Youtube Talk.

Twistor Space and Parabolic Structure

X: compact curve, $D = \{x\}$, $x \in X$

quasi-para: $0 = F_0 \subset F_1 \subset \cdots \subset F_{k-1} = F_k = E_x$

$-1 < a_1 < \cdots\cdots < a_k \leq 0$ "normalization"

"associated graded piece $\overset{a_i}{F_i \supset F_{i-1}}$" of the filtration.

(E, h), s.t. $\forall e \in \Gamma(E)$, s.t. $e_x \neq 0$, then:

$$|z| < |e|_h < |z|^{-\varepsilon}$$

$\downarrow$

adapt to parabolic if:

$$e(x) \in \underline{F_i - F_{i-1}}, \quad |z|^{-a_i+\varepsilon} < |e|_h < |z|^{-a_i-\varepsilon}$$

(values of sections growth approximation $|z|^{-a_i}$)

For harmonic metrics on tame Higgs bundle, $\exists !$ parabolic structure reflects the growth rate of the metric in this way.

Def: Tame Higgs Bundle is (E, φ)

$\varphi: E \to E \otimes K_X(D)$ residue fits parabolic structure

$\rightsquigarrow \; gr_x(E) \overset{\circ}{=} \bigoplus_{i=1}^{k} F_i / F_{i-1}$ $\therefore$ $Res_x(\varphi) \circlearrowright gr_x(E)$

strictly parabolic means: $Res_x(\varphi) = 0$

$\bigstar$ graded residue reflects monodromy of the associted flat connection. $\boxed{\text{constant unitary connection}}$

Example: (L, h)
$\downarrow$
Σ

$$d_A = d + \frac{a}{2}\left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z}\right), \quad a \in \mathbb{R}.$$

Consider

$$\phi = p\frac{dz}{z}, \quad p \in \mathbb{C} \quad \rightsquigarrow \text{ solution of Hitchin}$$

$$F^A + [\phi, \phi^*] = 0$$

$\downarrow$

Flat connection $D = d_A + \phi + \phi^* = \frac{a}{2}\left(\frac{d\bar z}{\bar z} - \frac{dz}{z}\right) + p\frac{dz}{z} + \bar p \frac{d\bar z}{\bar z}$.

$\rightsquigarrow \bar\partial_q = \bar\partial^0 + q\frac{d\bar z}{\bar z}$, then $f = (z\bar z)^{-q}$ satisfies

$$\bar\partial_q(f) = 0$$

"growth rate": $|f| = |z|^{-2\operatorname{Re} q}$.

If use $\bar\partial_q = (d_A)^{0,1}$, $q = a/2$, $f = (z\bar z)^{-a/2}$

$\rightsquigarrow |f| = |z|^{-a} \rightsquigarrow$ weight of parabolic bundle a.

$\rightsquigarrow$ Monodromy:

$$\exp\left(\oint (p - a/2)\frac{dz}{z} + (\bar p + a/2)\frac{d\bar z}{\bar z}\right) = \exp(2\pi i(p - a - \bar p))$$

$$= \exp(\operatorname{Im}(p)\cdot 2\pi - \underbrace{2\pi i a}_{\text{real-part}})$$

$\rightsquigarrow$ Monodromy has a rotation part by a.

scalar real component by $\operatorname{im}(P)$

λ- connection case

$D' = \partial_A + \phi^*$, $D'' = \bar\partial_A + \phi$, $D^\lambda = \lambda D' + D''$

$$= \lambda\partial + \bar\partial + \left((p - \lambda a/2)\frac{dz}{z} + (\lambda\bar p + a/2)\frac{d\bar z}{\bar z}\right)$$

