T^n-Invariant Calabi-Yau Manifold

Def: (Semi-flat) A calabi-Yau manifold (M, J, ω, g) called semi-flat if it admits a fibration by flat lagrangian tori.

$X \xrightarrow{\pi} B$, $\pi^{-1}(b)$ special lagrangian.

Notation: (Assumption)

$$M = D \times i\mathbb{R}^n \ (TD), \quad \Omega_M = dz_1 \wedge \cdots \wedge dz_n$$

$$\omega_M = i\partial\bar{\partial}\phi, \quad T^n \text{ invariant function: } \phi(x^j, y^j) = \phi(x^j)$$

Monge-Ampère:

$$\begin{cases} \det\left(\dfrac{\partial^2\phi}{\partial x^j \partial x^k}\right) = C & , \ \phi \text{ convex} \\ \phi|_{\partial D} = 0 \end{cases}$$

in this case, we get a T^n inva metric on M,

$$\begin{cases} g_M = \sum \phi_{jk} (dx^j dx^k + dy^j \otimes dy^k) \\ \omega_M = \dfrac{i}{2} \sum \phi_{jk} dz^j \wedge d\bar{z}^k. \ (\text{or } \sum \phi_{jk} dx^j \wedge dy^k) \end{cases}$$

(M is closed as $d\omega_M = d(\bar{\partial}\partial\phi) = 0$)

compact fiber $\rightsquigarrow D \times i\Lambda \subset TD$ special lagrang fibration.

Here assume D is affine:

transition function:

$$(x^j) \longrightarrow (\bar{x}^j) = (\underbrace{A^j_k x^k}_{\text{gives a line bundle } L} + B)$$

$\phi: D \to \mathbb{R}$, $i\partial\bar\partial\phi > 0 \rightsquigarrow \boxed{\exists d\phi} \in L^{\otimes 2}$ be solution.

D affine $\rightsquigarrow TD$ Complex manifold.

Legendre Transformation

Def: 给定 $f: \mathbb{R}^n \to \mathbb{R}$ convex, then one can produce $f^\vee: \mathbb{R}^{n\vee} \to \mathbb{R}$

Here: $\phi: \mathbb{R}^n \to \mathbb{R}$ convex $\rightsquigarrow (\mathbb{R}^n)^* \longrightarrow \mathbb{R}$

(i) change coordinate: $X_k = X_k(x^1, \dots, x^n)$, $\frac{\partial X_k}{\partial x^j} = \phi_{jk}$,

$(\phi^{jk})^{-1} = \phi_{jk}$

(ii) by Legendre-Transformation:

$\exists$ function $\psi: (X_1, \dots, X_n) \longrightarrow \mathbb{R}$, s.t. $\frac{\partial\psi}{\partial X_j} = x^j \rightsquigarrow \frac{\partial^2\psi}{\partial X_i \partial X_j} = \frac{\partial x^j}{\partial X_i} = \phi^{ij}$

$\rightsquigarrow \mathrm{Hess}\,\psi = (\phi^{ij})_{n\times n}$

☆☆:

We want to understand correspondence between:

$M = TD$		$W = T^\vee D$
(i) special lagrangian	$\longleftrightarrow$	complex submfd
(ii) complex structure	$\longleftrightarrow$	symplectic form
(iii) symplectic auto	$\longleftrightarrow$	Holomorphic isomorphis
(iv) Hard-Lefchetz	$\longleftrightarrow$	VHS

(v) Blaske Connection $\longleftrightarrow$ conjugate connection

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Structure of M, W.

1. M side

Def: $M = D \times iT$

$M \xrightarrow{\pi_1} D$, $(x^j, y^j) \longmapsto x^j$

we know $M \xrightarrow{\phi} \mathbb{R}$ with ϕ convex $\rightsquigarrow$ gives a metric on D

$\rightsquigarrow g_M = \phi_{ij}(dx^i dx^j + dy^i dy^j)$

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2. W side

Def: $W = D \times iT^* = T^*D$

$\rightsquigarrow g_W = \phi_{ij} dx^i dx^j + \phi^{ij} dy_i dy_j = \phi^{ij}(dx_i dx_j + dy_i dy_j)$

RK: $T^* = \mathrm{Hom}(S^1 \times \cdots \times S^1, \mathbb{R}) = \mathrm{Hom}(S^1 \times \cdots \times S^1, \mathbb{Z} \times S^1)$

$\cong \mathrm{Hom}(\mathbb{Z}^n, S^1) = \mathrm{Hom}(\pi_1(T), U(1))$

$\rightsquigarrow$ flat $U(1)$ connections on T.

