Moduli Space of "Flat Connection" on Punctured Riemann Surface.

Notation: $\mathcal{G}$ gauge-group with Lie-Algebra $\mathfrak{g}$.

$\mathcal{A}$: moduli space of connections on $\mathbb{C}^r \to V \to \Sigma$. $= \Omega^1(gl(r))$

The Symplectic Structure on $\mathcal{A}$:

$$\Omega_A\Big|_A = \int_\Sigma \mathrm{Tr}\, d^A \wedge d^A ,$$

$$\mathcal{G} \curvearrowright \Omega_A \xrightarrow{\mu} \Omega^2(\mathfrak{g}),$$

$A \longmapsto F(A)$ be the curvature

$\rightsquigarrow$ Moduli-Space: $\mu^{-1}(0)/\mathcal{G}$ Mardesen-Weinstein Quotient.

$\rightsquigarrow \mathcal{M}_g$

$\longrightarrow$ 推广这个过程,

$(\Sigma, \{p_1, \dots, p_n\})$

$\forall p_i$, give an coadjoint orbit in $\underset{B(a,\cdot)}{\mathfrak{g}^*} \overset{\text{killing}}{\underset{\text{form}}{\longleftrightarrow}} \underset{a}{\mathfrak{g}}$

$\because$ For $\mathfrak{g}^*$, the orbit is just conjugacy class:

$$G \curvearrowright \mathfrak{g}^* \longmapsto \mathfrak{g}^*$$

$$(g, \langle a,\cdot\rangle) \longmapsto (g \triangleright \langle a,\cdot\rangle)(b) = \langle a, \mathrm{Ad}_g b\rangle = \langle \mathrm{Ad}_g a, b\rangle$$

$$= \langle g^{-1} a g, b\rangle.$$

$\longrightarrow$ ✓.

(Kirillov - Form)

on $\mathcal{O}_D$ for $D \in \mathfrak{g}^*$, we have a natural symple-ctic form $\omega_D(\hat{X}, \hat{Y}) := -D([X,Y])$

the tangent space of $\mathcal{O}_D$ at D is

$$\{[D, X] \mid X \in \mathfrak{g}\}$$

(换言之: $\omega_D(\hat{X}, \hat{Y}) = B(D, [X,Y])$)

Def: (A punctured Riemann - surface:

$\Sigma \setminus \{p_1, \dots, p_n\}$, $\underbrace{\mathcal{O}_1, \dots, \mathcal{O}_n}_{\text{conjugacy class in } \mathfrak{g}}$ w.r.t p_i.

w.r.t $G \curvearrowright \mathfrak{g}$ $(v, D) \longmapsto v^{-1} D v$

Def: (Connection)

$$A = \sum A_i \frac{dz_i}{z_i - p_i} + \tilde{A}(z) \rightsquigarrow \mathcal{A}_{g,n}$$

$$\hat{A} \in \Omega^1(\mathfrak{g}), \quad A_i \in \mathcal{O}_i$$

Def: (Symplectic - Structure)

$$\mathcal{A}_{g,n} = \mathcal{A} \times \mathcal{O}_1 \times \cdots \times \mathcal{O}_n$$

$$\therefore \omega = \omega_A + \sum_{i=1}^{n} \omega_i$$

$\mathcal{G} \curvearrowright \mathcal{O}_i$

$g \triangleright (\nu_i^{-1} T_i \nu_i)$

$$\begin{cases} g \triangleright \nu_i = \nu_i \, g(\nu_i), \\ g \triangleright T_i = g^{-1} T_i \, g \end{cases}$$

$\therefore$ the moment map

$$\mu = \sum T_i \, \delta(z-z_i) - F(z)$$

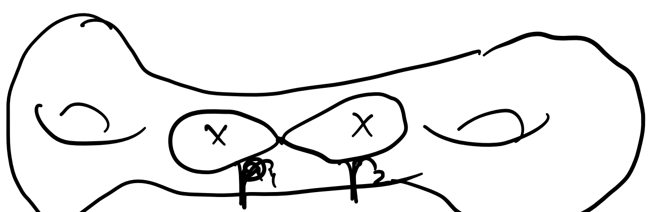
Thm:

