pf: define $\mathcal{J}(p,\lambda) = (p, -\bar{\lambda}^{-1})$, in S^2

cases, it sends $(x,y,z) \longmapsto (-x,-y,-z)$

then: $d\mathcal{J}: T_{\lambda_0}S^2 \longrightarrow T_{-\bar{\lambda}_0^{-1}}S^2$

(不依赖于 M 上的 hyperkahler-structure).

故: For $l \in \mathbb{C}$, we have:

$$d\mathcal{J}\left(l\frac{\partial}{\partial z}\right) = \frac{\bar{l}}{(\bar{\lambda}_0)^2}\frac{\partial}{\partial \bar{z}} = \bar{l}\, d\mathcal{J}\left(\frac{\partial}{\partial z}\right)$$

② Just check by definition.

RK: The map $\mathcal{J}$ is anti-holo.

(As $\mathcal{J}$ holomorphic $\Leftrightarrow$ $d\mathcal{J}$ is complex linear).

Def: $\mathcal{S} := \{ s \mid s: \mathbb{C}P^1 \to P$ is a $\overline{\text{holomor}}^{\text{real}}$ section$\}$

Que: What is the smooth structure of $\mathcal{S}$?

Def: (Real-Involution)

$$\mathcal{T}: \mathcal{S} \longrightarrow \mathcal{S}$$

$$s \longmapsto \mathcal{T}(s)\left(\lambda \longmapsto (s(-\bar{\lambda}^{-1}))\right)$$

$\therefore \mathcal{T} \circ \mathcal{T} = id,$

Def: (Real - Sections)

The real holomorphic sections $s \in \mathcal{T}$ satisfies : $\mathcal{T}(s) = s$ ($s(\lambda) = s(-\bar{\lambda}^{-1})$)

Rk:

For real-sections : $\mathcal{T}: \begin{array}{ccc} \mathcal{S}_{\mathbb{R}} & \longrightarrow & \mathcal{S}_{\mathbb{R}} \\ s & \longmapsto & s \end{array}$

$\therefore \mathcal{T}$ induces a map:

$$\begin{array}{ccc} T_s\mathcal{S} & \longrightarrow & T_s\mathcal{S} \\ \| & & \| \\ H^0(\mathbb{CP}^1, N_s) & & H^0(\mathbb{CP}^1, N_s). \end{array}$$

By formal discussion:

$\mathcal{J}$ induces an anti-complex linear tangent map.

Corallary: $\mathcal{M}$ is given by some subset of the space of real holo-sections.

Morever, we have the evaluation map: $\forall \lambda \in \mathbb{CP}^1$

$$\begin{array}{cccc} ev_\lambda : & \mathcal{S}_{\mathbb{R}} = \mathcal{M} & \longrightarrow & \mathbb{P}_\lambda \\ & S & & S(\lambda) \end{array}$$

which is a local diffeomorphism.

Moduli-Space of λ-Connections

(Hodge-Moduli-Space).

Def: A λ-connection on a cpx-vector bundle $\begin{matrix} V \\ \downarrow \\ \Sigma \end{matrix}$ is a triple $(\lambda, \bar\partial, D)$, s.t.

① $D(fs) = \partial f \cdot s + f \cdot Ds$

② $(\bar\partial + D)^2 = 0$

Rmk: Exterior-Derivative.

For $s \in \Gamma(V)$, $\omega \in \Omega^k(V)$, then:

$$\nabla(s \otimes \omega) = \nabla s \wedge \omega + (-1)^k s \wedge \nabla \omega.$$

$\square$

Rk: $(\bar\partial + D)^2$ is a tensoring.

$$(\bar\partial+D)^2(fs) = (\bar\partial+D)(\overset{0}{\bar\partial} f \cdot s + f\bar\partial s + \partial f \cdot s + fDs)$$

$$= \overset{0}{\partial} \cancel{f} \cdot \bar\partial s - \bar\partial \cancel{\partial} \overset{0}{f} \cdot s + \cancel{\bar\partial} \overset{0}{f} Ds - \cancel{f} \bar\partial Ds. = 0$$

$$+ \bar\partial \cancel{f} \cdot Ds - D \cancel{\bar\partial} \overset{0}{f} - \partial \cancel{f} \bar\partial s - f \cancel{D} \bar\partial s.$$

How to understand that $(\bar\partial+D)^2$ vanish?

Let $s_1 \sim s_r$ be a holomorphic frame on $\mathcal{U} \overset{\circ}{\subseteq} \Sigma$,

Write: $D = \lambda\partial^\circ + \Phi$ for some $\Phi \in \Gamma(K\,End(V))$

$\therefore 0 = (\bar\partial + D)^2 s_k = \bar\partial D s_k = \bar\partial(\lambda\partial^\circ s_k + \Phi s_k)$

$= \bar\partial(\Phi s_k) = 0$

$\therefore \Phi \in H^\circ(K\,End(V))$

$\begin{cases} \text{(i) } \bar\partial + \frac{1}{\lambda}D \text{ flat} \\ \text{(ii) } D \in H^\circ(K\,End\,V) \end{cases}$

Prop: Let $(\lambda, \bar\partial, D)$ be a λ-connection. if locally $D = \lambda\partial^\circ + \Phi$, then $\Phi \in H^\circ(K\,End\,V)$

Prop: The fiber at $\lambda = 0$ is the Higgs Pairs.

Solution: $\lambda = 0$ $\therefore D(fs) = Ds$ $\therefore D$ is a tensor

$\Phi \in H^\circ(K\,End\,V)$, $\therefore (0, \bar\partial, \Phi) \rightsquigarrow$ Higgs-Pair.

Prop: the fiber at $\lambda=1$ corresponds to flat-connection.

$(1, \bar{\partial}, D)$. take $\nabla = \bar{\partial} + D$, then $F^{\nabla} = \nabla \circ \nabla$

$= \bar{\partial} D + D \bar{\partial} = (\bar{\partial} + D)^2 = 0 \quad \therefore F^{\nabla} = 0$

$\therefore \nabla$ vanishes.

Prop: For $\lambda \neq 0$, the $\bar{\partial} + \frac{1}{\lambda} D$ is a flat connection.

Rmk:

The twistor space of M_{SD} should be the λ-connections.

Def: We call a λ-connection to be of type $SL(r,\mathbb{C})$ $\Leftrightarrow$ the induced λ-conn on $\Lambda^r V$ is trivial.

Check: $(\bar\partial, D, \lambda)$, on
$$\begin{array}{c}\mathbb{C}^r\\ \downarrow\\ V\\ \downarrow\\ \Sigma\end{array}$$
, then:

$$\bar\partial(s_1\wedge\cdots\wedge s_r) = \sum_i \cdots\wedge\bar\partial s_i\wedge\cdots$$

$$D(s_1\wedge\cdots\wedge s_r) = \sum_i \cdots\wedge D s_i\wedge\cdots$$

注: We impose that $\Lambda^r V = \mathbb{C}$ (Trivial)

Prop: A λ-connection is of type $SL(r,\mathbb{C})$

$\Leftrightarrow$ We find holomorphic-frames of V w.r.t $\bar\partial$, s.t.

① The cocycle $\in SL(r,\mathbb{C})$

② w.r.t to such holo-frame, D can be repre by a conn 1-form $D=\lambda\partial^\circ+\omega$, $\omega\in \Gamma(U, K\mathfrak{sl})$.

Remark: if $\nabla = d+\omega$ on $\begin{array}{c}\mathbb{C}^r\\ \downarrow\\ V\\ \downarrow\\ \Sigma\end{array}$, then the indu

connection $\nabla^{\wedge^r V} = d + \mathrm{tr}\,\omega$

Compute: $\nabla^{\wedge^r V}(s_1\wedge\cdots\wedge s_r) = \sum \cdots\wedge \nabla s_i \wedge \cdots$

$= \sum \cdots \wedge \omega s_i \wedge \cdots$

$\omega s_i = \omega_{ii} s_i$, $\therefore RHS = (\sum_i \omega_{ii})\, s_1\wedge\cdots\wedge s_r$.

