This paper aims to construct Deligne-Hitchin Moduli of λ-logarith connection of rank 1 via the Riemann-Hilbert Corresspondence.

Notation:

$G_m := (\mathbb{C}^*, \cdot)$ as a multiplication-group.

$Q := \{\pm\sqrt{-1}\}$, $C_Q: Q \to Q$ given by minus.

(Y, T_Y), T_Y involution, then for $Q \xrightarrow{f} Y$. we have 2-maps: $\begin{cases} f\circ C_Q \\ T_Y \circ f \end{cases}$

$Y^\perp := \{ f\in \mathrm{Hom}(Q,Y) \mid f\circ C_Q = T_Y\circ f \}$

Example:

$G_m^\perp := \{(x,y) \mid x^2+y^2=1,\ x,y\in\mathbb{C}\}$

check: $f\in G_m^\perp$. then $f(-i) = T_{\mathbb{C}^*}\circ f = \frac{1}{f(i)} \rightsquigarrow f(i)f(-i)=1$, set $x=\frac{1}{2}(f(i)+f(-i))$, $y=\frac{1}{2}(-if(i)+if(-i))$

$\rightsquigarrow x^2+y^2=1$.

$\exp^\perp: \mathbb{C} \to G_m^\perp : \theta \longmapsto (\cos(2\pi\theta), \sin(2\pi\theta))$

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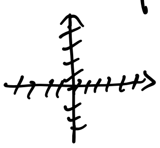
1. Logarithmic Connection

Def: normal-crossing divisor D, $\forall p\in D\subset X$. equation looks like $z_1z_2\cdots z_k=0$

E.g.  {real line} $\cup$ {image line}.

Def: (logarithmic-form)

$\Omega_X^1(\log D)$: Locally-free sheaf, s.t. if $U=(z_1,\dots,z_n)$, then $\Omega_X^1(\log D)(U)$ is spanned by $d\log z_1,\dots,d\log z_k, z_{k+1},\dots,z_n$.

(λ-Logarithmic-connection)

on $\mathbb{C}^r\to E\to X$, $\nabla: E\to E\otimes_{O_X}\Omega_X^1(\log D)$, s.t. $\nabla(fs)=\lambda df\,s+f\nabla s$.

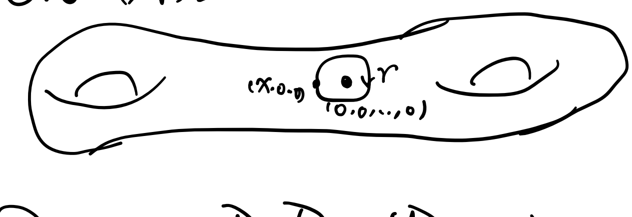
$\lambda=1$. usual connection / $\frac{\lambda^{-1}\nabla}{\lambda\neq 0}$: usual conn.

$\lambda=0$, Higgs field.

Riemann-Hilbert Connection:

$$\{\text{flat-Connections on } X\} \xleftrightarrow{\cong} \left\{\begin{matrix}\text{representation of} \\ \pi_1(X)\end{matrix}\right\}$$

On (X, D):



$\gamma(t) = (x e^{i 2\pi t}, 0, \dots, 0)$.

For $\rho: \pi_1(X, x) \to G \rightsquigarrow \rho^{\perp}: \pi_1(X, x)^{\perp} \to G^{\perp}$.