The moduli space is just:

$$\mathcal{M}_{g,n} = \mu^{-1}(0) / \mathcal{G} .$$

Fundamental - Group

For punctured Riemann surface $\Sigma \setminus \{p_1, \dots, p_n\}$



$\therefore$

$$\{ a_i, b_i, r_i \mid \prod_j r_j \cdot \prod_i [a_i, b_i] = 1 \}$$

Parabolic Higgs Bundle

§1. Linear Algebra (Parabolic Structure)

Def: For $E = \mathbb{C}^r$, λ is a partition $r = r_1 + \cdots + r_s$,

a flag is of type λ is:

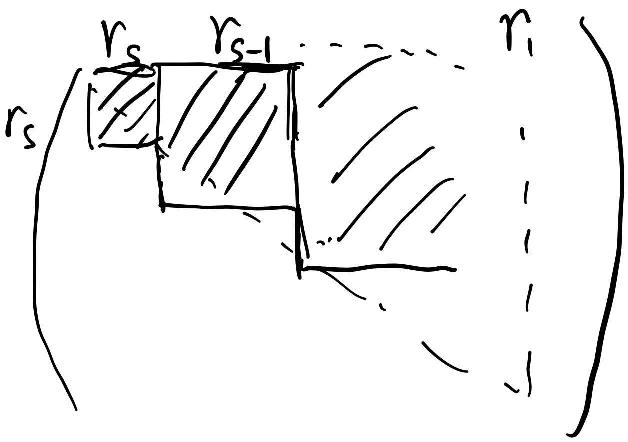
$$F = \{ E_s \subsetneq E_{s-1} \subsetneq \cdots \subsetneq E \mid \dim E_s = r_s, \ \cdots \}$$
$$E_{s-1} = r_s + r_{s-1}$$

RK: $GL(n,\mathbb{R}) \curvearrowright F_\lambda \longrightarrow F_\lambda$. given by:

$$g \triangleright (E_s \subsetneq E_{s-1} \subsetneq \cdots \subsetneq E) = (gE_s) \subsetneq (gE_{s-1}) \subsetneq \cdots \subsetneq (gE)$$

Def: The parabolic group w.r.t F is $P(F)$, the stablizer of this group action.

RK: In matrix repre, it looks like:



$\therefore$ $GL(n,\mathbb{R}) \longrightarrow F_\lambda$

$g \longmapsto gF$

$$GL(n,\mathbb{R}) / P(F) \cong F_\lambda.$$

Def: For $P(F) \subset GL(E)$,

$\underline{p(F)} = \{ L \in gl(E) \mid L(E_j) \subset E_j,\ j = 1 \sim s \}$

$\underline{n(F)} = \{ L \in gl(E) \mid L(E_j) \subset E_{j+1},\ j = 1 \sim s \}$

Prop: Given $\{\alpha_1, \dots, \alpha_s\}$ complex number, denote

$$S_W(E) = \{ g^{-1} W g \mid g \in GL(n, \mathbb{R}),\ W = \begin{pmatrix} \alpha_1 I_{r_1} & & \\ & \ddots & \\ & & \alpha_s I_{r_s} \end{pmatrix} \}$$

$$\exists\ \pi : S_W(E) \longrightarrow \mathcal{F}_\lambda = GL(n, \mathbb{R}) / P(F)$$

$$S \longmapsto E_j = \bigoplus_{k \geq j} \ker(S - \alpha_k I_r)$$

§2. Parabolic Vector bundle

Let Σ°: Σ with n-marked points $\{p_1, \dots, p_n\}$

$$\rightsquigarrow \pi_1(\Sigma^\circ) \cong \langle \begin{matrix} b_1 \cdots b_g \\ a_1 \cdots a_g \end{matrix},\ r_1 \sim r_n \mid \prod r_i \cdot \prod [a_i, b_i] = 1 \rangle$$

Given a unitary representation:

$$\rho : \pi_1(\Sigma^\circ) \longrightarrow U(r)$$

Def: The normal subgroup generated by $\gamma_1, \ldots, \gamma_n$ of $\pi_1(\Sigma^\circ)$ 称为 parabolic-generator.