$\omega_W := \sum dx^j \wedge dy_j$ (Canonical 1-form)

$\rightsquigarrow J_W := g_W^{-1} \omega_W$ gives the complex structure.

Because of change coordinate:

$\omega_W = \sum \phi^{kj} dx_k \wedge dy_j = \frac{i}{2} \sum \phi^{jk} dz_j \wedge d\bar{z}_k$

① Complex Structure on $M \longleftrightarrow$ Symplectic on W

Thm:

M be T^n- invariant CY, $W := M^\vee$, then:

$\{$moduli of complex structure on $M\}$

$\cong$

$\{$moduli of complexified symplectic structure on $W\}$

pf:

$g_M = \phi_{ij}(dx^i dx^j + dy^i dy^j)$, $J = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \rightsquigarrow \omega_M = \begin{pmatrix} & * \\ * & \end{pmatrix}$

$\therefore$ D is moduli of special lagrangian $\therefore$ $\{(L, \underset{\text{flat } U(1)}{\nabla})\} \cong W$

$\rightsquigarrow W$ moduli of flat $U(1)$-conn on Special-Lagrangian.

(A-cycle, $(M^n, \omega^{\mathbb{C}} = \omega + i\beta, \Omega)$ CY manifold, then:

(C, E^h) called A-cycle: ① C special-lagrangian

② $F^h + \beta = 0$

(C, E^h) called B-cycle: $\mathrm{Im}\, e^{i\theta}(\omega^{\mathbb{C}} + F)^m = 0$ for constant θ.

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idea: M is Calabi-Yau, then varying $J_M \Leftrightarrow$ varying $g_M^{-1}\omega_M$

so we fix ω_M or g_M, $\Leftrightarrow$ varying g_M / ω_M of M.

fix $\omega_M = \sum \phi_{jk} dx^j \wedge dy^k$, then vary g_M,

$$g_M = \sum (\phi_{jk}(x) + i\eta_{jk}(x))(dx^j dx^k + dy^j dy^k)$$

write $\theta_{jk}(x) = \phi_{jk}(x) + i\eta_{jk}(x)$ $(-i\partial\bar\partial\eta,\ \eta: M \to \mathbb{R})$

$$\rightsquigarrow g_M^{\mathbb{C}} = \sum \theta_{jk}(x)(dx^j \otimes dx^k + dy^j \otimes dy^k)$$

$$\omega_M^{\mathbb{C}} = \frac{i}{2} \sum \theta_{jk}(x)\, dz^j \wedge d\bar z^k$$

此时，对W而言：

$$dz_j = \sum \theta_{jk} dx^k = \sum (\phi_{jk} + i\eta_{jk}) dx^k$$

$$\therefore \mathrm{Re}\, dz_j = \phi_{jk} dx^k, \quad \mathrm{Im}\, dz_j = dy_j + \eta_{jk} dx^k,$$

$$\rightsquigarrow \omega_W^{\mathbb{C}} = \sum dx^j \wedge dy_j = \frac{i}{2} \sum \theta^{jk} dz_j \wedge d\bar z_k$$

$\rightsquigarrow J_W$ still fixed

$$\therefore g_W^{\mathbb{C}} = \sum \theta^{jk} dz_j \otimes d\bar z_k.$$

Infinitesimal Version of This Map.

$$T: H^1(M, T_M^*) \longrightarrow H^1(W, T_W)$$

(Symplec) (Complex)

if $\omega_M^{new} = \omega_M + \varepsilon \sum \xi_{ij} dx^i \wedge dy^j = \sum (\phi_{jk} + \varepsilon \xi_{jk}) dx^j \wedge dy^k$

By Monge-Ampére $\rightsquigarrow$ ξ is harmonic

$$dz_j^{new} = \sum (\phi_{jk} + \varepsilon \xi_{jk}) dx^k + i dy_j = \underline{\sum (\delta_j^l + \frac{\varepsilon}{2} \phi^{lk} \xi_{jk}) dz_l + \frac{\varepsilon}{2} \phi^{kl} \xi_{jk} d\bar z_l}$$

$\rightsquigarrow \ -\frac{1}{2}\sum \phi^{jk} g_{kl} \frac{\partial}{\partial z_l} \otimes d\bar{z}_j \in \Omega^{0,1}(W, TW)$

$\rightsquigarrow$ Harmonic means $H^1(W, TW)$.

