Gauge-Transformation: Let $X := \{$Spaces of all connection on $\mathbb{C}^r\}$

A gauge-trans $g: M \to GL(\mathbb{C}^r)$ is an action on X: $X \to X$

$\nabla \mapsto \nabla.g = g^{-1} \circ \nabla \circ g$

All the gauge-trans $\mathcal{G} \curvearrowright X$ forms an action.

RK: 1. For $\nabla = d + \omega$, then $\nabla.g := d + g^{-1}dg + g^{-1}\omega g$

2. For s be parallel. then $g^{-1}s$ be parallel for $\nabla.g$.

Thm:

If $\nabla = d + \omega$, $\omega \in \Omega^1(gl(r, \mathbb{C}))$ local admits r linear independ parallel sections $s_1 \sim s_r$, then $F^{\nabla} = 0$

pf: Let $\omega \in \Omega^1(gl(r, \mathbb{C}))$ w.r.t frame $e_1, \dots, e_r$, then: Let: $f = (s_1, \dots, s_r)$ be a matrix w.r.t $(e_1, \dots, e_r)$, for $\nabla.f$, we have: $\nabla.f(s) = ds + (f^{-1}df)s + (f^{-1}\omega f)s$, As we know:

$\nabla s_i = 0 = ds_i + \omega s_i$ $\therefore$ $\omega f = (\omega s_1, \dots, \omega s_r) = -df$

$\therefore$ $\nabla.f = d$ $\therefore$ $F^{\nabla.f} = 0$

In general: $F^{\nabla.g} = d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega} = d(f^{-1}df + f^{-1}\omega f)$
$+ (f^{-1}df + f^{-1}\omega f) \wedge (f^{-1}df + f^{-1}\omega f)$

$= g^{-1} F^{\nabla} g.$

($\Leftarrow$) solve the O.D.E ✓

RK: If $f = (s_1, \dots, s_r)$, $\tilde{f} = (\tilde{s}_1, \dots, \tilde{s}_r)$ be two parallel frames, then they differ by a constant matrix.

Choose any basis $(e_1 \sim e_r)$, then:

$\nabla.f = d$, if $\tilde{f} = fc$ $\nabla.(f).(c) = (\nabla f).c = d.c = d + c^{-1}dc = 0$

$\therefore c \equiv 0.$

How to understand the Gauge Transformations?

The gauge transformation between two connections

(V, ∇^1), (V, ∇^2) is a <u>parallel map</u> φ

(preserves the parallel-transport)

$$\Phi = \{\mathrm{Hom}(V,V) \mid \varphi \text{ is parallel}\}$$

Suppose: $\nabla^1 = \nabla^2 \varphi$, then $\varphi \nabla^1 = \nabla^2 \varphi$, if $\nabla^1 s = 0$,

We have: $\nabla^2 \varphi(s) = \varphi \nabla^1 s = 0$.

$\therefore \varphi \in \Phi$.

Conversely: not true!!, 取 local parallel frame: $(s_1, \dots, s_r)$ of ∇^1, $(\tilde{s}_1, \dots, \tilde{s}_r)$ of ∇^2

if: $\varphi \in \{A \in \mathrm{Hom}((V,\nabla^1),(V,\nabla^2)) \mid A \text{ is parallel}\}$, i.e.

$\varphi(s_1, \dots, s_r) = (\tilde{s}_1, \dots, \tilde{s}_r)\phi$ for the matrix ϕ, we

hope to show: $\varphi \nabla^1 = \nabla^2 \varphi$ 设 $(\tilde{s}_1, \dots, \tilde{s}_r) = (s_1, \dots, s_r)B$,

$$\varphi \nabla^1 (\sum f_i s_i) = \varphi(\sum_{i=1}^n df_i \cdot s_i) = \varphi(s_1, \dots, s_n) \begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix}$$

$$\nabla^2 \varphi(\sum f_i s_i) = \nabla^2 (\sum f_i \varphi(s_i)) = \varphi(s_1, \dots, s_n) \begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix}$$

So if φ is automorphism

$\hookrightarrow \varphi$ is a Gauge

Monodromy

Let $\gamma : [0,1] \to M$ be a curve, call $s(t) \in V_{\gamma(t)}$

from $\begin{array}{c} V \\ \downarrow \\ M \end{array}$ be parallel if $\nabla_{\frac{\partial}{\partial t}} s(t) = 0$

This gives an O.D.E:

$$\begin{cases} s'(t) + \omega\left(\frac{\partial}{\partial t}\right) s(t) = 0 \\ s(0) = s_0 \end{cases}$$

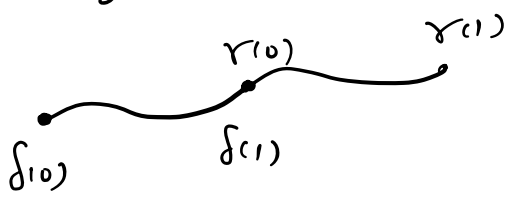
, 此即 Parallel Transport.

from $V_{\gamma(0)} \to V_{\gamma(1)}$, i.e.

$$\rho^{\nabla}(\gamma) : \begin{array}{ccc} V_{\gamma(0)} & \longrightarrow & V_{\gamma(1)} \\ s(0) & \longmapsto & s(1). \end{array}$$

Prop: (1) $\rho^{\nabla}(\gamma)$ is bijection + linear

(2) $\rho^{\nabla}(\gamma) \cdot \rho^{\nabla}(\delta) = \rho^{\nabla}(\delta * \gamma)$



(3) if f is a parallel frame on $\mathcal{U} \overset{\circ}{\subseteq} M$

s.t. $\mathrm{im}(\gamma) \subset \mathcal{U}$, then:

$$\rho^{\nabla}(\gamma) = f(\gamma(1)) \cdot f(\gamma(0))^{-1}$$

($s(t) = f(\gamma(t)) f^{-1}(\gamma(0)) s_0$ is just the solution)

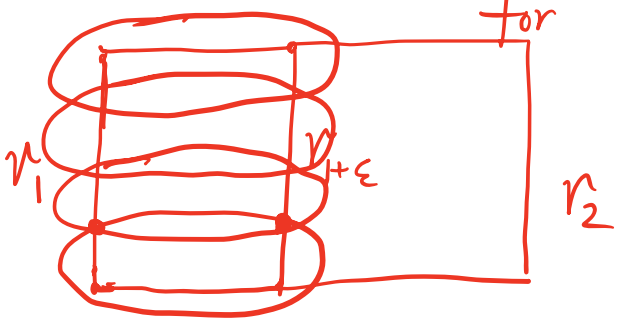
Thm:

For a flat connection $\begin{matrix}(E,\nabla)\\ \downarrow \\ M\end{matrix}$, the ρ^∇ induces a representation of $\pi_1(M,p)$, $\forall p\in M$.

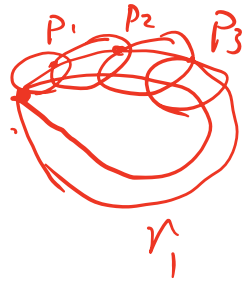
pf: 只须: $\rho^\nabla: \pi_1(M,p)\longrightarrow gl(E_p)$ 群同态.

即 $\rho^\nabla(r_1)=\rho^\nabla(r_2)$ if $r_1\simeq r_2$.

for ε be small enough, then:



$$\rho^\nabla(r_1)=\rho^\nabla(r_{1+\varepsilon})$$
$$\Downarrow$$



LHS: $=\rho^\nabla(r_1^0 * r_1^1 * r_1^2 * \cdots r_1^n)$, RHS: $=\rho^\nabla(r_{1+\varepsilon}^0 * \cdots * r_{1+\varepsilon}^n)$.

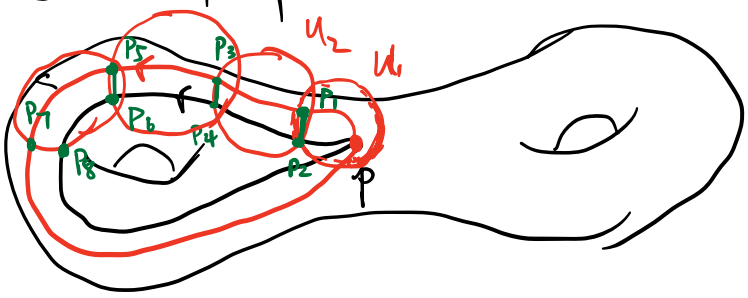
$$\| \qquad\qquad\qquad\qquad =$$

$$f'(p_n)\cdot \cancel{f'(p_{n-1}^{-1})}\cdot \cancel{f'(p_{n-1})}\cdot f'(p_{n-2}^{-1})\cdots$$

$\therefore$ This is well-def.

#.

进一步解释:



: why r_0 and r_ε induce the same monodromy.

Thm: (Riemann – Hilbert Correspondence)

For every representation $\rho: \pi_1(M,p) \to GL(r,\mathbb{C})$, $\exists$ is a bundle $\begin{matrix} V \\ \downarrow \\ M \end{matrix}$ with a flat connection ∇, s.t:

$$\rho^{\nabla} = \rho$$

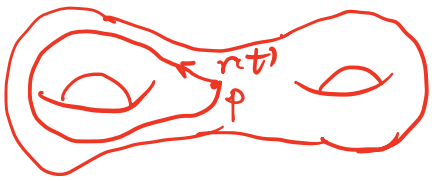
pf: Let $\tilde{M}$ denote the universal-covering of M. The vector bundle is defined by: $\tilde{M} \times \mathbb{C}^r / \sim$, where $(q, v) \sim (\tilde{q}, \tilde{v}) \Leftrightarrow \tilde{q} = q \cdot \gamma$, $\tilde{v} = \rho(\gamma^{-1}) v$

(若 ρ is the trivial-repre. then $\tilde{M} \times \mathbb{C}^r / \sim = M \times \mathbb{C}^r$)

We define the connection on this vector bundle by:

$s \in \Gamma(V)$ is parallel if s is const w.r.t any of trivilization.
$(\nabla s = 0)$

We show: $\rho^{\nabla} = \rho$, for $\gamma: [0,1] \to M$ be a curve, $\forall t \in [0,1]$, $\gamma|_{[0,t]}$ gives an element $q(t)$ on $\tilde{M}$



then for $v \in \mathbb{C}^r$, $(q(t), v)$ is a parrel section $s(t)$ over $r(t)$

$$s(1) = (q(1), v) = (q(0) \cdot \gamma, v) = (q(0) \cdot \gamma, \rho(\gamma^{-1}) \cdot \rho(\gamma) \cdot v)$$

$$= (q(0), \rho(\gamma) v)$$

$\therefore \rho^{\nabla} = \rho$. #

RK:

Two Connections $\nabla, \tilde{\nabla}$ are gauge equivalent

$\Leftrightarrow \rho^{\tilde{\nabla}} = c^{-1} \rho^{\nabla} c$ for some $c \in GL(r, \mathbb{C})$

let $f, \tilde{f}$ be two frames of $\nabla, \tilde{\nabla}$. then for $\forall$ $\gamma \in \pi_1(M,p)$,

$$\rho^{\tilde{\nabla}}(\gamma) = \tilde{f}(\gamma(1)) \cdot \tilde{f}(\gamma(0))^{-1}$$

$$= c f(\gamma(1)) \cdot f^{-1}(\gamma(0)) \cdot c^{-1}$$

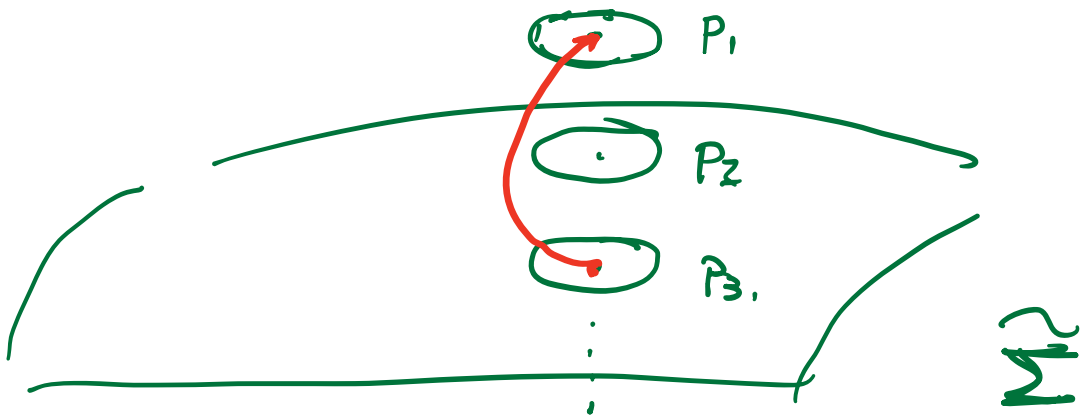
$\tilde{f} = c f$

$$= c \rho^{\nabla} c^{-1}$$

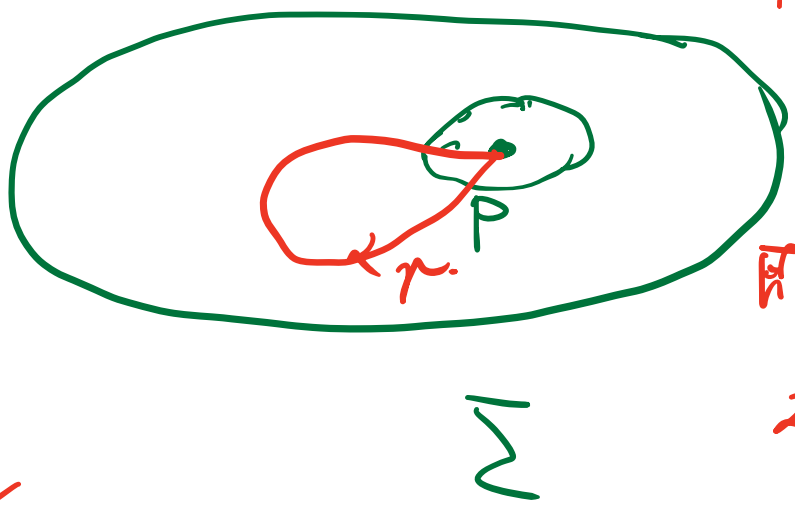
#

关于 Monodromy:

Let $\hat{\Sigma}$ be an univeral-covering of Σ.



