A SHORT REVIEW OF QUALITATIVE PROPERTIES FOR SEMILINEAR ELLIPTIC EQUATIONS

ZHENYE QIAN

Abstract. This paper presents a comprehensive review of key qualitative properties of solutions to *semilinear elliptic equations*, with particular emphasis on *symmetry* and *monotonicity*. Our discussion begins with the fundamental contributions of Berestycki, Caffarelli, and Nirenberg in the 1990s (notably [BCN97a]), which established the foundation for understanding these properties in more general domain settings. Building upon these classical results, we subsequently examine the proof of the De Giorgi conjecture in lower dimensions [GG98, AC00], highlighting how techniques from the earlier works naturally extend to address this celebrated problem. Finally, we elucidate the deep connections between these analytical methods and concepts from *geometric measure theory* and Γ-*convergence*, which ultimately inspired Savin's groundbreaking work [Sav09] that resolved the conjecture in higher dimensions.

Contents

1. Introduction

This paper investigates qualitative properties of solutions to elliptic partial differential equations, with primary emphasis on the semilinear case in both bounded and unbounded domains. Specifically, we examine the Dirichlet problem:

$$\begin{cases} \Delta u + f(u) = 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

where f is typically assumed to be C^1 for simplicity, though the original literature often considers *Lipschitz* ($C^{0,1}$) nonlinearities. The domain Ω may be bounded - frequently possessing *convexity* and *symmetry* properties such as the round ball

- or unbounded, including examples such as the *strip* $\mathbb{R}^{n-j} \times \omega$ (with ω bounded in $\mathbb{R}^j$), the half space $\mathbb{R}^n_+$, or the entire space $\mathbb{R}^n$.

It is natural to expect that the "shape" of solution u reflects some properties of the domain, such as monotonicity and symmetry, which are the main objects discussed at the beginning of this article, derived from [BN91] and earlier articles. In this category, we focus on such properties of solutions in bounded domains with certain symmetry. Furthermore, inspired by the results in [BCN97b], which was first motivated by *free boundary problems* earlier in [EL82], Berestycki, Caffarelli and Nirenberg [BCN97a, BCN97b] established some techniques for unbounded domains. In that case, they modified the classical methods in [BN91], namely the ϵ-step compactness argument presented in this paper, to solve a symmetry problem in a stronger sense (depending only on one variable) in half space, and furthermore, Farina and Sciunzi [FS16] solved cases for higher dimensions by modifying the original technique, called the *rotating plane method*. Additionally, the symmetry in the stronger sense is related to the *De Giorgi conjecture*, which states that the *entire* solution of the *Allen-Cahn equation*,

$$\Delta u + u - u^3 = 0, \quad \text{in } \mathbb{R}^n$$

depends only on one variable, namely symmetry, as seen in [DGS79]. We will use the techniques in [BCN97a], specifically arguments involving the *Schrödinger operator*, to prove this conjecture in 2 dimensions, see [GG98]; and [AC00] proved it by the same method in 3 dimensions. Last but not least, the method in [AC00] (or more generally in [AAC01]) is sharp, meaning that we cannot strengthen the estimate to prove the conjecture in higher dimensions; and the higher dimensional case was solved by O. Savin [Sav09], who not only introduced techniques from *geometric measure theory* and Γ-*convergence*, but also modified De Giorgi's *flatness convergence theorem*; however, we do not present the detailed procedure in this paper. The organization of this paper is as follows.

In Section 2, we will present the classical usage of the *moving plane* and *sliding* methods in general domains, which are derived from the *maximum principle*. The intuition behind the moving plane method can be described as a process of continuous reflection and comparison against a hyperplane. Thus, this method can be used to prove the symmetry of solutions. On the other hand, the sliding method can be regarded as a process of continuous translation and comparison along an interval. Thus, this method can be used to prove the monotonicity of solutions. However, the methods may require discussion about the corners of domains by modifying the *Hopf lemma*. Fortunately, Berestycki and Nirenberg [BN91] established a new idea to improve the maximum principle argument with an ϵ-step from the "*narrow*" domains to general bounded ones and avoid the discussion of corners.

In Sections 3 and 4, we will show some useful results from [BCN97a] (details in [BCN93, BNV94, BCN96]) which mainly focus on unbounded domains such as strips and half spaces. We are interested in two properties of solutions in such domains: *monotonicity* and symmetry. Furthermore, for the case of half space, we introduce a stronger concept also called symmetry, which means that the solution depends only on one variable. There are additional descriptions of this phenomenon, such as "the solution of ODE" or "flatness of level sets", both of which are connected to the equation and geometric intuition of the De Giorgi conjecture. The proof of the main result, Theorem 3.4, relies on several auxiliary results in semilinear elliptic equations, and we divide them into three parts. The first part is related to the monotonicity of the solution; the second part derives from a classical result in [BCN93], which involves the main argument of the proof. Finally, the third part is connected to the Schrödinger operator, which is helpful for the proof of the De Giorgi conjecture in lower dimensions.

In Section 5, we state the De Giorgi conjecture and briefly survey the proof of the conjecture under stronger assumptions, namely the *Gibbons conjecture*, which has been solved completely (for example, independently, [BBG00, BHM00, Far99]). In addition, we present the proof of the conjecture in 2 dimensions, which was established by Ghoussoub and Gui [GG98], motivated by the paper [BCN97a].

In Sections 6 and 7, we present a well-known observation by Cabré [AAC01], namely, the relation between the monotonicity assumption and local *minimality*; and the latter can modify the energy estimate in [BCN97a, GG98] to yield a sharp result,

$$\int_\Omega \frac{1}{2}|\nabla u|^2 \mathrm{d}x \leq C \cdot R^{n-1}$$

which extends the original estimate in [BCN97a, GG98] and implies the proof of the conjecture for $n \leq 3$. Additionally, in Cabré's paper [AAC01], they preview the proof of the conjecture in higher dimensions, namely the *asymptotic flatness* of the level sets of the solution u, which inspired the work of Savin, who proved the conjecture for $4 \leq n \leq 8$.

Finally, in Section 8 we will explain the relation of the De Giorgi conjecture with the *Bernstein problem* and preview the proof in [Sav09].

2. Classical Moving Plane and Sliding Methods

The *moving plane method* originated with Alexandroff [Ale62] in his work on constant mean curvature surfaces. Serrin [Ser71] considered the mixed boundary problem for the Poisson equation $\Delta u = c$ in Ω, with $u = 0$ and $\frac{\partial u}{\partial \nu} = 1$ on $\partial\Omega$, where c is a constant. The question is whether, for a suitable c, the domain Ω must be the unit round ball B_1. Furthermore, Gidas, Ni, and Nirenberg [GNN79] proved the monotonicity and symmetry of solutions vanishing on the boundary for the semilinear case $\Delta u + f(u) = 0$. Also, [GNN81] extended this method to the whole space $\mathbb{R}^n$. However, these papers required modifications of the *Hopf lemma* to address corners of domains. Later, the *sliding method* was introduced in [BN88, BN90] to establish monotonicity of solutions in infinite and finite cylinders, respectively.

First, we present the results from [GNN79].

Theorem 2.1 (Theorem 1, [GNN79])**.** Let $B_R := \{x \in \mathbb{R}^n : |x| < R\} \subset \mathbb{R}^n$ and let $u \in C^2(B_R)$ be a positive solution of

$$\Delta u + f(u) = 0, \quad \text{in } B_R; \quad u = 0, \quad \text{on } \partial B_R$$

where $f \in C^1$. Then u is radially symmetric and $u_r < 0$ for $0 < r < R$, where u_r denotes the radial derivative of u.

Actually, they proved a more general result but still required C^2 regularity, since previous arguments failed for domains with corners, such as cubes.

Theorem 2.2 ([GNN79])**.** Let $\Omega \subset \mathbb{R}^n$ be a general bounded domain with C^2 boundary, convex in the x_1-direction and symmetric with respect to $x_1 = 0$. Let u be a positive solution of

$$\Delta u + f(u) = 0, \quad \text{in } \Omega; \quad u = 0, \quad \text{on } \partial\Omega$$

where $f \in C^1$. Then u is symmetric with respect to x_1 and $u_1 < 0$ for $x_1 < 0$ in Ω, where $u_1 := \frac{\partial u}{\partial x_1}$.

However, Berestycki and Nirenberg [BN91] proved a result with weaker regularity assumptions on u and f using a new approach that avoids both the modified Hopf lemma and discussion of boundary points. We present the proof as an illustration of the moving plane method in this section.

Theorem 2.3 (Theorem 1.3, [BN91])**.** Under the same settings for Ω as in the previous theorem but dropping the regularity assumptions, let $u \in W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$ be a positive solution and $f \in C^{0,1}$. Then u is symmetric with respect to x_1 and $u_1 < 0$ for x_1 in Ω.

Since the moving plane method relies on various forms of the *maximum principle*, we begin the proof with necessary preliminaries. Consider a general elliptic operator $L := M + c = a_{ij}(x)u_{ij} + b_i(x)u_i + c(x)$, where the coefficients a_{ij}, b_i, c belong to $L^\infty(\Omega)$ and satisfy the uniform ellipticity condition $\lambda|\xi|^2 \le a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2$ for constants $\Lambda, \lambda > 0$. Additionally, we assume

$$\sqrt{\sum_i b_i^2}, |c| \le b, \quad \text{where } b \text{ is a positive constant.}$$

Meanwhile, functions acted upon by L are always in $W^{2,p}_{loc}$ in this section.

