Basic Structure of the Paper:

① $\mathcal{M}_{SD} := \{(A, \Phi) \mid F(A) + [\Phi, \Phi^*] = 0\} / \mathcal{G}$ (Chapter 1~4)

$\quad \quad$

$$\mathcal{M}_{SD} := \{(A, \Phi) \mid F(A) + [\Phi, \Phi^*] = 0\} / \mathcal{G}$$
$$\cong$$
$$\mathcal{M}_{Dol} := \{(\bar\partial, \Phi) \mid \text{stable Higgs Pair}\}$$

② A. Smooth-Structure of $\mathcal{M}_{SD}$.

(Coordinate, dimension, Complex Coordinate)

B. Riemann-Structure of $\mathcal{M}_{SD}$.

(Metric, Completeness, Hyperkahler-Structure).

C. Global-Topology of $\mathcal{M}_{SD}$.

(Morse function, non-compact, connected/simply-Connected)

D. Sympletic-Structure.

(Hitchin-System: $\mathcal{M} \longrightarrow H^0(M, K^2)$, $(\bar\partial, \Phi) \longmapsto \det \Phi$

non-Abelianzation / Spectral-Curve.

③ Application:

A. $\mathcal{M}_{SD} \cong \mathcal{M}_{dR} := \{\text{Flat-Conn}\}$

B. Real-Structure: $\sigma: (A, \Phi) \longmapsto (A, -\Phi)$

C. Teichmüller-Space: Underlying geometry.

$\mathcal{M}_{2g-2} \cong H^0(M, K^2) \cong \mathcal{T}_g$.

Principal Fibre Bundle.

(E,π,M) : (i) $E \triangleleft G$ free ; $\xrightarrow[\text{bundle}]{\text{Associated}}$ if $G \triangleright F$ $\rightsquigarrow P_F = P\times F/_{(p,f)\sim(p\triangleleft g,\, g^{-1}\triangleright f)}$

(ii) $\begin{matrix} E \\ \downarrow \pi \\ M \end{matrix} \cong \begin{matrix} E \\ \downarrow \rho \\ E/G \end{matrix}$. $\rightsquigarrow F$ - bundle.

Connection :

A. assignment $H \subset TP$, such-that.

$H\oplus V = TP$. $(\triangleleft g)_* H_p = H_{p\triangleleft g}$

B. 1-form $\omega \in \Omega^1(\mathfrak{g})$:

1. $(\triangleleft g)^*\omega|_p(X_p) = (Ad_{g^{-1}})_*(\omega_p(X_p))$

2. $\omega_p(X_p^A) = A$

C. Parallel-Transport,

$\gamma : [0,1] \to M \rightsquigarrow \gamma^{\uparrow} : [0,1] \to P$.

(horizontal-lift), the Paraller Trans :

$T_\gamma^{P_F} : \pi_F^{-1}(\gamma(0)) \longrightarrow \pi_F^{-1}(\gamma(1))$

$[p, f] \longmapsto [\gamma^{\uparrow}(1), f]$

给定 V, V' be 2 vector spaces admit G-action.

$\xmapsto[\text{Connection?}]{\text{怎么给一个}}$ $f : V \to V'$ be G-equiva

$(f(v\triangleleft g) = f(v)\triangleleft g)$

$\Leftrightarrow$ we have :

$P\times_G V \xrightarrow{[f]} P\times_G V'$

now :

$A \in \Omega^1(M)\otimes \mathfrak{su}(2)$

(Here : $V = \mathfrak{su}(2)$, $V' = End(\mathbb{C}^2)$),

$\mathfrak{su}(2) \longrightarrow End(\mathbb{C}^2)$ G-equiva

$B \longmapsto B\cdot S$ $\quad G = SU(2)$

$\rightsquigarrow$

$P\times_G \mathfrak{su}(2) \longrightarrow P\times_G End(\mathbb{C}^2)$

$A \longmapsto A\cdot v$.

Self-Duality Equation :

For $G \to P \to M$, $\Phi \in \Omega^{1,0}(adP\otimes\mathbb{C})$, $A \in \Omega^1(\mathfrak{su}(2))$ $\rightsquigarrow \begin{cases} F + [\Phi, \Phi^*] = 0 \\ d_A'' \Phi = 0, \end{cases}$

Example : (M, g) be Riemann-surface with metric

$g = \lambda dz d\bar{z}$. $\rightsquigarrow T^*_{\mathbb{C}}M = K\oplus\bar{K} \rightsquigarrow \nabla^{L.C.*}$ on $T^{\mathbb{C}}M \rightsquigarrow$ restricted

on $K \rightsquigarrow U(1)$-connection. $(\nabla^{L.C} = d+\omega \rightsquigarrow \nabla|_{K^{1/2}} = d + \frac{1}{2}\omega)$

Let $V = K^{1/2}\oplus K^{-1/2} \rightsquigarrow F = \begin{pmatrix} \frac{1}{2}F_0 & \\ & -\frac{1}{2}F_0 \end{pmatrix}$, $\Phi : K^{1/2}\oplus K^{-1/2} \to K(K^{1/2}\oplus K^{-1/2})$

Let $\Phi = \begin{pmatrix} 0 & 0 \\ \frac{1}{dz} & 0 \end{pmatrix} dz$. let $\Phi^* = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} d\bar{z}$, $a : K^{-\frac{1}{2}} \to K^{1/2} \rightsquigarrow a \in K$,

$g = \lambda dz d\bar{z} \rightsquigarrow \hat{g} = \frac{1}{\lambda}\frac{\partial}{\partial z}\otimes\frac{\partial}{\partial\bar{z}}$ on $T^{\mathbb{C}}M$.

$\therefore \sqrt{h} = a \cdot \frac{1}{\sqrt{h}} \rightsquigarrow a = h\,dz$

$\therefore [\Phi, \Phi^*] = \left[\begin{pmatrix} 0 & 0 \\ \frac{1}{dz} & 0 \end{pmatrix}, \begin{pmatrix} 0 & h\,d\bar{z} \\ 0 & 0 \end{pmatrix}\right] dz\,d\bar{z} = \begin{pmatrix} -h & \\ & h \end{pmatrix} dz\,d\bar{z}$

$\rightsquigarrow \begin{pmatrix} -\frac{1}{2}F_0 & \\ & \frac{1}{2}F_0 \end{pmatrix} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} h\,dz\,d\bar{z}.$

$\rightarrow F_0 = -2h\,dz\,d\bar{z} \rightsquigarrow K_G = -4.$ (Gauss Curvature).