Lift 之后 即从某个 $p_i \to p_j$



(Monodromy 即刻画 p_i 与 p_j 之差别)

$$\hat{\Sigma} \times \mathbb{C}^2 \Big/ (q, v) \sim (q * \gamma, \rho(\gamma^{-1}) v)$$

Structure Preserved by Flat Connection

Ex: Let $\nabla = d + \omega$ be flat, $\omega \in \Omega^1(M, sl(r, \mathbb{C}))$ with $\omega = g^{-1}dg$, then: $d(\det g) = tr(g^{-1}dg) = 0$

RK: Jacobi-Formula.

$$\frac{d}{dt}A(t) = \lim_{h\to 0} \frac{\det(A(t+h)) - \det A(t)}{h}$$

$$= \det A(t) \cdot \lim_{h\to 0} \frac{\det(A^{-1}(t)A(t+h)) - I}{h}$$

tangent at identity

And by $\frac{d}{dt}\det(I + tM) = trM$

$$\therefore RHS = \det A(t)\, tr\left(A^{-1}(t)\frac{dA(t)}{dt}\right)$$

#.

Ex: $\nabla = d + \omega$, $\omega \in \Omega^1(M, su(r))$, $F^{\nabla} = 0$, then:

$$\rho^{\nabla} \in Su(r)$$

choose $\forall \gamma \in \pi_1(M, p)$

$\rho^{\nabla}(\gamma): V_p \longrightarrow V_p$,

give the standard hermitian-metric

For s, t be local-parallel section:

$$d\langle s, t\rangle = \langle ds, t\rangle + \langle s, dt\rangle$$

$$= \langle ds + \omega s, t\rangle - \langle \omega s, t\rangle + \langle s, dt\rangle$$

$$\langle \omega s, t\rangle = \overline{(\omega s)}^t t = \bar{s}^t \bar{\omega}^t t = -\bar{s}^t \omega t = -\langle s, \omega t\rangle$$

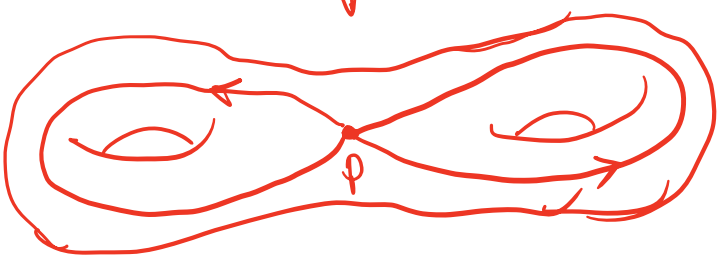
$$= \langle \nabla s, t\rangle + \langle s, \nabla t\rangle = 0$$

$\therefore \rho^{\nabla}(\gamma)$ preserves hermitian metric $\quad \therefore \rho^{\nabla}(\gamma) \in Su(r)$

Prop: A flat connection has SL-monodromy $\Leftrightarrow$ it admits a parallel determinant $\det \in \Gamma(\Lambda^r V)$.

pf: ($\Rightarrow$)

$$\begin{array}{c} M \times \mathbb{C}^r \\ \downarrow \pi \end{array}$$



$\rho^\nabla : \pi_1(M,p) \to SL(r,\mathbb{C})$

(Def: $\nabla \det(s_1,\dots,s_r) = d(\det(s_1,\dots,s_r)) - \sum_{i=1}^{r} \det(s_1,\dots,\nabla s_i,\dots,s_r)$)

and the det-form is given by: $V \times V \cdots \times V \to \mathbb{C}$ (即选定一组frame之后即可定义determinant.)

$Hol(\nabla) \subset SL(r,\mathbb{C})$

$\det \in (\Lambda^r V)^*$

$\det(f e_1 \wedge \cdots \wedge e_r) = f$

take $\ell = e_1 \wedge \cdots \wedge e_r$, then

$\det = d\ell$

to show $\nabla \det = 0$

只须: $Hol(\nabla) \det|_{x_0} = \det|_{x_0}$

$\rho^\nabla(\ell) = \det(SL(r,\mathbb{R}))\,\ell$

$= \ell$

$\therefore \nabla \det = 0$

Prop: A flat connection has a unitary monodromy $\Leftrightarrow$ it admits a parallel hermitian-metric.

$\rho^{\nabla} \subseteq U(n)$, then $Hol(\nabla) \subset U(n)$, take a hermitian-metric on any point, do parallel transport to any point which gives a global hermitian-metric.

Connection and Curvature

For $\mathbb{C}^r \to E \to M$. $\nabla = d + \omega$, $\omega = (\omega_{ij})_{r\times r}$, then ∇ induces a connection on E:

Let $\{e_1, \dots, e_r\}$ be the frame, such that:

$\nabla s = ds + \omega s$, then $F^{\nabla} = d\omega + \omega \wedge \omega$. For $\Lambda^r E$, we have:

$$\nabla \circ \nabla (e_1 \wedge \cdots \wedge e_r) = \nabla \left(\sum e_1 \wedge \cdots \wedge \nabla e_i \wedge \cdots \wedge e_r \right)$$

Assume ω be diagonal, $d\omega + \omega \wedge \omega = \begin{pmatrix} d\omega_1 + \omega_1 \wedge \omega_1 & & \\ & \ddots & \\ & & d\omega_r + \omega_r \wedge \omega_r \end{pmatrix}$

$$\begin{aligned} RHS &= \nabla \left(\sum e_1 \wedge \cdots \wedge \omega_i e_i \wedge \cdots \wedge e_r \right) \\ &= \sum e_1 \wedge \cdots \wedge \nabla e_j \wedge \cdots \wedge \omega_i e_i \wedge \cdots \wedge e_r + e_1 \wedge \cdots \wedge \nabla(\omega_i e_i) \wedge \cdots \wedge e_r \\ &= \underbrace{\left(\sum_{i,j} \omega_i \omega_j \right)}_{=0} e_1 \wedge \cdots \wedge e_r + \sum_i (d\omega_i + \omega_i \wedge \omega_i) e_1 \wedge \cdots \wedge e_r \\ &= tr(F^{\nabla}) e_1 \wedge \cdots \wedge e_r \end{aligned}$$

Thus: $F^{\Lambda^r \nabla} = tr(F^{\nabla})$

Computation of Chern-Class.

1. The dual bundle.

If L is a line bundle with Chern-class $c_1(L)$, then the dual line bundle L^* is isomorphic to the inverse line bundle, i.e.

$$c_1(L^*) = -c_1(L)$$

$$\begin{array}{ccc} L & & L^* \\ \downarrow & \swarrow & \\ \Sigma & & \end{array}$$

$R_h = -i\partial\bar{\partial}\log h_L$, then for L^*:

if: $h_L = h\, de_1 \otimes de_1$, then $h_{L^*} = \frac{1}{h} de_1^* \otimes de_1^*$

$\therefore R_{h_{L^*}} = -R_{h_L}$. now for any:

Vector Bundle V. let $V \cong \bigoplus_i L_i$. Thus:

$$c(V^*) = \prod_i (1 + c_1(L_i^*)) = \prod_i (1 - c_1(L_i))$$

$$\therefore c_1(V^*) = -c_1(V)$$

---

2. The Tensor Bundle

$V \cong \bigoplus_i L_i$. $V' \cong \bigoplus_i L_i'$, then: $V \otimes V' \cong \bigoplus L_i \otimes L_j'$

$$\therefore c(V \otimes V') = \prod (1 + c_1(L_i) + c_1(L_j'))$$

$$\therefore c_1(V \otimes V') = c_1(V)\dim V' + c_1(V')\dim V$$

3. The Exterior Power

$\Lambda(V\oplus W)=\Lambda(V)\otimes\Lambda(W)$, For $V\cong\bigoplus_i L_i$

$\therefore\ ch(\Lambda(V))=\prod_i ch(\Lambda(L_i))$

$\therefore\ c_1(\Lambda^2(V))=(\dim V-1)\,c_1(V)$

将此应用到 degree 之上:

$$\deg(V)=\frac{i}{2\pi}\int_\Sigma \mathrm{tr}(F^{\nabla^V})=\frac{i}{2\pi}\int_\Sigma c_1(V)=\frac{-i}{2\pi}\int_\Sigma c_1(V^*)$$

$$=-\deg(V^*)$$

类似: $\deg(V_1\otimes V_2)=\dim V_1\cdot\deg V_2+\deg V_1\cdot\dim V_2$

Prop: A vector bundle $\begin{array}{c} V \\ \downarrow \\ \Sigma \end{array}$ is $\mathbb{C}^\infty$-trivial iff $\deg(V) = 0$

pf: $(\Rightarrow)$ take $\nabla = d$ is OK

$(\Leftarrow)$ $V \cong \mathbb{C}^{r-1} \oplus \Lambda^r V$, $\therefore \deg(\Lambda^r V) = 0$, For line bundle, this is obviously $\mathbb{C}^\infty$-free, $\Rightarrow \Lambda^r V \cong \mathbb{C}$

$\therefore V \cong \mathbb{C}^r$.

Two Important Theorem.

Thm: (Serré-Duality)

For any holomorphic structure $\bar\partial$ on $\begin{array}{c} V \\ \downarrow \\ \Sigma \end{array}$, Σ is compact, then we have:

$$\underset{\omega}{H^0(V)} \cong \left(\underset{\eta}{H^1(KV^*)}\right)^*$$

The duality is given by:

$$\langle \omega, \eta \rangle \longmapsto \int_\Sigma \omega \wedge \eta.$$

Thm (Riemann-Roch-Theorem)

For holo vector bundle $\mathbb{C}^r \to V \to \Sigma$, we have:

$$h^0(V) - h^1(V) = \operatorname{rank}(V)(1-g) + \deg(V)$$

$h^0(V) = \dim H^0(V)$, $h^1(V) = \dim \Gamma(\bar K V) / \operatorname{im} \bar\partial : V \to \bar K V$

#.

pf: For $V = L_1 \oplus \cdots \oplus L_r$, $\bar\partial' = \begin{pmatrix} \bar\partial^{L_1} & & \\ & \ddots & \\ & & \bar\partial^{L_r} \end{pmatrix}$, then:

$$h^0(V) - h^1(V) = \deg(L_1) + 1 - g + \cdots + \deg(L_r) + 1 - g$$
$$= \deg V + (1-g)\operatorname{rank} V.$$

$$0 \longrightarrow \Gamma(V) \xrightarrow[\bar\partial']{\bar\partial} \Gamma(KV) \longrightarrow 0.$$

$\operatorname{ind}_{\bar\partial} = h^0(V) - h^1(V)$, notice: $\bar\partial' - \bar\partial$ is a compact-opera

$\therefore \operatorname{ind}_{\bar\partial} = \operatorname{ind}_{\bar\partial'}$

$\therefore \operatorname{ind}_{\bar\partial} = \deg(V) + \operatorname{rank}(V)(1-g)$

#

Def: $\Phi \in H^0(K \operatorname{End} V)$ is called a Higgs-Field.

