Focus on $SL(2;\mathbb{C})$ + trace free case.

X: cpt R.S. . genus$(X) = g$, $S = \{p_1, \dots, p_n\}$ marked points

Def: (Parabolic–Structure)

m_i 为相应的 dimension.

A vector bundle $\begin{matrix}\mathbb{C}^r \\ \downarrow \\ E \\ \downarrow \\ X\end{matrix}$ with weighted flags: $\underbrace{E_p = F_1 \supsetneq F_2}_{m_1} \supsetneq \cdots \supsetneq \underbrace{F_{s_p} \supsetneq 0}_{m_{s_p}}$ at $\forall p \in S$.

$0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_{s_p} < 1$

Def: (Parabolic Higgs Field)

$(\mathcal{E}_A, \Phi)$, $\Phi: \mathcal{E}_A \to K \otimes \mathcal{E}_A$ with simple pole at $p \in S$. s.t.

$$\operatorname{Res}_p \Phi(F_i(p)) \subsetneq F_i(p).$$

Def: (Pardegree)

$$\operatorname{pardeg} E := \deg E + \sum_{p \in S} \sum_{i=1}^{s_p} m_i \alpha_i(p), \quad \mu_\alpha(E) = \frac{\text{par-deg } E}{\text{rank } E}$$

类似定义 stability–condition.

Here: Fix determinant of Φ + trace free.

Assume: $\mu_\alpha(E) = 0$, $\alpha_2(p) = 1 - \alpha_1(p)$,

$$\operatorname{Res}_p \Phi = \begin{pmatrix} \sigma(p) & 0 \\ * & -\sigma(p) \end{pmatrix}$$

There is a natural–morphism:

semi-stable + weight α + par-deg$(E) = 0$

$$\pi: P_\alpha^0 \longrightarrow \mathbb{C}^n$$

$$(\mathcal{E}_A, \Phi) \longmapsto (\sigma(p_1), \dots, \sigma(p_n))$$

For $\sigma \in \mathbb{C}^n \rightsquigarrow \pi^{-1}(\sigma) := \mathcal{M}_{Dol}(\alpha, \sigma)$

Def: (α, σ) generic if no $(\mathcal{E}_A, \Phi) \in \mathcal{M}_{Dol}(\alpha, \sigma)$ which are not α–stable.

(i.e. semi-stable $\to$ stable)

(i.e. each element is α–stable)

Lemma: (α,σ) not generic $\Longleftrightarrow$ $\exists\, d\in\mathbb{Z}$, $e=(e_1,\dots,e_n)\in\{0,1\}^n$ s.t.

$$\begin{cases} d+\sum_i\left(e_i+(-1)^{e_i}\alpha_1(p_i)\right)=0 \\ \sum(-1)^{e_i}\sigma(p_i)=0 \end{cases}$$

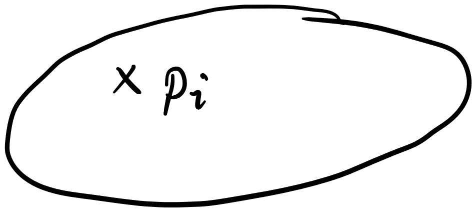
pf: $(\Rightarrow)$ if (α,σ) not generic.

$\exists\,(\mathcal{E}_A,\overline{\Phi})\in\mathcal{M}_{Dol}(\alpha,\sigma)$ stri semi-stable: $\exists\, L\hookrightarrow V$ presered by $\dots$, s.t. $\operatorname{pardeg}L=0$, $\rightsquigarrow$ $(\mathcal{E}_A,\overline{\Phi})=(L_1,\overline{\Phi}_1)\oplus(L_2,\overline{\Phi}_2)$

$e_i=\dim\left(L_1\cap F_2(p_i)\right)\in\{0,1\}$, L_1 has weight $e_i+(-1)^{e_i}\alpha_1(p_i)$

$$\operatorname{pardeg}L_1=\underline{\deg L_1+\sum\left(e_i+(-1)^{e_i}\alpha_1(p_i)\right)=0}$$

by Residue-thm ✓



$$\sum\operatorname{Res}\overline{\Phi}=0 \rightsquigarrow \sum(-1)^{e_i}\sigma(p_i)=0$$

$(\Leftarrow)$ we could construct L_1,L_2 ✓, $\overline{\Phi}_1,\overline{\Phi}_2$, s.t. $(L_1,\overline{\Phi}_1)\oplus(L_2,\overline{\Phi}_2)$ α-semi but not α-stable.