$\rightsquigarrow$ Jordan Canonical Form of

$\rho(\gamma_i)$ is determined by: $\alpha_{i_1} \sim \alpha_{i_s}$

$$\rho(\gamma_i) \sim \begin{pmatrix} \alpha_{i_1} I_{r_{i_1}} & & \\ & \ddots & \\ & & \alpha_{i_s} I_{r_{i_s}} \end{pmatrix}$$

Def: A quasi-parabolic structure on a vector bundle $E \longrightarrow \Sigma^\circ$, is a set:

$\{ F_i \in \mathcal{F}_{x_i}(E|_{z_i}) \mid i = 1, \ldots, n \}$

(在 singularity 处给一个 flag)

A parabolic structure is a quasi-para together with weights: $0 \leq \alpha_{i_1} \leq \cdots \leq \alpha_{i s_i} < 1$

multiplicities $r = r_{i_1} + \cdots + r_{i s_i}$ for $\forall i = 1 \sim n$

Def: (Parabolic Degree)

$$\text{par}\deg(E_*) = \deg E + \sum r_{ij} \alpha_{ij}$$

$$\mu(E_*) = \text{par}\deg(E_*) \,/\, \text{rank}(E).$$

Thm: (Mehta – Seshadri)

A parabolic $E_* \to \Sigma$ of degree 0 is stable

$\Longleftrightarrow$ $\exists$ irre unitary repre $\rho : \pi_1(\Sigma_0) \longrightarrow U(r)$

s.t. $E_* \cong E_*^{\rho}$

§3. Parabolic Higgs Bundle

Def: $\mathrm{Higgs}(E_*) := \{ \Phi \in H^0(\mathrm{End}\, E \otimes K_\Sigma(D)) \mid \mathrm{Res}_{z_i} \Phi \in \mathfrak{n}(F_i) \}$

Higgs Field 要有 Singularity.

(In rank 2 $\longrightarrow$ nilpotent)

类似定义 stability — Condition.

Thm:

Stable Parabolic Higgs bundle of $\underset{=0}{\text{pardeg}}$ posses a natural complex mfd structure.

$\mathcal{M}$, s.t.

$$\dim \mathcal{M} = 2(r^2(g-1)+1+\sum_{i=1}^{n} \dim \mathcal{F}_{\lambda_i}(V|_{z_i})$$

§4. Logrithmic Connection

Def: $\theta \in \Omega^1(E_*)$ 称 compatible with the parabolic structure if

$$\mathrm{Res}_{z_i}\theta \in \underline{P(E_i)}.$$

Def: A singular hermitian metric adapted to E_* if for small neighbor $\mathcal{U}_i \setminus \{z_i\}$, $\exists$ unitary frame $\{ e_j / |z-z_i|^{\alpha_{ij}} \}$ with $\{e_1, \dots, e_j\}$ holo frame for $E|_{\mathcal{U}_i}$ and $\{e_1(z_i), \dots, e_r(z_i)\}$ generating F_i.

Thm: (Parabolic Non-abelian Hodge),
A parabolic Higgs bundle (E_*, Φ) of degree 0 is stable $\Leftrightarrow$ it admits irre para harmo bundle.

$(E, \{E_{ij}\}, \{\alpha_{ij}\}, \{r_{ij}\})$ and a pair (ρ, H)

$\rho: \ \Gamma \cong \pi_1(\Sigma_0) \longrightarrow GL(r, \mathbb{C})$, s.t. $\rho(\gamma_i) \in \mathcal{O}(e^{2\pi i W_i})$.

$W_i = \begin{pmatrix} \lambda_{i1} Id & & \\ & \ddots & \\ & & \lambda_{is_i} Id \end{pmatrix}$, $H: \mathbb{H} \longrightarrow GL(r, \mathbb{C}) / U(r)$ is

a ρ- equi which is harmonic.

