(Loop Group Methods For the Non-Abelian Hodge Correspondence)

一. Deligne-Hitchin Moduli

We have $\mathbb{CP}^1 \cong S^2$.

$$\lambda \longmapsto \left(\frac{1-|\lambda|^2}{1+|\lambda|^2}, \frac{\lambda+\bar{\lambda}}{1+|\lambda|^2}, -\frac{i(\lambda-\bar{\lambda})}{1+|\lambda|^2} \right).$$

Given a hyperkähler manifold $(M, I, J, K) \rightsquigarrow$ Twistor-Space:

$M \times \mathbb{CP}^1$, where the complex-structure at (m, λ) is given by:

$$\mathcal{J} = \left(\frac{1-|\lambda|^2}{1+|\lambda|^2} I + \frac{\lambda+\bar{\lambda}}{1+|\lambda|^2} J + -\frac{i(\lambda-\bar{\lambda})}{1+|\lambda|^2} K,\ i \right)$$

In Hitchin's original Paper: $\mathcal{M}_{SD}$ hyperkahler-mfd. I, J, K.

(Given by the HK-Quotient.)

$\rightsquigarrow \mathcal{P} := \mathcal{M}_{SD} \times \mathbb{CP}^1$ As twistor-Space.

Prop: $\mathcal{J}: \mathcal{P} \longrightarrow \mathcal{P}$ is an anti-holomorphic map.

$$(m, \lambda) \longmapsto (m, -\bar{\lambda}^{-1})$$

($f: M \to N$ holomorphic means $df_p(T_p^{1,0} M) \subset T_{f(p)}^{1,0} N$).

Pick: $(X_1, X_2) \in T^{1,0}_{(m_0, \lambda_0)} \mathcal{P}$. then: $I_{\lambda_0} X_1 = i X_1$, notice that: $I_{-\bar{\lambda}_0^{-1}} = -I_{\lambda_0}$.

$$\therefore I_{-\bar{\lambda}_0^{-1}}(d\mathcal{J}(X_1)) = I_{-\bar{\lambda}_0^{-1}}(X_1) = -I_{\lambda_0}(X_1) = -iX_1 \in T^{0,1}\mathcal{P}.$$

On $\mathcal{M}_{SD}$, by Yau's thm $\rightsquigarrow$ Symplectic holomorphic Form ω

$\mathbb{CP}^1$, $\rightsquigarrow$ ✓

We can give a holomorphic-symplectic form

$$\hat{\omega} \in H^0(\mathcal{P}, \Lambda^2 V^* \otimes \mathcal{O}(-2))$$

$V = \ker d\pi$ w.r.t. $\pi: \mathcal{P} \longrightarrow \mathbb{CP}^1$, s.t. $\hat{\omega} = \bar{\omega} \otimes \lambda \frac{\partial}{\partial \lambda}$

$$(m, \lambda) \longmapsto \lambda$$

where $\overline{\omega} = \lambda^{-1}(\omega_J + i\omega_K) - 2\omega_I - \lambda(\omega_J - i\omega_K)$

Def: (Twistor-Line)

Let (∇, Φ) be solution of SD-equation, the twistor lines are given by the section $s: \mathbb{CP}^1 \longrightarrow \mathcal{P}$

$$\lambda \longmapsto ([\nabla, \Phi], \lambda)$$

Def: (Real-Section)

A section $s: \mathbb{CP}^1 \to \mathcal{P}$ is called real if:

$$\mathcal{J}(s(\lambda)) = s(-\overline{\lambda}^{-1})$$

Rk: the twistor lines are real.

The Delign-Hitchin Moduli

(A great way to compute NAH)

$$\begin{tikzcd}[column sep=small]
 & \underset{(\nabla, \Phi)}{\mathcal{M}_{SD}} \arrow[dl, leftrightarrow, "\cong"'] \arrow[dr, leftrightarrow, "\cong"] & \\
\underset{(\nabla, F^{\nabla}=0)}{\mathcal{M}_{dR}} & & \underset{(\bar\partial, \Phi)}{\mathcal{M}_{Dol}}
\end{tikzcd}$$

Def: $\lambda \in \mathbb{C}$, Σ be a R.S.

A λ-connection on $\begin{array}{c}\mathbb{C}^2 \\ \downarrow \\ V \\ \downarrow \\ \Sigma\end{array}$ is a pair $(\bar\partial^V, D)$, where:

①

$D: \Gamma(\Sigma, V) \longrightarrow \Omega^{1,0}(\Sigma, V)$, s.t.

$$D(fs) = \lambda \partial f s + f Ds$$

② Call λ-conn integrable: $\bar\partial^V D + D\bar\partial^V = 0$.

RK:

1. $\lambda=0$ means $D\in H^0(K\,\mathrm{End}(V))$.

