LECTURE NOTES ON ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Contents

1. Introduction to harmonic function and mean value properties

INTRODUCTION. In this section, we firstly introduce some historical viewpoints of the elliptic equations, especially in *physics* and *geometry.* Indeed, the motivation of studying Laplace operator Δ derived a famous functional, say *kinetic energy*, namely,

$$\frac{1}{2}\int_{\Omega}|\nabla u|^2\mathrm{d}x$$

, where u is the position function. Historically, the study of such critical point of energy - say *harmonic function* later - started from some seminal works in *Complex Analysis* [SS10], particularly, the *Cauchy integral formula.* Fortunately, as $n > 2$, the harmonic functions preserve certain mean value properties as the *holomorphic functions*, which will be presented in this section. Furthermore, we will show more "perfect" properties similar to the holomorphisms even in weak sense, such as Theorem 1.6.

Je m'entretenais avec L..... je le félicitais d'un ouvrage qu'il venait de publier et lui demandais comment le nom de Dieu, qui se reproduisait sans cesse sous la plume de Lagrange, ne s'était pas présenté une seule fois sous la sienne. C'est, me répondit-il, que je n'ai pas eu besoin de cette hypothèse.

– The Last Moments of Napoleon (1825)

1.1. **Introduction and Background.** At the beginning of the note, inspired by physical phenomenon, we consider the *Euler-Lagrange equation* of the *energy functional* E as follows

$$E[u] := \int_{\Omega}|\nabla u|^2\mathrm{d}x,$$

where $\Omega \subset \mathbb{R}^n$ is connected with C^1 boundary[1], say $\partial\Omega \in C^1$, and $E : C^2(\Omega) \cap C^1(\overline{\Omega}) \to \mathbb{R}$. For any $\eta \in C_c^{\infty}(\Omega),\ \epsilon \in (0,1)$, integrate by parts, it follows that

$$0 = \frac{\mathrm{d}}{\mathrm{d}\epsilon}|_{\epsilon=0}E[u+\epsilon\eta] = \int \nabla u \cdot \nabla\eta = -\int \eta\cdot\Delta u.$$

Using the arbitrary of η, we obtain

$$\Delta u = 0. \tag{1.1}$$

We say that $u \in C^2(\Omega)\cap C^1(\overline{\Omega})$ is **harmonic** and the equation (1.1) is **Laplace equation**. Furthermore, for some $f \in L^1(\Omega)$, $-\Delta u = f$ is called **Poisson equation**.

Besides, we introduce more different "such" partial differential equations by calculating the critical point of certain functionals.

Some geometrists may be interested in following the area functional A,

$$A[u] := \int_{\Omega}\sqrt{1+|\nabla u|^2}\mathrm{d}x,$$

[1] We always set these assumptions in this note.

which illustrates the area of the graph of u on the domain Ω. By immediate calculation, we obtain the following equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=0, \tag{1.2}$$

which is called the *minimal surface equation* (MSE). Someone familiar with the *complex analysis* knows that there are many different types of harmonic functions on the whole complex plane - if we regard the harmonic functions as the real part of *analytic* functions - or general $\mathbb{R}^n$, and we call them *entire functions*. One can see more details in *Weierstrass's theorem* in [SS10]. However, a well-known theorem - inspired from the *Bernstein problem* - says that the solution of (1.2) is "flat" in certain dimensions. We show this theorem, which may be presented in some books [GW84, CMI24].

Theorem 1.1 (Bernstein-Fleming-De Giorgi-Almgren-Simons-Bombieri-Giusti, 1915-1969)**.** The entire solution of equation (1.2) is only a plane when $n \leq 7$.

We call theorem 1.1 a certain *Liouville-type* theorem, which is named from a famous result by Liouville, that is, any bounded entire function is constant. Indeed, furthermore, we know that an unbounded entire function dominated by polynomials is definitely polynomial. We will prove this in the following sections by the gradient estimate. However, we don't explore more details in this theorem, which is related to the *minimal surfaces* [CMI24] and *geometric measure theory* [GW84, Sim14] and exceeds the category of PDEs.

We introduce another example by adding a nonlinear term in E. Someone call the following functional the *Ginzburg-Landau* energy functional, that is

$$G[u]:=\int_\Omega \frac{1}{2}|\nabla u|^2+W(u)\mathrm{d}x,$$

where $W(u)$ is usually called the *double-well potential*, when it has similar properties as $(1-u^2)^2$ [Bra02]. It is easy to show the Euler-Lagrange equation of the functional above, i.e.

$$-\Delta u+W'(u)=0.$$

Particularly, we call

$$\Delta u+u-u^3=0,\quad u\in[1,-1], \tag{1.3}$$

the *Allen-Cahn equation* (ACE). Someone familiar with the physical background of Ginzburg-Landau functional can obtain the following functional by scaling (*blow-down*) the solution [Sav10], that is

$$G_\epsilon[u]=\int_{B_1}\frac{\epsilon}{2}|\nabla u|^2+\frac{1}{\epsilon}W(u)\geq\int_{B_1}\sqrt{2W(u)}|\nabla u|\mathrm{d}x=\int_{-1}^1\sqrt{2W(u)}\mathrm{d}_{\mathcal{H}^{n-1}(u=s)}\mathrm{d}s.$$

where $\mathcal{H}^{n-1}(u=s)$ means the Hausdorff measure in $n-1$ dimension of the level set $\{u=s\}$. Therefore, the minimizer of G_ϵ is related to minimal surfaces. In the 1980s, De Giorgi [DGS79] conjectured that the solution of (1.3) shares the same properties as those of equation (1.2) under certain conditions - a conjecture now known as the De Giorgi Conjecture. In previous decades, the conjecture was solved. We refer the readers to professor Wei's survey [CW18] for more discussion and related topics.

Theorem 1.2 (De Giorgi-Ghoussoub-Gui-Cabré-Savin-del Pino–Kowalczyk–Wei, 1979-2011)**.** The level sets of bounded, entire and monotone solutions of the equation (1.3) are all flat when $n \leq 8$.

The third example is the *Monge–Ampère equation* (MAE), we show a simple version, i.e.

$$\det(D^2u) = f. \tag{1.4}$$

The geometric version is related to the *Minkowski problem* [CY76] - in real version, since the f above is the *Gauss curvature* of graph of u - and the *Calabi conjecture* [Yau77] - in complex version. On the other hand, a well-known application is the *optimal transportation* [Vil21].

In this series of lectures, we will show the basic theory of different types of such equations. Indeed, one can divide the equations into several categories,

- $-\Delta u = f$, linear equation
- $-\Delta u + W(u) = 0$, semilinear equation
- $\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$, div-type equation
- $\det(D^2u) = f$, totally nonlinear equation

And all of them are called elliptic equations, which will be defined in the following lectures.

Some may ask what the goals or principles of (elliptic) PDEs are. Indeed, we can merely construct certain types of solutions in (1.1) or rare kinds of solutions in (1.2) and (1.3). Thus, we show the systematic construction methods of the Poisson equation in the initial sections, that is, *Green's representation*, which is related to the *Cauchy integral formula* in complex analysis. Additionally, we will consider the well-posedness problem of elliptic equations, i.e., *existence* and *regularity*, in the following lectures.

1.2. **Mean value properties.** Initially, we introduce two equivalent mean value properties.

Definition 1.1. For $u \in C(\Omega)$, we say u satisfies the **first mean value property**, if

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x,r)} u(y) \mathrm{d}S_y,$$

where $B(x,r)$ is a ball centered at x with radius r, and the **second mean value property**, if

$$u(x) = \frac{n}{\omega_n r^n} \int_{B(x,r)} u(y) \mathrm{d}y,$$

where ω_n is the area of unit sphere S^{n-1} in $\mathbb{R}^n$.

One can easily show that the two properties are equal by integrating both sides of

$$u(x)\omega_n r^{n-1} = \int_{\partial B(x,r)} u(y)\mathrm{d}S_y.$$

Immediately, we show a useful consequence for functions with mean value properties.

Proposition 1.1 ((strong) maximum principle)**.** For $u \in C(\overline{\Omega})$ that satisfies mean value properties in Ω, and Ω is bounded, then it assumes its maximum or minimum on $\partial\Omega$ unless it is constant.

Proof. We use the *continuity method* and merely show the maximum version. Let

$$\mathcal{M} := \{x \in \Omega : u(x) = M = \max_{\overline{\Omega}} u\}.$$

Since Ω is bounded, $M < \infty$. Obviously, $\mathcal{M}$ is not empty and closed, thus it remains to show $\mathcal{M}$ is open. For all $x_0 \in \mathcal{M}$, let $B(x_0, r) \subset \Omega$, we obtain

$$M = u(x_0) = \frac{n}{\omega_n r^n} \int_{B(x,r)} u(y)\mathrm{d}y \leq M.$$

Thus, $B(x, r) \subset \mathcal{M}$, i.e., $\mathcal{M}$ is open. $\square$

Observed that the maximum version merely needs the one side of inequality, i.e., $u(x) \leq \frac{n}{\omega_n r^n} \int u(y)$, which is related to the *subharmonic functions*; we will define them in the following lectures. Indeed, in various situations, we have "mean value inequality" or merely in certain domains, however, we still have the maximum (minimum) principle.

Remark 1.1. We mention that the maximum principle above is called the *strong* maximum principle, since the maximum points can't be assumed both on the boundary and interior. Usually, in this case, giving two (sub)harmonic functions u, v agreeing with the same Dirichlet problem such that $u - v \geq 0$, then $u \equiv v$ or $u > v$, which will be used to prove the uniqueness of the Dirichlet problem in the following subsections.

Second consequence of the mean value properties is *Harnack inequality*, which tells that the inferior can dominate the superior.

Proposition 1.2 (Harnack)**.** For $u \in C(\overline{\Omega})$ that satisfies the mean value properties. For any compact $K \subset \Omega$, there is a constant $C = C(\Omega, K)$ such that if $u \geq 0$, then

$$\frac{1}{C}u(y) \leq u(x) \leq Cu(y), \quad x, y \in K$$

Proof. by mean value properties in $B(x, R) \subset B(x, 4R) \subset \Omega$, we obtain

$$\frac{1}{c}u(y) \leq u(x) \leq cu(y), \quad x, y \in B(x, R)$$

where c is only dependent on n. Now given K, we finitely cover K by $\{B(x_i, R)\}_{i=1}^N$ such that $4R < \mathrm{dist}(K, \partial\Omega)$. Then we let $C = c^N$. $\square$

Additionally, we can write the mean value properties by following the one-variable version,

$$u(x) = \frac{1}{\omega_n} \int_{|w|=1} u(x + rw)\mathrm{d}S_w,$$

and

$$u(x) = \frac{n}{\omega_n} \int_{|z|\le 1} u(x + rz)\mathrm{d}z.$$

Using this expression, we can prove a part of the following theorem.

Theorem 1.3. For $u \in C(\Omega)$, then u satisfies the mean value property if and only if u is harmonic.