$\rightsquigarrow (D^\lambda)^{0,1}$ has holomor frame $f = (z\bar z)^{-\varrho}$

$$\varrho = \left(\frac{a}{2} + \lambda\bar p\right)$$

having growth rate $|z|^{-b_\lambda}$ with weight

$$b_\lambda = a + 2\,\mathrm{Re}(\lambda\bar p)$$

$\to D^\lambda(f)^{1,0} = (-\lambda\varrho + p - \lambda a/2) f \frac{dz}{z}$

$\rightsquigarrow$ the resulting holo-bundle has a logarith-λ conn

with residue $B_\lambda = p - \lambda a - \lambda^2\bar p$

$\lambda = 0 \rightsquigarrow$ Higgs bundle case $b_0 = a$, $B_0 = p$

$\lambda = 1 \rightsquigarrow$ parabolic weight. $b_1 = a + 2\,\mathrm{Re}(p)$

residue $B_1 = 2i\,\mathrm{Im}(p) - a$

$\therefore \forall \lambda \in \mathbb{A}^1$, a loga λ-conn has form

$$D_h^\lambda = d + B^\lambda \frac{dz}{z} + \cdots, \quad |h| \sim |z|^{-2b_\lambda}$$

$b_\lambda = a + 2\,\mathrm{Re}(\lambda\bar p)$, $B_\lambda = p - \lambda a - \lambda^2\bar p$

$\therefore$ In rank 1 case

local struct governed by 3 real parameters.

$\rightsquigarrow$ 1 complex, 1 real

An explanation:

Twistor Space. $\longrightarrow$ Hodge Weight 2 but not 1.

"Preferred Sections, Twistor Space and so on ..."

e.g. $V = \mathbb{H}$
$\mathcal{V} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$

$\therefore \forall\, \mathbb{H}$ - module is just a direct sum of this $\rightsquigarrow$
$\mathcal{V}$ is going to be semi-stable of slope 1.

Also: $V = \Gamma(\mathbb{P}^1, \mathcal{V})^{6}$

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idea: $\forall\, x, y \in \mathbb{P}^1$, $\Gamma(\mathbb{P}^1, \mathcal{V}) \cong \mathcal{V}_x \oplus \mathcal{V}_y$ by evaluation map.

Choose antipodal point K, $6K$, $\exists!$ 6-inv section with a given value at a single point K

$\therefore \Gamma(\mathbb{P}, \mathcal{V})^{6} \cong \mathcal{V}_K$

$\mathcal{M}_{SD} \times \mathbb{CP}^1$

$\begin{cases} \lambda = 0 \rightsquigarrow \mathcal{M}_{Dol} \\ \lambda = 1 \rightsquigarrow \mathcal{M}_{dR}. \end{cases}$, $\lambda \neq 0, \infty \rightsquigarrow \lambda$ - connections.

$\mathcal{M}_{DH} \cong \mathcal{M}_{Hod}(X) \cup_{\varphi} \mathcal{M}_{Hod}(\overline{X})$ via $\lambda \to \bar{\lambda}^{-1}$

Thm:

$$\begin{tikzcd} \mathcal{M}_{DH} \arrow[dr] & \cong & \mathcal{M}_{SD} \times \mathbb{C}P^1 \arrow[dl] \\ & \mathbb{C}P^1 & \end{tikzcd}$$

$$s(\lambda): \lambda \longmapsto \left((E, \bar{\partial} + \lambda \varphi^*), (D^{\lambda} = \lambda\partial + \varphi) \right)$$

Thm: $\rho: \mathbb{P}^1 \to \mathcal{M}_{DH}$ twistor line

$\rightsquigarrow \mathcal{V} = \rho^* T\mathcal{M}_{DH}/_{\mathbb{P}^1}$ semi-stable + slope 1

over $\mathbb{P}^1$

"a property of weight 1"

$\longrightarrow$

"Weight 2" $\rightsquigarrow$ Puncture. property

$D(\lambda, a) = \lambda d + a \frac{dz}{z}$, $a \in \mathbb{C}$

(λ, a) equivalent to $(\lambda, a + k\lambda)$, $k \in \mathbb{Z}$.