2. $\lambda\neq 0$: integrable $(\bar\partial, D, \lambda)$ gives a flat conn: $\nabla=\bar\partial+\frac{1}{\lambda}D$.

3. the solution of SD-Equation: (∇,Φ) gives a family of λ-conn (integrable)

$(\nabla,\Phi)\longrightarrow \underline{(\bar\partial^\nabla+\lambda\Phi^*,\ \partial^\nabla+\lambda^{-1}\Phi,\ \lambda)}$.

★ (this is just constant section if fixing $\mathcal{P}=\mathcal{M}_{SD}\times\mathbb{C}P^1$)

Stability

A $SL(2;\mathbb{C})$ λ-conn $(\bar\partial_V, D)$ is called stable, if $\forall L\hookrightarrow V$ with $D(\Gamma(\Sigma,L))\subset\Omega^{1,0}(\Sigma,L)$, we have:

$$\deg L<0$$

类似: semi-stable / poly-stable.

RK:

1. For integrable $(\bar\partial_V, D)$, it must be semi-stable.

$\nabla=\frac{1}{\lambda}D+\bar\partial_V$. if $L\hookrightarrow V\rightsquigarrow\nabla=\begin{pmatrix}\nabla^L & x\\ 0 & x\end{pmatrix}$, $F^{\nabla^L}=0\rightsquigarrow\deg L=0$

2. $(\bar\partial, D)$ stable $\Leftrightarrow$ $\nabla=\frac{1}{\lambda}D+\bar\partial$ irreducible.

Def: (Hodge-Moduli)

$\mathcal{M}_{Hod}:=\{(\lambda,\bar\partial,D)\mid$ polystable $SL(2;\mathbb{C})$ λ-conn on $\Sigma\}$

$[\lambda,\bar\partial,D]_\Sigma$.

The delign-Hitchin moduli:

$$\mathcal{M}_{DH} := \mathcal{M}_{Hod}(\Sigma) \cup_g \mathcal{M}_{Hod}(\bar{\Sigma})$$

$$\boxed{[\lambda, \bar{\partial}, D]_{\Sigma} \cong \left[\tfrac{1}{\lambda}, \tfrac{1}{\lambda} D, \tfrac{1}{\lambda}\bar{\partial}\right]_{\bar{\Sigma}}}$$

Same as $\mathcal{M}_{SD}$, the stable points admit a smooth structure.

Real-Sections on $\mathcal{M}_{DH}$:

$$C: \mathcal{M}_{DH} \longrightarrow \mathcal{M}_{DH}$$
$$[\lambda, \bar{\partial}, D]_{\Sigma} \longmapsto [\bar{\lambda}, \bar{\bar{\partial}}, \bar{D}]_{\bar{\Sigma}} \sim [\bar{\lambda}^{-1}, \bar{\lambda}^{-1}\bar{D}, \bar{\lambda}^{-1}\bar{\bar{\partial}}]_{\Sigma}$$

$$N: \mathcal{M}_{DH} \longrightarrow \mathcal{M}_{DH}$$
$$[\lambda, \bar{\partial}, D]_{\Sigma} \to [-\lambda, \bar{\partial}, -D]_{\Sigma}.$$

then

$$\mathcal{J} = CN: \mathcal{M}_{DH} \longrightarrow \mathcal{M}_{DH}$$
$$[\lambda, \bar{\partial}, D]_{\Sigma} \longmapsto [-\bar{\lambda}^{-1}, \bar{\lambda}^{-1}\bar{D}, -\bar{\lambda}^{-1}\bar{\bar{\partial}}]_{\Sigma}.$$

Def: A section of $\mathcal{M}_{DH}$ is real if:

$$\mathcal{J}(s(\lambda)) = s(-\bar{\lambda}^{-1})$$

Prop: ★ The twistor lines are real.

pf: $\nabla = \nabla^* = \bar{\nabla} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightsquigarrow \bar{\partial}^{\nabla} = \overline{\partial^{\nabla}} \cdot \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, $\partial^{\nabla} = \overline{\bar{\partial}^{\nabla}} \cdot \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$,

$\Phi = \overline{\Phi^*} \cdot \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, $\Phi^* = \bar{\Phi} \cdot \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, thus:

$$\mathcal{J}(s(\lambda)) = \mathcal{J}([\lambda, \bar{\partial}^{\nabla} + \lambda \Phi^*, \lambda \partial^{\nabla} + \Phi]_{\Sigma})$$

$$= [-\bar{\lambda}^{-1}, \bar{\lambda}^{-1}(\bar{\lambda}\,\bar{\partial}^{\nabla} + \Phi), -\bar{\lambda}^{-1}(\overline{\bar{\partial}^{\nabla} + \lambda\Phi^{*}})]_{\Sigma}$$

$$= [-\bar{\lambda}^{-1}, \bar{\partial}^{\nabla} + \bar{\lambda}^{-1}\Phi, -\bar{\lambda}^{-1}\overline{\bar{\partial}^{\nabla}} - \Phi^{*}]_{\Sigma}$$