$\therefore$ We have a homomorphism of $H^1(M, T_M^*) \longrightarrow H^1(W, TW)$.

Thm: This map is holomorphic + isometry.

(推广到 differential-form 的 transformation)

$$T: \Omega^{p,q}(M) \longrightarrow \Omega^{n-p,q}(W) := \Omega^{0,q}(M, \wedge^{n-p} T^* W)$$

$$\Omega^{p,q}(M) = \Omega^{0,q}(M, \wedge^p T_M^*)$$

(i) $dx^j \longmapsto \phi^{jk} dx_k$ (Legendre-Transformation)

(ii) $dy^j \longmapsto \frac{\partial}{\partial y^j}$ (Fourier-Transform)

complex polarization

$$T(dz^j) = \frac{\partial}{\partial z_j}, \quad T(d\bar{z}^j) = \sum \phi^{jk} d\bar{z}^k$$

Actually, this T commute with $\bar{\partial}$-operator & $\bar{\partial}^*$-operator

$$\rightsquigarrow T: H^{p,q}(M) \longrightarrow H^{n-p,q}(W)$$

Hard Lefchetz $sl(2;\mathbb{C})$ — action

M is Kahler, then we have a $sl(2;\mathbb{C})$ action

$$sl(2;\mathbb{C}) \curvearrowright \Omega(M) \longmapsto \Omega(M)$$
$$(e \ , \ \omega) \longmapsto L_A \omega$$
$$(f \ , \ \omega) \longmapsto \Lambda_A \omega$$
$$(l \ , \ \omega) \longmapsto H_A \omega \ , \qquad H_A = (n-k) I$$

VHS $sl(2;\mathbb{C})$ — Action.

M is T^n invariant manifold

By deformation theory, for $\frac{dM_t}{dt} \in H^1(M, T_M)$, it sends

$$H^q(M, \Omega_M^p) \longrightarrow H^{q+1}(M, \Omega_M^{p-1})$$

$\rightsquigarrow$ We could assume : $\frac{dM_t}{dt} = \sum \frac{\partial}{\partial z^j} \otimes d\bar{z}^j$

$\therefore \Lambda_B = \sum \frac{\partial}{\partial \bar{z}^j} \otimes dz^j$

Thm:

On a T^n-invariant mfd M, if we define:

$$L_B = \sum \frac{\partial}{\partial z^j} \otimes d\bar{z}^j \ , \quad \Lambda_B = \sum \frac{\partial}{\partial \bar{z}^j} \otimes dz^j \ , \quad H_B = [L_B, \Lambda_B]$$

$\rightsquigarrow$ still a $sl(2;\mathbb{C})$ action

Thm:

This $sl(2;\mathbb{C})$ action commutes with hard lefsctchez $sl(2;\mathbb{C})$ action.

$$[L_A, L_B] = 0 \ , \quad [L_A, \Lambda_B] = 0$$

$$L_A = \omega_A \wedge \cdots = \phi_{ij} (dx^i \wedge dy^j) = \frac{i}{2} \phi_{ij} \, dz^i \wedge d\bar{z}^j$$

$L_B = \sum \frac{\partial}{\partial \bar{z}^j} \otimes d\bar{z}^j$, check !

Thm:

Let M, W be mirror T^n-invariant kahler mfd to each other. Then the mirror transformation T carries the hard Lefschetz $sl(2;\mathbb{C})$ action on M

to

VHS $sl(2;\mathbb{C})$ action on W.

pf:

$$T(d\bar{z}^j) = \sum \phi^{jk} dz_1 \cdots dz_n d\bar{z}_k$$

$$L_A T(d\bar{z}^j) = 0 \ , \ L_B(d\bar{z}^j) = 0 \rightsquigarrow L_A T(d\bar{z}^j) = T L_B(d\bar{z}^j)$$

Here:

For $\alpha = \sum \alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$

$$T(\alpha) = \sum \alpha_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} \phi^{k_1 j_1} \cdots \phi^{k_q j_q} \Omega_W \left(\frac{\partial}{\partial z_{i_1}}, \cdots, \frac{\partial}{\partial z_{i_p}}\right) d\bar{z}_{k_1} \wedge \cdots \wedge d\bar{z}_{k_q}$$