This theorem tells us that the mean value property is equal to the harmonic with the assumption of higher regularity (C^2). Thus, any harmonic function satisfies the maximum (minimum) principle. Firstly, we can easily calculate one side of the theorem. Indeed, for all $x \in \Omega$ and $B(x, \rho) \subset \Omega$, we derivative

$$\phi(\rho) := \frac{1}{\omega_n} \int_{|w|=1} u(x + \rho w)\mathrm{d}S_w$$

with respect to r. We obtain

$$\phi'(\rho) = \frac{1}{\omega_n} \int_{|w|=1} \nabla u(x+\rho w)\cdot w\mathrm{d}S_w = \frac{1}{\omega_n} \int_{\partial B(0,1)} \partial_\nu u\mathrm{d}S_w = -\frac{1}{\omega_n} \int_{|z|\le 1} \Delta u(x+\rho z)\mathrm{d}z = 0$$

i.e., $\phi'(\rho) = 0$, we have

$$u(x) = \lim_{\rho\to 0} \phi(\rho) = \frac{1}{\omega_n} \int_{|w|=1} u(x + \rho w)\mathrm{d}S_w$$

Secondly, to prove the remaining side, we demonstrate the following lemma through a process of modifying functions. We say $\varphi_\epsilon(x)$ is a *modification*, if $\varphi_\epsilon(x) = \frac{1}{\epsilon^n}\varphi(\frac{x}{\epsilon})$. Where $\varphi \in C_c^\infty(B_1)$, $\varphi(x) = \psi(|x|)$, where

$$\psi(r) = \begin{cases} C \cdot e^{\frac{1}{r^2-1}}, & r < 1 \\ 0, & r \ge 1 \end{cases}, \quad C \text{ is a constant}$$

and

$$\int_{B_1} \varphi = 1$$

It is clear that $u * \varphi_\epsilon \in C^\infty$, where

$$u * \varphi_\epsilon = \int_\Omega u(y) \cdot \varphi_\epsilon(x - y)\mathrm{d}y$$

We call it *convolution*. Since we have shown that if $u \in C^2$ then the mean value properties are equal to the harmonic in the previous proposition, we merely need to verify $u \in C^2$. Indeed, u is smooth.

Lemma 1.1. $u * \varphi_\epsilon = u$, thus u is smooth.

Proof. Let $\Omega^\epsilon = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \epsilon\}$, we consider the convolution at $x \in \Omega^\epsilon$

$$\begin{aligned} u * \varphi_\epsilon(x) &= \int_{\Omega^\epsilon} u(x+y)\cdot\varphi_\epsilon(y)\mathrm{d}y \\ &= \int_{|y|<\epsilon} u(x+y)\cdot\frac{1}{\epsilon^n}\varphi(\frac{y}{\epsilon})\mathrm{d}y \\ &= \int_{|z|<1} u(x+\epsilon z)\cdot\varphi(z)\mathrm{d}z \\ &= \int_0^1 r^{n-1}\cdot\psi(r)\mathrm{d}r\int_{|z|=1} u(x+\epsilon z)\mathrm{d}z \\ &= \frac{1}{\omega_n}\cdot\omega_n\cdot u(x) = u(x) \end{aligned}$$

Finally, let $\epsilon \to 0$, we obtain the smoothness in the whole domain. Therefore, we prove the theorem. □

Finally, we claim that the method of convolution is widely used in some approximation, that is

$$u * \varphi_\epsilon \to u, \quad \text{in certain topology}$$

where we can strengthen the approximation to a stronger topology (norm) such as L^p. And one can see more details in [SS09]. However, in the version of mean properties, we observed that u is invariant under normal convolution.

1.3. **Gradient estimate (toy version) and Liouville theorem.** We have known that the harmonic functions satisfy the maximum principle, which implies the uniqueness of the following *Dirichlet problem* in a bounded domain Ω, that is

$$\begin{cases} \Delta u = f, & x \in \Omega \\ u = \varphi, & x \in \partial\Omega \end{cases},$$

where f, φ are continuous. Indeed, assume there is another solution called v, we consider $w = u - v$. Obviously w is harmonic in Ω and vanishes on the boundary. Using the maximum principle, we find $w = 0$ in Ω. However, the Dirichlet problem lacks uniqueness in unbounded domains since the unbounded domain doesn't satisfy the condition of the maximum principle. For example, consider

$$\begin{cases} \Delta u = 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases},$$

where $\Omega := \{x \in \mathbb{R}^n : |x| > 1\}$. We can construct nontribyl solutions. For $n = 2$, we let $u(x) = \log|x|$, and note that $u \to \infty$ as $|x| \to \infty$, which is unbounded. However, when $n \geq 3$, we let $u = |x|^{2-n} - 1$, and note that u is bounded. Besides, we can consider a monotone solution in the half space $\mathbb{H}^n := \{x \in \mathbb{R}^n : x_n > 0\}$, that is $u(x) = x_n$.

However, we claim the entire function has some rigidities, using the following gradient estimates.

Lemma 1.2 (gradient estimate)**.** Let u a harmonic function such that $u \in C(\Omega)$, then

$$|\nabla u(x_0)| \leq \frac{n}{R} \max_{\overline{B_R}} |u|,$$

where $B_R := B(x_0, R) \subset \Omega$.

Proof. It is no hard to let $u \in C^1$, and obviously u_i is harmonic. by mean value properties and Stokes' formula, we obtain

$$u_i(x_0) = \frac{n}{\omega_n R^n} \int_{B_R} u_i(y) \mathrm{d}y = \frac{n}{\omega_n R^n} \int_{\partial B_R} u(y) \nu_i \mathrm{d}S_y.$$

Thus

$$|u_i(x_0)| \leq \frac{n}{\omega_n R^n} \cdot \max_{\partial B_R} |u| \cdot \omega_n R^{n-1} \leq \frac{n}{R} \max_{\overline{B_R}} |u|.$$

□

Therefore, we can show the well-known Liouville theorem.

Theorem 1.4 (Liouville)**.** A bounded entire function is definitely constant.

Proof. Let u be an entire function. by the gradient estimate, we have

$$|\nabla u(x_0)| \leq \frac{n}{R} \max_{\overline{B(x_0,R)}} |u|, \quad \forall x_0 \in \mathbb{R}^n.$$

let $R \to \infty$, since u is bounded, $|\nabla u(x_0)| \to 0$. We complete the proof. □

Furthermore, we demonstrate a higher gradient estimate.

Lemma 1.3. Same settings in lemma 1.2, for multiple index α such that $|\alpha| = m$, we have

$$|D^\alpha u(x_0)| \leq \frac{n^m e^{m-1} m!}{R^m} \max_{\overline{B_R}} |u|.$$

Proof. We use induction. For $m = 1$, the estimate is true; we assume m is true, and it remains to prove $m + 1$. However, be careful that we don't know the maximum point of $D^m u$. Thus, let a smaller ball B_r such that $r = (1 - \theta)R$, where $\theta \in (0, 1)$, we obtain

$$|D^{m+1} u(x_0)| \leq \frac{n}{r} \max_{\overline{B_r}} |D^m u|.$$

by hypothesis, we have

$$\max_{\overline{B_r}} |D^m u| \leq \frac{n^m e^{m-1} m!}{(R - r)^m} \max_{\overline{B_R}} |u|.$$

Thus

$$|D^{m+1} u(x_0)| \leq \frac{n^{m+1} e^{m-1} m!}{R^{m+1} \theta^m (1 - \theta)} \max_{\overline{B_R}} |u|.$$

Remain to show $1/\theta^m(1 - \theta) \leq e(m + 1)$, let $\theta = m/(1 + m)$. We complete the proof. □

Remark 1.2. Finally, we claim that harmonic functions are not only smooth but also analytic. Indeed, some smooth functions are analytic, such as

$$\varphi(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}.$$

Theorem 1.5. A harmonic function is analytic.

Proof. by Taylor expansion, we obtain

$$u(x+h) = u(x) + \sum_{i=1}^{m-1} \left[(h_1\partial_1 + \cdots + h_n\partial_n)^i u \right](x) + R_m(h), \quad |h| < R,$$

where

$$R_m(h) = \frac{1}{m!}(h_1\partial_1 + \cdots + h_n\partial_n)^m u(x_1 + \theta_1 h_1, \cdots, x_n + \theta_n h_n), \quad \theta_i \in (0,1).$$

Using a higher gradient estimate, we have

$$|R_m(h)| \leq \frac{1}{m!}|h|^m \cdot n^m \left(\frac{n^m e^{m-1} m!}{R^m} \right) \max_{\overline{B_{2R}}} |u| \leq \left(\frac{|h| n^2 e}{R} \right)^2 \max_{\overline{B_{2R}}} |u|.$$

Thus, let $|h| \leq n^2 eR/2$, we complete the proof. □

1.4. **Weyl lemma.** Now, we demonstrate another property of harmonic functions, which is usually called ”in a weak sense”.

Theorem 1.6 (Weyl)**.** Let $u \in C(\Omega)$ satisfies

$$\int_\Omega u \Delta\varphi \mathrm{d}x = 0, \quad \forall \varphi \in C_c^\infty(\Omega)$$

Then u is smooth; furthermore, u is harmonic.

Proof. We divide the proof into two main steps.

Step 1 We claim that if a sequence of harmonic functions $\{u_n\}_{n=1}^\infty$ such that $u_n \to u$ in L^1 sense, then u is harmonic.

by mean value properties, we fix any $x \in \Omega$ such that $B(x,r) \subset \Omega$,

$$|u_n(x) - u_m(x)| \leq \frac{n}{\omega_n r^n} \int_{B(x,r)} |u_n(y) - u_m(y)| \mathrm{d}y.$$

Thus, it is clear that $\{u_n(x)\}_{n=1}^\infty$ is Cauchy. Furthermore, $\{u_n\}_{n=1}^\infty$ is uniformally convergent to u in any compact subset of Ω, say $u = \lim_{n\to\infty} u_n$.[2] by the mean value property,

$$u_n = \frac{n}{\omega_n r^n} \int_{B(x,r)} u_n(y) \mathrm{d}y.$$

And let $n \to \infty$, we obtain

$$u(x) = \frac{n}{\omega_n r^n} \int_{B(x,r)} u_n(y) \mathrm{d}y.$$

[2]See the *compact convergence theorem*, [GT77] theorem 2.11.

Thus, u is harmonic.

Step 2 We start to construct such $\{u_n\}_{n=1}^{\infty}$ by using modification. With the same notations, we claim that $u_\epsilon := u * \varphi_\epsilon$ is desired. Indeed,

$$\Delta u_\epsilon(x) = \int_\Omega u(x-y) \cdot \Delta\varphi_\epsilon(y)\mathrm{d}y = 0.$$

On the other hand, since $C_c^\infty(\Omega)$ is dense in $L^1(\Omega)$, we consider $g \in C_c^\infty(\Omega)$ to approximate u, then

$$\|u_\epsilon - u\|_{L^2} \le \|(u-g) * \varphi_\epsilon\|_{L^2} + \|g * \varphi_\epsilon - g\|_{L^2} + \|g - u\|_{L^2},$$

where the first term, we recall *Young's inequality* [SS09], that is

$$\|(u-g) * \varphi_\epsilon\|_{L^1} \le \|u-g\|_{L^1} \cdot \|\varphi_\epsilon\|_{L^1}.$$

And the third term is tribyl. Finally, we show the second term,

$$\int_\Omega \left| \int_\Omega g(x-y)\varphi_\epsilon(y)\mathrm{d}y - g(x) \right| \mathrm{d}x.$$

As the standard trick in identity approximation [SS09], we divide the domain Ω into two part,

$$\begin{aligned} &\int_\Omega \left| \int_\Omega g(x-y)\varphi_\epsilon(y)\mathrm{d}y - g(x) \right| \mathrm{d}x \\ &\le \int_\Omega \left| \int_{\{|y|<\delta\}\cap\Omega} |g(x-y) - g(x)| \cdot \varphi_\epsilon + 2\|g\|_{L^\infty} \cdot \int_{\{|y \ge \delta|\}\cap\Omega} \varphi_\epsilon \right| \mathrm{d}x. \end{aligned}$$

Since g is uniformly continuous on $\overline{\Omega}$, we can dominate the $|g(x-y)-g(x)|$ through a sufficiently small δ, and the other term is dominated by letting $\epsilon \to \infty$.