"Change of trivilization"

RH - Correspod over $\lambda \neq 0$:

$$\mathbb{C}^* \times \mathbb{C} / \varphi \xrightarrow{\cong} \mathbb{C}^* \times \mathbb{C}^*$$

$$(\lambda, a) \longmapsto (\lambda, \exp(2\pi i a/\lambda))$$

gluing : $\mu = -\lambda^{-1}$

$$a/\lambda = -b/\mu \iff a = \lambda^2 b$$

$\rightsquigarrow \mathcal{M}_{DH} = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(2))/\mathcal{G}$

Also : σ antipodal involution

Lemma: $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))^{\sigma} \cong \mathbb{R}^3$, $\forall x \in \mathbb{P}^1$

"general weight 2" structure $\rightsquigarrow$ tangent bundle of $\mathbb{P}^1$ itself, as a way to determine the splitting.

困 : singularity of gauge group action.

need to define stability recover parabolic!

Talk 2:

"Twistor family of moduli space of local system"

weight 2 for local mono around punctures

$\rightsquigarrow$ relates to parabolic structure.

Let $\mathbb{H} = \mathbb{R}\langle 1, I, J, K\rangle$, $\mathcal{K} = xI + yJ + zK$, $(x,y,z) \in S^2$

$\rightsquigarrow K^2 = -1 \iff (x,y,z) \in S^2$ $\quad I \leftrightarrow 0 \in \mathbb{P}^1$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad J \leftrightarrow 1 \in \mathbb{P}^1$

V is $\mathbb{H}$-module, $\forall\, \mathcal{K} \in \mathbb{P}^1$. $\rightsquigarrow$ complex stru on V

$\rightsquigarrow \mathbb{C}$-vector space $V_{\mathcal{K}}$

$$\rightsquigarrow \mathcal{V} := V \times \mathbb{P}^1 \qquad\qquad \sigma: \lambda \longmapsto -\bar{\lambda}^{-1}$$

$$\downarrow \qquad\qquad\qquad (x,y,z) \longmapsto (-x,-y,-z)$$

$$\mathbb{P}^1$$

For $V = \mathbb{H} \rightsquigarrow \mathcal{V} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$

$\rightsquigarrow \mathcal{V}$ semi-stable of slope 1.

Thm.

If $(\mathcal{V}, \sigma)$ with weight 1

$$\rightsquigarrow \mathcal{V} = \Gamma(\mathbb{P}^1, \mathcal{V})^{\sigma}$$

$\mathrm{Tw}(M) := M \times \mathbb{P}^1$ "Penrose - Theory"

$\rightsquigarrow$ Delign-Hitchin Moduli: $\mathcal{M}_{DH}$

Alge-geome construction: $\mathcal{M}_{Hod} \cong \mathcal{M}_B \times \mathbb{G}_m := \mathbb{A}^1 - \{0\}$

(Riemann-Hilbert). 可以 gluing 下.

$$\rightsquigarrow \mathcal{M}_{DH} := \mathcal{M}_{Hod}(X) \cup \mathcal{M}_{Hod}(\bar{X})$$

$$\downarrow$$

$$\mathbb{P}^1$$

$M_{DH} \longrightarrow \mathbb{P}^1$ the "preferred sections"
via harmonic bundle $(E, \partial, \bar\partial, \varphi, \varphi^*)$

$\rho : \mathbb{P}^1 \longrightarrow M_{DH}$, the normal bundle of preferred section ν, then

$\rho^*(\nu)$: semistable vector bundle of slope 1 over $\mathbb{P}^1$

1. difficult to see quotient
(邓、胡 groupoid.)

2. Stability - Condition.

(extract the notion of parabolic-weight
不希望用这个 notion)

"locally finite-type"

3.

Y compact R.S. $D = \{y\}$

$(\lambda, E, \nabla, F, \beta)$

Twistor Geometry Basics.

(Hitchin: "Hyperkahler Geometry and Supersymmetry")

Let (M^{4n}, I, J, K) hyperkahler $\rightsquigarrow$ Twistor Space

$M \times \mathbb{CP}^1 := \mathcal{M}_{twis}$

with complex structure given by: at (m, λ)

$$\underline{I}_{(m,\lambda)} := \left(\frac{1-\lambda\bar{\lambda}}{1+|\lambda|^2} I + \frac{\lambda+\bar{\lambda}}{1+|\lambda|^2} J + i \frac{\lambda-\bar{\lambda}}{1+|\lambda|^2} K,\ I_0 \right)$$

Prop: $\underline{I}_{(m,\lambda)}$ is integrable.