Now,

$\mathcal{M}_{SD} \longleftrightarrow \mathcal{M}_{Dol}$. For $G \to P \to M$ principal-bundle,

$[\nabla, \Phi] \longmapsto (\bar{\partial}^{\nabla}, \Phi)$

(i) $W_2(P) = 0$, $A \rightsquigarrow A$ be connection on $\operatorname{ad}P \otimes \mathbb{C}$ (i.e. $\Lambda^2 V = \theta$)

(ii) $W_2(P) \neq 0$, $A \to B$ be connection on $\operatorname{ad}P \otimes \mathbb{C}$. $F(B) = F(A) + \frac{1}{2}F(A_0)$

where A_0 be a connection on $\Lambda^2 V$.

i.e. For Φ-invariant $L \hookrightarrow V$. $\deg L \leq \frac{1}{2}\deg(\Lambda^2 V) = \frac{1}{2}\deg V$.

pf: (key-Point: Wëzenbock-Formula)

$F(A) = A \circ A = (d_A' + d_A'')(d_A' + d_A'') = d_A' d_A'' + d_A'' d_A'$

$L \hookrightarrow V$ means (i) "$s \in H^0(M; L^* V)$ injective.

The Wëzenbock:

(ii) $d'' h(d_B' s, s) = h(d_B'' d_B' s, s) - h(d_B' s, d_B' s)$

$= h(F(B)s, s) - h(d_B' s, d_B' s).$

(iii) 积分: $\int_M h(F(B)s, s) = \int_M h(d_A' s, d_A' s). \geq 0.$

$F(B)s = -\deg L\,\omega s + F(A)s = -\deg L\,\omega s - [\Phi, \Phi^*]s.$

$\rightsquigarrow \deg L \leq 0$ (取等 $\partial^A s = 0 \rightsquigarrow \Phi^*(L) \subset \bar{K}L$).

$\rightarrow$ connect Leaves L invariant $\rightsquigarrow A$ reduces to $U(1)$.

关于 Stability 之 Structure:

Stable-Pairs: (V, Φ): $L \hookrightarrow V$ be Φ-invari, then $\deg L < \frac{1}{2}\deg \Lambda^2 V$.

Example: (i) $V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$, $\Phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in H^0(K\,\mathrm{End}_0 V) \rightsquigarrow \deg(K^{-\frac{1}{2}}) < 0$

$\rightsquigarrow (V, \Phi)$ stable.

(ii) In $\mathbb{P}^1$-case: $V = \mathcal{O}(m) \oplus \mathcal{O}(n)$, $m, n \in \mathbb{Z}$, $K \cong \mathcal{O}(-2)$.

$\Phi \in H^0(K\,\mathrm{End} V) \rightsquigarrow \Phi : \begin{pmatrix} 0 & \theta_1 \\ \theta_2 & 0 \end{pmatrix} \to H^0(\mathbb{P}^1, \underbrace{K}_{\deg = -2}) = 0$ ∴ no!

$m - n \geq 0$

$\theta_1 \in \mathcal{O}(m-n-2)$

$\theta_2 \in \mathcal{O}(n-m-2)$ $\rightsquigarrow \theta_2 = 0$

$\rightsquigarrow \deg \mathcal{O}(m) = m \geq \frac{1}{2}(m+n) = \frac{1}{2}\deg V$

∴ (V, Φ) unstable!

Bertini-Theorem.

★: For $\mathbb{C}^2 \to V \to M$, ∃ generic $\Phi \in H^0(M; K\,\mathrm{End}_0 V)$, with no $L \hookrightarrow V$ is preserved by Φ.

Prop: If (V_1, Φ_1), (V_2, Φ_2) stable pairs with $\Lambda^2 V_1 \cong \Lambda^2 V_2$, if: $\Psi \Phi_1 = \Phi_2 \Psi$.

then: (i) Ψ is an isomorphism: $\Psi: V_1 \longrightarrow V_2$

(ii) If $(V_1, \Phi_1) = (V_2, \Phi_2) \rightsquigarrow \Psi$ is scalar-multiplication.

proof: if not $\rightsquigarrow L_1 \subset \ker \Psi \subset V_1 \rightsquigarrow \Phi_1(L_1) \subset \ker\Psi \rightsquigarrow L_1$ preser $\to \deg L_1 < \frac{1}{2} d$.

take $L_2 \subset \mathrm{im}\Psi \hookrightarrow V_2$, $\Phi_2(L_2) = \Phi_2 \Psi(L_1^*) = \Psi \Phi_1(L_1^*) \hookrightarrow \mathrm{im}\Psi$.

$\Psi : V_1/L_1 \sim L_2 \in H^0((V_1/L_1)^* L_2) \rightsquigarrow \deg L_2 \geq \deg(V_1/L_1) \to$ 矛盾.

(ii) If $(V, \Phi) \xrightarrow{\Psi} (V, \Phi)$ $\underset{\text{decom}}{\text{eigen}}$ $V = L_1 \oplus L_2 \rightsquigarrow \det\Psi$ const $\rightsquigarrow \deg L_1 + \deg L_2 < \deg V$

与 stability 矛盾.

Inverse Direction

$\mathcal{M}_{Dol} \longrightarrow \mathcal{M}_{SD}$

$[\bar{\partial}, \Phi] \longmapsto [\nabla, \Phi]$

Yang-Mills-Theory.

Def: (killing-field)

(M,g) be Rieman-manifold, $X \in \Gamma(TM)$ called killing field if $\mathcal{L}_X g = 0$

Def: (Gradient)

For $f: (M,g) \to \mathbb{R}$: ∇f is defined by: $df_p(v) = g_p(v, \nabla f)$.

For $G \to P \to M$ be principal-bundle, the:

(i) $\mathrm{Ad}P = P \times_G G / (x,g) \sim (x \triangleleft f, f^{-1} g f)$

And we have the action: take $s \in \Gamma(\mathrm{Ad}P)$

$$\mathcal{G}(P): \quad P \longrightarrow P$$
$$x \longmapsto sx.$$

(与 adP 之关系) $\mathrm{Lie}\,\mathrm{Ad}P = \mathrm{ad}P$.

Curvature $\rightsquigarrow$ $F(A) \in \Omega^2(M; \mathrm{ad}P)$

Yang-Mills Functional : $\underset{\text{all-conn}}{\mathcal{A}(P)} \longrightarrow \mathbb{R}$

$\mathcal{A}(P) :: \Omega^1(\mathrm{ad}P)$

$$L(A) = \|F(A)\|^2 = \int_M \langle F(A), F(A) \rangle_{S^2} \,\mathrm{vol}.$$

(Metric-Given by killing-Form)

Uhlenbeck Compactness:

if $A_n \in \Omega^1(\mathrm{ad}P)$, s.t. $F(A_n) < C$, $\exists\, g_n \in \Gamma(\mathrm{Ad}P)$, s.t. $A_n . g_n$ has a weakly convergent subsequence.

$G \to P \downarrow M$ principal G-bundle, G compact Lie grp

$\Rightarrow \exists \langle , \rangle$ on $\mathfrak{g}$ by kill-form.

