$$\langle A, B\rangle = \frac{1}{2}\left(\langle A+B, A+B\rangle - \langle A, A\rangle - \langle B, B\rangle\right)$$

This defines an inner product on $\mathcal{H}$.

Prop: There is an <u>isometry</u> action of $SL(2,\mathbb{C})$ on $\mathcal{H}$ given by: (此 Action 亦 transitive)

$$SL(2,\mathbb{C}) \rhd \mathcal{H} \longrightarrow \mathcal{H}$$
$$(g, A) \longmapsto gA\bar{g}^t$$

$$\overline{gA\bar{g}^t}^t = (\bar{g}\bar{A}g^t)^t = g\bar{A}^t\bar{g}^t = gA\bar{g}^t$$

$$\langle g.A, g.A\rangle = -\det(g.A) = -\det(A)$$

$\therefore$ $\mathbb{H}^3$ 即可看成所有 hermitian-matrix 之子集.

In this case, $SL(2,\mathbb{C}) \longrightarrow SO^+(3,1)$

$$g \longmapsto \phi(g)$$

This is a 2:1 covering.

☆: Prop: $\mathbb{H}^3 := \{A \in \mathcal{H} \mid \det A = 1,\ A > 0\}$

Lemma: $T_{g\bar{g}^t} \mathbb{H}^3 = g\, i\,\mathfrak{su}(2)\, \bar{g}^T$

Q1

pf: For $id \in \mathbb{H}^3$, $T_{id} \mathbb{H}^3 = i\,\mathfrak{su}(2)$

Let $\gamma(t)$ be a curve, with $\gamma(0) = id$, then $\gamma(t)$ has form near identity to be:

$$\begin{pmatrix} 1 + a'(0)t + \cdots & b'(0)t + \cdots \\ \overline{b'(0)t + \cdots} & 1 + c'(0)t + \cdots \end{pmatrix} = M(t)$$

Thus: $\det M(t) = 1$.

$\therefore (\det M(t))' \Big|_{t=0} = 0 \rightsquigarrow a'(0) + c'(0) = 0$

$$\therefore M'(0) = \begin{pmatrix} a'(0) & b'(0) \\ \overline{b'(0)} & -a'(0) \end{pmatrix}$$

#

I think the tangent space should be trace-free hermitian-metric?

☆：研究$\mathbb{H}^3$中的 minimal-surface，由 lifting 看.

Def:（Lift）

The lift of $f: X \to \mathbb{H}^3$ $\subseteq \mathcal{H}$（行列式为1的厄米矩阵） is a map

$F: X \to SL(2,\mathbb{C})$, such that $f = F\bar{F}^t$.

precisely, F is called left lift,

$\bar{F}^t$ is called right lift.

Define: $\omega = F^{-1}dF \in \Omega^1(X, sl(2,\mathbb{C}))$

$\alpha = \frac{1}{2}(\omega + \bar{\omega}^t)$, $\beta = \frac{1}{2}(\omega - \bar{\omega}^t)$, $\omega = \alpha + \beta$.

(hermitian part)　　(skew-hermitian)

Prop: $\langle df(X), df(Y) \rangle = 2\,tr(\alpha(X)\alpha(Y))$

（是否应该有2？）

Lemma: $\mathcal{E}(f) = 2\int_X -\mathrm{tr}(\alpha \wedge * \alpha)$ Q2.

$= 2i\int_X \mathrm{tr}(\Phi \wedge \Phi^*)$

pf: by def: $\mathcal{E}(f) = \int_X \|df\|^2 \mathrm{vol} = \int_X (\langle f_x, f_x\rangle + \langle f_y, f_y\rangle)\, dx\, dy$

$= 2\int_X \mathrm{tr}(\alpha(X)\alpha(X) + \alpha(Y)\alpha(Y))\, dx \wedge dy$

but $\mathrm{tr}(\alpha \wedge * \alpha)(X, Y) = \mathrm{tr}(\alpha(X)\alpha(*Y) - \alpha(Y)\alpha(*X))$

$= -\mathrm{tr}(\alpha(X)\alpha(X) + \alpha(Y)\alpha(Y))$

$\therefore \mathcal{E}(f) = -2\int_X \mathrm{tr}(\alpha \wedge * \alpha) = 4i\int_X \mathrm{tr}(\Phi \wedge \Phi^*)$ #.