$\therefore$ $L_A T = T L_B$

$\Lambda_A T = T \Lambda_B$

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Holomorphic VS Symplectic Automorphism

Thm: For T^n-invariant CY mfd M, W, the mirror transfor induces isomorphi

$$\mathrm{Diff}(M,J)_{\lim} \xrightarrow{\cong} \mathrm{Diff}(W,\bar{\omega})_{\lim}$$
$$\mathrm{Diff}(M,\bar{\omega})_{\lim} \xrightarrow{\cong} \mathrm{Diff}(W,J)_{\lim}$$

and:

$$\mathrm{Diff}(M,J_\infty)_{\lim} \xrightarrow{\cong} \mathrm{Diff}(W,\omega)_{\lim}$$
$$\mathrm{Diff}(M,\omega)_{\lim} \xrightarrow{\cong} \mathrm{Diff}(W,J_\infty)_{\lim}$$

pf:

First of all: $(\cdot)_A$: $\mathrm{Diff}(D) \xrightarrow{\cong} \mathrm{Diff}(M,\omega)$

$$\mathrm{Diff}(D,\text{Affine}) \cong \mathrm{Diff}(M,\bar{\omega})_{\lim}$$

$(\cdot)_B$: $\mathrm{Diff}(D) \cong \mathrm{Diff}(M,J_\infty)_{\lim}$

$$\mathrm{Diff}(D,\text{Affine}) \cong \mathrm{Diff}(M,J)_{\lim}$$

$(\cdot)_B$: $f: D \to D \rightsquigarrow (f)_B: TD \to TD$ (f, df)

$$\therefore f_B(x^j + iy^j) = \left(f^k(x^j) + i\sum \frac{\partial f^k}{\partial x^j} y^j\right)$$

f_B holo $\Leftrightarrow \frac{\partial}{\partial \bar{z}^l} f_B = 0 \Leftrightarrow \frac{\partial^2 f^k}{\partial x^j \partial x^l} = 0 \Leftrightarrow f^k = Ax + b$

Affine

(if $t \to 0$ w.r.t. $dz^j(t) = t^{-1}dx^j + i\,dy^j \rightsquigarrow f_B$ holo $\checkmark$)

$\forall f : D \xrightarrow{\sim} D \quad \overset{\text{induces}}{\rightsquigarrow} \quad \hat{f} : D^* \leftarrow D^* \quad$ given by

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \Phi \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix} \longmapsto \Phi(f^{-1}\begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix}) = \Phi(\cdot \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}))$$

$\therefore$ $\hat{f}^*$ defines a Map from $T^*D^* \to T^*D^*$

$$\rightsquigarrow (\cdot)_A$$

$$(f)_A = \hat{f}^*(\hat{f}_j(x_j), y^j) \longmapsto (x_j, \sum \frac{\partial \hat{f}_k}{\partial x_j} y^k)$$

on $T^*D^* \rightsquigarrow \exists\, \bar{\omega} = \sum dx_j \otimes dy^j$

$$\therefore (f)_B(\bar{\omega}) = \bar{\omega} + \sum y^k \frac{\partial^2 \hat{f}_k}{\partial x_j \partial x_l} dx_j \otimes dx_l$$

$\therefore (f_A)(\bar{\omega}) = (\bar{\omega}) \iff \hat{f}$ affine

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pf: Now giving bijection :

Let $F \in \mathrm{Diff}(M, J)_{\bar{\omega}}$

$F = (F^1, \dots F^k)$ linear-along fiber :

$F^k = f^k(x) + i\sum g^k_l(x) y^l \rightsquigarrow \frac{\partial}{\partial \bar{z}} F^k = \frac{\partial f^k}{\partial x^j} - g^k_l \delta_{jk} + i \frac{\partial g^k_l}{\partial x^j} y^k$

$\therefore F$ preserves J_∞, $\mathrm{Re}\frac{\partial}{\partial \bar{z}} = 0 \rightsquigarrow g^k_j = \frac{\partial f^k}{\partial x^j}$

or $F = f_B$ if F preserves $J \rightsquigarrow 0 = \frac{\partial g^k_l}{\partial x^j}$

$$= \frac{\partial^2 f^k}{\partial x^j \partial x^l} \rightsquigarrow \checkmark$$