$\Rightarrow$ on $\mathrm{ad}(P)$.

Atiyah-Bundle: Fancy way describing TP/G.

(i) $TP \xrightarrow{\pi} P$, G-equi.

(ii) $P \times \mathfrak{g} \to TP$, G-equi.

$(p, A) \mapsto \frac{d}{dt}\Big|_{t=0} p e^{tA}$

(iii) $TP \xrightarrow{\pi_*} TM$, G-equi

$\rightsquigarrow 0 \to P \times \mathfrak{g} \to TP \to \pi^{-1}TM \to 0$

$\rightsquigarrow 0 \to \mathrm{ad}P \to At(P) \to TM \to 0$. $(*)$

(I. Biswas): $\pi^*(At(P)) \cong TP$.

Splitting of $(*)$:

$$0 \to \mathrm{ad}P \underset{\omega_A}{\overset{i}{\rightleftarrows}} At(P) \to TM \to 0$$

$\omega_A . \omega_{\underline{A}}$

$\omega_A - \omega_{\underline{A}} : TM \to \mathrm{ad}P. \rightsquigarrow \Omega^1(\mathrm{ad}P)$

$\therefore$ Conn of $G \to P \downarrow M$ $\iff$ Affine Space with: $\Omega^1(\mathrm{ad}P)$.

$\cong \Omega^{0,1}(\mathrm{ad}P \otimes \mathbb{C})$

Moment-Map:

$G \curvearrowright (N, \omega)$, $\mu: N \to \mathfrak{g}^*$, such that:

$$d\mu^X = i_{X^\#}\omega.$$

Example: (Atiyah-Bott)

$\mathcal{A} = \Omega^{0,1}(\mathrm{ad}P \otimes \mathbb{C})$, ω is given by metric:

$$g(\Psi, \Psi) = 2i \int_M \mathrm{Tr}(\Psi^* \Psi).$$

2-moment-Maps: $\mathcal{G} = \mathrm{Ad}\,P$.

(i) $\mathcal{G} \curvearrowright \mathcal{A} = \Omega^1(\mathrm{ad}P) \longrightarrow \mathcal{A}$,

$(g, A) \longmapsto g^{-1}Ag + g^{-1}dg$

$\rightsquigarrow$ moment-map $\underline{\mu_1(A) = F(A)}$

(ii) $\mathcal{G} \curvearrowright \Omega^{1,0}(\mathrm{ad}P \otimes \mathbb{C}) \hookrightarrow \Omega^{1,0}(\mathrm{ad}P \otimes \mathbb{C})$

$(g, \Phi) \longmapsto g^{-1}\Phi g$

$\rightsquigarrow$ moment-map: $\mu_2(\Phi) = [\Phi, \Phi^*]$,

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Now: give $(\bar\partial, \Phi) \in \mathcal{M}_{Dol} \rightsquigarrow$ Give $\mathcal{M}_{SD}$: $[\nabla, \Phi]$.

$\mathcal{M}_{Dol} \rightsquigarrow \Omega^{0,1}(\mathrm{ad}P \otimes \mathbb{C}) \times \Omega^{1,0}(\mathrm{ad}P \otimes \mathbb{C}) / \mathcal{G}$.

$\rightsquigarrow \mu(A, \Phi) = \mu_1 + \mu_2 = \mu_1(A) + \mu_2(\Phi)$ moment-map.

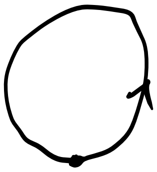
Prop: Pick Any metric on $\mathfrak{g}$ and hermitian-metric, then:

$\nabla \|\mu\|^2 = 0$ only if $\mu(x) = 0$.

(key: $\nabla \|\mu\|^2 = 2\langle \mu, \mathrm{grad}\,\mu \rangle$, $\mathrm{grad}_x \mu \neq 0$ (G's action free))

So our idea is clear:

Give: $(\bar\partial, \Phi) \in \Omega^{0,1}(\mathrm{ad}P\otimes\mathbb{C}) \otimes \Omega^{1,0}(\mathrm{ad}P\otimes\mathbb{C})$，考虑 orbit:



$(\bar\partial,\Phi)$

，在 orbit 上找 $\|\mu\|$ 的 critical-point 即可.

Let $f(A,\Phi) = \int_M \|F(A)+[\Phi,\Phi^*]\|^2$, pick (A_n,Φ_n) be a minimizing sequence of f:

$$\|F(A_n)+[\Phi_n,\Phi_n^*]\|_{L^2}^2 < m.$$

$\Phi \in H^0(K\,\mathrm{End}_0 V) \rightsquigarrow$ Weitzenböck Formula:

$$\int_M \langle d_B'\Phi, d_B'\Phi\rangle = \int_M \langle F(B)\Phi,\Phi\rangle = \underbrace{\int_M \langle F(A)\Phi,\Phi\rangle + (2g-2)\int_M \omega\langle\Phi,\Phi\rangle}_{\geq 0}$$

$\rightsquigarrow \|\Phi_n\|_{L^2}\leq K \rightsquigarrow \|F(A_n)\|_{L^2}\leq K.$

设 $A_n\cdot g_n \to A$; Now: (A,Φ) lies on same orbit

$\Phi_n\cdot g_n \to \Phi$;

Elliptic-Regularity: $d''_{A_n}\Phi_n = 0$

$d''_{AA_1} = d''_{A_nA_1} + \beta_n$ $\left(\beta_n \xrightarrow{L_1^2} 0\right) \rightsquigarrow L_2$-bound on $\{g_n\}$.

$\times$: $\|\psi_n\|_{L^2} = 1$ ✓.

$$\begin{pmatrix} d''_{A_nA_1} g_n = 0 \\ \|g_n\|_{L^2} = 1 \end{pmatrix}$$

$\rightsquigarrow \psi_n \to \psi$

If ψ not iso: $\psi: V\to L \rightsquigarrow L$ invariant, but (V,Φ) stable $\rightsquigarrow \times$.

but: $F(A)+[\Phi,\Phi^*]=0 \longrightarrow (A,\Phi)$ stable $\rightsquigarrow \psi$ iso.

