Affine Sphere / Affine Differential Geometry

Def: Affine Manifold: $\{(U_i, \varphi_i)\}$ local-coordinates, then:

$\varphi_j \circ \varphi_i^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ is Affine. (i.e. $GL(n, \mathbb{R})$)

$\boxed{\text{Affine immersion}}$

($f: M^n \to \mathbb{R}^{n+1}$, $d^2 f = (d_{f_* X}(f_* Y) - f_*(\nabla_X Y))\xi$ ∴ $d^2 f: TM \times TM \to \mathbb{R}^{n+1}/TM$)

$f: M^n \to \mathbb{R}^{n+1}$ be oriented hypersurface which is non-dege, i.e. 2^{nd} FF is non-dege. Let $T\mathbb{R}^{n+1} = TM \oplus L$, pick $\xi \in \Gamma(L)$, (transverse v.f.)

s.t. (i) $\nabla^L \xi = 0 = (d\xi)^L = 0$, (ii) $\det g = \det(df, \cdots, df, \xi)$

(a) L is called the affine normal line with affine normal vector ξ. (b) g called Blaschke Metric.

Def: Affine immersion means $\boxed{f \text{ is convex}}$ $\rightsquigarrow$ g is positive-def.

In this case, we have decomposition:

$$\begin{cases} d_X Y - \nabla_X Y = g(X,Y)\,\xi \\ d_X \xi = S(X) \end{cases}$$

S the Shape-Operator.

(又叫 Affine-Weingarten-Map) $S \in \Gamma(\mathrm{End}\,TM)$

规范地写,

$$d = \begin{pmatrix} \nabla & S\xi^* \\ g\xi & \nabla^L \end{pmatrix}$$

$$d^2 = 0 \rightsquigarrow \begin{pmatrix} \nabla & S\xi^* \\ g\xi & \nabla^L \end{pmatrix}\begin{pmatrix} \nabla & S\xi^* \\ g\xi & \nabla^L \end{pmatrix} = \begin{pmatrix} F^\nabla + S\wedge g & d^\nabla S \\ g\wedge\nabla + d\,g & g\wedge S + F^{\nabla^L} \end{pmatrix}$$

$\nabla^L \xi = 0$

$= d^\nabla g$

$\therefore F^\nabla + S\wedge g = 0$; $d^\nabla S = d^\nabla g = 0$

From Classical Diff-Geometry, we have:

S is self-adjoint.

Prop:

(i) $\nabla^{L.C.} = \nabla + \frac{1}{2} g^{-1} \circ \nabla g$. here $g: TM \to T^*M$ as isomorphism

here $(\nabla g)(X,Y) = d(g(X,Y)) - g(\nabla X, Y) - g(X, \nabla Y)$

(ii) The Pick form $C = -\frac{1}{2}\nabla g$ is symmetric $\in \Gamma(\mathrm{Sym}^3(TM,\mathbb{R}))$

(iii) The Pick form $C = -\frac{1}{2}\nabla g$ is trace-free.

pf: (ii) hope: $\nabla_X g(Y,Z) = \nabla_Y g(X,Z)$

i.e. $X(g(Y,Z)) - \cancel{g(\nabla_X Y, Z)} - g(Y, \nabla_X Z) - Y(g(X,Z)) + \cancel{g(\nabla_Y X, Z)} + g(X, \nabla_Y Z) = 0$

pick local integral coordi: (orthogonal) ✓

$d^\nabla g = 0 \rightsquigarrow X(g(Y,Z)) + g(\nabla_X Y, Z) = 0$ $\therefore \nabla_X g(Y,Z) = \nabla_Y g(X,Z)$

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Reconstruction and Monodromy.

Let (M, g, S, C, ∇) Riemann-Mfd with:

(i) $\nabla = \nabla^g - g^{-1} \circ C$ (iii) $g \circ S = S^* \circ g$

(ii) $d^\nabla g = d^\nabla S = 0$ (iv) $F^\nabla + S \wedge g = 0$

purpose: Construct affine immersion:

$$V = TM \oplus \mathbb{R} \quad \text{with:} \quad \begin{pmatrix} \nabla & S\xi^* \\ g\xi & d_{\mathbb{R}} \end{pmatrix} = d_V$$

then $\begin{array}{c}(V, d_V) \\ \downarrow \\ M\end{array}$ flat bundle . $\det_V = \det_g \wedge dt$ which

is parallel.

$(\mathbb{R}^{n+1}, d, \det)$ $(V, d_V, \det_V)$ quotient by monodromy ρ^{d_V}.

$$\begin{array}{ccc} (\mathbb{R}^{n+1}, d, \det) & & (V, d_V, \det_V) \\ \downarrow & & \downarrow \\ \tilde{M} & \xrightarrow{\pi} & M \end{array}$$

(RH-Correspondence: $\hat{M} \times \mathbb{R}^{n+1} / \rho$: $(x, v) \sim (\rho(\gamma) x, \rho(\gamma)^{-1} v)$)

Considering inclusion

$$i : T\hat{M} \hookrightarrow \mathbb{R}^{n+1}, \quad \because \hat{M} \text{ simply-conn}$$

$\therefore$ closed form means exact.

