Supplementary Material

A Large Dimensional Analysis of Least Squares Support Vector Machines

Appendix A
Proof of Theorem 1

Our key interest here is on the decision function of LS-SVM: $g(\mathbf{x}) = \boldsymbol{\alpha}^\mathsf{T}\mathbf{k}(\mathbf{x}) + b$ with $(\boldsymbol{\alpha}, b)$ given by

$$\begin{cases} \boldsymbol{\alpha} &= \mathbf{S}^{-1}\left(\mathbf{I}_n - \frac{\mathbf{1}_n^\mathsf{T}\mathbf{1}_n\mathbf{S}^{-1}}{\mathbf{1}_n^\mathsf{T}\mathbf{S}^{-1}\mathbf{1}_n}\right)\mathbf{y} \\ b &= \frac{\mathbf{1}_n^\mathsf{T}\mathbf{S}^{-1}\mathbf{y}}{\mathbf{1}_n^\mathsf{T}\mathbf{S}^{-1}\mathbf{1}_n} \end{cases}$$

and $\mathbf{S}^{-1} = \left(\mathbf{K} + \frac{n}{\gamma}\mathbf{I}_n\right)^{-1}$.

Before going into the detailed proof, as we will frequently deal with random variables evolving as n, p grow large, we shall use the extension of the $O(\cdot)$ notation introduced in [20]: for a random variable $x \equiv x_n$ and $u_n \geq 0$, we write $x = O(u_n)$ if for any $\eta > 0$ and $D > 0$, we have $n^D \mathrm{P}(x \geq n^\eta u_n) \to 0$. Note that under Assumption 1 it is equivalent to use either $O(u_n)$ or $O(u_p)$ since n, p scales linearly. In the following we shall use constantly $O(u_n)$ for simplicity.

When multidimensional objects are concerned, $\mathbf{v} = O(u_n)$ means the maximum entry of a vector (or a diagonal matrix) $\mathbf{v}$ in absolute value is of order $O(u_n)$ and $\mathbf{M} = O(u_n)$ means that the operator norm of $\mathbf{M}$ is of order $O(u_n)$. We refer the reader to [20] for more discussions on these practical definitions.

Under the growth rate settings of Assumption 1, from [20], the approximation of the kernel matrix $\mathbf{K}$ is given by

$$\mathbf{K} = -2f'(\tau)\left(\mathbf{P}\boldsymbol{\Omega}^\mathsf{T}\boldsymbol{\Omega}\mathbf{P} + \mathbf{A}\right) + \beta\mathbf{I}_n + O(n^{-\frac{1}{2}}) \tag{12}$$

with $\beta = f(0) - f(\tau) + \tau f'(\tau)$ and $\mathbf{A} = \mathbf{A}_n + \mathbf{A}_{\sqrt{n}} + \mathbf{A}_1$, $\mathbf{A}_n = -\frac{f(\tau)}{2f'(\tau)}\mathbf{1}_n\mathbf{1}_n^\mathsf{T}$ and $\mathbf{A}_{\sqrt{n}}$, $\mathbf{A}_1$ given by (18) and (19) at the top of next page, where we denote

$$t_a \triangleq \frac{\operatorname{tr}(\mathbf{C}_a - \mathbf{C}^\circ)}{\sqrt{p}} = O(1)$$
$$(\boldsymbol{\psi})^2 \triangleq [(\boldsymbol{\psi}_1)^2, \ldots, (\boldsymbol{\psi}_n)^2]^\mathsf{T}.$$

We start with the term $\mathbf{S}^{-1}$. The terms of leading order in $\mathbf{K}$, i.e.,$-2f'(\tau)\mathbf{A}_n$ and $\frac{n}{\gamma}\mathbf{I}_n$ are both of operator norm $O(n)$. Therefore a Taylor expansion can be performed as

$$\begin{aligned}\mathbf{S}^{-1} &= \left(\mathbf{K} + \frac{n}{\gamma}\mathbf{I}_n\right)^{-1} = \frac{1}{n}\Bigg[\mathbf{L}^{-1} - \frac{2f'(\tau)}{n} \\ &\left(\mathbf{A}_{\sqrt{n}} + \mathbf{A}_1 + \mathbf{P}\boldsymbol{\Omega}^\mathsf{T}\boldsymbol{\Omega}\mathbf{P}\right) + \frac{\beta\mathbf{I}_n}{n} + O(n^{-\frac{3}{2}})\Bigg]^{-1} \\ &= \frac{\mathbf{L}}{n} + \frac{2f'(\tau)}{n^2}\mathbf{L}\mathbf{A}_{\sqrt{n}}\mathbf{L} + \mathbf{L}\left(\mathbf{Q} - \frac{\beta}{n^2}\mathbf{I}_n\right)\mathbf{L} + O(n^{-\frac{5}{2}})\end{aligned}$$

with $\mathbf{L} = \left(f(\tau)\frac{\mathbf{1}_n\mathbf{1}_n^\mathsf{T}}{n} + \frac{\mathbf{I}_n}{\gamma}\right)^{-1}$ of order $O(1)$ and $\mathbf{Q} = \frac{2f'(\tau)}{n^2}\left(\mathbf{A}_1 + \mathbf{P}\boldsymbol{\Omega}^\mathsf{T}\boldsymbol{\Omega}\mathbf{P} + \frac{2f'(\tau)}{n}\mathbf{A}_{\sqrt{n}}\mathbf{L}\mathbf{A}_{\sqrt{n}}\right)$.