Prop: For $\begin{array}{c} \mathbb{C}^r \\ \downarrow \\ (V, \bar\partial) \\ \downarrow \\ \Sigma \ (R.S.) \end{array}$, ∇ is flat such that $\bar\partial^{\nabla} = \bar\partial$,

then: $\nabla + \Phi$ is also flat.

pf: $F^{\nabla+\Phi} = F^{\nabla} + d^{\nabla}\Phi = F^{\nabla} + \bar\partial^{\nabla}\Phi = 0$

#

Prop: $V \to \Sigma$ is a vector bundle of $\underline{\operatorname{rank} V = 2}$, $\deg V = 0$

and $\boxed{\Lambda^2 V = 0}$ $(V = L \oplus L^*)$ then $\exists$ holosubundle $L \hookrightarrow V$

of $\deg(L) \geq 1 - g$. $\boxed{(g(\Sigma) \geq 1)}$

Q: where we use this fact?

Def: A holomorphic line subbundle L is given by a holomorphic cocycle of V which is of the form $g_{\alpha\beta} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, and

L_p is just the image of the first coor w.r.t the correspondence holo-trivilization.

Another 2 equivalent def:

1. $\exists$ i injective holomorphic : $L \hookrightarrow V$, i.e. $i \in H^0(L^* V)$ without any zeros.

2. $L \hookrightarrow V$, such that : $\forall s \in \Gamma(L)$, $\bar\partial s \in \Gamma(\bar K L)$

Now we prove the prop:

idea: 先构造一个全纯子丛, 接着把其嵌入向量丛中.

Let $\tilde{L}$ be any holo-line (全纯) bundle of degree $-g = -\text{genus}(\Sigma)$,

$$h^0(\tilde{L}^* V) - h^1(\tilde{L}^* V) = 2(1-g) + \deg(\tilde{L}^* V)$$

$$\deg(\tilde{L}^* V) = 2\deg(\tilde{L}^*) + \deg(V)$$
$$= \deg(V) + 2g = 2g$$

$\rightsquigarrow h^0(\tilde{L}^* V) \geq 2$, let $s, t \in H^0(\tilde{L}^* V)$, be linear-independent, now we need to give injectivity. 考虑 $\text{span}\{s, t\}$. take $\hat{i} \in \text{span}\{s, t\}$, s.t. $\hat{i}(p) = 0$ at $p \in \Sigma$

Consider $L = \tilde{L} \otimes L(p)$, $L^* \otimes V = L(-p) \otimes \tilde{L}^* \otimes V$

$s(-p) \otimes \hat{i}$

This gives the construction.

Rk: Actually, we can make L as large as possible.

Prop: If ∇ is a flat $sl(2;\mathbb{C})$-connect on V with $\Lambda^2 V = \theta$. then for $L \hookrightarrow V$, we have

后会控制的吧).

$\deg(L) \leq g-1$ $\quad (g(\Sigma) \geq 1)$

pf: Let $V = L \oplus \tilde{L}$, then $\bar{\partial}^V = \begin{pmatrix} \bar{\partial}^L & \gamma \\ 0 & \bar{\partial}^{\tilde{L}} \end{pmatrix}$

$\gamma \in \Gamma(\bar{K}L^2)$, for ∇, we know:

$$\nabla = \begin{pmatrix} \nabla^L & \beta \\ \alpha & \nabla^{L^*} \end{pmatrix}, \quad \alpha \in \Gamma(KL^{*2}), \ \beta \in \Omega^1(L^2)$$

$\beta^{0,1} = \gamma$, As $F^\nabla = 0 \to \nabla \circ \nabla = 0$, by definition:

$$\begin{pmatrix} d\nabla^L & d\beta \\ d\alpha & d\nabla^{L^*} \end{pmatrix} + \begin{pmatrix} \nabla^L & \beta \\ \alpha & \nabla^{L^*} \end{pmatrix}\begin{pmatrix} \nabla^L & \beta \\ \alpha & \nabla^{L^*} \end{pmatrix}$$

$$= \begin{pmatrix} d\nabla^L & d\beta \\ d\alpha & d\nabla^{L^*} \end{pmatrix} + \begin{pmatrix} \nabla^L \cdot \nabla^L + \beta\wedge\alpha & \nabla^L \cdot \beta + \beta \nabla^{L^*} \\ \alpha \nabla^L + \nabla^{L^*}\alpha & \alpha\beta + \nabla^{L^*} \cdot \nabla^{L^*} \end{pmatrix}$$

$$= \begin{pmatrix} F^{\nabla^L} + \beta\wedge\alpha & d^{\nabla^L}\beta \\ d^{\nabla^L}\alpha & F^{\nabla^{L^*}} + \alpha\wedge\beta \end{pmatrix} = 0$$

$\therefore \bar{\partial}^{\nabla^L} \alpha = 0 \to \alpha \in H^0(KL^{*2})$ $\therefore \deg K \geq 2\deg(L)$

$\therefore \deg L \leq g-1$

Question: 事实上用到的是 $\underline{\Lambda^2 V = \theta}$. 并不是 (or $\rho^{\nabla} \in SL(2,\mathbb{C})$)

$\underline{\nabla \in \Omega^1(sl(2))}$?

#

RK:

If $\deg L = g-1$, then α has no zeros, $\therefore$ KL^{-2} is trivial.

Def: (Spin)

A holomorphic line bundle $L \hookrightarrow V$ s.t. $L^2 = K$ are called spin-bundle.

Thm: (Narasimhan – Seshadri)

∇ is flat unitary $sl_2(\mathbb{C})$ – connection, then:

$$\forall L \hookrightarrow V, \quad \deg(L) \leq 0$$

(If $\deg L = 0$, then $\nabla = \begin{pmatrix} \nabla^L & 0 \\ 0 & \nabla^{L^*} \end{pmatrix}$).

pf: Let $L \hookrightarrow V$, consider $L^\perp \subseteq V$, then for

$V = L \oplus L^\perp$, $\nabla = \begin{pmatrix} \nabla^L & -\alpha^* \\ \alpha & \nabla^{L^\perp} \end{pmatrix}$, $\alpha \in H^0(K L^{-2})$

$\therefore 0 = \begin{pmatrix} F^{\nabla^L} - \alpha^* \wedge \alpha & * \\ * & * \end{pmatrix}$ $\therefore \deg L = \frac{i}{2\pi} \int_\Sigma \alpha^* \wedge \alpha \leq 0$

#

Def: (Stable)

A holomorphic rank 2 bundle V, $\Lambda^2 V = \theta$

is called stable if $\forall L \hookrightarrow V$, $\deg L < 0$

(易知: unitary connection on (V, ∇)

$\Rightarrow V$ must be semi-stable.

#

Thm: (Weil)

A holomorphic $\begin{array}{c} V \\ \downarrow \\ \Sigma \end{array}$, $\operatorname{rank} V = 2$, $\Lambda^2 V = \theta$

admits a flat $sl(2;\mathbb{C})$ connection $\Leftrightarrow$

(a) Stable $\longrightarrow$ $H^0(\mathrm{End}_0 V)$ Vanishes.

(b) strictly semi-stable

(c) V is non-trivial extension of some holomorphic line bundle L^* of $\deg L^* < 0$ by its dual, i.e.

$$\bar{\partial}^{\nabla} = \begin{pmatrix} \bar{\partial}^{L} & \gamma \\ 0 & \bar{\partial}^{L^*} \end{pmatrix}, \quad \gamma \in \Gamma(\bar{K} L^2)$$

with $[\gamma] \neq 0$ in $H^1(L^2)$

pf: $(\Rightarrow)$ $\nabla = \begin{pmatrix} \nabla^{L} & \beta \\ \alpha & \nabla^{L^*} \end{pmatrix}$, $\alpha \in H^0(K L^{-2})$

if $\exists L$, $\deg L > 0$, then:

$$0 < \frac{i}{2\pi} \int_\Sigma F^{\nabla^L} = -\frac{i}{2\pi} \int_\Sigma \beta \wedge \alpha = -\frac{i}{2\pi} \int_\Sigma \beta^{0,1} \wedge \alpha$$

$\therefore \beta^{0,1} \neq 0$ in $H^1(L^2)$ (Serre-duality)

$(\Leftarrow)$

RK: 只要 $\Lambda^2 V=0$，就有：$\forall (V,\nabla)$ $\downarrow$ Σ，$F^{\nabla}\in\Omega^2(\mathrm{End}_0 V)$

Start with any $\tilde{\nabla}$, s.t

a) V is stable

$\bar{\partial}^V=\bar{\partial}^{\tilde{\nabla}}$, $F^{\tilde{\nabla}+\phi}=F^{\tilde{\nabla}}+d^{\tilde{\nabla}}\phi$, hope: $F^{\tilde{\nabla}}+d^{\tilde{\nabla}}\phi+\phi\wedge\phi=0$

take $\phi\in\Gamma(K\mathrm{End}_0\phi)$, 即 $F^{\tilde{\nabla}}+\bar{\partial}^{\tilde{\nabla}}\phi=0$

By Serre-duality: $F^{\tilde{\nabla}}=0$ in $H^1(K\mathrm{End}_0(V))$.

Notice: $H^1(K\mathrm{End}_0(V))\cong H^0(\mathrm{End}_0 V)$.

$\therefore \Leftrightarrow \forall \alpha\in H^0(\mathrm{End}_0 V)$, $\int_\Sigma \mathrm{tr}(\alpha\wedge F^{\tilde{\nabla}})=0$

Now we show: $H^0(\mathrm{End}_0 V)=0$.

let $A\neq 0$, $A\in H^0(\mathrm{End}_0(V))\to \det A\in H^0(\Sigma,\mathcal{O})$

1) $\det A\neq 0 \rightsquigarrow A$ has 2 eigen $\mu,-\mu$ $L,\tilde{L}$

$\therefore \deg L<0$, $\deg\tilde{L}<0$

$\Rightarrow \deg V<0$, contra

2) $\det A=0$ $\therefore A^2=0$, $\therefore$ Let $L\subseteq\ker A$ be kernal-bundle, consider $L^*=V/L$. $\tilde{A}: L^*\to L$

$\therefore \tilde{A}\in H^0(L^2)$ $\therefore \deg L\geq 0$, contradiction.

#

b) Strictly Semi-Stable

As a): $A \neq 0$ in $H^0(\mathrm{End}_0(V))$

1) $\det A \neq 0 \Rightarrow V = L \oplus \tilde{L}$ which is holomorphic subbundle. and $\deg L = \deg \tilde{L} = 0$.

RK:

For holomorphic $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ with $\deg E = 0$, if each indecompo part of E has $\deg E$ we have: E admits a flat connection,

In this case, we only need to show: if a holo-line-bundle has 0-degree, it must admits a flat-connection.

$$\begin{array}{c} \mathbb{C} \\ \downarrow \\ (L,\bar\partial) \\ \downarrow \\ \Sigma \end{array}$$

, give L any connection ∇ with $\bar\partial^{\nabla} = \bar\partial$. then

$0 = \int_\Sigma F^{\nabla} = 0$ this means $F^{\nabla} = \theta$ in $H^1(K\,\mathrm{End}_0(V))$

$\therefore F^{\nabla} = \bar\partial^{\nabla}\Phi$ for some $\Phi \in \Gamma(K\,\mathrm{End}_0(V))$, then

take $\nabla - \Phi = \tilde\nabla$

$$F^{\tilde\nabla} = F^{\nabla-\Phi} = F^{\nabla} + d^{\nabla}(-\Phi) + [\Phi,\Phi]$$
$$= F^{\nabla} - \bar\partial^{\nabla}\Phi = 0$$

This is what we want.

Big Picture of Monodromy and Stability

Monodromy: (holonomy-group)

For (V,∇) be flat, $\rho^{\nabla}: \pi_1(M,p) \to GL(r,\mathbb{C})$ 群表示, 具体来 on $\tilde M \times \mathbb{C}^r/\sim$

$V_p \longrightarrow V_p$

说: For $\gamma \in \pi_1(M,p)$, $\rho^{\nabla}(\gamma): (\gamma(0),v) \longmapsto (\gamma(t),v)$

ρ^{∇}受∇控制: $\nabla \in \Omega^1(\mathfrak{su}(r)) \to \rho^{\nabla} \in SU(r)$; 反之 $\rho^{\nabla} \in SL(r,\mathbb{C}) \to \nabla \det = 0$

$\rho^{\nabla} \in U(r) \to \nabla h = 0$

Stability: 一般 $V \to \Sigma$ with $\Lambda^2 V = 0$, 若 (V,∇) flat, $\forall L \hookrightarrow V$,

$\deg L \le g-1$ (若 flat unitary $\Rightarrow$ $\underline{\deg L \le 0}$) $\to$ semi-stable.