Def: A $SL(2,\mathbb{C})$ λ-connection is called stable $\Leftrightarrow$ $\forall$ invariant line subbundle $L\hookrightarrow V$ (L is parall w.r.t $\frac{1}{\lambda}D+\bar\partial$)

satisfies $\deg L<0$ 此即

Example: for $\lambda=0$: $(0, \bar\partial, \Phi)$

$\therefore L\hookrightarrow V$, with Φ-invariant

(此即 Hitchin 定义的 stable pair).

Prop: For $\lambda \neq 0$, $\forall$ λ-connection is semi-stable.

pf: $\forall L \hookrightarrow V \to \frac{1}{\lambda}D + \bar{\partial}$ flat. L is parallel

$\rightsquigarrow$ L is equipped with flat connection

$\therefore F^{D^L} = 0 = \deg L$

Rk: A λ-connection with $\lambda \neq 0$ is stable $\Leftrightarrow$ $\nabla := \frac{1}{\lambda}D + \bar{\partial}$ is irreducible

pf: So the only possibility is that no such L exists $\Leftrightarrow$ ∇ is irreducible.

Rmk:

Stability is needed to get good moduli-space.

1) If we don't impose stability, $\exists$ gauge

orbits which becomes close to each other. (stable-Higgs Pair 与 unstable one 交仿!)

2) If we allow unstable points like

$$\left(\begin{pmatrix} \bar{\partial}^{s^*} & 0 \\ q & \bar{\partial}^s \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} \right), \text{ then}$$

we'll usually get a non-trivial stablizer like $g = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}$, $\omega \in H^0(K)$

(Stablizer: For $\mathcal{G} \curvearrowright \mathcal{M}_{Hod}$, the stablizers are $g \in \mathcal{G}$, $\nabla \cdot g = \nabla$).

(Action 之后是自身) (会使 orbits 有不同的 dimension!)

(若 $\mathcal{M}/\mathcal{G}$ 是 mfd. 须要此作用是 proper 及 fixed point free 的) 产生 singularity.

Define:

The Hodge Moduli Space is:

$$\mathcal{M}^{s}_{Hod} := \{\text{all stable } SL(2,\mathbb{C})\ \lambda\text{-connections}\}/G.$$

$(\bar\partial, D, \lambda)$ can be seen as product of 3 Banach-Spaces.

$\bar\partial \in \Gamma(\bar K sl)$, we can equip $\Gamma(\bar K sl)$ a $\|\cdot\|_K$-norm:

$$\|s\|_K^2 = \int_\Sigma \left(|s|^2 + |\nabla s|^2 + \dots + |\nabla^k s|^2\right)$$

Thm:

$\mathcal{M}^{s}_{Hod}$ is $T_2 + A_2$.

pf: T_2,

if not, $\exists$ $\Phi_1 = (\bar{\partial}^1, D^1, \lambda_1)$ become
$\Phi_2 = (\bar{\partial}^2, D^2, \lambda_2)$

infinite close: (i) $\lambda_1 = \lambda_2$

(ii) $\exists\ \Phi^n \longrightarrow \Phi_1$, $g_n \in \mathcal{G}$,

s.t. $\Phi^n g_n \longrightarrow \Phi_2$.

$\bigstar$: If a sequence is bounded in Sobolev-Space, by Banach-Alaoglu,

$\exists$ subsequence is weak-convergent.

设 $\{g_n\} \to g \in \Gamma_{W^2}(\Sigma, SL)$

$\therefore (\bar{\partial}_2 \otimes \bar{\partial}_1^*) g = 0$ $\quad \therefore \bar{\partial}_1 \circ g = g \circ \bar{\partial}_2$

if g isn't invertible, take $L \hookrightarrow \ker g$

$\therefore \bar{\partial}_1$ leaves L invariant. which makes contradiction

Moduli-Space of "λ-Connection"

We have given its topology by:

$$I_{W^2}(\bar{K}_S L)\times I_{W^2}(K_S L)\times \mathbb{C} \longleftrightarrow \{\lambda\text{-Connection}\}$$

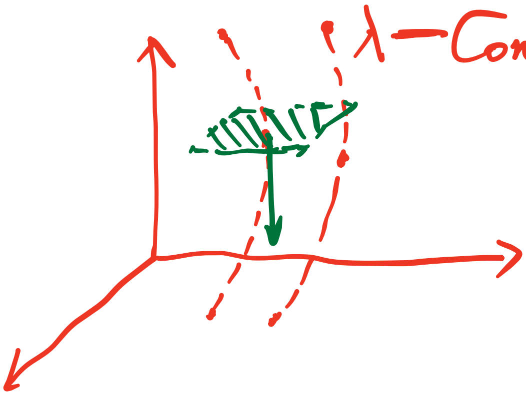
$$(\gamma,\ \omega,\ \lambda)\longmapsto(\bar{\partial}^{\circ}+\gamma,\ \lambda\partial+\omega,\ \lambda)$$

$$\mathcal{M}^{s}_{Hod} = \{\lambda\text{-Connections}\}\big/\mathcal{G}$$

(gauge quotient)

RK: (一般Lie-Group上给Manifold-Structure)

λ-Connection



① Look at tangent-Space

② Take Complementary Space

③ Use IFT to give a manifold structure.

Now Research $\mathcal{M}^{S}_{Hod}$:

先看 non-integrable 之情形

(i) tangent Space.

Let $g(t) \in \mathcal{G}$, $g(0) = id$. Given $\Phi = (\bar\partial, D, \lambda)$ be a "λ-connection", $g(t) = \exp_{id}(t\xi)$.

$$\frac{d}{dt}\Big|_{t=0} (\Phi . g(t)) = (\bar\partial\xi, D\xi, 0).$$

(用 $\bar\partial = \gamma$ 写的话: $\bar\partial\xi = \gamma\xi - \xi\gamma$).

(ii) Take Complementary Space

As a vector space, the original tangent is just $\bar{I}_{W^2}(\bar K sL) \times I_{W^2}(KsL)$, now do the quotient (看成 complex)

$$0 \longrightarrow I_{W^3}(\Sigma, sL) \underset{\mathcal{D}_1^*}{\overset{\mathcal{D}_1}{\rightleftarrows}} I_{W^2}(\bar K sL) \oplus I(KsL) \underset{\mathcal{D}_2^*}{\overset{\mathcal{D}_2}{\rightleftarrows}} I(K\bar K sL) \longrightarrow 0$$

$$\xi \longmapsto (\bar\partial\xi, D\xi)$$

Thus, the complementary space is just: $L^2_{W^2}(\overline{K}SL) \oplus L^2_{W^2}(KSL) / \operatorname{Im} \mathcal{D}_1$.

Each vector space is Hilbert $\therefore \operatorname{Im} \mathcal{D}_1 = (\ker \mathcal{D}_1^*)^\perp$

Let $\mathcal{H} := \ker \mathcal{D}_1^*$ a closed subspace

Then $\mathcal{G}$-action restricts to $\mathcal{H}$, $\exists$ neigh $\mathcal{U}_1 \subseteq \mathcal{H}$, $\mathcal{U}_2 \subseteq G_1$

s.t. $\mathcal{U}_1 \cdot \mathcal{U}_2 \longrightarrow \mathcal{M}^s_{Hod}$ local-diffeo

$(\bar{\partial}, D, \lambda) \cdot g \longmapsto [(\bar{\partial}, P, \lambda) \cdot g]$

claim: $\exists\, 0 \in \tilde{\mathcal{U}}_1 \subset \mathcal{U}_1 \times \mathbb{C}$, s.t.

$\tilde{\mathcal{U}}_1 \times G$ is global-diffeo if not: Any neighbor of D, $\exists D_1, D_2$ which is gauge-equivalent.

$\therefore \exists D_1^n, D_2^n \xrightarrow{(1)} D$, (2) $\exists g_n$ $D_2^n = D_1^n \cdot g_n$

$\therefore (D_2^n \otimes D_1^{n*}) g_n = 0$ $\therefore \exists$ limit $g_n \to g$, s.t.

$(D \otimes D^*) g = 0$ $\therefore g$ is smooth. As D is stable $\rightsquigarrow g = \pm id$ $\therefore \underline{g_n \longrightarrow id}$. (在$u_1 \cdot u_2$同胚)

Contradicting to the IFT for the local gauge group action.

$\rightsquigarrow \forall$ gauge-orbit of $\tilde{D}$ near D is repre by a unique element in $\hat{u}_1 \subseteq \mathcal{H} \oplus \mathbb{C}$

"Slice-Theorem"

我们由此给出了 all λ-connection 之 smooth structure.

