Basic Notations.

1.1 (Coherent Sheaf)

A sheaf of $\mathcal{O}_X$-module is called coherent if:

① $\forall x \in X$, $\exists\, x \in U \overset{\circ}{\subset} X$, s.t. $\mathcal{F}|_U$ is finite-generated;

② If $\varphi: \mathcal{O}_U^n \to \mathcal{F}(U)$, then $\ker\varphi$ is fini. gene $\mathcal{O}_X$-module.

2. Cartier-Divisor

Let D be a divisor on X, $D = \sum_{i=1}^{s} n_i V_i$, $n_i > 0$, $s \geq 1$

$\rightsquigarrow$ D is same as $\mathrm{Spec}(\mathcal{O}_X / I_D)$. (functions vanish on D)

($\forall p \in D$, $\exists$ local neigh U with coor $(z^1, \dots, z^k)$, s.t. $D = \{z^1 = \dots = z^l = 0\}$

$\therefore \mathcal{O}_X / I_D \cong k[z^1, \dots, z^l] \rightsquigarrow \mathrm{Spec}(\mathcal{O}_X / I_D) \cong (z^1, \dots, z^l) := \underset{D}{\overset{\shortparallel}{U}}$ which is what we want.)

3. Deformation-Theory of Vector Bundle.

Let $\mathbb{C}^r \to E \to X$ be vector bundle with transition function $\{g_{ij}\}$, deformation means changing the transition function:

$X_\varepsilon = X \times \mathrm{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2)$

If $\boxed{g_{ij}^{\varepsilon} = g_{ij} + \varepsilon h_{ij}}$ be the new functions $\underset{\text{s.t.}}{\rightsquigarrow}$ $\begin{cases} g_{ij}^{\varepsilon} g_{jk}^{\varepsilon} g_{ki}^{\varepsilon} = 0 \\ \varepsilon^2 = 0 \end{cases}$

$\rightsquigarrow h_{ij} g_{jk} g_{ki} + g_{ij} h_{jk} g_{ki} + g_{ij} g_{jk} h_{ki} = 0 \rightsquigarrow \{h_{ij}\} \in H^1(X, \mathrm{End}\, E)$

$\rightsquigarrow$ Generally, the deformation gives family of bundles E_ε on $X \times \mathrm{Spec}(k[\varepsilon]/\varepsilon^2)$ which restriction to $X \times \{0\}$ is E.

Dolbeault-thm: $\{h_{ij}\} \longrightarrow$ ( )

$H^1(X, \mathrm{End}\, E) \cong H^1_{\bar\partial}(X, \mathrm{End}\, E)$

4. Čech Resolution

Let X be a complex manifold, $\{\mathcal{F}^\bullet\}$ be a complex of sheaves on X,

(Voisin)

$$\mathcal{F}^0 \xrightarrow{d^0} \mathcal{F}^1 \xrightarrow{d^1} \mathcal{F}^2 \xrightarrow{d^2} \mathcal{F}^3 \xrightarrow{d^3} \cdots \longrightarrow X$$

$\Omega((A_1, B_1), (A_2, B_2))$

$= \int_\Sigma \mathrm{tr}(A_1 B_2 - A_2 B_1)$

Remark: $H^1(X, \mathrm{End}\,E) \overset{?}{\neq} H^1_{\bar\partial}(X, \mathrm{End}\,E)$

Notation: let $\mathcal{Z}(\mathrm{End}\,E)$ be holo-section of $\mathrm{End}\,E$.

$$\begin{cases} 0 \to \mathcal{Z}(\mathrm{End}\,E) \to \mathrm{End}\,E \xrightarrow{\bar\partial} \mathcal{Z}^{0,1}_{\bar\partial}(\mathrm{End}\,E) \to 0 \\ 0 \to \mathcal{Z}^{0,1}_{\bar\partial}(\mathrm{End}\,E) \to \Omega^{0,1}(\mathrm{End}\,E) \to 0 \end{cases}$$

then:

$$0 \to H^0(X, \mathcal{Z}(\mathrm{End}\,E)) \to H^0(X, \mathrm{End}\,E) \to H^0(X, \mathcal{Z}^{0,1}_{\bar\partial}(\mathrm{End}\,E))$$

$$\to H^1(X, \mathcal{Z}(\mathrm{End}\,E)) \to 0.$$

(As $\mathrm{End}\,E$ is fine.)