Blow Up

X be a complex manifold. $Y \subset X$ be closed submanifold, $Bl_Y(X)$ is a complex manifold with a proper holomorphic map $\sigma: \begin{array}{c}\hat{X}\\ \downarrow \\ X\end{array}$

Example: (Blow-Up of a point)

For $\mathcal{O}(-1)$ line bundle of $\mathbb{P}^n$, given as:

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, hook] \arrow[d] & \mathcal{O}(-1) \arrow[r, hook] \arrow[d, "\sigma"'] & \mathbb{P}^n \times \mathbb{C}^{n+1} \arrow[r] \arrow[d] & \mathbb{P}^n \\
\{0\} \arrow[r, hook] & \mathbb{C}^{n+1} \arrow[r, equal] & \mathbb{C}^{n+1} &
\end{tikzcd}$$

σ is another Projection of π:

For $z \in \mathbb{C}^{n+1}$, $z \neq 0$, $\sigma^{-1}(z)$: unique line l_z passes through $z \in \mathbb{C}^{n+1}$. i.e.

$\sigma^{-1}(z_0) = ([l_{z_0}], tz_0)$. $t \in \mathbb{C}$.

$\sigma^{-1}(0) \cong \mathbb{P}^n$. (zero section of $\mathcal{O}(-1) \to \mathbb{P}^n$)

$\therefore$ the Blow up $(Bl_0(\mathbb{C}^{n+1}), \sigma)$ is just

$\underline{(\mathcal{O}(-1), \sigma)}$ (如上)

Remark:

$\mathcal{O}(-1)$ can be seen as a variety of $\mathbb{P}^n \times \mathbb{C}^{n+1}$.

Denote: $(z_0, \dots, z_n)$ coordinate of $\mathbb{C}^{n+1}$

$[(x_0, \dots, x_n)]$ coordinate of $\mathbb{P}^n$.

Then $\mathcal{O}(-1)$ just be solution of equation:

$$\underline{z_i \cdot x_j = z_j \cdot x_i}$$

Blow-up along a Linear subspace.

Let $\mathbb{C}^m \subset \mathbb{C}^n$ As $z_{m+1} = \cdots = z_n = 0$

$\wr$

$(x_{m+1}, \dots, x_n) \in \mathbb{P}^{n-m-1}$ as coordinate.

$z_1 \sim z_m$ 自由变量.

$$Bl_{\mathbb{C}^m}(\mathbb{C}^n) := \{(x, \underline{z}) \mid x_i z_j = x_j z_i\} \subset \mathbb{P}^{n-m-1} \times \mathbb{C}^n.$$

$i, j = m+1 \sim n$.

$\rightsquigarrow \sigma: Bl_{\mathbb{C}^m}(\mathbb{C}^n) \longrightarrow \mathbb{C}^n$ is isomorphism over $\mathbb{C}^n \setminus \mathbb{C}^m$

For submanifold $Y^m \subset X^n$, pick atlas $\{u_i, \varphi_i\}$, then $\{Y \cap u_i, \varphi(u_i) \cap \mathbb{C}^m\}$ is the associated atlas on $Y^m \subset X^n$.

Let $\sigma: Bl_{\mathbb{C}^m}(\mathbb{C}^n) \to \mathbb{C}^n$ be the blow-up of $\mathbb{C}^n$ along $\mathbb{C}^m$.

$\rightsquigarrow \sigma_i : Z_i \longrightarrow \varphi_i(U_i)$ be the restriction

i.e. $Z_i = \sigma^{-1}(\varphi_i(U_i))$, $\sigma_i = \sigma|_{Z_i}$

Now: Each Z_i glues together.

$\rightsquigarrow$ We obtain:

$$\sigma^{-1}(Y) \cong \mathbb{P}(N_{Y/X})$$

Prop: $Y \subseteq X$ submanifold, then $\exists\ \hat{X} = Bl_Y(X)$, together with $\sigma: \hat{X} \to X$, s.t.

$$\sigma: \hat{X} - \sigma^{-1}(Y) \cong X - Y.$$

Prop: Pick $x \in X$. $Bl_x(X) \xrightarrow{\sigma} X$. denote E be $\sigma^{-1}(x)$. then:

$$K_{\hat{X}} \cong \sigma^* K_X \otimes \mathcal{O}_{\hat{X}}((n-1)E).$$

Prop: $x \in X$ be a point of a complex manifold, then: $Bl_x(X) \cong X \# \overline{\mathbb{P}}^n$.

(与 $\mathbb{P}^n$ 相反的 complex-structure).

§. 4. Betti-Moduli-Space.