Define: $\nabla = d + \beta$, β is the skew-hermitian part of $\omega = F^{-1} dF$. $\beta \in \Omega^1(\mathfrak{su}(2))$

Prop: ∇ is unitary connection on $\mathbb{C}^2$, but non-flat.

pf: As unitary connection is just the $\mathfrak{su}(2)$-matrix under 1 basis. So this is unitary

∇ also a connection on $gl(2,\mathbb{C})$-bundle given by $\nabla A = dA + [\beta, A]$ for some $A: X \to gl(2,\mathbb{C})$

Lemma: We have $d^{\nabla}\alpha = 0$ where $\alpha = \frac{1}{2}(\omega + \bar{\omega}^t)$

pf: $d^{\nabla}\alpha = d\alpha + [\beta, \alpha]$

For $\omega = F^{-1}dF \Rightarrow d\omega + \frac{1}{2}\omega \wedge \omega = 0$ (diff-forms sign变了)

$\boxed{\therefore\ d\bar{\omega}^T - \frac{1}{2}\bar{\omega}^T \wedge \bar{\omega}^T = 0}$ key-point

$\therefore\ d^{\nabla}\alpha = \frac{1}{2}d(\omega + \bar{\omega}^T) + \frac{1}{4}[\omega - \bar{\omega}^T, \omega + \bar{\omega}^T]$

$= \frac{1}{2}d\omega + \frac{1}{2}d\bar{\omega}^T + \frac{1}{4}[\omega, \omega] + \cancel{\frac{1}{4}[\omega, \bar{\omega}^T]} - \cancel{\frac{1}{4}[\bar{\omega}^T, \omega]}$

$- \frac{1}{4}[\bar{\omega}^T, \bar{\omega}^T]$

$= 0$

Now we could justify if a map into $\mathbb{H}^3$ is harmonic.

Energy - Functional:

$$\mathcal{E} : \begin{array}{ccc} C^\infty(\Sigma, \mathbb{H}^3) & \longrightarrow & \mathbb{R} \\ f & \longmapsto & \int_\Sigma \|df\|^2 vol. \end{array}$$

From above: $\mathcal{E}(f) = -2\int_\Sigma tr(\alpha \wedge * \alpha)$

Prop: Write $\gamma = F^{-1} \cdot \dot{F}$, then:

$$d_f \mathcal{E}(f) = \int_X tr\left((\gamma + \gamma^*) d^\nabla * \alpha\right)$$

here: $f = F \cdot \bar{F}^t$, α hermitian part

$dF = F\omega$, $\nabla = d + \beta \rightarrow$ skew-hermi part.

Pf: $d_f \mathcal{E}(f) = -2\int_\Sigma \mathrm{tr}(\dot{\alpha} \wedge * \alpha)$

$$= -\int_\Sigma \mathrm{tr}(2\dot{\alpha} \wedge * \alpha)$$

$2\dot{\alpha} = \dot{\omega} + \dot{\omega}^T$, $\omega = F^{-1} dF$. thus:

$\dot{\omega} = (\dot{F^{-1}}) dF + F^{-1} d\dot{F}$.

As $F \cdot F^{-1} = id$ $\therefore$ $\dot{F} \cdot F^{-1} + F \cdot \dot{F^{-1}} = 0$

$\therefore \dot{F^{-1}} = -F^{-1} \cdot \dot{F} \cdot F^{-1}$

$$\therefore \dot{\omega} = -\underline{F^{-1}\dot{F}} F^{-1} dF + F^{-1} d\dot{F}$$

$$= -\gamma \cdot F^{-1} dF + F^{-1} d\dot{F}$$

$$= -\gamma\omega + F^{-1} d\dot{F}$$

$\therefore \dot{\omega} + \dot{\omega}^* = -\gamma\omega + F^{-1} d\dot{F} - \omega^*\gamma^* + (d\dot{F})^*(F^{-1})^*$

$$\therefore d_f \mathcal{E}_f = -\int_\Sigma \mathrm{tr}\Big(\big(-\gamma\omega + F^{-1} d\dot{F} - \omega^*\gamma^* + (d\dot{F})^*(F^{-1})^*\big) \wedge * \alpha\Big)$$

by stokes-thm:

$$= \int_\Sigma tr((\gamma+\gamma^*) d^\nabla * \alpha)$$

so we could obtain that:

Thm:

$f: \Sigma \longrightarrow \mathbb{H}^3$ is harmonic $\Leftrightarrow$

$d^\nabla * \alpha = 0$

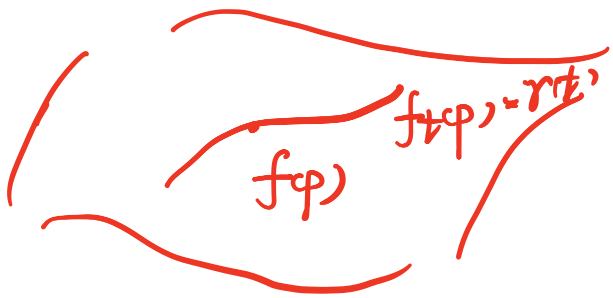
$(\nabla = \alpha + \beta$, α, β is her/Anti-her part of $\omega = F^{-1} dF$, where F is a Lifting of f),

借助 Lifting，可以更好刻画 harmonic density.

Prop: The energy functional $\mathcal{E}(f_t)$ is convex w.r.t variation with $\forall p \in X$, $t \longmapsto f_t(p)$ is a constant speed geodesic in $\mathbb{H}^3$. (关于测地变分是凸函数)

$$\{f_t\}: X \times [0,1] \longrightarrow \mathbb{H}^3$$

pf:



$\nabla_{\partial t} \dot{f}_t = 0$.

$$\frac{d^2}{dt^2}\mathcal{E}(f_t) = -\int_X g(\nabla \dot{f} \wedge * \nabla \dot{f}) - \int_X g(\dot{f}, tr(R(\dot{f}, df(\cdot))df(\cdot))) dA$$

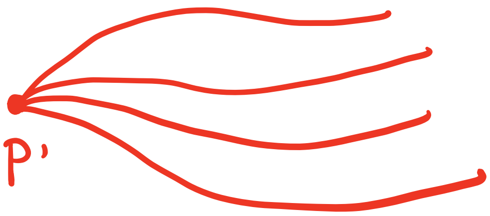
$$\geqslant 0$$

Corollary: There are no non-trivial harmonic map from a compact R.S. to $\mathbb{H}^3$.

pf: let $f_0: \begin{array}{l} X \longrightarrow \mathbb{H}^3 \\ x \longmapsto p \end{array}$ be const-map

Considering the variation: $f_t(x)=\exp_p(t\nu(x))$

$\therefore \frac{d^2}{dt^2}\mathcal{E}(f_t)\geq 0 \longrightarrow$ the energy is convex.



p'

if $f=f_1$ is harmonic

$\therefore t=1,0$ 均 critical point.

$\therefore \mathcal{E}(f_t)$ is constant $\therefore \mathcal{E}(f)=0$.

$\therefore f$ is constant. (每个f_1均可写成 $\exp_p(\nu(x))$)

(凸函数之 critical point 只可以有一个)

Def: A map $f:\tilde{X}\longrightarrow \mathbb{H}^3$ is ρ-equiva

if $\forall \gamma\in\pi_1(X,p)$. we have

$$f(\gamma\cdot x)=\rho(\gamma)\cdot f, \quad \rho \text{ is a } SL(2,\mathbb{C})$$

($\rho(\gamma)^* f \rho(\gamma)$) (故我们转而看 non-compact case)

— monodromy.

$$\left(\overline{\bar{A}^T H A}\right)^T = \bar{A}^T H A.$$

(各自有: $\pi_1(X,p) \curvearrowright \tilde{X}\xrightarrow{f}\mathbb{H}^3 \curvearrowleft SL(2,\mathbb{C})$)

Def: A left lift F of $f = F\bar{F}^t$ is called left-equivariant if:

$$\gamma^* F = \rho(\gamma)^{-1} F, \quad \forall \gamma \in \pi_1(X, p)$$

($F: \tilde{X} \longrightarrow SL(2, \mathbb{C})$)

形式上很像 R.H. Correspondence.