With the Sherman-Morrison formula we are able to compute explicitly $\mathbf{L}$ as

$$\begin{aligned}\mathbf{L} &= \left(f(\tau)\frac{\mathbf{1}_n\mathbf{1}_n^\mathsf{T}}{n} + \frac{\mathbf{I}_n}{\gamma}\right)^{-1} = \gamma\left(\mathbf{I}_n - \frac{\gamma f(\tau)}{1+\gamma f(\tau)}\frac{\mathbf{1}_n\mathbf{1}_n^\mathsf{T}}{n}\right) \\ &= \frac{\gamma}{1+\gamma f(\tau)}\mathbf{I}_n + \frac{\gamma^2 f(\tau)}{1+\gamma f(\tau)}\mathbf{P} = O(1).\end{aligned} \tag{13}$$

Writing $\mathbf{L}$ as a linear combination of $\mathbf{I}_n$ and $\mathbf{P}$ is useful when computing $\mathbf{L}\mathbf{1}_n$ or $\mathbf{1}_n^\mathsf{T}\mathbf{L}$, because by the definition of $\mathbf{P} = \mathbf{I}_n - \frac{\mathbf{1}_n\mathbf{1}_n^\mathsf{T}}{n}$, we have $\mathbf{1}_n^\mathsf{T}\mathbf{P} = \mathbf{P}\mathbf{1}_n = \mathbf{0}$.

We shall start with the term $\mathbf{1}_n^\mathsf{T}\mathbf{S}^{-1}$, since it is the basis of several other terms appearing in $\boldsymbol{\alpha}$ and b,

$$\begin{aligned}\mathbf{1}_n^\mathsf{T}\mathbf{S}^{-1} &= \frac{\gamma\mathbf{1}_n^\mathsf{T}}{1+\gamma f(\tau)}\left[\frac{\mathbf{I}_n}{n} + \frac{2f'(\tau)}{n^2}\mathbf{A}_{\sqrt{n}}\mathbf{L} + \left(\mathbf{Q} - \frac{\beta}{n^2}\mathbf{I}_n\right)\mathbf{L}\right] \\ &+ O(n^{-\frac{3}{2}})\end{aligned}$$

since $\mathbf{1}_n^\mathsf{T}\mathbf{L} = \frac{\gamma}{1+\gamma f(\tau)}\mathbf{1}_n^\mathsf{T}$.

With $\mathbf{1}_n^\mathsf{T}\mathbf{S}^{-1}$ at hand, we next obtain,

$$\begin{aligned}\mathbf{1}_n\mathbf{1}_n^\mathsf{T}\mathbf{S}^{-1} &= \frac{\gamma}{1+\gamma f(\tau)}\Bigg[\underbrace{\frac{\mathbf{1}_n\mathbf{1}_n^\mathsf{T}}{n}}_{O(1)} + \underbrace{\frac{2f'(\tau)}{n^2}\mathbf{1}_n\mathbf{1}_n^\mathsf{T}\mathbf{A}_{\sqrt{n}}\mathbf{L}}_{O(n^{-1/2})} \\ &+ \underbrace{\mathbf{1}_n\mathbf{1}_n^\mathsf{T}\left(\mathbf{Q} - \frac{\beta}{n^2}\mathbf{I}_n\right)\mathbf{L}}_{O(n^{-1})}\Bigg] + O(n^{-\frac{3}{2}})\end{aligned} \tag{14}$$

$$\begin{aligned}\mathbf{1}_n^\mathsf{T}\mathbf{S}^{-1}\mathbf{y} &= \frac{\gamma}{1+\gamma f(\tau)}\Bigg[\underbrace{c_2 - c_1}_{O(1)} + \underbrace{\frac{2f'(\tau)}{n^2}\mathbf{1}_n^\mathsf{T}\mathbf{A}_{\sqrt{n}}\mathbf{L}\mathbf{y}}_{O(n^{-1/2})} \\ &+ \underbrace{\mathbf{1}_n^\mathsf{T}\left(\mathbf{Q} - \frac{\beta}{n^2}\mathbf{I}_n\right)\mathbf{L}\mathbf{y}}_{O(n^{-1})}\Bigg] + O(n^{-\frac{3}{2}})\end{aligned}$$

$$\begin{aligned}\mathbf{1}_n^\mathsf{T}\mathbf{S}^{-1}\mathbf{1}_n &= \frac{\gamma}{1+\gamma f(\tau)}\Bigg[\underbrace{1}_{O(1)} + \underbrace{\frac{2f'(\tau)}{n^2}\frac{\gamma\mathbf{1}_n^\mathsf{T}\mathbf{A}_{\sqrt{n}}\mathbf{1}_n}{1+\gamma f(\tau)}}_{O(n^{-1/2})} \\ &+ \underbrace{\frac{\gamma}{1+\gamma f(\tau)}\mathbf{1}_n^\mathsf{T}\left(\mathbf{Q} - \frac{\beta}{n^2}\mathbf{I}_n\right)\mathbf{1}_n}_{O(n^{-1})}\Bigg] + O(n^{-\frac{3}{2}}).\end{aligned}$$