现在：把 integrable λ - connection 看成了其上的 submanifold.

注意到：$\mathcal{D}_2(\bar\partial^0+\gamma,\ \lambda\partial^0+\omega,\ \lambda)$

$$= (\lambda\partial^0+\omega)\gamma + (\bar\partial^0+\gamma)\omega$$

$$= \lambda\partial^0\gamma + \bar\partial^0\omega + [\omega,\gamma]$$

denote $F = \mathcal{D}_2$

Thus:

$$d_{\mathcal{D}}F(\dot\gamma,\dot\omega) = \lambda\partial^0\dot\gamma + \bar\partial^0\dot\omega + [\dot\omega,\gamma] + [\omega,\dot\gamma]$$

$$0 \longrightarrow \Gamma(\Sigma, sL) \longrightarrow \Gamma(\bar K sL)\oplus\Gamma(K sL) \longrightarrow \Gamma(K\bar K sL) \to 0$$

$$\xi \overset{G_*}{\longmapsto} (\bar\partial\xi, D\xi)$$

$$(\dot\gamma,\dot\omega) \overset{d_{\mathcal{D}}F}{\longmapsto} \lambda\partial^0\dot\gamma + \bar\partial^0\dot\omega + [\dot\omega,\gamma] + [\omega,\dot\gamma]$$

$$\chi(d^{\mathcal{D}}) \quad \mathrm{H}^0(d^{\mathcal{D}}) - \mathrm{H}^1(d^{\mathcal{D}}) + \mathrm{H}^2(d^{\mathcal{D}})$$

By deforming, this $\chi(d^{\mathcal{D}})$ is same as de-Rham complex.

$\therefore \chi(d^{\mathcal{D}}) = (2g-2)\cdot \dim(sl(r,\mathbb{C}))$
$= 6g-6+1$ (再加上$\mathbb{C}$-项)

$\therefore$ The dimension of stable integrable λ-connections is a complex-mfd of dim $6g-5$.
($d_{\mathcal{D}}\mathcal{F}$ is surjective)

Prop: $f: X \to \mathcal{M}_{Hod}$ is holomorphic
$\Leftrightarrow$ f admits locally a holomorphic lift.
(易见这个 Group-Action preserves complex structure.)

Rk: If we have a harmonic map:

$f: \Sigma \longrightarrow SU(2); \mathbb{H}^3$, then we get a family

of flat connections $\nabla^\lambda = \nabla + \lambda^{-1}\Phi + \lambda\Phi^*$. thus:

We obtain a map:

$$\mathbb{C} \longrightarrow \mathcal{M}_{Hod}$$

$$D(\lambda): \quad \lambda \longmapsto [\bar{\partial}^\nabla + \lambda\bar{\Phi}^*, \lambda\partial^\nabla + \Phi, \lambda]$$

Prop: $D(\lambda)$ is stable.

pf: By previous discussion, this is equivalent

to ∇^λ is irreducible.

For $\mathbb{H}^3$ – case, this ✓

$SU(2)$ – case, ✗

This curve is basically a twistor-line

associated with (∇, Φ, h) of solution.

(We need to give more structures to

$\mathcal{M}_{Hod}$ make it be a twistor-space)

Define: on the total space of all u.n.i λ-connections, the bilinear-form:

$$\Omega: (\mathcal{I}(\bar{K}SL)\oplus\mathcal{I}(KSL))\times(\cdots\cdots)\longrightarrow\mathbb{C}$$

$$((X_1,y_1),\ (X_2,y_2))=\int_{\Sigma}tr(X_1\wedge y_2-X_2\wedge y_1)$$

Prop: Ω is skew-symmetric + constant.

(不依赖坐标之选取)

$\downarrow$ push-down to moduli-space.

$$\Omega:\ T_{[\phi]}\mathcal{M}_{Hod}\times T_{[\phi]}\mathcal{M}_{Hod}\longrightarrow\mathbb{C}$$

Well-def: ① Let $\xi\in\mathcal{I}_{w^2}(SL)$, $\Omega(D_1\xi,(\gamma,\omega))$

$=\Omega((\bar{\partial}\xi,D\xi),(\gamma,\omega))=\int_{\Sigma}tr(\bar{\partial}\xi\wedge\omega-\gamma\wedge D\xi)$

stokes $-\int_{\Sigma}tr(\xi(\bar{\partial}\omega+D\gamma))=0$

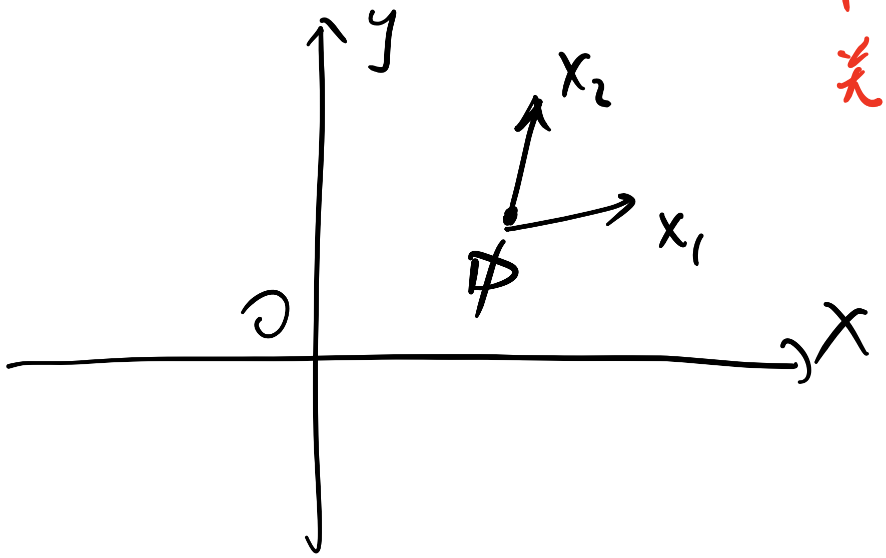
② For $g\in\mathcal{G}$ (gauge-group)

$$\Omega((X_1,y_1).g,\ (X_2,y_2).g)=\Omega((X_1,y_1),(X_2,y_2))$$

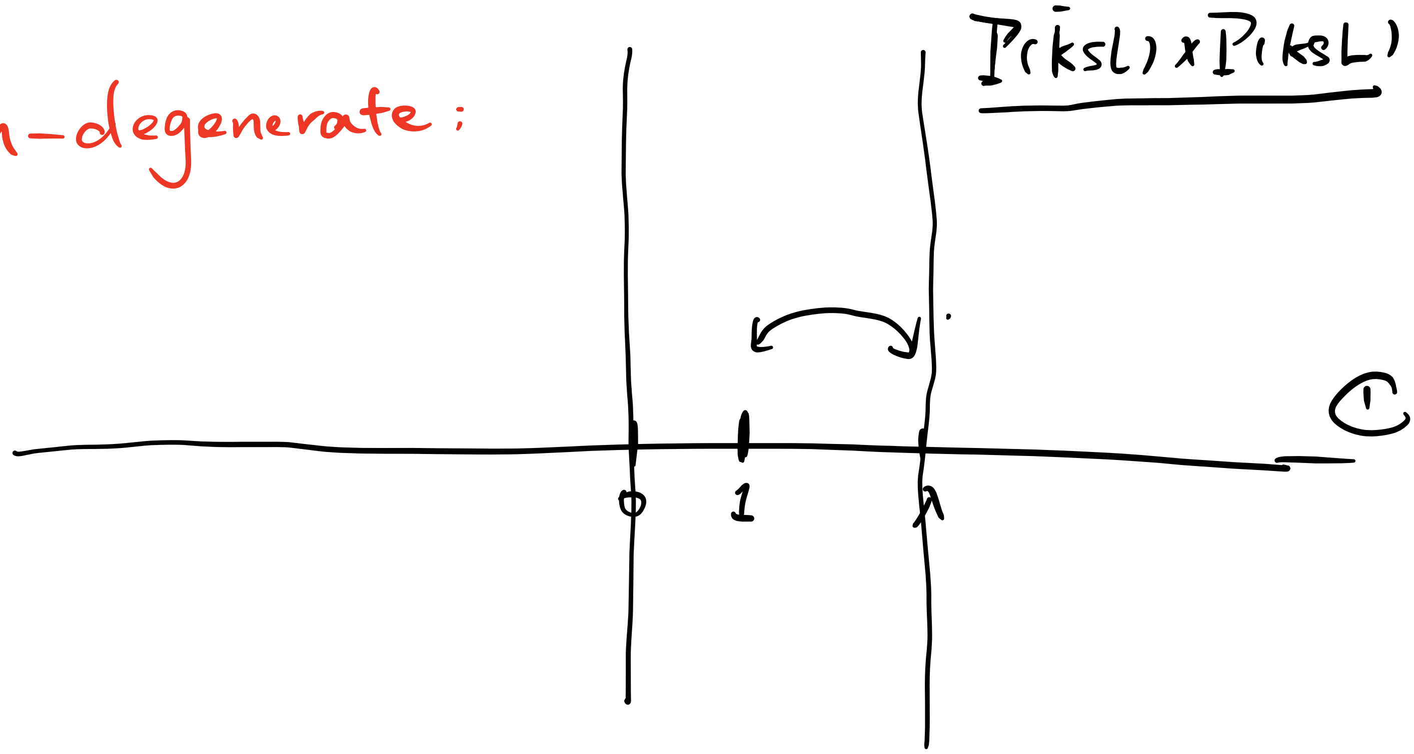
∵ trace 是可以交换的，故知以上成立.

