Construction of Minimal Surface from holo-sections of $\mathcal{M}_{DH} \to \mathbb{C}P^1$

For $f : \Sigma \to SU(2) = S^3$, $\rightsquigarrow$ Let $\theta = \frac{1}{2} f^{-1} df$ $\rightsquigarrow$

$\nabla^\lambda = \nabla + \lambda^{-1}\theta^{1,0} + \lambda\theta^{0,1}$ is family of flat-connections,

∇^λ gives rise to a section:

$$\mathbb{C}P^1 \longrightarrow \mathcal{M}_{DH}$$

$$s : \lambda \longmapsto [\bar{\partial}^\nabla + \lambda\theta^{0,1}, \lambda\partial^\nabla + \theta^{1,0}, \lambda]$$

s is real, $\rho(s(\bar{\lambda}^{-1})) = s(\lambda)$ where:

$$\rho : ([\bar{\partial}, D, \lambda]_\Sigma) \longmapsto [\bar{\partial}^*, D^*, \bar{\lambda}]_{\bar{\Sigma}}$$

then f is minimal $\Leftrightarrow d^\nabla * \alpha = 0 \Leftrightarrow \det\theta^{0,1} = 0$

and s passes through nilpotent-cone.

($\nabla^{\pm 1}$ is trivial)

Conversely, given a section $s : \mathbb{C}P^1 \to \mathcal{M}_{DH}$,

s.t. $\rho(s(\bar{\lambda}^{-1})) = s(\lambda)$, $\forall \lambda \in \mathbb{C}P^1$

$s(\lambda = \pm 1)$ is trivial.

then we can reconstruct a harmonic map which is minimal $s(0)$ passes throunge cone.

Def: The $NC = \{ (\bar\partial, \Phi) \in \mathcal{M}_{Dol} \mid \det \Phi = 0 \}$

RK:

f doesn't need to be an immersion.

$$(\bar\partial, \theta^{1,0}) = \left(\begin{pmatrix} \bar\partial^L & \gamma \\ 0 & \bar\partial^{L^*} \end{pmatrix}, \begin{pmatrix} 0 & \varphi \\ 0 & 0 \end{pmatrix} \right), \quad \varphi \in H^0(KL^2)$$

if $\varphi(p) = 0 \rightsquigarrow df|_p = 0$

And zero of φ are given by $\deg(KL^2)$

$= \operatorname{rank}(K) \cdot 2\deg(L) + \operatorname{rank}(L^2) \deg K$

$= 2g - 2 + 2\deg(L)$ $\quad \therefore f$ immersion $\Leftrightarrow \deg L = 1-g$.

Go from s to f:

1) We consider a lift

$$\hat{S}(\lambda) = \lambda^{-1}\Phi + \nabla + \lambda \Psi_1 + \cdots = \tilde\nabla^\lambda$$

reality $\rightsquigarrow (\tilde\nabla^{\bar\lambda^{-1}})^* = \nabla^\lambda g(\lambda) \rightsquigarrow \nabla^\lambda = \nabla^\lambda g(\lambda)(g(\bar\lambda^{-1})^*)^{-1}$

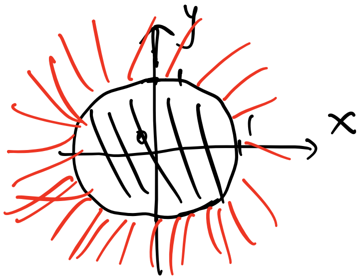
∇^λ is irre $\rightsquigarrow g(\lambda)(g(\bar\lambda^{-1})^*)^{-1} = \pm id$.

Fact: $\exists$ gauge g be $SL(2,\mathbb{C})$ along $\lambda \in S^1 \longrightarrow$

$g(\lambda) = g(\bar\lambda^{-1})^* \rightsquigarrow g$ is positive.

Do the Birkhoff - Factorization

$$g(\lambda) = g_+(\lambda)\, g_-(\lambda)$$



$$\rightsquigarrow \quad \tilde{\nabla}^{\lambda} . g_+(\lambda) = \left(\tilde{\nabla}^{\bar{\lambda}^{-1}}\right)^* . (g(\lambda))^{-1}$$

$$\| $$

$$\nabla + \lambda\Psi + \lambda^{-1}\Phi$$

$\rightsquigarrow$ It must be unitary along $\lambda \in S^1$

$\longrightarrow$ f is the gauge equi between

∇^{-1} and ∇^{1}!

Lawson Surface of genus g

Def: The underlying R.S. Σ is determined by

$$y^{g+1} = \frac{z^2 - i}{z^2 + i}$$

This determines a covering $\pi : \Sigma \to \mathbb{CP}^1$ totally

branched over $e^{\pm \pi i/4}$, $e^{\pm \frac{3}{4}\pi i}$

And there is a $\mathbb{Z}_{g+1}$ - fold symm acting on Σ

generated by μ:

$\mu: \Sigma \longrightarrow \Sigma$ for $f: \Sigma \longrightarrow SU(2)$
$(y,z) \longmapsto (e^{\frac{2\pi i}{g+1}} y, z)$

$$\rightsquigarrow \mu^* f = f \circ \mu = \bar{D}^{-1} f \bar{D} \quad \text{with} \quad \bar{D} = \begin{pmatrix} e^{\frac{2\pi i}{g+1}} & \\ & e^{-\frac{2\pi i}{g+1}} \end{pmatrix}$$

This implies for $\nabla^\lambda = \nabla + \lambda^{-1}\theta^{1,0} + \lambda \theta^{0,1}$, we have:

$$\mu^*(f^{-1}df) = \bar{D}^{-1} f^{-1} df \bar{D} \quad \therefore \quad \mu^*(\nabla^\lambda) = \nabla^\lambda \cdot \bar{D}$$

$\therefore$ The connections are equiva but not invariant.

Def:

Let $\begin{matrix} V \\ \downarrow \\ X \end{matrix}$ be a holomorphic v.b. on some R.S. X,

A connection ∇ on $V \to X \setminus \{p_1, \dots, p_k\}$ is called logarithmic connection with singular divisor $p_1 + \dots + p_k$ if $\bar{\partial}^\nabla = \bar{\partial}^V$ and ∇ can be written as $d + \omega$, ω is a meromorphic $\underline{sl}(V)$ — valued 1-form with at most first order pole at $p_1, \dots, p_k$.