The inverse of $\mathbf{1}_n^\mathsf{T}\mathbf{S}^{-1}\mathbf{1}_n$ can consequently be computed using a Taylor expansion around its leading order, allowing an error term of $O(n^{-\frac{3}{2}})$ as

$$\begin{aligned}\frac{1}{\mathbf{1}_n^\mathsf{T}\mathbf{S}^{-1}\mathbf{1}_n} &= \frac{1+\gamma f(\tau)}{\gamma}\Bigg[\underbrace{1}_{O(1)} - \underbrace{\frac{2f'(\tau)}{n^2}\frac{\gamma\mathbf{1}_n^\mathsf{T}\mathbf{A}_{\sqrt{n}}\mathbf{1}_n}{1+\gamma f(\tau)}}_{O(n^{-1/2})} \\ &- \underbrace{\frac{\gamma}{1+\gamma f(\tau)}\mathbf{1}_n^\mathsf{T}\left(\mathbf{Q} - \frac{\beta}{n^2}\mathbf{I}_n\right)\mathbf{1}_n}_{O(n^{-1})}\Bigg] + O(n^{-\frac{3}{2}}).\end{aligned} \tag{15}$$

$$\mathbf{A}_{\sqrt{n}} = -\frac{1}{2}\left[\boldsymbol{\psi}\mathbf{1}_n^\mathsf{T} + \mathbf{1}_n\boldsymbol{\psi}^\mathsf{T} + \left\{t_a\frac{\mathbf{1}_{n_a}}{\sqrt{p}}\right\}_{a=1}^2 \mathbf{1}_n^\mathsf{T} + \mathbf{1}_n\left\{t_b\frac{\mathbf{1}_{n_b}^\mathsf{T}}{\sqrt{p}}\right\}_{b=1}^2\right] \tag{18}$$

$$\begin{aligned}\mathbf{A}_1 = &-\frac{1}{2}\left[\left\{\|\boldsymbol{\mu}_a-\boldsymbol{\mu}_b\|^2\frac{\mathbf{1}_{n_a}\mathbf{1}_{n_b}^\mathsf{T}}{p}\right\}_{a,b=1}^2 + 2\left\{\frac{(\boldsymbol{\Omega}\mathbf{P})_a^\mathsf{T}(\boldsymbol{\mu}_b-\boldsymbol{\mu}_a)\mathbf{1}_{n_b}^\mathsf{T}}{\sqrt{p}}\right\}_{a,b=1}^2 - 2\left\{\frac{\mathbf{1}_{n_a}(\boldsymbol{\mu}_b-\boldsymbol{\mu}_a)^\mathsf{T}(\boldsymbol{\Omega}\mathbf{P})_b}{\sqrt{p}}\right\}_{a,b=1}^2\right] \\ &-\frac{f''(\tau)}{4f'(\tau)}\left[(\boldsymbol{\psi})^2\mathbf{1}_n^\mathsf{T} + \mathbf{1}_n[(\boldsymbol{\psi})^2]^\mathsf{T} + \left\{t_a^2\frac{\mathbf{1}_{n_a}}{p}\right\}_{a=1}^2\mathbf{1}_n^\mathsf{T} + \mathbf{1}_n\left\{t_b^2\frac{\mathbf{1}_{n_b}^\mathsf{T}}{p}\right\}_{b=1}^2 + 2\left\{t_at_b\frac{\mathbf{1}_{n_a}\mathbf{1}_{n_b}^\mathsf{T}}{p}\right\}_{a,b=1}^2 + 2\mathcal{D}\{t_a\mathbf{I}_{n_a}\}_{a=1}^2\boldsymbol{\psi}\frac{\mathbf{1}_n^\mathsf{T}}{\sqrt{p}}\right. \\ &\left.+2\boldsymbol{\psi}\left\{t_b\frac{\mathbf{1}_{n_b}^\mathsf{T}}{\sqrt{p}}\right\}_{b=1}^2 + 2\frac{\mathbf{1}_n}{\sqrt{p}}(\boldsymbol{\psi})^\mathsf{T}\mathcal{D}\{t_a\mathbf{1}_{n_a}\}_{a=1}^2 + 2\left\{t_a\frac{\mathbf{1}_{n_a}}{\sqrt{p}}\right\}_{a=1}^2(\boldsymbol{\psi})^\mathsf{T} + 4\left\{\operatorname{tr}(\mathbf{C}_a\mathbf{C}_b)\frac{\mathbf{1}_{n_a}\mathbf{1}_{n_b}^\mathsf{T}}{p^2}\right\}_{a,b=1}^2 + 2\boldsymbol{\psi}(\boldsymbol{\psi})^\mathsf{T}\right]\end{aligned} \tag{19}$$

$$\begin{aligned}\tilde{\mathbf{k}}(\mathbf{x}) = &f'(\tau)\left[\left\{\frac{\|\boldsymbol{\mu}_b-\boldsymbol{\mu}_a\|^2}{p}\mathbf{1}_{n_b}\right\}_{b=1}^2 - \frac{2}{\sqrt{p}}\left\{\mathbf{1}_{n_b}(\boldsymbol{\mu}_b-\boldsymbol{\mu}_a)^\mathsf{T}\right\}_{b=1}^2\boldsymbol{\omega}_\mathbf{x} + \frac{2}{\sqrt{p}}\mathcal{D}\left(\left\{\mathbf{1}_{n_b}(\boldsymbol{\mu}_b-\boldsymbol{\mu}_a)^\mathsf{T}\right\}_{b=1}^2\boldsymbol{\Omega}\right)\right] \\ &+\frac{f''(\tau)}{2}\left[\left\{\frac{(t_a+t_b)^2}{p}\mathbf{1}_{n_b}\right\}_{b=1}^2 + 2\mathcal{D}\left(\left\{\frac{t_a+t_b}{\sqrt{p}}\mathbf{1}_{n_b}\right\}_{b=1}^2\right)\boldsymbol{\psi} + 2\left\{\frac{t_a+t_b}{\sqrt{p}}\mathbf{1}_{n_b}\right\}_{b=1}^2\psi_\mathbf{x} + (\boldsymbol{\psi})^2 + 2\psi_\mathbf{x}\boldsymbol{\psi} + \psi_\mathbf{x}^2\mathbf{1}_n\right. \\ &\left.+\left\{\frac{4}{p^2}\operatorname{tr}(\mathbf{C}_a\mathbf{C}_b)\mathbf{1}_{n_b}\right\}_{b=1}^2\right]\end{aligned} \tag{20}$$