Thm: $\exists\ \Omega \in H^0(\mathcal{M}_{Hod},\ \Lambda^2(\ker d\pi)^*)$ which is fiberwise a closed holomorphic 2-form which is non-degenerate away from a codim 1 submanifold.

(这个 Ω 与 Φ 之选取毫无关系 $\rightsquigarrow$ Ω is constant + closed!



non-degenerate:



We have an action $\mathbb{C}^* \curvearrowright \Omega$ which is isomorphic $\rightsquigarrow$ 只须在一根 fiber上看即可

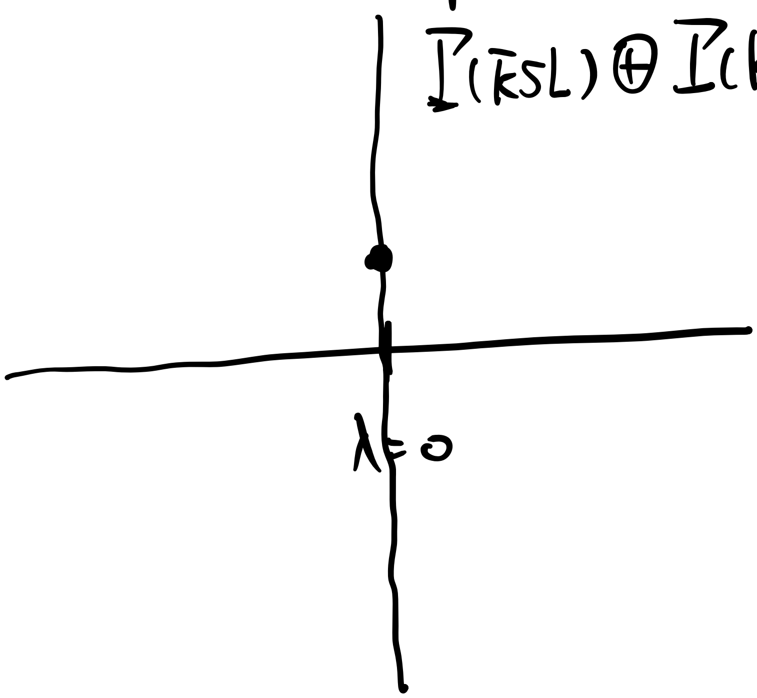
现在看:

$$\Omega^{(n)} = \underbrace{\Omega \wedge \cdots \wedge \Omega}_{n\text{-times}},$$

先算 $\Omega|_{\lambda=0}$: $\phi: [\bar\partial, \Phi, 0]$ stable Higgs Pair.

$$\mathcal{M}_{Hod}|_{\lambda=0} = \mathcal{M}_{DoL} = \pi^{-1}(\lambda=0)$$

Now we compute tangent Vector at $\lambda=0$.



At the point $(\bar\partial, 0, 0)$

$$T_{(\bar\partial,0,0)}\mathcal{M}_{Hod} = T_{(\bar\partial,0)}\mathcal{M}_{DoL}$$

$$0 \to \Gamma(SL) \to \Gamma(\bar{k}SL) \oplus \Gamma(kSL) \to \Gamma(k\bar{k}SL) \to 0$$

$$\xi \longmapsto (\bar{\partial}\xi, \ 0)$$

$$(\gamma, \ \omega) \longmapsto \bar{\partial}\omega$$

$\therefore T_{(\bar{\partial},0)} \mathcal{M}_{Dol} \simeq H^1(SL) \oplus H^0(KSL)$

And:

Ω is just the Serre-Duality.

Delign-Gluing.

Purpose: Glue: $\mathcal{M}_{Hod}(\Sigma)$ with $\mathcal{M}_{Hod}(\bar{\Sigma})$

$$\begin{array}{ccc} \mathcal{M}_{Hod}(\Sigma) & & \mathcal{M}_{Hod}(\bar{\Sigma}) \\ \downarrow & & \downarrow \\ \mathbb{C} & & \mathbb{C} \end{array}$$

$\bar{\Sigma}$: if (U, z) be holo-chart of Σ. then $(U, \bar{z})$ be $\bar{\Sigma}$'s holo-chart.

Example:

if ∇ be a irre unitary flat connection.

if $L \hookrightarrow V$, $\rightsquigarrow \deg L < 0$, then: $L^\perp$ has positive degree $\rightsquigarrow$ but:

$L^\perp \hookrightarrow V$ be a holomorphic subbundle for ∂^∇ on $V \to \bar{\Sigma}$, which is all $\deg(L^\perp) < 0$.

$\deg_{\bar{\Sigma}}(L^\perp) = -\deg_{\Sigma}(L^\perp) = \deg_{\Sigma}(L) < 0$.

In $V = L \oplus L^\perp$, $\bar{\partial}^\nabla = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, then by

Chern - equation: $\partial^\nabla = \begin{pmatrix} * & 0 \\ b^* & \boxed{*} \end{pmatrix} \to L^\perp$ is holomorphic.

Consider a $\overbrace{\lambda\text{-connection}}^{\text{integrable}}$

$$\cancel{D} = (\bar{\partial}, D, \lambda), \text{ then:}$$

$\bar{\partial} + \frac{1}{\lambda} D$ is flat on $\begin{matrix} V \\ \downarrow \\ \Sigma \end{matrix}$ and $\begin{matrix} V \\ \downarrow \\ \bar{\Sigma} \end{matrix}$, then:

$(\frac{1}{\lambda}D, \frac{1}{\lambda}\bar\partial, \frac{1}{\lambda})$ is a $\tilde\lambda = 1/\lambda$ connection

on $\begin{matrix} V \\ \downarrow \\ \bar\Sigma \end{matrix}$.

Define:

$$\mathcal{F}: \mathcal{M}_{Hod_\Sigma} \longrightarrow \mathcal{M}_{Hod_{\bar\Sigma}}$$

$$(\bar\partial, D, \lambda) \longmapsto (\tfrac{1}{\lambda}D, \tfrac{1}{\lambda}\bar\partial, \tfrac{1}{\lambda})$$

Prop: (1) $\mathcal{F}(\phi)$ is integral $\Leftrightarrow$ ϕ is integral.

(2) $\forall g \in \mathcal{G}$. $\mathcal{F}(\phi \cdot g) = \mathcal{F}(\phi) \cdot g$

(3) ϕ is stable on $\begin{matrix} V \\ \downarrow \\ \Sigma \end{matrix}$ $\Leftrightarrow$ $\mathcal{F}(\phi)$ is stable on $V \to \bar\Sigma$.

$L \hookrightarrow V$, $\deg L < 0 \rightsquigarrow L^\perp \hookrightarrow V_{\bar\Sigma}$, $\deg L^\perp < 0$.

Prop: The map $\mathcal{F}$:

$$\mathcal{M}_{Hod}|_{\mathbb{C}^*}(\Sigma) \longrightarrow \mathcal{M}_{Hod}|_{\mathbb{C}^*}(\bar\Sigma)$$

is holomorphic.