$$\therefore H^1(X, \mathcal{Z}(\mathrm{End}\,E)) \cong H^0(X, \mathcal{Z}^{0,1}_{\bar\partial}(\mathrm{End}\,E)) / \mathrm{Im}$$

$$= H^{0,1}_{\bar\partial}(X, \mathrm{End}\,E)$$

$$= H^1_{\bar\partial}(X, \mathrm{End}\,E)$$

$$\underset{\text{Serre-Duality}}{\cong} H^0(X, K_X \, \mathrm{End}\,E)$$

then we can Compute hypercohomology of X in this way :

$$\begin{tikzcd}
\vdots & & & & \\
\mathcal{F}^2 \arrow[r] & C^0\mathcal{F}^2 \arrow[r] & C^1\mathcal{F}^2 \arrow[r] & C^2\mathcal{F}^2 & \cdots \\
\mathcal{F}^1 \arrow[r] \arrow[u, "d^1"] & C^0\mathcal{F}^1 \arrow[r] \arrow[u] & C^1\mathcal{F}^1 \arrow[r] \arrow[u] & C^2\mathcal{F}^1 \arrow[u] & \cdots \\
\mathcal{F}^0 \arrow[r] \arrow[u, "d^0"] & C^0\mathcal{F}^0 \arrow[r] \arrow[u] & C^1\mathcal{F}^0 \arrow[r] \arrow[u] & C^2\mathcal{F}^0 \arrow[u] & \cdots
\end{tikzcd}$$

$= C^k(\{\mathcal{U}\}, \mathcal{F}^j)$

where $C^k\mathcal{F}^j := \bigoplus_{i_0 < \cdots < i_k} \mathcal{F}^j(\mathcal{U}_{i_0 \cdots i_k})$

$V \subset X$, $P(V) := C^k(\{\mathcal{U} \cap V\}, \mathcal{F}^j)$

5. Dolbeault Resolution

Let $\theta \in \Omega^{1,0}(\mathrm{End}\, E)$, D is a complex given by

$$E^0 \xrightarrow{\theta} \underset{(E^1)}{\Omega^1(E)} \longrightarrow \Omega^2(E) \longrightarrow \cdots$$

$\theta \in \Omega^1(E)$

$\rightsquigarrow \mathbb{H}^k(D)$'s computation :

$$\begin{tikzcd}
E^0 \arrow[r, "\theta"] \arrow[d] & E^1 \arrow[r, "\theta"] \arrow[d] & E^2 \arrow[r] \arrow[d] & \cdots \\
\Omega^{0,0}(E) \arrow[r] \arrow[d, "\bar\partial"] & \Omega^{1,0}(E) \arrow[r] \arrow[d, "\bar\partial"] & \Omega^{2,0}(E) \arrow[r] \arrow[d, "\bar\partial"] & \cdots \\
\Omega^{0,1}(E) \arrow[r] \arrow[d] & \Omega^{1,1}(E) \arrow[r] \arrow[d] & \Omega^{2,1}(E) \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

$\bar\partial \circ \bar\partial = 0$

$\eta = \bar\partial \gamma$

Def : (Resolution)

A resolution of a sheaf $\mathcal{F}$ on topo. space X is an exact seq of sheaves : $0 \to \mathcal{F} \to I^0 \to I^1 \to \cdots\cdots$ where each I^n is an injective sheaf.

(in derived means.)

二. Set-Up

Def: $\mathcal{M}_H(r,d)$:= moduli space of semi-stable framed Higgs bundles on X of rank r and degree d.