We say the (weak) **maximum principle** holds for L in Ω if:

$$\begin{cases} Lz \ge 0, & \text{in } \Omega \\ \limsup_{x \to \partial\Omega} z(x) \le 0, & \text{on } \partial\Omega \end{cases} \Longrightarrow z \le 0, \quad \text{in } \Omega. \tag{1}$$

Three well-known sufficient conditions for the maximum principle were listed in [BN91]:

(i) $c \le 0$ (see Theorem 9.6 in [GT77]);
(ii) If there exists a positive function $g \in W^{2,\infty}(\Omega) \cap C(\overline{\Omega})$ such that $Lg \le 0$, then there exists a new elliptic operator $\tilde{L}$ with nonpositive $\tilde{c}$ such that $\tilde{L}(z/g) \ge 0$;

(iii) In a "*narrow*" domain $\Omega := \{x_1 \in (\alpha, \alpha + \epsilon)\}$, we can construct $g(x_1)$ satisfying (ii), i.e., $a_{11}\ddot{g} + b_1\dot{g} + cg \leq 0$ in Ω.

An obstacle in applying the maximum principle is the sign requirement for c, which must be nonpositive. However, through an observation by Varadhan, [BCN97a] established a general maximum principle in narrow domains without sign restrictions on c.

Lemma 2.1 (Varadhan)**.** Let diam $\Omega \leq d$. There exists $\delta = \delta(n, d, \lambda, b) > 0$ such that the maximum principle holds for L in Ω if $|\Omega| < \delta$, where $|\cdot|$ denotes the *measure* of the set.

Note that for a given diameter d of Ω, the upper bound δ on the measure requires Ω to be "narrow", such as a thin finite cylinder $(0, d) \times (0, \delta/d)$, whereas the "finite" condition can be dropped under additional conditions, as shown in the following section.

Proof. First, recall the well-known Alexandroff-Bakelman-Pucci estimate (Theorem 9.1 in [GT77]): If $c \leq 0$ and $Lz \geq f$ with $\limsup_{z \to \Omega} \leq 0$, then

$$\sup_{\Omega} z \leq \overline{C}(n, \lambda, b, d)\|f\|_{L^n}.$$

We decompose c into two parts: $c^+ := \max\{c, 0\}$, yielding

$$(M - c^-)z \geq -c^+ z^+$$

. Using the inequality above, we obtain

$$\sup_{\Omega} z^+ \leq \overline{C}\|c^+ z^+\|_{L^n} \leq \overline{C} b|\Omega|^{1/n} \cdot \sup_{\Omega} z^+$$

. Thus, the conclusion $z \leq 0$ holds provided $\delta < (1/\overline{C}b)^n$. □

Now we prove Theorem 2.3.

Proof of Theorem 2.3. Let $x = (x_1, y) \in \Omega$. We will prove that $u_1 > 0$ for $x_1 < 0$, and

$$u(x_1, y) < u(x_1', y), \quad \text{if } x_1 < x_1' \text{ and } x_1 + x_1' < 0. \tag{2}$$

Indeed, $u_1 > 0$ is implied by the inequality above (via the Hopf lemma), and symmetry is also covered. By continuity, letting $x_1' \to x$ in the inequality yields

$$u(x_1, y) \leq u(-x_1, y), \quad \text{for } x_1 < 0,$$

and interchanging x_1 and $-x_1$ by domain symmetry gives the symmetry of u. Thus, it remains to prove inequality (2).

Set $-a := \inf_{x \in \Omega} x_1$, and define the plane $T_\lambda := \{x_1 = \lambda\}$ and $\Sigma(\lambda) := \{x_1 < \lambda\} \cap \Omega$ for $-a < \lambda < 0$. Meanwhile, define

$$v(x) := u(2\lambda - x_1, y), \quad w_\lambda(x) := v(x) - u(x).$$

Since v satisfies $\Delta v + f(v) = 0$, the function w_λ satisfies the linearized equation with Dirichlet conditions:

$$\begin{cases} \Delta w_\lambda + c_\lambda w_\lambda = 0, & \text{in } \Sigma(\lambda), \quad c_\lambda(x) = \frac{f(v)-f(u)}{v-u} \\ w_\lambda \geq\not\equiv 0, & \text{on } \partial\Sigma(\lambda) \end{cases}$$

Note that:

(i) c_λ is bounded by the Lipschitz constant of f, say b;
(ii) $w_\lambda = 0$ on T_λ and $u = 0$ on $\partial\Omega$.

We begin by applying the maximum principle in a narrow domain.

Step 1 (Moving plane in narrow domain) Using the maximum principle in a narrow domain, when $a + \lambda > 0$ is sufficiently small and $\partial\Omega$ is convex in the x_1-direction, we obtain

$$w_\lambda > 0, \quad \text{in } \Sigma(\lambda).$$

The next step was first established in [BN91].

Step 2 (ϵ-step compactness argument) Let $(-a, \mu)$ be the largest interval such that $w_\mu > 0$ holds; it remains to show $\mu = 0$. We proceed by contradiction. Assume $\mu < 0$; we claim there exists $\epsilon > 0$ such that $\mu + \epsilon$ satisfies the maximum principle, implying $\mu = 0$.

Precisely, choose a compact subset $K \subset \Sigma(\mu)$ such that $|\Sigma(\mu) - K| < \delta/2$, where δ is as in Lemma 2.1. By compactness and continuity, $w_\mu > 0$ in K. By continuity, $|\Sigma(\mu+\epsilon_0) - K| \leq \delta$ for sufficiently small ϵ_0. Thus, for all $0 < \epsilon < \epsilon_0$, $w_{\mu+\epsilon} > 0$ in $\Sigma(\mu+\epsilon)$. Indeed, $w_{\mu+\epsilon} > 0$ in K by continuity. Moreover,

$$\begin{cases} \Delta w_{\mu+\epsilon} + c_{\mu+\epsilon} w_{\mu+\epsilon} = 0, & \text{in } \Sigma(\mu+\epsilon) \\ w_{\mu+\epsilon} \geq\not\equiv 0, & \text{on } \partial\Sigma(\mu+\epsilon) \end{cases}$$

leads to a contradiction, completing the proof.

□

Finally, we present the sliding method to conclude this section.

Theorem 2.4 (Theorem 1.4, [BN91]). Let Ω be an arbitrary bounded domain convex in the x_1-direction, and let $u \in W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$ satisfy

$$\begin{cases} \Delta u + f(u) = 0, & \text{in } \Omega \\ u = \varphi, & \text{on } \partial\Omega \end{cases}$$

where $f \in C^{0,1}$. Assume there exist three points $x' = (x_1', y), x = (x_1, y), x'' = (x_1'', y)$ with $x_1' < x_1 < x_1''$ lying on the same segment parallel to the x_1-axis and $x', x'' \in \partial\Omega$, such that

$$\varphi(x') < u(x) < \varphi(x''), \quad \text{for } x \in \Omega,$$

and

$$\varphi(x') \leq \varphi(x) \leq \varphi(x''), \quad \text{for } x \in \partial\Omega.$$

Then u is monotone with respect to x_1, i.e.,

$$u(x_1+\tau, y) > u(x_1, y), \quad \text{for } (x_1+\tau, y) \in \Omega,\ \tau > 0.$$

Furthermore, if $f, \varphi \in C^1$, then $u_1 > 0$.

We sketch the proof, which is similar to that of the previous theorem.

Sketch of proof. The approach is similar to Theorem 2.3, but we compare values at different points via translation rather than reflection. Set $\Omega^\tau := \Omega - \tau e_1$ and $u^\tau(x) := u(x_1+\tau, y)$. It suffices to show

$$w^\tau(x) := u^\tau(x) - u(x) > 0, \quad \forall \tau > 0,\ \text{in } D^\tau := \Omega^\tau \cap \Omega.$$

We divide the proof into two steps:

Step 1 (Sliding in narrow domain) Let $\tau_0 := \sup\{\tau > 0 :\ D^\tau \neq \emptyset\}$ and take $\tau_0 - \tau > 0$ sufficiently small so that D^τ is narrow. Then

$$\begin{cases} \Delta w^\tau + c^\tau(x) w^\tau = 0, & \text{in } D^\tau \\ w^\tau \geq\not\equiv 0, & \text{on } \partial D^\tau \end{cases}$$

satisfies the maximum principle, so $w^\tau > 0$ in D^τ.

Step 2 (ϵ-step compactness argument) Similarly, choose a compact subdomain K such that $|D^\tau - K| < \delta/2$, then $|D^{\tau-\epsilon_0} - K| \leq \delta$. Thus,

$$\begin{cases} \Delta w^{\tau-\epsilon} + c^\tau(x) w^{\tau-\epsilon} = 0, & \text{in } D^{\tau-\epsilon} \\ w^{\tau-\epsilon} \geq\not\equiv 0, & \text{on } \partial D^{\tau-\epsilon} \end{cases}, \quad \forall 0 < \epsilon < \epsilon_0.$$

The proof is completed similarly. □

3. The main results in [BCN97a] and refinements

In this section, we present the main results from the milestone paper [BCN97a], which contains significant contributions to the study of monotonicity and symmetry of solutions to Dirichlet problems in unbounded domains. First, we recall the type of semilinear elliptic equations and the settings under consideration. Our primary focus is on the following Dirichlet problem:

$$\begin{cases} \Delta u + f(u) = 0,\ u > 0 & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{3}$$

with the following characteristics:

- Ω is unbounded, such as a *slab-like* domain $\mathbb{R}^{n-j} \times \omega$, where $\omega \subset \mathbb{R}^j$ is bounded; the *half space* $\mathbb{R}^n_+$; or the *whole space* $\mathbb{R}^n$.
- f is typically C^1, except in some cases where it is $C^{0,1}$.
- We study the *monotonicity* and *symmetry* of the solution u.

Remark 1. (1) The term "symmetry" has two distinct meanings in this paper. First, it refers to the classical notion of symmetry with respect to a certain axis, such as x_1; second, it means that the solution depends only on one variable, such as $u = u(x_n)$. We will carefully distinguish between these different meanings in the subsequent results.