Part2:

Manifold Structure.

$$\mathcal{M}_{SD} :\triangleq \left\{ (A,\Phi) \;\middle|\; \begin{array}{l} F(A)+[\Phi,\Phi^*]=0 \\ d_A''\Phi=0 \end{array} \right\}$$

Key-Point: (i) Linearization $\rightsquigarrow$ (ii) Slice Theory

$\rightsquigarrow$ $(\dot{A},\dot{\Phi}) \in \Omega^1(\mathrm{ad}P) \oplus \Omega^{1,0}(\mathrm{ad}P\otimes\mathbb{C})$

$$\begin{cases} d_A\dot{A}+[\dot{\Phi},\Phi^*]+[\Phi,\dot{\Phi}^*]=0 \\ d_A''\dot{\Phi}+[\dot{A}^{0,1},\Phi]=0 \end{cases}$$

$\mathcal{G} \curvearrowright \Omega^1(\mathrm{ad}P)\oplus\Omega^{1,0}(\mathrm{ad}P\otimes\mathbb{C})$, Let $g(0)=\dot{\psi}$

$g(t)$

$$(g(t))^{\#}\Big|_{(A,\Phi)} = (d_A\dot{\psi},\ [\Phi,\dot{\psi}])$$

看切空间

$$\Omega^0(\mathrm{ad}P) \longrightarrow \Omega^1(M;\mathrm{ad}P)\oplus\Omega^{1,0}(M;\mathrm{ad}P\otimes\mathbb{C}) \longrightarrow \Omega^2(\mathrm{ad}P)\oplus\Omega^2(\mathrm{ad}P\otimes\mathbb{C})$$

$$(\dot{\psi}) \xmapsto{d_1} (d_A\dot{\psi},\ [\Phi,\dot{\psi}])$$

$$(\dot{A},\ \dot{\Phi}) \xmapsto{d_2} (d_A\dot{A}+[\dot{\Phi},\Phi^*]+[\Phi,\dot{\Phi}^*],\ d_A''\dot{\Phi}+[\dot{A}^{0,1},\Phi])$$

$$\rightsquigarrow T_{(A,\Phi)}\mathcal{M}_{SD} = \ker d_2/\mathrm{im}\, d_1$$

Index - Theory

De-Rham - Operator

$$\chi = \operatorname{rank} V \, (2-2g)$$

Dolbeault - Operator

$$\chi = \operatorname{rank}(V)\,(1-g)$$

$$\therefore \chi = 3(2-2g) + 6(1-g) = 12(1-g)$$

$$\dim H^0 - \dim H^1 + \dim H^2 = 12(1-g)$$

$H^0 \equiv 0$

($\because$ 此时意味: $[A,\Phi].\, e^{t\dot\psi} = [A,\Phi]$ infini $\rightsquigarrow$ $e^{t\dot\psi} \equiv \mathrm{id}\ \lambda$ in t very small

$\rightsquigarrow \dot\psi = 0$).

$\underline{H^0 \cong H^2}$ $\rightsquigarrow \dim H^2 = 0$

$$\longrightarrow \dim H^1 = \dim T_{[A,\Phi]} \mathcal{M}_{SD} = 12(g-1).$$

Slice - Thm:

$\mathcal{G} \curvearrowright \mathcal{M}_{SD}$: free + properly $\rightsquigarrow$ $\mathcal{M}_{SD}/\mathcal{G}$ manifold. $:= \mathcal{B}$

$$\mathcal{M}_{SD} \cong \mathcal{M}_{DoL} \longrightarrow \mathrm{Iso}(A,\Phi) = \mathrm{Id}\ !!$$

Riemannian Structure of $\mathcal{M}_{SD}/\mathcal{M}_{DOL}$.

$$0 \to \Omega^0(\mathrm{ad}P) \underset{d_1^*}{\overset{d_1}{\rightleftarrows}} \Omega^1(\mathrm{ad}P) \oplus \Omega^{1,0}(\mathrm{ad}P\otimes\mathbb{C}) \xrightarrow{d_2} \Omega^2(\mathrm{ad}P) \oplus \Omega^{1,1}(\mathrm{ad}P\otimes\mathbb{C}) \longrightarrow 0$$

$T_m\mathcal{M} = \ker d_2 / \mathrm{im}\, d_1$, notice : $(\mathrm{im}\, d_1)^\perp = \ker d_1^* \rightsquigarrow T_m\mathcal{M} = \ker d_2 \cap \ker d_1^*$

take $[\dot{A}, \dot{\Phi}] \in T_m\mathcal{M} \rightsquigarrow$ $d_2(\dot{A}, \dot{\Phi}) = 0$ (in L^2-sense)

$d_1^*(\dot{A}, \dot{\Phi}) = 0$.

$\Omega^{0,1}(\mathrm{ad}P\otimes\mathbb{C})$
$\wr$

在 $\Omega^1(\mathrm{ad}P) \oplus \Omega^{1,0}(\mathrm{ad}P\otimes\mathbb{C})$ 上给 metric :

$$g((\psi, \Phi), (\psi, \Phi)) = 2i\int_M \mathrm{Tr}(\psi^*\psi + \Phi\Phi^*).$$

$\rightsquigarrow$ invariant under unitary-transformation.

<u>Thm</u>:

M compact R.S., $g(M)>1 \rightsquigarrow \mathcal{M}_{SD}$ is complete.

pf:

Let $\gamma: [0,1] \to \mathcal{M}$ geodesic : $\{s\in\mathbb{R} \mid \gamma(s)\in\mathcal{M}_{SD}\}$: $\begin{array}{c}\mathcal{G} \to \overline{\mathcal{M}} \\ \downarrow \\ \mathcal{M}_{SD}\end{array}$ $\rightsquigarrow$ horizantal lifting $\hat{\gamma}$.

$\rightsquigarrow d(\hat{\gamma}(s_n), \hat{\gamma}(s_0)) = d(\gamma(s_n), \gamma(s_0)) \geq \underline{|s_n - s_0|} = d((A_n, \Phi_n), (A_0, \Phi_0))$

$\rightsquigarrow \|A(s_n) - A(s_0)\|_{L^2}^2 + \|\Phi(s_n) - \Phi(s_0)\|_{L^2}^2 \leq M$ as $s_n \to \overline{s}$.

$\rightsquigarrow \|\Phi(s_n)\|_{L^2} \leq M \rightsquigarrow L^2$-bound on $(A(s_n), \Phi(s_n)) \xrightarrow{U.C.} (A, \Phi)$.

#

Hyperkahler Structure.

Motivation: $d_A''\Phi = 0$ 而为 moment-map.