X compact R.S., $g_X :=$ genus(X), $D=\sum_{i=1}^{s} n_i x_i$, $n_i \geq 1$, $\{x_i\}_{i=1}^{s} \in X$

$D=p$ $\quad \delta: E_p \to \mathbb{C}^r$

Def: (frame)

A frame on E is an isomorphism $\delta: E_D \to \mathbb{C}^r_D$, a vector bundle with a frame (E,δ) called a framed bundle.

Def: "rigidity" , (E,ϕ)

A Higgs field on a framed bundle (E,δ) is $\Phi \in H^0(\mathrm{End}E \otimes K_X(D))$, a framed higgs bundle is a triple (E,δ,θ).

(Framing gives rigidity for moduli-problem)

Prop: Let (E,θ), (E',θ') semi-stable Higgs bundle with $\mu(E)=\mu(E')$, if $E \xrightarrow{h} E'$ s.t.

- $\theta' \circ h = (h \otimes \mathrm{Id}_{K_X(D)}) \circ \theta$
- $\exists\, x_0 \in X$, s.t. $h(x_0)=0$

then: $h \equiv 0$

$E/\ker h \cong \mathrm{Im}\, h$

Pf: $E \xrightarrow{h} \underset{\underset{I}{\parallel}}{h(E)} \oplus E'/I \overset{丛代}{\rightsquigarrow} I' \oplus E'/I'$, $\mu(I') \leq \mu(E') = \mu(I)$

$\rightsquigarrow I = I' \rightsquigarrow$ 与 $h(x_0)=0$ 矛盾.

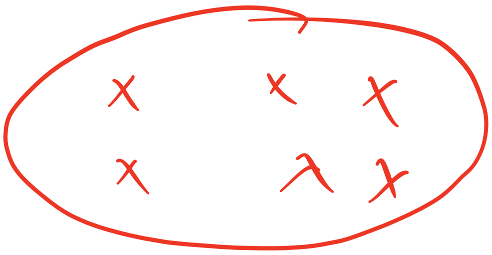
Def: (Gauge-Equivalent Condition)

Let (E,δ,θ), (E',δ',θ') framed higgs bundle:

称 gauge-equiva if ① $\theta' = \theta \circ h$, ② $h \circ \delta = \delta'$
$h: E \to E'$

$$\begin{tikzcd} E_D \arrow[d, "h"] \arrow[r, bend right, "\delta^{-1}"'] & \mathcal{O}_D^{\oplus r} \arrow[l, "\delta"'] \\ E'_D \arrow[r, bend left, "\delta'"] & \mathcal{O}_D^{\oplus r} \arrow[l, "\delta'"] \end{tikzcd}$$

(D,g)



X

$\delta : E_D| \longrightarrow \mathbb{C}_D^r$

$\delta' : E_D \longmapsto \mathbb{C}_D^r$

if $\delta' \circ \delta^{-1} \in H_x$ $\qquad H_x \subsetneq GL(r, \mathbb{C})$

$H = \{H_{x_1}, H_{x_2}, \dots\}$

$[E, \delta, \boxed{H}]$ $\qquad \nabla$ $\quad Res_x(\nabla) \in [\delta]\left(\boxed{\mathfrak{h}_x^{\perp}}\right)$

example : "trivial"

"diagonal"

$(E, \theta) \longrightarrow$ "parabolic" ?

h

Coro: A semi-stable framed higgs bundle (E, δ, θ) doesn't admit any non-trivial automorphism. $\theta \in H^0(X, \mathrm{End}\,E \otimes K_X(D))$

Pf: $E \xrightarrow{h} E$, $\boxed{\begin{array}{ccc} E_D & \xleftarrow{\delta} & \mathcal{O}_D^{\oplus r} \\ {\scriptstyle h}\downarrow & & \\ E_D & \xleftarrow{\delta} & \mathcal{O}_D^{\oplus r} \end{array}}$, $\theta . h = \theta \rightsquigarrow$ Consider $h - \mathrm{Id}_E$

$\boxed{h \circ \delta = \delta}$

$\rightsquigarrow$

$\rightsquigarrow (h - \mathrm{Id}_E)|_D \equiv 0 \rightsquigarrow h \equiv \mathrm{Id}_E.$ #