Lemma:
For $\forall$ ρ-equivariant $f: \tilde{X} \to \mathbb{H}^3$, there is a left/right equivariant lift F or F^*.

Pf: For ρ be trivial: $f(\gamma . p) = f(p)$

$\therefore$ f can be decended to $X \to \mathbb{H}^3$.

$f^{\frac{1}{2}} = \exp(\frac{1}{2} \log f) = (g^{-1})^T \begin{pmatrix} \sqrt{a} & \\ & 1/\sqrt{a} \end{pmatrix} g$ where

$$f = (g^{-1})^{T} \begin{pmatrix} a & \\ & 1/a \end{pmatrix} g$$

「Left-Lift 之好处」: Energy 可以 decend to Σ」

For $f: \hat{\Sigma} \to \mathbb{H}^3$ with equivariant lift

$F: \hat{\Sigma} \to SL(2,\mathbb{C})$, then:

$-tr(\alpha \wedge * \alpha)$ is well-def on Σ.

pf: Only need to show:

$F^{-1}dF = \omega$ can be decend to Σ.

$$\gamma^{*}(F^{-1}dF) = (F\circ\gamma)^{-1}\gamma^{*}dF = (F\circ\gamma)^{-1} d\rho(\gamma)^{-1}\cdot F$$

$F\circ\gamma = \rho(\gamma)^{-1}\cdot F \quad \therefore \quad (F\circ\gamma)^{-1} = F^{-1}\rho(\gamma)$

$\therefore$ RHS $= F^{-1}dF$. $\longrightarrow$ this is invariant.

Def: ρ-equivariant $f: \tilde{X} \to \mathbb{H}^3$. the energy $\mathcal{E}(f) = \int_X \mathrm{tr}(\alpha \wedge * \alpha)$

故之前关于论证 harmonicity 之判据仍成立:

$$d^{\nabla} * \alpha = 0 \quad !!!$$

Thm: ★★★:

Let $f: \hat{\Sigma} \to \mathbb{H}^3$ be ρ-equivariant,

then:

$\bar{\Phi}$: $(1,0)$-part of $F^{-1} dF$, where $\gamma^* F = \rho(\gamma) \cdot F$

$$f \text{ is harmonic}$$

$$\Updownarrow$$

$$\nabla^{\lambda} \triangleq \nabla + \lambda^{-1} \bar{\Phi} + \lambda \bar{\Phi}^* \text{ is flat. } \forall \lambda \in \mathbb{C}^*.$$

ρ is the monodromy of $\nabla^{\lambda=1}$.

Conversely, for any family of flat $SL(2,\mathbb{C})$ connections $\nabla^\lambda = \nabla + \lambda^{-1}\Phi + \lambda\Phi^*$, then:

$\exists$ a $\rho(\nabla^{\lambda=1})$-equi harmonic-map f whose associated family is ∇^λ.

pf: First show harmonic $\Rightarrow$ ∇^λ flat

$$F^{\nabla^\lambda} = \cancel{F^{\nabla} + [\Phi, \Phi^*]} + \cancel{\lambda^{-2}\Phi\wedge\Phi} + \cancel{\lambda^{2}\Phi^*\wedge\Phi^*} + \lambda^{-1}\partial^{\nabla}\Phi + \lambda\,\partial^{\nabla}\Phi^*$$

$$d^{\nabla}\alpha = 0, \quad d^{\nabla}{*\alpha} = 0 \longrightarrow d^{\nabla}\Phi = 0, \quad d^{\nabla}\Phi^* = 0.$$

故才有 $F^{\nabla^\lambda} = 0$

F^{-1} is parallel frame for $\nabla^1 = d + \omega$

as.