(2) For the half space, [BCN97b] established a general result for Lipschitz graphs, that is,

$$\Omega := \{x \in \mathbb{R}^n : \ x_n > \varphi(x_1, \cdots, x_{n-1})\}$$

where φ is a Lipschitz function. However, Esteban and Lions [EL82] proved the case where $\lim_{|x'|\to\infty} \varphi(x) = \infty$ with $\varphi \in C^\infty$ using the moving plane method. Notably, they did not need to extend the maximum principle to unbounded domains due to this special property of φ, and the method from the previous section can handle nonsmooth φ.

Slab-like domains. First, we consider slab-like domains with the setting $(x, y) \in \mathbb{R}^{n-j} \times \omega$.

Theorem 3.1 (Theorem 1.2 [BCN96]; Theorem 1.1 (special case) [BCN97a])**.** In problem (3), assume ω is a ball $B_R \subset \mathbb{R}^j$, $f \in C^1$; and suppose $j \geq 2$ or $j = 1$ with $f(0) \geq 0$. Then u is radially symmetric in $\mathbb{R}^j$ and $u_\rho < 0$ for the radial coordinate ρ in $\mathbb{R}^j$ with $0 < \rho < R$.

In particular, if $\Omega = \mathbb{R}^{n-1} \times (0, h)$ and $f(0) \geq 0$, then $\partial_y u > 0$ and

$$u(x, y) < u(2\lambda - x, y), \quad \forall x \in \mathbb{R}^{n-1}, y \in (0, h/2),$$

where u is a solution of $\Delta u + f(u) = 0$ with $u(\cdot, 0) = 0$. Furthermore, u is symmetric with respect to $y = h/2$ if $u = 0$ on both boundaries.

A more general case, similar to the statement in the previous section, is as follows.

Theorem 3.2 (Theorem 1.2' [BCN96])**.** Under the same settings as above, but with ω convex in the y_1-direction and symmetric with respect to $y_1 = 0$, we have $u_{y_1} > 0$ for $j \geq 2$ or $j = 1$ with $f(0) \geq 0$.

In [BCN97a], the condition $f(0) \geq 0$ for $n = 2$ was dropped, and the case for $n \geq 3$ and $j = 1$ remained open until [FS16].

Theorem 3.3 (Theorem 1.1' ($n = 2$) [BCN97a]; Theorem 1.3 ($n \geq 3,\ j = 1$) [FS16])**.** The condition "$f(0) \geq 0$" can be dropped for all dimensions.

Remark 2. The technique in [FS16] is derived from [BCN97a] and is called the *rotating plane method* in two dimensions; we will illustrate it as an example.

Half space domains. Second, we present results for the half space, which can be viewed as a corollary of the strip case by moving the boundary to infinity. Here, we focus on more subtle properties of monotone solutions, which not only connect to *stability* and *minimizers* in *phase transitions* and Γ-*convergence*, but also relate to the *De Giorgi Conjecture*, as seen in [AAC01, Sav10]. The term "symmetry" here means that the solution depends only on one variable, such as x_n, which some authors refer to as the solution of an ODE.

We now present a deeper property of the solution u—symmetry in this stronger sense.

Theorem 3.4 (Theorem 1.5 ($n = 2, 3$) [BCN97a])**.** In problem (3), let $\Omega = \mathbb{R}^n_+$, $f \in C^1$, and u be bounded. Then:

i) for $n = 2$, u is symmetric;
ii) for $n = 3$ with $f(0) \geq 0$, u is symmetric.

We now outline the key arguments in the proof of Theorem 3.4, which form the core of [BCN97a] and this article. Furthermore, parts of this proof inspire the proof of the De Giorgi Conjecture in dimensions 2 and 3, as seen in [GG98, AC00].

We divide the proof of this theorem into three parts.

PART I (Monotonicity from Theorem 3.3) By extending h to infinity, we obtain monotonicity in the half space.

Corollary 3.1 ((n=2) [BCN97a]; ($n \geq 3$) [FS16])**.** In problem (3), let $\Omega = \mathbb{R}^n_+$; then $u_n > 0$ in Ω.

Remark 3. In fact, the condition $f(0) \geq 0$ can be dropped in the $n = 3$ case due to this corollary.

PART II (The main argument) This part contains the core of the proof. To clarify, we present the following result from [BCN93]:

Theorem 3.5 ([BCN93])**.** Let u be a bounded solution of problem (3) with $\Omega := \mathbb{R}^n_+$ and $M = \sup_\Omega u$. If $f(M) \leq 0$, then u is symmetric, i.e., $u = u(x_n)$, and monotone, i.e., $u_n > 0$. Furthermore, necessarily, $f(M) = 0$.

Remark 4. The condition $f(M) \leq 0$ cannot be dropped. Consider the following important model:

$$\begin{cases} \Delta u + u - 1 = 0, & \text{in } \mathbb{R}^n_+, \quad \text{with } 0 < u < \sup u = M < \infty \\ u = 0, & \text{on } \partial\mathbb{R}^n_+ \end{cases} \tag{4}$$

where $f(M) > 1$, noting that $M > 2$ [BCN97a]. A conjecture raised in that paper has been partially solved in their work and in [FS16], but remains open for $n \geq 4$.

Conjecture 1 ([BCN97a])**.** There are no solutions to problem (4).

Note that there is a classical solution from ODE theory, namely $u = 1 - \cos x_n$, which is periodic and vanishes at $x_n = 2k\pi$ for $k \in \mathbb{Z}^+$. We present related results from [BCN97a, FS16].

Theorem 3.6 (Proposition 1.6 [BCN97a]; Theorem 1.2 [FS16])**.** Under the same settings:

[BCN97a] $u(x', 2\pi) \equiv 0$ for all $x' \in \mathbb{R}^{n-1}$;
[FS16] If $u \in C^2(\overline{\mathbb{R}^2_+})$ is a nonnegative solution, then $u(x, y) = 1 - \cos y$.

PART III (Schrödinger operator) The final ingredient is the *Schrödinger operator*, whose relevance will be explained at the end of this section.

We present two results from [BCN97a] that inspire the proof of the De Giorgi Conjecture in lower dimensions and other semilinear elliptic equations. Let $L := -(\Delta + q)$ be a Schrödinger operator in $\mathbb{R}^m$, where $m = n-1$, $q \in L^\infty_{loc}(\mathbb{R}^m)$, and the function acted upon is $\psi \in W^{2,p}$ with $p > m$.

Theorem 3.7 (Theorem 1.7 [BCN97a]). Suppose ψ is a changing-sign solution of L satisfying

$$\psi(x) \sim O(|x|^{1-m/2}), \quad \text{as } |x| \to \infty. \tag{5}$$

Then L has a negative spectrum, i.e.,

$$\int |\nabla\zeta|^2 - q\zeta^2 < 0, \quad \text{for some } \zeta \in C_c^\infty(\mathbb{R}^m).$$

In the proof of Theorem 3.4 for $m = n-1 \leq 2$, we only need ψ to be bounded, which satisfies the decay condition (5). However, for higher dimensions, we either need a more precise prior estimate for ψ or must modify the decay condition. Thus, [BCN97a] raised the following question:

Question 1 ([BCN97a]). Can the decay condition (5) for ψ be weakened to L^∞ by assuming $q \in L^\infty$ or even smooth?

Unfortunately, this question has been answered negatively, even for smooth q.

Proposition 3.1 (Proposition 2.6 [GG98]). For $n \geq 7$, there exists a smooth bounded q such that $Lu = 0$ has a bounded changing-sign solution as well as a positive solution (i.e., $\lambda_1(L) = 0$).

Proposition 3.2 (Theorem 3 (b) [Bar98]). For $n \geq 3$, there exists a smooth bounded q such that $\lambda_1(L) = 0$, where λ_1 is the principal eigenvalue.

Additionally, the estimate for the solution fails; see [AC00, AAC01]. Therefore, we can only prove symmetry for general semilinear equations in lower dimensions using this method.

On the other hand, Theorem 3.7 is established by the following rigidity result, known as the "Liouville property for divergence form operators" in [Bar98], which is helpful in proving the De Giorgi Conjecture.

Theorem 3.8 (Theorem 1.8 [BCN97a]). Let φ be a positive function in $W^{2,p}_{loc}(\mathbb{R}^m)$ with $p > m$ such that

$$(\Delta + q)\varphi \leq 0. \tag{6}$$

Suppose $0 \not\equiv \psi \in W^{2,p}_{loc}$ satisfies

$$\psi(\Delta + q)\psi \geq 0$$

and the decay condition (5). Then $\psi = c\varphi$ for some constant c, and furthermore, equality holds in (6).

Finally, we sketch the proof of the main theorem (Theorem 3.4), assuming for simplicity that $f \in C^1$.

The proof reduces to the main argument (PART II): if $f(M) = 0$, then u (which is monotone by PART I and bounded) is symmetric. Thus, the goal is to prove $f(M) = 0$, which is related to the following observation: if $\nabla u \cdot \xi$ does not change sign for any $\xi \in \mathbb{R}^{m=n-1}$, then $f(M) = 0$. Hence, it remains to show that $\psi := \nabla u \cdot \xi$, which satisfies the *linearized* equation

$$-L(\psi) = \Delta\psi + q\psi = 0,$$

does not change sign. By contradiction, if ψ changes sign, then Theorem 3.7 (PART III) applies, leading to a contradiction with classical results from [BNV94].

4. Refinements and Components of the Main Proof

PART I. We begin by recalling a classical result from [BCN97a].