Remark: $\mathrm{Res}_{p_k}(\nabla) \in \mathrm{End}(V_{p_k})$ is well-defined.
(Residue Theorem)

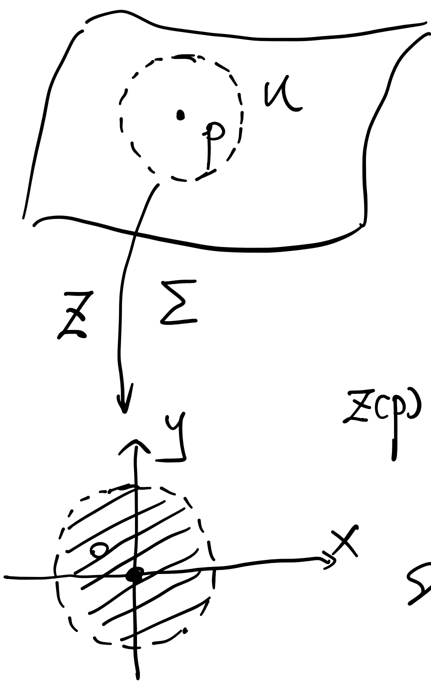
Example:

A Fuchsian System with singular divisor $p_1 + \dots + p_n$ is a connection on $\mathbb{CP}^1 \setminus \{p_1, \dots, p_n\}$ of the form

$\nabla = d + \sum A_k \frac{dz}{z-p_k}$, $\sum A_k = 0$

Lemma:

Let $z: \mathcal{U} \subseteq \Sigma \to \mathbb{C}$ be holo-coor centered at p, $z(p)=0$. Then, $\exists$ positive gauge $g_+(\lambda)$ over $\mathcal{U} \simeq$ Disk with $\nabla^\lambda . g_+(\lambda) = d + \eta^{nor}$, where:



$z(p)=0$

$$\eta^{nor} = \begin{pmatrix} & \lambda^{-1} f(z,0) \\ q(z)/h(z,0) & \end{pmatrix} dz$$

is the so-called Dor normalized potential,

and $g = h(z,\bar{z})\, dz\, d\bar{z}$, the metric of minimal surface $f: \Sigma \to S^3 = SU(2)$,

$q(z)(dz)^2 = Q$, the Hopf-diff, i.e. $(2,0)$-part of the second f.f.

Corallary: Near $p_1, \ldots, p_n$, one of the fixed points of the $\mathbb{Z}_{g+1}$-Symm such that:

$(\nabla^\lambda . g)$ is invariant under $\mathbb{Z}_{g+1}$-symm and has a log-singu at $z=0$

We should be able to construct Lawson Surface from Fuchsian-Systems:

$$\nabla^t = d + t\left(A_1 \frac{dz}{z-p_1} + \cdots + A_4 \frac{dz}{z-p_4}\right)$$

- ∇^t has unitary monodromy for $\lambda \in S^1$
- A_k has specific conjugacy class

$$A_k \sim \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad \forall k = 1 \sim 4$$

+ additional symmetry:

$$\sigma(z) = -z \; : \; \delta^* \nabla^t = \nabla^t . D, \quad D = \begin{pmatrix} i & \\ & -i \end{pmatrix}$$

Corresp to the notation around a circle

$$\tau(z) = \frac{1}{z} \rightsquigarrow \tau^* \nabla^t = \nabla^t . C, \quad C = \begin{pmatrix} & i \\ i & \end{pmatrix}$$

$$\xi(z) = \bar{z} \longrightarrow \overline{\xi^* \eta(\bar{\lambda})} = \eta(\lambda)$$

This implies $A_1 = \begin{pmatrix} i x_1 & x_2 + i x_3 \\ x_2 - i x_3 & -i x_1 \end{pmatrix}$

$$A_2 = \begin{pmatrix} -i x_1 & -x_2 + i x_3 \\ -x_2 - i x_3 & i x_1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} i x_1 & -(x_2 + i x_3) \\ -(x_2 - i x_3) & -i x_1 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} -i x_1 & -(-x_2 + i x_3) \\ -(-x_2 - i x_3) & i x_1 \end{pmatrix}$$

These $x_1 \sim x_3$, they need to be chosen in such a way that the mono of ∇^c becomes unitary.

If these holds for $t = \frac{1}{2g+2}$, then we can pull back ∇^t to $g+1$ - fold covering $\Sigma \to \mathbb{C}P^1$ and produce a family of flat conn asso to a harmonic - map!

Monodromy Problem.

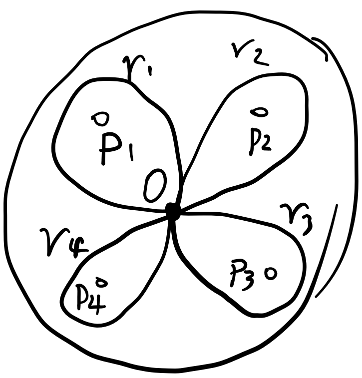
$X = \mathbb{C}P^1 \setminus \{p_1, p_2, p_3, p_4\}$ $\quad p_1 = e^{\frac{\pi i}{4}}$, $p_2 = e^{\frac{3}{4}\pi i}$

$p_3 = e^{-\frac{\pi}{4}i}$, $p_4 = e^{-\frac{3}{4}\pi i}$

Consider $d + A_1 \dfrac{dz}{z-p_1} + \cdots + A_4 \dfrac{dz}{z-p_4} = d + \eta$

$A_1 = \begin{pmatrix} iX_1 & X_2 + iX_3 \\ X_2 - iX_3 & -iX_1 \end{pmatrix}$, $A_2, \dots, A_4 \ (X_1, X_2, X_3)$

We try to find $X_1 \sim X_3$ to solve our monodromy problem.



For $M_1 \sim M_4$ being the local monodromy of $d\Phi = \Phi \cdot \eta$ with $\Phi(0) = id$, $\Phi(\hat{o}^k) = \underline{M_k}$

(obtained from γ_k)

Appendix: Advanced Gauge.

Intro: Given a holo vector bundle on Σ, compact R.S. of $g \geq 2$

Def: (Slope) For $E \to \Sigma$

$$\text{slope}(E) = \frac{\deg(E)}{Rk(E)} := \mu(E)$$

Def: We say E is stable (semi-) if $\forall F \subset E$ subbundle, $\mu(F) < \mu(E)$ ($\leq$).

Def: $\mathcal{N}_{R,D}$:= Moduli Space of rank R and degree d bundles isomorph classes of semi-stable bundle.

Remark: For $(D,R)=1$, $\mathcal{N}_{R,D}$ is a smooth space of dim $R^2(g-1)+1$

Consider its cotangent space.