Combing (14) with (15) we deduce

$$\begin{aligned}\frac{\mathbf{1}_n\mathbf{1}_n^\mathsf{T}\mathbf{S}^{-1}}{\mathbf{1}_n^\mathsf{T}\mathbf{S}^{-1}\mathbf{1}_n} = &\underbrace{\frac{\mathbf{1}_n\mathbf{1}_n^\mathsf{T}}{n}}_{O(1)} + \underbrace{\frac{2f'(\tau)}{n^2}\mathbf{1}_n\mathbf{1}_n^\mathsf{T}\mathbf{A}_{\sqrt{n}}\left[\mathbf{L} - \frac{\gamma\frac{\mathbf{1}_n\mathbf{1}_n^\mathsf{T}}{n}}{1+\gamma f(\tau)}\right]}_{O(n^{-1/2})} \\ &+\underbrace{\mathbf{1}_n\mathbf{1}_n^\mathsf{T}\left(\mathbf{Q} - \frac{\beta}{n^2}\mathbf{I}_n\right)\left[\mathbf{L} - \frac{\gamma\frac{\mathbf{1}_n\mathbf{1}_n^\mathsf{T}}{n}}{1+\gamma f(\tau)}\right]}_{O(n^{-1})} + O(n^{-\frac{3}{2}})\end{aligned} \tag{16}$$

and similarly the following approximation of b as

$$\begin{aligned}b = &\underbrace{c_2 - c_1}_{O(1)} - \underbrace{\frac{2\gamma}{\sqrt{p}}c_1c_2f'(\tau)(t_2-t_1)}_{O(n^{-1/2})} - \underbrace{\frac{\gamma f'(\tau)}{n}\mathbf{y}^\mathsf{T}\mathbf{P}\boldsymbol{\psi}}_{O(n^{-1})} \\ &\underbrace{-\frac{\gamma f''(\tau)}{2n}\mathbf{y}^\mathsf{T}\mathbf{P}(\boldsymbol{\psi})^2 + \frac{4\gamma c_1c_2}{p}[c_1T_1 + (c_2-c_1)D - c_2T_2]}_{O(n^{-1})} \\ &+O(n^{-\frac{3}{2}})\end{aligned} \tag{17}$$

where

$$D = \frac{f'(\tau)}{2}\|\boldsymbol{\mu}_2-\boldsymbol{\mu}_1\|^2 + \frac{f''(\tau)}{4}(t_1+t_2)^2 + f''(\tau)\frac{\operatorname{tr}\mathbf{C}_1\mathbf{C}_2}{p}$$

$$T_a = f''(\tau)t_a^2 + f''(\tau)\frac{\operatorname{tr}\mathbf{C}_1\mathbf{C}_2}{p}$$

which gives the asymptotic approximation of b.

Moving to $\boldsymbol{\alpha}$, note from (13) that $\mathbf{L} - \frac{\gamma}{1+\gamma f(\tau)}\frac{\mathbf{1}_n\mathbf{1}_n^\mathsf{T}}{n} = \gamma\mathbf{P}$, and we can thus rewrite:

$$\begin{aligned}\frac{\mathbf{1}_n\mathbf{1}_n^\mathsf{T}\mathbf{S}^{-1}}{\mathbf{1}_n^\mathsf{T}\mathbf{S}^{-1}\mathbf{1}_n} = &\frac{\mathbf{1}_n\mathbf{1}_n^\mathsf{T}}{n} + \frac{2\gamma f'(\tau)}{n^2}\mathbf{1}_n\mathbf{1}_n^\mathsf{T}\mathbf{A}_{\sqrt{n}}\mathbf{P} \\ &+\gamma\mathbf{1}_n\mathbf{1}_n^\mathsf{T}\left(\mathbf{Q} - \frac{\beta}{n^2}\mathbf{I}_n\right)\mathbf{P} + O(n^{-\frac{3}{2}}).\end{aligned}$$

At this point, for $\boldsymbol{\alpha} = \mathbf{S}^{-1}\left(\mathbf{I}_n - \frac{\mathbf{1}_n\mathbf{1}_n^\mathsf{T}\mathbf{S}^{-1}}{\mathbf{1}_n^\mathsf{T}\mathbf{S}^{-1}\mathbf{1}_n}\right)\mathbf{y}$, we have

$$\begin{aligned}\boldsymbol{\alpha} = \mathbf{S}^{-1}\Big[&\mathbf{I}_n - \frac{2\gamma f'(\tau)}{n^2}\mathbf{1}_n\mathbf{1}_n^\mathsf{T}\mathbf{A}_{\sqrt{n}} \\ &-\gamma\mathbf{1}_n\mathbf{1}_n^\mathsf{T}\left(\mathbf{Q} - \frac{\beta}{n^2}\mathbf{I}_n\right)\Big]\mathbf{P}\mathbf{y} + O(n^{-\frac{5}{2}}).\end{aligned}$$

