Random Matrix Theory and Its Applications in Large-scale Systems

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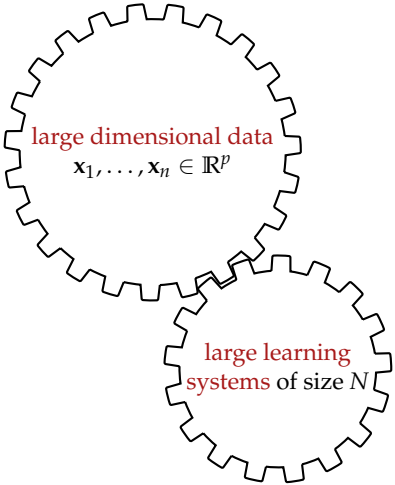
Large-scale systems for high-dimensional and massive data



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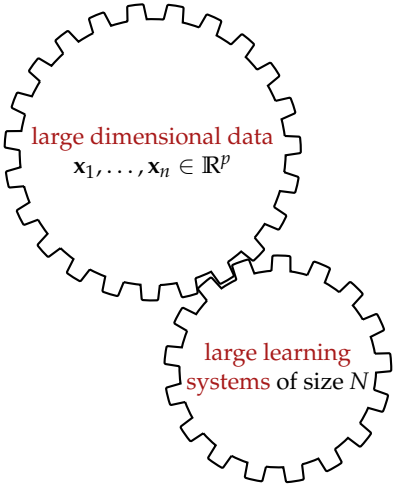

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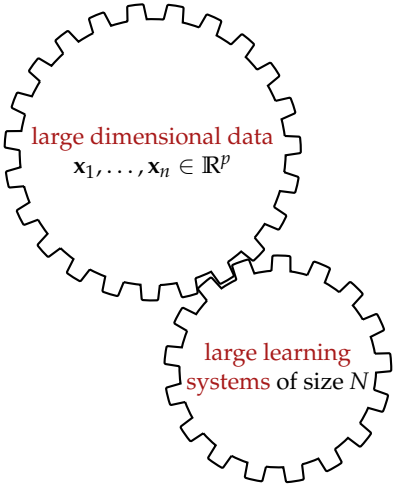

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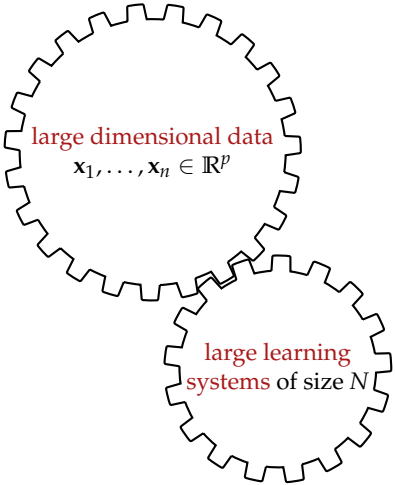

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- complete change of understanding of many methods
- **RMT** provides the tools!

Outline

1. Sample covariance matrix for large dimensional data
2. Application of RMT to large-scale telecommunication
3. Application of RMT to large-scale signal processing
4. Application of RMT to large-scale machine learning

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Sample covariance matrix in the large n, p regime

- For $\mathbf{x}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$, estimate population covariance $\mathbf{C} \in \mathbb{R}^{p \times p}$ from n data samples $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n] \in \mathbb{R}^{p \times n}$.

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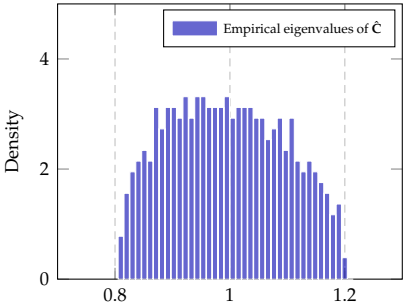

Figure: Eigenvalue distribution of $\hat{\mathbf{C}}$ versus Marčenko-Pastur law, $p = 500$, $n = 50\,000$.