Take b a repre of a class, $T_v^*\mathcal{N}$

$:= H^1(\Sigma, \mathrm{End}_0 V)$

By Serre, RHS $\cong$ $H^0(\Sigma, K\,\mathrm{End}_0 V)$.

怎么看？

Def: A Higgs Bundle (E, Φ) on Σ is a pair where E is a holomorphic v.b., $\Phi \in H^0(\Sigma, K\,\mathrm{End}_0 V)$

Example: $E = K^{1/2} \oplus K^{-1/2}$, fix a choice $K^{1/2}$.

$$\Phi : K^{1/2} \oplus K^{-1/2} \longrightarrow (K^{1/2} \oplus K^{-1/2}) \otimes K$$

$$\left(\begin{array}{c|c} 0 & \omega \\ \hline \underset{\mathrm{Id}}{1} & 0 \end{array}\right) \qquad \omega \in H^0(\Sigma, \underset{\text{Quadratic Diff}}{K^2}). \qquad (*)$$

Family of rank 2 Higgs-Bun

PAR by ω

To build moduli-space, need stability.

We say a subbundle $F \subset E$ is preserved by Φ (Φ-invariant) if $\Phi(F) \subset F \otimes K$.

针对Higgs-Field

Def: We say (E, Φ) stable (semi) if $\forall F \subset E$ which is Φ-invariant, We have $\mu(F) < \mu(E)$ ($\leq$)

In (*), $\omega \neq 0$, no subbundle preserved.

$\omega = 0$, $K^{-1/2}$ is preserved. $\mu(K^{-1/2}) = \underset{<0}{-g+1} < \mu(E) \to \Phi$ is stable.

Def:

Polystable: If we can write $(E, \Phi) = (F_1, \Phi_1) \oplus \cdots \oplus (F_k, \Phi_k)$, s.t. $\mu(F_i) = \mu(E)$, where each F_i is stable.

$\mathcal{M}_{B,D}$ = moduli-space of isomor-classes of semi-stable Higgs Bundle of fixed rank and degree.

Remark: If (E, Φ) is stable, then $(E, \lambda\Phi)$ for $\lambda \neq 0$, $\lambda \in \mathbb{C}$ is stable.

$(E, \alpha^*\Phi)$ for α automorphism of E is stable.

$G_{\mathbb{C}}$: complex Lie grp,

Def: A $G_{\mathbb{C}}$ – Higgs bundle (P, Φ) is formed of P a principal $G_{\mathbb{C}}$-bundle,

$\Phi \in H^0(\Sigma, K \otimes \mathrm{Ad}P)$

Rk: For $G_{\mathbb{C}} = A, B, C, D$; a $G_{\mathbb{C}}$ Higgs Bundle can be expressed as a pair (E, Φ) as before + condition reflecting nature of the grp.

Example: $G_{\mathbb{C}} = SL(n, \mathbb{C})$, then an $SL(n,\mathbb{C})$ Higgs is a pair (E, Φ) for E holo vector bundle of Rank m,

s.t. $\Lambda^m(E) \cong \mathcal{O}$

$\Phi : E \to E \otimes K$, s.t. $tr(\Phi) = 0$

$\mathcal{M}_{G_{\mathbb{C}}}$ = moduli of isomorphism classes of semi-stable $G_{\mathbb{C}}$-Higgs bundle.

It's hyperkahler.

(Dolbeault - Moduli - Space) reducti of repre

$\mathcal{M}_{G_{\mathbb{C}}}$ = $G_{\mathbb{C}}$-Higgs Bundle $\mathcal{M}_{\text{Betti-Moduli}}$ of: $\pi_1(\Sigma) \to G_{\mathbb{C}}$

$\wr$ NAHC $\wr$ Riem-Hil

$\mathcal{M}_{dR}$ = de Rham moduli of flat-conn

Core, Simpson, Hitchin, Dona

G: real Lie grp

Recall: a real form of $G_{\mathbb{C}}$ is a real G, s.t. $G^{\mathbb{C}} = G_{\mathbb{C}}$, or equiva. an anti-holo involution $\sigma: G_{\mathbb{C}} \to G_{\mathbb{C}}$, $G_{\mathbb{C}}^{\sigma} = G$.

Take G real Lie grp and H maximal compact.

$SL(2p, \mathbb{C}) = G_{\mathbb{C}}$, $G = SO(p,p)$

$H = SU(p) \times SU(p)$

with Cartan-Decompo of $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$

ortho comple of $\mathfrak{h}$

$$\therefore \mathfrak{h} = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$$

$$\mathfrak{m} = \left(\begin{array}{c|c} 0 & \xi \\ \hline \xi & 0 \end{array}\right)$$

The isotropy Repre:

$$i: H \longrightarrow GL(\mathfrak{m})$$

$$g \longmapsto (X \longmapsto Ad_g X)$$

$i_{\mathbb{C}}: H^{\mathbb{C}} \longrightarrow GL(m^{\mathbb{C}})$, consider vector bund

$E(m^{\mathbb{C}}) = E \times_i m^{\mathbb{C}}$

Def: A G-Higgs bundle is a pair

(E, Φ).

- For E a holo $H^{\mathbb{C}}$-bundle.
- $\Phi \subset H^0(\Sigma, E(m^{\mathbb{C}}) \otimes K)$

Hitchin - Fibration 1987

Σ : cpt R.S., $g \geq 2$

$G_{\mathbb{C}}$: cpx Lie Grp G is a real-form.

$$\begin{tikzcd} \mathcal{M}_{G_{\mathbb{C}}} : \text{(Higgs Bundle Moduli)} \arrow[r, squiggly] \arrow[d, "h"] & \text{Rep of } \pi_1(\Sigma) \arrow[d] \\ \Omega(\Sigma) : \text{Differential Form} & G_{\mathbb{C}} \end{tikzcd}$$

Consider the ring of invariant polynomials of $\mathfrak{g}_{\mathbb{C}}$ and take $P_1, \dots, P_k$ homogeneous basis of invariant poly of degree $d_1 \sim d_k$.

Def: (Hitchin Fibration)

$$h : \mathcal{M}_{G_{\mathbb{C}}} \longrightarrow \bigoplus H(\Sigma, K^{d_i})$$

$$(E, \Phi) \longmapsto (P_1(\Phi), \dots, P_k(\Phi))$$

Ex: $TR(\Phi^i)$ (not for $SO(2m, \mathbb{C})$)

Hitchin Map: (Independent of basis)

- is a proper map
- generic fibers are compact abelian

varieties.

- Makes $\mathcal{M}_{G_{\mathbb{C}}}$ into an integrable system.

Abelianzation of Higgs Bundle.