□

Remark 1.3. Finally, by the proof, we claim that the theorem 1.3 and 1.6 hold for $u \in L^1_{loc}$.

2. Fundamental solution and Green representation

INTRODUCTION. In this section, we will show more similarities as the holomorphisms in complex analysis, such as the Cauchy integral formula - see Green's representation, which illustrates that the value of a harmonic function may be determined by surrounding values - and Riemannian removable singularity theorem - see Theorem 2.4. Additionally, we refer the readers to [GS54] for more discussions about the singularities. On the other hand, we will present Perron's seminal work on the existence of Dirichlet problem, see subsection 2.4 or [Per23].

... important parts of mathematics were influenced by Dirichlet. His proofs characteristically started with surprisingly simple observations, followed by extremely sharp analysis of the remaining problem. With Dirichlet began the golden age of mathematics in Berlin.

– Koch (1998)

2.1. **Fundamental solution.** Firstly, we want to find a radial solution of $\Delta u = 0$, namely, one that only depends on radius r instead of rotation. Say $u(x) = v(r)$, where $r = |x-a|$, $a \in \mathbb{R}^n$. Calculate

$$u_{ii} = v'' \cdot \frac{x_i^2}{|x-a|^2} + v' \cdot \left(\frac{1}{|x-a|} - \frac{x_i^2}{|x-a|^3} \right),$$

thus,

$$\Delta u = v'' + (n-1)\frac{v'}{r}.$$

It is clear that the solution of the equation above is

$$v = \begin{cases} c_1 + c_2 \log|a-x|, & n = 2 \\ c_3 + c_4 r^{2-n}, & n \geq 3 \end{cases}.$$

Let $c_1 = c_3 = 0$, and $\int_{\partial B_r} \partial_r v = -1$, Hence,

$$\Gamma(a,x) := v(r) = \begin{cases} -\frac{1}{2\pi} \log|a-x|, & n = 2 \\ -\frac{1}{(2-n)\omega_n} r^{2-n}, & n \geq 3 \end{cases}, \quad a \in \mathbb{R}^n.$$

We call Γ the **fundamental solution**[3].

Remark 2.1. (1) $\Gamma(a,x)$ is harmonic at $x \neq a$, that is $\Delta_x \Gamma = 0, \quad x \neq a$.

(2) It is easy to verify
- If $a \in \overline{\Omega}$, then $\int_{\partial\Omega} \frac{\partial \Gamma}{\partial \nu} = -1$;
- If $a \notin \overline{\Omega}$, we consider two conditions: firstly, if $\partial\Omega \in C^1$, then $\int_{\partial\Omega} \frac{\partial\Gamma}{\partial\nu} = -1/2$, but if $\partial\Omega$ is just continuous, such as a is 1/4-corner, then $\int_{\partial\Omega} \frac{\partial\Gamma}{\partial\nu} = -1/4$.

(3) For any $r > 0$, $\Gamma(a,x) \in L^1(B(a,r))$, but $\Gamma(a,x) \notin L^1(\mathbb{R}^n)$.

Recall the *Cauchy integral formula* in complex analysis [SS10], we have a general formula for any C^2 function.

Theorem 2.1 (Green representation)**.** Suppose Ω a bounded domain in $\mathbb{R}^n$, and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Then for any $a \in \Omega$, we have

$$u(a) = -\int_\Omega \Gamma(a,x)\Delta u(x)\mathrm{d}x + \int_{\partial\Omega} \Gamma(a,x)\frac{\partial u}{\partial \nu_x} - u(x)\frac{\partial \Gamma}{\partial \nu_x}(a,x)\mathrm{d}S_x \tag{2.1}$$

Next, we introduce two basic tools.

Lemma 2.1 (gradient estimate of fundamental solution)**.** (1)

$$|\Gamma(a,x)| \leq \begin{cases} C|\log r|, & n = 2 \\ C \cdot r^{-(n-2)}, & n \geq 3 \end{cases}.$$

(2)

$$|D^k \Gamma(a,x)| \leq \frac{C}{r^{n-2+k}},$$

where $k \geq 1$.

[3]One may see more explanations about this name in [Eva22, Zho05].

Lemma 2.2 (Green)**.** Let $\Omega \subset \mathbb{R}^n$ bounded and open with $\partial\Omega \in C^1$, and $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$, then

$$\int_\Omega u\Delta v - v\Delta u \mathrm{d}x = \int_{\partial\Omega} u\frac{\partial v}{\partial\nu} - v\frac{\partial u}{\partial\nu}\mathrm{d}S_y.$$

Now, we start to prove the theorem.

Proof of theorem 2.1. It is no hard to let $a = 0$, and we denote $\Gamma(x, y)$ as $\Gamma(x-y)$.

Firstly, since Γ has a singularity at 0, we delete a small ball of 0. by lemma 2.2, we have

$$\begin{aligned}
&\int_{\Omega - B(x,\epsilon)} u(y)\Delta_y\Gamma(x-y) - \Gamma(x-y)\Delta_y u(y)\mathrm{d}y \\
&= \int_{\partial\Omega} u(y)\frac{\partial\Gamma}{\partial\nu}(x-y) - \Gamma(x-y)\frac{\partial u(y)}{\partial\nu}\mathrm{d}S_y \\
&- \int_{\partial B(x_\epsilon)} u(y)\frac{\partial\Gamma}{\partial\nu}(x-y) - \Gamma(x-y)\frac{\partial u(y)}{\partial\nu}\mathrm{d}S_y.
\end{aligned}$$

Secondly, the first term on the left side is tribyl, and we estimate the second term.

$$\int_{\Omega - B(x,\epsilon)} \Gamma(x-y)\Delta_y u(y)\mathrm{d}y = \int_\Omega \Gamma(x-y)\Delta_y u(y)\mathrm{d}y - \int_{B(x,\epsilon)} \Gamma(x-y)\Delta_y u(y)\mathrm{d}y,$$

and the second term tends to zero as $\epsilon \to 0$ by lemma 2.1. Then we estimate the third and fourth terms. Consider

$$-\int_{\partial B(x,\epsilon)} u(y)\frac{\partial\Gamma}{\partial\nu}(x-y)\mathrm{d}S_y = u(x) - \int_{\partial B(x_\epsilon)} [u(y) - u(x)] \cdot \frac{\partial\Gamma}{\partial\nu}(x-y).$$

The second term on the right side, say A, we estimate

$$|A| \le \int_{\partial B(x,\epsilon)} |u(y) - u(x)| \cdot \left|\frac{\partial\Gamma}{\partial\nu}(x-y)\right| \mathrm{d}S_y \le \Omega_n \cdot \epsilon^{n-1} \cdot \epsilon \cdot \epsilon^{-(n-2)},$$

which is dominated by ϵ from lemma 2.1. Finally, similarly, the fourth term can be estimated by

$$\left|\int_{\partial B(x_\epsilon)} \Gamma(x-y)\frac{\partial u(y)}{\partial\nu}\mathrm{d}S_y\right| \le \omega_n \cdot \epsilon^{n-1} \cdot \epsilon^{-(n-2)} \cdot C_0,$$

where C_0 is from continuity of Du. Therefore, we complete the proof. $\square$

Examining formula 2.1, we observe that the value surrounding it determines the value of the harmonic function at a single point, such as the value of the boundary. However, one can neglect the boundary term by a specific method of compact support, which we refer to as the *cutoff* function. In unit ball B_1, let $\varphi \in C_c^\infty(B_1)$, precisely, $\varphi = 1$ in $B(r)$ and $\varphi = 0$ outside the B_R, where $0 < r < R < 1$. Then using the same proof above to u and $\varphi\Gamma(a, x)$ in, we obtain

$$u(a) = \int_{r<|x|<R} u(x) \cdot \Delta(\varphi(x)\Gamma(a, x))\mathrm{d}x, \quad a \in B_r.$$

Thus, we can use another method (without using the mean value properties) to prove the following "Sobolev embedding" and gradient estimate, which are

$$\|u\|_{L^\infty(B_{1/2})} \le C\|u\|_{L^p(B_1)}$$

and

$$[\|\nabla u\|_{L^\infty(B_{1/2})} \le C\|u\|_{L^\infty(B_1)}.$$

Furthermore, using theorem 2.1, we can solve the equation $-\Delta u = f(x)$, where $f \in C_c^2(\Omega)$, that is

$$u = \Gamma * f.$$

However, we can no longer deal with the boundary term, which will be addressed in the next subsection.

2.2. **Green functions.** Firstly, we demonstrate the Dirichlet problem,

$$\begin{cases} -\Delta u = f(x), & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega, \end{cases}$$

where $f \in C(\overline{\Omega}),\ g \in C(\partial\Omega)$. Recall theorem 2.1, we discover

$$u(x) = -\int_\Omega \Gamma(x-y)\Delta u(y)\mathrm{d}y + \int_{\partial\Omega} \Gamma(x-y)\frac{\partial u(y)}{\partial\nu} - u(y)\frac{\partial\Gamma}{\partial\nu}(x-y)\mathrm{d}S_y, \tag{2.2}$$

where we lack information in Du on the boundary, except for the Neumann boundary. Thus, we introduce a new harmonic function to delete this term. Let $\Phi(x,y)$ such that $\Phi(x,\cdot) \in C^2(\Omega)$ and

$$\begin{cases} -\Delta_y \Phi(x,y) = 0, & y \in \Omega \\ \Phi(x,y) = \Gamma(x,y), & y \in \partial\Omega \end{cases}.$$

Apply lemma 2.2 to u and $\Phi(x,y)$, we obtain

$$\int_\Omega u\Delta_y\Phi(x,y) - \Phi(x,y)\Delta_y u\mathrm{d}x = \int_{\partial\Omega} u\frac{\partial\Phi}{\partial\nu}(x,y) - \Phi(x,y)\frac{\partial u}{\partial\nu}\mathrm{d}S_y.$$

Add it to equation (2.2), we have

$$u(x) - \int_\Omega \Phi(x,y)\Delta_y u = -\int_\Omega \Gamma(x-y)\Delta_y u + \int_{\partial\Omega} u(y)\left[\frac{\partial\Phi}{\partial\nu}(x,y) - \frac{\partial\Gamma}{\partial\nu}(x-y)\right]\mathrm{d}S_y.$$

Let $G(x,y) := \Gamma(x-y) - \Phi(x,y)$, which is called the **Green function**. We have

$$u(x) = -\int_{\partial\Omega} g\frac{\partial G}{\partial\nu}(x,y) - \int_\Omega fG(x,y).$$

Obviously, $G(x,y) = 0$ when $y \in \partial\Omega$ and G is unique for a given domain by the maximum principle.

Now, we are going to show a nontrivial property of Green functions, which is related to the *self-adjoint* property of the Laplace operator [Con19].