分类定理: flat connection with $\Lambda^2 V = \theta$ $\Longleftrightarrow$ (i) V stable (ii) V semistable (iii) $\bar\partial = \begin{pmatrix} \bar\partial^L & \gamma \\ 0 & \bar\partial^{L^*} \end{pmatrix}$, $\gamma \neq \theta$.

Harmonic — Map

(Thm: Birkhoff - Grothendick)

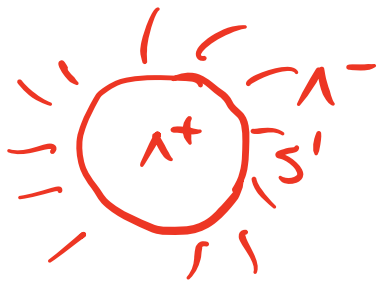
$\mathbb{C}P^1$上向量丛之分类.

Every holo-vector-bundle $V \to \mathbb{C}P^1$, we have

$V \simeq L_1 \oplus \cdots \oplus L_r$, $L_i \cong \mathcal{O}(d_i)$, and the degree is invariant.

Rk: For $\mathbb{C}^2 \to V \to \mathbb{C}P^1$ with $\Lambda^2 V = \theta$, then:

$V \cong \mathcal{O}(d_i) \oplus \mathcal{O}(-d_i)$.



Def: (Loop Group)

$\Lambda SL(2,\mathbb{C}) = \{ g : S^1 \to SL(2;\mathbb{C})$ real-analytic $\}$

$\Lambda^+ SL(2,\mathbb{C}) = \{ g \in \Lambda SL(2;\mathbb{C}) \mid g$ extends holo to $\mathbb{D} \}$

$\Lambda^- SL(2,\mathbb{C}) = \{ g \in \Lambda SL(2;\mathbb{C}) \mid g$ extends holo to $\mathbb{C}P^1 \setminus \mathbb{D} \}$

$\mathrm{Diag} := \{ g(\lambda) = \begin{pmatrix} \lambda^k & \\ & \bar{\lambda}^k \end{pmatrix} \mid$ for some $k \in \mathbb{Z} \}$

Thm:

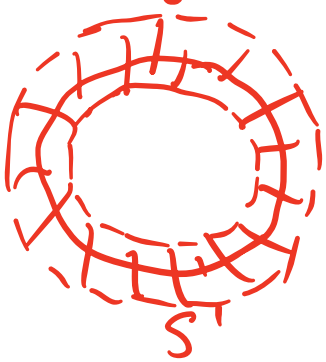
$$\forall g \in \Lambda SL(2,\mathbb{C}),\quad g = g_+ D g_- = b_- D b_+$$

看成$\mathbb{CP}^1$上的 holo-vector-bundle.

Remark:

If f is analytic on $\mathbb{R}$, $\exists$ open neigh $\Omega \subset \mathbb{C}$ and g holomorphic on Ω, s.t. $\mathbb{R} \subset \Omega \subset \mathbb{C}$ and $g|_{\mathbb{R}} = f$.

看法 $g \in \Lambda SL(2,\mathbb{C}) \rightsquigarrow$ holomorphic-vector-bundle.



对 g 作 holo extension

$g_{\infty,0}|_A$

$A = \{ r < |z| < \frac{1}{r} \}$

$\therefore g$ defines a holo-vector bundle $V \simeq \begin{pmatrix} \lambda^k & \\ & \lambda^{-k} \end{pmatrix}$ $\overset{f}{\overset{\shortparallel}{V(\mathbb{D})}}$

$V(g)|_{U_0} \xrightarrow{\Phi_0} U_0 \times \mathbb{C}^2$ $\quad \therefore \Phi_\infty \circ \Phi_0^{-1} = g$

$V(g)|_{U_\infty} \xrightarrow{\Phi_\infty} U_\infty \times \mathbb{C}^2$. By $V \simeq \begin{pmatrix} \lambda^k & \\ & \lambda^{-k} \end{pmatrix}$.

$\exists\ \lambda_0 : U_0 \to SL(2,\mathbb{C})$, s.t. $g = \lambda_\infty \begin{pmatrix} \lambda^k & \\ & \lambda^{-k} \end{pmatrix} \lambda_0$

$\lambda_\infty : U_\infty \to SL(2,\mathbb{C})$

RK: 此分解不唯一，但我们可以加入normalization 使之唯一.

Iwasawa Decomposition

There is a unique decomposition:

$$\Lambda SL(2,\mathbb{C}) = \Lambda^{+,b} \cdot \Lambda SU(2)$$

$$\Lambda^{+,b} = \left\{ g \in \Lambda^{+} SL(2,\mathbb{C}) \;\middle|\; g(0) = \begin{pmatrix} r & b \\ 0 & \frac{1}{r} \end{pmatrix},\ r>0,\ b \in \mathbb{C} \right\}$$

pf: (uniqueness)

if $g = \underset{\Lambda^{+,b}}{u} \cdot \underset{\Lambda SU(2)}{p} = \tilde{u} \cdot \tilde{p}$.

$\therefore\ \tilde{u}^{-1} u = \tilde{p} \cdot p^{-1} = k$

$k : S^1 \longrightarrow SU(2) \in \Lambda^{+,b}$

S^1

claim: $k = id.$

k can extend to whole $\mathbb{CP}^1 \to k$ const, unitary

of the form $\underbrace{\begin{pmatrix} r & * \\ 0 & \frac{1}{r} \end{pmatrix}}_{A}$, $A \cdot \bar{A}^t = 1 \Rightarrow k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Families of Holo-structure

X is 1-dim complex-mfd, a 1-dim family of holo-structures is a map:

$$\lambda \in X \longmapsto \bar{\partial}^{\lambda} \text{ on } \begin{array}{c} V \\ \downarrow \\ \Sigma \end{array}, \quad \Sigma \text{ is compact}$$

Example: $\mathbb{C} \longrightarrow \Omega^{0,1}(V)$

$\lambda \longmapsto \bar{\partial} + \lambda \Phi^*$ for $\Phi^* \in \Gamma(\bar{K} \mathrm{End}_0(V))$

Thm:

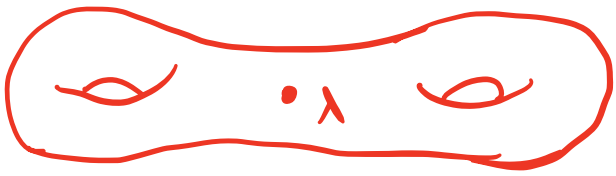
For a 1-dim family of holomorphic-structures:

$X \longrightarrow \bar{\partial}^{\lambda}$ on compact R.S. $V \to \Sigma$, $\exists$ a holo bundle $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ with

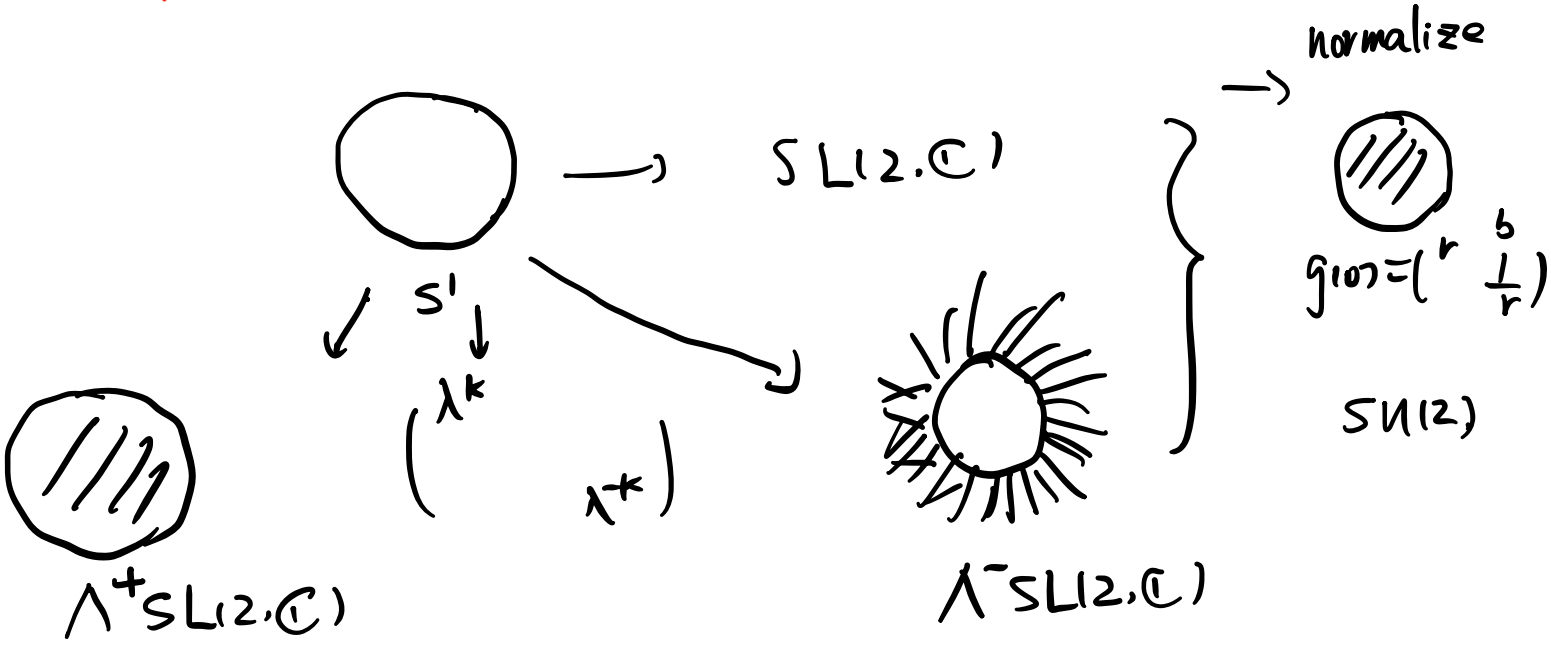
1) $L_\lambda \subset \mathbb{P}(V)$
2) Away from discrete set $S \subseteq X$, $\mathcal{L}_\lambda = \ker(\bar{\partial}^{\lambda})$

$$X \longrightarrow \Omega^{0,1}(\mathrm{End}\, V) \quad , \quad \begin{matrix} V \\ \downarrow \\ \Sigma \end{matrix}$$

solution:



$\forall \lambda \in X$, the fiber is given by $\ker(\bar{\partial}^{\lambda})$, as $\bar{\partial}^{\lambda}$ is Fredholm $\Rightarrow$ This is a v.b.



Harmonic-Map is S^3

1. Structure of S^3

$S^3 := \{ x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \} \subset \mathbb{R}^4$, in this case,

We can induce S^3 with $\mathbb{R}^4$'s Riemann-Metric.

Prop 1: $S^3 \cong SU(2)$

pf: The $SU(2)$ is given by: $= \underbrace{\left\{ \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \middle| \, |x|^2 + |y|^2 = 1 \right\}}_{SU(2)\text{之基本形式}}$

同构为 $\begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \end{pmatrix} \in S^3 \subseteq \mathbb{C}^2 = \mathbb{R}^4.$

so we could S^3 as a Lie Group.

Corallary: $T_{id} SU(2) = \mathfrak{su}(2) = \{ A + \bar{A}^t = \theta, \ A \in \mathfrak{gl}(2) \}$

which has the form $\begin{pmatrix} ir & -s+it \\ s+it & -ir \end{pmatrix}$

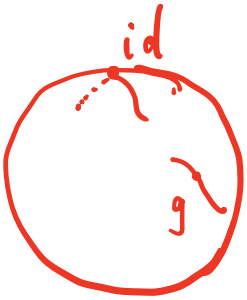
Prop: The Riemann-Metric on $T_{id} SU(2)$ is given by:

$\langle A, B \rangle = -\frac{1}{2} tr(AB) = \frac{1}{2} tr(A\bar{B}^t)$, $\forall A, B \in T_{id} SU(2)$.