三. Tangent Space

Infinitesimal - Deformation:

Lemma: The infinitesimal-deformation of framed bundle (E, δ) is $H^1(X, \mathrm{End}(E)(-D))$. $\cong H^1_{\bar\partial}(X, \mathrm{End}(E)(-D))$

Pf: infinitesimal of $E \rightsquigarrow H^1(X, \mathrm{End}\,E)$, As δ should be preserved: if $\{\mathrm{Id} + \varepsilon s_{jk}\}$ new-deformation $\rightsquigarrow$ we shouldn't give new identi $\rightsquigarrow s_{jk}(D) \equiv 0 \rightsquigarrow \{s_{jk}\} \in \mathrm{End}\,E(-D)$

$\delta : E|_D \to \mathbb{C}^r$ $\quad$ $\delta : \cancel{E|_D} \to \mathbb{C}^r$

Def ($\bar\delta$-stability)

$E|_D$, δ

D effective-divisor, $\bar\delta = \deg(D)$. then (E, δ) is $\bar\delta$-stabili

$\Leftrightarrow \forall F \subset E$. s.t. $\mu(F) - \bar\delta \frac{1}{rk(F)} < \mu(E) - \bar\delta \frac{1}{rk(E)}$.

~~Thm~~: The infinitesimal-deforma of (E, δ, θ) can be identified by the complex $\mathcal{C}_\bullet$: $\mathrm{End}\,E(-D) \xrightarrow{f_\theta} \mathrm{End}\,E \otimes K_X(D)$

$$s \longmapsto \theta \circ s - s \circ \theta$$

, i.e. $T_{(E,\delta,\theta)} \mathcal{M}_H(r,d) \cong \mathbb{H}^1(\mathcal{C}_\bullet)$

$\underline{\mathbb{CP}^1}$: $\begin{cases} \mathbb{C} : U \\ \mathbb{C}_\infty - \{0\} : V \end{cases}$ $\quad U \cap V = \mathbb{C}$

$H^1(X, \mathrm{End}(E)(-D))$:

$H^1(\mathbb{CP}^1, \mathrm{End}\, E(-D))$:

$0 \to C^0(U) \oplus C^0(V) \to C^1(U \cap V) \to 0$

$(f, g) \longmapsto (g - f)$

$(1 - \varepsilon h_{ij})(\theta + \varepsilon \gamma_j)(1 + \varepsilon h_{ij}) = \theta + \varepsilon \gamma_i$

$(\theta + \varepsilon \gamma_j - \varepsilon h_{ij}\theta)(1 + \varepsilon h_{ij}) = \cancel{\theta} + \varepsilon \theta h_{ij} + \varepsilon \gamma_j - \varepsilon h_{ij} \theta$

$= \cancel{\theta} + \varepsilon \gamma_i$

$\to [h_{ij}, \theta] = \gamma_i - \gamma_j$

pf: The complex is given by

$$\begin{tikzcd}
\mathrm{End}\,E \otimes K_X(D) \arrow[r] & C^0(\mathrm{End}\,E \otimes K_X(D)) \arrow[r] & C^1(\mathrm{End}\,E \otimes K_X(D)) \arrow[r] & \cdots \\
\mathrm{End}\,E(-D) \arrow[r] \arrow[u, "f_\theta"] & C^0(\mathrm{End}\,E(-D)) \arrow[r] \arrow[u, "\{\gamma_i\}"] & C^1(\mathrm{End}\,E(-D)) \arrow[r] \arrow[u, "\{h_{ij}\}"] & \cdots
\end{tikzcd}$$

$U_i \quad U_j$

$\mathbb{H}^1(\mathcal{C}_\cdot) = H^1(\mathrm{Tot}(C^p\mathcal{F}^q))$ which is given by a pair