Theorem 2.2 (Symmetry of Green functions)**.** For any $x_1, x_2 \in \Omega$, we have

$$G(x_1,x_2) = G(x_2,x_1).$$

Proof. Firstly, we consider domain $\Omega_\epsilon := \Omega - B(x_1,\epsilon) \cup B(x_2,\epsilon)$, where $B(x_1,\epsilon) \cap B(x_2,\epsilon) = \emptyset$. And let $G_1(y) = G(x_1, y),\ G_2(y) = G(x_2, y)$, for $y \in \Omega_\epsilon$. Remain to prove $G_1(x_2) = G_2(x_1)$.

by lemma 2.2, we obtain

$$\begin{aligned}
0 &= \int_{\Omega_\epsilon} G_1(y)\Delta_y G_2(x-y) - G_2(x-y)\Delta_y G_1(y)\mathrm{d}y \\
&= \int_{\partial\Omega} G_1(y)\frac{\partial G_2}{\partial \nu}(y) - G_2(y)\frac{\partial G_1(y)}{\partial \nu}\mathrm{d}S_y \\
&- \int_{B(x_1,\epsilon)} G_1(y)\frac{\partial G_2}{\partial \nu}(y) - G_2(y)\frac{\partial G_1(y)}{\partial \nu}\mathrm{d}S_y \\
&- \int_{B(x_2,\epsilon)} G_1(y)\frac{\partial G_2}{\partial \nu}(y) - G_2(y)\frac{\partial G_1(y)}{\partial \nu}\mathrm{d}S_y.
\end{aligned}$$

Obviously, the first and second terms on the right side are tribyl. By symmetry, merely estimate the third and fourth terms on the right side.

Firstly, we consider the third term,

$$\begin{aligned}
\left|\int_{\partial B(x_1,\epsilon)} G_1 \frac{\partial G_2}{\partial \nu}\mathrm{d}S_y\right| &\le \int_{\partial B(x_1,\epsilon)} |G_1| \cdot \left|\frac{\partial G_2}{\partial \nu}\right| \mathrm{d}S_y \\
&\le C_0 \int_{\partial B(x_1,\epsilon)} |\Gamma(x-y)| + |\Phi(x,y)|\mathrm{d}S_y \\
&\le C_0 \left[\int_{\partial B(x_1,\epsilon)} |\Gamma(x-y)| + C_1 \int_{\partial B(x_1,\epsilon)}\right] \\
&\le C_0[\epsilon^{n-1} \cdot \epsilon^{-(n-2)} + C_1\epsilon^{n-1}],
\end{aligned}$$

which is trivial as $\epsilon \to 0$. Finally, the fourth term,

$$\int_{\partial B(x_1,\epsilon)} G_2 \frac{\partial G_1}{\partial \nu}\mathrm{d}S_y = \int_{\partial B(x_1,\epsilon)} G_2 \left[\frac{\partial \Gamma}{\partial \nu}(x-y) + \frac{\partial \Phi}{\partial \nu}(x,y)\right] \to G_2(x)$$

as $\epsilon \to 0$. □

Finally, we show some simple properties for Green functions.

(1) $\Delta_y G(x,y) = 0, \quad x \neq y;$
(2) As $y \to x$, we have estimate

$$G(x,y) \sim \Gamma(x-y) \to \infty;$$

(3) For all $x \in U$, where $U \subset \Omega$, we have

$$\int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x,y)\mathrm{d}S_y = -1,$$

if $x \notin \overline{\Omega}$, then

$$\int_{\partial U} \frac{\partial G}{\partial \nu}(x,t)\mathrm{d}S_y = 0.$$

2.3. **Poisson Kernel.** Now, we want to calculate some specific Green functions.

Half space $\mathbb{R}^n$. Recall $G(x,y) = \Gamma(x-y) - \Phi(x,y)$,

$$\begin{cases} -\Delta_y \Phi(x,y) = 0, & y \in \mathbb{R}^n_+ \\ \Phi(x,y) = \Gamma(x,y), & y \in \mathbb{R}^{n-1} \end{cases}.$$

Let $\tilde{x} := (x_1, \cdots, -x_n)$, and observed that $\Gamma(x-y) = \Gamma(\tilde{x}-y)$ when $y \in \mathbb{R}^{n-1}$, we merely let $\Phi(x,y) = \Gamma(x-y)$. Then

$$G(x,y) = \begin{cases} -\frac{1}{2\pi}\left[\log|x-y| - \log|\tilde{x}-y|\right], & n = 2 \\ -\frac{1}{(2-n)\omega_n}\left[\frac{1}{(x-y)^{n-2}} - \frac{1}{(\tilde{x}-y)^{n-2}}\right], & n \geq 3 \end{cases}.$$

Remain to calculate $\partial_\nu G$, where $\nu = (0, \cdots, -1)$. When $x \in \mathbb{R}^n_+,\ y \in \mathbb{R}^{n-1}$, we have

$$\frac{\partial G}{\partial \nu} = \frac{-1}{\omega_n} \cdot \frac{2x_n}{|x-y|^n}.$$

Finally, let $K(x,y) = -\partial_\nu G$, we obtain

$$u(x) = K *_{\mathbb{R}^{n-1}} g = \frac{2x_n}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{g(y)}{|x-y|^n} \mathrm{d}S_y. \tag{2.3}$$

We call K the **Poisson kernel** for the half space.

Now, we merely show certain solutions for $\mathbb{R}^n_+$, but we remain to show all solutions like formula (2.3). Furthermore, the formula (2.3) tells us the way to *expand* the harmonic function on one "line" $\mathbb{R}^{n-1}$ to the half "plane" $\mathbb{R}^n_+$.

Theorem 2.3. Let $f = 0,\ g \in C(\partial\Omega)$, then the formula (2.3) completely determines the solution for the Dirichlet problem

$$\begin{cases} -\Delta u = 0, & \text{in } \mathbb{R}^n_+ \\ u = g, & \text{on } \mathbb{R}^{n-1} \end{cases}$$

with the following properties,

(1) $u \in C^\infty(\mathbb{R}^n_+)$ and bounded;
(2) $\Delta u = 0, \quad x \in \mathbb{R}^n_+$;
(3) For all $x \in \mathbb{R}^{n-1}$, we have $\lim_{x \to x_0,\ x \in \mathbb{R}^n_+} u(x) = g(x)$.

The properties above are all easy to verify, except for the last one, which we refer readers to use the identity approximation since Poisson kernels are "good kernels" [SS09].

Unit ball B_1. Here we introduce the **Kelvin transformation**

$$\kappa : B_1 - \{O\} \to \mathbb{R}^n - \overline{B_1}, \quad \text{by } x \to x^* := \frac{x}{|x|^2}$$

which implies $|x-y| = |x| \cdot |x^* - y|$. Thus let $\Phi(x,y) = \Gamma(|x| \cdot |x^* - y|)$, one can calculate the Poisson kernel for unit ball,

$$K(x,y) = \frac{1 - |x|^2}{\omega_n |x-y|^n}$$

Same consequence for the ball case in the theorem 2.3, and we neglect the statement. Meanwhile, we call the form

$$u(x) = K *_\Omega g$$

The **Poisson integral formula**.

Remark 2.2. For general radius R, we have

$$K = \frac{1}{\omega_n R} \cdot \frac{R^2 - |x|^2}{|x-y|^n},$$

letting $x = 0$, the mean value property reappears,

$$u(0) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R} g(y) \mathrm{d}S_y.$$

We refer the readers to [Zho05] for more details.

We can use the Poisson integral formula above to show the Harnack inequality in a quantitative version.

Proposition 2.1 (Harnack)**.** Suppose u is a nonnegative harmonic function in $B(x_0, R)$, then

$$\left(\frac{R}{R+r}\right)^{n-2} \cdot \frac{R-r}{R+r} \cdot u(x_0) \leq u(x) \leq \left(\frac{R}{R-r}\right)^{n-2} \cdot \frac{R+r}{R-r} \cdot u(x_0),$$

where $r = |x - x_0| < R$.

Proof. We assume $x_0 = O,\ u \in C(\overline{B_R})$, by Poisson integral formula and $R - |x| \leq |x-y| \leq R + |x|$ for $|y| = R$, we obtain

$$\frac{1}{\omega_n R} \cdot \frac{R-|x|}{R+|x|} \left(\frac{1}{R+|x|}\right)^{n-2} \int_{\partial B_R} u(y) \mathrm{d}S_y \leq u(x) \leq \frac{1}{\omega_n R} \cdot \frac{R+|x|}{R-|x|} \left(\frac{1}{R-|x|}\right)^{n-2} \int_{\partial B_R} u(y) \mathrm{d}S_y.$$

Using the mean value property above, we complete the proof. $\square$

Looking at the coefficients of $u(x_0)$, which tends to 1 as $R \to \infty$, we observe the identification between any two points of the entire function, which is Liouville's theorem that we have shown in the first section.

Recalling Riemann's work on complex functions, he described the type of removable singularity for meromorphic functions, as seen in [SS10]. However, here we demonstrate a similar observation by regarding the fundamental solution as a "critical point".

Theorem 2.4 (removable singularity)**.** Suppose u is harmonic in $B_R - \{O\}$, and such that

$$u(x) \sim o(\Gamma(x)), \quad \text{as } |x| \to 0$$

then u can be defined at O such that u is C^2 and harmonic in B_R.

Proof. The idea is to construct a harmonic function v that solves the same Dirichlet problem as u (by uniqueness) and agrees with u in $B_R - \{O\}$.

Assume u is continuous in $0 < |x| \le R$, and v solves

$$\begin{cases} \Delta v = 0, & x \in B_R \\ v = u, & x \in \partial B_R \end{cases}.$$

We will prove that $u = v$ in $B_R - \{O\}$. Set $w = v - u$ in $B_R - \{O\}$, and $M_r := \sup_{\partial B_r} |w|$. For $n \ge 3$ (we left the case $n = 2$ to the readers), obviously,

$$|w(x)| \le M_r \cdot \frac{r^{n-2}}{|x|^{n-2}}, \quad x \in \partial B_r.$$

Note that $w, |x|^{2-n}$ are harmonic in $B_R - B_r$. Using the maximum principle, we obtain

$$|w(x)| \le M_r \cdot \frac{r^{n-2}}{|x|^{n-2}}, \quad x \in B_R - B_r.$$

Finally, we observe

$$M_r \le \max_{\partial B_r} |v| + \max_{\partial B_r} |u| \le M + \max_{\partial B_r} |u|,$$

where $M = \max_{\partial B_R} |v|$. Hence, fixed $x \ne O$,

$$|w(x)| \le \frac{r^{n-2}}{|x|^{n-2}} \cdot M + \frac{1}{|x|^{n-2}} \cdot r^{n-2} \cdot \max_{\partial B_r} |u| \to 0$$

as $r \to 0$. Therefore, $w = 0$ in $B_R - \{O\}$. □

Remark 2.3. Additionally, we refer the readers to [GS54] for more discussions.

2.4. **Perron's method and Capacity.** At the end of this section, we demonstrate a useful tool established by Perron in 1923 [Per23], which can also be considered a methodology to seek a possible solution to the equation. As an application, we will show how Perron's method solves the Dirichlet problem,

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases},$$

where we assume $g \in C(\partial\Omega)$ and $\partial\Omega$ will be equipped with a certain "functional" property, called *regular*. The philosophy behind Perron's method is that we collect all possible "harmonic-like" functions nearly lower than g on the boundary, and select a suitable subsequence to converge to the possible solution, which has a certain "Gap" to g on the boundary. Finally, we will show that the Gap can be neglected due to a certain property of the boundary.