$A = \begin{pmatrix} ir_1 & -s_1+it_1 \\ s_1+it_1 & -ir_1 \end{pmatrix}$, $B = \begin{pmatrix} ir_2 & -s_2+it_2 \\ s_2+it_2 & -ir_2 \end{pmatrix}$, then:

$-\frac{1}{2} tr(AB) = -\frac{1}{2}(-2r_1r_2 - 2s_1s_2 - 2t_1t_2) = r_1r_2 + s_1s_2 + t_1t_2.$

Prop: The Riemann-Metric at $T_g SU(2)$ is also given by: $\forall A, B \in T_g SU(2)$, $\langle A, B\rangle_g = \frac{1}{2} tr(A\bar{B}^t)$



$T_g SU(2) = g\, T_{id} SU(2) = g\, \mathfrak{su}(2) = \mathfrak{su}(2)\, g$

then $\langle A, B\rangle_g = \langle Ag^{-1}, Bg^{-1}\rangle_{id} = \frac{1}{2} tr(Ag^{-1}(\bar{g}^{-1})^t \bar{B}^t)$

$= \frac{1}{2} tr(A\bar{B}^t)$

Prop: The Riemann-Metric on $SU(2)$ is invariant under the left/right multiplication.

pf: $L_g : \begin{array}{l} SU(2) \longrightarrow SU(2) \\ h \longmapsto gh \end{array}$, it is enough to show:

$\forall A, B \in T_{id} SU(2)$,

$$\langle A, B\rangle_{id} = \langle (L_g)_* A, (L_g)_* B\rangle_g = \langle gA, gB\rangle_g$$
$$= \langle A, B\rangle_{id}$$

#

Prop: There admits a natural map:

$$SU(2) \times SU(2) \longrightarrow SO(4)$$
$$\Phi : (h_1, h_2) \longmapsto \Phi(h_1, h_2)$$

pf: Define: $\mathbb{R}SU(2) \cong \mathbb{R}^4$

$$r\begin{pmatrix} x & -\bar{y} \\ x_1+ix_2 \; y & \bar{x} \\ y_1+iy_2 \end{pmatrix} \longmapsto (rx_1, rx_2, ry_1, ry_2).$$

the inner product is also $\frac{1}{2} tr(A\bar{B}^t)$

then: $\Phi(h_1, h_2)(rg) = h_1 r g h_2^{-1}$

$\therefore \langle h_1 r g h_2^{-1}, h_1 r g h_2^{-1}\rangle = \langle rg, rg\rangle = \frac{1}{2} tr(r^2 g \cdot \bar{g}^t)$

$$LHS = \frac{1}{2} tr(h_1 r g \cancel{h_2^{-1}} \cdot \cancel{h_2^{-1}}{}^t \cdot (\overline{rg})^t \cdot \overline{h_1}^t) = \frac{1}{2} tr(h_1 r g \cdot \overline{g}^t \cdot \overline{r}^t \cdot \overline{h_1}^t)$$

$$= \frac{1}{2} tr(r^2 g \cdot \overline{g}^t)$$

#

Prop: The map:

$SU(2) \times SU(2) \longrightarrow SO(4)$ is 2:1 covering.

pf: This map is open, as $SU(2)$ compact

$\therefore$ This map is closed $\therefore$ surjective.

Energy Functional

For (M^m, g), (N^n, h) be two Riemann-manifold, and vol be the volumn-form on M^m, then the energy

$$\mathcal{E}(f) = \int_M \underline{\|df\|_h^2 vol}.$$

Q: 这个structure 感觉很奇怪, $\|df\|_h^2$ 是在 (f^*V, f^*h) 上做么?

$\|\cdot\|$ is the Hilbert-Schmidt norm given as:

$$\|d_p f\|_h^2 = \sum_{i=1}^m \|d_p f(e_i)\|_h^2, \text{ where } \underline{e_1 \sim e_m}$$

正交基

is an orthonormal basis of T_pM.

$$\mathcal{E}: C^\infty(M, N) \longrightarrow \mathbb{R}$$
$$f \longmapsto \int_M \|df\|^2 vol.$$

Example: (Σ, g) be a Riemann surface with compatible metric g, i.e. $g(Jx, Jy) = g(x, y)$.

Then for $g = e^{2\lambda}(dx^2 + dy^2)$ locally where $\lambda : U \to \mathbb{R}$

then $e_1 = e^{-\lambda}\frac{\partial}{\partial x}$, $e_2 = e^{-\lambda}\frac{\partial}{\partial y}$ is orthonoral-basis.

then:

$$\|d_p f\|_h^2 = \|d_p f(e_1)\|_h^2 + \|d_p f(e_2)\|_h^2$$

$df = f_x dx + f_y dy$, RHS $= \|e^{-\lambda} f_x\|_h^2 + \|e^{-\lambda} f_y\|_h^2$

$$= e^{-2\lambda}(\|f_x\|_h^2 + \|f_y\|_h^2)$$

the $vol_g = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n$. the matrix $\begin{pmatrix} e^{2\lambda} & \\ & e^{2\lambda} \end{pmatrix}$

$$= e^{2\lambda} dx \wedge dy$$

$\therefore$ Locally: $\int_U \|df\|^2 vol = \int_U e^{-2\lambda}(\|f_x\|_h^2 + \|f_y\|_h^2) \cdot e^{2\lambda} dx \wedge dy$

$$= \int_U (\|f_x\|_h^2 + \|f_y\|_h^2)\, dx \wedge dy.$$

Rk: $h(*df \wedge df) = \|d_p f\|_h^2 vol$

Prop: The Energy of harmonic-map 不依赖于 g, 只依赖于 h 之选取.

Def. Energy is a functional: $\mathcal{E}$

$$C^\infty(M,N) \longrightarrow \mathbb{C}$$

$$\mathcal{E}: \quad f \longmapsto \int_M \|df\|^2 vol$$ 看成一个泛函.

The critical points of $\mathcal{E}$ are called harmonic map.

(即对1变分: $[0,1] \longrightarrow f_t \in C^\infty(M,N)$,
满足: $\frac{d}{dt}\Big|_{t=0} \mathcal{E}(f_t) = 0$)

$f: \Sigma \to S^3$ as a harmonic-map. 若M是R.S.且isometry
$\to$ 移M minimal.

Thm: $f: \Sigma \to S^3$ is a harmonic map
$\iff d^\nabla * \theta = 0$, $\theta = \frac{1}{2} f^{-1} df$.

$(C^\infty(\Sigma, SU(2)))$

把S^3看成$SU(2)$, let $\theta = \frac{1}{2} f^{-1} df \in \Omega^1(\Sigma, \mathfrak{su}(2))$, then:

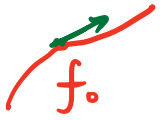
$d\theta + [\theta,\theta] = 0$, we have $d(f^{-1}f) = (df^{-1}) \cdot f + f^{-1} df = 0$

$\to df^{-1} = -f^{-1} df \cdot f^{-1} \Rightarrow d\theta = \frac{1}{2} df^{-1} \wedge df = -\frac{1}{2} f^{-1} df \cdot f^{-1} df$

$[\theta,\theta] = \frac{1}{2} f^{-1} df f^{-1} df \Rightarrow d\theta + [\theta,\theta] = 0$

In this case, take $\nabla = d + \theta$, we have $d^\nabla \theta = 0$.

pf: $(\Rightarrow)$



consider a variation $f_t: (-\varepsilon,\varepsilon) \times \Sigma \to SU(2)$

where $f_0 = f$, denote: $\dot{f} \triangleq \frac{d}{dt} f_t \in \Omega^1(\mathfrak{gl}(2,\mathbb{C}))$

It is easy to see: $f_0^{-1} \cdot \dot{f} \in \Omega^1(\mathfrak{su}(2,\mathbb{C}))$.

and : $\theta_t = \frac{1}{2} f_t^{-1} df_t$, then:

$$\dot{\theta} = (f^{-1}\dot{df})/2 = (-f^{-1}\dot{f}f^{-1}df + f^{-1}\dot{df})/2$$

now: $\varepsilon(f) = \int_M \|df\|_h^2 \, vol$ thus for volumn-form $dx\,dy$,

we have : $\|df\|_h^2 vol = (\langle f_x, f_x\rangle|_p + \langle f_y, f_y\rangle|_p)\, dx\wedge dy$

$$= \frac{-1}{2}\left[tr(f^{-1}f_x(f^{-1}f_x)) + tr(f^{-1}f_y(f^{-1}f_y))\right] dx\wedge dy$$

$\theta = \frac{1}{2} f^{-1}df = \frac{1}{2}(f^{-1}f_x dx + f^{-1}f_y dy)$. $\quad *dx = -dy$, $*dy = dx$

$\therefore *\theta = \frac{1}{2}(f^{-1}f_y dx - f^{-1}f_x dy)$

$$\therefore \theta\wedge *\theta = \frac{-1}{2}(f^{-1}f_x f^{-1}f_x + f^{-1}f_y f^{-1}f_y)\, dx\wedge dy$$

$$\varepsilon(f) = \int_M \|df\|_h^2 vol = \int_M tr(\theta\wedge *\theta)$$

thus:

$$d_f\varepsilon(f) = 2\int_M tr(\dot{\theta}\wedge *\theta) = 2\int_M tr\left(-f^{-1}\dot{df}f^{-1}df\wedge *\theta + f^{-1}\dot{df}\wedge *\theta\right)$$

By stokes': $d\,tr(f^{-1}\dot{f}\wedge *\theta) = tr\left(-f^{-1}\dot{df}f^{-1}df\wedge *\theta + f^{-1}\dot{df}\wedge *\theta\right) + tr(f^{-1}\dot{f}\wedge d*\theta)$

$\therefore RHS = -2\int_M tr(f^{-1}\dot{f}\wedge d*\theta)$, also we have:

$\theta\wedge *\theta = 0 \quad \therefore RHS = -2\int_M tr(f^{-1}\dot{f}\wedge d^{\nabla}*\theta)$

$\therefore d_f\varepsilon(f) = 0 \iff d^{\nabla}*\theta = 0$.

拆分: $\theta = \Phi + \Phi^*$, $\Phi \in \Gamma(K_{sl(2,\mathbb{C})})$, $\Phi^* \in \Gamma(\bar{K}_{sl(2,\mathbb{C})})$

$\Phi = \frac{1}{2}(\theta - i*\theta)$, $\Phi^* = \frac{1}{2}(\theta + i*\theta)$

Thus: $\mathcal{E}(f) = \int_M tr(\theta \wedge *\theta) = \int_M tr((\Phi+\Phi^*)\wedge(i\Phi - i\Phi^*))$

$$= i\int_M tr(\Phi^2) - i\int_M tr(\Phi^{*2}) + \underbrace{i\int_M tr(\Phi\wedge\Phi^*) + tr(\Phi^*\wedge\Phi)}_{= 2i\int_M tr(\Phi\wedge\Phi^*)}$$

Rk: $\mathcal{E}(f) = \int_M \|df\|^2 vol = \int_M tr(\theta \wedge *\theta)$

$\theta = \Phi + \Phi^*$. $\Sigma \xrightarrow{f} SU(2)$, the induced metric is given by f^*h, we can compute that:

$$f^*h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = h(f_x, f_x) = \frac{1}{2}(f^{-1}f_x f^{-1}f_x)$$

$$\Rightarrow f^*h = -\frac{1}{2}(f^{-1}f_x f^{-1}f_x)\, dx\otimes dx + (\cdots) - \frac{1}{2}(f^{-1}f_y f^{-1}f_y)\, dy\otimes dy.$$

$\theta = \frac{1}{2}f^{-1}f_x dx + \frac{1}{2}f^{-1}f_y dy$ $\quad\therefore\ -2\theta\otimes\theta^* = f^*h.$

$$g_f = -2tr(\theta^2)$$

$$= -2tr\left(\frac{\Phi^2 + \Phi\Phi^* + \Phi^*\Phi + \Phi^*\Phi^*}{4}\right) = -\frac{1}{2}\Big(tr(\Phi^2) + \underbrace{tr(\Phi\Phi^*) + tr(\Phi^*\Phi)}_{tr(\Phi\wedge\Phi^*)} + tr(\Phi^*\Phi^*)\Big)$$

$\therefore$ HW: g_f is compatable $\Leftrightarrow tr(\Phi^2) = 0$.

pf: $g_f(J\cdot, J\cdot) = -\frac{1}{2}\Big(tr(\Phi^2)\cdot(-1) + tr(\Phi\Phi^*) + tr(\Phi^*\Phi) + tr(\Phi^*\Phi^*)\cdot(-1)\Big)$

$$= g_f(\cdot,\cdot)$$

$\therefore \Rightarrow tr(\Phi^2) = 0$, 反之亦然

#

Def: (M, g), (N, h) 为 Riemann-metric, 称

$f: M \to N$ is conformal if $g = f^* h$.

RK: $f: \Sigma \to SU(2)$ is conformal if $tr(\Phi \otimes \Phi) = 0$

where $\theta = \Phi + \Phi^* = \frac{1}{2} f^{-1} df$.

Def: f is a minimal if: $(M^2, g) \to (N, g)$ is:

(1) harmonic

(2) conformal

RK: For $f: \Sigma \to SU(2)$, f is harmonic $\Leftrightarrow d^{\nabla} * \theta = 0$

f is conformal $\Leftrightarrow tr(\Phi^2) = 0$ $\quad \left(\mathcal{E}(f) = \frac{-i}{2} \int_\Sigma tr(\Phi \wedge \Phi^*) \right)$

$\Leftrightarrow \det \Phi = 0$

$\Leftrightarrow \Phi$ is nilpotent.