(h_{ij}, γ_k), compatability means that

v.b. $\quad$ higgs field

$$(1+\varepsilon h_{ij})^{-1}(\theta+\varepsilon\gamma_j)\big|_{U_j}(1+\varepsilon h_{ij}) = \theta+\varepsilon\gamma_i\big|_{U_i}$$

$$\begin{cases} \theta|_{U_i} = \theta_i \\ \theta|_{U_j} = \theta_j \end{cases} : \mathrm{End}(E)\otimes K_X(D)$$

$E\ \{t_{ij}\}$: $t_{ij}\theta_j = \theta_i$

i.e. $\theta h_{ij} - h_{ij}\theta = \gamma_i - \gamma_j \quad \rightsquigarrow f_\theta(h_{ij}) = \gamma_i - \gamma_j \to (h_{ij}, \gamma_k) \in \mathbb{H}^1(\mathcal{C}_\cdot)$

$t_{ij}\cdot\theta_j\cdot t_{ij}^{-1} = \theta_i$

Remark: This deformation is unobstructed ? $(\theta \in H^0(\mathrm{End}\,E \times (-D)))$

frame 'semi-stable higgs'

pf: $\mathbb{H}^0(\mathcal{C}_\cdot) = 0$ by rigidity, $\longrightarrow \mathbb{H}^2(\mathcal{C}_\cdot) = 0$ by isomor duality $\mathcal{C}_\cdot \cong \mathcal{C}^\vee$

$$\left\{\frac{\mathcal{O}\oplus\mathcal{O}}{\mathcal{O}(-1)\oplus\mathcal{O}(1)}\right. \qquad \mathbb{C}^r \to X\times\mathbb{C}^r \to X \quad \rightsquigarrow \begin{cases} H^1(X, \mathrm{End}\,x(-D)) \\ H^0(X, K\,\mathrm{End}\,x) \end{cases}$$

$\{0, 1, \infty\}$ with $\int_{S^1(1,\mathbb{C})}$ coordinate, #

Lemma.

Let $\mathcal{M}(r,d) :=$ moduli of stable framed bundle of rank r degree d

then $\exists$ tautological - Embedding:

$$\iota^*: T^*\mathcal{M}(r,d) \hookrightarrow \mathcal{M}_H(r,d)$$

pf: $T_{(E,S)}\mathcal{M}(r,d) = H^1(\mathrm{End}\,E(-D)) \rightsquigarrow T^*_{(E,S)}\mathcal{M}(r,d) = H^1(\mathrm{End}\,E(-D))^*$

$$= \underbrace{H^1(\mathrm{End}\,E(D))\otimes K_X}$$

framed higgs field.

$\therefore$ it maps $\Phi \longmapsto [E, S, \Phi]$

#

$$\Theta = \left(\quad \right)\frac{dz}{z} + \left(\quad \right)\frac{dz}{z-1}, \qquad \delta : SL(2;\mathbb{C}) \times SL(2;\mathbb{C}) \times SL(2;\mathbb{C})$$

$$\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$$

$$\mathcal{G} : H^0(\mathrm{End}\,E(-0)) : SL(2;\mathbb{C}) \otimes (0,1)$$

$\rightsquigarrow (\Theta, \delta)/\mathcal{G}$

$\rightsquigarrow$ Tangent Space

$\rightsquigarrow$ Symplectic Form

Lemma: Let $n = \sum_{i=1}^{s} n_i$, then $\dim \mathcal{M}_H(r,d) = 2r^2(g_X + n - 1)$

pf: We use Dolbeault resolution to compute:

$$\dim \mathcal{M}_H(r,d) = \dim \mathbb{H}^1(\mathcal{C}_\bullet) = -\chi(\mathcal{C}_\bullet) = -\chi(\mathrm{End}\,E(-D)) + \chi(\mathrm{End}\,E \otimes K_X D)$$

by Dolbeault-resolution + Riemann-Roch

$$-\chi(\mathrm{End}\,E(-D)) \overset{\text{Serre}}{\underset{\text{Duality}}{=}} -\chi(\mathrm{End}\,E\, K_X(D)) = r^2(g_X + n - 1)$$

$$\therefore \dim \mathcal{M}_H(r,d) = 2r^2(g_X + n - 1)$$

#

IV. Symplectic Geometry

The key point is to give the explicit expression for $\mathbb{H}^1(\mathcal{C}^\bullet) \simeq \mathbb{H}^1(\mathcal{C}_\bullet)^*$.