$$= [-\bar{\lambda}^{-1}, \bar{\partial}^{\nabla} + \bar{\lambda}^{-1}\Phi, -\bar{\lambda}^{-1}\overline{\bar{\partial}^{\nabla}} - \Phi^{*}].\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}]_{\Sigma}$$

$$= [-\bar{\lambda}^{-1}, \overline{\partial^{\nabla}} - \bar{\lambda}^{-1}\Phi^{*}, \Phi - \bar{\lambda}^{-1}\partial^{\nabla}]_{\Sigma}$$

$$= S(-\bar{\lambda}^{-1})$$

#

(Negative / Positive)

Let $S(\lambda) : \mathbb{C}P^1 \longrightarrow \mathcal{M}_{DH}$ be a real section, then $\mathcal{J}(S(\lambda)) = S(-\bar{\lambda}^{-1})$

Locally $S(\lambda)$ looks like:

$$[\lambda, \bar{\partial}(\lambda), \lambda\partial + \Phi(\lambda)] \rightsquigarrow \nabla^{\lambda} = \frac{1}{\lambda}(\lambda\partial + \Phi(\lambda)) + \bar{\partial}(\lambda).$$

$$= \partial^{\circ} + \bar{\partial}(\lambda) + \lambda^{-1}\Phi(\lambda) \sim \lambda^{-1}\Phi + \nabla + \cdots$$

We can make such $(\bar{\partial}^{\nabla}, \Phi)$ stable.

Reality — means: $\overline{\nabla^{-\bar{\lambda}^{-1}}} = \nabla^{\lambda}.g(\lambda)$

pf: Let $S(\lambda) = [\lambda, \bar{\partial}(\lambda), D(\lambda)]$

$\mathcal{J}(S(\lambda)) = [-\bar{\lambda}^{-1}, \bar{\lambda}^{-1}\bar{D}(\lambda), -\bar{\lambda}^{-1}\bar{\bar{\partial}}(\lambda)] \rightsquigarrow \nabla^{\mathcal{J}(S(\lambda))} = \frac{1}{-\bar{\lambda}^{-1}}(-\bar{\lambda}^{-1}\bar{\bar{\partial}}(\lambda)) + \bar{\lambda}^{-1}\bar{D}(\lambda)$

$$= \overline{\tfrac{1}{\lambda}D + \bar{\partial}} = \overline{\nabla^{\lambda}} \sim \nabla^{-\bar{\lambda}^{-1}}$$

$$\therefore \overline{\nabla^{-\bar{\lambda}^{-1}}} = \nabla^{\lambda}.g(\lambda).$$

$\therefore \nabla^{\lambda}_{\cdot} g(\lambda)\overline{g(-\bar{\lambda}^{-1})} = \nabla^{\lambda}$.

stability means:

$$g(\lambda)\overline{g(-\bar{\lambda}^{-1})} = \pm Id.$$

The sign "$\pm$" is independent of lifting.

Def: A stable section of $\mathcal{M}_{DH}$ is called posi/negative depending on the sign.

(Admissable negative real sections $\longrightarrow$ Self-Duality.)

(The gauge $g(\lambda)$ 取决于 lifting, admiss 意味 globally-def).

$\overline{\hat{\nabla}^{-\bar{\lambda}^{-1}}} = \tilde{\nabla}^{\lambda}_{\cdot} g(\lambda)$. 做 Loop-group decompo $g(\lambda) = g_+(\lambda)\cdot g_-(\lambda)$

Self-Duality Equa 只须让 $g(\lambda)$ looks like $\delta = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

negative : $\overline{g(-\bar{\lambda}^{-1})}^{-1} = -g(\lambda)$

$\because \lambda \mapsto -\bar{\lambda}^{-1}$ 交换 $(\ )^+$ 与 $(\ )^-$ 部分:

$$\begin{cases} g^+(\lambda) = -\overline{g(-\bar{\lambda}^{-1})}^{-1} \bar{B}^{-1} \\ g^-(\lambda) = B\,\overline{g^+(-\bar{\lambda}^{-1})}^{-1} \end{cases}, \quad \bar{B}B = -Id. \ B: \Sigma \to SL(2,\mathbb{C})$$

$\therefore B$ 与 $\delta = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ same conjugacy

$\rightsquigarrow \nabla^{\lambda} = \tilde{\nabla}^{\lambda}_{\cdot}(g^{+}(\lambda), G)$; 满足 $\overline{\nabla^{-\bar{\lambda}^{-1}}} = \nabla^{\lambda}_{\cdot}\delta$.

$\to V$.

Goldman's Symplectic Form.