Here again, we use $\mathbf{1}_n^\mathsf{T}\mathbf{L} = \frac{\gamma}{1+\gamma f(\tau)}\mathbf{1}_n^\mathsf{T}$ and $\mathbf{L} - \frac{\gamma}{1+\gamma f(\tau)}\frac{\mathbf{1}_n\mathbf{1}_n^\mathsf{T}}{n} = \gamma\mathbf{P}$, to eventually get

$$\begin{aligned}\boldsymbol{\alpha} = &\underbrace{\frac{\gamma}{n}\mathbf{P}\mathbf{y}}_{O(n^{-1})} + \underbrace{\gamma^2\mathbf{P}\left(\mathbf{Q} - \frac{\beta}{n^2}\mathbf{I}_n\right)\mathbf{P}\mathbf{y}}_{O(n^{-2})} \\ &\underbrace{-\frac{\gamma^2}{1+\gamma f(\tau)}\left(\frac{2f'(\tau)}{n^2}\right)^2\mathbf{L}\mathbf{A}_{\sqrt{n}}\mathbf{1}_n\mathbf{1}_n^\mathsf{T}\mathbf{A}_{\sqrt{n}}\mathbf{P}\mathbf{y}}_{O(n^{-2})} + O(n^{-\frac{5}{2}}).\end{aligned} \tag{21}$$

Note here the absence of a term of order $O(n^{-3/2})$ in the expression of $\boldsymbol{\alpha}$ since $\mathbf{P}\mathbf{A}_{\sqrt{n}}\mathbf{P} = 0$ from (18).

We shall now work on the vector $\mathbf{k}(\mathbf{x})$ for a new datum $\mathbf{x}$, following the same analysis as in [20] for the kernel matrix $\mathbf{K}$, assuming that $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}_a, \mathbf{C}_a)$ and recalling the random variables definitions,

$$\begin{aligned}\boldsymbol{\omega}_\mathbf{x} &\triangleq (\mathbf{x} - \boldsymbol{\mu}_a)/\sqrt{p} \\ \psi_\mathbf{x} &\triangleq \|\boldsymbol{\omega}_\mathbf{x}\|^2 - \mathbb{E}\|\boldsymbol{\omega}_\mathbf{x}\|^2\end{aligned}$$

we show that the j-th entry of $\mathbf{k}(\mathbf{x})$ can be written as

$$[\mathbf{k}(\mathbf{x})]_j = \underbrace{f(\tau)}_{O(1)} + f'(\tau)\Bigg[\underbrace{\frac{t_a+t_b}{\sqrt{p}} + \psi_x + \psi_j - 2(\boldsymbol{\omega}_{\mathbf{x}})^\mathsf{T}\boldsymbol{\omega}_j}_{O(n^{-1/2})} + \underbrace{\frac{\|\boldsymbol{\mu}_b-\boldsymbol{\mu}_a\|^2}{p} + \frac{2}{\sqrt{p}}(\boldsymbol{\mu}_b-\boldsymbol{\mu}_a)^\mathsf{T}(\boldsymbol{\omega}_j-\boldsymbol{\omega}_{\mathbf{x}})}_{O(n^{-1})}\Bigg] + \frac{f''(\tau)}{2}\Bigg[\underbrace{\left(\frac{t_a+t_b}{\sqrt{p}} + \psi_j + \psi_{\mathbf{x}}\right)^2 + \frac{4}{p^2}\operatorname{tr}\mathbf{C}_a\mathbf{C}_b}_{O(n^{-1})}\Bigg] + O(n^{-\frac{3}{2}}). \tag{22}$$

Combining (21) and (22), we deduce

$$\boldsymbol{\alpha}^\mathsf{T}\mathbf{k}(\mathbf{x}) = \underbrace{\frac{2\gamma}{\sqrt{p}}c_1c_2f'(\tau)(t_2-t_1)}_{O(n^{-1/2})} + \underbrace{\frac{\gamma}{n}\mathbf{y}^\mathsf{T}\mathbf{P}\tilde{\mathbf{k}}(\mathbf{x})}_{O(n^{-1})} + \underbrace{\frac{\gamma f'(\tau)}{n}\mathbf{y}^\mathsf{T}\mathbf{P}(\boldsymbol{\psi} - 2\mathbf{P}\boldsymbol{\Omega}^\mathsf{T}\boldsymbol{\omega}_{\mathbf{x}})}_{O(n^{-1})} + O(n^{-\frac{3}{2}}) \tag{23}$$

with $\tilde{\mathbf{k}}(\mathbf{x})$ given in (20).

At this point, note that the term of order $O(n^{-\frac{1}{2}})$ in the final object $g(\mathbf{x}) = \boldsymbol{\alpha}^\mathsf{T}\mathbf{k}(\mathbf{x}) + b$ disappears because in both (17) and (23) the term of order $O(n^{-1/2})$ is $\frac{2\gamma}{\sqrt{p}}c_1c_2f'(\tau)(t_2-t_1)$ but of opposite signs. Also, we see that the leading term $c_2 - c_1$ in b will remain in $g(\mathbf{x})$ as stated in Remark 2.