Example: (Hopf-Differential)

For $f: \Sigma \to SU(2)$ minimal, then $\det \Phi = 0$, consider $L \hookrightarrow V$

be the kernal bundle of Φ.

then $\Phi: V/L \cong KL$, $\Phi = (\theta)^{1,0} = (f^{-1} df / 2)^{(1,0)}$

$\boxed{= \frac{f^{-1} f_z}{2} dz.}$

As f immersion $\Rightarrow \Phi$ is non-vanish.

$\therefore \Phi \in H^0(KL^2)$ without zeros.

$\therefore KL^2$ is trivial-bundle

let $S = L^*$, then $S^2 = K \to L^*$ is a spin-bundle.

For $\nabla : S \oplus S^* \to \Omega^1(\Sigma) \otimes (S \oplus S^*)$

$$\nabla = \begin{pmatrix} \nabla^S & -q^* \\ q & \nabla^{S^*} \end{pmatrix}$$

q 称为 hopf-diff of the minimal surface f.

$(1,0)$-part of the second fundamental-form.

(The Associated Family of Flat Connections)

For $f : \Sigma \to SU(2)$ with $\theta = \frac{1}{2} f^{-1} df$, we have:

$$f \text{ is harmonic} \Leftrightarrow \nabla^\lambda = \nabla + \lambda^{-1}\Phi + \lambda \Phi^* \text{ is flat.}$$

pf: $(\Leftarrow)$ $F^{\nabla^\lambda} = F^\nabla + d^\nabla(\lambda^{-1}\Phi) + d^\nabla(\lambda \Phi^*) + [\lambda^{-1}\Phi + \lambda\Phi^*, \lambda^{-1}\Phi + \lambda\Phi^*]$

$= F^\nabla + \lambda^{-1} d^\nabla \Phi + \lambda d^\nabla \Phi^* + [\Phi, \Phi^*] + [\Phi^*, \Phi]$

$= \lambda^{-1} d^\nabla \Phi + \lambda d^\nabla \Phi^*$ $\quad \therefore F^{\nabla^\lambda} = 0 \Rightarrow d^\nabla \Phi = d^\nabla \Phi^* = 0.$

$\therefore d^\nabla * \theta = 0.$

$(\Rightarrow)$ f harmonic $\Rightarrow d^\nabla \Phi = d^\nabla \Phi^* = 0$

#

Flat Connection 与 Harmonic-Map 之关系

首先：$f: \Sigma \to S^3 = SU(2)$ harmonic $\Rightarrow$ 可以得到一族 flat-connections

$\nabla^\lambda = \nabla + \lambda^{-1}\Phi + \lambda\bar{\Phi}^*$，$\nabla = d + \frac{1}{2}f^{-1}df$，$\frac{1}{2}f^{-1}df = \Phi + \bar{\Phi}^*$

$$F^{\nabla^\lambda} = \underbrace{F^\nabla}_{=0} + \lambda^{-1}\underbrace{d^\nabla\Phi}_{0} + \lambda\underbrace{d^\nabla\bar{\Phi}^*}_{0} + [\lambda^{-1}\Phi + \lambda\bar{\Phi}^*, \lambda^{-1}\Phi + \lambda\bar{\Phi}^*]$$

$d^\nabla\theta = 0$

$d^{\nabla *}\theta = 0$

$\nabla^{-1} = d$，$\nabla^1 = \nabla^{-1}f \rightsquigarrow F^{\nabla^1} = 0$ $\therefore F^{\nabla + \Phi + \bar{\Phi}^*} = F^\nabla + [\Phi + \bar{\Phi}^*, \Phi + \bar{\Phi}^*] = 0$

反之：给一族特殊的 flat-connections，亦可以构造 harmonic $f: \Sigma \to SU(2)$.

Thm:

If $\lambda \in \mathbb{C}^* \longrightarrow \nabla^\lambda = \nabla + \lambda^{-1}\Phi + \lambda\Psi$ is a $\mathbb{C}^*$-family of flat conn

s.t.h:

1. $\forall \lambda \in S^1$, ∇^λ is unitary
2. ∇^λ is trivial for $\lambda = \pm 1$

unitary

$\therefore \exists$ a harmonic f, s.t. ∇^λ 为 f 生成的一族 flat-conn.

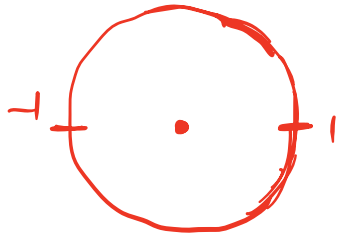
Condition 的含义:

1. $\forall \lambda \in S^1$, ∇^λ is unitary $\Longrightarrow \Psi = \bar{\Phi}^*$

$\Downarrow$

the gauge-trans between $(f: \Sigma \to SU(2))$

∇^1, ∇^{-1} 是 $SU(2)$-valued.



$\nabla^{-1} = d + \frac{1}{2}f^{-1}df + (-\Phi - \bar{\Phi}^*)$

$\nabla^1 = \nabla^{-1}f = \nabla^{-1} + f^{-1}df = d + f^{-1}df$

这上是缺细节的：Why $\Psi = \bar{\Phi}^*$?

Theorem 6. *Let $\lambda \in \mathbb{C}^* \mapsto \tilde{\nabla}^\lambda$ be a holomorphic family of flat $\mathrm{SL}(2,\mathbb{C})$-connections on a rank 2 bundle $V \to M$ over a compact Riemann surface M of genus $g \geq 2$ such that*

- *the asymptotic at $\lambda = 0$ is given by*
$$\tilde{\nabla}^\lambda \sim \lambda^{-1}\Psi + \tilde{\nabla} + \ldots$$
where $\Psi \in \Gamma(M, K\,\mathrm{End}_0(V))$ is nowhere vanishing and nilpotent;
- *for all $\lambda \in S^1 \subset \mathbb{C}$ there is a hermitian metric on V such that $\tilde{\nabla}^\lambda$ is unitary with respect to this metric;*
- *$\tilde{\nabla}^\lambda$ is trivial for $\lambda = \pm 1$.*

Then there exists a unique (up to spherical isometries) minimal surface $f\colon M \to S^3$ such that its associated family of flat connections ∇^λ and the family $\tilde{\nabla}^\lambda$ are gauge equivalent, i.e., there exists a λ-dependent holomorphic family of gauge transformations g which extends through $\lambda = 0$ such that $\nabla^\lambda \cdot g = \tilde{\nabla}^\lambda$.

pf: $\tilde{\nabla}^\lambda$ is asymptotic means $(\tilde{\nabla}^\lambda)''$ is stable, hence $\tilde{\nabla}^\lambda$ is irreducible for generatic $\lambda \in \mathbb{C}^*$. Thus, the hermitian-metric for $\tilde{\nabla}^\lambda$ is unique up to constant multiplication. If requiring the unit det $(\ ,\ )^\lambda$ is real-analytic on generatic $\lambda \in \mathbb{C}^*$.

W.L.O.G. on $V = X \times \mathbb{C}^2$, given the standard hermitian metric $h = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$, thus: $(\ ,\ )^\lambda$ is given by a metric $[A(\lambda)] \in SL/SU(2)$, where $(\ ,\ )^\lambda = \overline{A(\lambda)}^t A(\lambda)$

$\therefore$ We can have $h \in \Gamma(S^1 \times M, SL(2,\mathbb{C}))$. $h(\lambda) = A(\lambda)$

do the Iwasawa-Decompo for $A(\lambda)^{-1}$:

$$A(\lambda)^{-1} = B(\lambda)F(\lambda), \quad B \in \Lambda^{+,b}SL, \quad F \in \Lambda SU(2)$$

We obtain a new gauging: $\nabla^\lambda = \tilde{\nabla}^\lambda \cdot B(\lambda)$

claim: ∇^λ is unitary respect to $(\ ,\ )$, the

standard hermitian-metric.

Then:

$$dh(s_1, s_2) = dh_\lambda(A(\lambda)^{-1}s_1, A(\lambda)^{-1}s_2)$$

$$= h_\lambda(\tilde{\nabla}^\lambda(A(\lambda)^{-1}s_1), A^{-1}(\lambda)s_2) + h_\lambda(A(\lambda)^{-1}s_1, \tilde{\nabla}^\lambda(A(\lambda)^{-1}s_2))$$

Well, $h_\lambda(\tilde{\nabla}^\lambda(A(\lambda)^{-1}s_1), A^{-1}(\lambda)s_2) = h(A(\lambda)\tilde{\nabla}^\lambda(A(\lambda)^{-1}s_1), s_2)$

$\tilde{\nabla}^\lambda = B \circ \nabla^\lambda \circ B^{-1}$, $\therefore A(\lambda)\tilde{\nabla}^\lambda(A(\lambda)^{-1}s_1) = A(\lambda)B\nabla^\lambda B^{-1}A^{-1}s_1$

$$= F^{-1}\nabla^\lambda(Fs_1)$$

$\therefore RHS = h(F^{-1}\nabla^\lambda(Fs_1), s_2) + h(s_1, F^{-1}\nabla^\lambda(Fs_2))$

As $F \in ASU(2) \longrightarrow F^{-1} = F^*$

$\therefore RHS = h(\nabla^\lambda(Fs_1), Fs_2) + h(Fs_1, \nabla^\lambda(Fs_2))$

$= dh(s_1, s_2)$

$\therefore dh(F^{-1}s_1, F^{-1}s_2) = dh(s_1, s_2) = h(\nabla^\lambda s_1, s_2) + h(s_1, \nabla^\lambda s_2)$

$\therefore$ By Schwarz-reflection for ∇^λ

$(\nabla^{\bar{\lambda}^{-1}})^* = \nabla^\lambda \longrightarrow$ compare the coefficient, we (for $\nabla^\lambda = \nabla + \lambda^{-1}\Phi + A(\lambda)\Psi$) can deduce the harmonicity.

Rk: 在这里，gauging的意义在于 $\hat{\nabla}^\lambda$ is unita 是针对不同的 hermitian-metric，故须给出一个 new metric 使得 new ∇^λ 关于此以及一个 hermitian metric unitary 即可.

☆: Hitchin 给出了一个这样的一个 idea:

把所有的从 manifold X 到 compact Lie Group G 的映射看成 X 上的 Principal G-bundle with a field.

故我们得到了一组 "1-1" correspondence:

$$\{\text{Harmonic Maps From R.S. into } S^3\}$$

$$\updownarrow \quad ☆$$

$$\left\{\begin{array}{l} \mathbb{C}^*\text{-family of flat connections, s.t.} \\ \quad (1)\ \text{harmonicity}: \nabla^\lambda = \nabla + \lambda^{-1}\Phi + \lambda\Psi \\ \quad (2)\ \text{intrining closing}: \nabla^\lambda \text{ unitary for } \lambda \in S^1 \\ \quad (3)\ \text{extrising closing}: \nabla^{\pm 1} \text{ is trivial.} \end{array}\right\}$$

Thm: 若 ☆ 中的 flat-connections 均 trivial, $\forall \lambda \in \mathbb{C}^*$, then

$$f: \Sigma \to SU(2) = S^3 \quad \text{conformally}.$$

Pf: To show conformal: $tr(\Phi^2) = 0 \Leftrightarrow \det(\Phi) = 0$ (As $\Phi \in \mathfrak{su}(2)$)

$\mathbb{C}^* \longrightarrow$ Connections

$\lambda \longmapsto \nabla^\lambda$

For this 1-dim Fredholm-operator, we obtain a line bundle L on $\mathbb{C}^*$, $L_\lambda = \ker(\bar\partial^{\nabla^\lambda})$

希望延拓定义到 $z=0$

Q: 为什么可以延拓定义至 $z=0$?

Let $s \in H^0(U, L)$, $0 \in U$, $s(0) \neq 0$

expand:

$s(\lambda) = s_0 + \lambda s_1 + \cdots$

$\therefore \bar\partial^\lambda s(\lambda) = 0$, $\forall \lambda \in \mathbb{C}$, $s(\lambda)$ is parallel

$(\partial + \lambda^{-1}\bar\Phi) s(\lambda) = 0$

$\therefore \bar\Phi(s_0) = 0$ $\quad \therefore \det(\bar\Phi) = 0$ $\quad \therefore \bar\Phi^2 = 0$

$\bar\partial^\nabla s_0 = 0$

$\partial s_0 + \bar\Phi s_1 = 0$

define $L \subseteq \ker \bar\Phi$, then: $L = \mathrm{Span}\{s_0\}$

$\therefore \nabla s_0 = \partial^\nabla s_0 + \bar\partial^\nabla s_0 = -\bar\Phi s_1 \in L \otimes \Omega^1$

As ∇ is unitary $\rightarrow$ $\nabla = \begin{pmatrix} \nabla^L & \\ & \nabla^{L^\perp} \end{pmatrix}$ w.r.t. $L \oplus L^\perp$

$\bar\Phi = \begin{pmatrix} 0 & \varphi \\ 0 & 0 \end{pmatrix}$

$\therefore \nabla^\lambda = \begin{pmatrix} \nabla^L & \\ & \nabla^{L^\perp} \end{pmatrix} + \lambda^{-1}\begin{pmatrix} 0 & \varphi \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 \\ \varphi^* & 0 \end{pmatrix}$

证毕. #

Thm: Locally there are many harmonic maps from $\mathbb{D}$ into S^3.

