For $SL(N,\mathbb{C})$ Higgs Bundle $(E, \bar\partial_E, \varphi)$, define the conformal limit: $\hbar \in \mathbb{C}^*$, $R \in \mathbb{R}^+$

$\nabla_{\hbar,R} = D_{h(R)} + \hbar^{-1}\varphi + \hbar R^2 \varphi^{*h(R)}$ where $h(R)$ is the harmonic metric w.r.t. $(E, \bar\partial_E, R\varphi)$

(Gaiotto Conjecture)

For $(E, \bar\partial_E, \varphi_{\vec u}) \in$ Hit Section, then:

$$\lim_{R\to 0} \nabla_{\hbar,R,\vec u} = D_{\boxed{h_\zeta}} + \hbar^{-1}\varphi_{\vec u} + \hbar\,\varphi_0^{*h_\zeta}$$

(key point: uniformalzation of R.S.)

pf:

Hitchin Section Means $E = K^{N-1} \oplus K^{N-3} \oplus \cdots \oplus K^{1-N}$

for $\vec u = (\phi_2, \phi_3, \ldots, \phi_N) \rightsquigarrow \varphi_{\vec u} = \begin{pmatrix} 0 & \phi_2 & \cdots & \phi_N \\ \sqrt{r_1} & 0 & \ddots & \vdots \\ & \ddots & \ddots & \phi_2 \\ 0 & & \sqrt{r_{N-1}} & 0 \end{pmatrix}$

By Uniformazation thm: Let $h(R) = e^{-\frac{1}{2}\bar x^T} h_\zeta e^{\frac{1}{2}x} = h_\zeta e^x$, then:

$$D_{h(R)} = h(R)^{-1} d h(R) = e^{-x} \cdot h_\zeta^{-1}(d h_\zeta \cdot e^x + h_\zeta d e^x)$$
$$= e^{-x} D_{h_\zeta} e^x$$

For $\varphi^{*h(R)} = h(R)^{-1}\, \bar\varphi^T \cdot h(R)$

$$= e^{-X} h_{\mathfrak{h}}^{-1} \overline{\varphi}^{T} \cdot h_{\mathfrak{h}} e^{X} = e^{-X} \varphi^{*h_{\mathfrak{h}}} \cdot e^{X}$$

$$\therefore \nabla_{\hbar, R, \vec{u}} = e^{-X} D_{h_{\mathfrak{h}}} e^{X} + \hbar^{-1} \varphi_{\vec{u}} + \hbar R^{2} e^{-X} \varphi_{\vec{u}}^{*h_{\mathfrak{h}}} e^{X}$$

$$= e^{-X} D_{h_{\mathfrak{h}}} e^{X} + \hbar^{-1} \varphi_{\vec{u}} + \hbar\, e^{-X} \varphi_{R^2 u}^{*h_{\mathfrak{h}}} e^{X}$$

$\therefore$ to decide $\lim\limits_{R \to 0} \nabla_{\hbar, R, \vec{u}}$, it is equivalent to see

$\lim\limits_{R \to 0} X(R, \vec{u})$.

☆ (The Most Technical Part): $X(R, \vec{u}) = O(R^4)$.

Considering:

$$N(X, R) = F_{D_h} + [R\varphi_{\vec{u}} \cdot R\varphi_{\vec{u}}^{*h(R)}]$$

$$= [\bar{\partial}_E, e^{-X} \circ \partial_E^{h_{\mathfrak{h}}} \circ e^{X}] + [\varphi_{\vec{u}}, e^{-X} \circ \varphi_{R^2 u}^{*h_{\mathfrak{h}}} \circ e^{X}]$$

☆: Prop:

The harmonic metric for $(E, \bar{\partial}_E, \varphi_0)$ with parameter

$$\varphi_0 = \begin{pmatrix} 0 & & 0 \\ \sqrt{r_1} & \ddots & \\ 0 & \sqrt{r_{n-1}} & 0 \end{pmatrix}$$

R is $h_{\mathfrak{h}}(R)$.

pf: For R.S. Σ, if $\omega = -\frac{i}{2} h\, dz \wedge d\bar{z}$, then Gauss

curvature $k = -\dfrac{\Delta \log h}{h^2}$

$\therefore$ Let $g_{\mathfrak{h}} = \lambda^2 dz d\bar{z} \rightsquigarrow \Delta \log \lambda_{\mathfrak{h}} - \lambda_{\mathfrak{h}}^2 = 0$ (Const. Curvature)

$$\therefore h_G(R) = \begin{pmatrix} R^{N-1}\lambda_G^{1-N} & & \\ & \ddots & \\ & & R^{1-N}\lambda_G^{N-1} \end{pmatrix}$$

$$\therefore F_{D_{h_G(R)}} = \bar{\partial}(h_G^{-1}\partial(h_G)) = \bar{\partial}\partial(\log h_G) = (\Delta \log \lambda_G)\cdot H$$

$$\therefore F_{D_{h_G(R)}} + R^2[\varphi_0, \varphi_0^{*h_G(R)}] = (\Delta \log \lambda_G H + [X_-, \lambda_G^2 X_+])dz\wedge d\bar{z}$$

$$= (\Delta \log \lambda_G - \lambda_G^2) H dz\wedge d\bar{z} = 0$$

※: harmonic-Metric 就来自于 Conformal-Metric.

$$\therefore N(0,0) = [\bar{\partial}_E, \partial_E^{h_G}] + [\varphi_u, \varphi_0^{*h_G}]$$

$$\therefore \text{if } \varphi_u = \varphi_0 \rightsquigarrow N(0,0) = 0$$

$$\text{But}: [\varphi_u - \varphi_0, \varphi_0^{*h_G}] = 0 \rightsquigarrow N(0,0) = [\bar{\partial}_E, \partial_E^{h_G}] + [\varphi_0, \varphi_0^{*h_G}] = 0$$

Linearization for $N(X,R)$, we have:

$$L_u(\dot{X}) := D_X N_{\vec{u}}\Big|_{(0,0)}(\dot{X}) = \bar{\partial}_E \partial_E^{h_G}\dot{X} + [\varphi_u, [\varphi_0^{*h_G}, \dot{X}]]$$

$\rightsquigarrow$ isomorphism

$$\therefore X(R, \vec{u}) = \sum_{i=1}^{\infty} R^i f_i, \quad \text{insert } N(X,R) = 0$$

$\therefore \chi = O(R^4)$

$$\therefore \lim_{R \to 0} \nabla_{\hbar, R, \vec{u}} = D_{h_g} + \hbar^{-1} \varphi_{\vec{u}} + \hbar \varphi_0^{*h_g}$$

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