$$\begin{array}{ccccc}
(\mathcal{C}_\bullet \otimes \mathcal{C}_\bullet)_0 & \xrightarrow{f_0 \otimes Id + Id \otimes f_0} & (\mathcal{C}_\bullet \otimes \mathcal{C}_\bullet)_1 & \xrightarrow{Id \otimes f_0 - f_0 \otimes Id} & (\mathcal{C}_\bullet \otimes \mathcal{C}_\bullet)_2 \\
\| & & \| & & \| \\
\mathrm{End}\,E(-D) \otimes \mathrm{End}\,E(-D) & & (\mathrm{End}\,E \otimes K_X(D) \otimes \mathrm{End}\,E(-D)) & & (\mathrm{End}\,E\,K_X(D)) \otimes (\mathrm{End}\,E \otimes K_X(D)) \\
 & & \oplus & & \\
 & & (\mathrm{End}\,E \otimes K_X(D) \otimes \mathrm{End}\,E(-D)) & &
\end{array}$$

Define the map:

$$\rho: (\mathrm{End}\,E \otimes K_X(D) \otimes \mathrm{End}(-D)) \oplus (\mathrm{End}\,E \otimes K_X(D) \otimes \mathrm{End}\,E(-D)) \longrightarrow K_X$$

$$((a_1 \otimes b_1), (a_2 \otimes b_2)) \longmapsto \mathrm{Tr}(a_1 \circ b_1 + a_2 \circ b_2)$$

this gives the homomorphism of complexes:

$$\begin{array}{ccccc}
(\mathcal{C}_\bullet \otimes \mathcal{C}_\bullet)_0 & \longrightarrow & (\mathcal{C}_\bullet \otimes \mathcal{C}_\bullet)_1 & \longrightarrow & (\mathcal{C}_\bullet \otimes \mathcal{C}_\bullet)_2 \\
\downarrow & & \downarrow \rho & & \downarrow \\
0 & \longrightarrow & K_X & \longrightarrow & 0
\end{array}$$

$\therefore$ ρ induces a map

$$\mathbb{H}^1(\mathcal{C}_\bullet)\otimes\mathbb{H}^1(\mathcal{C}_\bullet) = \mathbb{H}^2(\mathcal{C}_\bullet\otimes\mathcal{C}_\bullet)_1 \xrightarrow{\rho'} \mathbb{H}^2\left(\begin{array}{c}0\\ \downarrow\\ k_x\\ \downarrow\\ 0\end{array}\right) = H^1(X, k_x) = \mathbb{C}$$

(künneth 公式)

$$\boxed{\Psi_\theta : \mathbb{H}^1(\mathcal{C}_\bullet)\otimes\mathbb{H}^1(\mathcal{C}_\bullet) \longrightarrow \mathbb{C}}$$

(this is what we want)

then Ψ_θ is non-dege + symmetric.

Liouville 1-Form.

Consider short exact sequence of complexes:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
\mathcal{D}_\bullet : & 0 \arrow[r] \arrow[d] & \operatorname{End} E\otimes k_x(D) \arrow[d] \\
\mathcal{C}_\bullet : & \operatorname{End} E(-D) \arrow[r, "f_\theta"] \arrow[d] & \operatorname{End} E\otimes k_x(D) \arrow[d] \\
\mathcal{E}_\bullet : & \operatorname{End} E(-D) \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

$u \longmapsto (E, S, \theta + tu)$

$$H^0(\operatorname{End} E\otimes k_x(D)) = \mathbb{H}^1(\mathcal{D}_\bullet)$$

$$\rightsquigarrow 0 \to \mathbb{H}^0(\mathcal{C}_\bullet) \to \mathbb{H}^0(\mathcal{E}_\bullet) \to \mathbb{H}^1(\mathcal{D}_\bullet) \xrightarrow{a} \mathbb{H}^1(\mathcal{C}_\bullet)$$

$$\xrightarrow{b} \underbrace{\mathbb{H}^1(\mathcal{E}_\bullet)}_{= H^1(X, \operatorname{End} E(-D))} \longrightarrow \mathbb{H}^2(\mathcal{D}_\bullet) \longrightarrow \mathbb{H}^2(\mathcal{C}_\bullet) \longrightarrow 0$$