Lemma 4.1 ([BCN96])**.** Let $\Omega := \mathbb{R}^{n-j} \times \omega$, where ω is bounded in $\mathbb{R}^j$, and consider an operator $L := \Delta + q(X)$ with $\|q\|_{L^\infty(\Sigma)} \le b$. The maximum principle holds for the function class $\mathcal{C}_\mu$ under the action of L in Ω when $|\omega| < \delta$ for sufficiently small δ, where

$$\mathcal{C}_\mu := \{z \in C^2(\Sigma) \cap C(\overline{\Sigma}) : \ z(x,y) \le Ce^{\mu|x|}\}.$$

Busca [Bus99] extended this result to fully general uniformly elliptic operators, i.e., $L = a_{ij}\partial_{ij} + b_i\partial_i + c$ with $b_i, c \in L^\infty$.

Thus, by the lemma above, we can establish the maximum principle in a very narrow infinite cylinder since the exponential estimate still holds as shown in [BCN96]. However, the main difficulty lies in extending the "narrow" condition to cylinders of arbitrary width, which was solved in [BCN97a] for the two-dimensional case.

Proof of Theorem 3.3 in 2 *dimensions.* We divide the proof into two steps.

Step 1 (Moving plane in an infinite narrow cylinder) Consider $\mathbb{R} \times (0,h)$. Our goal is to prove that

$$\begin{cases} \Delta u + f(u) = 0, \ u > 0 & \text{in } \mathbb{R} \times (0,h) \\ u = 0, & \text{on } \{y = 0\} \end{cases} \Longrightarrow \frac{\partial u}{\partial y} > 0, \quad \text{in } \Sigma(h/2) := \{(x,y) : \ 0 < y < h/2\}.$$

Indeed, $u(x,y) < u(x, 2\lambda - y)$ for all $\lambda \in (0, h/2)$.

Let $w_\lambda(x) = u(x, 2\lambda - y) - u(x,y)$; then

$$\begin{cases} \Delta w_\lambda + c_\lambda w_\lambda = 0, & \text{in } \Sigma(\lambda) \\ w_\lambda \ge\not\equiv 0, & \text{on } \partial\Sigma(\lambda) \end{cases}$$

From Theorem 1.1 in [BCN96], $f \in C^{0,1}$ implies $u < Ce^{\mu|x|}$ for $y \in [0, h_0]$, where $0 < h_0 < h$. Taking $h_0 = h/2$ and applying the lemma above, we find that the maximum principle holds for the Dirichlet problem above for sufficiently small $\lambda > 0$, so that

$$w_\lambda > 0, \quad \text{in } \Sigma(\lambda), \text{ for } \lambda \in (0, \sigma),$$

where σ is sufficiently small.

Step 2 (Rotating plane method in 2 dimensions) Our goal is to prove:

Let $(0, \mu)$ be the largest interval such that the maximum principle holds; then there exists $\epsilon > 0$ such that $(0, \mu + \epsilon)$ also satisfies the maximum principle.

We discover an observation implied by the C^1 regularity of u.

Lemma 4.2. Let $0 < \rho < \mu$; there exists $\epsilon > 0$ sufficiently small such that for $-\epsilon \leq \theta \leq \epsilon$ and all $\rho \leq \lambda \leq \mu + \epsilon$,

$$u(0, y) < u(S_{\lambda,\theta}(0, y)), \quad \forall y, \ 0 \leq y < \lambda,$$

where $S_{\lambda,\theta}(0, y)$ is the reflection of $(0, y)$ with respect to the line $T_{\lambda,\theta}$, and $T_{\lambda,\theta}$ is the line with slope ϵ intersecting the y-axis at λ; see Figure 1.

We prove this by contradiction. If not, suppose there exists a sequence $\{((0, y_n); \ \theta_n; \ \lambda_n)\}$ such that

$$u(0, y_n) \geq u(S_{\lambda_n,\theta_n}(0, y_n)) \tag{7}$$

with

$$\begin{cases} (0, y_n) \to (0, \overline{y}), & \forall 0 \leq \overline{y} \leq \mu \\ \theta_n \to \theta & \\ \lambda_n \to \lambda, & \rho \leq \lambda \leq \mu. \end{cases}$$

Taking the limit, we obtain

$$u(0, \overline{y}) \geq u(0, 2\lambda - \overline{y}).$$

But since $u(0, y) < u(0, 2\lambda - y)$ for $0 < y < \lambda$, it follows that $\overline{y} = \lambda$. By inequality (7), there exists a sequence

$$\begin{cases} \xi_n \to e_2 = (0, 1) \\ q_n \subset \overline{(0, y_n), \ S_{\lambda_n,\theta_n}(0, y_n)} \end{cases}$$

where $\overline{(0, y_n), \ S_{\lambda_n,\theta_n}(0, y_n)}$ denotes the segment from $(0, y_n)$ to S_{λ_n,θ_n}, such that $\nabla u(q_n) \cdot \xi_n$ (using the C^1 regularity of u). Taking the limit, we obtain

$$\frac{\partial u}{\partial y}(0, \lambda) \leq 0,$$

which leads to a contradiction.

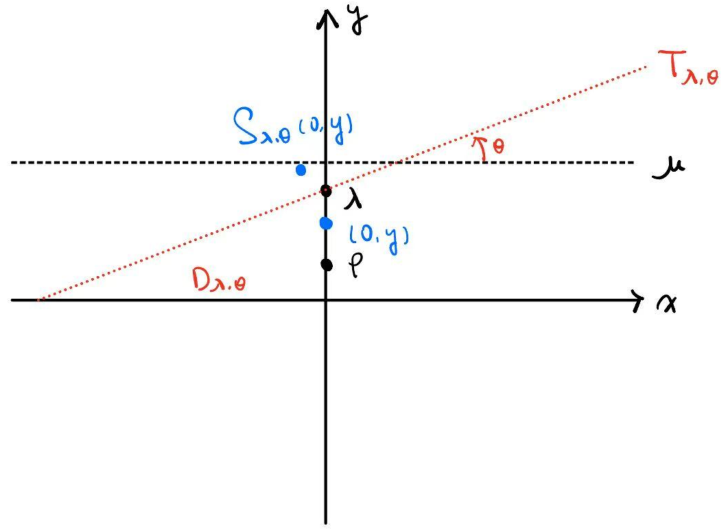

FIGURE 1. Illustration of the rotating plane method

Now we proceed to prove the theorem in $2D$ by comparing points through rotation of the plane (line) $T_{\lambda,\theta}$. Consider

$$w_{\lambda,\theta}(x) := u(S_{\lambda,\theta});$$

then

$$\begin{cases} \Delta w_{\lambda,\theta} + c_{\lambda,\theta} w_{\lambda,\theta} = 0, & \text{in } D_{\lambda,\theta} \\ w_{\lambda,\theta} \geq\not\equiv 0, & \text{on } \partial D_{\lambda,\theta} \end{cases}, \quad 0 < \lambda \leq \mu + \epsilon.$$

Note that $D_{\lambda,\theta}$ is narrow (finite in 2 dimensions) when ϵ is sufficiently small. Thus, the Dirichlet problem above satisfies the maximum principle for $-\epsilon \leq \theta \leq \epsilon$. We have

$$w_{\lambda,\theta} > 0, \quad \text{in } \Sigma(\lambda)$$

for $0 < \lambda \leq \mu + \epsilon$, $-\epsilon \leq \theta \leq \epsilon$. Finally, letting $\theta \to 0$, we obtain $u(x, y) \leq u(x, 2\lambda - y)$, and by the usual maximum principle (choosing a suitable finite subdomain at each comparison point), we conclude

$$u(x, y) < u(x, 2\lambda - y).$$

□

PART II. We now present the main argument, simplifying it by considering only the case $f \in C^1$. Recall the Dirichlet problem under consideration:

$$\begin{cases} \Delta u + f(u) = 0, \ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}, \quad \text{where } \Omega = \mathbb{R}^n_+, \ n = 2, 3.$$

From Corollary 3.1 (or from Theorem 3.3 with $f(0) \geq 0$) and Theorem 3.5, it suffices to show:

If u is bounded and monotone (with $u_n > 0$ as shown in Corollary 3.1), then
$$f(M) = 0.$$

Recall the observation mentioned in the previous section.

Lemma 4.3 (From ODE theory)**.** Suppose u is a bounded solution of $\Delta u + f(u) = 0$ in $\mathbb{R}^p$, $p \geq 1$. Assume $\nabla u \cdot \xi$ does not change sign for all $\xi \in \mathbb{R}^p - \{0\}$. Then $f(M) = 0$.

Proof. We proceed by induction. For $p = 1$, this is an elementary ODE problem where u is a bounded and monotone solution. Assuming $M = u(+\infty)$, the equation easily gives
$$f(M) = 0.$$
Now suppose the statement holds for $p - 1 \geq 1$, and consider the case p. Let $\xi = e_p = (0, \cdots, 1) \in \mathbb{R}^p$ and $\partial_\xi u > 0$; then
$$w(x_1, \cdots, x_{p-1}) := \lim_{x_p \to \infty} u(x_1, \cdots, x_p)$$
is also a solution of $\Delta w + f(w) = 0$ in $\mathbb{R}^{p-1}$, with
$$M = \sup_{\mathbb{R}^{p-1}} w = \sup_{\mathbb{R}^p} u.$$
By elliptic theory, $u \to w$ in the C^1 sense, so $\nabla u \cdot \eta$ does not change sign for all $\eta \in \mathbb{R}^p - \{0\}$. By the induction hypothesis, $f(M) = 0$. $\square$

We now present the main argument to complete the proof of Theorem 3.4.

Proof of Theorem 3.4. Let
$$z(x_1, \cdots, x_{n-1}) := \lim_{x_n \to \infty} u(x_1, \cdots, x_n),$$
which is well-defined by the boundedness and monotonicity of u. Then we also have
$$\begin{cases} \Delta z + f(z) = 0, \ z > 0 \\ \sup_{\mathbb{R}^{m=n-1}} z = M \end{cases}, \quad \text{in } \mathbb{R}^m.$$
By the observation above, we only need to show that $\nabla z \cdot \xi$ does not change sign. Let $\psi = \nabla z \cdot \xi$; then
$$\Delta \psi + f'(z)\psi = 0, \quad \text{in } \mathbb{R}^m.$$
We proceed by contradiction. If not, recall Theorem 3.7:

i) ψ changes sign;
ii) ψ is bounded for $m = 1, 2$.