Def: (Delign - Hitchin Moduli).

$$\mathcal{M}_{DH} := \mathcal{M}_{Hod}(\Sigma) \cup_{\mathcal{f}} \mathcal{M}_{Hod}(\bar{\Sigma})$$

with projection:

$$\pi((\bar{\partial}, D, \lambda)_{\Sigma}) = \lambda$$

$$\pi((\partial, \bar{D}, \tilde{\lambda})_{\bar{\Sigma}}) = 1/\tilde{\lambda}$$

Thm:

$\mathcal{M}_{DH} \longrightarrow \mathbb{C}P^1$ is a holomorphic fibration of dim $6g-5$.

(In case of genus$(\Sigma) = g$, $SL(2, \mathbb{C})$ surface)

Prop:

$\exists$ a holomorphic $\Omega_\lambda \in H^0(\mathcal{M}_{DH}, \overset{\lambda\text{-fiber}}{\Lambda^{(2,0)}}(\ker d\pi)^* \otimes \mathcal{O}(2))$

with:

$$\Omega_\lambda \big|_{\mathcal{M}_{Hod}(\Sigma)} = \Omega \otimes \frac{1}{\lambda}\frac{\partial}{\partial\lambda}$$

The Real-Structure of $\mathcal{M}_{DH}$

First, Let $V := \begin{array}{c} \Sigma \times \mathbb{C}^2 \\ \downarrow \\ \Sigma \end{array}$ with fixed hermitian metric h_D.

For λ-connections $D: \Gamma(V) \to \Gamma(KV)$, $\exists!$ operator

$D^*: \Gamma(V) \longrightarrow \Gamma(\bar{K}V)$, s.t.

(i) $\lambda \partial^\circ (s, t) = (Ds, t)_{h_D} + (s, D^* t)_{h_D}$.

(ii) $D^*(fs) = \bar{\lambda}(\bar{\partial} f) \otimes s + f \otimes D^* s$.

pf: $\begin{cases} D = \lambda \partial^\circ + A \\ D^* = \bar{\lambda} \bar{\partial}^\circ - A^* \end{cases}$

In this case, we obtain a map:

$$\hat{\rho}: (\bar{\partial}, D, \lambda)_\Sigma \longrightarrow (\bar{\partial}^*, D^*, \bar{\lambda})_{\bar{\Sigma}}$$

Prop: $\hat{\rho}$ is compatible with gauge-group action

it maps integrable λ-connection on Σ to

integrable $\bar{\lambda}$-connection on $\bar{\Sigma}$.

proof:

Restricted to $\lambda=1$: $(\nabla.g)^* = \nabla^* g^{*-1}$,

this is because:

$$d(s,t) = (\nabla s, t) + (s, \nabla t) = (\nabla s - (\nabla.g)s, t) + ((\nabla.g)s, t)$$

$$+ (s, \nabla t - \nabla^* g^{*-1} t) + (s, \nabla^* g^{*-1} t)$$

$$\nabla s - (\nabla.g)s = \nabla s - g^{-1}\nabla g s,$$

$$\nabla t - \nabla^* g^{*-1} t = \nabla t - g^* \nabla^* g^{*-1} t$$

$$\nabla s - g^{-1}\nabla g s = -g^{-1}(g\nabla - \nabla g)s = -g^{-1}(\nabla g)(s),$$

$$\nabla t - \nabla^* g^{*-1} t = (\nabla^* g^*)(g^{*-1} t)$$

$$\therefore -(g^{-1}(\nabla g)s, t) + (s, (\nabla^* g^*)(g^{*-1} t))$$

$= 0$ by definition.

$\therefore$ For $[\bar\partial . D, 1] . g = [\bar\partial . g, D . g . 1]$

$\hat\rho([\bar\partial . D, 1] . g) = [\bar\partial^* . g^{*^{-1}}, D^* . g^{*^{-1}}, 1]$

$\rightsquigarrow$ well-defined.

$\Big\}$ decent.

#

We have a well-defined by:

$$\rho : \mathcal{M}_{DH} \longrightarrow \mathcal{M}_{DH} \text{ covering } \lambda \longrightarrow \bar\lambda^{-1}.$$

This map is smooth + antiholomorphic.

$$\rho^* J = -J$$

Define: $\mathcal{N} : \mathcal{M}_{DH} \longrightarrow \mathcal{M}_{DH}$ given by action of

"$\mu = -1$": $\mathcal{N}(\bar\partial . D, \lambda)_\Sigma = (\bar\partial, -D, -\lambda)_\Sigma$

check: $\mathcal{N}$ is well-defined

It is only need to show that:

$$\mathcal{N} \circ \mathcal{H} = \mathcal{H} \circ \mathcal{N}$$

$$\mathcal{N} \circ \mathcal{H}(\bar\partial, D, \lambda) = \mathcal{N}\left(\frac{1}{\lambda}D, \frac{1}{\lambda}\bar\partial, \frac{1}{\lambda}\right) = \left(\frac{1}{\lambda}D, -\frac{1}{\lambda}\bar\partial, -\frac{1}{\lambda}\right)$$

$$\mathcal{H} \circ \mathcal{N}(\bar\partial, D, \lambda) = \mathcal{H}(\bar\partial, -D, -\lambda) = \left(\frac{1}{\lambda}D, -\frac{1}{\lambda}\bar\partial, -\frac{1}{\lambda}\right)$$

$\rightsquigarrow$ this is well-defined.

Define:

$$\mathcal{T} : \mathcal{M}_{DH} \longrightarrow \mathcal{M}_{DH}$$

$$[\bar\partial, D, \lambda] \longmapsto [\bar\partial^*, -D^*, -\bar\lambda]$$

$\mathcal{T} \equiv \mathcal{N} \circ \rho$

claim: $\mathcal{T} = \rho \circ \mathcal{N}$

$$\rho \circ \mathcal{N}(\bar\partial, D, \lambda) = \rho(\bar\partial, -D, -\lambda) = (\bar\partial^*, -D^*, -\bar\lambda)$$

$$\mathcal{N} \circ \rho(\bar\partial, D, \lambda) = \mathcal{N}(\bar\partial^*, D^*, \bar\lambda) = (\bar\partial^*, -D^*, -\bar\lambda)$$

Rk: $\mathcal{T}$ is an anti-holo involution covering

$$\lambda \longmapsto -\bar\lambda^{-1}$$

Narasimhan–Seshadri Theorem, (Delign–Hitchin Moduli 之 argument 终于来了)

Thm:

Every stable holomorphic structure on a rank 2 bundle with trivial determinant admits an unique irre unitary flat conn.

$$\begin{array}{c} \mathbb{C}^2 \\ \downarrow \\ (V, \bar\partial) \rightsquigarrow (\nabla, h) \\ \downarrow \\ \Sigma \end{array}$$

Remark: Let $\begin{array}{c} \mathbb{C}^2 \\ \downarrow \\ (V, \underline{h}, \bar\partial) \end{array}$

pf: Assume $g(\Sigma) > 1$, then $\exists$ unitary irreducible $SU(2)$ — representation:

$$\rho: \pi_1(\Sigma, p) \longrightarrow SU(2)$$

$\pi_1(\Sigma, p) = \langle A_1, B_1, \dots, A_k B_k \mid A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_k B_k A_k^{-1} B_k^{-1} = id \rangle$

just pick A_1, B_1 / A_2, B_2 diagonal / off-diag , $A_k, B_k = id$, $k \geq 2$

$\therefore$ By RH-Correspondence $\rightsquigarrow$ ρ induces a flat $SU(2)$-Conn. which is irreducible.

Because the representation irre $\rightsquigarrow$ $\bar{\partial}^{\nabla}$ is stable and admits a flat-connection.

($\exists$ stable holo-structure on a rank 2 bundle admits a unique unitary flat-conn)

Prop: A flat connection has a unitary monodromy $\Leftrightarrow$ it admits a parallel hermitian-metric.

$\rho^{\nabla} \subseteq U(n)$, then $Hol(\nabla) \subset U(n)$, take a hermitian-metric on any point, do parallel transport to any point which gives a global hermitian-metric.

(2) Now we show the openess, closedness, connectedness.

a. Openess.