(monodromy)

Decompose $\begin{cases} D = D_1 + \dots + D_k \rightsquigarrow \text{irre-component}. \\ \mathcal{G} = \mathbb{Z}^k \end{cases}$

(L, ∇), $\mathcal{G} \curvearrowright (L, \nabla) \longmapsto (L^g, \nabla^g)$, $\begin{cases} L^g = L(g_1 D_1 + \dots + g_k D_k) \\ \nabla^g \text{ (Same as } \nabla\text{)}. \end{cases}$

$(\operatorname{Res}_{D_i} \nabla^g = \operatorname{Res}_{D_i} \nabla - \lambda g_i \rightsquigarrow \nabla^g = \nabla \cdot z^{-g} = d + a \frac{1}{z} dz - \lambda \frac{g}{z} dz)$

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2. Delign – Gluing

a. Compact Case

For $\mathcal{M}_{Hod}(X) \triangleq \{(\lambda, \bar\partial, D) \mid \bar\partial D + D \bar\partial = 0\}$

$$\mathcal{M}_{DH} := \mathcal{M}_{Hod}(X) \cup_{\mathcal{f}} \mathcal{M}_{Hod}(\overline{X})$$
$$[\lambda, \bar\partial, D]_X \qquad [\tfrac{1}{\lambda}, \tfrac{1}{\lambda} D, \tfrac{1}{\lambda} \bar\partial]_{\overline{X}}$$

The real structure on $\mathcal{M}_{DH}$ is given by:

$$\mathcal{T}: [\lambda, \bar\partial, D]_{\Sigma} \longmapsto [-\bar\lambda^{-1}, \bar\lambda^{-1} \overline{D}, -\bar\lambda^{-1} \overline{\bar\partial}]_{\Sigma}$$

twistor line: $s: \mathbb{CP}^1 \to \mathcal{M}_{DH}$. s.t. $\mathcal{T}(s(\lambda)) = s(-\bar\lambda^{-1})$.

Prop:

On $\mathbb{CP}^1$, the $\mathcal{O}_{\mathbb{CP}^1}(d)$ is given by homogeneous poly of degree d.

Thm:

If X compact, $\begin{matrix} \mathbb{C} \\ \downarrow \\ L \\ \downarrow \\ X \end{matrix}$, then: $\Gamma(\mathbb{P}^1, \mathcal{M}_{DH}(X))^{\sigma} \cong \mathcal{M}_{DH}(X)|_p$
$$s \longmapsto s(p).$$

Lemma:

Let $\alpha, \beta, \omega, \eta \in H^0(K_X)$, then $\exists$ holo-section $s: \mathbb{CP}^1 \to \mathcal{M}_{DH}$ defined by $s(\lambda) := [\lambda, \bar\partial + \bar\omega + \lambda \bar\eta, \lambda(\partial + \alpha) + \beta]$, Conversely, each holo-sections have this form.

Pf: ① — simply-connected, $\exists\, \omega_1(\lambda) \in \overline{H^0(K_X)}$, s.t. $\phi \circ s(\lambda) = [\bar\partial + \omega_1(\lambda)]$

Let $\beta = s(0)$ $\rightsquigarrow$ a lift of s is given by $\hat{s}(\lambda) = [\lambda, \bar\partial + \omega_1(\lambda), \beta + \lambda\partial + \lambda\alpha_1(\lambda)]$

$\alpha_1 \in H^0(K_X)$ by integrability. $\rightsquigarrow$ $\hat\nabla^\lambda = d + \lambda^{-1}\beta + \bar\omega_1(\lambda) + \alpha_1(\lambda)$

over $\mathbb{CP}^1 - \{0\}$, we also have: $\hat{s}(\mu) = [\mu, \partial + \alpha_2(\mu), \mu(\bar\partial + \bar\alpha_2(\mu)) + \bar\eta]$

gluing $\rightsquigarrow$ $\hat\nabla^\lambda . g(\lambda) = \tilde\nabla^\lambda$ $\rightsquigarrow$ $\tilde\nabla^\lambda - \hat\nabla^\lambda = d\log$ $\rightsquigarrow$ Lattice point

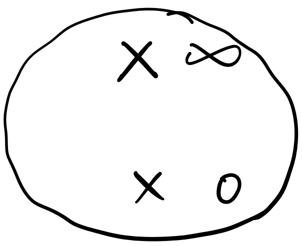
$\therefore$ χ 与 λ 无关, by const. $\hat\nabla^\lambda$ has 1-pole at $\lambda = \infty$

$\deg \bar\omega'(\lambda) \leq 1$

$\therefore$ $\alpha_1(\lambda)$ 与 λ 无关.