#

Let $W=(0,\frac{1}{2})^n$, For $\forall d\in\mathbb{Z}$, $(e_1,\dots,e_n)$, denote $\Delta(\sigma)$ solution of $\left(\tilde{\sum}(-1)^{e_i}\sigma(p_i)=0\right)$

generic-set: (σ fixed)

$$W-\bigcup_{(d,e)\in\Delta(\sigma)}H_{d,e}:\ \text{weights }\alpha,\ \text{s.t. }(\alpha,\sigma)\text{ generic}.$$

$$\begin{array}{c} \mathbb{C}^r \\ \downarrow \\ V \\ \downarrow \\ \Sigma \end{array}$$ smooth parabolic vector bundle,

Def: (Regular D-module)

Flat conn on $X-S=\{p_1,\dots,p_n\}$), s.t.

(Project)

(i) If $D=d'+d''$, d'' extends through S.

(ii) $D=d+A\frac{dz}{z}$ in local-trivi $C(1,0)$

(iii) $Res_{p_i}(D)(V_i(p))\subset V_i(p)$

Def: (degree + slope)

Just as parabolic higgs field.

Here: $\mu_\beta(V)=0$, $\underline{\beta_2(p)=1-\beta_1(p)}$ weight.

$$Res_p(D)=\begin{pmatrix} \underline{\Gamma(p)} & 0 \\ * & -\underline{\Gamma(p)} \end{pmatrix}$$ eigenvalue.

$\mathcal{M}_{dR}(\beta,\Gamma)$: β-semistable filtered regular D-module.

Def: (β,Γ) generic same as before.

Prop: (β,Γ) not generic $\Leftrightarrow$ $\exists d\in\mathbb{Z}$, s.t.

$$\begin{cases} d + \sum \left(e_i + (-1)^{e_i} \beta_1(p_i) \right) = 0 \\ \sum (-1)^{e_i} \tau(p_i) = 0 \end{cases}$$

$$\rightsquigarrow W \,|\, \bigcup_{(d,e)\in\Delta(\tau)} H_{d,e}$$

Analytic Construction.

$$\begin{tikzcd} \mathbb{C}^r \arrow[d] \\ E \arrow[d] \\ M \end{tikzcd}$$

, here $h(e_i,e_i) = |z|^{2\alpha_i(p)} \to H = \begin{pmatrix} |z|^{2\alpha_1(p)} & \\ & |z|^{2\alpha_2(p)} \end{pmatrix}$

parabolic higgs bundle, h hermition over E with singularity at each marked points $p_1, p_2, \dots, p_n \in M$:

For $\forall p \in S$. let $\{e_1, e_2\}$ local-frame, $e_2(p) = F_2(p) \leftarrow$ flag.

define H. s.t. $\{|z|^{-\alpha_j(p)} e_j\}_{j=1,2}$ be local unitary frame.

$\rightsquigarrow$ Chern – Connection:

$$d_A = d + \begin{pmatrix} \alpha_1(p) & \\ & \alpha_2(p) \end{pmatrix} i\,d\theta + O(|z|^{\alpha_1(p) - \alpha_2(p)})$$

How to compute?

extend to whole X.

$E = G_1 \oplus G_2$

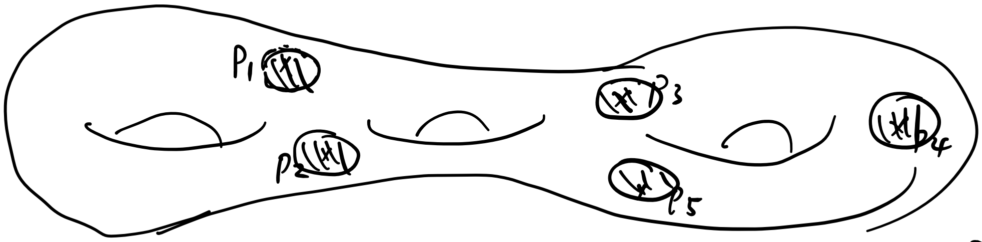
$G_1 = \mathrm{span}\{e_1\}$, $G_2 = \mathrm{span}\{e_2\}$

$$\rightsquigarrow \operatorname{End} E = \underset{\substack{\| \\ (\bigoplus_i \operatorname{End} G_i)}}{E_D} \oplus E_H := \bigoplus_{i\neq j} \operatorname{Hom}(G_i, G_j).$$

Pick u be local section $\rightsquigarrow u = u_D + u_H$, then

def: $$\|u\|_{D^p_k} := \sum_{i+j\le k} \left\{ \int \left|\frac{\partial^i}{\partial z^i}\frac{\partial^j}{\partial \bar z^j} u_D\right|^p + \left| |z|^{i+j-k} \partial^i \bar\partial^j u_H\right|^p \right\}$$

local trivi is taken w.r.t $\{|z|^{-\alpha_i(p)} e_j\}$



outside these neigh put usual L^p-metric.

