Name ____________

Student number ____________

You have 1 hour and 40 minutes to complete the exam. There are 6 problems for 120 points total. If you score over 100 points, you will get 100 points.

Grades:

Problem 1 (15 pts): ______________

Problem 2 (20 pts): ______________

Problem 3 (25 pts): ______________

Problem 4 (20 pts): ______________

Problem 5 (25 pts): ______________

Problem 6 (15 pts): ______________

Total: ____________________

Problem 1. (15 pts)

Let $\mathbb{P}$ be the uniform probability distribution on $[0,1]$ where $\mathbb{P}((a,b]) = b - a$ for $a, b \in [0,1]$.

(a) (8 pts) By the basic properties of probability measure, prove that for any $x \in [0,1]$, $\mathbb{P}(\{x\}) = 0$.

(b) (7 pts) Since $\{x\} \cap \{x'\} = \emptyset$ for $x \neq x'$ and $\cup_{x \in [0,1]}\{x\} = [0,1]$ by applying the probability formula $\mathbb{P}(\cup_n A_n) = \sum_n \mathbb{P}(A_n)$ for disjoint events $\{A_n\}$, we have that

$$\mathbb{P}([0,1]) = \mathbb{P}(\cup_{x \in [0,1]}\{x\}) = \sum_{x \in [0,1]} \mathbb{P}(\{x\}) = 0. \tag{1}$$

However, we know that $\mathbb{P}([0,1]) = 1$. Why does equation (1) give the wrong result?

Problem 2. (20 pts)
Let $A_1, \ldots, A_n$ be events in a common probability space. For a set $\mathcal{I} \subseteq \{1, \ldots, n\}$, define $A_{\mathcal{I}} = \bigcap_{j \in \mathcal{I}} A_j$.

(a) (6 pts) Prove

$$\begin{aligned}\mathbb{P}\left(\bigcap_{j\in S} A_j \bigcap_{i\notin S} A_i^c\right) \quad &= \quad \mathbb{P}(A_S) - \sum_{i\in S^c} \mathbb{P}(A_{S\cup\{i\}}) + \sum_{\substack{i<j\\ i,j\in S^c}} \mathbb{P}(A_{S\cup\{i,j\}}) - \cdots \\ &\quad \cdots + (-1)^{n-|S|}\mathbb{P}(A_{S\cup S^c}),\end{aligned}$$

where $A_i^c = \Omega \setminus A_i$ and $S^c = \{1, ..., n\} \setminus S$.

Hints: you can leverage the inclusion/exclusion formula, and

$$\mathbb{P}(A) - \mathbb{P}(A \cap B^c) = \mathbb{P}(A \cap B), \quad \textit{where } A, B \textit{ are events in a common probability space}$$

to prove the about formula.

(b) (14 pts) Let B_k denote the event that exactly k (not more, not less) of the events $A_1, \ldots, A_n$ occur. Show the following identity (known as Waring's formula):

$$\mathbb{P}(B_k) = \sum_{i=0}^{n-k} (-1)^i \binom{k+i}{k} \rho_{k+i}$$

where we define the quantities

$$\rho_j = \sum_{i_1<i_2<\cdots<i_j} \mathbb{P}(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_j})$$

Notice: the sum $\sum_{i_1<i_2<\cdots<i_j}$ *indicates summation with respect to all* j*-tuples of distinct events.*

Hint: in order to get started, notice that

$$\mathbb{P}(B_k) = \sum_{\substack{S\subseteq\{1,\ldots,n\}\\ |S|=k}} \mathbb{P}\left(\bigcap_{j\in S} A_j \bigcap_{i\notin S} A_i^c\right)$$

Problem 3: (25 pts)

(a) (10 pts) Assume variables X and Y have joint pdf $f_{X,Y}(x,y)$. Let $W = X + Y$. Prove that

$$f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, w-x)dx = \int_{-\infty}^{\infty} f_{X,Y}(w-y, y)dy.$$

(*Hint: use the invertible affine transformations.*)

(b) (10 pts) Let

$$f_{X,Y}(x,y) = \begin{cases} 8xy, & \text{if } 1 \le x \le y \le 2, \\ 0, & \text{else,} \end{cases} \tag{2}$$

amd $W = X + Y$. Note that $W \in [0,2]$. Compute PDF $f_W(w)$. *Hint: Consider two regions of* W, $[0,1]$ *and* $[1,2]$, *respectively.*

(c) (5 pts) Furthermore, define $Z = 2Y$. Compute the joint PDF $f_{W,Z}(w,z)$.

Problem 4: (20 pts) Let X be a standard Gaussian random variable with mean 0 and variance 1 and let $S \in \{+1, -1\}$ be a uniformly distributed random variable.

(a) (5 pts) Compute the probability density function (pdf) of $Y = SX$ and show that it is a Gaussian random variable.

(b) (5 pts) Show that $\mathbb{E}[XY] = 0$ but X and Y are not independent.

(c) (10 pts) Does (X, Y) have a joint Gaussian distribution? Why?

Problem 5: (25 pts)

We have a communication channel which the error probability to transmit a bit is p. We do as follows: we transmit n i.i.d. bits and define random variables $X_i \in \{0, 1\}$, $i = 1, 2, \dots, n$ where $X_i = 1$ if the i-th bit is received in error. We consider the following estimator $\widehat{p} = \frac{\sum_{i=1}^{n} X_i}{n}$ for p.

(a) (5 pts) Compute the mean and variance of X_i.

(b) (5 pts) Assume n is large, what is the approximate distribution of $\sqrt{n}(\widehat{p} - p)$?

(b) (15 pts) How many number of bits n are approximately required such that $\mathbb{P}(0.99p \leq \widehat{p} \leq 1.01p) \geq 0.9974$? (Again assume n is large)

Problem 6: (15 pts)
Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with a Cauchy distribution with pdf $f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}$ over $x \in \mathbb{R}$ and let $S_n = \frac{\sum_{i=1}^n X_i}{n}$.

(a) (8 pts) Prove that S_n has a Cauchy distribution. (*Hint: use the fact that the characteristic function of a Cauchy distribution is given by* $\phi_X(\omega) = e^{-|\omega|}$.)

(b) (7 pts) From part (a), we have that the sample mean S_n does not converge to a deterministic number. Is this in contradiction with the law of large numbers? why?