(forgetful map)

Def: the 1-form is given by $\Phi_\theta : \mathbb{H}^1(\mathcal{C}_\bullet) \longrightarrow \mathbb{C}$, $v \longmapsto \theta(b(v))$

Prop:

For the map $\iota: T^*\mathcal{M}(r,d) \hookrightarrow \mathcal{M}_H(r,d)$ the $\iota^*\Phi$ is just the canonical symplectic form on cotangent space.

pf: Let the map $p: T^*\mathcal{M}(r,d) \longrightarrow \mathcal{M}(r,d)$ be the projection. For $(E,\delta,\theta) \in T^*\mathcal{M}(r,d)$, considering the differential: $dp_{(E,\delta,\theta)}: T_{(E,\delta,\theta)}T^*\mathcal{M}(r,d) \longrightarrow T_{(E,\delta)}\mathcal{M}(r,d)$

then $dp|_{(E,\delta,\theta)} = b$ as forgetful-map

$\therefore$ by definition of Liouville-1-form: $\left(\begin{matrix}\text{base part as Identi}\\ \text{fiber part be } 0\end{matrix}\right)$

$\therefore$ $\iota^*\Phi$ is just Liouvill 1-form. and $d\iota^*\Phi = \iota^*\Psi$.

(See next page)

Thm:

Ψ is symplectic, s.t. $\Psi = d\Phi$ on $\mathcal{M}_H(r,d)$,

(7.3) Proposition. *On the closed point $c \in \operatorname{Spec} R$, the evaluation of the 2-form $d\Theta(c)$ coincides with $\Phi(c)$.*

Proof. We need some general facts about Higgs bundles. Let (E, θ) be any Higgs bundle. Define $\Omega^{p,q}(\operatorname{End}(E)) := C^\infty(X, \operatorname{End}(E))$, i.e. the space of all smooth (p, q)-forms with values in $\operatorname{End}(E)$. Let $\bar\partial_E$ be the Dolbeault operator which defines the holomorphic structure on $\operatorname{End}(E)$. Using ~~the Dolbeault resolution of the complex~~ $C.$ as in (3.6), the i-th hypercohomology $\mathbb{H}^i(C.)$ can be computed as the cohomology of the following complex

$$\sum_{j=0}^{i-1} \Omega^{j,i-j-1}(\operatorname{End}(E)) \xrightarrow{\bar\partial_E + \theta} \sum_{j=0}^{i} \Omega^{j,i-j}(\operatorname{End}(E)) \xrightarrow{\bar\partial_E + \theta} \sum_{j=0}^{i+1} \Omega^{j,i-j+1}(\operatorname{End}(E)) \,.$$

The given family of bundles $\bar E$ parametrized by $\operatorname{Spec} R$ is C^∞ trivial, in other words there is an C^∞ isomorphism $f: \bar E \to p^* E$, where $p: \bar X \to X$ is the natural projection. Using this isomorphism, the Dolbeault operator that defines the holomorphic structure on $\operatorname{End}(\bar E)$ can be expressed in the following form

$$\bar\partial_E + A_1 \varepsilon_1 + A_2 \varepsilon_2 + B_1 \varepsilon_1^2 + B_2 \varepsilon_2^2 + C \varepsilon_1 \varepsilon_2 \,,$$

where A_i, B_i and C are smooth sections of $\operatorname{End}(E) \otimes \Omega_X^{0,1}$.