$\because \nabla$ is torsion free $\therefore$ i is closed.

$\boxed{\therefore\ i = df \quad \text{for} \quad f : \tilde{M} \to \mathbb{R}^{n+1}}$ ☆☆☆

non-degenerate!!

We also have: ($\forall X \in T\hat{M}$, $X(df) = X(f) = i(X)$) $\rightsquigarrow$ f immersion)

$$(\rho, T) : \pi_1(M, p_0) \longrightarrow SL_{n+1}(\mathbb{R}) \oplus \mathbb{R}^{n+1}$$
$$\gamma \longmapsto \left(\rho(\gamma), \int_\gamma df\right)$$

s.t. $\gamma^* f = \rho_\gamma f + T_\gamma$

Affine Sphere.

Def: $f : M^n \to \mathbb{R}^{n+1}$ be affine immersion, then it is called a affine sphere:

1. All ξ meets at 1 point

(内部刻移 M: elliptic affine sphere;
外部刻移 M: hyperbolic affine sphere.)

2. All ξ are parallel.

Prop: For case 1, $\xi = f$ or $-f$, $S = I$ or $-I$.

case 2, $S = 0$

pf: $D_X\xi = L_X\xi = \lim_{t\to 0}\frac{(P_{-t})_*\xi(t) - \xi(0)}{t} = \lim_{t\to 0}\frac{p(t)-p(0)}{t} = X \quad \therefore S = I/-I$.

ξ constant $\longrightarrow S = 0$

For Parabolic Affine Sphere:

Choose transversal hyperplane $E \pitchfork \{\xi\}$, then

$\underline{f(p) = p + \phi(p)\xi}$, $M \subset E$ any n-mfd for smooth convex ϕ

(immersion)

$\mathrm{Hess}\, g = \nabla d\phi \quad \therefore \det_g = \det(df, \dots, df, \xi) \Leftrightarrow \det \nabla d\phi = 1$

$\therefore$ constructing parabolic affine sphere $\Leftrightarrow$ Constructing Mongé Ampére Metric.

$(M^n, g, C, \boxed{S = HI})$, $H = \pm 1$, $d_V = \begin{pmatrix} \nabla & HI \\ g & d_{\mathbb{R}} \end{pmatrix}$

$\nabla = \nabla^g - g^{-1}\circ C$

$$\begin{array}{c} V \\ \downarrow \\ M \end{array}$$

Considering $B = M \times (0,1) \xrightarrow{\pi_1} M$ by $(p, r) \longmapsto p$

"coning"

$V = TM \oplus \underline{\mathbb{R}}$, canonically: $\pi^* V \overset{?}{\cong} (TB, \tilde{\nabla})$

Let $F: TB \to \pi^* V$ given by $(v, \mu\partial r) \longmapsto (rv, \mu H)$

then $\tilde{\nabla} = F^{-1}\circ \pi^* d_V \circ F$ on TB.

Lemma: B is affine flat $n+1$-manifold, the function $\phi : B \to \mathbb{R}$, $\phi(r) = -H\int_0^r (1-H\rho^{n+1})^{\frac{1}{n+1}}\, d\rho$ is convex and s.t. $\det_B \tilde{\nabla} d\phi = 1$ which defines the Monge-Ampère Metric $\tilde{\nabla} d\phi$ on B.

(对于 hyperbolic / elliptic affine sphere M, 可以在 $M\times(0,1) = B$ 上构造 Monge-Ampère Metric)

Comment: (由 elliptic / hyperbolic affine sphere $\rightsquigarrow$ parabolic affine sphere)

$$\tilde{\nabla} d\phi = (1-Hr^{n+1})^{-\frac{n}{n+1}} r^n dr^2 + r(1-Hr^{n+1})^{\frac{1}{n+1}} \pi^* g$$

Two-dimensional Affine Sphere and Higgs Bundle

$(M^2, g, \nabla) \xrightarrow{f} \mathbb{R}^3$, orientation + metric $\rightsquigarrow$ R.S. with Metric g' conformal to g.

$$\begin{aligned} F^{\nabla^g} = \nabla^g \circ \nabla^g &= (\nabla - g^{-1}C)(\nabla - g^{-1}C) \\ &= F^{\nabla} - \nabla g^{-1}C + g^{-1}C \wedge g^{-1}C \\ &= -HI\wedge g + g^{-1}C\wedge g^{-1}C \end{aligned}$$