The development of $\mathbf{y}^\mathsf{T}\mathbf{P}\tilde{\mathbf{k}}(\mathbf{x})$ induces many simplifications, since i) $\mathbf{P}\mathbf{1}_n = \mathbf{0}$ and ii) random variables as $\boldsymbol{\omega}_{\mathbf{x}}$ and $\boldsymbol{\psi}$ in $\tilde{\mathbf{k}}(\mathbf{x})$, once multiplied by $\mathbf{y}^\mathsf{T}\mathbf{P}$, thanks to probabilistic averaging of independent zero-mean terms, are of smaller order and thus become negligible. We thus get

$$\frac{\gamma}{n}\mathbf{y}^\mathsf{T}\mathbf{P}\tilde{\mathbf{k}}(\mathbf{x}) = 2\gamma c_1c_2f'(\tau)\left[\frac{\|\boldsymbol{\mu}_2-\boldsymbol{\mu}_a\|^2 - \|\boldsymbol{\mu}_1-\boldsymbol{\mu}_a\|^2}{p} - 2(\boldsymbol{\omega}_{\mathbf{x}})^\mathsf{T}\frac{\boldsymbol{\mu}_2-\boldsymbol{\mu}_1}{\sqrt{p}}\right] + \frac{\gamma f''(\tau)}{2n}\mathbf{y}^\mathsf{T}\mathbf{P}(\boldsymbol{\psi})^2 + \gamma c_1c_2f''(\tau)\left[2\left(\frac{t_a}{\sqrt{p}} + \psi_{\mathbf{x}}\right)\frac{t_2-t_1}{\sqrt{p}} + \frac{t_2^2-t_1^2}{p} + \frac{4}{p^2}\operatorname{tr}(\mathbf{C}_a\mathbf{C}_2 - \mathbf{C}_a\mathbf{C}_1)\right] + O(n^{-\frac{3}{2}}). \tag{24}$$

This result, together with (23), completes the analysis of the term $\boldsymbol{\alpha}^\mathsf{T}\mathbf{k}(\mathbf{x})$. Combining (23)-(24) with (17) we conclude the proof of Theorem 1.

Appendix B
Proof of Theorem 2

This section is dedicated to the proof of the central limit theorem for

$$\hat{g}(\mathbf{x}) = c_2 - c_1 + \gamma\left(\mathfrak{P} + c_{\mathbf{x}}\mathfrak{D}\right)$$

with the shortcut $c_{\mathbf{x}} = -2c_1c_2^2$ for $\mathbf{x} \in \mathcal{C}_1$ and $c_{\mathbf{x}} = 2c_1^2c_2$ for $\mathbf{x} \in \mathcal{C}_2$, and $\mathfrak{P}, \mathfrak{D}$ as defined in (7) and (8).

Our objective is to show that for $a \in \{1,2\}$, $n(\hat{g}(\mathbf{x}) - G_a) \xrightarrow{d} 0$ with

$$G_a \sim \mathcal{N}(\mathrm{E}_a, \mathrm{Var}_a)$$

where E_a and Var_a are given in Theorem 2. We recall that $\mathbf{x} = \boldsymbol{\mu}_a + \sqrt{p}\boldsymbol{\omega}_{\mathbf{x}}$ with $\boldsymbol{\omega}_{\mathbf{x}} \sim \mathcal{N}(0, \mathbf{C}_a/p)$.

Letting $\mathbf{z}_{\mathbf{x}}$ such that $\boldsymbol{\omega}_{\mathbf{x}} = \mathbf{C}_a^{1/2}\mathbf{z}_{\mathbf{x}}/\sqrt{p}$, we have $\mathbf{z}_{\mathbf{x}} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_n)$ and we can rewrite $\hat{g}(\mathbf{x})$ in the following quadratic form (of $\mathbf{z}_{\mathbf{x}}$) as

$$\hat{g}(\mathbf{x}) = \mathbf{z}_{\mathbf{x}}^\mathsf{T}\mathbf{A}\mathbf{z}_{\mathbf{x}} + \mathbf{z}_{\mathbf{x}}^\mathsf{T}\mathbf{b} + c$$

with

$$\mathbf{A} = 2\gamma c_1c_2f''(\tau)\frac{\operatorname{tr}(\mathbf{C}_2-\mathbf{C}_1)}{p}\frac{\mathbf{C}_a}{p}$$

$$\mathbf{b} = -\frac{2\gamma f'(\tau)}{n}\frac{(\mathbf{C}_a)^{\frac{1}{2}}}{\sqrt{p}}\boldsymbol{\Omega}\mathbf{P}\mathbf{y} - \frac{4c_1c_2\gamma f'(\tau)}{\sqrt{p}}\frac{(\mathbf{C}_a)^{\frac{1}{2}}}{\sqrt{p}}(\boldsymbol{\mu}_2-\boldsymbol{\mu}_1)$$

$$c = c_2 - c_1 + \gamma c_{\mathbf{x}}\mathfrak{D} - 2\gamma c_1c_2f''(\tau)\frac{\operatorname{tr}(\mathbf{C}_2-\mathbf{C}_1)}{p}\frac{\operatorname{tr}\mathbf{C}_a}{p}.$$

Since $\mathbf{z}_{\mathbf{x}}$ is (standard) Gaussian and has the same distribution as $\mathbf{U}\mathbf{z}_{\mathbf{x}}$ for any orthogonal matrix $\mathbf{U}$ (i.e., such that $\mathbf{U}^\mathsf{T}\mathbf{U} = \mathbf{U}\mathbf{U}^\mathsf{T} = \mathbf{I}_n$), we choose $\mathbf{U}$ that diagonalize $\mathbf{A}$ such that $\mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\mathsf{T}$, with $\boldsymbol{\Lambda}$ diagonal so that $\hat{g}(\mathbf{x})$ and $\tilde{g}(\mathbf{x})$ have the same distribution where

$$\tilde{g}(\mathbf{x}) = \mathbf{z}_{\mathbf{x}}^\mathsf{T}\boldsymbol{\Lambda}\mathbf{z}_{\mathbf{x}} + \mathbf{z}_{\mathbf{x}}^\mathsf{T}\tilde{\mathbf{b}} + c = \sum_{i=1}^n\left(z_i^2\lambda_i + z_i\tilde{b}_i + \frac{c}{n}\right)$$

and $\tilde{\mathbf{b}} = \mathbf{U}^\mathsf{T}\mathbf{b}$, λ_i the diagonal elements of $\boldsymbol{\Lambda}$ and z_i the elements of $\mathbf{z}_{\mathbf{x}}$.