Let $(\bar{\partial}, D)$ be a λ-conn, $(A, B) \in \Omega^{0,1} \oplus \Omega^{1,0}$ be tangent vector : $0 = \bar{\partial}B + DA$

the infini gauge trans $\xi \in \Gamma(\Sigma, \underline{sl}(2;\mathbb{C}))$

$$\frac{d}{dt}\Big|_{t=0}(\bar{\partial}.\exp(t\xi), D.\exp(t\xi)) = (\bar{\partial}\xi, D\xi).$$

Def: (the Symplec-form):

$$\Theta^{\lambda}((A_1, B_1), (A_2, B_2)) = 4\int_{\Sigma} tr(A_1 \wedge B_2 - A_2 \wedge B_1)$$

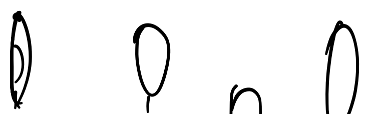
$\wr|$

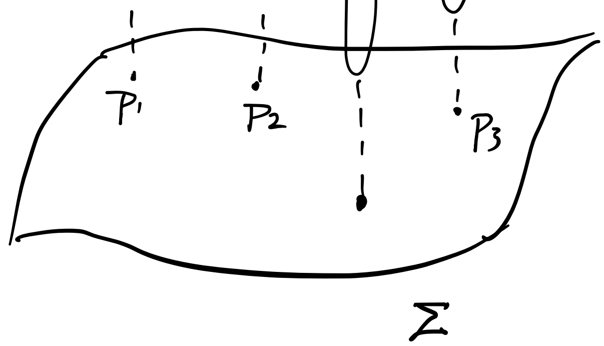
$\bar{\omega} = \lambda^{-1}(\omega_J + i\omega_K) - 2\omega_I - \lambda(\omega_J - i\omega_K)$

Logarithmic Connection

Σ be compact R.S. , $p_1, \dots, p_n \in \Sigma$

$$\Sigma^0 = \Sigma - \{p_1, \dots, p_n\}$$





在每个 p_i 处 choose 一个 Conjungacy-Class. $\mathcal{O}_i \in \mathfrak{g}^*$

$$\nabla = d + \sum A_i \frac{dz}{z-p_i} + \text{smooth part.}$$

On $\mathcal{O}_i$, we have a symplec-form:

$\omega_D(\hat{X}, \hat{Y}) = -D([X,Y])$, tangent Space of $\mathcal{O}_i$ at D is $\{[D,X] \mid X \in \mathfrak{g}\}$

$$\rightsquigarrow \omega_\nabla = \omega_{str} + \sum \omega_i$$

进一步算的办法

(不split去看)

Lemma: $X \in \Omega^1(\Sigma^o, sl)$ tangent to $\nabla \in \mathcal{A}$, then:

$\exists\, \xi \in \Gamma(\Sigma, sl)$, $\hat{X} \in \Omega^1(\Sigma, sl)$, s.t.

$$X = d^\nabla \xi + \hat{X}, \quad d^\nabla \hat{X} = 0$$

pf: Write $\nabla = A_j \frac{dz_j}{z_j} + \text{smooth conn}$

Let $X \in \Omega^1(\Sigma^\circ; \mathfrak{sl})$ tangent to $\nabla \rightsquigarrow$ (flatness) $\frac{d}{dt}\Big|_{t=0} (\nabla + tX)(\nabla + tX) = d^\nabla X = 0$

$X = \hat{X} + [A_j, \xi_j] \frac{dz_j}{z_j}$, pick $\xi \in \Gamma(\mathfrak{sl})$, $\xi(p_j) = \xi_j$

$\rightarrow X = d^\nabla \xi + (X - d^\nabla \xi)$.

Lemma:

Let $\nabla \in \mathcal{A}$, $X = d^\nabla \xi + \hat{X}$, $Y = d^\nabla \mu + \hat{Y} \in T_\nabla \mathcal{A}$, then:

$$\int_{\Sigma^\circ} tr(X \wedge Y) = \int_{\Sigma} tr(\hat{X} \wedge \hat{Y}) - 2\pi i \sum_j tr(A_j [\xi(p_j), \mu(p_j)])$$

pf: $\int_{\Sigma^\circ} tr(X \wedge Y) = \int_{\Sigma^\circ} tr((d^\nabla \xi + \hat{X})(d^\nabla \mu + \hat{Y}))$

$$\int_{\Sigma^\circ} tr(d^\nabla \xi \wedge \hat{Y}) = -\int_{\Sigma^\circ} d\, tr(\xi \wedge \hat{Y}) + \underbrace{\int_{\Sigma^\circ} tr(\xi\, d^\nabla \hat{Y})}_{=0}$$

$\underline{\text{stokes}}$ $\int_{\partial \Sigma^\circ} tr(\xi \wedge \hat{Y}) = 0$

$$\rightsquigarrow \int_{\Sigma^\circ} tr(d^\nabla \xi \wedge d^\nabla \mu) = \int_{\Sigma^\circ} d\, tr(\xi\, d^\nabla \mu) = -\lim_{t \to 0} \sum_j \int_{r_t} tr(\xi\, d^\nabla \mu)$$

$$= -2\pi i \sum_j tr(\xi(p_j) [A_j, \mu(p_j)]) = -2\pi i \sum_j tr(A_j [\xi(p_j), \mu(p_j)]).$$

☆.