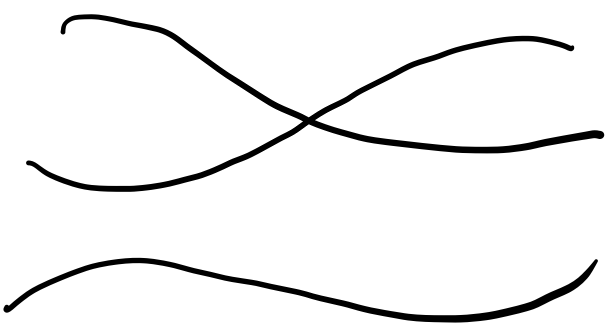
Consider the basis of invariant poly appering as coeffient of a charac polynomial.

$$\det(\Phi - \lambda \, Id) = \lambda^{k} + a_{k-1}\lambda^{k-1} + \cdots + a_1$$

$$a_i \in H^0(\Sigma, K^{d_i})$$

Idea: The point in the base $\mathcal{A}_{G_{\mathbb{C}}}$ can be thonght as eigenvalues of Φ.

curves of eigenvalue



"Spectral – curves" Σ

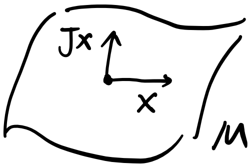
Fiber: Eigen – Spaces

Hyperkahler Manifold

(Constructions, Examples)

(M, g) Riemann-Manifold

Add complex-structure: If M cpx mfd, we obtain $J: TM \to TM$ (a linear bundle map), s.t. $J^2 = -Id$



(Almost cpx-structure!)

(M, g, J) 称 kahler: (1) (M, g) Riemann (2) J almost cpx. $g(J\cdot, J\cdot) = g$ (compatible)

(3) $\omega(X, Y) = g(JX, Y)$ is alternating 2-form which is closed.

Example: $\mathbb{C}^n$. $g = dz_i \otimes d\bar{z}_i$, J: multiply by i.

$(z_1, \dots, z_n)$

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j$$

Q: How could you find more examples about kahler mfd?

$\to$ subjects!

Question: How do get quotient on Kahler?

Suppose (M, g, J, ω) kahler, $G \curvearrowright M$ action: G is a cpx Lie-grp.

then: $M/G_{\mathbb{C}}$ as a kahler-mfd?

(应该 preserves metric: take $G < G_{\mathbb{C}}$ preserves metric!)

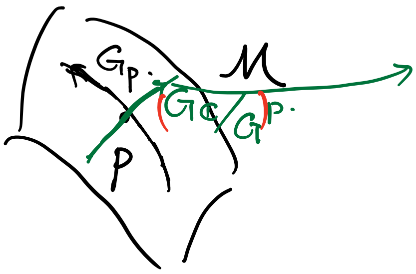
Ex: $M = \mathbb{C}^n$

$\mathbb{C}^* \curvearrowright M \longrightarrow M$, then $U(1) < \mathbb{C}^*$ preserves metric.

$(\lambda, (z_1, \dots, z_n)) \longmapsto (\lambda z_1, \dots, \lambda z_n)$

then:

$M/U(1)$, $[g] \longrightarrow$ 不一定 complex (40)



1. take submfd of M that doesn't have $G_{\mathbb{C}}/G$ direction in tangent-space (必须 Quotient 整个 G 才 complex)

then Quotient by G.

Def: An action of a Lie grp G on a kahler M is called Hamiltonion if: $\exists\, \mu: M \to \mathfrak{g}^*$, $\forall X \in \mathfrak{g}$, the following holds: (moment-map)

$$\underline{i_X \omega} = -d\mu(X) \qquad 40{:}00$$

Idea: $\mu^{-1}(\xi) \subset M$ for $\forall\, \xi \in (\mathfrak{g}^*)^G$ is a candidate. submfd "gets rid of the extra directions".

Ex: $S^1 \simeq U(1) \curvearrowright \mathbb{C}^n$ given by $e^{i\theta}(z_1, \dots, z_n)$

$$\mu: \begin{aligned} \mathbb{C}^n &\longrightarrow \mathbb{R} \\ (z_1, \dots, z_n) &\longmapsto \tfrac{1}{2}|z|^2 = \tfrac{1}{2}\sum |z_j|^2 \end{aligned}$$ is a moment-map

Thm: For a Hamiltonion-action of $G \curvearrowright (M, g, J, \omega)$, for $\forall\, \xi \in (\mathfrak{g}^*)^G$, the quotient $\mu^{-1}(\xi)/G$ is a kahler-mfd. if ξ is a regular-value and.

Check: $G \curvearrowright M$, $\mu: M \longrightarrow \mathfrak{g}^*$, s.t.

$$i_X \omega = d(\mu(X)).$$

Example: $U(1) \curvearrowright \mathbb{C}^n \longrightarrow \mathbb{C}^n.$

$$(e^{i\theta}, z_1, \dots z_n) \longmapsto (e^{i\theta} z_1, \dots, e^{i\theta} z_n)$$

Thus: $\forall X \in \mathfrak{g} = \mathbb{R}$, the vector-field

$\therefore \omega = \frac{i}{2} \sum dz_k \wedge d\bar{z}_k$, thus:

$\gamma(t) = (e^{itX} z_1, \dots, e^{itX} z_n) \longrightarrow$ For $e^{itX} z_1 = e^{itX}(x_1 + i y_1)$

$i_X \omega = \omega(X, \cdot)$, then $\frac{d}{dt}\Big|_{t=0} e^{itX} z_1 = -y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1}.$

$$\therefore i_X \omega = d(\mu(X))$$

G acts freely on $\mu^{-1}(\xi)$.

Ex: $\mu^{-1}(\xi)/U(1) = S^{2n+1}/\text{radius} \cong \mathbb{C}P^{n-1}$.

Def: A hyper-kahler is (M, g, I, J, K):

(i) (M, g) Riemann.

(ii) I, J, K: Almost cpx-structure, s.t.

$I^2 = J^2 = K^2 = -Id$. $\quad IJ = -JI, \quad IJ = K$

(iii) (M, g, I) kahler.
$\qquad\quad J$
$\qquad\quad K$

$(\omega_I, \omega_J, \omega_K)$

For: For any hyperkahler M, the Ricci-Curvature $R_{\mu\nu} \equiv 0$

(2-order 条件).

(Ricci-Flat)

(interesting geometric P.D.E) "generalitivity".

Fact: no explicity solution besides flat tori.

