The Dynamics of Learning: A Random Matrix Approach

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Zhenyu Liao, Romain Couillet

L2S, CentraleSupélec, Université Paris-Saclay, France
GSTATS IDEX DataScience Chair, GIPSA-lab, Université Grenoble-Alpes, France.

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- (explicit or implicit) regularization: early stopping, l_2-penalization

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Problem Setup

A toy model of binary classification:

Gaussian mixture data

Consider data $\mathbf{x}_i$ drawn from a two-class Gaussian mixture model: for $a = 1, 2$

$$\mathbf{x}_i \in \mathcal{C}_a \Leftrightarrow \mathbf{x}_i = (-1)^a \boldsymbol{\mu} + \mathbf{z}_i$$

with $\mathbf{z}_i \sim \mathcal{N}(\mathbf{0}_p, \mathbf{I}_p)$. With label $y_i = -1$ for $\mathcal{C}_1$ and $+1$ for $\mathcal{C}_2$.

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Random Matrix Theory is the answer!

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$\Rightarrow$ Network performance at any time is in fact deterministic and predictable!

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Proposed analysis framework

Resolvent and deterministic equivalents

Consider an $n \times n$ Hermitian random matrix $\mathbf{M}$. Define its resolvent $\mathbf{Q}_{\mathbf{M}}(z)$, for $z \in \mathbb{C}$ not eigenvalue of $\mathbf{M}$

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However, for more sophisticated functionals of $\mathbf{M}$:

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almost surely as $n \to \infty$, with $\mathbf{A}, \mathbf{a}, \mathbf{b}$ of bounded norm (operator and Euclidean).

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However, for more sophisticated functionals of $\mathbf{M}$:

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with γ a positively oriented path circling around all the eigenvalues of $\mathbf{M}$.

Proposed analysis framework

Resolvent and deterministic equivalents

Consider an $n \times n$ Hermitian random matrix $\mathbf{M}$. Define its resolvent $\mathbf{Q}_{\mathbf{M}}(z)$, for $z \in \mathbb{C}$ not eigenvalue of $\mathbf{M}$

$$\mathbf{Q}_{\mathbf{M}}(z) = (\mathbf{M} - z\mathbf{I}_n)^{-1}.$$

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Not really understandable, nor interpretable...

Simplification: "break" the contour integration

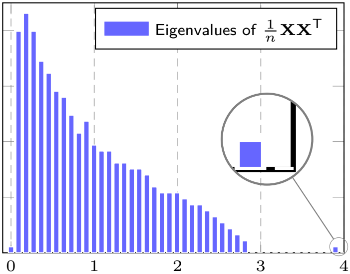

Figure: Eigenvalue distribution of $\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T}$ for $\boldsymbol{\mu} = [1.5; \mathbf{0}_{p-1}]$, $p = 512$, $n = 1\,024$ and $c_1 = c_2 = 1/2$.

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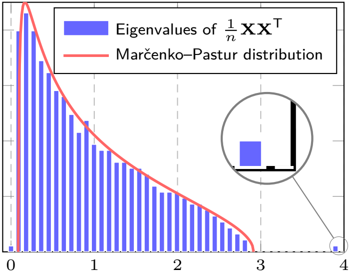

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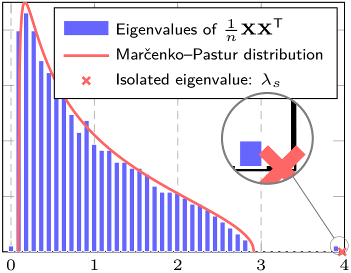

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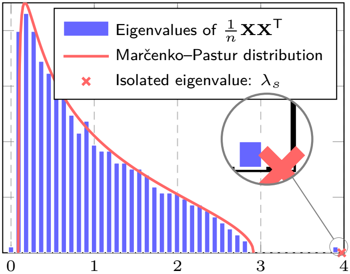

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3. How much we over-fit? As $t \to \infty$, the performance drop by a factor $\sqrt{1 - \min(c, c^{-1})}$, with $p/n \to c \in (0, \infty)$.

Numerical validations

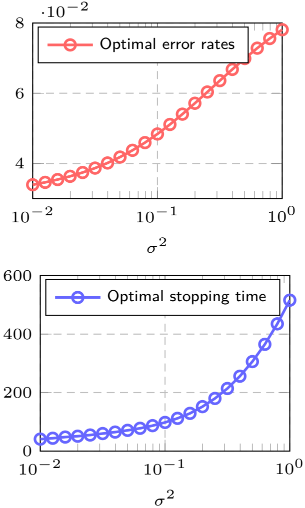
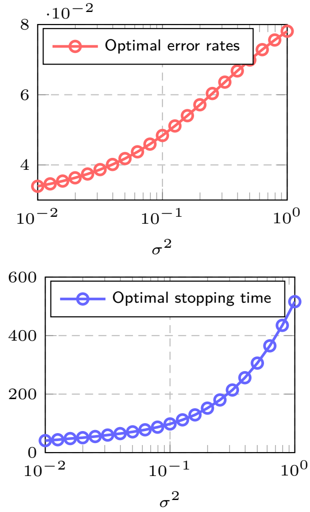


Figure: Optimal performance and stopping time as function of σ^2 with $c = 1/2$, $\|\boldsymbol{\mu}\|^2 = 4$ and $\alpha = 0.01$.



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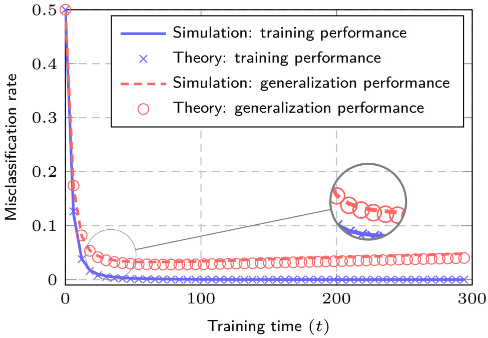

Figure: Training and generalization performance for MNIST data (number 1 and 7) with $n = p = 784$, $c_1 = c_2 = 1/2$, $\alpha = 0.01$ and $\sigma^2 = 0.1$. Results averaged over 100 runs.

Outline

Summary

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  Cauchy's integral formula + technique of **deterministic equivalent**

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Summary

Take-away messages:

- RMT framework to understand and predict learning dynamics:

  Cauchy's integral formula + technique of **deterministic equivalent**

- easily extended to more elaborate data models: e.g., Gaussian mixture model with different means and covariances
- a byproduct: choose the initialization variance σ^2 even smaller!

Thank you!

Any question? Poster # 189!