Pick form: $C \in \Gamma(\mathrm{Sym}^3(TM,\mathbb{R})) \rightsquigarrow C = Q + \bar{Q}$, $Q \in \Gamma(K^3)$

(Trace free + Symmetric: $C = Q_{(3,0)} + \bar{Q}_{(3,0)}$)

$\because F^{\nabla} + S\wedge g = 0 \Leftrightarrow F^{\nabla^g} + g^{-1}C\wedge g^{-1}C + HI\wedge g = 0$

$$F^{\nabla} + S \wedge g = 0 \,, \quad d^{\nabla} g = d^{\nabla} S = 0 \,, \quad \nabla = \nabla^{g} + g^{-1} C$$

$$C = -\frac{1}{2} \nabla g \,, \quad \nabla^{g} C \in \Gamma(\mathrm{Sym}^{4}(TM, \mathbb{R}))$$

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then: $\nabla^{g} C \in \Gamma(\mathrm{Sym}^{4}(TM, \mathbb{R})) \rightsquigarrow Q \in H^{0}(K^{3})$

write $g = e^{2u} \underline{g_0}$ background metric

$$F^{\nabla^{g_0}} + H \wedge I e^{2u} g_0 - e^{-4u} (Q_0 + \overline{Q}_0)^{2} = 0$$

$$g_0 = |dz|^2 \rightsquigarrow g^{-1} = e^{-2u} \left(\frac{\partial}{\partial z}^{2} + \frac{\partial}{\partial \bar{z}}^{2} \right)$$

Pick $Q = q\,dz^3$: $\Delta u + 8 |q|^2 e^{-4u} - \frac{1}{2} H e^{2u} = 0$,

this same as flatness of $d_V = \begin{pmatrix} \nabla & HI \\ g & d_{\mathbb{R}} \end{pmatrix}$

connection is given by $\nabla^{g} \oplus d_{\mathbb{R}}$

$$\therefore d_V = \begin{pmatrix} \nabla^{g} & \\ & d_{\mathbb{R}} \end{pmatrix} + \underline{\begin{pmatrix} g^{-1}C & 0 \\ 0 & 0 \end{pmatrix}} + \underbrace{\begin{pmatrix} & HI \\ g & \end{pmatrix}}_{A}$$

A is self-adjoint:

$$g(A\xi, X) = g(X, X) = g(\xi, g^{2}(X)\xi) = g(\xi, AX)$$

$\therefore$ the last 2-terms self-adjoint, first term is Levi-Civita.

$V \otimes \mathbb{C} = (TM \oplus \mathbb{R}) \otimes \mathbb{C} = T^{\mathbb{C}}M \oplus \underline{\mathbb{C}} = (K^{-1} \oplus \overline{K^{-1}}) \oplus \mathbb{C}$

$= K^{-1} \oplus \mathbb{C} \oplus \overline{K^{-1}} = K^{-1} \oplus \mathbb{C} \oplus K$

$h = g \oplus H dt^2 \oplus g^{-1}$ be the induced hermitian-metric

$d_{V \otimes \mathbb{C}} = D + \Phi + \Phi^*$, $\Phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ Q & 0 & 0 \end{pmatrix}$ 1 here means constant section (M compact)

$\therefore$ Flatness $\Leftrightarrow$ $\begin{cases} F^D + [\Phi, \Phi^*] = 0 \\ \bar{\partial}^D \Phi = 0 \end{cases}$, i.e. Self-Duality Equation.

$\nabla^g C \in \Gamma(\mathrm{Sym}^4(TM, \mathbb{R})) \rightsquigarrow Q \in H^0(K^3)$

Lemma: (M^2, Q, g), $V = K^{-1} \oplus \underline{\mathbb{C}} \oplus K$ with $h = g \oplus H dt^2 \oplus g^{-1}$ where $H = \pm 1$, M complex curve

(1) g and Q Blaschke metric & Pick Differential, for an affine sphere immersion $f: \hat{M} \hookrightarrow R^3$ w.r.t a monodromy $\rho: \pi_1 \to SL_3(\mathbb{R})$;

(2) g, Q satisfies Tzitzéica Equation;

(3) $(D = \nabla^g \oplus d_{\mathbb{R}}, \Phi)$ satisfies Self-Duality Equation.

Set-Up of Question:

$\left(\Sigma = \mathbb{C}P^2 - \{p_1, p_2, p_3\} \ , \quad Q \in H^0(K^3) \text{ with quadratic poles} \right)$

Step1: Extending the bundle.

(In order to obtain the right asympototic behavior tensor K with $\mathcal{O}(D)$, K^{-1} with $\mathcal{O}(-D)$)

$$\underset{\Sigma}{V} = K^{-1} \oplus \mathbb{C} \oplus K \xrightarrow{\text{extending}} V_{S^2} = \mathcal{O}(-1) \oplus \mathbb{C} \oplus \mathcal{O}(1)$$

$$\Phi \quad \rightsquigarrow \quad \Phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ Q & 0 & 0 \end{pmatrix} \text{ has}$$

simple pole.