(Holomorphic Symplectic Form)

$$\mathcal{O}(X, Y) \longmapsto \int_{\Sigma^\circ} tr(X \wedge Y) + 2\pi i \sum_j tr(Res_{p_j}(\nabla)[Res_{p_j}(X), Res_{p_j}(Y)])$$

Ansatz and Initial Condition at $t=0$.

Focus our attention to $\Sigma_p = \mathbb{C}P^1 - \{p, -p, 1/p, -1/p\}$, then:

for $\mathcal{M}^{t}_{DH}(\Sigma_p)$, $\forall\, \nabla^\lambda$ has the form: (1-order pole at $\lambda=0$)

$$\nabla^\lambda . g(\lambda) = d + \xi^\lambda, \quad \xi^\lambda = \sum A_i \frac{dz}{z-p_i}, \quad \det A_i = -t^2$$

$$\underline{M_j \leftrightarrow \rho^{\nabla^\lambda}(r_j)} \longrightarrow tr(M_j) = 2\cos(2\pi t)$$

$\rightsquigarrow$ considering $\eta^\lambda_t = t\,\xi^\lambda = \sum_{i=1}^{4} t A_i^t(\lambda) \frac{dz}{z-p_i}$, $(\eta^\lambda_t \to 0,\ t \to 0)$

which is a Fuchsian System: $\sum A_i^t(\lambda) \equiv 0$.

Because of the symmetry, we only need to considering

$$A_1^t = \begin{pmatrix} x_1^t & x_2^t - i x_3^t \\ x_2^t + i x_3^t & -x_1^t \end{pmatrix}, \quad \det A_1^t = -1$$

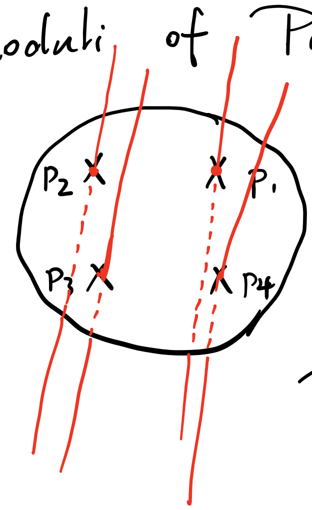
the reality condition:

$$\nabla^\lambda . g(\lambda) = \overline{\nabla}^{\bar\lambda^{-1}} \quad \text{means:}$$

$$\rho(\nabla^\lambda) = \rho(\overline{\nabla}^{\bar\lambda^{-1}}) \rightsquigarrow \text{this means:} \begin{cases} x_1^* = x_1 \\ x_2^* = x_2 \\ x_3^* = x_3 \end{cases} \text{in Betti-Moduli Space.}$$

Then using implicit function theory !!!

Moduli of Parabolic Higgs Field.



$\eta^\lambda_t = \sum t A_i^t \frac{dz}{z-p_i}$. Let $\Psi = Res_{\lambda=0}\, \eta^\lambda_t$.

$\det A_i^t = -1 \rightsquigarrow \det(Res_{\lambda=0} A_i^t) = 0$.

$\rightsquigarrow$ give a line at each $p_1 \sim p_4$

with weight t

$0 \subsetneq L_1 \subsetneq \mathbb{C}^2, \quad 0 \subsetneq L_2 \subsetneq \mathbb{C}^2, \quad \cdots$

$t \quad -t$

Lemma: the Parabolic Higgs Field $\Phi \sim A_1$, $\hat{\Phi} \sim g^{-1} A_1 g$ are gauge equivalent iff $g \in \langle C, D \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Lemma:

Pick $(u,v) \in \mathbb{C}^2 - \{0\}$, with:

$$A_1^{(-1)} = \begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix}$$

$\rightsquigarrow (\bar{\partial}^0, \Phi)$ is semi-stable.

(strictly stable unless $uv = 0$, $u^2 = v^2$, $u^2 = -v^2$)

pf: Let $L \hookrightarrow V$ preserved by Φ, we need to show:

$\deg L \leq 0$, if not: $\text{par-deg}(L) = \deg(L) + 4t > 0$

$\therefore \deg L > -1 \quad \therefore \deg L \geq 0 \to L$ must be constant.

$\therefore L$ can only ① pass one of
$L_1 = \mathbb{C}(u,v) \quad L_4 = \mathbb{C}(v,u)$
$L_2 = \mathbb{C}(-v,u)$
$L_3 = \mathbb{C}(-u,v)$

$\therefore \text{par-deg} L = 0 + t - 3t < 0$.