$$\nabla^1 F^{-1} = d + F\omega F^{-1} + F dF^{-1}, \quad \omega = F^{-1}\alpha F$$

$$= d + dF\cdot F^{-1} - dF\cdot F^{-1} = d$$

so ∇^1 has monodromy: ρ

下证：∇' 之 monodromy 即 ρ：

choose $\gamma \in \pi_1(\Sigma, p)$, and pick any lift $\gamma^\uparrow : [0,1] \longrightarrow \hat{\Sigma}$, then:

by Riemann-Hilbert Correspondence:

$$\begin{array}{c} \hat{\Sigma}\times\mathbb{C}^2 / (q,v)\sim(q\cdot\gamma, \rho(\gamma)^{-1}v) \\ \downarrow \\ \Sigma \end{array}$$

We have:

$$\rho^{\nabla'}(\gamma) = F^{-1}(\gamma^\uparrow(1)) \cdot F(\gamma^\uparrow(0))$$

$$= F^{-1}(\gamma^\uparrow(1)) \cdot F(\gamma^\uparrow(1) * \gamma)$$

$$= F^{-1}(\gamma^\uparrow(1)) \cdot F(\gamma^\uparrow(1)) \cdot \rho(\gamma)$$

$$= \rho(\gamma)$$

故 $\rho^{\nabla'}(\gamma) = \rho(\gamma)$

Conversely, give such family of flat-conn

Let $F^{t}: \tilde{\Sigma} \to Sl(2,\mathbb{C})$ be parallel-frame due of ∇^{t} to simply-connected of $\tilde{\Sigma}$.

Define $f = F\bar{F}^{t}: \tilde{\Sigma} \to \mathbb{H}^3$, then:

f is ρ-equivariant:

$$f(\gamma * p) = F(\gamma * p)\bar{F}^{t}(\gamma * p)$$

$$= \bar{\rho}^{\nabla^{t}}(\gamma) \cdot f(\gamma) \cdot \bar{\rho}^{\nabla^{t}}(\gamma)^{*}$$

(because for parallel frame: $F(\gamma * p) = \rho^{\nabla^{t}}(\gamma)^{-1} \cdot F$)

Also: $d^{\nabla}\Phi$; $d^{\nabla}\Phi = 0$, 又 $\nabla^{t} F = 0$. 故: $F^{-1}dF = \alpha + \beta$

$\alpha = \Phi + \Phi^{*}$

$\to f$ is harmonic.

#

Higgs Field and Hitchin Equation

Def: A Higgs-Field is a pair $(\bar\partial, \Phi)$, $\bar\partial$ holomorphic-structure, $\Phi \in H^0(\Sigma, K\,\mathrm{End}_0(V))$

Def: The Hitchin-Self-Duality equation is a triple (∇, Φ, h), s.t.

$$\begin{cases} \bar\partial^{\nabla}\Phi = 0 \\ F^{\nabla} + [\Phi, \Phi^{*,h}] = 0 \end{cases}$$

Example: Every solution of SD-equation gives a Higgs-Field.

Rk: 这是stable的, as for flat-unitary $SL(2,\mathbb{C})$-connection, $\forall L \hookrightarrow V$, $\deg L \leq 0$.

Example: Every solution of SD-Equation gives a family of flat-connections.

$$\nabla^\lambda = \nabla + \lambda^{-1}\Phi + \lambda\Phi^{*,h}$$

computing:

$$F^{\nabla^\lambda} = F^\nabla + d^\nabla(\lambda^{-1}\Phi + \lambda\Phi^*) + (\lambda^{-1}\Phi + \lambda\Phi^*)(\lambda^{-1}\Phi + \lambda\Phi^*)$$

$$\boxed{d^\nabla\Phi = \bar\partial^\nabla\Phi = 0 \longrightarrow \partial^\nabla\Phi^* = 0}$$

$$= F^\nabla + \Phi\wedge\Phi^* + \Phi^*\wedge\Phi = F^\nabla + [\Phi, \Phi^*] = 0.$$

故: Equivariant Harmonic Map;

Family of Flat connections;

Self-Duality Equations.

They describe the same objects.

Example:

If (∇, Φ, h) is a solution of SD-Equ., then: $\forall g \in \mathcal{G}$ (gauge group), we have $(\nabla.g, g^{-1}\Phi g, g^*h)$ is another solution.