$i\xi\zeta^{2}\!/_{z}$: $W = \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O}(1)$. $Q \in H^0(K^3\mathcal{O}(2D)) = \mathcal{O}$

$$\therefore \mathrm{Res}_{p_k} \Phi = \begin{pmatrix} & \mathrm{Res}_{p_k} 1 & \\ & & \mathrm{Res}_{p_k} 1 \\ & & \end{pmatrix} \in \mathfrak{sl}(W_{p_k})$$

this is standard model for Harmonic-Metric

$\therefore$ Asympototically, the hermitian Metric is

$$\tilde{h} = |\log r|^{-2} \oplus 1 \oplus |\log r|^{2}$$

Prop:

Let M compact R.S, $D = P_1 + \cdots + P_n \in M$,

$\mathring{M} = M - D$, (W, Φ) strongly parabolic Higgs bundle of degree 0 & trivial flag weights,

then $\rho : \pi_1(M) \to SL_r(\mathbb{C}) \cong \bar{\rho} : \pi_1(M) \to SL_r(\mathbb{C})$

$\Leftrightarrow (W, \Phi) \cong (W^*, \Phi^*)$

pf: $(\bar{W}, \bar{\Phi}^*)$ gives same flat connection $\rightsquigarrow$

its monodromy is $\bar{\rho}$

We have $\overline{(W, \Phi)} \xrightarrow{h} (W, -\Phi)^*$

$$\overset{\bar{\rho}}{\rotatebox{90}{$\cong$}}$$

(because $\Phi - \Phi^*$ is hermitian) (W^*, Φ^*)

$\therefore \rho \cong \bar{\rho} \Leftrightarrow (W, \Phi) \cong (W^*, \Phi^*)$

Prop: (1) The tame harmonic h is globally diagnal w.r.t $W = \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O}(1)$,

$h = h_{-1} \oplus h_0 \oplus h_1$

$h_{-1} = h_1^{-1}$, $h_0 = dt^2$

(2) The identify $K = K^{-1}$ gives a Rieman metric g on Σ satisfying Tzitzéica equation for hyperbolic affine sphore, $H=1$

pf: Φ is cyclic Higgs bundle

$\rightarrow [(\bar{\partial}, e^{2\pi i/3}\Phi)] \sim [(\bar{\partial}, \Phi)].\mathrm{diag}(e^{2\pi i/3}, 1, e^{4\pi i/3})$

$\therefore c^* h c = h$, $D.C = D$

$\rightsquigarrow h$ is diagnal !!

For the pairing $b: (W, \Phi) \cong (W^*, \Phi^*)$

by put $\mathcal{O}(-1)$ to $\mathcal{O}(1)$ $\therefore b = b^*$ $\therefore bh^{-1} = hb$.

$\rightsquigarrow h = h_1^{-1} \oplus dt \oplus h_1$, $\rho \in SL_3(\mathbb{R})$

pf for (2):

Notice that on S^2: $K^{-1} = \mathcal{O}(2) = \mathcal{O}(-1)\mathcal{O}(\underset{S}{D})$

$\therefore$ on $\Sigma = S^2 - \{p_1, p_2, p_3\}$, we have an isomorphism:

$a: \mathcal{O}(-1)|_{\Sigma} \xrightarrow{\cong} K^{-1}|_{\Sigma}$, given by multiply s.

$W = \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O}(-1)$, gauging (W, Φ) by $\begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix}$

we obtain $K^{-1} \oplus \underline{\mathbb{C}} \oplus K|_{\Sigma}$ with higgs field $\Phi = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$.

with metric $g \oplus dt^2 \oplus g^{-1}$ from h by gauging $a \oplus 1 \oplus a^{-1}$

$g = \pi^* g_B \oplus \pi^* g_B \longrightarrow (T(TB), g, J)$ kahler

and fibers lagrangian.

$\omega = g(J\cdot,\cdot) = \begin{pmatrix} & -\pi^* g_B \\ \pi^* g_B & \end{pmatrix}$ $\therefore \omega^n = (2i)^n \det(\nabla d\phi)\, dz_1 \wedge \cdots \wedge d\bar{z}_n$

$= 2\, \pi^* \nabla d\phi$

$= 2\, \pi^* (i\partial\bar{\partial}\phi)$

$\therefore \Omega$ has constant length of $K_{T(TB)}$

$\therefore K_{T(TB)} \cong \underline{\mathbb{C}}$, where $\Omega = \pi^* \det B \oplus i J^* \pi^* \det B$

Now we hope to descent to $X/\Lambda \longrightarrow B$

this is equivalent to $\rho^\nabla \in SL(3,\mathbb{Z})$!

pf:

Pick $p \in B$, if $\rho \in SL(3,\mathbb{Z})$, $\exists$ corresp basis $\{s_1, s_2, s_3\}$

then parallel transport $\rightsquigarrow$ full lattice defined

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Thm: $\Sigma = S^2 \setminus \{p_1, p_2, p_3\}$, $B = \Sigma \times (0,1)$, $Y: \{\{p_k\} \times (0,1)\}$

then: $\exists$ $\mathbb{C}$-family para by $Q \in H^0(K\Theta(D))$ on S^2

of non-isometric semi flat CY metric, s.t. the

fiber $\pi: X \to B$ are special lagrangian.