Then L has a negative spectrum, i.e., there exists $\zeta \in C_c^\infty(\mathbb{R}^m)$ such that
$$-\delta := \int_{\mathbb{R}^m} |\nabla \zeta(x)|^2 - f'(z)\zeta(x) \mathrm{d}x < 0.$$
However, this contradicts results from [BNV94].

Lemma 4.4 ([BNV94])**.** For any bounded subdomain $\Omega' \subset \Omega$, the principal eigenvalue of $-\Delta - q$ in Ω' is positive.

Thus, it remains to show that the principal eigenvalue of L is negative in a special subdomain. Consider

$$D_{a,h} := \{|x'| < R,\ a < x_n < a+h\} = B_R \times (a, a+h).$$

[BCN97a] proved that $D_{a,h}$ has a negative principal eigenvalue when h is sufficiently large. □

PART III. Finally, we prove Theorem 3.7. We begin with a useful tool.

Proof of Theorem 3.8. Set $\sigma = \psi/\varphi$; it suffices to show that σ is constant. Observe that

$$\sigma \nabla \cdot (\varphi^2 \nabla \sigma) \geq 0.$$

Indeed, from $\psi(\Delta\psi + q\psi) \geq 0$, we have

$$\sigma \nabla \cdot (\varphi^2 \nabla \sigma) + \sigma^2 \varphi(\Delta\varphi + q\varphi) \geq 0,$$

and note that φ is positive and $\Delta\varphi + q\varphi \leq 0$.

Using a cutoff function, let ζ be smooth with $\zeta \equiv 1$ in B_1 and $\zeta \equiv 0$ outside B_2. Define $\zeta_R(x) := \zeta(|x|/R)$. By Stokes' theorem, we obtain

$$\begin{aligned} \int \zeta^2 \varphi^2 |\nabla \sigma|^2 &\leq 2 \int \varphi^2 \zeta \sigma \nabla \zeta_R \cdot \nabla \sigma \\ &\leq 2 \left(\int_{|x|<2R} \zeta^2 \varphi^2 |\nabla \sigma|^2 \right)^{1/2} \cdot \left(\int_{R<|x|<2R} \varphi^2 \sigma^2 |\nabla \zeta_R|^2 \right)^{1/2}. \end{aligned}$$

Recalling the decay of $\psi = \sigma\varphi$, we find

$$\int_{|x|<2R} \zeta^2 \varphi^2 |\nabla \sigma|^2 \leq C_R.$$

Letting $R \to \infty$, we complete the proof. □

Proof of Theorem 3.7. We proceed by contradiction. Let λ_R be the principal eigenvalue in B_R and assume $\lambda_R > 0$; consider the eigenfunction φ_R satisfying

$$\begin{cases} L\varphi_R = \lambda_R \varphi_R, & \text{in } B_R \\ \varphi_R = 0, & \text{on } \partial B_R. \end{cases}$$

By the maximum principle (which clearly applies), we have $\varphi_R > 0$. Recall classical results, such as those in [BNV94], that λ_R is decreasing with respect to R. It suffices to show that λ_R is negative for some $R > 0$. Let $\overline{\lambda} := \lim_{R\to\infty} \lambda_R$.

By the *Krylov-Safonov Harnack inequality* (see [GT77]),

$$\delta_R^{-1} < \varphi_R < \delta_R, \text{ in } B_{R/2}, \quad \exists \delta_R > 0.$$

Then, by L^p estimates,

$$\|\varphi_R\|_{W^{2,p}} \leq C_R, \quad p > n,$$

and by *Morrey's embedding* (or *Sobolev embedding* in general; see [Eva22]), there exists a sequence $\varphi_{R_j} \to \varphi > 0$ in $C^{1,\alpha}$ such that

$$(\Delta + q)\varphi = -\overline{\lambda}\varphi \leq 0, \quad \text{in } \mathbb{R}^m,$$

which contradicts the fact that $\psi = c\varphi$ since ψ changes sign. □

5. A short survey on De Giorgi conjecture and proof in 2 dimension

In this section, we present the statement of the *De Giorgi conjecture* and provide a brief survey of the proof of the conjecture in lower dimensions.

The De Giorgi conjecture was originally formulated by De Giorgi in 1978 [DGS79].

Conjecture 2 ([DGS79])**.** Let $u : \mathbb{R}^n \to (-1,1)$ be a smooth *entire* solution of the *Allen-Cahn equation*:

$$\Delta u + u - u^3 = 0, \quad \text{in } \mathbb{R}^n$$

with the monotonicity condition $\partial_n u > 0$. Then all level sets of u, denoted by $\{u = s\}$, are hyperplanes for $n \leq 8$.

Here, "flatness" (i.e., the hyperplane condition) means that u is symmetric in the stronger sense discussed in Section 3. Indeed, for $n = 1$, the ODE

$$\ddot{u} + u - u^3 = 0, \quad \text{in } \mathbb{R}$$

has a solution

$$u(x) = \tanh\left(\frac{x}{\sqrt{2}} + c\right)$$

where c is a constant. Based on arguments from classical models in [DGS79, AAC01, Sav10], we expect the solution to take the form

$$u(x) = \tanh\left(\frac{1}{\sqrt{2}} x \cdot a + b\right)$$

where $a, b \in \mathbb{R}^n$ are constant vectors.

For $n = 2$, the conjecture was proved by Ghoussoub and Gui [GG98], inspired by techniques in [BCN97a]; the case $n = 3$ was proved by Ambrosio and Cabré [AC00] using modified techniques. Specifically, the proof in [GG98] was inspired by part of the proof of a classical theorem in [BCN97a], presented in Section 3 of this paper, which involves the argument of Schrödinger operators—or the "Liouville property for divergence form operators" from [Bar98]. Additionally, [GG98] showed that this method fails for $n \geq 7$, and [Bar98] demonstrated its failure for $n \geq 3$.

Indeed, [GG98] established more general results for semilinear equations:

$$\Delta u - F'(u) = 0, \quad \text{in } \mathbb{R}^n \tag{8}$$

where $F \in C^2$ is typically referred to as the potential in certain functionals, and [AAC01, AC00] extended these results to higher dimensions. We summarize them in the following theorem.

Theorem 5.1 (Theorem 1.1 [GG98]; Theorem 1.1 [AAC01] (also in Theorem 1.2 [AC00]))**.** Assume $F \in C^2$. Let u be a bounded solution of equation (8). If $n = 2, 3$, then all level sets of u are flat, i.e., there exist $a \in \mathbb{R}^n$ and $g \in C^2$ such that

$$u(x) = g(a \cdot x), \quad \forall x \in \mathbb{R}^n.$$

In particular, taking $F' = u^3 - u$, we obtain the De Giorgi conjecture in dimensions 2 and 3.

Before presenting the proof, we provide a brief survey of the conjecture's proof in lower dimensions or under additional hypotheses. First, the monotonicity assumption on u, i.e., $\partial_n u > 0$, is not only technical (as seen in the proof for $n = 2$) but also implies an implicit geometric intuition (indeed, the monotonicity condition naturally leads to the concept of *local minimality*, discussed in the next section). Typically, we add the assumption:

$$\lim_{x_n \to -\infty} u(x', x_n) = \inf u; \quad \lim_{x_n \to \infty} u(x', x_n) = \sup u \tag{9}$$

which was proved for $n = 2$ in [GG98] and for $n = 3$ in [AC00]; Savin [Sav09] resolved the case for $n \leq 8$ using more advanced techniques from *geometric measure theory* and Γ-*convergence*, which are not presented in this paper.

Note that the direction of a in Theorem 5.1 is not known a priori, since we can rotate the coordinates such that u still satisfies the equation, which also holds under the additional assumption (9). In contrast, if we assume the convergence in (9) is uniform, then the direction of a is determined, specifically $a \cdot x = x_n$, and furthermore, the level sets of u are contained between two parallel hyperplanes. In this uniform case, the conjecture is also known as the "*Gibbons conjecture*", which has been completely and independently solved by several authors; see [BBG00, BHM00, Far99].

We now present the proof for $n = 2$. Define

$$\varphi := \partial_n u > 0, \quad \sigma_i := \frac{\partial_i u}{\partial_n u} = \frac{\partial_i u}{\varphi}, \ \forall i = 1, \cdots, n-1,$$

where the latter is well-defined. The goal is to prove that σ_i is constant. This implies $\nabla u = a|\nabla u|$, where a is a unit vector in $\mathbb{R}^n$, and thus the level sets of u are hyperplanes orthogonal to a.

We claim that $\varphi^2 \sigma_i$ is divergence-free in the proof of Theorem 3.8. Indeed,

$$\varphi^2 \nabla \sigma_i = \varphi \nabla \partial_i u - \partial_i u \nabla \varphi,$$

and since both φ and $\partial_i u$ satisfy the same linear equation $\Delta v - F''(u)v = 0$, we have

$$\operatorname{div}(\varphi^2 \nabla \sigma_i) = \nabla \cdot (\varphi^2 \nabla \sigma_i) = \varphi \Delta \partial_i u - \partial_i u \Delta \varphi = 0.$$

We restate the observation from [BCN97a] and Theorem 3.8 as follows:

Lemma 5.1. Let $\varphi \in L^\infty_{loc}(\mathbb{R}^m)$ be a positive solution. Assume that $\sigma \in H^1_{loc}(\mathbb{R}^m)$ satisfies

$$\sigma \nabla \cdot (\varphi^2 \nabla \sigma) \geq 0, \quad \text{in } \mathbb{R}^m \tag{10}$$

in the distributional sense, and that

$$\int_{B_R} (\varphi \sigma)^2 \leq C \cdot R^2, \quad \forall R > 1,$$

where C is independent of R. Then σ is constant.