$(\underline{I}\theta = \theta i)$

pf: By Nirenberg, $\Longleftrightarrow$ $\forall$ $(1,0)$-form θ, we have $d\theta = \underset{\substack{\downarrow \\ (1,0)}}{\theta_i} \wedge \underset{\substack{\downarrow \\ 1\text{-form}}}{\alpha_i}$.

—. What is $(1,0)$-form for the complex structure $\underline{I}$?

Prop: if $\varphi_1, \dots, \varphi_{2n}$ local basis of $(1,0)$ form for

I, then: $\varphi_i + \lambda K \varphi_i$, $d\lambda$ gives a basis for $(1,0)$ forms of M_{tw}.

Pf:

Let φ be $(1,0)$ form. $\underline{I}\varphi = i\varphi$, set

$\theta = \varphi + \lambda K \varphi \rightsquigarrow (1+\lambda\bar{\lambda})\underline{I}\theta = \left((1-|\lambda|^2)I + (\lambda+\bar{\lambda})J + i(\lambda-\bar{\lambda})K\right)\theta$

$= i(1+|\lambda|^2)\theta.$

For the integrability:

$$d\theta = d(\varphi + \lambda K \varphi) = dx^i \wedge \underbrace{\nabla_{\frac{\partial}{\partial x^i}}(\varphi + \lambda K \varphi)}_{(1,0)\text{-form}} + \underbrace{d\lambda}_{(1,0)\text{-form}} \wedge K\varphi$$

$\therefore$ By Niven-berg ✓

In this case: $\begin{matrix} M_{tw} \\ \downarrow p \\ \mathbb{CP}^1 \end{matrix}$ holomorphic and

(m,λ) is holosection. ← "twistor-line.

$$\underline{I} = \frac{1}{1+|\lambda|^2}\begin{pmatrix} i(1-|\lambda|^2) & 2\lambda \\ -2\lambda & -i(1-|\lambda|^2) \end{pmatrix}$$

Now we compute holomorphic-tangent vector

At (m_0, λ)

$$\rightsquigarrow \begin{pmatrix} v \\ i\lambda v \end{pmatrix}, \quad \text{change of coordinate:}$$

$$\tilde{\lambda} = \frac{1}{\lambda} \rightsquigarrow \begin{pmatrix} -i\tilde{\lambda} w \\ w \end{pmatrix}$$

$\therefore$ transition function: $\begin{pmatrix} i\lambda & & & \\ & i\lambda & & \\ & & \ddots & \\ & & & i\lambda \end{pmatrix}$

$\rightsquigarrow \mathcal{O}(1)^{2d}$.

This gives that:

Prop:

Normal-bundle of twistor lines $\cong \mathcal{O}(1)^{\oplus 2d}$.

Symplectic Form.

$$\frac{1}{2}\omega_t = \sum \varphi_i \wedge \varphi_{n+i}$$

$$= \sum (\varphi_i + \lambda K \varphi_i) \wedge (\varphi_{n+i} + \lambda K \varphi_{n+i})$$

$$= \sum \varphi_i \wedge \varphi_{n+i} + \lambda\underline{(K\varphi_i \wedge \varphi_{n+i} + \varphi_i \wedge K\varphi_{n+i})} + \lambda^2 \underline{K\varphi_i \wedge K\varphi_{n+i}}$$

记作 (*): $2(*)(X,Y) = 2g(IX,Y) = 2\omega_I(X,Y)$

$\left\{ 2k\varphi_i \wedge k\varphi_{n+i} = -(\omega_2 - i\omega_3) \right.$

$\therefore\ \omega = (\omega_J + i\omega_K) + 2\lambda\omega_I - \lambda^2(\omega_J - i\omega_K)$

Real-Sections.

$T: \mathcal{M}_{tw} \longrightarrow \mathcal{M}_{tw}$

$(m,\lambda) \longmapsto (m, -\bar{\lambda}^{-1})$