Define a new symplectic-form:

$$\omega((\Psi_1,\Phi_1),(\Psi_2,\Phi_2)) = \int_M \mathrm{Tr}(\Phi_2\Psi_1 - \Phi_1\Psi_2)$$

(i) $d\omega=0$ (不依赖于 base-point)

(ii) non-degenerate ✓.

$$\mathcal{G} := \Gamma(\mathrm{Ad}\,P) \curvearrowright \mathcal{M} \xrightarrow{\mu} \mathfrak{g}^* = \Gamma(\mathrm{ad}\,P) \xrightarrow{\langle X,\cdot\rangle} \mathbb{R}.$$

$[A,\Phi]$ $\qquad\qquad \psi$

$$\psi^\#\big|_{(A,\Phi)} = (d_A''\psi, [\Phi,\psi])$$

$$\rightsquigarrow \omega(\psi^\#, (\dot{A}^{0,1},\dot{\Phi})) = \int_M \mathrm{Tr}(\dot{\Phi}\cdot d_A''\psi - [\Phi,\psi]\dot{A}^{0,1})$$

$$\overset{\text{stokes}}{=\!=} \int_M \mathrm{Tr}\Big((d_A''\dot{\Phi})\psi + \psi[\dot{A}^{0,1},\Phi]\Big)$$

$$= df(\dot{A}^{0,1},\dot{\Phi}), \qquad f = \int_M \mathrm{Tr}(d_A''\Phi\cdot\psi)$$

$$\therefore \mu^\psi = \int_M \mathrm{Tr}(d_A''\Phi\cdot\psi).$$

$$\rightsquigarrow \omega = \omega_2 + i\omega_3$$

$\rightsquigarrow (\omega_1,\omega_2,\omega_3)$ kahler-forms $\rightsquigarrow$ Hyperkähler ✓

$$\rightsquigarrow \mathcal{M} = \bigcap_{i=1}^{3} \mu_i^{-1}(0)/\mathcal{G}$$

Prop: Any compact kahler mfd with a holomorphic symplectic form is hyperkahler.

$(\underline{M, I, \omega})$: ω^n holo-trivi $\underset{\text{Yau}}{\rightsquigarrow}$ $\exists$ kahler metric g with

vanishing Ricci-Tensor. $\rightsquigarrow$ $Hol(M) \subseteq Sp_n$.

Topology of $\mathcal{M}_{SD}$.

(Morse-Theory)

$$S^1 \curvearrowright \mathcal{M}_{SD} \xrightarrow{\mu} u(1)^* \xrightarrow{\langle 1, \cdot \rangle} \mathbb{R}.$$

$$1^{\#}\big|_{(A,\Phi)} = (0, i\Phi) \rightsquigarrow \omega(1^{\#}, Y) = g(I1^{\#}, Y) = g(-(0,\Phi), Y)$$
$$= -\frac{1}{2} dg(\Phi,\Phi)(Y)$$

$\therefore \mu = -\frac{1}{2}\|\Phi\|^2_{L^2}$.

$$\mathcal{M} = \left\{ [A, \Phi] \;\middle|\; \begin{array}{l} F(A) + [\Phi, \Phi^*] = 0 \\ d''_A \Phi = 0 \end{array} \right\} : \quad V \text{ rank } 2 \text{ of odd-degree.}$$

$$\mathcal{M} \xrightarrow{\mu} \mathbb{R}$$
$$(A, \Phi) \longmapsto \|\Phi\|^2_{L^2} = 2i \int_M \operatorname{Tr}(\Phi, \Phi^*)$$

Prop:

(i) μ is proper;

(ii) critical-value 0, $(d-\frac{1}{2})\pi$, $0 < d \leq g-1$;

(iii) $\mu^{-1}(0)$: critical mfd of index 0

$\quad \mathcal{N}$: stable rank 2-bundle. ($\mu^{-1}(0) \cong \mathcal{N}$)

(iv) $\mu^{-1}((d-\frac{1}{2})\pi)$ non-dege critical mfd of index $2(g+2d-2)$.

pf: (i): $\mu^{-1}([0,k])$ compact $\rightsquigarrow$ U.C.

(ii) $d\mu = \omega(\underline{1}^{\#}, \cdot)$ $\therefore$ $d\mu \equiv 0 \Leftrightarrow \underline{1}^{\#} = 0$

$\therefore$ A. $\Phi = 0 \rightsquigarrow F = 0 \rightsquigarrow NS \rightsquigarrow \mathcal{N}$: stable rank 2-bundle.

B. $e^{i\theta} \curvearrowright (A, \Phi) \sim (A, \Phi)$:

$$\begin{cases} g(\theta)^{-1} \Phi g(\theta) = \Phi \\ g(\theta)^{-1} d_A g(\theta) = d_A \end{cases} \rightsquigarrow \nabla^A g'(0) = \nabla^A \rightsquigarrow \text{take Eigen-bundle } L$$

$\rightsquigarrow V = L \oplus L^* \Lambda^2 V \rightsquigarrow \Phi = \begin{pmatrix} 0 & 0 \\ \phi & 0 \end{pmatrix}$ (只须令 $\Phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, $g = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$ 即可)

$\rightsquigarrow 0 = F + [\Phi, \Phi^*] = \begin{pmatrix} F_1 - \phi\phi^* & \\ & -F_1 + \phi\phi^* \end{pmatrix}$, $F(L) = F_1 + \frac{1}{2} F(\Lambda^2 V)$

$$\rightsquigarrow \deg(L) = \frac{i}{2\pi} \int_M F_1 + \frac{1}{2} F(\Lambda^2 V) = \frac{i}{2\pi} \underbrace{\int \mathrm{Tr}\, \Phi \Phi^*} + \frac{1}{2} \deg(V)$$

$\deg(L) = d$.

$$= \frac{1}{\pi} \cdot \mu + \frac{1}{2} \deg(\Lambda^2 V)$$

$\rightsquigarrow \mu = \pi(d - \frac{1}{2})$ / $\deg(L^{-2} \otimes \Lambda^2 V^*) \leq 2g-2 \rightsquigarrow \underline{d \leq g-1}$

$(\because \dim H^0(KL^{-2} \otimes \Lambda^2 V) \geq 1)$

(iii) ✓

(iv) $\Phi = \begin{pmatrix} 0 & 0 \\ \phi & 0 \end{pmatrix}$, $\phi \in H^0(M, KL^{-2} \otimes \Lambda^2 V)$

$\rightsquigarrow V(\phi)$ divisor of degree $2g-2d-1$.

反之: $\forall$ posi-divi of degree $2g-2d-1$ in M

$\hookrightarrow \exists$ holo-line-bundle U of degree $2d$, with section of $U^{-1}K(1)$ vanishes on this divisor.

$\rightsquigarrow$ 一共 2^{2g}-choice of L.

$\mathrm{Pic}(X) \cong \mathrm{Jac}(X) \cong \mathbb{C}^g/\mathbb{Z}^{2g} \cong (\underbrace{S^1 \times \cdots \times S^1}_{2g})$

$\rightsquigarrow L'$ 有 2^{2g}-choice, s.t. $(L')^2 = 1 \rightsquigarrow L \otimes L'$ ✓

ϕ is derter by a const-multi by its divisor,

Considering $\mathbb{C}^* \rhd V \longrightarrow V$

$\lambda \quad (a,b) \longmapsto (\lambda a, \lambda^{-1} b)$

$\therefore \mathbb{C}^*$ takes $\phi \sim \lambda^2 \phi \rightsquigarrow (V, \Phi) \lneq (V, \lambda^2 \Phi)$ gauge

- equivalent.