Firstly, we explain the meaning of "harmonic-like" functions. The goal of such a concept is to restrict a suitable range of functions to approximate g. We say that a continuous function u is **subharmonic (in the C^0 sense)** in the domain Ω, if for any ball $B \subset\subset \Omega$ and h harmonic in B such that $u \le h$ on ∂B, then $u \le h$ throughout B. We show three basic properties of the subharmonic function, say u:

I (strong maximum principle) If v is superharmonic on Ω, such that $v \geq u$ on the boundary, then $v > u$ or $v \equiv u$ throughout Ω.

II (harmonic lift) Given a ball $B \subset\subset \Omega$, We say

$$U(x) := \begin{cases} \overline{u}(x), & x \in B \\ u(x), & x \in \Omega - B \end{cases}$$

a *harmonic lift* of u in ball B, where $\overline{u}$ is Poisson integral of u in B. We say U is also subharmonic, and it is obviously continuous.

III Let $u_1, \cdots, u_N$ be finite subharmonic in Ω, then

$$u(x) := \max\{u_1(x), \cdots, u_N(x)\}$$

is also subharmonic in Ω.

III is trivial; we merely show the proof of I and II.

Proof of I. We use contradiction by observing that $w := u - v$ is subharmonic. Assume there is $x_0 \in \Omega$ such that

$$w(x_0) = \sup_{\Omega} w =: M \geq 0$$

and there is a ball $B = B(x_0, r) \subset \Omega$ such that $w \not\equiv M$.

Let $\overline{u}, \overline{v}$ be Poisson integral of u, v on ∂B, then $\overline{u}, \overline{v}$ are harmonic in B. By the maximum principle and the definition of subharmonic functions, we have

$$M \geq \sup_{\partial B}(\overline{u} - \overline{v}) \leq (\overline{u} - \overline{v})(x_0) \leq (u - v)(x_0) = M.$$

Thus $(\overline{u} - \overline{v}) \equiv M$ in B, then $w \equiv M$, which leads to contradiction. □

Proof of II. Using the definition, for all balls $B' \subset\subset \Omega$, we want to show: for all g harmonic in B' such that $U \leq g$ on $\partial B'$, then $U \leq g$ in B'. Obviously, $u \leq g$ in B', i.e. $U \leq g$ in $B' - B$. On the other hand, U is harmonic in $B' \cap B$. by the maximum principle, $U \leq g$ in $B' \cap B$. □

Lemma 2.3 (Main)**.** Let $u(x) := \sup_{v \in S_g} v$, then u is harmonic, where S_g is the *subfunction* set of g about Ω. Where the subfunction v means that v is subharmonic and $v \leq g$ on $\partial\Omega$.

Proof. The main idea is to find a suitable sequence $v_n \to v$ and verify $v = u$. We divide the proof into two parts.

Step 1 Construct point-wise convergence. For any fixed $y \in \Omega$, there is a sequence $\{v_n\} \subset S_g$ such that $v_n(y) \to u(y)$ with above bound for the sake of $v_n \leq g$ on $\partial\Omega$. Indeed $\{v_n\}$ can have lower bound by replacing by $\max\{v_n, \inf g\}$. Thus $v_n(y) \to u(y)$ is well-defined.

Then we lift v_n to V_n in ball $B := B_R(y) \subset \Omega$ and $V_n(y) \to u(y)$. Obviously $\{V_n\}$ is bounded. by *compact convergence theorem* (theorem 2.11 in [GT77]), there is a subsequence $\{V_{n_k}\}$ uniformally converges to v in B_r, $r < R$, where v is harmonic.

Step 2 We claim that $v = u$ in B. If not, there is a point $z_0 \in B$ such that $v(z_0) < u(z_0)$, then there is $\overline{u} \sin S_g$ such that $v < \overline{u}$. Let $w_k := \max\{\overline{u}, V_{n_k}\}$ and lift them to W_k. By the same procedure, we have $W_k \to w$. However, $v \leq w \leq u$ and $v(y) = w(y) = u(y)$, and noting that v, w are harmonic, thus $v \equiv w$ in B by maximum principle, which leads to contradiction.

□

We call such u a Perron solution of the Dirichlet problem, and if the problem is solvable, then u is definitely the unique solution.

The last step of Perron's process is to verify $u(\xi) = g(\xi)$ for boundary point ξ, which can be fulfilled by $u(x) \to g(\xi),\ x \to \xi$, and that is the "Gap" we mentioned at the beginning of this subsection. Primarily, we demonstrate a functional concept. We say $\xi \in \partial\Omega$ is *regular*, if there is a function w in Ω such that

(i) w is superharmonic in Ω;
(ii) $w > 0$ in Ω except for ξ, and assume 0 at ξ merely.

where w will be called the **barrier** of the domain Ω at the boundary point ξ.

Lemma 2.4 (Gap)**.** Let u be a Perron solution. If ξ is regular, g is continuous at ξ, then $u(x) \to g(\xi),\ x \to \xi$.

Proof. For all $\epsilon > 0$, by continuity of g at ξ, there is a $\delta > 0$, such that $|g(x) - g(\xi)| < \epsilon$, as $|x - \xi| < \delta$. Let $M = \sup g$, there is a $k > 0$, w.t. $|kw(x)| \geq 2M$, as $|x - \xi| \geq \delta$. It is clear that $g(\xi) - \epsilon - kw(x)$ and $g(\xi) + \epsilon + kw(x)$ are subfunction and superfunction, respectively. Thus, by the definition of u, we obtain

$$g(\xi) - \epsilon - kw(x) \leq u(x) \leq g(\xi) + \epsilon + kw(x)$$

i.e. $|g(\xi) - u(x)| \leq \epsilon + kw(x)$. Finally, let $x \to \xi$, we complete the proof. □

Theorem 2.5. The classical Dirichlet problem in a bounded domain is solvable if and only if $\partial\Omega$ is regular.

Proof. $\Leftarrow$: By the lemma above. $\Rightarrow$: for all $\xi \in \partial\Omega$, then define $g(x) := |x - \xi|$, which induces a harmonic solution $w(x)$ in Ω that is barrier of ξ. □

Remark 2.4. It is a natural question to ask what the "regular" property (in a smooth sense) of a regular domain is. Indeed, there is C^0 boundary that is not regular, as seen in the references in section 2.7 in [GT77].

On the other hand, we introduce an *interior sphere condition* for the boundary of the domain, that is, for all $\xi \in \partial\Omega$, there is a ball $B = B(y, R)$ such that $\overline{B} \cap \partial\Omega = \xi$. So that we can choose the barrier at ξ as follows,

$$w(x) := \begin{cases} \log \frac{|x-y|}{R}, & n = 2 \\ R^{2-n} - |x - y|^{2-n}, & n \geq 3 \end{cases}.$$

It is clear that a C^2 boundary satisfies the exterior sphere condition; thus, the Dirichlet problem is solvable for C^2 domains.

Finally, to end this section, we add a physical concept, called the capacity of a domain, which has been used in *electric conductors* and *potential theory*. Let

$\Omega \subset \mathbb{R}^n$ a bounded domain with smooth boundary, $n \geq 3$, and u be a harmonic function on $\mathbb{R}^n - \Omega$ such that

$$u \equiv 1, \text{ on } \partial\Omega; \quad \lim_{|x|\to\infty} u(x) =,$$

Then we define the capacity of Ω,

$$\text{Cap } \Omega := \int_{\mathbb{R}^n - \Omega} |\nabla u|^2,$$

which is well defined, and we left the proof to the readers. Indeed, you need to verify that u is unique and the decay of $|\nabla u|$ makes the energy integral finite [hint: use the theorem 2.4 and Kelvin transformation]. Furthermore, a description of regularity is shown as follows without proof,

Theorem 2.6 (Wiener Criterion)**.** $\xi \in \partial\Omega$ is regular if and only if

$$\sum_{j=0}^{\infty} \frac{C_j}{\lambda^{j(n-2)}} = \infty$$

where $C_j := \text{Cap } \{x \notin \Omega : |x - \xi| \leq \lambda^j\}$.

Remark 2.5. See more details in section 2.8 of [GT77].

3. Maximum principle and Prior estimate

INTRODUCTION. In this section, we will prove the maximum principle with more prior methods, instead of the explicit formula. In fact, the former one may be extended to more general elliptic equations, however, there exists some technical difficulties, such the *barrier* for *perturbation* and the argument of sign of functions, which will be shown in the last subsection, called the maximum principle in the version *Hopf* compared with the version of *Alexandroff* [4]. On the other hand, we will show the simple attempts of the *energy method* and the *iteration* by combined with the *Sobolev imbedding.*

Mathematicians for a long time have confined themselves to the finite or algebraic integration of differential equations, but after the solution of many interesting problems the equations that can be solved by these methods have to all intents and purposes been exhausted, and one must either give up all further progress or abandon the formal point of view and start on a new analytic path. The analytic trend in the theory of differential equations has only recently become established; and only seven years ago the late Professor Korkin in a conversation with me spoke scornfully of the "decadence" of Poincaré's work.

– Bernstein (1913)

[4] We will present it in the following lectures.

3.1. **The maximum principle in simple version.** Now we will use the maximum principle to derive the interior gradient estimate and the Harnack inequality, except for the mean value properties of harmonic functions or Laplace operator.

Firstly, we employ a prior method to derive the maximum principle, rather than the Poisson-type formula, to facilitate the preparation of general elliptic equations.

Theorem 3.1 ((Weak) maximum principle)**.** Suppose $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be subharmonic function, that is $-\Delta u \leq 0$, then

$$\sup_{B_1} u \leq \sup_{\partial B_1} u.$$

Proof. In this case, we deliberately don't use the mean value properties. One suitable approach is to test the consequences of touching the maximum in the interior. For instance, if u touches the maximum point x_0 in Ω, then $D^2u(x_0) \leq 0$. Then $\Delta u \leq 0$. However, we can't obtain the contradiction for the case $\Delta u = 0$. Therefore, we need to *perturb* u to u_ϵ such that $\Delta u_\epsilon > 0$ and go back to u.

We call u_ϵ the *barrier function*, and let it be $u + \epsilon|x|^2$, for $\epsilon > 0$. Easily, $\Delta u_\epsilon = \Delta u + 2n\epsilon > 0$. by contradiction, we obtain

$$\sup_{B_1} u_\epsilon \leq \sup_{\partial B_1} u_\epsilon.$$

Then

$$\sup_{B_1} u \leq \sup_{B_1} u_\epsilon \leq \sup_{\partial B_1} u_\epsilon = \sup_{\partial B_1} u + \epsilon.$$

Finally, let $\epsilon \to 0$, we complete the proof. □

Remark 3.1.
(1) We can replace B_1 by any bounded domain Ω.
(2) We also find that the key observation in contradiction is that

$$D^2u = \operatorname{diag}\{\lambda_1, \cdots, \lambda_n\} \geq 0, \quad \text{at } x_0,$$

which implies $\sigma_k(D^2u) \geq 0$, where $\sigma_k := \sum_{i_1<\cdots<i_k} \lambda_{i_1} \cdots \lambda_{i_k}$ is the Newton polynomial. Meanwhile, the barrier function shows

$$\sigma_k(D^2(\epsilon|x|^2)) = \epsilon \cdot (\text{positive number}) > 0.$$

Thus u_ϵ also works. Then we obtain the maximum principle for such *k-Hessian* equations, particularly, the Monge-Ampere equation.
(3) If we add a free term $c(x)u \leq 0$, such that $-\Delta u - c(x)u \leq 0$, we still have the maximum principle for nonnegative u.