To apply this lemma to our proof, note that $\varphi\sigma_i = \partial_i u$, so we need to show

$$\int_{B_R} |\nabla u|^2 \leq C \cdot R^2, \quad \forall R > 1. \tag{11}$$

This estimate is easily obtained from general gradient estimates, specifically $|\nabla u| \in L^\infty$ (see [GT77]). Thus, the estimate (11) holds for $n = 2$, completing the proof.

Remark 5. We offer the following comments on the above argument:

(i) [AAC01] established a sharp estimate:

$$\int_{B_R} |\nabla u|^2 \leq C \cdot R^{n-1}, \quad \forall R > 1. \tag{12}$$

(ii) Thus, the conjecture would be proven if we could strengthen the lemma to:

$$\int_{B_R} (\varphi\sigma)^2 \leq C \cdot R^{\gamma_n} \Longrightarrow \sigma \text{ is constant}, \tag{13}$$

assuming equality in condition (10). If $\gamma_n \geq n - 1$, then the conjecture follows. Hence, finding the optimal γ_n for which (13) holds is an interesting problem. [GG98] proved that $\gamma_n < n$ for $n \geq 3$, and [Bar98] proved that $\gamma_n < 2 + 2\sqrt{n-1}$ for $n \geq 7$.

6. Monotonicity, Stability, and Local Minimality

In this section, we clarify the connection with the *monotonicity* assumption and a new concept called *local minimality*, which derived from *geometric measure theory* and Γ-*convergence*.

Firstly, we say the entire solution u of equation (8) is *stable*, if

$$\int_{\mathbb{R}^n} |\nabla\zeta|^2 + F''(u)\zeta^2 \mathrm{d}x \geq 0, \quad ,\forall\zeta \in C_c^\infty(\mathbb{R}^n)$$

where the inequality above says the *second derivative* of the functional

$$\mathcal{F}[u] := \int_{\mathbb{R}^n} \frac{1}{2}|\nabla u|^2 - F(u)\mathrm{d}x$$

is nonnegative, noting that the equation (8) is the critical point of it. Particularly, as $F(u) = (1-u^2)^2/4$, say *double-well potential*, the functional is called *Ginzburg-Landau functional*. We claim that u is stable if and only if there exists a positive solution of the linearized equation $\Delta\varphi = F''(u)\varphi$. Particularly, if u is monotone, say $\partial_n u > 0$, then $\varphi := \partial_n u$ is the solution desired. Hence, if u is monotone, then u is stable.

Proposition 6.1. The entire solution u of (8) is stable if and only if there exists a continuous positive function φ, such that $\Delta\varphi = F''(u)\varphi$ in the sense of distribution.

Proof. $\Longrightarrow$: Firstly, the condition says that the first eigenvalue λ_1 of the Schrödinger operator $L := -\Delta + F''(u)$ is nonnegative in all B_R. Since λ_1 is decreasing with respect to R, all these first eigenvalues are positive. Thus, L is definite positive, which implies that, by the *Fredholm alternative theorem*, for all $c_R > 0$, there is a unique solution φ_R of

$$\begin{cases} \Delta\varphi = F''(u)\varphi_R, & \text{in } B_R \\ \varphi_R = c_R, & \text{on } \partial B_R \end{cases}$$

Furthermore, $\varphi_R > 0$ by the maximum principle. Then, by the *Krylov-Safonov Harnack* inequality, φ_R is bounded locally. Thus, there exists a subsequence of $\{\varphi_R\}$ converges to $\varphi > 0$ locally, such that $\Delta\varphi = F''(u)\varphi$.

$\Longleftarrow$: Multiply ζ^2/φ on both sides of the equation. We obtain the following inequality by integrating by parts and the Hölder inequality,

$$\int F''(u)\zeta^2 \mathrm{d}x \geq \int \frac{\zeta^2}{\varphi^2}|\nabla\varphi|^2 - 2\left(\int \frac{\zeta^2}{\varphi^2}|\nabla\varphi|^2\right)^{1/2} \cdot \left(\int |\nabla\zeta|^2\right)^{1/2}$$

Set

$$t = \left(\int \frac{\zeta^2}{\varphi^2}|\nabla\varphi|^2\right)^{1/2}, \quad s = \left(\int |\nabla\zeta|^2\right)^{1/2}$$

and observed that

$$\int F''(u)\zeta^2 \mathrm{d}x \geq (t-s)^2 - s^2 \geq -s^2$$

We complete the proof. □

We call u is a *local minimizer* of $\mathcal{F}$, if

$$\mathcal{F}(u,\Omega) \leq \mathcal{F}(v,\Omega), \quad \text{whenever } \{u \neq v\} \subset \Omega \subset\subset \mathbb{R}^n$$

where

$$\mathcal{F}(w,\Omega) := \int_\Omega \frac{1}{2}|\nabla w|^2 - F(w)\mathrm{d}x$$

Thus, it is obvious that if u is a local minimizer, then u is stable.

Question 2. Does the stability imply local minimality?

Indeed, [AAC01] presents the relationship when the comparing function v is in a certain class. Hence, we firstly introduce two simple notations, which are not only related to the additional assumption in the De Giorgi conjecture, see setting (9) section 5; but also a more general result in section 7, motivated by the sliding method. For a bounded and monotone solution u, we set

$$\underline{u}(x') := \lim_{x_n \to -\infty} u(x', x_n), \quad \overline{u}(x') := \lim_{x_n \to \infty} u(x', x_n)$$

Meanwhile, we define the sliding of u along the x_n-axis,

$$u^t(x) := u(x', x_n + t)$$

obviously, $\underline{u}(x') < u^t(x) < \overline{u}(x')$ in all compact subdomains $\overline{\Omega}$ in $\mathbb{R}^n$. Moreover, we discover the existence and uniqueness of the value $u^t(x)$ in the whole infinite cylinder $\mathbb{R}^n \times (\underline{u}, \overline{u})$; and denote the desired class to be

$$\mathcal{B} := \{v \in C^1(\overline{\Omega}) : \ \underline{u}(x') < v(x) < \overline{u}(x'), \ \forall x \in \overline{\Omega}\}$$

and

$$B(\overline{\Omega}) := \{(x,s) \in \overline{\Omega} \times \mathbb{R} : \ x \in \overline{\Omega}, \ \underline{u}(x') < s < \overline{u}(x')\} \subset \overline{\Omega} \times \mathbb{R}$$

Hence, for all $v \in \mathcal{B}$ at any point $x \in \overline{\Omega}$, there is a unique pair $(x,s,t) \in B(\overline{\Omega}) \times \mathbb{R}$ such that $v(x) = s = u^t(x)$.

Now, we want to show that, if u is stable, then u is a local minimizer in any bounded subdomain Ω in class $\mathcal{B}$, namely,

$$\mathcal{F}(u, \Omega) \le \mathcal{F}(v, \Omega), \quad \forall v \in \mathcal{B}$$

Thus, we introduce the *calibration* with respect to u, say $\mathcal{C}$, which is a functional defined by following properties:

(a) $\mathcal{C}(u, \Omega) = \mathcal{F}(u, \Omega)$;
(b) $\mathcal{C}(v, \Omega) \le \mathcal{F}(v, \Omega), \ \forall v \in \mathcal{B}$;
(c) $\mathcal{C}$ is *null-lagrangian*, namely, $\mathcal{C}$ is determined merely by the boundary value, i.e.

$$\mathcal{C}(v, \Omega) = \mathcal{C}(\tilde{v}, \Omega), \quad \forall v.\tilde{v} \in \mathcal{B}, \text{ with } v \equiv \tilde{v} \text{ on } \partial\Omega$$

To simplify, we ignore the symbol Ω in brackets of functionals. Hence, it suffices to find the calibration. Indeed, if there is a calibration, we obtain

$$\mathcal{F}(u) = \mathcal{C}(u) = \mathcal{C}(v) \le \mathcal{F}(v), \quad \forall v \in \mathcal{B}, \text{ with } u \equiv v \text{ on } \partial\Omega$$

where the first equality is from (a), the second equality is from (c), and the inequality is from (b). We now turn to the calibration and motivation derived from [AAC01], which will be presented in more detail in our forthcoming paper.

Now, we state the result and provide its proof.

Theorem 6.1 (Monotonicity implies local minimality; Theorem 4.4 [AAC01])**.** Let u be a bounded and entire solution of 8 with monotonicity, namely, $\partial_n u > 0$. Then u is local minimizer in domain Ω in the class $\mathcal{B}$, i.e.

$$\int_\Omega \frac{1}{2}|\nabla u|^2 + F(u)\mathrm{d}x \le \int_\Omega \frac{1}{2}|\nabla v|^2 + F(v)\mathrm{d}x, \quad \forall v \in \mathcal{B}$$

Proof. We start to define the calibration. Define a vector field $\phi := (\phi^x, \phi^s) : B(\overline{\Omega}) \to \mathbb{R}^n \times \mathbb{R}$; and ∇ is always the gradient in $\mathbb{R}^n$, where

$$\phi^x(x,s) := \nabla u^t, \quad \phi^t(s) := \frac{1}{2}|\nabla u^t(x)|^2 - F(s)$$

with $t = t(x,s)$ is the unique real number such that

$$u^t(x) = u^{t(x,s)}(x) = s \tag{14}$$

Then we define

$$\begin{aligned}\mathcal{C}(v) &:= \int_\Omega \phi^x(x, v(x)) \cdot \nabla v - \phi^s(x, v(x))\mathrm{d}x \\ &= \int_\Omega \nabla u^t \cdot \nabla v - \frac{1}{2}|\nabla u^t| + F(v)\mathrm{d}x\end{aligned}$$

for $v \in \mathcal{B}$, where $t = t(x, v(x))$ is defined by $u^t(x) = v(x)$, which is well defined as the argument above.