1) $\nabla.g$ is unitary for g^*h.

$$d\, g^*h(s_1, s_2) = dh(gs_1, gs_2) = h(\nabla(gs_1), gs_2) + h(gs_1, \nabla(gs_2)) = h(g.g^{-1}\nabla.(gs_1), gs_2) + h(gs_1, g.g^{-1}\nabla g s_2)$$

$$= g^*h((\nabla.g)s_1, s_2) + g^*h(s_1, (\nabla.g)s_2)$$

2) $\bar{\partial}^{\nabla.g}(g^{-1}\Phi g) = 0$

3) $$g^*h(g^{-1}\Phi g s_1, s_2) = h(\Phi(gs_1), gs_2)$$

$$= h(gs_1, \Phi^* g s_2) = h(s_1, (g^{-1}\Phi^* g)s_2)$$

$\therefore (g^{-1}\Phi g)^{*,g^*h} = g^{-1}\Phi^{*,h} g.$

RK: take $\lambda=1$ and considering the parallel Frame, this gives us a harmonic-map，故人有时又称，harmonic-metric.

Thm: $\tilde{\Sigma}\times\mathbb{C}^2 \to \tilde{\Sigma} \to \Sigma$, let (∇,Φ,h) on $\tilde{\Sigma}\times\mathbb{C}^2$ be a solution of Self-Duality Equation, with harmonic map f, then: (Energy-Bound)

(如果只把它给定在$\Sigma\times\mathbb{C}^2$上，怎么办)

$$\mathcal{E}(f) \leq 2\pi(2g-2) + 4\int_\Sigma |\det\Phi|$$

$(\nabla,\Phi,h) \rightsquigarrow \nabla^1 = \nabla + \Phi + \Phi^*$ on $\tilde{\Sigma}\times\mathbb{C}^2$, then 看$\nabla^1$之parallel-frame，记作 $F^{-1}: \tilde{\Sigma}\to SL(2,\mathbb{C})$, $\underline{\omega = F^{-1}dF}$

$f = F\bar{F}^t$

Pf: $\mathcal{E}(f) = 2i\int_\Sigma \mathrm{tr}(\Phi\wedge\Phi^*)$

(关于 $\det\Phi$: if $\Phi = A\,dz$, $\det\Phi = (\det A)\,dz = \omega^2 dz$

then $|\det\Phi| = i|\omega|^2 dz\wedge d\bar{z}$)

① $\Phi = 0 \rightarrow \mathcal{E}(f) = 0$ ✓

② $\Phi \neq 0$, but $\det\Phi = 0$ $\therefore \exists\, L \overset{\text{holomorphic}}{\hookrightarrow} V$, such that

$\Phi(L) = 0$, Let $\Rightarrow V = L\oplus L^\perp$. write

$$\nabla = \begin{pmatrix} \nabla^L & \gamma \\ \underset{(1,0)}{-\gamma^*} & \nabla^{L^*} \end{pmatrix}, \quad \Phi = \begin{pmatrix} 0 & \varphi \\ 0 & 0 \end{pmatrix}, \ \varphi \in H^0(KL^2)$$

$\therefore 2g-2+2\deg L \geq 0$

$\therefore \deg L \geq 1-g$, by flatness: $F^\nabla + [\Phi, \Phi^*] = 0$

$\therefore F^{\nabla^L} - \gamma\wedge\gamma^* + \varphi\wedge\varphi^* = 0$

$\therefore \mathcal{E}(f) = 2i\int_\Sigma(\Phi\wedge\underset{(0,1)}{\Phi^*}) = 2i\int_\Sigma \varphi\wedge\varphi^*$

$= -2i\left(\int_\Sigma F^{\nabla^L} + \int_\Sigma \gamma\wedge\gamma^*\right) \leq 2\cdot 2\pi(g-1)$

$= 2\pi(g-2)$

③ If $\det\Phi \neq 0$

(Hitchin-Curve) (谱曲线)

simple-zeros.