Next, we show a beautiful method established by Bernstein in 1910, called the *cutoff method*, which has been used in many rigidity results of solutions of elliptic equations. And we will demonstrate a toy model as follows.

Proposition 3.1 (Bernstein, 1910)**.** Suppose u is harmonic in B_1, we have following "tribyl" gradient estimate

$$\sup_{B_{1/2}} |\nabla u| \leq c(n) \cdot \sup_{\partial B_1} |u|.$$

Proof. The idea has been used in many cases. Indeed, we need to estimate the Laplacian of the energy term, $|\nabla u|^2$, that is (*Bochner* formula)

$$\frac{1}{2}\Delta(|\nabla u|^2) = \sum_i \partial_i(\nabla u_i \cdot \nabla u) = |D^2 u|^2 + \nabla\Delta u \cdot \nabla u \geq |D^2 u|^2 \geq 0,$$

thus, $|\nabla u|^2$ is subharmonic. Obviously,

$$\sup_{B_1} |\nabla u| \leq \sup_{\partial B_1} |\nabla u|.$$

Here, we want to compare the term $|\nabla u|$ with $|u|$, and observe that $\Delta(1/2u^2) = |\nabla u|^2$. Thus, it is possible to consider $\Delta(|\nabla u|^2 * u^2)$, where the notation $*$ is not convolution, but some (linear) combination of $|\nabla u|^2$ and u^2.

Firstly, we introduce a small perturbation called $\varphi \in C_c^\infty(B_1)$ to the Laplacian and $\varphi \equiv 1$ in $B_{1/2}$. Then

$$\begin{aligned}\Delta(\varphi|\nabla u|^2) &= \Delta\varphi|\nabla u|^2 + 4\sum_{i,j}\varphi_i u_j u_{ij} + 2\varphi\nabla\Delta u \cdot \nabla u + 2\varphi\sum_{i,j} u_{ij}^2 \\ &\geq \Delta\varphi|\nabla u|^2 + 4\sum_{i,j}\varphi_i u_j u_{ij}.\end{aligned}$$

Observed that u_{ij} is 2-term, we want to delete this one by integration by parts by letting φ be the square of the function, say $\varphi = \eta^2$. Then

$$\begin{aligned}\frac{1}{2}\Delta(\eta^2|\nabla u|^2) &\geq (\eta\Delta\eta + |\nabla\eta|^2)|\nabla u|^2 + 4\eta\sum_{i,j}\eta_i u_j u_{ij} \\ &\geq -3|\nabla\eta|^2 \cdot |\nabla u|^2 \\ &\geq -\tilde{c}(n) \cdot |\nabla u|^2.\end{aligned}$$

Thus

$$\Delta(\eta^2|\nabla u|^2 + \tilde{c}(n)u^2) \geq 0,$$

which is a subharmonic function. by the maximum principle,

$$\sup_{B_1}(\eta|\nabla u|^2 + \tilde{c}(n)u^2) \leq \sup_{\partial B_1}(\eta^2|\nabla u|^2 + \tilde{c}(n)u^2).$$

Noting that $\eta|_{\partial B_1} \equiv 0$, we obtain

$$\sup_{B_{1/2}} |\nabla u| \leq c(n) \cdot \sup_{\partial B_1} |u|.$$

□

The philosophy behind the estimate above is that

$$\| - \|_{\dot{W}^{1,\infty}(B_{1/2})} \leq c(n) \cdot \| - \|_{L^\infty(B_1)}$$

which is nontrivial or rare for general functions. However, we claim that the loss of domain (say $B_{1/2}$) can be modified by a more elaborate estimate, noting that we neglect the Hessian term in the Bochner formula.

Lemma 3.1 (gradient estimate: cutoff version)**.** Suppose u is nonnegative harmonic function in B_1, then

$$\sup_{B_{1/2}} |\nabla \log u| \leq C(n).$$

Proof. It is a stronger estimate than Bernstein's, since we need to deal with the Hessian term.

Firstly, set $v = \log u$ by assume $u > 0$, then $\Delta v = -|\nabla v|^2$. Then let $w = |\nabla v|^2$. Before we calculate the Hessian term, we observe that

$$\sum_{i,j} v_{ij}^2 \geq \sum_i v_{ii}^2 \geq \frac{1}{n}(\Delta v)^2 = \frac{|\nabla v|^4}{n} = \frac{w^2}{n}.$$

Similarly, by integration by parts, we obtain

$$\begin{aligned}\Delta(\varphi w) + \nabla v \cdot \nabla(\varphi w) &\geq \Delta\varphi |\nabla v|^2 + 4\sum_{i,j} \varphi_i v_j v_{ij} + 2w\nabla\varphi \cdot \nabla v + \varphi\frac{w^2}{n} \\ &\geq -C(n,\eta) \cdot \frac{|\nabla \varphi|^2}{\varphi} \cdot |\nabla v|^2 - 2|\nabla\varphi||\nabla v|^3 + \frac{\varphi}{n} \cdot |\nabla v|^4.\end{aligned}$$

Choose $\varphi = \eta^4$, such that $|\nabla\varphi|^2/\varphi$ bounded in B_1, then

$$\Delta(\eta^4 w) + \nabla v \cdot \nabla(\eta^4 w) \geq \frac{1}{n} \cdot \eta^4 |\nabla v|^4 - C(n)|\nabla\eta|^2 \cdot \eta^3 |\nabla v|^3 - C(n)|\nabla \eta|^2 \cdot \eta^2 |\nabla v|^2.$$

We write

$$\Delta(\eta^4 w) + \nabla v \cdot \nabla(\eta^4 w) \geq \frac{1}{n} \cdot \eta^4 |\nabla v|^4 - \tilde{C}(n, \eta, |\nabla v|^3)$$

Finally, we apply the maximum principle to $\eta^4 w$. Assume $x_0 \in \Omega$ is the maximum point, then $\Delta(\eta^4 w) + \nabla v \cdot \nabla(\eta^4 w) \leq 0$, thus

$$\eta^4 w^2 \leq \tilde{C}(n, \eta, w^{3/2})$$

It is sufficient to consider $w(x_0) > 1$, we write

$$\eta^4 w^{1/2} \leq \frac{\tilde{C}(n, \eta, w^{3/2})}{w^{3/2}},$$

which leads to $w^{1/2}$ is bounded. We complete the proof. □

So far, we have obtained the weak maximum principle, but we can't claim that the maximum point merely appears on the boundary, since $\partial\Omega$ and Ω can share the maximum. Here we give an intuition of a harmonic function, i.e. $\Delta u = 0$, which means that there is a pair of eigenvalues, say λ, Λ, that have opposite directions. Thus, the harmonic function has certain monotonicity.

Lemma 3.2 (Hopf)**.** Suppose $u \in C(\overline{B_1})$ is harmonic in B_1. If $u(x) < u(x_0)$, where $x_0 \in \partial B_1$, for any $x \in \overline{B_1} - \{x_0\}$, then

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$

Precisely,

$$\frac{\partial u}{\partial \nu}(x_0) \geq C \cdot (u(x_0) - u(0)) > 0.$$

Proof. We need to perturb u. It is a natural question to ask how to choose a suitable barrier function. We list two principles in this case:

(i) Preserve the condition: $h_\epsilon(x) := u(x) - u(x_0) + \epsilon v(x)$ obtains the strictly maximum at x_0 throughout B_1;

(ii) v has strict monotonicity at x_0, say $\partial v/\partial \nu(x_0) < 0$.

Indeed, the (i) can nearly be replaced by the subharmonic property, because if h_ϵ assumes a value at x_0, then the maximum principle says that x_0 must be the maximum point instead of any monotonicity test. Therefore, we drop the principles into

(i) v is subharmonic;

(ii) v has strict monotonicity at x_0, say $\partial v/\partial \nu(x_0) < 0$.

Recall the *heat equation*, we discover that the decay of the heat kernel in exp sense to the boundary. Thus we consider

$$v(x) := e^{-\alpha|x|^2} - e^{-\alpha},$$

where α will be determined by $|x|$ and the Laplacian. Calculate

$$\Delta v(x) = e^{-\alpha|x|^2}(4\alpha^2|x|^2 - 2\alpha n).$$

Unfortunately, Δv can't be positive throughout B_1, so we cut the domain and consider $B_1 - B_{1/2}$. Thus, we discover that $\Delta v > 0$ if $|x| > 1/2,\ \alpha > 2n+1$. On the other hand, we have

$$\frac{\partial v}{\partial \nu}(x_0) = -2\alpha\epsilon e^{-\alpha} > 0.$$

Now, we find the barrier function v; however, it is crucial to compare the inner boundary $|x| = 1/2$ and exterior boundary point x_0 to ensure that x_0 is the maximum point. Indeed,

$$h_\epsilon(x) = u(x) - u(x_0) + \epsilon(e^{-\alpha|x|^2} - e^{-\alpha}).$$

For $|x| = 1/2$, we let

$$\epsilon < \frac{u(x_0) - u(x)}{e^{-\alpha/4} - e^{-\alpha}}.$$

Thus, we complete the qualitative part of the proof.

Furthermore, we will show the quantitative part by linking with $u(x_0) - u(x)$. Observed that we can set

$$\epsilon \geq \frac{u(x_0) - \max_{B_{1/2}} u(x)}{e^{-\alpha/4} - e^{-\alpha}}.$$

By the Harnack inequality above, we have

$$\inf_{B_{1/2}} (u(x_0) - u(x)) \geq C(n) \cdot (u(x_0) - u(0)),$$

i.e.,

$$u(x_0) - \max_{B_{1/2}} u \geq C(n) \cdot (u(x_0) - u(0)).$$

We complete the proof. $\square$

Finally, we demonstrate a Hölder estimate for the boundary term to end this subsection.

Lemma 3.3 (Hölder)**.** Suppose $u \in C(\overline{B_1})$ a harmonic function in B_1, with $u = g$ on ∂B_1. If $g \in C^{0,\alpha}$, then $u \in C^{0,\alpha/2}$. Moreover,

$$\|u\|_{C^{0,\alpha/2}} \leq C(n, \alpha) \cdot \|g\|_{C^{0,\alpha}}.$$

Proof. We claim that

$$\sup_{x \in B_1} \frac{|u(x) - u(x_0)|}{|x - x_0|^{\alpha/2}} \leq 2^{\alpha/2} \cdot \sup_{\partial B_1} \frac{|g(x) - g(x_0)|}{|x - x_0|^{\alpha}},$$

which leads to the desired inequality.

Indeed, for any $x, y \in B_1$, set $d_x : \mathrm{dist}(x, \partial B_1),\ d_y : \mathrm{dist}(y, \partial B_1)$, with $d_y \leq d_x$. Take $x_0, y_0 \in \partial B_1$, such that $\mathrm{dist}(x, x_0) = d_x,\ \mathrm{dist}(y, y_0) = d_y$.

CASE1 If $|x - y| < d_x/2$, then $x, y \in \overline{B_{d_x/2}(x)} \subset B_{d_x}(x) \subset B_1$. by gradient estimate,

$$d_x^{\alpha/2} \cdot \frac{|u(x) - u(y)|}{|x - y|^{\alpha/2}} \leq C|u(x) - u(y)| \leq C \cdot d_x^{\alpha/2} \|g\|_{C^{0,\alpha}(\partial B_1)}.$$

The last "$\leq$" is from the claim.