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What about $n = 100p$? For $\mathbf{C} = \mathbf{I}_p$, as $n, p \to \infty$ with $p/n \to c \in (0, \infty)$: the Marčenko–Pastur law

$$\mu(dx) = (1 - c^{-1})^+ \delta(x) + \frac{1}{2\pi c x} \sqrt{(x - E_-)^+ (E_+ - x)^+} dx$$

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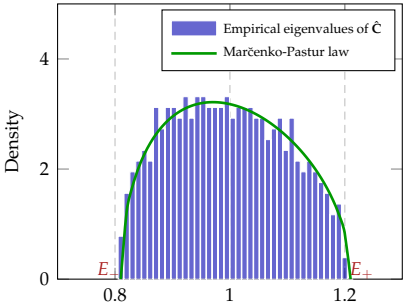

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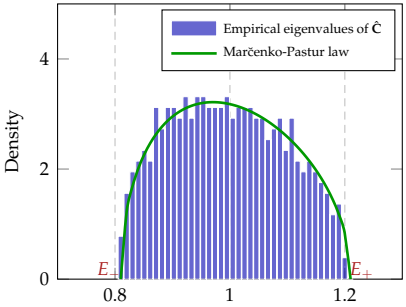

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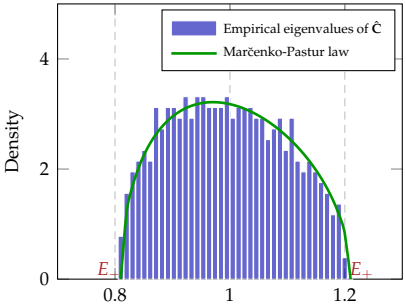

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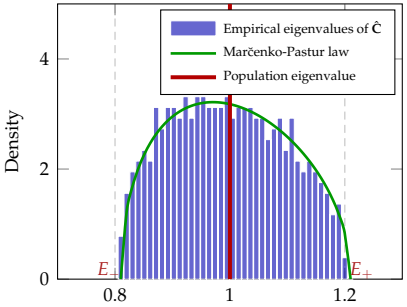

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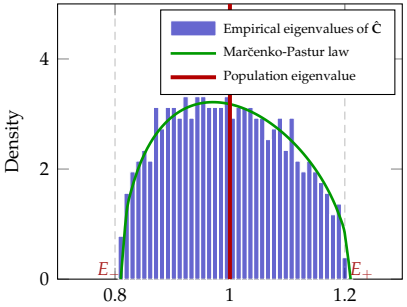

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- **Question**: how to evaluate the **capacity** (max achievable transmission data rate) of CDMA network? (which clearly depends on pre-coding strategy)
- **Answer**: to use random i.i.d. codes, that is (pseudo-)random CDMA! (that can be shown to outperform TDMA and fully orthogonal CDMA)

For **orthogonal** CDMA, assume:

- frequency flat channel conditions for all users; and
- channel stability over a large number of successive symbol periods;

then the rates achieved in the up-link (from user terminals to APs) are maximal when the orthogonal codes are as long as the number of users n, and we have, for a noise power σ^2, system capacity given by

$$C_{\text{orth}}(\sigma^2) = \frac{1}{n} \log \det \left(\mathbf{I}_n + \frac{1}{\sigma^2} \mathbf{W}\mathbf{G}\mathbf{G}^{\mathsf{H}}\mathbf{W}^{\mathsf{H}} \right), \tag{1}$$

with $\mathbf{W} \in \mathbb{C}^{n \times n}$ having its columns the **orthogonal** CDMA codes (and $\mathbf{W}$ being unitary in this case), and $\mathbf{G} \equiv \text{diag}\{g_i\}_{i=1}^{n}$ the diagonal matrix of the channel **gains** of the users $1, \ldots, n$.

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It can be easily checked (with Sylvester's identity and $\mathbf{W}$ unitary with $\mathbf{W}^{\mathsf{H}} \mathbf{W} = \mathbf{I}_n$) that

$$C_{\text{orth}}(\sigma^2) = \frac{1}{n} \log \det \left(\mathbf{I}_n + \frac{1}{\sigma^2} \mathbf{G} \mathbf{G}^{\mathsf{H}} \right) = \frac{1}{n} \sum_{i=1}^{n} \log \left(1 + \frac{|g_i|^2}{\sigma^2} \right) = C_{\text{TDMA}}(\sigma^2). \tag{2}$$

This justifies the equivalence between TDMA and **orthogonal** CDMA rate performance.