$\rightsquigarrow \mu^{-1}((d-\frac{1}{2})\pi) \cong$ covering of $(2g-2d-1)$-divisor on M

$\mathcal{M}$ is foliated by Lagrangian submanifold :

$\Omega^{0,1}(\mathrm{ad} \otimes \mathbb{C}) \oplus \Omega^{1,0}(\mathrm{ad}\, P \otimes \mathbb{C}) / \mathcal{G}$

Fix A, let Φ varies $\rightsquigarrow \dim = \dim H^0(K \mathrm{End}_0 V) - \dim \mathrm{Aut}_0 V$

$= 3g-3$

(A, Φ)

且 $\omega|_{\mathcal{M}} \equiv 0$.

$\therefore \forall m \in T_m\mathcal{M} \rightsquigarrow W \cong H^1(k\,\mathrm{End}_0V) / H^0(\mathrm{End}_0V).\Phi$

(isotropic).

Let $m \in \mu^{-1}((d-\frac{1}{2})\pi)$: $V \cong L \oplus L^*(1)$

$$\Phi = \begin{pmatrix} & \\ \phi & \end{pmatrix}$$

$$g = \begin{pmatrix} e^{-i\theta/2} & \\ & e^{i\theta/2} \end{pmatrix}$$

$\mathrm{End}_0V \cong L^{-2}(1) \oplus L^2(-1) \oplus \mathcal{O}$

$g(\theta)$ acts as $(e^{i\theta},\ e^{-i\theta},\ 1)$

$\left(e^{in\theta} \rightsquigarrow \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} \rightsquigarrow n>0 \text{ . positive-weight} \right).$

$\dim H^0(M; kL^2(-1)) = h^1 + (1-g) + 2g-2+2d-1 = g+2d-2.$

$S^1 \curvearrowright T_m \rightsquigarrow T_m.$

看 $S^1 \curvearrowright W^*$: $(e^{i\theta})^*w = e^{i\theta}w.$

$\therefore$ action on W^* becomes: $(1,\ e^{2i\theta},\ e^{i\theta})$

$\rightsquigarrow$ subspace of T_m acts trivially becomes:

$\dim H^0(M, k) - \dim H^0(L^2(1)) + \dim H^0(L^{-2}(1)k) - \dim H^0(\mathcal{O})$

$= 2g-2d-1.$

$\rightsquigarrow$ index $=$ (real-dim of negative space)

$2(g+2d-2).$

Thm:

$\mathcal{M}_{DOL}$

(i) non-compact ;

(ii) connected + simply-connected

(i) highest critical-value : $d-1$,

$\rightsquigarrow$ ind $= 6g-8$ nullity 2

$\therefore$ μ has no maximal at all.

(ii) index :

$2(g+2d-1)$ $\rightsquigarrow$ non-connected $\xrightarrow{\text{Morse}}$ Simply-Conn .

Hitchin - Fibration.

$$\det : \mathcal{M}_{SD} \longrightarrow H^0(M; K^2)$$
$$(A, \Phi) \longmapsto \det \Phi$$

Complete - Hamiltonion - System

$(\mathcal{M}^{2n}, \omega)$ Symplectic, $h : \mathcal{M}^{2n} \to \mathbb{R}^n$ given by

$(h_1, \dots, h_n)$, s.t.

1) X^{h_i} linear - independ ; $\longrightarrow (\mathcal{M}^{2n}, \omega, h)$

2) $\{h_i, h_j\}_\omega = 0$ 称 complete Hamiltonion System.

Prop: $h^{-1}(r)$ lagrangian - submfds for regular r.

$T_m h^{-1}(r) = \mathrm{Span}\{X^{h_1}, \dots, X^{h_n}\} \rightsquigarrow \omega_x(X^{h_i}, X^{h_j}) = 0$.

Arnold - Liouville.

$r \in \mathbb{R}^n$ regular, (M, ω, h) complete Hamilton system $\rightsquigarrow$

$h^{-1}(r)$ embedded n-torus.

Thm:

$\mathcal{M}_{Dol}$, then:

(i) det is proper;

(ii) det is surjective;

(iii) If $\alpha_1 \sim \alpha_{3g-3} \in H^0(M; K^2)^*$ be basis $\rightsquigarrow f_i = \alpha_i \circ \det$ satisfies $\{f_i, f_j\}_\omega = 0$

$H^0(M;K^2)^* = H^1(M, K^{-1})$

(iv) If $q \in H^0(M; K^2)$ with simple zeros $\rightsquigarrow \det^{-1}(q)$ be a Prym-Variety.

pf:

(i) Just Uhlen. Compact 即可.

(iii) 直接算 X^{f_i}:

$$f_i = \alpha_i \circ \det = \int_M \alpha_i \det \Phi = -\int_M \beta_i \operatorname{Tr} \Phi^2$$

$$df_i(\dot{B}, \dot{\Psi}) = \iota_{X_{f_i}} \omega(\dot{B}, \dot{\Psi}), \quad \text{设 } X^{f_i}|_{(A,\Phi)} = (\dot{A}, \dot{\Phi})$$

$$= \omega((\dot{A}, \dot{\Phi}), (\dot{B}, \dot{\Psi}))$$

$$= \int_M \operatorname{Tr}(\dot{\Psi}\dot{A} - \dot{\Phi}\dot{B})$$

$$= -2\int_M \beta_i \operatorname{Tr}(\Phi \dot{\Psi})$$

$\rightsquigarrow$ Poisson Commute.

$$\therefore (\dot{A}, \dot{\Phi}) = (-2\beta_i \Phi, 0) = X^{f_i}$$

$$\therefore \omega(X^{f_i}, X^{f_j}) = 0. \checkmark$$

(iv) $\det^{-1}(q) \cong \{ L \in H^1(M; \mathcal{O}^*) \mid \sigma(L) = L^{-1} \}$ → Prym-Variety

$(\rightarrow)$ 设 $\det \Phi = -q \in H^0(M, K^2)$ simple-zero → non-singular

$\hat{M} = \{ a_x \in K_x \mid a_x^2 = q(x) \in K_x^2 \}$

且有 $\sigma : \tilde{M} \rightarrow \tilde{M} \quad a_x \mapsto -a_x$, $\chi(\hat{M}) = n\chi(M) + \sum_p (e_p - 1)$

$2\hat{g} - 2 = 2(2g-2) + 2(2g-2)$

$\rightarrow \hat{g} = 4g-3$.

$$\begin{array}{ccc} p^*V & & V \\ \downarrow & & \downarrow \\ \tilde{M} & \xrightarrow{p} & M \end{array} \rightsquigarrow \sqrt{q} \in H^0(\hat{M}; p^*K_M)$$

$\rightsquigarrow$ 2 rank 1 subbundle of p^*V

$L_1 \subseteq \ker(\tilde{\Phi} + \sqrt{q}\, Id)$, $L_2 \subseteq \ker(\tilde{\Phi} - \sqrt{q}\, Id)$.