Open-Variety: $(\mathbb{CP}^1 - \{0, \infty\})$

$\mathbb{P}^1 - \{0, \infty\}$



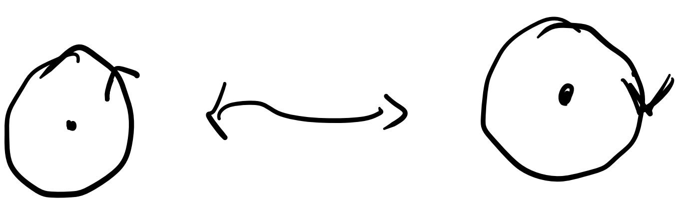
Logarithmic-Connection $\nabla^\lambda = \lambda d + \alpha \frac{dz}{z}$

$\rightsquigarrow M_{Hod}(X, D) = \mathbb{A}^1 \times \mathbb{A}^1 = \{(\lambda, \alpha)\}$

$\rightsquigarrow$ monodromy $\rho^{\nabla^\lambda} = \rho^{(d + \frac{\alpha}{\lambda}\frac{dz}{z})} = \exp(2\pi i \frac{\alpha}{\lambda}) = \left(\cos\left(\frac{2\pi\alpha}{\lambda}\right), \sin\left(\frac{2\pi\alpha}{\lambda}\right)\right)$

Now turn-around:

For $\gamma := e^{2\pi i x}$, if $X \longleftrightarrow \bar{X}$ $\rightsquigarrow$ $\varphi(\gamma) = \gamma^{-1}$



X

Now we give the Delign-Glueing:

For $\mathcal{M}_{Hod}(X)$: (λ, α), $\lambda^{-1}\nabla$

$\Big\updownarrow$ Riemann-Hilbert.

$\mathcal{M}_{Betti}(X)$: $\rho^{\lambda^{-1}\nabla}: \pi_1(X, x) \longmapsto \mathbb{G}_m := \mathbb{C}^*$

$\rightsquigarrow$ on $X \cong \overline{X}$

$$\rho^{\lambda^{-1}\nabla}(\gamma') = \rho^{\lambda^{-1}\nabla}(\gamma^{-1}).$$ ← inverse of a curve.

$\Big\updownarrow$

$\mathcal{M}_{Hod}(\overline{X})$: $\overline{\Phi} = d - \frac{\alpha}{\lambda}\frac{dz}{z}$.

delign-gluing means:

$$d : (\lambda, E, \nabla) \longmapsto (\lambda^{-1}, E, \lambda^{-1}\nabla).$$

$$\therefore d(\lambda, \alpha) \longmapsto (\lambda^{-1}, -\lambda^{-2}\alpha).$$

$\rightsquigarrow \mathcal{M}_{EDH}(\mathbb{P}^1 - \{0, \infty\}) \cong \mathcal{O}_{\mathbb{P}^1}(2)$.

In this case:

$\rightsquigarrow \Gamma(\mathbb{P}^1, \mathcal{M}_{DH}(X, \log D)) \cong \mathbb{C}^3$.

Thm:

For $(a, \alpha) \in \mathbb{R} \times \mathbb{C}$, the real sections have the form

$$p^{\lambda} := \alpha - a\lambda - \overline{\alpha}\lambda^2.$$

the real structure is given by:

$$\sigma(\lambda, \alpha) = (-\bar{\lambda}^{-1}, -\bar{\lambda}^{-2}\bar{\alpha})$$

so just check by def.