Thm: (Calabi-Yau)

Ricci-flat metric on compact kahler exists.

if $C_1(X) = 0$

Example: $\mathbb{H} = \mathbb{R}\langle 1, i, j, k\rangle$

$\mathbb{H}^n$ — flat hyper-kahler

⚠ Try to find more examples by Quotient.

Thm: If G is compact Lie Grp:

$G \curvearrowright (M, g, I, J, K)$ hyperkahler in

tri Hamiltonions actions,

i.e. $\exists \mu_I, \mu_J, \mu_K : M \to \mathfrak{g}^*$ $\quad d\mu_I(X) = -i_X \omega$

$\vdots$

then: $\forall\, \xi_I, \xi_J, \xi_K \in (\mathfrak{g}^*)^G$ regular, then

$\mu_I^{-1}(\xi_I) \cap \cdots \cap \mu_K^{-1}(\xi_K) \,/\, G$ is hyper-kahler-mfd.

The proof, using Theorem 2, is direct. Each form ω_i descends to a form $\bar{\omega}_i$ just as in the symplectic case. What remains to be checked is that the quaternionic algebraic relations between $\bar{\omega}_1$, $\bar{\omega}_2$ and $\bar{\omega}_3$ are still satisfied.

THEOREM 2. *Let M^{4n} be a manifold with 2-forms ω_1, ω_2, ω_3 whose stabilizer in $GL(4n, \mathbf{R})$ at each point $m \in M$ is conjugate to $Sp(n)$. Then the forms define a hyperkähler structure if and only if they are closed.*

This theorem, which is a straightforward consequence of the Newlander-Nirenberg theorem, places the theory of hyperkähler manifolds firmly within the context of symplectic geometry.

Check: $\mathbb{H}$ is a hyperkahler - manifold.

(i) $\mathbb{H} \xrightarrow{\cong} \mathbb{R}^4$, as

$$\rho(x,y) = xy^*, \quad \text{For } x = a + b\vec{i} + c\vec{j} + d\vec{k}$$

$$x^* = a - b\vec{i} - c\vec{j} - d\vec{k}, \quad xx^* = a^2 + b^2 + c^2 + d^2$$

(ii) $\mathbb{H} \cong \mathbb{C} \oplus \mathbb{C}$

$$a + bi + cj + d\vec{k} = a + bi + cj + dij = (a+bi) + (c+di)j$$

$$\therefore (z_1, z_2) \longleftrightarrow z_1 + z_2 j \in \mathbb{H}$$

Almost Complex Structure.

This is just given by

$$J_1 = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}, \quad J_2 = \begin{pmatrix} & & -1 & \\ & & & -1 \\ 1 & & & \\ & 1 & & \end{pmatrix}$$

#

Appendix 3: Uhlenbeck Compactness.

Appendix 4: Deformation Theory.

Def: (Smooth Family)

A smooth family of complex-manifolds is a proper holomorphic map $f: M \to B$, such that:

(1) M, B non-empty complex manifold with B connect.

(2) $f_*: T_x M \to T_{f(x)} B$ is surjective.

RK: f is proper + holomorphic means: f is open + closed + surjective.

Def: (Equivalence)

Two family with same base are called equivalent if $\exists$ biholomorphic φ. s.t. the diagram commutes.

$$\begin{tikzcd} M \arrow[rr, leftrightarrow, "\varphi"] \arrow[dr, "f_M"'] & & N \arrow[dl, "f_N"] \\ & B & \end{tikzcd}$$

RK: For $\begin{array}{c} M \\ \downarrow f \\ B \end{array}$ be smooth family, if $V \subset B$, then $\begin{array}{c} f^{-1}(V) \\ \downarrow f \\ V \end{array}$ is also a smooth family.

Def: (Trivial-Family)

$\begin{array}{c} M \\ \downarrow f \\ B \end{array}$ be a smooth family which is called trivial if

$$\begin{tikzcd} M \arrow[rr, "\cong"] \arrow[dr, "f"'] & & M_b \times B \arrow[dl, "\pi_2"] \\ & B & \end{tikzcd}$$

(Locally-Trivial)

$\exists$ open covering $B = \bigcup_{\alpha} U_\alpha$, such that:

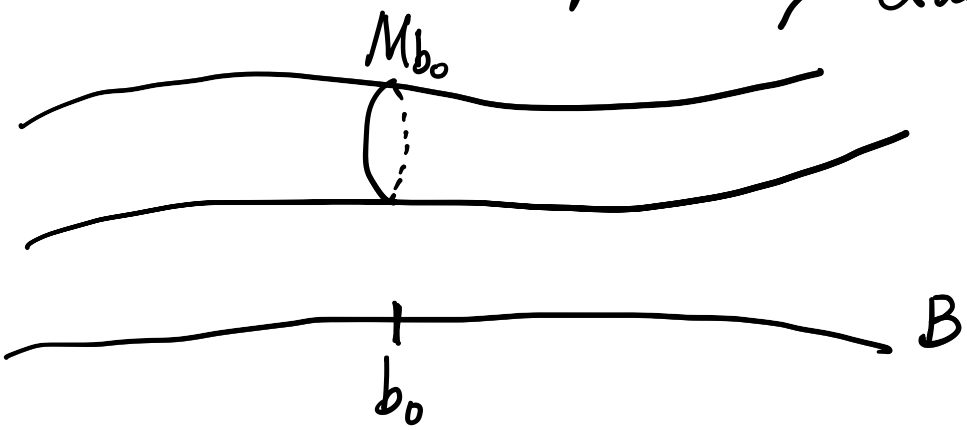
$\begin{array}{c} f^{-1}(U_\alpha) \\ \downarrow f \\ U_\alpha \end{array}$ is trivial.

Deformation Functor

Def: $(\text{Germ})^{sm}$ - category

Obj: $\{$ pointed complex-manifold $(M, x)\}$

Mor: $Mor((X,x),(Y,y)) := \{ f: X \to Y \mid f(x) = y$ which is holomorphic $\} /$ Quotient.



Deformation: Let $(B, b_0) \in Ob(Gem)^{sm}$, a (Smooth-Family $\begin{matrix} M \\ \downarrow f \\ B \end{matrix}$ 天然看成 $M_b \to (B, b)$)

deformation of compact complex manifold M_0 over (B, b_0) is a diagram: (i deformation)

$$\begin{tikzcd} M_0 \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{M} \arrow[d, "f"] \\ \{b_0\} \arrow[r, "\text{inclusion}"] & B \end{tikzcd}$$

, s.t. $i: M_0 \cong f^{-1}(b_0)$ and

$\exists$ neighbor $\mathcal{U}$ of b_0, s.t. $f: f^{-1}(\mathcal{U}) \to \mathcal{U}$ is a proper smooth family.

