IE 529 HW3 Zhenye Na Zna2

1. Use induction to prove Jensen's Inequality, as follows:

$$\sum_{i=1}^{n} \alpha_i f(x_i) \le f\left[\sum_{i=1}^{n} \alpha_i x_i\right]$$

Base case:

For $n=1,2$, the equality is true.

$$\alpha_1 f(x_1) + \alpha_2 f(x_2) \le f(\alpha_1 x_1 + \alpha_2 x_2)$$

Induction steps:

Assume that $n=m$, the inequality stays true.

$$\sum_{i=1}^{m} \alpha_i f(x_i) \le f\left(\sum_{i=1}^{m} \alpha_i x_i\right)$$

Then for $n=m+1$:

$$f\left(\sum_{i=1}^{m} \alpha_i x_i\right) = f\left[\alpha_{m+1} x_{m+1} + \sum_{i=1}^{m} \alpha_i x_i\right)$$

$$= f\left[\alpha_{m+1} x_{m+1} + (1-\alpha_{m+1}) \sum_{i=1}^{m} \frac{\alpha_i}{1-\alpha_{m+1}} x_i\right)$$

$$\ge \alpha_{m+1} f(x_{m+1}) + (1-\alpha_{m+1}) f\left[\sum_{i=1}^{m} \frac{\alpha_i}{1-\alpha_{m+1}} x_i\right.$$

as α_{m+1} is not related to i, so $\rightarrow$ $\alpha_{m+1} f(x_{m+1}) + f\left[\sum_{i=1}^{m} \alpha_i x_i\right]$

$$\ge \alpha_{m+1} f(x_{m+1}) + \sum_{i=1}^{m} \alpha_i f(x_i)$$

$$= \sum_{i=1}^{m+1} \alpha_i f(x_i)$$

So, with the method of mathematical Induction, Jensen's Inequality has been proved ※

2. We have noticed that $f(x) = \log(x)$ is concave on $(0, \infty)$

So, we take log on the left-hand side of the inequality.

$$\log\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} = \frac{1}{n}(\log x_1 + \log x_2 + \cdots + \log x_n) = \sum_{i=1}^{n} \frac{1}{n} \log x_i \le \log\left(\sum_{i=1}^{n} \frac{1}{n} x_i\right)$$

Because $\{x_i\}$ is a non-negative set, $f(x) = \log(x) \nearrow$

So, $\left(\prod_{i=1}^{n} x_i\right)^{1/n} \le \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)$ ※

3. $Y = \beta x + e$

(a)

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_n \end{bmatrix}$$

and $\beta = [\beta]$, $Y = X\beta + e$

The least-squares line minimizes

$$Q(\beta) = \sum_{i=1}^{n} (y_i - \beta x_i)^2 = (Y - X\beta)^T (Y - X\beta)$$

The least squares estimate $\hat{\beta}$ solves the first order equation : $\frac{\partial Q(\beta)}{\partial \beta} = 0$

and is given by

$$\hat{\beta} = (X^T X)^{-1} X^T Y = \sum_{i=1}^{n} (x_i^2)^{-1} \sum_{i=1}^{n} x_i y_i = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

(b) $\hat{\beta} = \sum_{i=1}^{n} w_i y_i$, where $w_i = \frac{x_i}{\sum_{j=1}^{n} x_j^2}$. It can be considered as a weighted sum of the independent normal random variables : $y_i \sim N(x_i\beta, \sigma^2)$, which is a normal distribution. So next step is to figure out μ_β and σ_β^2.

$$E(\hat{\beta}) = E\left(\sum_{i=1}^{n} w_i y_i\right)$$

$$= \sum_{i=1}^{n} w_i E(y_i) = \sum_{i=1}^{n} w_i \cdot (x_i \beta)$$

$$= \sum_{i=1}^{n} \frac{x_i}{\sum_j x_j^2} \cdot x_i \beta$$

$$= \beta \cdot \sum_{i=1}^{n} \frac{x_i^2}{\sum_j x_j^2} = \beta$$

$$Var(\hat{\beta}) = Var\left(\sum_{i=1}^{n} w_i y_i\right)$$

$$= \sum_{i=1}^{n} w_i^2 \cdot Var(y_i) = \sum_{i=1}^{n} w_i^2 \cdot \sigma^2$$

$$= \sigma^2 \times \sum_{i=1}^{n} \left[\left(\frac{x_i^2}{\sum_{j=1}^{n} x_j^2}\right)^2\right]$$

$$= \sigma^2 \times \frac{\sum_{i=1}^{n} x_i^2}{(\sum_{j=1}^{n} x_j^2)^2}$$

$$= \sigma^2 \times \frac{1}{\sum_{j=1}^{n} x_j^2}$$

So, $\hat{\beta} \sim N(\beta, \sigma_\beta^2)$ where $\sigma_\beta^2 = \sigma^2 \times \frac{1}{\sum_{j=1}^{n} x_j^2}$

(c). $SS_R = \sum_{i=1}^{n} (y_i - x_i\hat{\beta})^2 \sim \sigma^2 \chi^2_{(n-1)}$

(d)

A significance level γ test of H_0 :

reject H_0 if $\sqrt{\frac{(n-2) S_{xx}}{SS_R}} |B| > t_{\frac{\gamma}{2}, n-1}$

accept H_0 otherwise

rejecting H_0 if the desired significance level is at least as large as:

$$p\text{-value} = P\{|T_{n-1}| > v\}$$

$$= 2P\{T_{n-1} > v\}$$