Gauge Transformation.

$$\mathbb{Z} \circlearrowright M_{Hod} \longmapsto M_{Hod}$$

$$(g, (\lambda, \alpha)) \longmapsto (\lambda, \alpha - g\lambda)$$

$$\therefore M_{DH}(X, \log D) \cong \mathcal{O}_{\mathbb{P}^1}(2) / (1,0)\cdot\mathbb{Z}.$$

General Rank 1 Case.

X smooth projective-variety, $D = D_1 + \cdots + D_k$ normal-crossing divisor,

$U \triangleq X - D$

$M_{Hod}(\lambda, L, D)$, For $\mathbb{C} \to L \to X$, $C_1(L)_{\mathbb{Q}} \in \mathbb{Q}[D_1] + \cdots + \mathbb{Q}[D_k]$

$\in H^2(X, \mathbb{Q}^L)$

$$\nabla : L \longrightarrow L \otimes_{\mathcal{O}_X} \Omega^1_X(\log D)$$

For $\lambda \neq 0 \rightsquigarrow \lambda C_1(L) = -\sum_i \mathrm{Res}(\nabla; D_i)[D_i] \in H^2(X, \mathbb{C}^L)$

Gauge Action

$$G := \mathbb{Z}^k \circlearrowright M_{Hod}(X, \log D) \longmapsto M_{Hod}(X, \log D)$$

$$(g_1, \dots, g_k) \rhd (\lambda, L, \nabla) \longmapsto (\lambda, L^g, \nabla^g)$$

$$\mathrm{Res}(\nabla^g; D_i) = \mathrm{Res}(\nabla; D_i) - \lambda g_i$$

Delign- Glueing

$$\mathcal{M}_{Hod}(X,\log D) \rightsquigarrow \mathcal{M}_{DR}(X,\log D) \sim \mathcal{M}_{B}(X,\log D)$$

$$\wr\wr$$

$$(\rho, res(\nabla, D_1), \ldots, res(\nabla, D_k))$$

delign gluing same as before:

$$(\lambda, \alpha_1, \ldots, \alpha_k) \longmapsto (\lambda^{-1}, -\lambda^{-2}\alpha_1, \ldots, -\lambda^{-2}\overline{\alpha_k})$$

$$\begin{tikzcd} \rightsquigarrow \mathcal{M}_{DH}(X,\log D) \arrow[d] \\ \underline{\mathbb{P}^1} \end{tikzcd}$$

Residue & Parabolic Structure

$0 \subsetneq \mathbb{C}$

Δ (weight)

$\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$ encodes the data of residues and parabolic weights for a harmonic bundle.

$$Res_{D_i,p}: \mathcal{M}_{Hod}(X,\log D)_p \longrightarrow \mathbb{C} \cong T(1,\log)_p$$

$$\rightsquigarrow$$

(compatible with gauge)

$$Res_{D_i,p}: \mathcal{M}_{DH}(X,\log D)_p \longrightarrow T(1,\log)_p.$$

Let $(\mathcal{E}^\lambda, \mathbb{D}^\lambda)$ be parabolic vector bundle, for variety D_i, Let u be local unit section, if near singularity, $|u|_h \sim |z|^{-b_i} \longrightarrow$ "b_i" is the weight.

Thm:

D_i be divisor component $p \in \mathbb{A}^1 \subset \mathbb{P}^1$, if $\mathcal{E} = (E, D', D'', h) \in \mathcal{M}_{Har}(n)$ with rank 1,

then:

$$(\overline{\omega}_p, res_p)^{g}_{D_i}(\mathcal{P}(\mathcal{E})) \in \frac{\mathbb{R} \times \mathbb{C}}{(1,-p) \cdot \mathbb{Z}} : \Gamma(\mathbb{P}^1, T\mathbb{P}^1)^6$$

is the parabolic weight and residue of the parabolic λ-connection.

Rk:

$$\overline{\omega}_p : (a, \alpha) = a + \alpha \overline{p} + \overline{\alpha} p.$$