$\mathcal{M}$: 称为 total space of deformation.

(B, b_0): base germ space.

称 two-deformation: $M_0 \xrightarrow{i} \mathcal{M} \xrightarrow{f_M} B$, $M_0 \xrightarrow{i} N \xrightarrow{f_N} B$ are equivalent iff

$\exists$ biholomorphic $\psi: \mathcal{M} \cong N$, s.t. the diagram commutes:

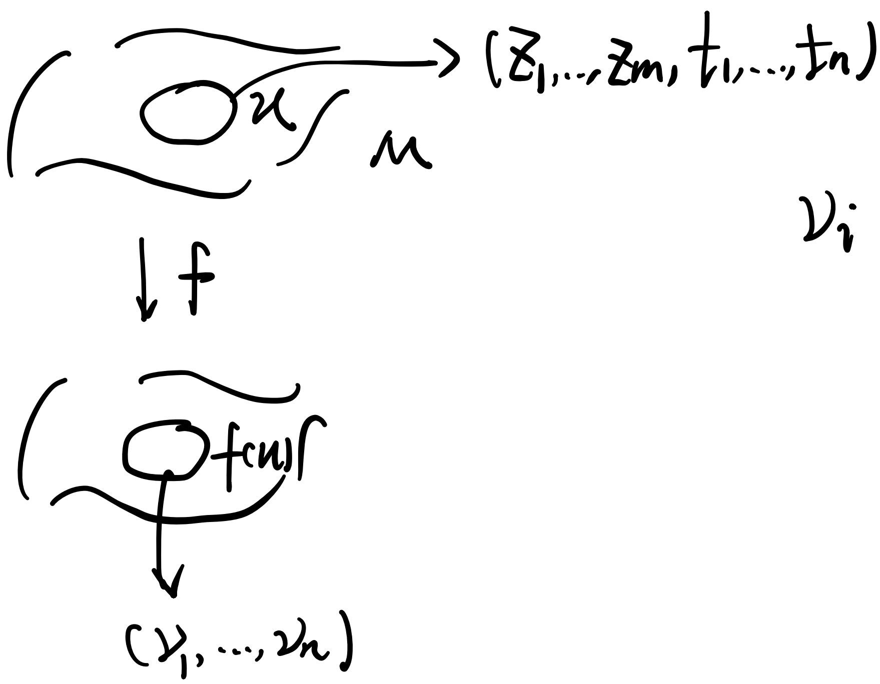
$$\begin{tikzcd} & & N \arrow[ddl, bend left, "f_N"] \\ M_0 \arrow[r, "i_M"] \arrow[urr, bend left, "i_N"] \arrow[d] & \mathcal{M} \arrow[ur, dashed, "\exists \cdot \psi"] \arrow[d, "f_M"] & \\ \{b\} \arrow[r] & B & \end{tikzcd}$$

Kodaira–Spencer Map.

$\begin{array}{c} \mathcal{M} \\ \downarrow f \\ B \end{array}$ is a smooth family of compact complex manifolds, $\dim B = n$, $\dim \mathcal{M} = m+n$; for $b \in B$,

Let $\mathcal{M}_b = f^{-1}(b)$

Def: (Admissible Coordinate).



$$\nu_i \circ f = t_i.$$

For smooth family satisfies f_* surjective. we know $\mathcal{M}$ must be covered admissible coordinates $\mathcal{U} = \{U_\alpha\}$.

Lemma: The differential-map on $\Gamma(T\mathcal{M})$

$$f_* : \Gamma(T\mathcal{M}) \longrightarrow \Gamma(TB) \text{ is surjective.}$$

pf: As $v_i \circ f = t_i$ $\therefore f_*(\frac{\partial}{\partial t_i}) = \frac{\partial}{\partial v_i}$. Just use partition of unity: $\sum \rho_\alpha \eta_\alpha$, we can obtain everything.

Notation: $T_f \subset TM$ be the vertical vector subbundle.

Def: (Kodaira–Spencer Map):

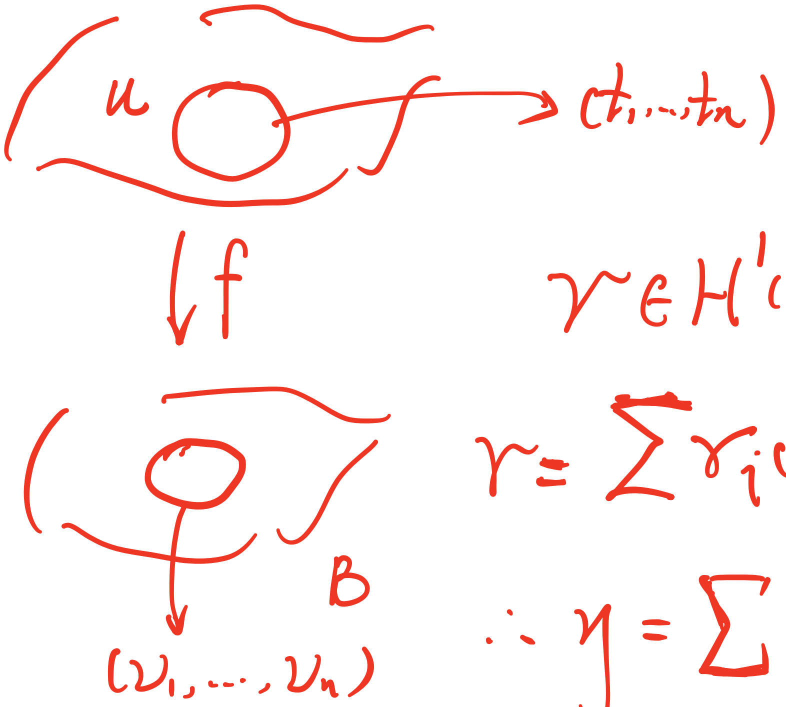
For a smooth family $\begin{matrix} M \\ \downarrow f \\ B \end{matrix}$, the map is defined by: $V \subset M$

$$KS(V)_f : H^0(V, TB) \longrightarrow H^1(f^{-1}(V), T_f)$$

For $\gamma \in H^0(V, TM)$, let $f_*\eta = \gamma$, then

$$KS(V)_f(\gamma) = [\bar\partial \eta] \in H^1(f^{-1}(V), T_f)$$

Computation: The Map is well-def.