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- for example, via **signal sensing/detection**

Hypothesis testing in a signal-plus-noise model for cognitive radios

System model: let $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n] \in \mathbb{R}^{p \times n}$ be a matrix with i.i.d. columns $\mathbf{x}_i \in \mathbb{R}^p$ received by a large array of p sensors, the decision problem is formulated as the following binary hypothesis test:

$$\mathbf{X} = \begin{cases} \sigma \mathbf{Z}, & \mathcal{H}_0 \\ \mathbf{a}\mathbf{s}^{\mathsf{T}} + \sigma \mathbf{Z}, & \mathcal{H}_1 \end{cases}$$

where $\mathbf{Z} = [\mathbf{z}_1, \ldots, \mathbf{z}_n] \in \mathbb{R}^{p \times n}$ with $\mathbf{z}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_p)$, $\mathbf{a} \in \mathbb{R}^p$ deterministic with unit norm $\|\mathbf{a}\| = 1$, signal $\mathbf{s} = [s_1, \ldots, s_n]^{\mathsf{T}} \in \mathbb{R}^n$ with s_i i.i.d. random scalars, and $\sigma > 0$. We also denote $c = p/n \in (0, \infty)$.

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- If the parameters $\mathbf{a}$, σ as well as the statistics of s_i are known, a mere Neyman-Pearson test allows one to discriminate between $\mathcal{H}_0$ and $\mathcal{H}_1$ with optimal detection probability: decide on the genuine hypothesis according to the ratio of posterior probabilities

$$\frac{\mathbb{P}(\mathbf{X} \mid \mathcal{H}_1)}{\mathbb{P}(\mathbf{X} \mid \mathcal{H}_0)} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \alpha \tag{4}$$

for some $\alpha > 0$ controlling the desired Type I and Type II error rates (that is, the probability of false positives and of false negatives).

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- under both a Gaussian noise and signal s_i assumption, the GLRT has an explicit expression as a monotonous increasing function of $\|\mathbf{X}\mathbf{X}^\mathsf{T}\| / \operatorname{tr}(\mathbf{X}\mathbf{X}^\mathsf{T})$. so the test is equivalent to

$$T_p \equiv \frac{\left\| \frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T} \right\|}{\frac{1}{p}\operatorname{tr}\left(\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T}\right)} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} f(\alpha),$$

for some known monotonously increasing function f (the normalizations $1/p$ and $1/n$ so that both the numerator and denominator are of order $O(1)$ as $n, p \to \infty$).

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- to evaluate the power of GLRT above, we need to assess the max and mean eigenvalues of SCM $\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T}$

Hypothesis testing in a signal-plus-noise model via GLRT

As a consequence, in order to set a maximum false alarm rate (or false positive, or Type I error) of $r > 0$ in the limit of large n, p, one must choose a threshold $f(\alpha)$ for T_p such that

$$\mathbb{P}(T_p \leq f(\alpha)) = r,$$

that is, such that

$$\mu_{\mathrm{TW}_1}((-\infty, A_p]) = r, \quad A_p = (f(\alpha) - (1+\sqrt{c})^2)(1+\sqrt{c})^{-\frac{4}{3}} c^{\frac{1}{6}} n^{\frac{2}{3}} \tag{5}$$

with μ_{TW_1} the Tracy-Widom distribution.

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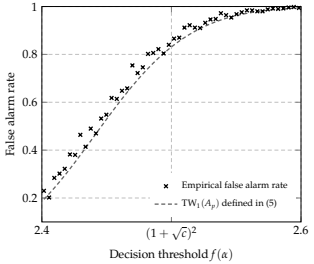

Figure: Comparison between empirical false alarm rates and $\mathrm{TW}_1(A_p)$ for A_p of the form in (5), as a function of the threshold $f(\alpha) \in [(1+\sqrt{c})^2 - 5n^{-2/3}, (1+\sqrt{c})^2 + 5n^{-2/3}]$, for $p = 256$, $n = 1\,024$ and $\sigma = 1$. Results obtained from 500 runs.