CAST2 If $d_y \leq d_x \leq 2|x - y|$, we consider

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u(x_0)| + |u(x_0) - u(y_0)| + |u(y_0) - u(y)| \\ &\leq C \cdot (d_x^{\alpha/2} + |x_0 - y_0|^{\alpha/2} + d_y^{\alpha/2}) \cdot \|g\|_{C^{0,\alpha}(B_1)} \\ &\leq C \cdot |x - y|^{\alpha/2} \|g\|_{C^{0,\alpha/2}}. \end{aligned}$$

The last "$\leq$" is from $|x_0 - y_0| \leq d_x + |x - y| + d_y \leq 5|x - y|$.

Finally, we prove the claim. Let $K = \sup_{x \in B_1} |g(x)|/|x|^{\alpha}$. We want to show $|u(x)| \leq 2^{\alpha/2} K \cdot |x|^{\alpha/2}$. For any $x \in \partial B_1$, noting that $|x|^2 = x_1$, we have

$$g(x) \leq K|x|^{\alpha} \leq 2^{\alpha/2} K \cdot x_1^{\alpha}$$

and

$$\Delta v = 2^{\alpha} \cdot \frac{\alpha}{2}(\frac{\alpha}{2} - 1) \cdot x_1^{\alpha/2 - 2} < 0.$$

By the maximum principle, we have $|u(x)| \leq v(x) = 2^{\alpha/2} K \cdot x_1^{\alpha/2}$. $\square$

3.2. **Energy method.** In this subsection, we will show a rare estimate for general functions, but natural for harmonic functions. Firstly, we introduce the concepts of elliptic operators. Say

$$Lu := \sum_{i,j} a_{ij}(x)u_{ij} + \sum_i b_i(x)u_i + c(x)u, \quad a_{ij} = a_{ji},$$

is an **elliptic operator**, if the eigenvalues of $\{a_{ij}\}$ are positive. We set the least and largest eigenvalues of $\{a_{ij}\}$ be λ and Λ. Say L is **uniformly elliptic**, if Λ/λ is bounded, and we assume L is uniformly elliptic and $b_i, c = 0$ in this subsection. Meanwhile, we consider the following weak sense,

$$\int a_{ij}u_i\varphi_j = 0, \quad \forall\varphi \in C_c^\infty(B_1).$$

Lemma 3.4 (Caccipolli)**.** Suppose $u \in C_c^\infty(B_1)$, such that

$$\int a_{ij}u_i\varphi_j = 0, \quad \forall\varphi \in C_c^\infty(B_1)$$

then,

$$\int \eta^2|\nabla u|^2 \le C(\lambda, \Lambda) \cdot \int |\nabla\eta|^2 u^2$$

Proof. Let $\varphi = \eta^2 u$, then

$$0 = \int a_{ij}u_i(\eta^2 u)_j = \int 2a_{ij}uu_i \cdot \eta\eta_j + \int a_{ij}\eta^2 u_i u_j$$

By integration by parts,

$$\lambda \int \eta^2|\nabla u|^2 \le C \cdot \Lambda \int |\nabla\eta|^2 u^2$$

we complete the proof. □

By choosing a suitable η, such as $\eta \equiv 1$ in B_r, and vanishes outside B_R, as $0 < r < R < 1$. Furthermore, we can estimate the gradient of η be $|\nabla\eta| \le 2/(R-r)$, then we obtain

Corollary 3.1. Same settings in the lemma above, we have

$$\int_{B_r} |\nabla u|^2 \le \frac{C(\lambda, \Lambda)}{(R-r)^2} \cdot \int_{B_R} u^2$$

Remark 3.2. Indeed, the estimate is nontribyl and rare for general functions or equations, because we obtain a converse embedding

$$L^2(B_R) \hookrightarrow H^2(B_r)$$

with "cost" on measure. The classical *Sobolev imbedding* says $H^2 \hookrightarrow L^{\frac{2n}{n-2}}$, that is

$$\|u\|_{L^{\frac{2n}{n-2}}} \le C\|\nabla u\|_{L^2},$$

which is common for general functions. Indeed, we can combine two inequalities to lead to the iteration.

Lemma 3.5. Same settings, for $0 < R < 1$, then

$$\int_{B_{R/2}} u^2 \leq \theta \int_{B_R} u^2; \quad \int_{B_{R/2}} |\nabla u|^2 \leq \theta \int_{B_R} |\nabla u|^2,$$

where $\theta \in (0,1)$ is up to λ, Λ, n.

Proof. Take $\eta \equiv 1$ in $B_{R/2}$, and $|\nabla \eta| \leq 2/R$, then

$$\int_{B_R} \eta^2 |\nabla u|^2 \leq C \cdot \int_{B_R} |\nabla \eta|^2 u^2 \leq \frac{C}{R^2} \int_{B_R - B_{R/2}} u^2$$

But the *Poincaré* inequality says

$$\int_{B_R} (\eta u)^2 \leq C(n) \cdot R^2 \int_{B_R} |\nabla (\eta u)|^2 \leq C \cdot \int_{B_R - B_{R/2}} u^2 + CR^2 \int_{B_R} \eta^2 |\nabla u|^2.$$

Thus

$$\int_{B_{R/2}} u^2 \leq C \cdot \int_{B_R - B_{R/2}} u^2,$$

that is

$$\int_{B_{R/2}} u^2 \leq \left(\frac{C}{1+C} \right) \cdot \int_{B_R} u^2.$$

Another inequality is proved by more tricks. Observed that the lemma 3.4 can be proved by $u - a$, $a \in \mathbb{R}$, i.e.

$$\int_{B_R} \eta^2 |\nabla u|^2 \leq C \cdot \int_{B_R} |\nabla \eta|^2 (u-a)^2 \leq \frac{C}{R^2} \int_{B_R - B_{R/2}} (u-a)^2,$$

by Poincaré inequality (see [Eva22] p274),

$$\int_{B_R - B_{R/2}} (u-a)^2 \leq c(n) \cdot R^2 \cdot \int_{B_R - B_{R/2}} |\nabla u|^2.$$

Thus

$$\int_{B_{R/2}} |\nabla u|^2 \leq C \int_{B_R - B_{R/2}} |\nabla u|^2.$$

Similarly, we complete the proof. □

This lemma says the Liouville theorem in a certain weak sense. One can easily prove that a harmonic function in $\mathbb{R}^n$ with finite L^2 norm or finite Dirichlet energy is definitely 0 or constant.

Finally, we show a precise estimate by iteration.

Lemma 3.6. Same settings,

$$\int_{B_r} u^2 \leq c \cdot \left(\frac{r}{R} \right)^n \int_{B_R} u^2$$

and

$$\int_{B_r} (u - u_r)^2 \leq c \cdot \left(\frac{r}{R} \right)^{n+2} \int_{B_R} (u - u_R)^2,$$

where u_r is defined by the average integral of u in B_r.

Proof. Without loss of generality, by dilation, we set $R = 1$. Meanwhile, it is sufficient to prove the case for $r \in (0, 1/2]$. We claim:

$$\|u\|^2_{L^\infty(B_{1/2})} + \|\nabla u\|^2_{L^\infty(B_{1/2})} \leq c(\lambda, \Lambda) \int_{B_1} u^2,$$

which leads to two desired inequalities.

Indeed,

$$\int_{B_r} u^2 \leq c \cdot r^n \|u\|^2_{L^\infty(B_{1/2})} \leq c \cdot r^n \int_{B_1} u^2$$

and

$$\int_{B_r} (u - u_r)^2 \leq \int_{B_r} (u(x) - u(0))^2 \leq r^{n+2} \|\nabla u\|^2_{L^\infty(B_{1/2})} \leq c \cdot r^{n+2} \int_{B_1} (u - u_1)^2$$

Finally, we present two methods to show the claim.

METHOD1 (Coefficient-freezing: Toy version) By rotating coordinates, the equation becomes

$$\sum_i \lambda_i u_{ii} = 0, \quad x \in B_1,$$

where $0 < \lambda \leq \lambda_i \leq \Lambda$. Change the coordinates

$$x_i \mapsto y_i := \frac{x_i}{\sqrt{\lambda_i}},$$

set $v(y) = u(x)$. Then $\Delta v = 0$ in $\{y : \sum_i \lambda_i y_i^2 < 1\}$. For any $x_0 \in B_{1/2}$. We want to show

$$|v(y_0)|^2 + |\nabla v(y_0)|^2 \leq c(\lambda, \Lambda) \cdot \int_{\{y:\ \sum_i \lambda_i y_i^2 < 1\}} v^2.$$

The inequality will hold if we show there is an $r_0 > 0$ such that $B(y_0, r_0) \subset \{y : \sum_i \lambda_i y_i^2 < 1\}$. Indeed, for any $x_0 \in B_{1/2}$, there is an $r_0 > 0$ such that the rectangle $\{x : \frac{x_i - x_{0i}}{\sqrt{\lambda_i}} < r_0\}$ contained in B_1, then $B(y_0, r_0) \subset \{y : \sum_i \lambda_i y_i^2 < 1\}$.

METHOD2 (Sobolev imbedding) By derivativing the gradient estimate in a weak sense, we obtain

$$\|u\|_{H^k(B_{1/2})} \leq c(k, \lambda, \Lambda) \cdot \|u\|_{L^2(B_1)}.$$

For sufficiently large $k \geq 1$ with respect to. n, we recall the Sobolev imbedding $H^k \hookrightarrow C^1$ (see [Eva22] p268). Then

$$\|u\|_{L^\infty(B_{1/2})} + \|\nabla u\|_{L^\infty(B_{1/2})} \leq c(\lambda, \Lambda) \cdot \|u\|_{L^2(B_1)}.$$

□

3.3. **Hopf's maximum principle.** Now we are going to extend the maximum principle for simple Laplace operator (in subsection 3.1) to general linear elliptic operators. Set Ω be a bounded and connected domain in $\mathbb{R}^n$, and L be an elliptic operator in Ω defined in subsection 3.2 without uniform property.

Lemma 3.7 (Weak maximum principle)**.** Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $Lu \geq 0$ and $c(x) \leq 0$ in Ω. Then u attains on $\partial\Omega$ its nonnegative maximum in $\overline{\Omega}$.

Proof. We use the same method as the proof of theorem 3.1 with a simple trick but not essential. We introduce a perturbation $u_\epsilon(x) := u(x) + \epsilon e^{\alpha x_1}$, then

$$Lu_\epsilon = Lu + \epsilon e^{\alpha x_1}(a_{11}\alpha^2 + b_1\alpha + c).$$

Since b_1, c are bounded, $\alpha \geq \lambda > 0$, one may choose a suitable (large) α such that

$$a_{11}(x)\alpha^2 + b_1(x)\alpha + c(x) > 0, \quad \forall x \in \Omega.$$

Thus $Lu_\epsilon > 0$.