Vogt-Coordinate.

Let $G = SL(2;\mathbb{C})$，希望给出 $\mathrm{Hom}(\pi, G)$ 之坐标.

先看 simple-Case:

$H = SL(2;\mathbb{C}) \times SL(2;\mathbb{C})$, $G \curvearrowright H \longrightarrow H$

$g \cdot (\xi, \eta) \longmapsto (g\xi g^{-1}, g\eta g^{-1})$

$\chi(x, y, z) = x^2 + y^2 + z^2 - xyz - 2$.

Thm (Vogt, Fricke) (Main-Result)

(i) $f: H \to \mathbb{C}$ be a regular-function invariant under G action, then $\exists\, F(x, y, z) \in \mathbb{C}[x, y, z]$, s.t.

$$f(\xi, \eta) = F(tr(\xi), tr(\eta), tr(\xi\eta))$$

(ii) $\forall (x, y, z) \in \mathbb{C}^3$, $\exists (\xi, \eta) \in H$, s.t.

$$(x, y, z) = (tr(\xi), tr(\eta), tr(\xi\eta))$$

(iii) if $(tr(\xi), tr(\eta), tr(\xi\eta)) = (tr\xi', tr\eta', tr(\xi'\eta'))$

$\rightsquigarrow (\xi', \eta') = g \cdot (\xi, \eta)$ for some $g \in G$.

freely group generated by X, Y

Rk: $\pi := \langle X, Y \rangle$,

then we have:

$$\mathrm{Hom}(\pi, G) \cong H = G \times G$$

$$\rho \longmapsto (\rho(X), \rho(Y))$$

$$\therefore \boxed{\mathrm{Hom}(\pi, G)/G} \cong H/G \cong \mathbb{C}^3,$$

$$\rho \longmapsto (\rho(X), \rho(Y)) \longmapsto (\mathrm{tr}(\rho(X)), \mathrm{tr}(\rho(Y)), \mathrm{tr}(\rho(XY)))$$

Betti-Moduli

Key-Technical:

Cayley-Hamilton-Thm: $\xi \in M_{2\times 2}$, then:

$$\xi^2 - \mathrm{tr}(\xi) \cdot \xi + \det(\xi)\,\mathbb{I} = 0.$$

$$\overset{SL(2,\mathbb{C})}{\rightsquigarrow} \xi - \mathrm{tr}(\xi)\,\mathrm{Id} + \underset{\overset{\shortparallel}{1}}{\underline{\det\xi}} \cdot \xi^{-1} = 0.$$

$$\rightsquigarrow \xi + \xi^{-1} = \mathrm{tr}(\xi)\,\mathrm{Id}.$$

$$\longrightarrow \mathrm{tr}(\xi) + \mathrm{tr}(\xi^{-1}) = 2\,\mathrm{tr}(\xi)$$

Prop:

(1) $tr(\xi) = tr(\xi^{-1})$

(2) $tr(\xi\eta) + tr(\xi\eta^{-1}) = tr(\xi) \cdot tr(\eta)$

(3) Let $\pi = \langle x, y \rangle$, $G = SL(2, \mathbb{C})$, then:

$\rho \in Hom(\pi, G)$ is irreducible $\Leftrightarrow$ $\mathcal{K}(x, y, z) \neq 2$.

For Rank3 Case:

$\pi := \langle X, Y, Z \rangle$. then:

$$Hom(\pi, G) \sqsubset \mathbb{C}^8$$

坐标分别为:

$tr(X)$, $tr(Y)$, $tr(Z)$

$tr(XY)$, $tr(XZ)$, $tr(XZ)$

In Rank4 - Case:

$\pi = \langle M_1, M_2, M_3, M_4 \rangle$ $\mathbb{C}^7$

enough for:

$$x = tr(M_1M_2), \quad y = tr(M_2M_3), \quad z = tr(M_1M_3)$$

及 $\mathrm{tr}(M_1)$, $\mathrm{tr}(M_2)$, $\mathrm{tr}(M_3)$, $\mathrm{tr}(M_4)$

Affine GIT Quotient for Betti-Moduli

Purpose: Classifying Problem: $(\mathcal{A}, \sim)$

Find an algebraic variety $\mathcal{M}$ to describle $\mathcal{A}/\sim$.