② pass 2 of $L_1 \sim L_4$, then

$\text{par-deg} L = 2t - 2t = 0$

Conversely, for $\begin{matrix} \mathbb{C}^2 \\ \downarrow \\ (V, \bar\partial^0) \\ \downarrow \\ \Sigma_p \end{matrix}$, each parabolic Higgs Field is gauge equivalent to a symm one.

Thm: $\mathcal{M}^{\text{para}}_{\text{Higgs}} \cong Bl_0(\mathbb{C}^2/\mathbb{Z}_2)/\mathbb{Z}_2\times\mathbb{Z}_2$

$\cong T^*\mathbb{C}P^1/\mathbb{Z}_2\times\mathbb{Z}_2$

At $t=0$, denote:

$$\eta^\lambda_t = t\sum A_i^t(\lambda)\frac{dz}{z-p_i}$$

, then the initial-data is just:

$$\underline{A}_1 \triangleq A_i^{t=0}(\lambda)$$

$$\| $$

$$\begin{pmatrix} \underline{x}_1 & \underline{x}_2 - i\underline{x}_3 \\ \underline{x}_2 + i\underline{x}_3 & -\underline{x}_1 \end{pmatrix}$$

Initial Condition

$$\begin{cases} \det \underline{A}_1 = -1 \\ \underline{A}_1 = \underline{A}_1^* \end{cases} \longrightarrow \begin{cases} \underline{x}_1^2 + \underline{x}_2^2 + \underline{x}_3^2 = 1 \\ \underline{x}_j^* = \underline{x}_j \end{cases}$$

this equation is easy to be solved:

Let $\underline{x}_j = \underline{x}_{j,-1}\lambda^{-1} + \underline{x}_{j,0} + \underline{x}_{j,1}\lambda$.

then: $\underline{x}_j^* = \underline{x}_j \sim \overline{\underline{x}_j(-\bar{\lambda}^{-1})} = \underline{x}_j(\lambda)$

$\longrightarrow \underline{x}_{j,-1} = -\overline{\underline{x}_{j,1}}$, $\underline{x}_{j,0} \in \mathbb{R}$.

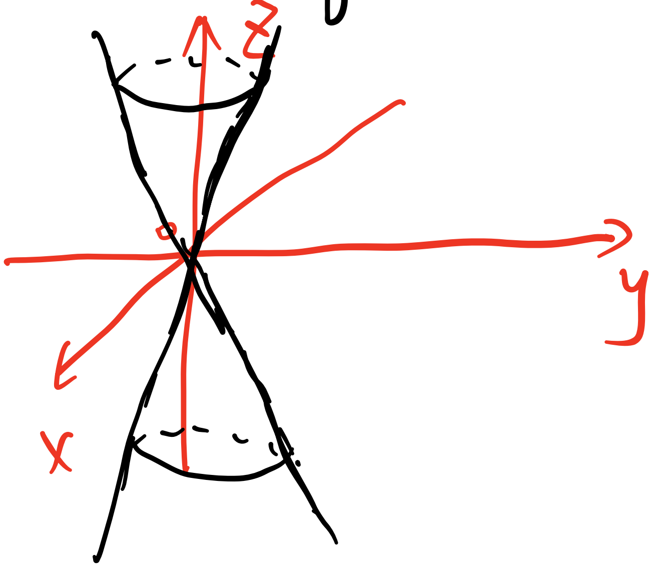
$\longrightarrow \underline{x}_j = \underline{x}_{j,-1}\lambda^{-1} + \underline{x}_{j,0} - \overline{\underline{x}_{j,-1}}\lambda$!

$$\begin{cases} \sum_j \underline{x}_{j,-1}^2 = 0 & (1) \\ \sum \underline{x}_{j,-1}\underline{x}_{j,0} = 0 & (2) \\ \sum \underline{x}_{j,0}^2 - 2|\underline{x}_{j,-1}|^2 = 1. & (3)! \end{cases}$$

(In chapter 5, this is just NAH at t=0)

In Algebraic — Geometry :

(1) $\longrightarrow$ quadric $\{x^2+y^2+z^2=0\}$



the solution looks like:

$\underline{X}_{1,-1} = uv$, $\underline{X}_{2,-1} = \frac{1}{2}(v^2-u^2)$, $\underline{X}_{3,-1} = \frac{i}{2}(u^2+v^2)$