Then, similarly, if u_ϵ assumes the nonnegative maximum at $x_0 \in \Omega$, then $D^2u_\epsilon(x_0) \leq 0$. Since the matrix $\{a_{ij}(x_0)\}_{ij} > 0$, then

$$\{a_{ij}\partial^2_{ij}u(x_0)\}_{ij} \leq 0.$$

Noting that $b_i\partial_i u(x_0) = 0$, hence, by $c(x_0)u^+_\epsilon(x_0) \leq 0$, $Lu_\epsilon(x_0) \leq 0$, leading to the contradiction. □

Remark 3.3. Similar to (3) in remark 3.1, if $c(x) \equiv 0$, then the requirement of u - nonnegative - can be dropped, which holds for the rest discussion in this section. Additionally, the one order term $b_i\partial_i$ can be neglected.

Similarly, by the maximum principle, we obtain the uniqueness of following Dirichlet problem with $c(x) \leq 0$, namely,

$$\begin{cases} Lu = f, & \text{in } \Omega \\ u = \varphi, & \text{on } \partial\Omega \end{cases},$$

where $u \in C^2(\Omega) \cap C(\overline{\Omega})$, f, φ are continuous. Similarly, also, Ω is required to be bounded. Besides, there exists counterexamples for $c(x) > 0$. Consider

$$\begin{cases} \Delta u + 2u = 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases},$$

where $\Omega := \{(x, y) \in \mathbb{R}^2 : \ 0 < x, y < 2\pi\}$. Indeed, there exists a nontrivial solution $u = \sin x \sin y$.

Next, we will lift the maximum principle in weak sense to strong sense by following observation.

Theorem 3.2 (Hopf lemma)**.** Let B be an open ball in $\mathbb{R}^n$ with $x_0 \in \partial B$. Suppose $u \in C^2(B) \cap C(B \cup \{x_0\})$ such that $Lu \geq 0$ and $c(x) \leq 0$ in B. Assume in addition that

$$u(x) < u(x_0), \quad \text{for any } x_0 \in B \text{ and } u(x_0) \geq 0.$$

Then for each exterior vector ν at x_0 with $\nu \cdot \mathbf{n} > 0$ such that

$$\liminf_{t\to 0^+} \frac{1}{t}[u(x_0) - u(x_0 - t\nu)] > 0,$$

where $\mathbf{n}$ is the exterior vector of ∂B. Particularly, if $u \in C^1(B \cup \{x_0\})$, then

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$

Proof. It is no hard to let $B = B(0, r)$ and $u \in C(\overline{B})$. Indeed, one may choose a small ball B' in B, touching at x_0. Similar to the proof of lemma 3.2, we perturb u by

$$u_\epsilon(x) := u(x) + \epsilon h(x),$$

where $h(x) := e^{-\alpha|x|^2} - e^{-\alpha r^2}$. Set $\Sigma = B \cap B(x_0, r/2)$. It remains to show two principles - (i) and (ii) in Σ - in proof of lemma 3.2.

(i)

$$\begin{aligned} Lh &= e^{\alpha|x|^2}\left\{4\alpha^2 \sum_{ij} \alpha_{ij}x_ix_j - 2\alpha \sum_i \alpha_{ii} - 2\alpha \sum_i b_ix_i + c]\right\} - ce^{-\alpha r^2} \\ &\geq e^{\alpha|x|^2}\left\{4\alpha^2 \sum_{ij} \alpha_{ij}x_ix_j - 2\alpha \sum_i [\alpha_{ii} + b_ix_i] + c\right\}. \end{aligned}$$

Noting that

$$\sum_{ij} \alpha(x)x_ix_j \geq \lambda|x|^2 \geq \lambda\left(\frac{r}{2}\right)^2 > 0,$$

we choose α large enough, then $Lh > 0$.

(ii) It is similar to the argument of lemma 3.2 by choosing small sufficiently ϵ, which is left to the readers.

Hence, by two principles, we say u_ϵ assumes the maximum at x_0 in Σ. Therefore, we obtain

$$\frac{u(\epsilon(x_0)) - u(x_0 - t\nu)}{t} \geq 0, \quad \text{for small } t > 0.$$

Take $\liminf$, we complete the proof. $\square$

Remark 3.4. Recall subsection 2.4, the Hopf lemma holds for $x_0 \in \partial\Omega$, if x_0 satisfies the interior sphere condition, moreover, holds for C^2 domain. Historically, some authors dealt with the boundary with corners by modifying the Hopf lemma, such as [Ser71, GNN79].

Now, we obtain the maximum principle in strong sense.

Theorem 3.3 (Strong maximum principle)**.** Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $Lu \geq 0$ and $c(x) \leq 0$ in Ω. Then the nonnegative maximum of u in $\overline{\Omega}$ can be attained only on $\partial\Omega$, unless u is constant.

Proof. We apply the continuity method. Let M be nonnegative maximum of u in Ω and set $\Sigma := \{x \in \Omega : \ u(x) = M\}$. It remains to show $\Sigma = \Omega$.

If not, Σ is a proper subdomain of Ω, then there exists an open ball B in $\Omega - \Sigma$ such that $\partial B \cap \Sigma \neq \emptyset$ and $\partial B \cap \partial\Omega = \emptyset$. Then, obviously,

$$u(x) < u(x_0), \quad \forall x \in B \text{ and } u(x_0) = M > 0.$$

By Hopf lemma, $\frac{\partial u}{\partial \mathbf{n}} > 0$. But since x_0 is the interior maximum point of Ω, then $Du(x_0) = 0$, leading to the contradiciton. □

Corollary 3.2 (Comparison principle)**.** Given same settings in the theorem above, if $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ in Ω. In fact, either $u < 0$ in Ω or $u \equiv 0$ in Ω.

We call the corollary above the *comparison principle*, because, if we set two functions u, v satisfies the conditions above, and $u \leq v$ on the boundary, then $u < v$ or $u \equiv v$ in interior. In other words, u will never touch the v in the interior.

On the other hand, we can use the maximum principle to show the uniqueness of *Newman problem*, or generally, the *Robin problem*, see [Zho05]. Consider

$$\begin{cases} Lu = f, & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \alpha(x) u = \varphi, & \text{on } \partial\Omega \end{cases}, \tag{3.1}$$

where f, φ are continuous. Assume in addition that $c(x) \leq 0$ and $\alpha(x) \geq 0$, then

a) If $c \not\equiv 0$ or $\alpha \not\equiv 0$, then the problem (3.1) has the unique solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

b) If $c \equiv 0$ and $\alpha \not\equiv 0$, then the problem (3.1) has the unique solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ up to additive constants.

Proof. It suffices to consider the following equation

$$\begin{cases} Lu = 0, & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \alpha(x) u = 0, & \text{on } \partial\Omega \end{cases}.$$

Proof of a) We need to show $u \equiv 0$. By the strong maximum principle, there are merely two cases. Firstly, if $u \equiv C > 0$, then it leads to contradiction with $c \not\equiv 0$ or $\alpha \not\equiv 0$ on the boundary. Otherwise, there is a $x_0 \in \partial\Omega$ such that $\frac{\partial u}{\partial \nu}(x_0) > 0$, which leads to contradiction with the boundary value. Hence, $u \equiv 0$.

Proof of b) We need to show $u \equiv C$. If u is not constant, then the maximum may be attained on the boundary, say $x_0 \in \partial\Omega$. Thus, by Hopf lemma, $\frac{\partial u}{\partial \nu}(x_0) > 0$, which leads to the contradiction.

□

Finally, we summarize the different types of the maximum principle in the sense of Hopf, which are inspired by Berestycki and Nirenberg [BN91]. Generally, we say the (weak) maximum principle holds for the elliptic operator L in Ω, if

$$\begin{cases} Lz \geq 0, & \text{in } \Omega \\ \limsup_{x \to \partial\Omega} z(x) \leq 0, & \text{on } \partial\Omega \end{cases} \Longrightarrow z \leq 0, \quad \text{in } \Omega. \tag{3.2}$$

There are three well-known sufficient conditions for the maximum principle, which are listed in [BN91]:

(i) $c \leq 0$;
(ii) If there exists a positive function $g \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that $Lg \leq 0$, then there exists a new elliptic operator $\tilde{L}$ with nonpositive $\tilde{c}$ such that $\tilde{L}(z/g) \geq 0$;
(iii) In a "*narrow*" domain $\Omega := \{x_1 \in (\alpha, \alpha + \epsilon)\}$, simply, we can construct $g(x_1)$ satisfying (ii), i.e., $a_{11}\ddot{g} + b_1\dot{g} + cg \leq 0$ in Ω.

The first simple model derives from Serrin.

Proposition 3.2 (Serrin)**.** Suppose $z \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that $Lz \geq 0$. If $z \leq 0$ in Ω, then either $z < 0$ or $z \equiv 0$ in Ω.

Proof. We present two methods.

METHOD 1 Suppose $z(x_0) = 0$ for some $x_0 \in \Omega$. Write $c(x) = c^+ - c^-$, then

$$a_{ij}\partial_{ij}z + b_i\partial_i z - c^- z \geq -c^+ z \geq 0$$

By the previous argument, $z \equiv 0$.

METHOD 2 Set $v = z \cdot e^{-\alpha x_1}$, then

$$Lz = a_{ij}\partial_{ij}v + [\alpha(a_{1i} + a_{i1}) + b_i]\partial_i v + (a_{11}\alpha^2 + b_1\alpha + c)v \geq 0$$

Hence

$$a_{ij}\partial_{ij}v + [\alpha(a_{1i} + a_{i1}) + b_i]\partial_i v \geq 0.$$

Similarly, $v < 0$ or $v \equiv 0$.

□

Proposition 3.3 ((ii))**.** If there exists a positive function $g \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that $Lg \leq 0$, then there exists a new elliptic operator $\tilde{L}$ with nonpositive $\tilde{c}$ such that $\tilde{L}(z/g) \geq 0$. Furthermore, the maximum principle holds for z.

Proof. Set $w = z/g$, then

$$a_{ij}\partial_{ij}w + \left(b_i + \frac{2}{g}a_{ij}\partial_{ij}g\right)\partial_i w + \left(\frac{Lg}{g}\right)w \geq 0.$$

We discover the desired $\tilde{L}$ and $\tilde{c}$. □

Remark 3.5. Indeed, the maximum principle is strong - so does the proposition as follows, since it satisfies the condition in the theorem 3.3.

Proposition 3.4 ((iii))**.** Let $d = d(b_i, c^+) > 0$ be small, and $\mathbf{e}$ be a unit vector such that $|(y - x) \cdot \mathbf{e}| < d$ for any $x, y \in \Omega$. Then the maximum principle holds for L in Ω.

Proof. It is no hard to let $\mathbf{e} = (1, 0, \cdots, 0)$, then $\Omega \subset \{0 < x_1 < d\}$. Assume that $b_i, c^+ \leq M$, where M is a positive constant. We claim that $g := e^{\alpha d} - e^{\alpha x_1} > 0$ is desired, where $\alpha = \alpha(d)$ will be determined at the end of proof.

Indeed,

$$Lv = -(a_{11}\alpha^2 + b_1\alpha)e^{\alpha x_1} + c(e^{\alpha d} - e^{\alpha x_1}) \leq -(a_{11}\alpha^2 + b_1\alpha) + Me^{\alpha d}.$$

Choose α large enough such that $a_{11}\alpha^2 + b_1\alpha \geq \lambda\alpha^2 - M\alpha \geq 2M$. Hence,

$$Lv \leq -2M + Me^{\alpha d} = M(e^{\alpha d} - 2) \leq 0,$$

if d is small enough. □

Remark 3.6. The proposition holds for unbounded domain, however, it is constrained to certain geometric (narrow) shape.

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