Considering the map:

$f : \mathcal{M}_{dR}^{irr} := \{\text{irre flat unitary}\} \longrightarrow \mathcal{M}_{Dol}$

$$\nabla \longmapsto \bar{\partial}^{\nabla}$$

$\dim \mathcal{M}_{dR}^{irr} = \dim \mathcal{M}_{Dol} = 6g-6$

the differential $f_*|_\nabla : T_\nabla \mathcal{M}_{dR}^{irr} \longrightarrow T_{\bar{\partial}^\nabla} \mathcal{M}_{Dol}$

$$-\bar{\Phi}^* + \bar{\Phi} \longmapsto -\bar{\Phi}^*$$

which is bijective, so by IFT ✓

(3) Closedness.

Let $\bar{\partial}^n \to \bar{\partial}$ with property $\bar{\partial}^n = \bar{\partial}^{\nabla^n}$, $\bar{\partial}$ is stable.

We have a sequence of monodromy:

$\rho(\nabla^n) : \pi_1(\Sigma, p) \longrightarrow SU(2)$. As $SU(2)$ compact

$\therefore \exists$ converging subsequence.

let limit $\rho^\infty : \pi_1(\Sigma, p) \longrightarrow SU(2) \rightsquigarrow \bar{\partial}^\infty$

claim: $\bar{\partial}^\infty = \bar{\partial}$. follows by Uhlenbeck Compactness,

$\exists g_n$. s.t. $\bar{\partial}^n \cdot g_n \to \bar{\partial}$, which means: $\exists g$, s.t. $(\bar\partial^* \otimes \bar\partial^\infty). g = 0$

If g is invertible ✓ if not: $g: V/L_1 \to L_2$

which contradicts to stability.

(4) Connected.

Recall that: For $\begin{array}{c}\mathbb{C}^2\\ \downarrow\\ V\\ \downarrow\\ \Sigma\end{array}$ with $\Lambda^2 V = 0$, $\exists L \hookrightarrow V$,

s.t. $\bar\partial^V = \begin{pmatrix} \bar\partial^L & \gamma \\ 0 & \bar\partial^{L^*} \end{pmatrix}$, $\gamma \in \Gamma(KL^{-2})$, $\deg L \geq 1-g$

We look at such extension $V = L \oplus L^\perp$ with

$\bar\partial^V = \begin{pmatrix} \bar\partial^L & \gamma \\ 0 & \bar\partial^{L^*} \end{pmatrix}$, $\gamma \in \Gamma(KL^{-2})$, $\deg L \in \{1-g, 2-g, \dots, -1\}$

$[\gamma] \neq 0$, or $L^* \hookrightarrow V \rightsquigarrow$ contra to stability.
in $H^1(L^{-2})$

$\dim H^1(L^{-2}) = \dim H^0(KL^2)$

$\longrightarrow [\gamma]$ has choice $\dim H^1(L^{-2}) - 1$ as we

identify the bundles

$\begin{pmatrix} \bar\partial^L & \gamma \\ 0 & \bar\partial^{L^\perp} \end{pmatrix}$ and $\begin{pmatrix} \bar\partial^L & c\cdot\gamma \\ 0 & \bar\partial^{L^\perp} \end{pmatrix}$

Lemma: For $\deg L \in \{1-g, 2-g, \dots, -1\}$, generic $\gamma \in H^1(L^2)$ there is no $L' \hookrightarrow V$ with $\deg L' \geq 0$ $\rightsquigarrow$ so this $(V, \begin{pmatrix} \bar\partial^L & \gamma \\ 0 & \bar\partial^{L^*} \end{pmatrix})$ is stable.

away from codim 1 subset.

proof:

(Case I) $\deg L = -1$, if $\exists\ L' \hookrightarrow V$, $\deg L' \geq 0$, then $\exists$ a holo-bundle line E of degree 0 with:

$$\exists\, 0 \neq f : E \to V \text{ holomorphically.}$$

then: $f : \begin{pmatrix} a \\ b \end{pmatrix} : E \to L \oplus L^*$ satisfy $\bar\partial^{L^*} b = 0$

$b \in H^0(E^* L^*)$, $\bar\partial a + b\gamma = 0$

$b \neq 0$, otherwise a holo gives $\deg(E) = 0 > \deg L = -1$

$\rightsquigarrow b = s_p$ for some $p \in \Sigma$, $\because \bar\partial a + b\gamma = 0 \Rightarrow$
$\in H^0(L_p)$

$\gamma \cdot s_p \in \Gamma(K L^2 \cdot L^* E^*) = \Gamma(K L E^*)$ gives

$0 = [\gamma \cdot s_p] \in H^1(L E^*)$ by Serre-Duality.

This equn tells us that the subspace $\gamma^{\perp} \subseteq H^0(K L^{-2})$ of codimension 1 is of the form $S_p \otimes \underbrace{H^0(K L^{-2} L(-p))}$

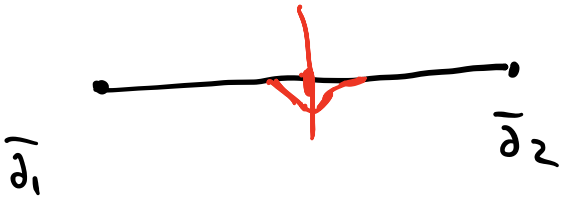
This space has co dimension 1 from R-R-T

The dimension of subspaces of hyperplanes of the form $S_p \otimes H^0(K L^{-2} L(-p))$ has dimen 1 (as it is para by $p \in \Sigma$)

but the dim of the space of all hyperplanes in $H^0(K L^{-2})$ is $\dim H^0(K L^{-2}) - 1 = g+1-1 = g > 1$

$\therefore$ For γ away that $\overset{\text{special complex}}{\text{analytic}}$, we obtain a stable holomorphic structure.

So we can obtain that, generically, $(V, \bar{\partial})$ is stable. Let $(\bar{\partial}_1)$, $(\bar{\partial}_2)$ be 2 stable structures, obviously we have a line conn $(\bar{\partial}_1)$ and $(\bar{\partial}_2)$,



if we meet a unstable point, by generic property, we can obviously avoid that.

Key Point: (a) dimension of moduli of non-trivial V with $L \hookrightarrow V$ is $\dim H^1(L^{-2})-1$ (b) this moduli $\cong \gamma^{\perp} \subset H^0(KL^{-2})$
c) if V not stable $\rightsquigarrow \gamma^{\perp}$ has form $S_p \otimes H^0(KL^{-2}(-p))$ d): $p \to S_p \otimes H^0(KL^{-2}(-p))$ is a 1-dim curve in $\mathbb{P}_g'$. ✓

Uniformazation:

S, $S^2=K$, i.e. S is holomorphic spin.

consider $V = S \oplus S^*$

$$\left(\bar\partial = \begin{pmatrix} \bar\partial^S & \\ & \bar\partial^{S^*} \end{pmatrix}, \quad \Phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \lambda = 0 \right)$$

stable Higgs Pair.

$\rightsquigarrow$ We find a solution (∇, Φ, h) w.r.t $(V, \bar\partial)$,

Prop: h is diagonal w.r.t. $V = S \oplus S^*$

Assume this is write, this means:

$S \hookrightarrow \mathbb{C}^2$ and $S^* = S^\perp$ w.r.t h_0

$\therefore \nabla = \begin{pmatrix} \nabla^S & \\ & \nabla^{S^*} \end{pmatrix}$, $\Phi^* = \begin{pmatrix} 0 & 1^* \\ 0 & 0 \end{pmatrix}$, $1^* = I(K\bar{K})$

$\therefore \nabla^\lambda = \nabla + \lambda^{-1}\Phi + \lambda\Phi^* = \nabla^{\lambda=1} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$

Towards NAH Correspondence.