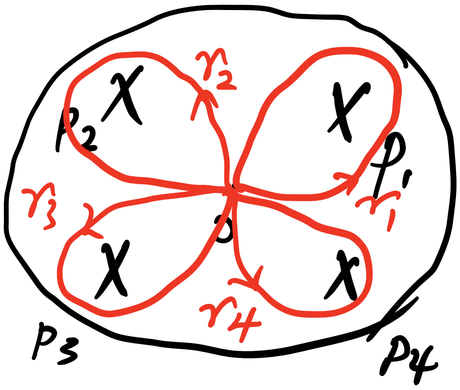
$\underline{X}_{1,0} = \rho(|u^2|-|v^2|)$, $\underline{X}_{2,0} = 2\rho\,\mathrm{Re}(u\bar{v})$, $\underline{X}_{3,0} = 2\rho\,\mathrm{Im}(u\bar{v})$

$$\rho^2 = \frac{1+(|u|^2+|v|^2)^2}{(|u|^2+|v|^2)^2} \rightsquigarrow \rho = -\sqrt{1+(|u|^2+|v|^2)^{-2}}$$

Constructing Real Holomorphic Section

Real means:

$$[d-\eta\lambda] = [d-\overline{\eta^{\bar{\lambda}^{-1}}}]$$



Monodromy — Equation:

$$\begin{cases} d_z \Phi_t = \eta_t \Phi_t \\ \Phi_t(z=0) = Id. \end{cases}$$

Define : $S_k := \operatorname{trace}(M_k)$

$S_{kl} := \operatorname{trace}(M_k M_l)$ Fricke-Voigt

For $S_1 = S_2 = S_3 = S_4 = S$, the coordinate of Betti-Moduli is defined by

$$(S, \underset{\underset{u}{\shortparallel}}{S_{12}}, \underset{\underset{v}{\shortparallel}}{S_{13}}, \underset{\underset{w}{\shortparallel}}{S_{23}}). \qquad (1)$$

The following equation holds:

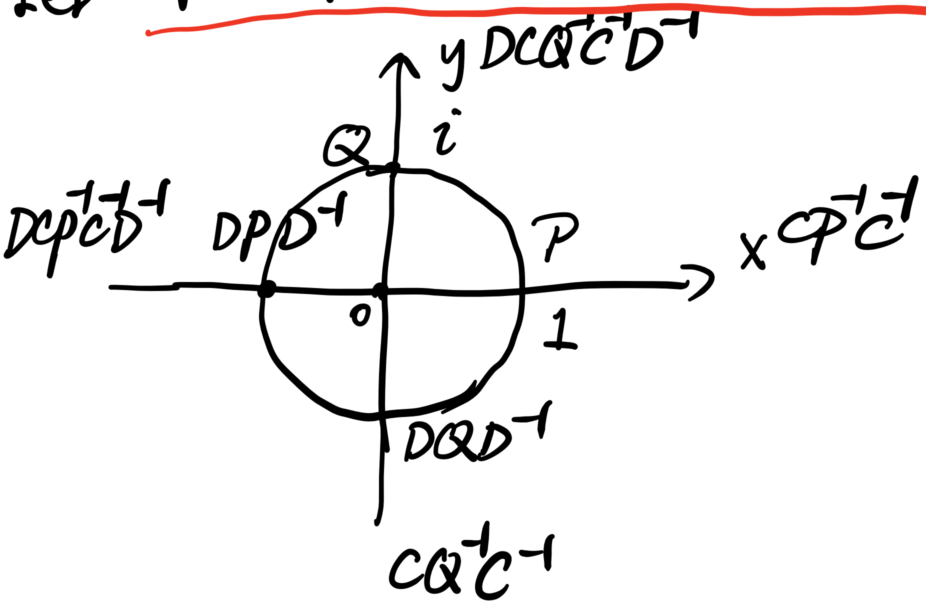
$$u^2 + v^2 + w^2 + uvw - 2s^2(u+v+w) + 4(s^2-1) + s^4 = 0. \qquad (2)$$

反之: (1)+(2) defines a monodromy.

$$\begin{cases} S^{*}_{jk} = S_{jk}, & (j,k) \in \{(1,2),(1,3),(2,3)\} \\ \sum_{j=1}^{3} x_j^{t\,2}(\lambda) = 1 \end{cases}$$

Let $P = \Phi(z=1)$, $Q = \Phi(z=i)$

(We can directly compute M_i w.r.t P, Q.)



then:
$$\begin{cases} S_{12} = 2 - 4\underline{p}^2 \\ S_{23} = 2 - 4\underline{q}^2 \\ S_{13} = 2 - 4\underline{\tau}^2 \end{cases}$$

$$\underline{p} = P_{11}P_{21} + P_{12}P_{22}, \quad \underline{q} = i(Q_{11}Q_{21} - Q_{12}Q_{22})$$

$$\underline{\tau} = \frac{i}{2}(P_{22}Q_{11} + P_{12}Q_{21})^2 + \frac{i}{2}(P_{22}Q_{12} + P_{12}Q_{22})^2 - \frac{i}{2}(P_{21}Q_{11} + P_{11}Q_{21})^2 - \frac{i}{2}(P_{21}Q_{12} + P_{11}Q_{22})^2.$$

Easy to Compute:

At $t=0$ $\quad p(0) = q(0) = \tau(0) = 0$

$P'(0) = 2\pi \underline{X}_3$, $\underline{q}'(0) = 2\pi \underline{X}_2$, $\Gamma'(0) = 2\pi \underline{X}_1$.