§.1. Linear Algebraic Group.

Def: G. an algebraic group admits variety structure, s.t.

(1) $m: \begin{array}{l} G \times G \to G \\ (g, h) \mapsto gh \end{array}$

(2) $i: \begin{array}{l} G \to G \\ g \mapsto g^{-1} \end{array}$

be morphism of variety.

Prop: Alge. grp must be non-singular.

Example:
$$GL(n,k) := \{ x \in M_{n\times n} \mid \det(x) \neq 0 \}$$
$$\cong \{ (g, \lambda) \in M_{n\times n} \times k^* \mid \det g \cdot \lambda = 1 \}$$
$$\subseteq A_k^{n^2+1}$$

$$G_m = (k^*, \times) \cong GL(1,k)$$

Def: $\rho : \pi_1(X, x) \to GL(n, k)$ semi-simple means:

(1) $\rho = \bigoplus_{i=1}^{\ell} \rho_i$, each ρ_i is irreducible.

(2) G be a reductive group means:

$\forall$ fini-dim repre of G is complete-reduci. (or semi-simple)

($\Leftrightarrow$ (Weyl, Nagata, Mumford). unipotent radical is trivial)

Def: G-variety is an algebraic variety X, s.t. $G \curvearrowright X$ (i.e. $G \times X \to X$ also morphism of variety).

Def: (Categorical Quotient)

A catego-Quoti of $G \curvearrowright X$ is (Y, φ), s.t. $\varphi : X \to Y$ is G-inv morphism of varieties which is universal.

$$\left(\begin{tikzcd} & Y \arrow[d, dashed, "\exists ! g"] \\ X \arrow[ur, "\varphi"] \arrow[r, "f"'] & Z \end{tikzcd}\right).$$

Def: (Good Quotient)

$G \curvearrowright X$ is G-inv morphism $\varphi : X \to Y$, s.t.

(1) φ surject + affine

(2) $\mathcal{O}_Y(U) \cong \mathcal{O}_X(\varphi^{-1}(U))^G$

(3) $\forall W \subset X$ G-inv closed $\Rightarrow$ $\varphi(W) \subset Y$ closed

(4) $\forall W_1, W_2 \subset X$, G-invar & closed, s.t. $W_1 \cap W_2 = \phi$

$\longrightarrow \varphi(W_1) \cap \varphi(W_2) = \phi$

Def: (Geometric Quotient)

$G \curvearrowright X$, $\varphi: X \to Y$, s.t. Y is the orbit-space.

$\star$: $G \curvearrowright X$

Def: The affine GIT Quotient is $\varphi: X \to X /\!/ G = \mathrm{Spec}(k[X]^G)$

induced from $k[X]^G \hookrightarrow k[X]$.

Thm: (Hilbert - Mumford)

$x \in X$, $\mathcal{O}_x \subset \varphi^{-1}(\varphi(x))$, then it is a unique closed orbit iff

$\exists$ 1-para $\lambda: \mathbb{G}_m \longrightarrow G$. s.t.

$\lim_{t \to 0} \lambda(t) \cdot x$ exists $\in \mathcal{O}_x$

Now giving the GIT Quotient for $\mathcal{M}_{Betti}$

§ Local System: Representation of Fundamental - Group.

Def:

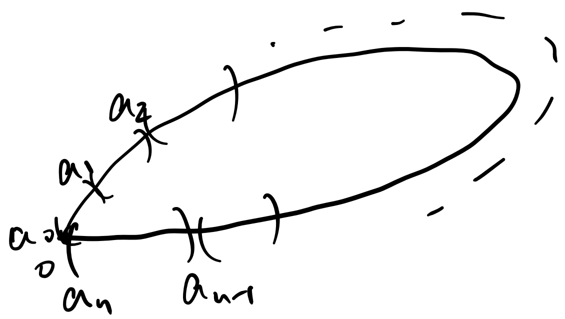
A k-local System of rank n is a rank n locally constant sheaf of k-vector space $\tilde{\mathcal{F}}$.

$\mathcal{C}_{Loc}(X,n)$: k-local system of rank n on X.

