sample variance of the $\{X_i\}$.

a. Show that the random variable defined as

$$V := \frac{(n-1)s^2}{\sigma^2} \quad \text{is } \chi^2_{(n-1)} - \text{distributed.}$$

b. Show that the random variable defined as

$$U := \frac{\overline{x} - \mu}{s/\sqrt{n}} \quad \text{is } T_{(n-1)} - \text{distributed.}$$

(Note: $\overline{x}$ denotes the sample mean).

VIII. Let $X_1, X_2, \ldots, X_{n_1}$ be a set of **i.i.d.** random variables, with $X_i \in \mathcal{N}(\mu_1, 1)$, and let $Y_1, Y_2, \ldots, Y_{n_2}$ be a set of **i.i.d.** random variables, with $Y_i \in \mathcal{N}(\mu_2, 1)$, and further suppose X_i and Y_j are independent for all i, j. Define a new random variable V by

$$V := \sum_{i=1}^{n_1} (X_i - \overline{x})^2 \; + \; \sum_{i=1}^{n_2} (Y_i - \overline{y})^2.$$

a. Determine and describe the distribution of V (i.e., what type of random variable is V?).

b. Compute the expected value and variance for the r.v. V, i.e., $E(V)$ and $\mathrm{Var}(V)$.