Combining with (2), everything becomes:

$$\begin{cases} \underline{P} = \underline{P}^* \\ \underline{q} = \underline{q}^* \\ \sum_j x_j^2 = 1 \end{cases}$$

solving the equation:

at $t=0$, this is obviously true.

(Implicit Function Theory)

Let X, Y, Z be Banach-Space, $\Omega \doteq X \times Y$,

$F \in C^r(\Omega, Z)$, $r \geq 0$, if $\partial_x F$ has a bounded

inverse function, then $\exists$ open neigh

of (x_0, y_0), $g: V \to U$, $g(y_0) = x_0$,

s.t. $F(g(y), y) = F(x, y)$ in this neighbor.

Thm 4.

For $(u,v)\neq 0$ which dermines $\underline{x}_i$, $i=1\sim 3$, $\exists\, \varepsilon_0>0$, $a>1$, s.t. $\exists !$ values of the para $x=(x_1,x_2,x_3)\in(\mathcal{W}_a^{\geq 0})^3$ in a neigh of $\underline{x}$ with prescribed $x_{j,-1}(t)=\underline{x}_{j,-1}$

depending analytically on (t,p,u,v)

solveing equations with $x(t=0)=\underline{x}$.

Thm:

Actually: Let $u=r\hat{u}$, $v=r\hat{v}$, then the

solution can extend to $r=0$, s.t.

$\exists\, u\in SL(2,\mathbb{C})$, s.t.

$$\begin{cases} u M_K u^{-1}\in SU(2,\mathbb{C}). \\ x_1^2+x_2^2+x_3^2=1 \end{cases}$$

$$x_1=-(|\hat{u}|^2-|\hat{v}|^2),\quad x_2=-2\,\mathrm{Re}(\hat{u}\,\overline{\hat{v}})$$
$$x_3=-2\,\mathrm{Im}(\hat{u}\,\overline{\hat{v}}).$$

Non-Abelian-Hodge at $t=0$

the Correspondence is given by:

$$\begin{tikzcd} & \mathcal{M}_{SD} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \mathcal{M}_{dR} \arrow[rr, phantom, "\cong"] & & \mathcal{M}_{Dol} \end{tikzcd}$$

In Simpson's explanation, it is the evaluation map of

twistor lines of $\mathcal{M}_{DH}$

at $\lambda=0$, $\lambda=1$

$\therefore$ for t near 0 $\rightsquigarrow$ the solution

$x_i(t, p, \hat{u}, \tilde{v}, r)$ gives twistor lines

$s(\lambda) = d - \eta_\lambda^t$, then

$$s(1) \longleftrightarrow s(0)$$

for $t=0$:

(i) $(u,v) \neq 0 \longrightarrow$ nilpotent orbits.

$\cong SL(2;\mathbb{C})$ orbit through $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$- \{\text{hermitian}\}$

(ii) $(u, v) \longrightarrow 0$

$Bl_0(\mathbb{C}^2/\mathbb{Z}_2) \longleftrightarrow$ Full orbit through $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

"limit Non-Abelian Hodge at $t=0$"

Rescalled Metric at $t=0$

$$d - \frac{1}{t}\eta^t_\lambda \Big|_{t=0} = \nabla^\lambda.$$

$\Sigma_k \subsetneq \mathbb{C}P^1$ given by

$$\left\{ (y, z) \,\middle|\, y^k = \frac{(z-p_1)(z-p_3)}{(z-p_2)(z-p_4)} \right\}$$

$$\Sigma_k \longrightarrow \Sigma_p$$

$$(y, z) \longmapsto z.$$

$\therefore$ $\pi^* D^\lambda$ complex-submfd of

$$\left(\mathcal{M}_{SD}(\Sigma_k), I, J, K\right)$$

记作 $\mathcal{M}^{\frac{L}{k}}$

---

holo-symplectic-form.

(By Goldman).

$$\omega_\eta(X_1, X_2) = 2\pi i k^3 \sum_j tr\left(Res_{p_j}(\eta)\left[Res_{p_j}(X_1), Res_{p_j}(X_2)\right]\right)$$

Here: $d + \sum B_j \frac{dz}{z-p_j}$,

"twist holo-symplec form":

$$\overline{\omega} = \frac{2\pi i}{t^3} \sum_j tr\left(B_j (dB_j \wedge dB_j)\right)$$

on Moduli Space of Fuchsian System.

代入

$$A_1^{t=0}(\lambda) = \lambda^{-1}\begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix} + \rho\begin{pmatrix} & \\ & \end{pmatrix} + \lambda \Xi^*$$

ω_I, ω_J, ω_K