Outline

"Curse of dimensionality": loss of relevance of Euclidean distance

- Binary Gaussian mixture classification $\mathbf{x} \in \mathbb{R}^p$:

$$\mathcal{C}_1 : \mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}_1, \mathbf{C}_1), \text{ versus } \mathcal{C}_2 : \mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}_2, \mathbf{C}_2);$$

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- Neyman-Pearson test: classification is possible only when [CLM18]

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for some constants $C_{\boldsymbol{\mu}}, C_{\mathbf{C}} > 0$.

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  for some constants $C_{\boldsymbol{\mu}}, C_{\mathbf{C}} > 0$.

- In this non-trivial setting, for $\mathbf{x}_i \in \mathcal{C}_a, \mathbf{x}_j \in \mathcal{C}_b$:

$$\boxed{\max_{1 \leq i \neq j \leq n} \left\{ \frac{1}{p} \|\mathbf{x}_i - \mathbf{x}_j\|^2 - \frac{2}{p} \operatorname{tr} \mathbf{C}^\circ \right\} \xrightarrow{a.s.} 0}$$

  as $n, p \to \infty$ (i.e., $n \sim p$), for $\mathbf{C}^\circ \equiv \frac{1}{2}(\mathbf{C}_1 + \mathbf{C}_2)$, regardless of the classes $\mathcal{C}_a, \mathcal{C}_b$! (In fact even for $n = p^m$.)

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Loss of relevance of Euclidean distance in large dimensions: visual representation

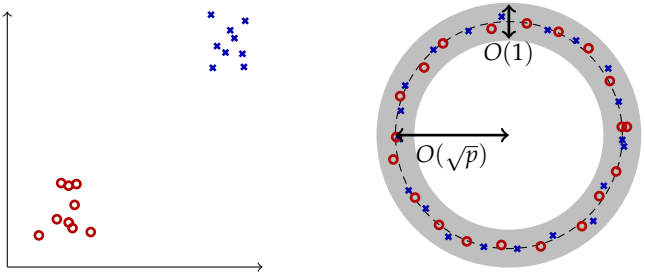

Figure: Visual representation of classification in (**left**) small and (**right**) large dimensions.

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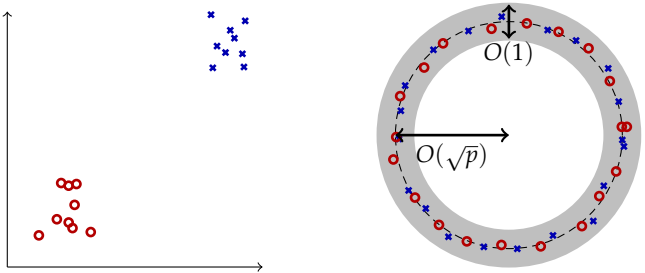
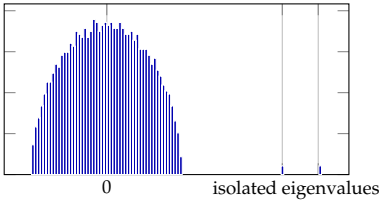


Figure: Visual representation of classification in (**left**) small and (**right**) large dimensions.

$\Rightarrow$ Direct consequence to various distance-based machine learning methods (e.g., kernel spectral clustering)!

Reminder on kernel spectral clustering

Two-step classification of n data points based on distance kernel matrix $\mathbf{K} \equiv \{f(\|\mathbf{x}_i - \mathbf{x}_j\|^2/p)\}_{i,j=1}^n$:



Reminder on kernel spectral clustering

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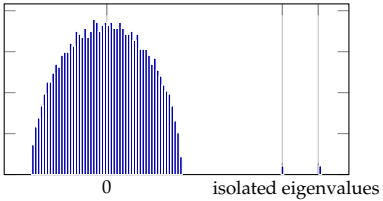
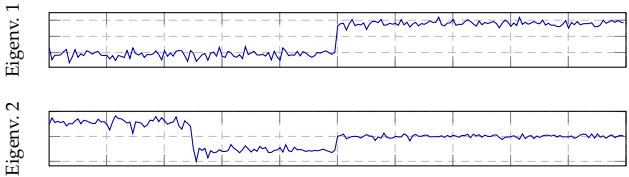