Thm: In above, $\exists$ infinite series of non-isometric semi-flat CY metrics on $\pi: X/\Lambda \to B$ fibered by Special lagrangian,

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The Betti-Moduli Space

Our purpose is to construct $SL(3,\mathbb{Z})$ - monodromy.

Here: $\Phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ Q & 0 & 0 \end{pmatrix} \rightsquigarrow \mathrm{Res}_{p_k}\Phi = \begin{pmatrix} & 1 & \\ & & 1 \\ & & \end{pmatrix}$

As $\underset{0}{0} \subsetneq \underset{0}{\mathbb{C}} \subsetneq \underset{0}{\mathbb{C}^2} \subsetneq \underset{0}{\mathbb{C}^3} \rightsquigarrow$ the monodromy: $\begin{pmatrix} \cancel{k} & 1 & 0 \\ 0 & \cancel{k} & 1 \\ 0 & 0 & \cancel{k} \end{pmatrix}$

So our whole space is:

$$\mathcal{M}_B = \{\rho : \pi_1(\Sigma, p_0) \to SL_3(\mathbb{C}) \mid \rho \text{ unipotent}\} / SL_3(\mathbb{C})$$

i.e.

$$\mathcal{M}_B = \{(A_1, A_2) \mid A_1, A_2, (A_1 A_2)^{-1} \text{ unipotent}\} / SL_3(\mathbb{C})$$

Def:

$$\chi : \mathcal{M}_B \longrightarrow \mathbb{C}^3,$$

$$\rho \longmapsto (\mathrm{tr}(\rho(\gamma_1)\rho(\gamma_2)^{-1}), \mathrm{tr}(\rho(\gamma_1)^{-1}\rho(\gamma_2)), 1)$$

$p: \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ just like magic

$$(x,y,z) \longmapsto 414-108x+x^3-108y+21xy+y^3 - (51-9x-9y+xy)z+z^2$$

denote $\mathcal{F} = p^{-1}(0)$

Thm4:

(i) The cubic $\mathcal{F} = p^{-1}(0) \subset \mathbb{C}^3$ is smooth away from $(3,3,3) \in \mathcal{F}$ and has representation:

$$\psi: \mathbb{C}^2 \to \mathcal{F}, \quad (s,t) \longmapsto \left(3+\frac{(3+3s+t)^2}{st-s^3-1},\ 3+s\frac{(3+3s+t)^2}{st-s^2-1},\ 3+t\frac{(3+3s+t)^2}{st-s^3-1}\right)$$

(ii) $\mathcal{X}: \mathcal{M}_B \to \mathbb{C}^3$ surjects to $\mathcal{F}$. i.e. $p(\mathcal{X}(\mathcal{M}_B))=0$

$\mathcal{X}: \mathcal{M}_B^{ps} \xrightarrow{\cong} \mathcal{F}$ homeomorphism variety, each one is irredu except $\mathcal{X}_{Id}$.

$\mathcal{X}: \mathcal{M}_B^{s} \xrightarrow{\cong} \mathcal{F}$ biholomorphism.

(iii) $\mathcal{F} \cap \mathbb{R}^3$ is just $SL(3,\mathbb{R})$ representation,

and $\mathcal{F} \cap \mathbb{R}^3 = C_1 \cup C_2$ (2 connected components)

one contains trivial representation, another one is $SL_3(\mathbb{R})$ Hitchin component contains uniformization representation.

C_2 is smooth.

Prop: (1) $A \in SL_3(\mathbb{C})$ s.t. $\det(A-\lambda Id) = (\lambda-1)^3 \Leftrightarrow trA = trA^2 = 3$;

(2) $A \in SL_3(\mathbb{C})$ with $trA = 3$, then $tr(A^{-1}) = 3 \Leftrightarrow trA^2 = 3$.