$v_i \circ f = t_i$

$\gamma \in H^1(f^{-1}(V), T_f)$, Let

$\gamma = \sum \gamma_i(v) \frac{\partial}{\partial v_i}$, $\gamma_i(v)$ holo

$\therefore \eta = \sum \eta_i(z,t) \frac{\partial}{\partial z_i} + \gamma_i(t) \frac{\partial}{\partial t_i}$

$\therefore f_* \eta = \gamma$, if $f_* \eta = f_* \hat{\eta}$, $\eta - \hat{\eta} \in \Gamma(f^{-1}(V), T_f)$,

$\therefore [\bar{\partial} \eta] = [\bar{\partial} \hat{\eta}]$ in $H^1(f^{-1}(V), T_f)$.

#

We can think $KS(V)_f$ in terms of Čech-Cohomology: $\check{H}^1(f^{-1}(V), T_f)$

$[\bar{\partial}\eta]$怎么塞进去 $\nearrow$

$(z_1^\alpha, \dots, z_m^\alpha, t_1^\alpha, \dots, t_n^\alpha)$

on $U_\alpha \cap U_\beta$:

$(v^1, \dots, v^n)$

$$\begin{cases} z_i^\beta = g_{\beta\alpha}^i (z_i^\alpha, \dots, t_n^\alpha) \\ t_i^\beta = t_i^\alpha \end{cases}$$

Let $\frac{\partial}{\partial t_h} \in \Gamma(f^{-1}(V), TM)$, then: if

$f_* \eta = \frac{\partial}{\partial t_h} \to \eta = \sum \eta_i^\alpha (z^\alpha, t^\alpha) \frac{\partial}{\partial z^\alpha} + \frac{\partial}{\partial t_h^\alpha}$

$\therefore \eta - \frac{\partial}{\partial t_h^\alpha} \in \Gamma(f^{-1}(V), T_f)$

$\therefore KS(V)_f \left(\frac{\partial}{\partial t_h}\right)_{b,a} \underset{\text{Čech}}{\overset{\text{in}}{=}} \left(\eta - \frac{\partial}{\partial t_h^b}\right) - \left(\eta - \frac{\partial}{\partial t_h^a}\right)$

$$= \frac{\partial}{\partial t_h^\alpha} - \frac{\partial}{\partial t_h^\beta}$$

$$\frac{\partial}{\partial t_h^\alpha} = \sum \frac{\partial z_i^\beta}{\partial t_h^\alpha} \frac{\partial}{\partial z_i^\beta} + \underline{\sum \frac{\partial t_i^\beta}{\partial t_h^\alpha} \frac{\partial}{\partial t_i^\beta}} \qquad \frac{\partial}{\partial t_i^\beta}.$$

$$= \sum \frac{\partial g_{\beta\alpha}^i}{\partial t_h^\alpha} \frac{\partial}{\partial z_i^\beta} \qquad \therefore RHS = \sum \frac{\partial g_{\beta\alpha}^i}{\partial t_h^\alpha} \frac{\partial}{\partial z_i^\beta}$$

Infinitesimal Deformation

We consider a special differentible family of compact complex manifolds:

$\mathcal{M}$ be a differentiable manifolds. B a domain of $\mathbb{R}^m$, $\bar{\omega}: \mathcal{M} \to B$ is a C^∞-map.

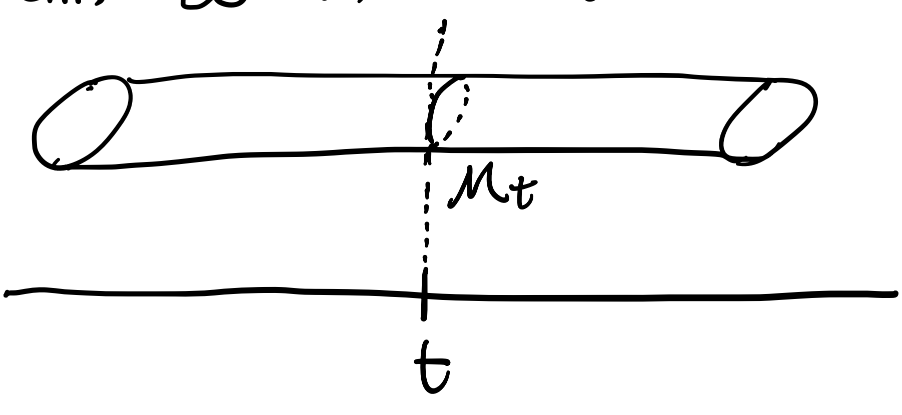
Def: Suppose given a compact complex manifold $M_t = M_t^n$ for $\forall t \in B \subset \mathbb{R}^m$, then: $\{M_t \mid t \in B\}$ called differentiable family of compact complex mfds if $\exists$ a smooth manifold $\mathcal{M}$ and $\bar{\omega}: \mathcal{M} \to B$, s.t.

(i) $\bar{\omega}$ is proper,

(ii) $(\bar{\omega})_*$ is surjective;

(iii) $\bar{\omega}^{-1}(t) = M_t$

#



Rk:

$\exists \{\mathcal{U}_j\}$ open covering of M and $z_j^1(p), \dots,$ $z_j^n(p)$, $j = 1, 2, \dots$, defined on $\mathcal{U}_j$, s.t.

$(\mathcal{U}_j, z_j^1, \dots, z_j^n)$ $\quad \{\mathcal{U}_j \cap M_+ \mid (z_j^1, \dots, z_j^n)\}$ forms a complex coordinate on M_+.

$x_j^{2\alpha-1} + i\, x_j^{2\alpha}$

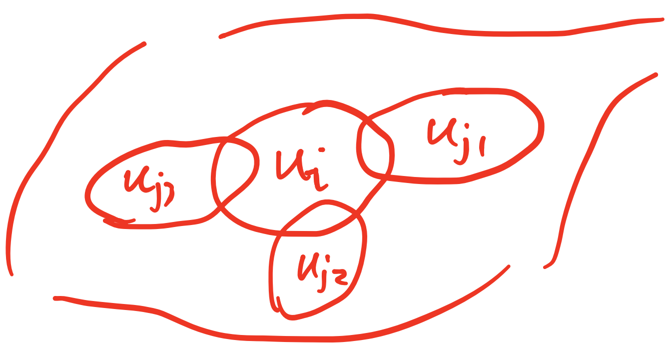
$\omega(p) = (t_1, \dots, t_m)$.