Def: $D^p_k\Omega^0(sl(2;\mathbb{C}))$ be completion of $C^\infty_0(X-S, \operatorname{End} E)$.

Def: $\mathcal{G}^{\mathbb{C}} := \{ g \in D^p_2\Omega^0(\operatorname{End} E) \mid \det g \equiv 1 \}$

$$\mathcal{A}_{Dol} := \{ (\bar\partial_A, \Phi) \mid \bar\partial_A \Phi = 0,\ \bar\partial_A - \bar\partial_{A_0} \in D^p_1\Omega^{0,1},\ \Phi - \Phi_0 \in D^p_1\Omega^{1,0} \}$$

Tangent Space at $(\bar\partial_A, \bar\Phi)$:

complex:

$$0 \to D_2^p \Omega^0(sl(E)) \longrightarrow D_1^p \Omega^{0,1}(sl(E)) \oplus D_1^p \Omega^{1,0}(sl(E)) \longrightarrow D_0^p \Omega^{1,1}(sl(E))$$

$$S \longmapsto (\bar\partial_A S, \bar\Phi S)$$

$$(\bar\partial_A', \bar\Phi') \longmapsto \bar\partial_A \bar\Phi' + \bar\Phi \bar\partial_A'$$

For generic $(\alpha, 6)$

$$\longrightarrow \mathcal{M}_{Dol}(\alpha, 6) \text{ complex-manifold}$$

s.t.

$$\mathcal{M}_{Dol}(\alpha, 6) \overset{S}{\cong} \mathcal{A}_{Dol} / \mathcal{G}^{\mathbb{C}}.$$

D-Modules. Case.

Let $p \in S$ with local frame $\{e_1, e_2\}$

A filtered regular D-module:

$$D = d + \begin{pmatrix} T(p) & 0 \\ * & -T(p) \end{pmatrix} \frac{dz}{z}$$

(flat connection)

Put hermitian-metric H with

$$\left\{ |z|^{-\beta_1-(\Gamma(\varphi)-\overline{\Gamma(\varphi)})/2} e_1 ,\ |z|^{-\beta_2-\frac{(\Gamma(\varphi)-\overline{\Gamma(\varphi)})}{2}} e_2 \right\}$$

Local unitary frame why?

$$\rightsquigarrow D = d + \begin{pmatrix} \frac{3\Gamma}{4}+\frac{\overline{\Gamma}}{4}-\frac{\beta_1}{2} & 0 \\ * & 1-\frac{5}{4}\Gamma+\frac{\overline{\Gamma}}{4}-\frac{\beta_2}{2} \end{pmatrix}\frac{dz}{z}$$

$$+ \begin{pmatrix} -\frac{\Gamma-\overline{\Gamma}}{4}-\frac{\beta_1}{2} & 0 \\ * & -\frac{\Gamma-\overline{\Gamma}}{4}-\frac{\beta_1}{2} \end{pmatrix}\frac{d\bar{z}}{\bar{z}}.$$

$$\mathcal{G}^{\mathbb{C}} := \{ g \in D_2^p\Omega^0(\mathrm{End}(V)) \mid \det g \equiv 1 \}$$

$$\mathcal{A}_{DR} := \{ D \mid D - D_0 \in \Omega^1(sl(V)) \}$$

Tangent-Complex:

$$D_2^p\Omega^0(sl\,E) \xrightarrow{D} D_1^p\Omega^1(sl\,E) \xrightarrow{D} D_0^p\Omega^2(sl\,E)$$

Harmonic Bundles and Hÿperkahler Structure

Let $(\bar\partial^A, \Phi)$ be parabolic higgs bundle given hermitian metric h

$\rightsquigarrow$ Chern-Conn $D = d_A = \partial^A + \bar\partial^A$

$(\bar\partial^A, \Phi)$

Def (harmonic):

Trace free part of curvature.

(i.e. $R^\perp + [\Phi, \Phi^*] = 0$)

$$\frac{1}{2}(F^D - F^{D^*}) + [\Phi, \Phi^*] = 0$$

$\underline{\mathfrak{sl}}\, E = \underline{\mathfrak{su}}\, E \oplus i\, \underline{\mathfrak{su}}\, E$

For D regular D-module

$D = \underline{d_A} + \underline{\Psi} : \Omega^1(\underline{\mathfrak{su}}(E)) \oplus \Omega^1(i\,\underline{\mathfrak{su}}(E))$

anti-hermitian part

hermitian

Def: $d_A = \partial^A + \bar\partial^A$, $\Psi = \Phi + \Phi^*$, then call

D is harmonic if $\bar\partial^A \Phi = 0$

Prop: D is harmonic D-module

$\Leftrightarrow (\bar\partial^A, \Phi)$ harmonic higgs bundle.