Conditioning on $\boldsymbol{\Omega}$, we thus result in the sum of independent but not identically distributed random variables $r_i = z_i^2\lambda_i + z_i\tilde{b}_i + \frac{c}{n}$. We then resort to the Lyapunov CLT [33, Theorem 27.3].

We begin by estimating the expectation and the variance

$$\mathbb{E}[r_i|\boldsymbol{\Omega}] = \lambda_i + \frac{c}{n}$$
$$\mathrm{Var}[r_i|\boldsymbol{\Omega}] = \sigma_i^2 = 2\lambda_i^2 + \tilde{b}_i^2$$

of r_i, so that

$$\sum_{i=1}^n\mathbb{E}[r_i|\boldsymbol{\Omega}] = c_2 - c_1 + \gamma c_{\mathbf{x}}\mathfrak{D} = \mathrm{E}_a$$

$$\begin{aligned} s_n^2 &= \sum_{i=1}^n\sigma_i^2 = 2\operatorname{tr}(\mathbf{A}^2) + \mathbf{b}^\mathsf{T}\mathbf{b} \\ &= 8\gamma^2c_1^2c_2^2\left(f''(\tau)\right)^2\frac{\left(\operatorname{tr}\left(\mathbf{C}_2-\mathbf{C}_1\right)\right)^2}{p^2}\frac{\operatorname{tr}\mathbf{C}_a^2}{p^2} \\ &\quad + 4\gamma^2\left(\frac{f'(\tau)}{n}\right)^2\mathbf{y}^\mathsf{T}\mathbf{P}\boldsymbol{\Omega}^\mathsf{T}\frac{\mathbf{C}_a}{p}\boldsymbol{\Omega}\mathbf{P}\mathbf{y} \\ &\quad + \frac{16\gamma^2c_1^2c_2^2(f'(\tau))^2}{p}(\boldsymbol{\mu}_2-\boldsymbol{\mu}_1)^\mathsf{T}\frac{\mathbf{C}_a}{p}(\boldsymbol{\mu}_2-\boldsymbol{\mu}_1) \\ &\quad + O(n^{-\frac{5}{2}}). \end{aligned}$$

We shall rewrite $\boldsymbol{\Omega}$ into two blocks as:

$$\boldsymbol{\Omega} = \left[\frac{(\mathbf{C}_1)^{\frac{1}{2}}}{\sqrt{p}}\mathbf{Z}_1, \quad \frac{(\mathbf{C}_2)^{\frac{1}{2}}}{\sqrt{p}}\mathbf{Z}_2\right]$$

where $\mathbf{Z}_1 \in \mathbb{R}^{p\times n_1}$ and $\mathbf{Z}_2 \in \mathbb{R}^{p\times n_2}$ with i.i.d. Gaussian entries with zero mean and unit variance. Then

$$\boldsymbol{\Omega}^\mathsf{T}\frac{\mathbf{C}_a}{p}\boldsymbol{\Omega} = \frac{1}{p^2}\begin{bmatrix}\mathbf{Z}_1^\mathsf{T}(\mathbf{C}_1)^{\frac{1}{2}}\mathbf{C}_a(\mathbf{C}_1)^{\frac{1}{2}}\mathbf{Z}_1 & \mathbf{Z}_1^\mathsf{T}(\mathbf{C}_1)^{\frac{1}{2}}\mathbf{C}_a(\mathbf{C}_2)^{\frac{1}{2}}\mathbf{Z}_2 \\ \mathbf{Z}_2^\mathsf{T}(\mathbf{C}_2)^{\frac{1}{2}}\mathbf{C}_a(\mathbf{C}_1)^{\frac{1}{2}}\mathbf{Z}_1 & \mathbf{Z}_2^\mathsf{T}(\mathbf{C}_2)^{\frac{1}{2}}\mathbf{C}_a(\mathbf{C}_2)^{\frac{1}{2}}\mathbf{Z}_2\end{bmatrix}$$

and with $\mathbf{P}\mathbf{y} = \mathbf{y} - (c_2 - c_1)\mathbf{1}_n$, we deduce

$$\mathbf{y}^\mathsf{T}\mathbf{P}\mathbf{\Omega}^\mathsf{T}\frac{\mathbf{C}_a}{p}\mathbf{\Omega}\mathbf{P}\mathbf{y} = \frac{4}{p^2}\left(c_2^2\mathbf{1}_{n_1}^\mathsf{T}\mathbf{Z}_1^\mathsf{T}(\mathbf{C}_1)^{\frac{1}{2}}\mathbf{C}_a(\mathbf{C}_1)^{\frac{1}{2}}bZ_1\mathbf{1}_{n_1}\right.$$
$$-2c_1c_2\mathbf{1}_{n_1}^\mathsf{T}\mathbf{Z}_1^\mathsf{T}(\mathbf{C}_1)^{\frac{1}{2}}\mathbf{C}_a(\mathbf{C}_2)^{\frac{1}{2}}\mathbf{Z}_2\mathbf{1}_{n_2}$$
$$\left.+c_2^2\mathbf{1}_{n_1}^\mathsf{T}\mathbf{Z}_2^\mathsf{T}(\mathbf{C}_2)^{\frac{1}{2}}\mathbf{C}_a(\mathbf{C}_2)^{\frac{1}{2}}\mathbf{Z}_2\mathbf{1}_{n_2}\right).$$