$\mathcal{C}_{Rep}(X,n)$: funda group repre $\rho: \pi_1(X,x) \to GL(n,k)$

Thm: $\mathcal{C}_{Loc}(X,n) \cong \mathcal{C}_{Rep}(X,n)$

pf: $(\Rightarrow)$ $\mathcal{F}$, $\forall \gamma: [0,1] \longrightarrow X$



$$\begin{tikzcd} \mathcal{F}|_{a_0} \cong \cdots \cong \mathcal{F}|_{a_n} \end{tikzcd}$$
$$\mathcal{F}|_{a_0} \xrightarrow{\rho(\gamma)} \mathcal{F}|_{a_n}$$

$(\Leftarrow)$ $E = \tilde{X} \times_\rho k^n / (x, \ell) \sim (\gamma \cdot x, \rho(\gamma)^{-1}\ell)$

$\mathcal{E}(E)$: sheaf.

X smooth, non-singular, irreducible, proj-varie $/k$

fix $x \in X$

$\rightsquigarrow \pi_1(X,x) := \langle \gamma_1, \dots \gamma_g, \gamma_1', \dots, \gamma_g' : \begin{matrix} r_1(\gamma_1, \dots, \gamma_l) = 0 \\ \vdots \\ r_m(\gamma_1, \dots, \gamma_l) = 0 \end{matrix} \rangle$

(Variety of $GL(n,k)^{2g}$)

Def: $\mathcal{U}(X,x,n) := \mathrm{Hom}(\pi_1(X,x), GL(n,k))$

$$= \{\rho : \pi_1(X,x) \to GL(n,k)\}$$

Embedding:

$$\mathcal{U}(x,n) \hookrightarrow GL(n,k)^{\ell}$$

$$\rho \longmapsto (\rho(\gamma_1), \dots, \rho(\gamma_\ell))$$

$GL(n,k) \curvearrowright \mathcal{U}(X,n)$ by conjugation

$$g \triangleright \rho := g \cdot \rho \cdot g^{-1}$$

$\longrightarrow$ Affine GIT Quotient

$$\varphi : \mathcal{U}(X,n) \longrightarrow \mathcal{U}(X,n) /\!/ GL(n,k)$$

$$:= \mathrm{Spec}(k[\mathcal{U}(X,n)]^{GL(n,k)})$$

$$:= \mathcal{M}_B(X,n)$$

For $\dim_k X = 1$

(1) $g=0$, $\mathcal{M}_B(X,n) = \begin{cases} 0, & n \geq 2 \\ \{pt\}, & n=1 \end{cases}$

(2) $g \geq 2$, $\mathcal{M}_B(X,n) \cong (\mathbb{C}^*)^{2g}$

Def: (Stability)

a. ρ is polystable if $\boxed{GL(n,k)\cdot\rho \subset \mathcal{M}(X,n)}$ closed

($\Leftrightarrow$ $\lim_{t\to 0}\lambda(t)\cdot\rho$ 不存在)

stable if ① ✓

$+$

② $\dim(GL(n,k)\rho) = \dim(GL(n,k)/\mathrm{stab})$

b. $\rho_1 \sim_s \rho_2$ if $\overline{GL(n,k)\rho_1} \cap \overline{GL(n,k)\rho_2} \neq \phi$

Lemma:

(1) ρ is stable $\Leftrightarrow$ ρ is irredu

(2) ρ is polystable $\Leftrightarrow$ ρ is completely reducible.

Lemma. (1) ρ is stable $\Leftrightarrow$ ρ is irredu

(2) ρ is poly $\Leftrightarrow$ ρ is comple reduci.

pf: (Hilbert–Mumford):

For $\rho \in \mathcal{M}(X, n)$, $\exists$ 1-ps $\lambda: \mathbb{G}_m \to GL(n,k)$, s.t.

$$\lim_{t\to 0} \lambda(t)\rho := \lim_{t\to 0} \lambda(t)\rho\lambda(t)^{-1} \text{ exists } \& \in \mathcal{O}_\rho.$$

(unique closed orbit inside).

In parti, ρ stable $\Leftrightarrow$ $\lim_{t\to 0}\lambda(t)\cdot\rho$ 不存在. (for $\forall$ non-tri $\lambda: \mathbb{G}_m \to GL(n,k)$).