(⇒) $(\bar\partial^A, \Phi)$ harmonic iff

$$R^\perp + [\Phi, \Phi^*] = 0$$

D flat ⟺ $(d^A + \Psi)(d^A + \Psi) = 0$

⟺ $(d^A + \Phi + \Phi^*)(d^A + \Phi + \Phi^*) = 0$

$$\partial^A \bar\partial^A + \bar\partial^A \partial^A + [\Phi, \Phi^*] = 0 \quad (*)$$

$(\bar\partial^A, \Phi)$ harmonic

$\therefore\ \bar\partial^A \Phi = 0,\ \partial^A \Phi^* = 0$

Correspondence: (between harmonic bundle higgs ⟷ harmonic D-module)

Let $(\bar\partial^A, \Phi) \in \mathcal{M}_{Dol}^{s}(\alpha, b)$, in local frame $\{|z|^{-\alpha_i(p)} e_i\}$. we know:

$$\begin{cases} d_A = d + \begin{pmatrix} \alpha_1(p) & \\ & \alpha_2(p) \end{pmatrix} i\,d\theta + E_1, & E_i \in D_1^p \Omega^1(\mathfrak{su}E). \\ \Phi = \begin{pmatrix} b(p) & \\ & -b(p) \end{pmatrix} \dfrac{dz}{z} + E_2 & \end{cases}$$

$\rightsquigarrow D = d_A + \Phi + \Phi^*$

If $D \in \mathcal{M}^S_{dR}$: in the frame

$$\left\{ |z|^{-\beta_1 - (\pi(p) - \overline{\pi}(p))/2} e_1,\ |z|^{-\beta_2 - (\pi(p) - \overline{\pi}(p))/2} e_2 \right\}$$

$\rightsquigarrow$

$$d_A = d + \begin{pmatrix} \frac{\pi(p) + \overline{\pi}(p)}{2} & 0 \\ 0 & 1 - \frac{\pi(p) + \overline{\pi}(p)}{2} \end{pmatrix} i\,d\theta + E_3$$

$$\Phi = \begin{pmatrix} \frac{\pi(p) - \beta_1(p)}{2} & \\ & \frac{1 - \pi(p) - \beta_2(p)}{2} \end{pmatrix} \frac{dz}{z} + E_4$$

$$\therefore\ \alpha_1(p) = \frac{\pi(p) + \overline{\pi}(p)}{2}, \quad \sigma(p) = \frac{\pi(p) - \beta_1(p)}{2}$$

Hyperkahler Structure on harmonic parabolic Higgs Bundle.

(The Moduli of harmonic parabolic higgs bundle is given by hyperkähler - construction:)

$$\mathcal{G} \overset{\text{def}}{=} \{ g \in D_2^p \Omega^0(\operatorname{End} E) \mid g \in SU(E_x),\ \forall x \in X \}$$

$$\mathcal{A} :\overset{\text{def}}{=} \{ (d_A, \Phi) \mid d_A \in \Omega^1(\underline{su} E),\ \Phi \in \Omega^{1,0}(\underline{sl} E) \}$$

$$\mu_I : \mathcal{A} \longrightarrow \Omega^{1,1}$$
$$(d_A, \Phi) \longmapsto R_A^{\perp} + [\Phi, \Phi^*]$$

$$\mu_J + i\mu_K : (d_A, \Phi) \longmapsto -2\bar{\partial}_A \Phi$$

$\rightsquigarrow$ Higgs Moduli:

$$\mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0) / \mathcal{G}$$

where Riemann-Metric given by L^2-inner product

$$I : (\xi, \phi) \longmapsto (i\xi, i\phi)$$
$$J : (\xi, \phi) \longmapsto (i\phi^*, -i\xi^*)$$
$$K : (\xi, \phi) \longmapsto (-\phi^*, \xi^*)$$

Thm: We have 2 diffeomorphism:

$$(\mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0) / \mathcal{G}, I) \cong \mathcal{M}_{Dol}(2, 6)$$

$$(\mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0) / \mathcal{G}, J) \cong \mathcal{M}_{dR}(\beta, T)$$

Thm: (NAH)

$$\mathcal{M}_{Dol}(\alpha, \sigma) \cong \mathcal{M}_{dR}(\beta, \tau)$$

For $\forall (\alpha, \sigma)$, (β, τ).