For second: $9 = (trA)^2 = tr(A^2) + 2tr(A^{-1})$

first one: $\det(\lambda Id - A) = \lambda^3 - trA\,\lambda^2 + \frac{1}{2}((trA)^2 - tr(A^2))\lambda + \det A$

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pf of theorem:

$\therefore \mathcal{M}_B := \{\rho(\gamma_1), \rho(\gamma_2) \mid tr(\rho(\gamma_i)) = tr(\rho(\gamma_i)^2) = 3,\ i=1,2$
$tr(\rho(\gamma_1)\rho(\gamma_2)) = tr(\rho(\gamma_1)\rho(\gamma_2))^2 = 3\}$

Now coming back:

$\rho(\gamma_1)$ has 3 types: (after conjugation)

$$A_1 = \begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix}, \quad \tilde{A}_1 = \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}, \quad Id.$$

Observation:

(1) $g^{-1}A_1 g = A_1$ for $g \in SL_3(\mathbb{C}) \Leftrightarrow g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ & g_{11} & g_{12} \\ & & g_{11} \end{pmatrix}$

(2) $\tilde{g}^{-1}\tilde{A}_1\tilde{g} = \tilde{A}_1$ for $\tilde{g} \in SL_3(\mathbb{C}) \Leftrightarrow \tilde{g} = \begin{pmatrix} \tilde{g}_{22}^{-2} & & \tilde{g}_{13} \\ \tilde{g}_{21} & \tilde{g}_{22} & \tilde{g}_{12} \\ & & \tilde{g}_{22} \end{pmatrix}$

$A_2 = (a_{ij})$, $\tilde{A}_2 = (\tilde{a}_{ij})$, g, $\tilde{g}$ has the form above

(1) $a_{31} \neq 0$, $\exists$ unique g, s.t. $g^{-1}A_2 g = \begin{pmatrix} & b_{12} & b_{13} \\ & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$

(2) $a_{31}=0$, $\exists$ 1-dim invariant subspace of A_1, A_2.
or $\exists$ a unique g such that:

$$g^{-1}A_2 g = \begin{pmatrix} & b_{12} & b_{13} \\ b_{21} & b_{22} & \\ & b_{32} & b_{33} \end{pmatrix}$$

(3) $\hat{a}_{32} \neq 0$, $\exists ! \tilde{g}$, s.t.

$$\tilde{g}^{-1} \hat{A}_2 \tilde{g} = \begin{pmatrix} \tilde{b}_{11} & & \tilde{b}_{13} \\ \tilde{b}_{21} & & \tilde{b}_{23} \\ & \tilde{b}_{32} & \tilde{b}_{33} \end{pmatrix}$$

(4) If $\hat{a}_{32}=0$, $\exists$ common invariant subspace of $\hat{A}_1$ and $\hat{A}_2$ of dim 1 or 2, or $\exists ! \tilde{g}$ such that:

$$\tilde{g}^{-1} \hat{A}_2 \tilde{g} = \begin{pmatrix} & \tilde{b}_{12} & \tilde{b}_{13} \\ & & \tilde{b}_{23} \\ 1 & & \tilde{b}_{33} \end{pmatrix}$$

Observation:
let $[\rho] \in \mathcal{M}_B$, then ρ is irreducible $\Leftrightarrow$ $X_\rho \neq (3,3,3)$

$\therefore$ $\rho = Id$ is the only completely reducible presenta.

pf:
W.L.O.G., let $\rho(\gamma_1) = A_1 = \begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$. ($\hat{A}_1$ is more simple)
so we can fix A_1 and the gauge group becomes:

$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ & g_{11} & g_{12} \\ & & g_{11} \end{pmatrix} = g$, by observation 3 $\rightsquigarrow$ $A_2 := \rho(\gamma_2)$ can be considered in 2 types :

$a_{31} = 0$ or $a_{31} \neq 0$

① $a_{31} \neq 0$, $\rho(\gamma_2) = \begin{pmatrix} & b_{12} & b_{13} \\ & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$

$\rho(\gamma_1) = \begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$

$\therefore$ no invariant subspace.

$\therefore$ unimodolue $\rightsquigarrow \begin{cases} b_{22} + b_{33} + b_{32} = 3 = tr(\rho(\gamma_2)\rho(\gamma_1)) \\ b_{22} + b_{33} = 3 \end{cases}$

$\therefore b_{32} = 0$, but then $tr(\rho(\gamma_1)^{-1}\rho(\gamma_2)) = tr\rho(\gamma_2) + b_{31} = 3 + b_{31} \neq 3$

$\therefore \chi_\rho \neq (3, 3, 3)$

② $a_{31} = 0$,

$\begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix} = \rho(\gamma_1)$, $\begin{pmatrix} & b_{12} & b_{13} \\ b_{21} & b_{22} & \\ & b_{32} & b_{33} \end{pmatrix} = \rho(\gamma_2)$

$tr(\rho(\gamma_1)\rho(\gamma_2)) = b_{21} + b_{32} + b_{22} + b_{33} = 3 = tr(\gamma_2) = b_{22} + b_{33}$