Using f, $\bar\theta$ is of the form

$$\theta + \alpha_1 \varepsilon_1 + \alpha_2 \varepsilon_2 + \beta_1 \varepsilon_1^2 + \beta_2 \varepsilon_2^2 + \gamma \varepsilon_1 \varepsilon_2 \,,$$

where $\alpha_i, \beta_i, \gamma \in \Omega^{1,0}(\operatorname{End}(E))$. For $\phi \in \Omega^{0,1}(\operatorname{End}(E))$ and $\psi \in \Omega^{1,0}(\operatorname{End}(E))$, define $(\phi, \psi) := \int_X \operatorname{tr}(\phi \wedge \psi) L^{n-1}$, where the $(1, 1)$-form $\operatorname{tr}(\phi \wedge \psi)$ is defined using the trace map. Recall that if $\bar\partial_V + h\varepsilon$ is the holomorphic structure on a family over $\mathbb{C}[\varepsilon]/\varepsilon^2$ then, h represents the element of $H^1(X, V)$ which corresponds to this infinitesimal deformation. The 1-form Θ on $\operatorname{Spec} R$ is

$$(A_1, \bar\theta)\, d\varepsilon_1 + (A_2, \bar\theta)\, d\varepsilon_2 + (B_1, \bar\theta)\, d(\varepsilon_1^2) + (B_2, \bar\theta)\, d(\varepsilon_2^2) + (C, \bar\theta)\, d(\varepsilon_1 \varepsilon_2) \,.$$

Taking exterior derivation we get

$$\begin{aligned} d\Theta = {} & (A_1, \alpha_2 + 2\beta_2 \varepsilon_2 + \gamma \varepsilon_1)\, d\varepsilon_2 \wedge d\varepsilon_1 + (A_2, \alpha_1 + 2\beta_1 \varepsilon_1 + \gamma \varepsilon_2)\, d\varepsilon_1 \wedge d\varepsilon_2 \\ & + (B_1, \alpha_2 + 2\beta_2 \varepsilon_2) 2\varepsilon_1\, d\varepsilon_2 \wedge d\varepsilon_1 + (B_2, \alpha_1 + 2\beta_1 \varepsilon_1) 2\varepsilon_2\, d\varepsilon_1 \wedge d\varepsilon_2 \\ & + (C, \alpha_2 + \gamma \varepsilon_1) \varepsilon_2\, d\varepsilon_2 \wedge d\varepsilon_1 + (C, \alpha_1 + \gamma \varepsilon_2) \varepsilon_1\, d\varepsilon_1 \wedge d\varepsilon_2 \,. \end{aligned}$$

Hence $d\Theta(c) = [-(A_1, \alpha_2) + (A_2, \alpha_1)]\, d\varepsilon_1 \wedge d\varepsilon_2$. From the definition (1.4) it is easy to check that this is the same as $\Phi(c)$. This completes the proof. □

$$\begin{tikzcd}
0 \arrow[r] & \begin{array}{c} \mathrm{End}\,E(-1) \\ \oplus \\ \mathrm{End}\,E(-1) \end{array} \arrow[r, "f_0 \otimes \mathrm{Id}", "\mathrm{Id} \otimes f_0"'] \arrow[d] & \begin{array}{c} \mathrm{End}\,E(1)k_x \otimes \mathrm{End}\,E(-1) \\ \oplus \\ \mathrm{End}\,E(-1) \otimes \mathrm{End}\,E(1)k_x \end{array} \arrow[r, "\mathrm{Id} \otimes f_0", "-f_0 \otimes \mathrm{Id}"'] \arrow[d] & \begin{array}{c} \mathrm{End}\,E(1)k_x \\ \otimes \\ \mathrm{End}\,E(1)k_x \end{array} \arrow[d] \\
& I^0 \arrow[r] \arrow[d, "\partial"] & I^0 \arrow[r] \arrow[d, "-\partial"] & I^0 \arrow[d] \\
& I^1 \arrow[r] \arrow[d] & I^1 \arrow[r, "\mathrm{Id} \otimes f_0", "-f_0 \otimes \mathrm{Id}"'] \arrow[d] & I^1 \arrow[d] \\
& 0 & 0 & 0
\end{tikzcd}$$

$(H^2, e_\cdot \otimes e_\cdot)$