Reminder on kernel spectral clustering



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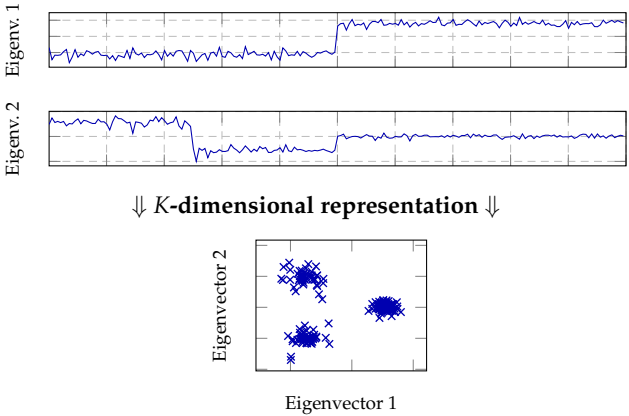

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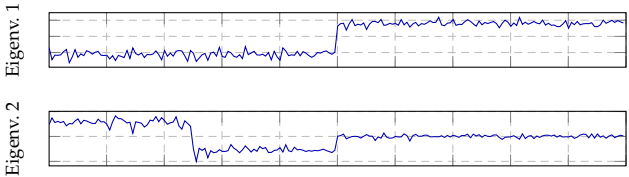
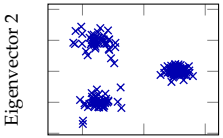


$\Downarrow$ K-**dimensional representation** $\Downarrow$



$\Downarrow$

EM or k-means clustering.

(Three classes/clusters in this example.)

Visualization of kernel matrices for large dimensional Gaussian data

Objective: "cluster" Gaussian data $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbf{R}^p$ into $\mathcal{C}_1$ or $\mathcal{C}_2$.

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(a) $p = 5, n = 500$ (b) $p = 250, n = 500$

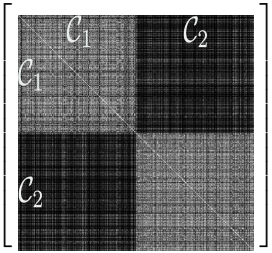
$\mathbf{K} =$ 

Figure: Kernel matrices $\mathbf{K}$ and the second top eigenvectors $\mathbf{v}_2$ for small (left, $p = 5, n = 500$) and large (right, $p = 250, n = 500$) dimensional data.

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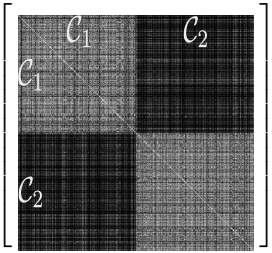

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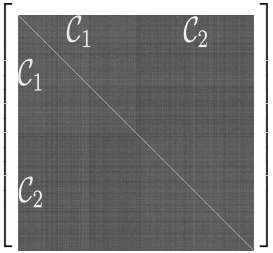

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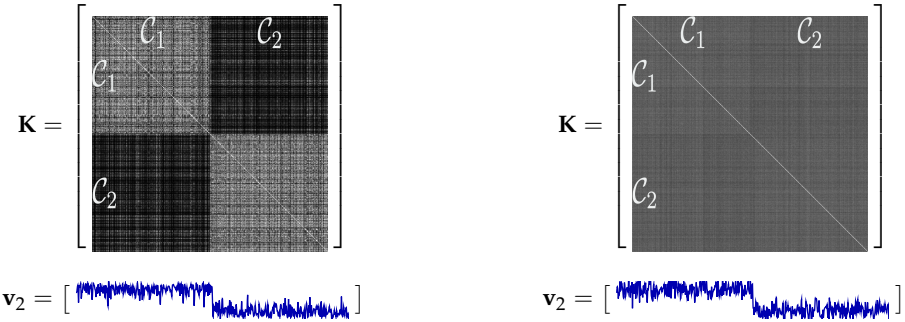

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Kernel matrices for large dimensional real-world data

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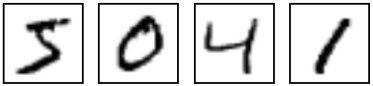
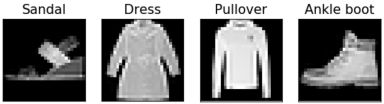