$\rightsquigarrow b_{32} = -b_{21}$, $\because b_{32} = -b_{21} \neq 0$

$\rightsquigarrow \chi_\rho = (3 - b_{21}^2, 3, 3 + 3b_{21}^2 - b_{21}^3) \neq (3, 3, 3)$

Now giving the proof of part 2:

$$\chi: \mathcal{M}_B := \{\rho(\gamma_1), \rho(\gamma_2) \mid \ldots\} \longrightarrow \mathbb{C}^3$$
$$\rho \longmapsto (tr(\rho(\gamma_1)\rho(\gamma_2)^{-1}), tr(\rho(\gamma_1)^{-1}\rho(\gamma_2)), 3)$$

$$P: \mathbb{C}^3 \longrightarrow \mathbb{C}$$
$$(x,y,z) \longmapsto 414 - 108x + x^3 - 108y + 21xy + y^3 - (51 - 9x - 9y + xy)z + z^2$$

$$\underline{\chi: \mathcal{M}_B^{ps} \cong P^{-1}(0)}$$
$$\underline{\mathcal{M}_B^{s} \cong P^{-1}(0) \setminus \{(3,3,3)\}}$$

Key point

pf: Let $\chi_\rho = (x, y, z) \in \mathbb{C}^3$

(I) $y \neq 3 \rightsquigarrow \rho(\gamma_1) = A_1$, $\rho(\gamma_2)$ satisfies $b_{32} = 0$

$y = tr(\rho(\gamma_1)^{-1}\rho(\gamma_2))$, $tr(\rho(\gamma_2)) = 3 \implies b_{33} = 3 - b_{22}$
$\qquad b_{31} = y - 3$

$tr(\rho(\gamma_2))^2 = 3 \to b_{13} = \dfrac{3 - 3b_{22} + b_{22}^2}{3 - y}$

$tr(\rho(\gamma_1)\rho(\gamma_2))^2 = 3 \iff (b_{12} + b_{22} + b_{23})\overset{(y-3)}{} = 0$

$\therefore \tilde{x} = x \det\rho(\gamma_2) = \det\rho(\gamma_2)\, tr(\rho(\gamma_1)\rho(\gamma_2)^{-1})$ for b_{22}

$\rightsquigarrow b_{22} = -\dfrac{\tilde{x} - 3}{y - 3}$, $b_{12} = \dfrac{3\tilde{x} + 3y + \tilde{z} - 21}{(y-3)^2}$

$\therefore \det\rho(\gamma_2) = 1 \iff (\tilde{x}, y, \tilde{z}) \in \mathcal{L}$ and $\tilde{x} = x$

$Z = \mathrm{tr}(\rho(\gamma_1)\rho(\gamma_2)\rho(\gamma_1)^{-1}\rho(\gamma_2)^{-1})$

gives $\tilde{z} - \det\rho(\gamma_2)\, z = 3\, \dfrac{p(\tilde{x}, y, \tilde{z})}{(y-3)^2}$

$\therefore\ z = \tilde{z}$

$\therefore\ \forall\,(x, y, z) \in \tilde{\mathcal{F}}$ with $y \neq 3$

$$\rho(\gamma_2) = \begin{pmatrix} & \frac{3x+3y+z-21}{(y-3)^2} & -\frac{63+x^2-15x-27y+3xy+3y^2}{(y-3)^2} \\ & & \\ y-3 & & \star \end{pmatrix}$$

Case II : $y = 3,\ x \neq 3$

$$\rightsquigarrow\ \rho(\gamma_2) = \begin{pmatrix} & 3-\frac{3}{b_{21}}-b_{21} & \frac{(b_{21}-1)^3}{b_{21}^2} \\ b_{21} & 3-b_{21} & 0 \\ & -b_{21} & b_{21} \end{pmatrix} \quad \star$$

Case III : $y = 3,\ x = 3$

then $p(x, y, z) = 0 \rightsquigarrow z = 3 \rightsquigarrow \rho = Id.$

$\therefore\ \mathcal{F}(\mathbb{R}) = \tilde{\mathcal{F}} \cap \mathbb{R}^3$ is just $Sl_3(\mathbb{R})$ — representation

$\tilde{\mathcal{F}}(\mathbb{R}) = \Psi : \mathbb{R}^2 \setminus \{st - s^3 - 1 = 0\} \longrightarrow \tilde{\mathcal{F}}(\mathbb{R})$

$\therefore\ \mathrm{im}\,\Psi \cup \underset{\{x=3\}}{D_1} \cup \underset{\{y=6-x,\ z=3\}}{D_2}$

$\Psi(S_2) = C_2 = \{x>3, y>3\}$

$C_1 = \Psi(S_1) \cup \Psi(\hat{S}_1) \cup D \subset \mathcal{F}(\mathbb{R})$

Key Part: Find integer representation.