$L_1 = L_2$ at $q|_{zero} = 0$.

$\rightsquigarrow L_1 \otimes L_2 \cong p^*(\Lambda^2 V) \otimes p^* K_M^{-1} \rightsquigarrow \sigma(L_1) \cong L_1^* \otimes p^*(\Lambda^2 V) \otimes p^* K_M^{-1}$

$s_1 \otimes s_2 \mapsto (s_1 \wedge s_2) \otimes (\sqrt{q})^{-1}$.

$p^*(\Lambda^2 V \otimes K_M^{-1}) \cong L_0 \otimes \sigma(L_0)$

$\rightsquigarrow \sigma(\underbrace{L_1 \otimes L_0^{-1}}_{L}) \cong \underline{L_1^{-1} \otimes L_0}$. $\rightarrow L^* \rightsquigarrow L \in \mathrm{Prym}(\hat{M} \xrightarrow{\pi} M)$

反之

Abelian Variety and Prym Variety

$\pi: \hat{C} \to C$ double covering

$\rightsquigarrow Nm_\pi: J\hat{C} \to JC$

$[\sum n_i p_i] \mapsto [\sum n_i \pi(p_i)]$

$\rightsquigarrow$ Prym - Varie $\#P$ $[\ker Nm_\pi]^\circ$

Def: (Abelian - Variety)

Compact Complex Lie group. X, $V = Lie(X) = T_{id}X$.

Prop: (a) X commutative;

(b) exp is surjective;

(c) X torus

(d) $H^i(X, \mathbb{Z}) \cong Hom(\Lambda^i M, \mathbb{Z})$.

pf: (a) Ad holo $\rightsquigarrow$ const $\rightsquigarrow$ V commu $\longrightarrow$ X commu.

(b) $\exp: V \to X$, $U = \exp(V)$ subgrp + open $\rightsquigarrow$ X/U discre

+ conn $\rightsquigarrow$ point $\to$ $X = U$. $\rightsquigarrow$ $\ker(\exp) = M$ lattice:

$$V/M \cong X$$

(c)

(d) $\Lambda^r H^1(X, \mathbb{Z}) \overset{\text{cup-product.}}{\cong} H^r(X, \mathbb{Z})$

$\quad\;\; \wr\!\wr$ $\qquad\qquad\qquad \Lambda^r X^* \times \Lambda^r X \to \mathbb{Z}$

$Hom(\Lambda^r X, \mathbb{Z})$ by: $(f_1 \wedge \cdots \wedge f_r, e_1 \wedge \cdots \wedge e_r) \longmapsto \det(f_i(e_j))$

Thm: (Appell – Hormbert)

$$0 \longrightarrow \mathcal{P}^{\circ} \longrightarrow \mathcal{P} \longrightarrow \mathcal{R} \longrightarrow 0$$

$\mathcal{P}^{\circ} = \mathrm{Hom}(M, U(1))$

$\mathcal{P} = \{(H,\alpha) \mid H \in \mathcal{R},\ \alpha: M \to U(1),\ \text{s.t. } \alpha(x+y) = e^{i\pi E(x,y)}\alpha(x)\alpha(y)\}$

$\mathcal{R} = \{\text{Rie-form on } V\}$ (hermi-form H s.t. $E = \mathrm{im} H$ takes value in M)

$$(H,\alpha) \rightsquigarrow L(H,\alpha) \rightsquigarrow V\times\mathbb{C} \big/ (v,z) \sim (v+\lambda,\ \alpha(\lambda)e^{\pi H(v,\lambda)+\frac{\pi}{2}H(\lambda,\lambda)}\cdot z),$$

$L(H,\alpha)$ ample $\Leftrightarrow$ H positive.

Jacobian – Variety.

X: Rieman–Surface, $g(X) = g$

Def: $\mathrm{Jac}(X) = H^{\circ}(X, K_X)^{*} / H_1(X,\mathbb{Z}) = H^1(X,\mathcal{O}) / H_1(X,\mathbb{Z})$

Prop: $T_0(\mathrm{Jac}(X)) = V^{*} = H^1(X,\mathcal{O})$

$H^{\circ}(\mathrm{Jac}(X), \Omega^1) \cong V = H^{\circ}(X, K_X)$,

$H_1(\mathrm{Jac}(X), \mathbb{Z}) = H_1(X,\mathbb{Z})$

Prop: $\exists$ natural: $\mathrm{Jac}(X) \cong \mathrm{Pic}^{\circ}(X)$

$$0 \longrightarrow H^1(X,\mathcal{O}) / H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}^{*}) \longrightarrow H^2(X,\mathbb{Z}) \longrightarrow 0$$

$H^1(X,\mathcal{O}^{*}) = \mathrm{Pic}$, $\quad \mapsto \xrightarrow{\deg}$

$(0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0)$

Real Structure.

$$S^1 \curvearrowright \mathcal{M} \longrightarrow \mathcal{M}$$
$$e^{i\theta} \quad (A, \Phi) \longmapsto (A, e^{i\theta}\Phi)$$

考虑 $\sigma = -1$'s action: $\begin{cases} \sigma^* \omega_2 = -\omega_2 \\ \sigma^* \omega_3 = -\omega_3 \end{cases}$

Hyperkahler: $J: (A, B) \longmapsto (iB^*, -iA^*)$

$\therefore \sigma^* J = -J \rightsquigarrow$ anti-holo

$$\sigma^*(\omega_1 + i\omega_3) = \omega_1 - i\omega_3$$

★ Real-Points:

$$\sigma(A, \Phi) = (A, \Phi)$$

(i) $\Phi = 0 \rightsquigarrow F(A) = 0 \qquad \mathcal{M}_0$

(ii) $(A, \Phi) \sim (A, -\Phi)$, $\therefore \exists g \in \Gamma(\mathrm{Ad}P)$

$\begin{cases} A \cdot g = A \\ \Phi \cdot g = -\Phi \end{cases} \longrightarrow \quad V = L \oplus L^* \Lambda^2 V$

$$\sigma = \begin{pmatrix} i & \\ & -i \end{pmatrix} \pm 1 \qquad \text{(stability)}$$

$\rightsquigarrow \Phi = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$. $\mathcal{M}_{2d-1}$ $(1 \le d \le g-1)$

Thm:

(i) $\mathcal{M}_0$ stable rank 2 bundle

(ii) $\mathcal{M}_{2d-1}$ vector bundle.

$$\begin{array}{c} \mathcal{M}_{2d-1} \\ \downarrow \\ S^{2g-2d-1}M \end{array}$$

pf: (i) ✓

(ii): $\mathcal{M}_{2d-1}$

$\Phi = \begin{pmatrix} & b \\ c & \end{pmatrix}$, $b \in H^0(KL^2\Lambda^2V^*)$

$c \in H^0(KL^{-2}\Lambda^2V)$

$\underline{c \neq 0}$ (stabi) $\rightsquigarrow$ a Point in $\mathbb{P}(H^0(M, KL^{-2}\otimes\Lambda^2V))$

$\rightsquigarrow x \in S^{2g-2d-1}M$

Fiber over $x \rightsquigarrow b$.