We aim to show: $\forall\, \not{D} \in \mathcal{M}_{DH}$, $\exists\, (\nabla, \Phi, h_0)$ solution of SD-Equation. s.t. $\not{D} = (\bar\partial^\nabla + \lambda \Phi^*, \lambda \partial^\nabla + \Phi, \lambda)$, $\lambda = \pi(\not{D})$

$$\begin{array}{c} \mathcal{M}_{DH} \\ \downarrow \pi \\ \mathbb{CP}^1 \end{array}$$

Def: A tangent vector (ξ, φ) at (∇, Φ, h_0) must satisfy:

a) $0 = \frac{d}{dt}\Big|_{t=0} \left(F^{\nabla + t\xi} + [\Phi + t\varphi, \Phi^* + t\varphi^*] = 0 \right)$

$= d^\nabla \xi + [\Phi, \varphi^*] + [\varphi, \Phi^*]$

b) $0 = \frac{d}{dt} \left(\bar\partial^{\nabla + t\xi} (\Phi + t\varphi) \right) = \bar\partial^\nabla \varphi + [\xi^{0,1}, \Phi]$

The Infinitismal Variation:

$$\Omega^0(\Sigma, \underline{su(2)}) \xrightarrow{d_1} \Omega^1(su(2)) \oplus \Omega^{1,0}(sl(2,\mathbb{C})) \to \Omega^{1,1}(su(2)) \oplus \Omega^{1,1}(sl(2,\mathbb{C}))$$

$$\mu \longmapsto (d^\nabla \mu, [\Phi, \mu])$$

$$(\xi, \varphi) \longmapsto (d^\nabla \xi + [\Phi, \varphi^*] + [\Phi^*, \varphi], \bar\partial \varphi + [\xi^{0,1}, \Phi])$$

We consider inner-product on:

$$\Omega^{0,1}(su(2))^{\otimes \mathbb{C}} \oplus \Omega^{1,0}(sl(2,\mathbb{C}))$$

$$(\xi, \varphi)$$

$$\langle (\xi,\varphi), (\hat{\xi},\hat{\varphi}) \rangle = \int_\Sigma tr(\xi \wedge * \hat{\xi}) + i \int_\Sigma tr(\varphi \wedge \hat{\varphi}^* + \hat{\varphi} \wedge \varphi^*)$$

$\therefore$ $\langle \cdot,\cdot \rangle$ is a real-valued inner-product.

Thm:

All formal tangent to space to solution of SD can be represented by $(\xi,\varphi) \in \Omega^{0,1}(gl(\mathbb{C})) \oplus \Omega^{1,0}(sl(2,\mathbb{C}))$

s.t.

1) $\bar{\partial}^\nabla + [\bar{\Phi}, \xi^{0,1}] = 0$

2) $d^\nabla \xi + [\bar{\Phi}, \varphi^*] + [\bar{\Phi}^*, \varphi] = 0$

3) $\underline{d^\nabla * \xi + i[\bar{\Phi}, \varphi^*] - i[\bar{\Phi}^*, \varphi] = 0}$ (this is adjoint map of d_1)

There are natural complex-structure on each formal tangent space:

$$\ker d_2 / \operatorname{im} d_1 .$$

pf: We identify $\Omega^{0,1}(sl(2,\mathbb{C})) \sim \Omega^1(su(2,\mathbb{C}))$

$$\eta \longmapsto \eta - \eta^*$$

So define the $\boxed{\text{complex-structure}}$ to be

$$I: (\eta, \varphi) \longmapsto (i\eta, i\varphi)$$

SD-Equation $\gtrsim$ Hyperkahler Structure.

$J: (\eta, \varphi) \longmapsto (i\varphi^*, -i\eta^*)$

显然: $IJ = -JI$, so define $K = IJ$ ✓

this can also be defined on $\mathcal{M}_{DH}\big|_{\lambda=0,\ \lambda=1}$?

Def: The evaluation map is defined by

$T_{[\nabla, \Phi, h_0]}\mathcal{M}_{SD} \longrightarrow T_{s(\lambda)}\mathcal{M}_{DH}$, 其中:

$ev: \mathcal{M}_{SD} \longrightarrow \mathcal{M}_{DH}$

$(\nabla, \Phi, h_0) \longmapsto (\bar{\partial}^{\nabla} + \lambda\Phi^*, \lambda\partial^{\nabla} + \Phi, \lambda)$

Prop: This tangent map is injective for $\forall \lambda \in \mathbb{CP}^1$.

Thm: (Open-Neighbor-Thm)

$\forall \not{D} \in \mathcal{M}_{DH}$ with $\not{D} = s(\pi(\not{D}))$ for some twistor-line s, $\exists$ open neighbor of $\not{D} \in \mathcal{M}_{DH}$

s.t.:

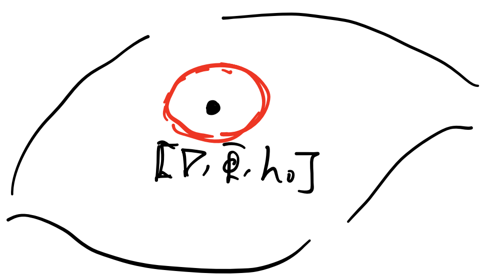
$\forall \tilde{\not{D}} \in \mathcal{U}$, $\exists$ $\hat{s}$-twistor line with $s(\pi(\tilde{\not{D}})) = \tilde{\not{D}}$

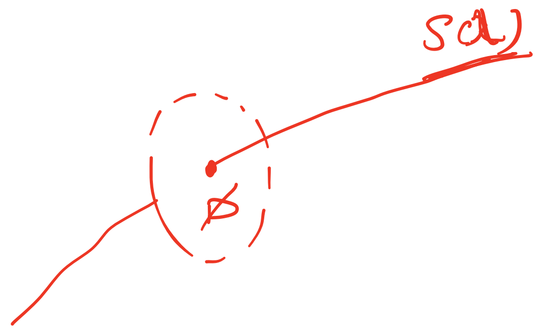
pf: $T_{[\nabla,\Phi,h_0]}\mathcal{M}_{SD} \longrightarrow T_{s(\lambda)}\mathcal{M}_{DH}$

X

$s(\lambda) = (\bar\partial^\nabla + \lambda^{-1}\Phi^{*,h_0}, \partial^\nabla + \Phi\cdot\lambda)$

1) Let $X \in T_{[\nabla,\Phi,h_0]}\mathcal{M}_{SD}$ which is a harmonic represent of the complex.





As injective $\rightsquigarrow$ $\exists$ neigh of ϕ ✓

We start with stable holo-structure $\rightsquigarrow$ connected

Thm: Every stable Higgs Pair admits a solution of SD-Equation. Show every unstable Higgs Pair can be connected with that!

pf: $\Phi = 0 \rightsquigarrow$ This is Narasimhan–Seshadri.

Let $\bar\partial$ be not stable, then: $\exists\, L \hookrightarrow V$, $\deg L > 0$

s.t. $\bar\partial = \begin{pmatrix} \bar\partial^L & \gamma \\ 0 & \bar\partial^{L^*} \end{pmatrix}$, $\Phi = \begin{pmatrix} a & b \\ \varphi & -a \end{pmatrix}$, $\varphi \neq 0$ in $H^0(KL^{-2})$

as $(\bar\partial, \Phi)$ is stable. (rescale Φ, change limit)

Clearly: $\lim_{\mu\to 0}(\bar\partial, \mu\bar\Phi) = (\bar\partial, 0)$ is unstable, let

$g_\mu = \begin{pmatrix} 1 & \\ & \mu \end{pmatrix}$ w.r.t $V = L \oplus L^\perp$, then:

$\bar\partial . g_\mu = \begin{pmatrix} \bar\partial^L & \mu\gamma \\ 0 & \bar\partial^{L^*} \end{pmatrix}$, $\mu . \bar\Phi . g_\mu = \begin{pmatrix} \mu a & \mu^2 b \\ \varphi & -\mu a \end{pmatrix}$

and we obtain a stable-limit for $\mu \to 0$.

This stable pair is:

$\begin{pmatrix} \bar\partial^L & 0 \\ 0 & \bar\partial^{L^*} \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}$, we claim it admits solution.

Now, considering $\bar\partial = \begin{pmatrix} \bar\partial^L & 0 \\ \beta & \bar\partial^{L^*} \end{pmatrix}$ with $\Phi = \begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}\mu$

this is stable for generic $\beta \neq 0$ and this pair admits a solution of SD-Equation for all the $\mu \in \mathbb{C}$ in above analysis.

If $\mu \to \infty$, we obtain our previous Higgs Bundle

$\begin{pmatrix} \bar\partial^L & \\ & \bar\partial^{L^*} \end{pmatrix}$, $\begin{pmatrix} & \\ \varphi & \end{pmatrix}$ as a limit after gauging with

$\begin{pmatrix} 1/\mu & 0 \\ 0 & 1 \end{pmatrix}$.