Pf: Start with $\tilde\Phi \in H^0(\mathfrak{ks}(12, \mathbb{C}))$, $\tilde\Phi \neq 0$. $\tilde\nabla^\lambda = d + \lambda^{-1}\tilde\Phi$ flat $\lambda \in \mathbb{C}^*$

Let $F(\lambda)$ be a holo-family of parallel frame of $\tilde\nabla^\lambda$, $F_{p_0}(\lambda) = id$.

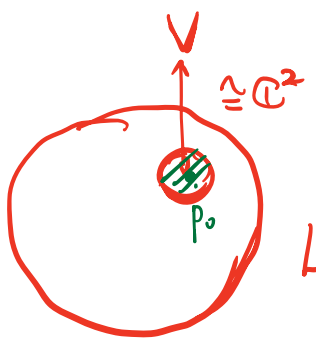
(局部上可parallel-frame)

(局部上给出满足(12)(2)(3)的flat-connection)

对parallel-frame作Iwasawa-Decompo.

Then $F(\lambda) = p(\lambda)\cdot u(\lambda)$ $\Lambda\, su(2,\mathbb{C})$

$\Lambda^{+,b} sL(2,\mathbb{C})$



then: $\tilde{\nabla}^{\lambda}\cdot F(\lambda) = d$

$\therefore\ \tilde{\nabla}^{\lambda}\cdot p(\lambda) = d\cdot u(\lambda)^{-1}$

$LHS = d + p(\lambda)^{-1}\cdot\lambda^{-1}\tilde{\Phi}\cdot p(\lambda) + p(\lambda)^{-1}d(p(\lambda)) = d\cdot u(\lambda)^{-1}$

$= d + \lambda^{-1}p(\lambda)^{-1}\tilde{\Phi}\,p(\lambda) + p(\lambda)^{-1}d(p(\lambda)) = d\cdot u(\lambda)^{-1}$ (2) unitary on $\lambda\in S^1$

Take $\Phi = p(0)^{-1}\tilde{\Phi}\,p(0)$, then:

$p(\lambda)$: S^1

(1) harmonicity $LHS = d + \lambda^{-1}\Phi + \cdots$ (just对λ Taylor 展开)

$\because D$ simply connected $\therefore \exists$ gauge between ∇^{-1} and ∇!

HW: There are infinitely many "f" from above construction.

notice that: $\nabla^{-1}\cdot f = \nabla^{1}$. $\therefore\ f^{-1}df = 2(\Phi + \Phi^{*})$

As $\Phi = p(0)^{-1}\tilde{\Phi}\,p(0)$ $\therefore \det\Phi = \det\tilde{\Phi}$, if f' is conjugate with f

$\therefore$ pick $\tilde{\Phi}\in H^0(ksl)$, just multiply $\tilde{\Phi}$ with different complex number.

Q: 这个习题的意义到底在哪里? $\nabla^{\lambda} \xrightarrow{\text{produce}} f$.

take $\tilde{\nabla}^{\lambda} = \nabla^{\lambda}h \xrightarrow{\text{produce}} h^{-1}fh$, In this case, we can give infinte such harmonic just by let $h\in SU(2)$ ✓

Thm: Let ∇^λ be associ family of flat connections given by some harmonic $f: \Sigma \to SU(2)$, and assume that:

a) Σ is compact

b) ∇^λ is irreducible for some $\lambda \in \mathbb{C}^*$

c) ∇^{λ_0} is reducible for some $\lambda_0 \in \mathbb{C}^*$

Then, $\exists$ a different harmonic $\tilde{f}: \Sigma \to SU(2)$ with asso family $\tilde{\nabla}^\lambda$, s.t. $\tilde{\nabla}^\lambda$ and ∇^λ are gauge-equivalent, $\forall \lambda \in \mathbb{C}^* \setminus \{\lambda_0, \bar{\lambda}_0^{-1}\}$

pf: Let L be the parallel subbundle for ∇^{λ_0} by c), $V = L \oplus L^\perp$

Define: $d(\lambda) = \Pi_L + \frac{1-\bar{\lambda}_0^{-1}}{1-\lambda_0} \cdot \frac{\lambda-\lambda_0}{\lambda-\bar{\lambda}_0^{-1}} \Pi_{L^\perp}$, then:

(1) $d(\lambda)$ is unitary along $\lambda \in S^1$:

$$\frac{1-\bar{\lambda}_0^{-1}}{1-\lambda_0} \cdot \frac{\lambda-\lambda_0}{\lambda-\bar{\lambda}_0^{-1}} \cdot \frac{1-\lambda_0^{-1}}{1-\bar{\lambda}_0} \cdot \frac{\bar{\lambda}-\bar{\lambda}_0}{\bar{\lambda}-\lambda_0^{-1}}$$

$$= \frac{\bar{\lambda}_0 - 1}{\lambda\bar{\lambda}_0 - 1} \cdot \frac{\lambda-\lambda_0}{1-\lambda_0} \cdot \frac{\lambda_0 - 1}{\bar{\lambda}\cdot\lambda_0 - 1} \cdot \frac{\bar{\lambda}-\bar{\lambda}_0}{1-\bar{\lambda}_0} = \frac{|\lambda|^2 - \lambda\bar{\lambda}_0 - \lambda_0\bar{\lambda} + |\lambda_0|^2}{(\lambda\bar{\lambda}_0 - 1)(\bar{\lambda}\lambda_0 - 1)}$$

For $\lambda = e^{i\theta}$, RHS $= 1$ $\therefore$ $d(\lambda)$ is unitary.

在 $\lambda = \lambda_0$ 处对 ∇^λ 做 Taylor 展开:

$$\nabla^\lambda \underset{\lambda_0}{\sim} \nabla^{\lambda_0} + (\lambda - \lambda_0) \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} + \cdots\cdots$$

Here:

$$\nabla^{\lambda_0} = \begin{pmatrix} \nabla^L & \omega \\ 0 & \nabla^{L^*} \end{pmatrix} \quad \text{in} \quad V = L \oplus L^\perp$$

take $c(\lambda) = \dfrac{1-\bar{\lambda}_0^{-1}}{1-\lambda_0} \cdot \dfrac{\lambda-\lambda_0}{\lambda-\bar{\lambda}_0^{-1}}$

then $\nabla^\lambda \cdot d(\lambda) = \begin{pmatrix} \nabla^L & 0 \\ \frac{\gamma(\lambda-\lambda_0)}{c(\lambda)} & \nabla^{L^*} \end{pmatrix} + (\lambda-\lambda_0)(\cdots)$

$\therefore$ this gauge is globally defined.

denote $\tilde{\nabla}^\lambda = \nabla^\lambda \cdot d(\lambda) \longrightarrow$ We obtain a new family of flat connections.

(1) $\tilde{\nabla}^\lambda = \hat{\nabla} + \lambda^{-1}\hat{\Phi} + \cdots$

(2) $\tilde{\nabla}^\lambda$ unitary for $\lambda \in S^1$

(3) $\tilde{\nabla}^{\pm 1}$ trivial

$\Rightarrow$ We obtain a new harmonic-map $\tilde{f}$.

claim: $\tilde{f} \neq f$.

Argue: if $\tilde{f} = f$ (up to conjugation) $\rightarrow \tilde{\nabla}^\lambda = \nabla^\lambda \cdot g$ for some constant g.

Def: (Gauge-Bundle)

If (V, ∇_1), (V, ∇_2) are same vector bundle with different connections ∇_i, the Gauge bundle is defined to be

$$P = \{\phi \in \mathrm{End}(V) \mid \nabla_2(\phi s) = \phi \nabla_1 s\}$$

Prop: The irreducibility is an open condition:

For some $\lambda_1 \in \mathbb{C}^*$, ∇^{λ_1} is irreducible, then: for generatic $\lambda \in \mathbb{C}^*$, ∇^{λ} is irreducible.

Prop: The gauge between 2 irreducible gauge equivalent connections ∇^{λ} and $\tilde{\nabla}^{\lambda}$ is unique up to a constant.

否则: h_1, h_2 be the gauge: $\nabla^{\lambda} h_1 = \tilde{\nabla}^{\lambda}$, $\nabla^{\lambda} h_2 = \tilde{\nabla}^{\lambda}$, $h_1 - ch_2$ has non-trivial kernal: $\therefore$ kernal is V. ✓

For generic $\lambda \in S^1$, ∇^λ is irreducible (and unitary)

$\therefore$ for generic $\lambda \in \mathbb{C}$, $\bar\partial^\lambda := \bar\partial^\nabla - \lambda \bar\varphi^*$ is stable.

Q: 只能找到generatic $\lambda \in S^1$, $\bar\partial^\nabla - \lambda\bar\varphi^*$ is stable?

(semi-stable + irreducible $\Rightarrow$ stable)

否则由之前的HW, $V = L \oplus L$, 可以给 connection compactible with $\bar\partial^\nabla$, s.t. irreduci fails.

$\therefore$ $\mathcal{L}$ can extend to $\mathbb{C}P^1$.

$\downarrow$

$\mathbb{C}^*$

claim: $\deg \mathcal{L} = -1$

pf: d has 1 pole, 0 zeros.

$\therefore$ there cannot be constant g, s.t.

$$\tilde\nabla^\lambda = \nabla^\lambda g$$

#.

Thm: Let $f: \Sigma \to SU(2)$ harmonic $\rightsquigarrow$ ∇^λ flat-connect

1) Σ is compact

2) ∇^λ is irredu for some $\lambda_0 \in \mathbb{C}^*$ $\rightsquigarrow$ ① semi stable 不合 $\rightsquigarrow$ stable. ② $\mathrm{Hom}((V,\bar\partial^1),(V,\bar\partial^2))$ has dim 1.

3) $\nabla^{-\lambda} \cong \nabla^\lambda$ for generatic $\lambda \in \mathbb{C}^*$

Then: $\exists\, \hat f$ harmonic: $\Sigma \to SU(2)$, s.t. $\tilde\nabla^\lambda \cong \nabla^\lambda$ for generatic $\lambda \in \mathbb{C}^*$.

pf: $\nabla^{-\lambda} \cong \nabla^\lambda g$ $\qquad \nabla^\lambda \cong \nabla^{-\lambda} g \cong \nabla^\lambda g^2$. $\therefore g^2 = id$.

($g^2 \in \mathrm{Hom}((V,\nabla^\lambda),(V,\nabla^\lambda))$ which is parallel, as (V,∇^λ) stable,

$\therefore \mathrm{Hom}(V,\nabla^\lambda),(V,\nabla^\lambda)$ stable only has 1-dim.

$\therefore g^2 = \pm id$

if $g^2 = id \to g = -id$, $\nabla^\lambda = \nabla^{-\lambda}$, $\forall \lambda \in \mathbb{C}^*$

$\therefore \nabla^1 = \nabla^{-1}$ $\therefore \Phi = 0 \Rightarrow f =$ constant-map.

if $g^2 = -id$

Define the gauge-line-bundle $\mathcal{L} \to \mathbb{CP}^1$, then $\deg \mathcal{L} \le 0$

by applying dressing, we can increase the degree, after finite step, we get the 1st-situation.

HW: Show that f has same energy as $\tilde{f}$.

Check:

$$\mathcal{E}(f) = \int_M tr(\theta \wedge *\theta) = \frac{1}{4}\int_M tr(\Phi \wedge \Phi^*)$$

notice that:

Φ is the first term of $\nabla' = \nabla + \lambda^{-1}\Phi + \lambda\Phi^*$

as $\tilde{\nabla}' = \nabla' d(\lambda)$

$\therefore \tilde{\Phi} = d(\lambda)^{-1}\Phi d(\lambda)$

$\therefore tr(\tilde{\Phi}) = tr(d(\lambda)^{-1}\Phi d(\lambda)) = tr(\Phi)$

$\therefore \mathcal{E}(f) = \mathcal{E}(\tilde{f})$

#.