(1) $\rho: \pi_1(X,x) \to GL(n,k)$ stable $\Leftrightarrow$ $\lim_{t\to 0}\lambda(t)\rho$ 不存在.

$\overset{\text{Borel}}{\Longleftrightarrow}$ up to conju, $\lambda(t) = \begin{pmatrix} t^{a_1} & & \\ & \ddots & \\ & & t^{a_n}\end{pmatrix} \in GL(n,k)$.

$a_1 \geq a_2 \geq \cdots \geq a_n \in \mathbb{Z}$ w.r.t $\{e_1, \ldots, e_n\}$.

write $a_1 = a_2 = \cdots = a_{n_1} > a_{n_1+1} = \cdots = a_{n_1+n_2} > \cdots\cdots = a_{n_1+\cdots+n_m} = a_n$.

write $\rho(\gamma) = \begin{pmatrix} \rho_{11} & \cdots & \rho_{1n} \\ \vdots & \ddots & \vdots \\ \rho_{n1} & \cdots & \rho_{nn}\end{pmatrix}$ $\Rightarrow \lambda(t)\rho = \lambda(t)\rho\lambda(t)^{-1}$

$$= \begin{pmatrix} \rho_{11} & t^{a_1-a_2}\rho_{12} & \cdots & t^{a_1-a_n}\rho_{1n} \\ \vdots & & \ddots & \vdots \\ t^{a_n-a_1}\rho_{n1} & & \cdots & \rho_{nn}\end{pmatrix}$$

$\Rightarrow \lim_{t\to 0}\lambda(t)\rho\ \exists \Leftrightarrow \rho_{ij} = 0,\ a_i < a_j$

$\Leftrightarrow \rho(\gamma) = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{n1} \\ \vdots & \vdots & & \vdots \\ \rho_{n_1,1} & \vdots & & \vdots \\ 0 & \rho_{n_1+n_2,1} & & \\ \vdots & 0 & & \\ 0 & 0 & & \end{pmatrix} = \begin{pmatrix} 0 & & & \\ \vdots & \vdots & & \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

$\Leftrightarrow \rho$ preserves a flag $0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_m = k^n$.

$W_i = \text{span}\langle e_1, \ldots, e_{n_1+\cdots+n_i}\rangle \rightsquigarrow$ reduci

$\therefore \rho$ stable $\Leftrightarrow \rho$ irredu

(2) polystable $\Leftrightarrow$ comple reducible.

(limit-pt 若不在 orbit 之上 $\rightsquigarrow$ not orbit closed).

$(\Leftarrow)$: ρ c.r., $\exists$ non-trivial flag $0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_m = k^n$ preserved by ρ.

$\rho = \begin{pmatrix} & * \\ 0 & \end{pmatrix}$, $\rho' = \lim_{t\to 0}\lambda(t)\rho = \begin{pmatrix} & 0 \\ 0 & \end{pmatrix}$ $\to \exists$ complement $W_1^\perp$ of W_1

which is ρ-invariant. $\rho \& \rho'$ act on $W_1 \& W_1^\perp$ in same way

$\therefore \rho \sim \rho'$

$$\longrightarrow \quad \mathcal{M}_B(X,n)(k) \cong \mathcal{U}^{ss}(X,n)(k) / GL(n,k)$$

$$\cong \mathcal{U}(X,n)(k) / \sim_S$$

☆:

$\forall\ \overline{GL(n,k)\,\rho} \supset GL(n,k)\,\underline{\rho'}$ polystable + unique

ρ': semi-simple repre of ρ (称半单化)

Lemma:

$\forall \rho \in \mathcal{U}(X,n)$ admits a Jordan-filtration,

$0 \subsetneq \rho_1 \subsetneq \cdots \subsetneq \rho_m = \rho$, s.t. $g_{n_i} = \rho_i / \rho_{i-1}$ is

irreducible.

$$\therefore \quad \mathcal{M}_B(X,n) \xleftarrow{\ \varphi\ } \mathcal{U}(X,n)$$

$$[gr(\rho)] \longleftarrow\!\!\mapsfrom \rho$$

半单化