Since $\mathbf{Z}_i\mathbf{1}_{n_i} \sim \mathcal{N}(\mathbf{0}, n_i\mathbf{I}_{n_i})$, by applying the trace lemma [39, Lemma B.26] we get

$$\mathbf{y}^\mathsf{T}\mathbf{P}\mathbf{\Omega}^\mathsf{T}\frac{\mathbf{C}_a}{p}\mathbf{\Omega}\mathbf{P}\mathbf{y} - \frac{4nc_1^2c_2^2}{p^2}\left(\frac{\operatorname{tr}\mathbf{C}_1\mathbf{C}_a}{c_1} + \frac{\operatorname{tr}\mathbf{C}_2\mathbf{C}_a}{c_2}\right) \xrightarrow{\text{a.s.}} 0. \tag{25}$$

Consider now the events

$$E = \left\{\left|\mathbf{y}^\mathsf{T}\mathbf{P}\mathbf{\Omega}^\mathsf{T}\frac{\mathbf{C}_a}{p}\mathbf{\Omega}\mathbf{P}\mathbf{y} - \rho\right| < \epsilon\right\}$$
$$\bar{E} = \left\{\left|\mathbf{y}^\mathsf{T}\mathbf{P}\mathbf{\Omega}^\mathsf{T}\frac{\mathbf{C}_a}{p}\mathbf{\Omega}\mathbf{P}\mathbf{y} - \rho\right| > \epsilon\right\}$$

for any fixed ϵ with $\rho = \frac{4nc_1^2c_2^2}{p^2}\left(\frac{\operatorname{tr}\mathbf{C}_1\mathbf{C}_a}{c_1} + \frac{\operatorname{tr}\mathbf{C}_2\mathbf{C}_a}{c_2}\right)$ and write

$$\mathbb{E}\left[\exp\left(iun\frac{\tilde{g}(\mathbf{x}) - \mathrm{E}_a}{s_n}\right)\right] = \mathbb{E}\left[\exp\left(iun\frac{\tilde{g}(\mathbf{x}) - \mathrm{E}_a}{s_n}\right)\middle|E\right]$$
$$\mathrm{P}(E) + \mathbb{E}\left[\exp\left(iun\frac{\tilde{g}(\mathbf{x}) - \mathrm{E}_a}{s_n}\right)\middle|\bar{E}\right]\mathrm{P}(\bar{E}) \tag{26}$$

We start with the variable $\tilde{g}(\mathbf{x})|E$ and check that Lyapunov's condition for $\bar{r}_i = r_i - \mathbb{E}[r_i]$, conditioning on E,

$$\lim_{n\to\infty}\frac{1}{s_n^4}\sum_{i=1}^n\mathbb{E}[|\bar{r}_i|^4] = 0$$

holds by rewriting

$$\lim_{n\to\infty}\frac{1}{s_n^4}\sum_{i=1}^n\mathbb{E}[|\bar{r}_i|^4] = \lim_{n\to\infty}\sum_{i=1}^n\frac{60\lambda_i^4 + 12\lambda_i^2\tilde{b_i}^2 + 3\tilde{b_i}^4}{s_n^4} = 0$$

since both λ_i and $\tilde{b}_i$ are of order $O(n^{-3/2})$.

As a consequence of the above, we have the CLT for the random variable $\tilde{g}(\mathbf{x})|E$, thus

$$\mathbb{E}\left[\exp\left(iun\frac{\tilde{g}(\mathbf{x}) - \mathrm{E}_a}{s_n}\right)\middle|E\right] \to \exp(-\frac{u^2}{2}).$$

Next, we see that the second term in (26) goes to zero because $\left|\mathbb{E}\left[\exp\left(iun\frac{\tilde{g}(\mathbf{x})-\mathrm{E}_a}{s_n}\right)\middle|\bar{E}\right]\right| \leq 1$ and $\mathrm{P}(\bar{E}) \to 0$ from (25) and we eventually deduce

$$\mathbb{E}\left[\exp\left(iun\frac{\tilde{g}(\mathbf{x}) - \mathrm{E}_a}{s_n}\right)\right] \to \exp(-\frac{u^2}{2}).$$

With the help of Lévy's continuity theorem, we thus prove the CLT of the variable $n\frac{\tilde{g}(\mathbf{x})-\mathrm{E}_a}{s_n}$. Since $s_n^2 \to \mathrm{Var}_a$, with Slutsky's theorem, we have the CLT for $n\frac{\tilde{g}(\mathbf{x})-\mathrm{E}_a}{\sqrt{\mathrm{Var}_a}}$ (thus for $n\frac{\hat{g}(\mathbf{x})-\mathrm{E}_a}{\sqrt{\mathrm{Var}_a}}$), and eventually for $n\frac{g(\mathbf{x})-\mathrm{E}_a}{\sqrt{\mathrm{Var}_a}}$ by Theorem 1 which completes the proof.