Question:

∇^λ irreducible 到底意味着什么?

首先若 $\nabla^\lambda \cong \tilde{\nabla}^\lambda$ irreduci, then $Hom((V,\nabla^\lambda),(V,\tilde{\nabla}^\lambda))$ 是 exactly 1-dim 的 $\rightsquigarrow L_\lambda := Hom((V,\nabla^\lambda),(V,\tilde{\nabla}^\lambda))$

那么对 $\lambda \in \mathbb{C}^*$ reducible, L_λ 是什么呢?, dressing 的意义在哪儿呢?

① 若 ∇^λ irreduci, 那么 $\nabla^\lambda \cdot d(\lambda)$ 还是 irreducible 么?

② 若 ∇^{λ_0} reducible $\Rightarrow$ $\nabla^{\lambda_0} \cdot d(\lambda)$ 就 irreducible 了么?

Theorem 7. *Let $f, \tilde{f}\colon M \to S^3$ be two conformal minimal immersions from a compact Riemann surface of genus $g \geq 2$ together with their associated families of flat connections ∇^λ and $\hat{\nabla}^\lambda$. Assume that ∇^λ is gauge equivalent to $\tilde{\nabla}^\lambda$ for generic $\lambda \in \mathbb{C}^*$. Then there exists a meromorphic map*

$$g\colon \mathbb{CP}^1 \to \Gamma(M, \mathrm{End}(V))$$

such that $\nabla^\lambda \cdot g = \tilde{\nabla}^\lambda$. This map g is holomorphic and takes values in the invertible endomorphisms away from those $\lambda_0 \in \mathbb{C}^*$ where ∇^{λ_0} or equivalently $\tilde{\nabla}^{\lambda_0}$ is reducible.

The space of such dressing deformations of surfaces $f \mapsto \tilde{f}$ is generated by simple factor dressing, i.e., by maps $d\colon \mathbb{CP}^1 \to \Gamma(M, \mathrm{End}(V))$ of the form

$$d(\lambda) = \pi^L + \frac{1-\bar{\lambda}_0^{-1}}{1-\lambda_0}\frac{\lambda-\lambda_0}{\lambda-\bar{\lambda}_0^{-1}}\pi^{L^\perp},$$

where L is an eigenline bundle of the connection ∇^{λ_0} and $L^\perp$ is its orthogonal complement.

Proof. We first show that ∇^λ and $\tilde{\nabla}^\lambda$ are gauge equivalent away from those $\lambda_0 \in \mathbb{C}^*$ where ∇^{λ_0} or $\tilde{\nabla}^{\lambda_0}$ is reducible. The gauge between two irreducible gauge equivalent connections ∇^λ and $\tilde{\nabla}^\lambda$ is unique up to a constant multiple of the identity. Moreover, multiples of this gauge are the only parallel endomorphisms with respect to the connection $\nabla^\lambda \otimes (\tilde{\nabla}^\lambda)^*$. As the connections depend holomorphic on λ there exists a holomorphic line bundle $\mathcal{G} \to \mathbb{C}^*$ whose line at $\lambda \in \mathbb{C}^*$ is a subset of the parallel endomorphisms (and coincides with it whenever ∇^λ or equivalently ∇^λ is irreducible). A non-vanishing section $g \in \Gamma(U, \mathcal{G})$ around $\lambda \in U \subset \mathbb{C}^*$ gives rise to the gauge between ∇^λ and $\tilde{\nabla}^\lambda$ as long as g^λ is an isomorphism. This can fail only in the case where ∇^λ or equivalently $\tilde{\nabla}^\lambda$ is reducible. We need to prove that g extends holomorphically to an isomorphism at $\lambda = 0$. Then, as g is unitary along the unit circle, g also extends holomorphically to an isomorphism at $\lambda = \infty$ by the Schwarz reflection principle. Note that locally around $\lambda = 0$ all connections are irreducible and all holomorphic structures are stable away from $\lambda = 0$. Then, as above, there exists a family of gauge transformations g^λ which extend to a holomorphic endomorphism g^0 with respect to $\bar{\partial}^0 \otimes (\tilde{\bar{\partial}}^0)^*$. From the fact that the connections around $\lambda = 0$ are gauge equivalent and the expansions of the two families one deduces that g^0 is a holomorphic endomorphism between the stable pairs $(\tilde{\bar{\partial}}, \tilde{\Phi})$ and $(\bar{\partial}, \Phi)$, i.e., $\Phi \circ g^0 = g^0 \circ \tilde{\Phi}$. Therefore Proposition (3.15) of [H1] implies that g^0 is an isomorphism.

In order to find the globally defined dressing $g\colon \mathbb{CP}^1 \to \Gamma(M, \mathrm{End}(V))$ we first investigate the bundle $\mathcal{G} \to \mathbb{C}^*$. We have seen that it extends to $\lambda = 0$ holomorphically and by switching to anti-holomorphic structures, one can also show that it extends holomorphically to $\lambda = \infty$. Therefore there exists a meromorphic section $\tilde{g} \in \mathcal{M}(\mathbb{CP}^1, \mathcal{G})$ whose only poles are at $\lambda = \infty$. As $\mathcal{G}$ is a holomorphic subbundle of $\mathbb{C} \times \Gamma(M; \mathrm{End}(V))$ the determinant is a holomorphic map

$$\det\colon \mathcal{G} \to \mathbb{C}.$$

Consider the holomorphic function

$$h\colon \mathbb{C} \to \mathbb{C},\ h(\lambda) = \det(\tilde{g}^\lambda).$$

my understanding

若 ∇^{λ_0} reducible,

则 $\mathrm{Hom}((V, \nabla^{\lambda_0}), (V, \tilde{\nabla}^{\lambda_0}))$

不是 1-dim 的

但可以 extends to the pt

还没有处理 reducible pt,

延拓过去了没啥,

但延拓的 $\tilde{g}_{\lambda_0}$ 长什么样呢?

Note that we may assume that h is non-vanishing along the unit circle as $\tilde{g}$ is a complex multiple of a unitary gauge there. The Iwasawa decomposition $h = h_+ h_u$ determines functions $h_+ \colon \{\lambda \mid \lambda\bar{\lambda} \leq 1\} \to \mathbb{C}^*$ and $h_u \colon \mathbb{C}^* \to \mathbb{C}$ which satisfies $\| h_u(\lambda) \| = 1$ for $\lambda \in S^1$. These are unique up to unimodular constants. The square root $\sqrt{h_+}$ is then well-defined on $\{\lambda \mid \lambda\bar{\lambda} \leq 1\}$ and we define

$$g = \frac{1}{\sqrt{h_+}} \tilde{g} \in H^0(\{\lambda \mid \lambda\bar{\lambda} \leq 1\}, \mathcal{G}).$$

The determinant $\det g$ is unimodular along the unit circle, and therefore, g is unitary along the unit circle. By the Schwarz reflection principle, we obtain a meromorphic map $g \in \mathcal{M}(\mathbb{CP}^1, \mathcal{G})$ which satisfies $\nabla^\lambda \cdot g = \tilde{\nabla}^\lambda$ by construction.

It is shown in [BDLQ] that a simple factor dressing $\nabla^\lambda \mapsto \nabla^\lambda \cdot d$ makes the associated family of a new minimal surface. We want to show by induction that any g as above is the product of simple factor dressings. Note that $\det g \colon \mathbb{CP}^1 \to \mathbb{CP}^1$ is a rational function. If its degree is 0, then $\det g$ is a non-zero constant, and g is constant in λ. As it is unitary on the unit circle, g acts as a spherical isometry on the surface. Assume that $\det g$ has a zero at λ_0. As we have seen $\lambda_0 \in \mathbb{C}^* \setminus S^1$. By multiplying with (a power of) $\frac{1-\bar{\lambda}_0^{-1}}{1-\lambda_0} \frac{\lambda-\lambda_0}{\lambda-\bar{\lambda}_0^{-1}} \operatorname{Id}$ we can also assume that $g^{\lambda_0} \neq 0$. As g^{λ_0} is a non-zero parallel endomorphism with respect to $\nabla^{\lambda_0} \otimes (\tilde{\nabla}^{\lambda_0})$ we see that the line bundle $L \to M$, which is given by $L_p = \ker g_p^{\lambda_0}$ at generic points $p \in M$, is an eigenline bundle of $\tilde{\nabla}^{\lambda_0}$. As a consequence of the unitarity of $\tilde{\nabla}^\lambda$ along the unit circle, $L^\perp$ is an eigenline bundle of $\tilde{\nabla}^{\bar{\lambda}^{-1}}$. We can apply the simple factor dressing

$$d(\lambda) = \pi^{L^\perp} + \frac{1-\lambda_0}{1-\bar{\lambda}_0^{-1}} \frac{\lambda - \bar{\lambda}_0^{-1}}{\lambda - \lambda_0} \pi^L$$

to $\tilde{\nabla}^\lambda$. Then, the product gd is again a meromorphic family of gauge transformations which extends holomorphically through λ_0, and the degree of the rational function $\det(gd)$ is the degree of the rational function $\det(g)$ minus 1. $\square$

Harmonic Maps into $\mathbb{H}^3$

$\mathbb{R}^{3,1}$ is defined by $(\mathbb{R}^4, \cdot)$, where:

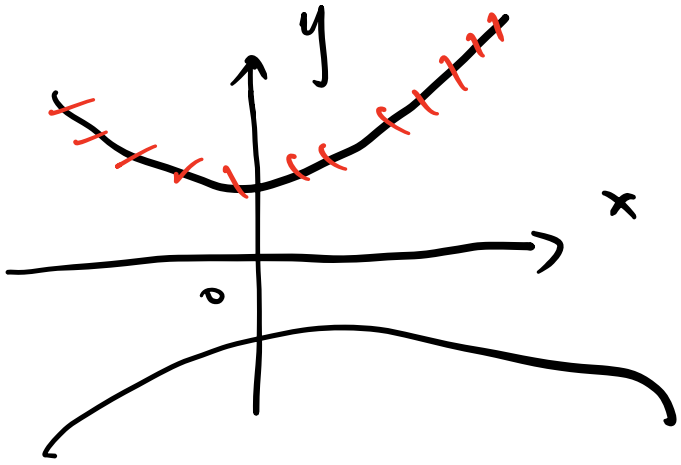
$$(\cdot, \cdot) = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$$

Def: ($\mathbb{H}^3$)

$$\mathbb{H}^3 := \{ p \in \mathbb{H}^3 \mid (p, p) = -1,\ p_0 > 0 \}$$

$$p_0^2 = p_1^2 + p_2^2 + p_3^2 + 1$$

$$(y^2 = x^2 + 1)$$



$T_p\mathbb{H}^3 := \{ v \mid \langle v, p \rangle = 0 \}$, then:

$(\mathbb{H}^3, \langle \cdot, \cdot \rangle)$ 是一个 Riemann - Manifold.

Def: $(SO(3,1)) := \{ g \in \boxed{SL}(4, \mathbb{R}) \mid \langle gv, gw \rangle = \langle v, w \rangle,$
$\forall v, w \in \mathbb{R}^{3,1} \}$

conformal map.

then:

Prop: $SO^+(3,1)$ acts isometry on $\mathbb{H}^3$.

pf: $SO^+(3,1) \triangleright \mathbb{H}^3 \longrightarrow \mathbb{H}^3$

$(g, r) \longmapsto gr.$

then $g_*(X_p) = \frac{d}{dt}\Big|_{t=0} g \cdot \gamma(t)$, $\gamma'(0) = X_p$.

$\therefore \langle g_*(X_p), g_*(X_p)\rangle = \langle \frac{d}{dt}\Big|_{t=0} g \triangleright \gamma(t), \frac{d}{dt}\Big|_{t=0} g \triangleright \gamma(t)\rangle$

$= \langle X_p, X_p\rangle.$

Prop: The action $SO^+(3,1) \triangleright \mathbb{H}^3$ is transitive.

($\because$ $\mathbb{H}^3$ is a homogeneous-space)

Hermitian-Matrix (将$\mathbb{H}^3$视作Matrix Lie Group)

$\mathcal{H} := \{ A \in gl(2,\mathbb{C}) \mid A^* = A \}$

$\therefore$ $\mathcal{H}$ has the form $\begin{pmatrix} x_0 & x_2+ix_3 \\ x_2-ix_3 & x_1 \end{pmatrix}$, $x_0 \sim x_3 \in \mathbb{R}$

$\rightsquigarrow \begin{pmatrix} x_0+x_1 & x_2+ix_3 \\ x_2-ix_3 & x_0-x_1 \end{pmatrix}$

Define: (quatradic-form) $\langle A, A\rangle = -\det A$
$= -x_0^2+x_1^2+x_2^2+x_3^2$