$\dim H^0(KL^2\Lambda^2V^*) = g+2d-2$.

Teichmüller Space

$\mathcal{M}_{2g-2}$ (real component of $\deg L = g-1$).

$W_2(p) = 0$

$\rightsquigarrow V = L \oplus L^*$, $\Phi = \begin{pmatrix} & a \\ b & \end{pmatrix}$, $b \neq 0 \rightsquigarrow L^2 \cong K \rightsquigarrow b = 1$

$\rightsquigarrow V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$, $a \in H^0(K \otimes K^{\frac{1}{2}} \otimes K^{\frac{1}{2}}) = H^0(M, K^2)$

$\rightsquigarrow \mathcal{M}_{2g-2} \cong H^0(M, K^2) \cong \mathbb{C}^{3g-3}$

$\therefore F + [\Phi, \Phi^*] = 0 \rightsquigarrow F_1 = 2(1 - \|a\|^2)\,\omega.$

(Vortex-equation).

---

check: $g = h\,dz\,d\bar{z} \rightsquigarrow (K_M, \underset{\substack{\uparrow \\ \text{Levi-Civita}}}{\nabla}) \rightsquigarrow$ on $K^{\frac{1}{2}}$ $F = \begin{pmatrix} \frac{1}{2}F_1 & \\ & -\frac{1}{2}F_1 \end{pmatrix}$

$$\Phi = \begin{pmatrix} & a \\ 1 & \end{pmatrix} = \begin{pmatrix} & \frac{a}{dz} \\ \frac{1}{dz} & \end{pmatrix} dz \rightsquigarrow \Phi^* = \begin{pmatrix} & h\,dz \\ \frac{a}{h\,dz} & \end{pmatrix} d\bar{z}$$

$$\rightsquigarrow [\Phi, \Phi^*] = \left(\begin{pmatrix} & \frac{a}{dz} \\ \frac{1}{dz} & \end{pmatrix} \begin{pmatrix} & h\,dz \\ \frac{a}{h\,dz} & \end{pmatrix} - \begin{pmatrix} & h\,dz \\ \frac{a}{h\,dz} & \end{pmatrix} \begin{pmatrix} & \frac{a}{dz} \\ \frac{1}{dz} & \end{pmatrix} \right) dz\,d\bar{z}$$

$$= \begin{pmatrix} a^2 - h & \\ & h - a^2 \end{pmatrix} dz\,d\bar{z}$$

$$\rightsquigarrow \frac{1}{2} F_1 = (1 - \|a\|^2)\,\omega \rightsquigarrow F_1 = 2(1 - \|a\|^2)\,\omega.$$

Thm:

M R.S., $g(M) \geq 2$, $V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$, $\Phi = \begin{pmatrix} & a \\ 1 & \end{pmatrix}$, h origial metric

(1) $\forall a \in H^0(M, K^2) \rightsquigarrow$

$$\hat{h} = a + (h + \frac{a\bar{a}}{h}) + \bar{a} \in \Omega^0(M; S^2T^* \otimes \mathbb{C})$$

gives a Riemann-Metric on M.

(2) $K_G(\hat{h}) = -4$.

(3) All const curvature -4 on $M \cong \hat{h}$ (some a)

pf: (1) 易见: $\hat{h}$ is real

$\therefore$ 只须 non-degenerate: $\frac{1}{4}(h + \frac{a\bar{a}}{h})^2 - a\bar{a} > 0$

即: $(h - \frac{a\bar{a}}{h})^2 > 0 \iff (1 - \|a\|^2)\omega > 0$

$\therefore$ 只须: $\|a\|^2 < 1$ (严格小于 1)

$d''\langle d'a, a\rangle = \langle Fa, a\rangle - \langle d'a, d'a\rangle$.

$$\rightsquigarrow d''d'\|a\|^2 \overset{(dzd\bar{z})}{=} 4(1-\|a\|^2)(\|a\|^2\omega) \overset{(dzd\bar{z})}{-} \|d'a\|^2\omega \quad (dzd\bar{z})$$

$$\mathcal{L}(1-\|a\|^2) = -|\partial a|^2 < 0$$

$$\mathcal{L} = \frac{1}{2h}(\partial_{xx} + \partial_{yy}) - 4|a|^2$$

Strong-Maximal-Principal:

$L = a^{ij}D_iD_j + b_iD_i - c$: (i) $\{a_{ij}\}$ positive. (ii) $c \geq 0$.

$\Rightarrow$ if $Lu \equiv 0$

$\Longrightarrow u > 0$ or u constant.

(2) $$\hat{h} = h(\underbrace{dz + \frac{\bar{q}\,d\bar{z}}{h}}_{u})(d\bar{z} + \frac{q\,dz}{h})$$

$$u = dz(1 + \bar{q}\,d\bar{z}/\text{vol})$$

$\hat{h}(u,u) = h^{-1}$, 记 $\hat{\nabla} u = u \otimes \theta \to$

$$du = \theta \wedge u = d(dz + (\bar{q}/h)\,d\bar{z})$$
$$= -\frac{\bar{q}}{h^2}\frac{\partial h}{\partial z}\,dz \wedge d\bar{z}$$

$$d(\hat{h}(u,u)) = d(h^{-1}) = (\theta + \bar{\theta})h^{-1}$$

$$\begin{cases} -\frac{1}{h}\,dh = \theta + \bar{\theta} \\ -\theta^{0,1} + \frac{\bar{q}}{h}\theta^{1,0} = -\frac{\bar{q}}{h^2}\frac{\partial h}{\partial z} \end{cases}$$

$$\longrightarrow \frac{h}{\bar{q}}\theta^{0,1} = -\bar{\theta}^{0,1} \rightsquigarrow q\bar{q} = h^2$$

$h>0 \rightsquigarrow \theta^{1,0} = -h^{-1}\frac{\partial h}{\partial z}$

$\rightsquigarrow \hat{F} = -d''d'\log h$

$= 2(h - \frac{a\bar{a}}{h})\,dz\,d\bar{z}$

$\hat{\omega} = h(u)\wedge\bar{u} = (h - \frac{a\bar{a}}{h})\,dz\,d\bar{z}$

$\cancel{= 2\hat{\omega}}.$

$\Rightarrow \hat{F} = 2\hat{\omega} \rightsquigarrow K_G = -4.$