$\therefore$ put: $\quad x_j(p) = (x_j^1(p), \dots, x_j^{2n}(p), t_1, \dots, t_m)$

This forms a system of local coordinate on

$\mathcal{M}$: $\quad x_j : \mathcal{U}_j \longrightarrow \mathcal{U}_j \times B$

怎么理解一个 Complex-Manifolds:

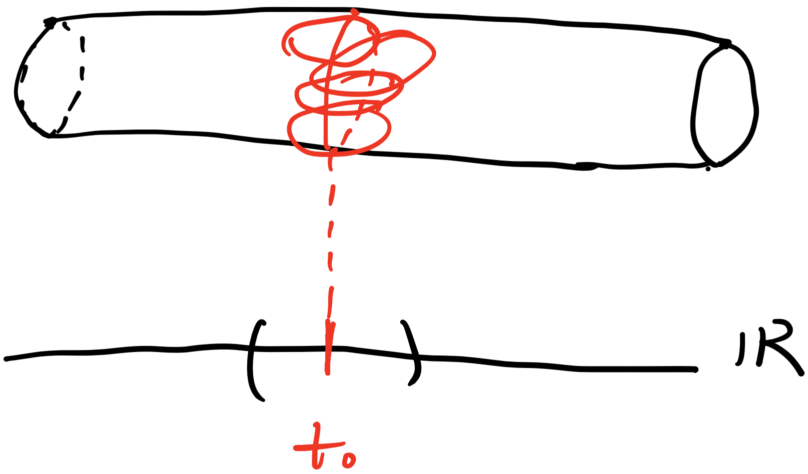


看成 coordinate $(\mathcal{U}_i, z_i)$ with transition functions: $z_k \circ z_i^{-1}$, 记作:

$$z_j = f_{jk}(z_k)$$

Now: Considering a smooth family:

$(\mathcal{M}, \mathbb{R}, \bar{\omega})$



We want to know if the complex structure on $M_t = \bar{\omega}^{-1}(t)$ varifies w.r.t $t_0 \in \mathbb{R}$.

Pick the admissible coordinate on $\mathcal{M}$, denote it to be $\{\mathcal{U}_j\}$, then: $\{\mathcal{U}_j \cap M_t\}$ is the complex-coordinates on M_t.

Idea: The complex-data of M_t is contained in a Čech-Coho group:

$$\check{H}^1(M_t, \Theta_t), \quad \Theta_t: \text{Holomorphic-Vector-field.}$$

Because of admissible-coordinate,

$$f_{ik}^{\alpha}(z_k,t) = f_{ij}^{\alpha}(f_{jk}^{1}(z_k,t),\ldots,f_{jk}^{n}(z_k,t),t)$$

$$\rightsquigarrow \frac{\partial f_{ik}^{\alpha}}{\partial t} = \frac{\partial f_{ij}^{\alpha}}{\partial t} + \sum_{\beta=1}^{n} \frac{\partial f_{ij}^{\alpha}}{\partial z_j^{\beta}} \cdot \frac{\partial f_{jk}^{\beta}}{\partial t}$$

$$\frac{\partial}{\partial z_j^{\beta}} = \frac{\partial z_i^{\alpha}}{\partial z_j^{\beta}} \frac{\partial}{\partial z_i^{\alpha}},$$

$$\therefore \sum_{\alpha} \frac{\partial f_{ik}^{\alpha}}{\partial t} \frac{\partial}{\partial z_i^{\alpha}} = \sum_{\alpha} \frac{\partial f_{ij}^{\alpha}}{\partial t} \frac{\partial}{\partial z_i^{\alpha}}$$

$$+ \sum_{\beta} \frac{\partial f_{jk}^{\beta}}{\partial t} \frac{\partial}{\partial z_j^{\beta}}$$

Let $\theta_{ij} = \sum_{\alpha} \frac{\partial f_{ij}^{\alpha}(z^j,t)}{\partial t} \frac{\partial}{\partial z_i^{\alpha}}$

$\therefore \theta_{ij} : \mathcal{U}_i \cap \mathcal{U}_j \longrightarrow H^0(M_t, TM_t)$

$$\begin{array}{c} H^0(M_t, TM_t) \\ \parallel \\ H^0(M_t, \Theta_t) \end{array}$$

$\therefore \theta_{ik} = \theta_{ij} + \theta_{jk}$ $\therefore \{\theta_{ij}\}$ is a cocycle of $\check{H}^1(M_t, \Theta_t)$

$\therefore$ For Smooth family $(\mathcal{M}, \mathbb{R}, \overline{\omega})$, we have a:

$$\theta(t) \in H^1(M_t, \Theta_t).$$

Def: $\theta(t)$ is called the infinitesimal deformation of $M(t)$ w.r.t. $(\mathcal{M}, \mathbb{R}, \overline{\omega})$.

denote: $\dfrac{d M_t}{dt} = \theta(t)$.

Rk：不取决于 admissible-coordinate 之选取.

Thm: If each $\mathcal{M}_t$ is biholomorphic to a fixed M, then: $(\mathcal{M}, \mathbb{R}, \bar{\omega})$ is locally-trivial

solution: Locally trivial means transition-function 与 t 无关 $\to \theta(t)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0$

Thm: If $\dim H^1(\mathcal{M}_t, \Theta_t)$ 与 t 无关,

$\theta(t)=0 \Longrightarrow (\mathcal{M}, \mathbb{R}, \bar{\omega})$ locally trivial.

$H^0(\mathbb{R}, T\mathbb{R}) \longrightarrow H^1(\mathcal{M}, T_{\bar{\omega}}) \ni \theta(t)$

Def: $\rho_t : \frac{\partial}{\partial t} \longmapsto \frac{\partial \mathcal{M}_t}{\partial t}$

貌似就是 Kodaira-Spencer Map.

$\therefore \rho_t \equiv 0 +$ dim $H^1(M_t, \Theta_t)$ 与 t 无关

$\Rightarrow (\mathcal{M}, \mathbb{R}, \varpi)$ locally-tri.

Denote $\theta = \theta(0) \in H^1(M, \Theta)$.

then for $(\mathcal{M}, \mathbb{R}, \varpi) \rightsquigarrow \theta$

反之: give $\theta \in H^1(M, \Theta)$, 是否

$\exists (\mathcal{M}, \mathbb{R}, \varpi)$, s.t.

$$\mathcal{M}_0 \cong M \, ?$$

(Theorem of Existence)

Let M be a compact complex mfd and $H^2(M, \Theta) = 0$, then there exists $(\mathcal{M}, B, \varpi)$ with

$0 \in B \subset \mathbb{C}^m$, s.t.

(i) $\varpi^{-1}(0) = M$;

(ii) $\rho_0 : \frac{\partial}{\partial t}\Big|_{t=0} \longrightarrow \frac{\partial M_t}{\partial t}\Big|_{t=0}$ is an isomorphism of $T_0 B$ to $H^1(M, \Theta)$