Now, we turn to prove that $\mathcal{C}$ is the calibration desired. The properties (a) and (b) are easy to verify, since $t(x, u(x)) \equiv 0$ and (b) is from the *Cauchy-Schwarz* inequality. Next, we prove $\mathcal{C}$ satisfies (c).

We claim that ϕ is div-free, namely $\mathrm{div}\phi = 0$, which implies $\mathcal{C}$ is null-Lagrangian. To verify $\mathrm{div}\phi = 0$, firstly, note that $u \in C^2(\mathbb{R}^n)$. Indeed, by $W^{2,p}$ estimates from $\Delta\partial_i u = F''(u)\partial_i u$, we obtain $u \in W^{3,p} \subset C^2$, where $p > n$. Thus, by the inverse mapping theorem, $t = t(x, s)$ is $C^2(B(\overline{\Omega}))$. Thus, we differentiate the relation (14),

$$\partial_t u^t \partial_s t = 1, \quad \nabla u^t + \partial_t u^t \nabla_x t = 0 \tag{15}$$

Then, since $\phi \in C^1$, we compute

$$\mathrm{div}_x \phi^x = \Delta u^t + \nabla\partial_t u^t \cdot \nabla_x t = \Delta u^t - \nabla\partial_t u^t \cdot \partial_s t \nabla u^t$$

by (15), and

$$\partial_s \phi^s = \nabla u^t \cdot \partial_s t \nabla \partial_t u^t$$

Thus,

$$\mathrm{div}\phi = \mathrm{div}_x \phi^x + \partial_s \phi^s = \Delta u^t - F(s) = 0$$

Finally, we will show $\mathcal{C}$ is null-Lagrangian. Indeed, for all $v, \tilde{v} \in \mathcal{B}$ with $v \equiv \tilde{v}$ on $\partial\Omega$, define $s = \zeta = \tilde{v} - v$. Then $v^\tau := v + \tau s = v + \tau\zeta = v + \tau(\tilde{v} - v),\ \tau \in [0, 1]$ satisfies the condition of the theorem. We consider

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{C}(v^\tau) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\int_\Omega \phi^x(x, v^\tau) \cdot \nabla v^\tau - \phi^s(x, v^\tau)\mathrm{d}x \\ &= \int_\Omega \partial_s\phi^x(x, v^\tau) \cdot \nabla v^\tau \cdot \zeta + \phi^x(x, v^\tau) \cdot \nabla\zeta - \partial_s\phi^s(x, v^\tau)\zeta\mathrm{d}x\end{aligned}$$

integrating by parts in the second term above, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{C}(v^\tau) &= \int_\Omega \partial_s\phi^x(x, v^\tau) \cdot \nabla v^\tau \cdot \zeta + \mathrm{div}_x\phi^x(x, v^\tau)\zeta - \partial_s\phi^x(x, v^\tau) \cdot \nabla v^\tau \cdot \zeta - \partial_s\phi^s(x, v^\tau)\zeta\mathrm{d}x \\ &= -\int_\Omega \mathrm{div}\phi(x, v^\tau)\mathrm{d}x = 0\end{aligned}$$

Therefore, $\mathcal{C}(v) = \mathcal{C}(\tilde{v})$. □

In this theorem, we not only know the relation of the monotonicity assumption - implying the graphical property of all level sets of u, see [AAC01, Sav09, Sav10], and we postpone this observation to our next paper - with the local minimality in class, say $\mathcal{B}$, which is related to certain *compactness* in geometric measure theory and Γ-convergence; but also establish a simple energy comparison between the local minimizer and any functions in such class, which will be used in next section. Indeed, it is essential that each x-slice of $B(\overline{\Omega})$ is an *interval*, which ensures that the graph of any perturbation, say v^τ, is contained in the class $\mathcal{B}$.

7. Proof of De Giorgi conjecture in 3 dimension and related aspects

Now, we turn to prove the De Giorgi conjecture in 3 dimension. Firstly, we present a result by Modica [Mod89], which illustrates the monotonicity of a certain *measure density* derived from the energy (the functional in the previous sections), without the monotonicity assumption of u actually. Recall our goal, namely, the energy estimate following,

$$\int_{B_R} |\nabla u|^2 \leq C \cdot R^{n-1}, \quad \forall R > 1$$

which is from

$$\int_{B_R} \frac{1}{2}|\nabla u|^2 + F(u) - c_u \mathrm{d}x \leq C \cdot R^{n-1}, \quad \forall R > 1 \tag{16}$$

where $c_u := \min\{F(s) : \inf u \leq s \leq \sup u\}$. In contrast, [Mod89] has shown the converse estimate.

Theorem 7.1 (Monotonicity formula and lower bounds; [Mod89])**.** Let u be a bounded entire solution of (8), then

$$\Theta_{\mathcal{C}}(R) := \frac{\int_{B_R} \frac{1}{2}|\nabla u|^2 + F(u) - c_u \mathrm{d}x}{R^{n-1}}$$

is nondecreasing in $(0, +\infty)$. Particularly, if u is not a constant, then there is a positive constant c, such that

$$\int_{B_R} \frac{1}{2}|\nabla u|^2 + F(u) - c_u \mathrm{d}x \geq c \cdot R^{n-1}, \quad \forall R > 1$$

Additionally, [Mod85] proved a pointwise gradient estimate

$$|\nabla u|^2 \leq 2(F(u) - c_u), \quad \text{in } \mathbb{R}^n$$

where both results are without the monotonicity assumption.

Finally, we show the proof of the De Giorgi conjecture in 3 dimension, but also the proof of the general case, i.e., the remaining part in theorem 5.1. In fact, [AC00] (the earlier paper) has proved such an estimate for a special class of F, which is based on a "sliding" argument by letting $u^t(x) = u(x', x_n + t)$. But in [AAC01], they use the local minimality property of u - see the previous section - to extend the estimate to any $F \in C^2$. We show the following estimate, which implies the theorem 5.1.

Theorem 7.2 (Upper bound; Theorem 5.2 [AAC01])**.** Let u be a bounded entire solution of (8), with the monotonicity, namely, $\partial_n u > 0$. If $n \leq 3$ or u satisfies the condition (9) - the Gibbons conjecture, then the energy estimate

$$\int_{B_R} \frac{1}{2}|\nabla u|^2 + F(u) - c_u \mathrm{d}x \leq C \cdot R^{n-1}, \quad \forall R > 1$$

holds for some constant C independent of R.

Proof. Let $m = \inf u,\ M = \sup u$, and set $s \in [m, M]$ such that $c_u = F(s)$.

CASE 1 If u satisfies (9), then $\underline{u} \equiv m,\ \overline{u} = M$, where we can use the energy comparison argument from theorem 6.1. Indeed, choose the cutoff function $\phi_R \in C_c^\infty(\mathbb{R}^n)$, such that

$$\phi_R \equiv 1 \text{ in } B_{R-1}, \quad \phi_R \equiv 0 \text{ outside } B_R$$

with $|\nabla \phi_R| \leq 2$. Then consider

$$v_R := (1 - \phi_R)u + \phi_R s$$

which satisfies the condition in theorem 6.1, then we obtain

$$\begin{aligned}
\int_{B_R} \frac{1}{2}|\nabla u|^2 + F(u) - c_u \mathrm{d}x &\leq \int_{B_R} \frac{1}{2}|\nabla v_R|^2 + F(v_R) - c_u \mathrm{d}x \\
&= \int_{B_R - B_{R-1}} \frac{1}{2}|\nabla v_R|^2 + F(v_R) - c_u \mathrm{d}x \\
&\leq C|B_R - B_{R-1}| \leq C \cdot R^{n-1}, \quad \forall R > 1
\end{aligned}$$

Thus, we have the desired estimate.

CASE 2 The condition (9) dropped but $n \leq 3$. Our goal is to find a new bound to go back to CASE 1. Let

$$\tilde{m} = \sup \underline{u}, \quad \tilde{M} = \inf \overline{u}$$

We claim that $\tilde{m} < \tilde{M}$ as $n \leq 3$. Indeed, it is obvious that $\tilde{m}, \tilde{M} \in [m, M]$; then by Lemma 3.1 and Lemma 3.2 in [AC00], $\underline{u}, \overline{u}$ are either constant or monotone in $\mathbb{R}^2$. Thus, it remains to show that the only case for $\underline{u}, \overline{u}$ are not constant. Indeed, by the ODE argument - see [AC00], we obtain

$$F > F(m) = F(\tilde{m}), \quad \text{in } (m, \tilde{m})$$

$$F > F(M) = F(\tilde{M}), \quad \text{in } (\tilde{M}, M)$$

thus, $\tilde{m} < \tilde{M}$. Therefore, we obtain the new bound, namely,

$$\tilde{m} \leq u \leq \tilde{M}$$

which leads to the CASE 1.

$\square$

8. Higher-Dimensional Extensions and the Blow-Down Approach

We end this paper by previewing the proof of De Giorgi conjecture in higher dimensions, which has been solved by O.Savin [Sav09]; and we will show more details in our forthcoming paper.