Finally, if $\bar\partial$ is strictly semi-stable, we can show that it $(\bar{\partial}, \Phi)$ can be approximated by $(\bar{\partial}^n, \Phi^n)$

with $\bar{\partial}^n$ stable, $\det \Phi^n = \det \Phi$

Big Picture for now:

$$\begin{tikzcd}[column sep=small] & \mathcal{M}_{SD} \cong \{\text{components of the space of real holo sections } \mathcal{M}_{DH} \to \mathbb{CP}^1\} \arrow[dl, <->] \arrow[dr, <->] & \\ \mathcal{M}_{Dol} = \pi^{-1}(\lambda = 0) & & \mathcal{M}_{dR} = \pi^{-1}(\lambda = 1) \end{tikzcd}$$

Riemann-Metric g on $\mathcal{M}_{SD}$ is given by:

$$g((\delta_1, \varphi_1), (\delta_2, \varphi_2)) = \int_\Sigma tr(\delta_1 \wedge * \delta_2) + i \int_\Sigma tr(\varphi_1 \varphi_2^* + \varphi_2 \varphi_1^*)$$

this is well-def if we restricted to harmonic-repre.

(Hyperkahler + compact $\Leftrightarrow$ Kahler + Symplectic holo)

$$\Omega|_{\lambda=0} = \omega_J + i\omega_K$$

Lemma: For $\mathcal{E} = |\Phi|^2 = 2i \int_\Sigma tr(\Phi \wedge \Phi^*)$, we have:

$d\mathcal{E} = -2\omega_1(X, \cdot)$, X is generated by S^1-action

$(\bar\partial, \Phi).e^{i\theta} = (\bar\partial, e^{i\theta}\Phi)$.

Then: $\mathcal{E}$ is the moment-map, i.e. $d\mathcal{E}^X(Y) = i_X\omega(Y)$

$S^1 \curvearrowright \mathcal{M}_{Dol}$ $\mathcal{E}: \mathcal{M}_{Dol} \longrightarrow \underline{u(1)}^* \xrightarrow{\langle\cdot, 1\rangle} \mathbb{R}$

$$d\mathcal{E}^1(Y) = i_1\omega(Y) = \omega(1^{\#}|_{(\nabla,\Phi)}, Y)$$

$$= \omega((0, i\Phi), Y) = g(I(0, i\Phi), Y)$$

$$= g((0, -\Phi), Y) = -\frac{1}{2} d\,g((0,\Phi),(0,\Phi))(Y)$$

✓

Thm: The energy $\mathcal{E}$ gives a kahler-potential w.r.t J, meaning $\partial_J \bar\partial_J \mathcal{E} = \omega_J$, up to some constant c.

Actually, for a complex manifold (M, I) complex-structure

$\partial_I = \frac{1}{2}(d - i*d)$ $\bar\partial_I = \frac{1}{2}(d + i*d)$

$*$ is the operator induced from $I: T_pM \to T_pM$

$I(\frac{\partial}{\partial z}) = i\frac{\partial}{\partial z}$, $I(\frac{\partial}{\partial \bar z}) = -i\frac{\partial}{\partial \bar z}$

$*dz = i\,dz$, $*d\bar z = -i\,d\bar z$

Thm.

Let $s: \mathbb{CP}^1 \to \mathcal{M}_{DH}$ be a <u>twistor line</u> which admits a lift of the form

$$\hat{s}(\lambda) = (\bar\partial + \lambda\Psi_1 + \lambda^2\Psi_2 + \cdots, \underbrace{\tilde{\bar\Phi} + \lambda\partial + \lambda^2\bar\Phi_1 + \cdots}_{\bar\Phi})$$

then:

$$|\bar\Phi|^2 = \mathcal{E}(s) = 2i\int_\Sigma \mathrm{tr}(\tilde{\bar\Phi}\wedge\Psi_1)$$

proof:

$$\begin{tikzcd} \mathcal{M}_{DH} \arrow[d, shift left, "\pi"] \\ \mathbb{CP}^1 \arrow[u, shift left, "s"] \end{tikzcd}$$

$\hat{s} = s^{SD}(\lambda)\cdot g(\lambda)$ where $s^{SD}(\lambda) = (\bar\partial^\nabla + \lambda\Phi^*, \lambda\partial^\nabla + \Phi)$, For the s^{SD} lifts, the statement is obviously fail, as

$$\mathcal{E}(s) = 2i\int_\Sigma \bar\Phi\wedge\Phi^*.$$

By assumption, $\hat{s}$ is a gauge trans of s^{SD} by $g = g(\lambda)$ which extends holo to $\lambda = 0$.

Write $g = g_0(id + \lambda\xi + \cdots)$ $\therefore$ the state is true for $s^{SD}\cdot g_0$.

It remains to compute $g(\lambda) = id + \lambda\xi + \cdots$

As $(\bar\partial + \lambda\Psi_1 + \cdots, \tilde{\bar\Phi} + \lambda\partial + \cdots).g(\lambda) = (\bar\partial + \lambda(\Psi_1 + \bar\partial\xi), \tilde{\bar\Phi} + \lambda(\cdots))$

So it remains:

$$\int_\Sigma \mathrm{tr}(\bar\Phi\wedge\Psi_1) = \int \mathrm{tr}(\bar\Phi\wedge(\Psi_1 + \bar\partial\xi)) = \int \mathrm{tr}\,\tilde{\bar\Phi}\wedge\Psi_1 - \int d\,\mathrm{tr}(\tilde{\bar\Phi}\xi)$$

Thm: $\mathcal{M}_{DH}$ is the twistor space of $(\mathcal{M}_{SD}, I, J, K, g)$

pf: Hitchin, Sypersymmetry ....

Thm: (Teichmüllar Theory)

The fixed points of $[\bar\partial, \Phi] \longrightarrow [\bar\partial, -\Phi] \in \mathcal{M}_{Dol}$ are exactly these whose corresponds NAH to real or unitary representation

$$\rho^{\nabla}: \pi_1(\Sigma) \longrightarrow \begin{matrix} SL(2,\mathbb{R}) \\ SU(2) \end{matrix}$$

pf: We know $(\bar\partial, -\Phi)$ correspo to rep ∇^{-1} which is gauge equiva $(\nabla')^*$.

Denote: $\nabla' = \nabla$ and obtain:

if $(\bar\partial, \Phi) \cong (\bar\partial, -\Phi)$ also $\nabla' \cong \nabla^{-1} \rightsquigarrow \nabla^* = \nabla . g$ for some $g: \Sigma \to SL(2,\mathbb{C})$.

$\nabla = (\nabla^*)^* = (\nabla . g)^* = \nabla . g g^{*-1} \rightsquigarrow g g^{*-1} = \pm id$

if $g g^{*-1} = id \rightsquigarrow g = \pm h \cdot h^*$

implies $(\nabla . h) = \nabla^* . h^{*-1} = (\nabla . h)^*$ is unitary

If $gg^{*-1} = -\mathrm{id}$, look at $\rho(D) :\triangleq \rho$, this satisfies: $g^{-1}\rho g = \rho^{*-1} = \delta\bar{\rho}\delta^{-1}$, where $\delta = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

$\longrightarrow \delta^{-1}g^{-1}\rho g\delta = \bar{\rho}$ for $h = g\delta$ we obtain

$h\cdot\bar{h} = \mathrm{id}$.

We can write those h's as $h = f\cdot\bar{f}^{-1}$ to obtain $f^{-1}\rho f = \bar{f}^{-1}\bar{\rho}\bar{f}$ showing that ρ can be conjugated into an $SL(2,\mathbb{R})$-repre.

Conversely:

For $\rho : \pi_1(\Sigma, p) \longrightarrow SL(2,\mathbb{C}) \rightsquigarrow \nabla^{-1} \triangleq \nabla' \rightsquigarrow$

$(\bar{\partial}, \Phi) \cong (\bar{\partial}, -\Phi)$

Example:

$\begin{pmatrix} \bar{\partial}^S & \\ & \bar{\partial}^{S^*} \end{pmatrix}, \begin{pmatrix} & q \\ 1 & \end{pmatrix}$ satisfies the fixed point property, $\forall\, q \in H^0(\Sigma, K)$.