(b) Fashion-MNIST



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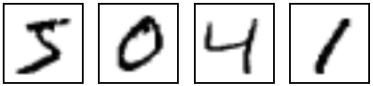
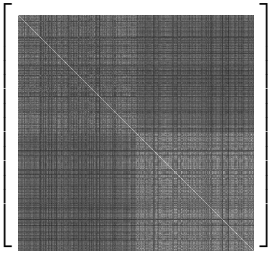


$\mathbf{K} =$



$\mathbf{v}_2 =$

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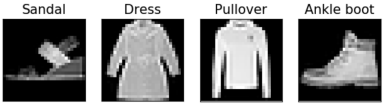

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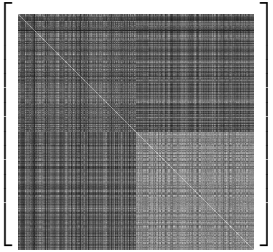

$\mathbf{v}_2 =$

A spectral viewpoint of large kernel matrices in large dimensions

- "local" linearization of *nonlinear* kernel matrices in large dimensions, e.g., Gaussian kernel matrix $\mathbf{K}_{ij} = \exp(-\|\mathbf{x}_i - \mathbf{x}_j\|^2/2p)$ with $\mathbf{C}_1 = \mathbf{C}_2 = \mathbf{I}_p$ (e.g., $\mathcal{C}_1 : \mathbf{x}_i = \boldsymbol{\mu}_1 + \mathbf{z}_i$ versus $\mathcal{C}_2 : \mathbf{x}_j = \boldsymbol{\mu}_2 + \mathbf{z}_j$) so that

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Therefore

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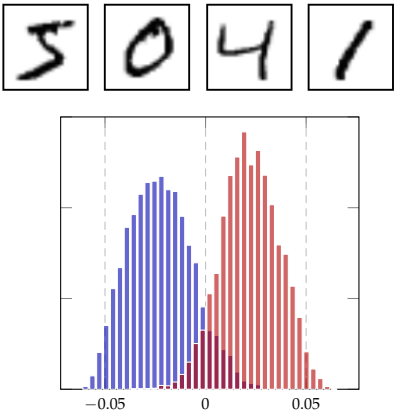
$\Rightarrow$ With **RMT**, we understand kernel spectral clustering for large dimensional data!

Numerical results



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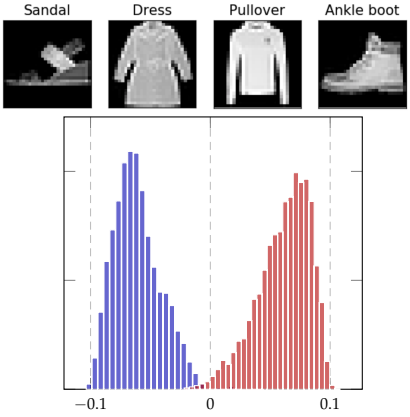

(b) Fashion-MNIST

Figure: Empirical histogram of LS-SVM soft output versus RMT prediction, $n = 2\,048$, $p = 784$, $\gamma = 1$ with Gaussian kernel, for MINST (**left**, 7 versus 9) and Fashion-MNIST (**right**, 8 versus 9) data. Results averaged over 30 runs.

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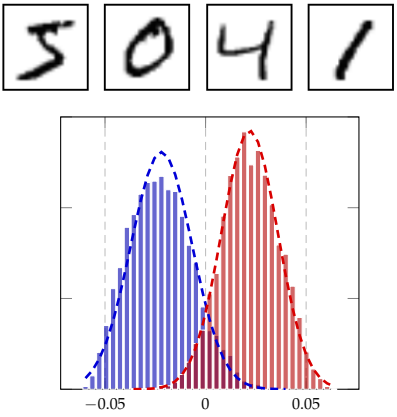

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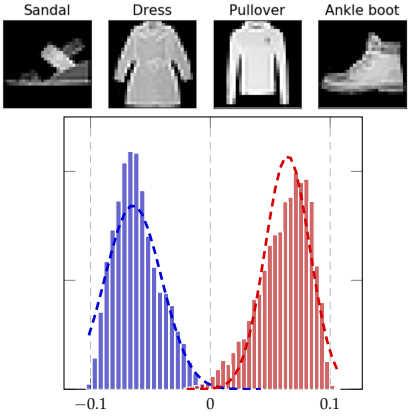

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Thank you! And some more information

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Thank you! Q & A?