$\Psi: \mathbb{R}^2 \longrightarrow \mathcal{F}(\mathbb{R})$, $\Psi(s,t) = \left(3+\frac{(3+3s+t)^2}{st-s^2-1},\ 3+s\frac{(3+3s+t)^2}{st-s^2-1},\ 3+t\frac{(3+3s+t)^2}{st-s^3-1}\right)$

$\Psi(n,n^2) \subsetneq \mathcal{F}(\mathbb{Z}) \in C_1$

$$\rightsquigarrow \rho(r_1) = \begin{pmatrix} 1 & 3+3n+n^2 & \\ & 1 & 3+3n+n^2 \\ & & 1 \end{pmatrix}$$

$$\rho(r_2) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 3+n \\ -n & -1 & 3 \end{pmatrix}$$

For Hitchin-Component Part: For $k, l \in \mathbb{N}$

$(s,t) = \left(\frac{1}{k}, \frac{l}{k}\right)$, $k>0$, $l > k^2 + \frac{1}{k}$

has $st - s^3 - 1 > 0$

$$\therefore \Psi(s,t) = \left(3+\frac{k(3k+l+3)^2}{kl-k^3-1},\ 3+\frac{(3k+l+3)^2}{kl-k^3-1},\ 3+\frac{l(3k+l+3)^2}{kl-k^3-1}\right)$$

$\in C_2$

$\therefore \Psi(s,t) \in \hat{\mathcal{F}}(\mathbb{Z}) \iff \exists\, m \in \mathbb{N}$, s.t.

$$(3k+l+3)^2 = m(kl-k^3-1)$$

Overleaf ✓

The Dolbeault Moduli Space

具体构造:

$$(\mathcal{O}(-1)\oplus\mathcal{O}\oplus\mathcal{O}(1),\ \overline{\Phi}^{Q})\ \text{to}\ SL_3(\mathbb{R})\ \text{Hitchin}$$

trivial flag and weight

component $C_2 \subset \mathcal{M}_{Betti}$

Lemma3:

let $(W,\overline{\Phi})$ rk 3 stable parabolic of deg 0 over S^2 with trivial flag, weight at $p_k \in S^2$, then:

$$W = \mathcal{O}^{\oplus 3} \quad \text{or} \quad W = \mathcal{O}(-1)\oplus\mathcal{O}\oplus\mathcal{O}(1)$$

pf: let $W = \mathcal{O}(-k-l)\oplus\mathcal{O}(k)\oplus\mathcal{O}(l)$

$$\rightsquigarrow \operatorname{ord}\overline{\Phi} = \begin{pmatrix} -2 & -2l-k-2 & -l-2k-2 \\ 2l+k-2 & -2 & l-k-2 \\ 2k+l-2 & k-2-l & -2 \end{pmatrix}$$

$\overline{\Phi}$ has at most simple pole

$$\Rightarrow -l-2k-2 < -2l-k-2 < -3$$

$\therefore \mathcal{O}(l)\oplus\mathcal{O}(k)$ invariant subbundle with $k+l>0$

$\therefore$ only 2 other cases.

lemma:

let $[(W,\Phi)] \in \mathcal{M}_B^s$ with $W = \mathcal{O}(-1)\oplus\mathcal{O}\oplus\mathcal{O}(1)$, then $\exists !$ $Q \in H^0(K^3\mathcal{O}(2D))$ with quadric poles, s.t.

$$\Phi = \begin{pmatrix} & 1 & \\ & & 1 \\ Q & & \end{pmatrix}$$ is what we want.

pf: Write $\Phi = (\Phi_{ij}) \rightsquigarrow \Phi_{13} = 0$

if $\Phi_{23} = 0$ or $\Phi_{12} = 0 \rightsquigarrow \mathcal{O}(1)$ or $\mathcal{O}\oplus\mathcal{O}(1)$ are Φ-inv subbundle of positive degree $\rightsquigarrow$ $\lightning$ stability 矛盾

$\therefore \Phi_{12} = \Phi_{23} = 1$

$\exists !$ $a \in H^0(SL(W))$, $a = \begin{pmatrix} 1 & & \\ a_{21} & 1 & \\ a_{31} & a_{32} & 1 \end{pmatrix}$, s.t.

$a^{-1}\Phi a$ has form ✓

即解代数方程 $\rightarrow$ $\Phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \Phi_{31} & \Phi_{32} & 0 \end{pmatrix}$

by Maxi-nilpotent, $tr(Res\,\Phi)^2 = tr(Res\,\Phi)^3 = 0$

$\rightsquigarrow$ $\Phi_{31} \in \mathbb{C} = H^0(K^3\mathcal{O}(2D))$, $\Phi_{32} = 0$

lemma:

For $W = \mathcal{O}^{\oplus 3}$, $K = \ker Res_{p_k}\Phi$, we could give classification of Φ: ✓