Indeed, the De Giorgi conjecture is related to the *Bernstein problem*, which says that the entire minimal graph is definitely a plane under certain dimensions. In this section, we give a simple illustration of this relation. Indeed, we rewrite the Ginzburg-Landau functional with the double-well potential in a unit ball, i.e.,

$$\mathcal{F}(u, B_1) = \int_\Omega \frac{1}{2}|\nabla u|^2 + F(u)\mathrm{d}x$$

where $F(u) := (1-u^2)^2/4$. Introduce the blow-down of u, say $u_\epsilon(x) := u(x/\epsilon)$ which implies a new scaling energy, say

$$\mathcal{F}_\epsilon(v, B_1) = \int_\Omega \frac{\epsilon}{2}|\nabla v|^2 + \frac{1}{\epsilon}F(v)\mathrm{d}x$$

We require this energy to be minimal as $\epsilon \to 0$. Note that the *kinetic energy* term $|\nabla v|$ can be determined by ϵ, however, the *potential* term $F(v)$ may tend to ∞ by the process of ϵ; thus, v must assume $+1$ on nearly "half" of the ball B_1 and assume -1 on the other part. Therefore, v will sharply drop from $+1$ to -1 through a very narrow family of hypersurfaces $\{v = s\}_{-1<s<+1}$. It is natural to ask the shape of all these hypersurfaces, which is the statement of De Giorgi conjecture.

On the other hand, we explain the relation with the Bernstein problem. Indeed, using the Cauchy-Schwarz inequality, we obtain

$$\mathcal{F}_\epsilon(v, B_1) = \int_\Omega \frac{\epsilon}{2}|\nabla v|^2 + \frac{1}{\epsilon}F(v)\mathrm{d}x \geq \int_{B_1} \sqrt{2F(v)}|\nabla v|$$

and by the co-area formula, namely,

$$|\nabla v|\mathrm{d}x = \mathrm{d}_{\mathcal{H}^{n-1}(\{v=s\})}\mathrm{d}s$$

where $\mathcal{H}^{n-1}(\{v = s\})$ is the *Hausdorff measure* of the level set $\{v = s\}$. Then we obtain

$$\mathcal{F}_\epsilon(v, B_1) \geq \int_{-1}^{1} \sqrt{2F(s)}\mathcal{H}^{n-1}\{v = s\}\mathrm{d}s$$

Thus, the level sets $\{v = s\}$ all need to have the minimal area, which is related to the minimal surfaces.

Thus, in the process of blow-down, we hope the sequence $\{u_\epsilon\}$ can tend to the desired solution with flat level sets, [Mod78] proved such convergence in L^1_{loc} sense, which is a weak result but inspires us to modify it to stronger sense, such as L^∞. Fortunately, Caffarelli and Cordoba [CC95] proved an upper bound of energy, which implies the *asymptotic flatness* at ∞ of level sets (also shown in [AAC01]). Furthermore, Savin modified a well-known result by De Giorgi (see Theorem 5.1 in [Sav10]) by Harnack inequality (see Theorem 5.3 in [Sav10]) to the following rough statement, namely, if the level set lies in a nearly flat cylinder, then it lies in a flatter cylinder, which implies the result of the conjecture.

References

[AAC01] Giovanni Alberti, Luigi Ambrosio, and Xavier Cabré. On a long-standing conjecture of e. de giorgi: symmetry in 3d for general nonlinearities and a local minimality property. *Acta Applicandae Mathematica*, 65(1-3):9–33, 2001. 1, 3, 3, 5, 5, 5.1, 5, 6, 6.1, 6, 7, 7.2, 8

[AC00] Luigi Ambrosio and Xavier Cabré. Entire solutions of semilinear elliptic equations in $\mathrm{r}^3 andaconjectureofdegiorgi.JournaloftheAmericanMathematicalSociety, 13(4) : 725--739, 2000.(document), 1, 3, 3, 5, 5, 5.1, 5, 7, 7$

[Ale62] Aleksandr D Alexandrov. A characteristic property of spheres. *Annali di Matematica Pura ed Applicata*, 58(1):303–315, 1962. 2

[Bar98] Martin T Barlow. On the liouville property for divergence form operators. *Canadian Journal of Mathematics*, 50(3):487–496, 1998. 3.2, 3, 5, 5

[BBG00] Martin T Barlow, Richard F Bass, and Changfeng Gui. The liouville property and a conjecture of de giorgi. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 53(8):1007–1038, 2000. 1, 5

[BCN93] Henri Berestycki, Luis A Caffarelli, and Louis Nirenberg. Symmetry for elliptic equations in a half space. *Boundary value problems for partial differential equations and applications*, 29:27–42, 1993. 1, 3, 3.5

[BCN96] H Berestycki, Luis A Caffarelli, and L Nirenberg. Inequalities for second-order elliptic equations with applications to unbounded domains i. *Duke Math. J.*, 85(1):467–494, 1996. 1, 3.1, 3.2, 4.1, 4

[BCN97a] Henri Berestycki, Luis A Caffarelli, and Louis Nirenberg. Further qualitative properties for elliptic equations in unbounded domains. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 25(1-2):69–94, 1997. (document), 1, 2, 3, 3.1, 3, 3.3, 2, 3.4, 3, 3.1, 4, 1, 3, 3.6, 3, 3.7, 3, 1, 3.8, 4, 4, 4, 5, 5

[BCN97b] Henri Berestycki, Luis A Caffarelli, and Louis Nirenberg. Monotonicity for elliptic equations in unbounded lipschitz domains. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 50(11):1089–1111, 1997. 1, 2

[BHM00] H Berestycki, F Hamel, and R Monneau. One-dimensional symmetry of bounded entire solutions of some elliptic equations. *Duke Math. J.*, 104(1):375–396, 2000. 1, 5

[BN88] Henri Berestycki and Louis Nirenberg. Monotonicity, symmetry and antisymmetry of solutions of semilinear elliptic equations. *Journal of Geometry and Physics*, 5(2):237–275, 1988. 2

[BN90] Henri Berestycki and Louis Nirenberg. Some qualitative properties of solutions of semilinear elliptic equations in cylindrical domains. In *Analysis, et cetera*, pages 115–164. 1990. 2

[BN91] Henri Berestycki and Louis Nirenberg. On the method of moving planes and the sliding method. *Boletim da Sociedade Brasileira de Matemática-Bulletin/Brazilian Mathematical Society*, 22(1):1–37, 1991. 1, 2, 2.3, 2, 2, 2.4

[BNV94] Henri Berestycki, Louis A Nirenberg, and SR Srinivasa Varadhan. The principal eigenvalue and maximum principle for second-order elliptic operators in general domains. *Communications on Pure and Applied Mathematics*, 47(1):47–92, 1994. 1, 3, 4, 4.4, 4

[Bus99] Jérôme Busca. Existence results for bellman equations and maximum principles in unbounded domains. *Communications in partial differential equations*, 24(11-12):2023–2042, 1999. 4.1

[CC95] Luis A Caffarelli and Antonio Córdoba. Uniform convergence of a singular perturbation problem. *Communications on Pure and Applied Mathematics*, 48(1):1–12, 1995. 8

[DGS79] Ennio De Giorgi and Sergio Spagnolo. Convergence problems for functionals and operators. *Ennio De Giorgi*, 487, 1979. 1, 5, 2, 5

[EL82] Maria J Esteban and Pierre-L Lions. Existence and non-existence results for semilinear elliptic problems in unbounded domains. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 93(1-2):1–14, 1982. 1, 2

[Eva22] Lawrence C Evans. *Partial differential equations*, volume 19. American Mathematical Society, 2022. 4

[Far99] Alberto Farina. Symmetry for solutions of semilinear elliptic equations in r^ n and related conjectures. *ATTI-ACCADEMIA NAZIONALE DEI LINCEI RENDICONTI LINCEI CLASSE DI SCIENZE FISICHE MATEMAICHE E NATURALI SERIE 9 MATEMATICA E APPLICAZIONI*, 10(4):255–266, 1999. 1, 5

[FS16] Alberto Farina and Berardino Sciunzi. Qualitative properties and classification of nonnegative solutions to $-\delta u = f(u)$ in unbounded domains when $f(0) < 0$. *Revista Matemática Iberoamericana*, 32(4):1311–1330, 2016. 1, 3, 3.3, 2, 3.1, 4, 3, 3.6

[GG98] Nassif Ghoussoub and Changfeng Gui. On a conjecture of de giorgi and some related problems. *Mathematische Annalen*, 311(3):481–491, 1998. (document), 1, 3, 3.1, 5, 5.1, 5, 5

[GNN79] Basilis Gidas, Wei-Ming Ni, and Louis Nirenberg. Symmetry and related properties via the maximum principle. *Communications in mathematical physics*, 68(3):209–243, 1979. 2, 2.1, 2.2

[GNN81] Basilis Gidas, Wei-Ming Ni, and Louis Nirenberg. Symmetry of positive solutions of nonlinear elliptic equations in r$^n, parta.AdvancesinMath.Suppl.Studies'A, pages369--402, 1981.2$

[GT77] David Gilbarg and Neil S Trudinger. *Elliptic partial differential equations of second order*, volume 224. Springer, 1977. 2, 2, 4, 5

[Mod78] Luciano Modica. Convergence to minimal surfaces problem and global solutions of u= 2 (u3- u). In *Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978)*, pages 223–244, 1978. 8

[Mod85] Luciano Modica. A gradient bound and a liouville theorem for nonlinear poisson equations. *Communications on pure and applied mathematics*, 38(5):679–684, 1985. 7

[Mod89] Luciano Modica. Monotonicity of the energy for entire solutions of semilinear elliptic equations. In *Partial Differential Equations and the Calculus of Variations: Essays in Honor of Ennio De Giorgi Volume 2*, pages 843–850. 1989. 7, 7, 7.1

[Sav09] Ovidiu Savin. Regularity of flat level sets in phase transitions. *Annals of Mathematics*, pages 41–78, 2009. (document), 1, 5, 6, 8

[Sav10] Ovidiu Savin. Phase transitions, minimal surfaces and a conjecture of de giorgi. In *Current developments in mathematics*, volume 2009, pages 59–114. International Press of Boston, 2010. 3, 5, 6, 8

[Ser71] James Serrin. A symmetry problem in potential theory. *Archive for Rational Mechanics and Analysis*, 43(4):304–318, 1971. 2