$\Phi = \begin{pmatrix} A_1 dz & A_2 dz \\ A_3 dz & A_4 dz \end{pmatrix}$

$\hat{\Sigma} := \{\omega \in K_\Sigma \mid -\omega^2 = \det\Phi\}$

$\det\Phi = A_1 A_4 dz^2 - A_2 A_3 dz^2.$

应该写成 tensor-Equation

This a smooth R.S which gives a branced double-covering of Σ by $\pi: \hat{\Sigma} \to \Sigma$ which has
$(\omega_p) \mapsto p.$

an involution $\delta: \hat{\Sigma} \to \hat{\Sigma}$.
$\omega \mapsto -\omega$

then: $\hat{\Sigma}/\delta \cong \Sigma.$

Consider $\pi^*\Phi \in H^0(\hat{\Sigma}, \pi^* K_\Sigma \otimes \mathrm{End}_0(V))$

with eigenvalues $\pm\omega \in H^0(\hat{\Sigma}, \pi^* K_X)$

s.t. $-\omega^2 = \det\Phi.$ $\lambda \in K_\Sigma$ ✓

☆ $\mathrm{Char}(\Phi) = \det(\Phi - \lambda \mathrm{Id}) = \lambda^2 \mathrm{Id} - (\mathrm{tr}\Phi)\lambda + \det\Phi.$

Consider eigen-line-bundle L of $\pi^*\Phi$ w.r.t $H^0(\hat{\Sigma}, \pi^*(K_\Sigma \otimes \mathrm{End}_0(V)))$

(借此看 $\deg L$)(理解成: image preserves in L).

$\delta^* L$ is the eigenvalue bundle

Consider the natural-map on $\pi^* V$.

$$d: L \oplus \delta^* L \xrightarrow{\det} \Lambda^2 \pi^* V = \pi^* \mathcal{O} = \mathcal{O}$$

claim: the vanishing divisor (d) is exactly branch divisor of $\pi: \hat{\Sigma} \to \Sigma$

pf of claim: If $\omega \in \hat{\Sigma}$, $\omega(p) \neq 0$, then $\pi(\omega) = p$

we have 2 different values $\omega, -\omega$, therefore d doesn't vanish at those points.

$\therefore$ the claim follows if we can show the eigenline L, $\delta^* L$ fall together in first order w.r.t some holo-frame,

$$\Phi = \begin{pmatrix} a & 1 \\ b & -a \end{pmatrix} dz, \quad \det \Phi = -z\,dz^2.$$

conjugate with $\begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} \to \tilde{\Phi} = \begin{pmatrix} 0 & 1 \\ a^2+b & 0 \end{pmatrix} dz$

$$= \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} dz.$$

Use the coordinate y on $\hat{\Sigma}$ with $y^2 = z$, and the eigenlines (w.r.t above frame) become $\begin{pmatrix} 1 \\ y \end{pmatrix}$, $\begin{pmatrix} 1 \\ -y \end{pmatrix}$. which fall toge in 1^{st}-order. at $y=0$

$\deg(L \otimes \delta^* L) = 4-4g = -\deg(K_\Sigma \otimes K_\Sigma \to \Sigma)$

$= \#$ zeros of $\det \Phi$.

with $\deg L = \deg \delta^* L \Rightarrow \deg L = 2-2g$

We now pull back self-duality (P, Φ, h) to $\hat{\Sigma}$,

i.e. $\pi^* \Phi \in H^0(\hat{\Sigma}, K_{\hat{\Sigma}} \otimes \operatorname{End} \mathcal{L})$

We have: $\pi^* \Phi = \Phi \circ \pi \otimes d\pi$.

so that L is still an eigenline of $\pi^* \Phi$

w.r.t the eigenvalue $\omega \otimes d\pi := \hat{\omega} \in H^0(\hat{\Sigma}, K_{\hat{\Sigma}})$

satisfying: $-\hat{\omega}^2 = \pi^* \det \Phi$

Write $\pi^*\nabla = \begin{pmatrix} \nabla^L & \overset{(0,1)}{\gamma} \\ \underset{(1,0)}{-\gamma^*} & \nabla^{L^\perp} \end{pmatrix}$ w.r.t $V = L \oplus L^\perp$, $L \hookrightarrow V$

$$\pi^*\Phi = \begin{pmatrix} \hat{\omega} & \alpha \\ 0 & -\hat{\omega} \end{pmatrix}, \quad \alpha \in \Gamma(K_{\hat{\Sigma}} L^2)$$

and the vanishing $0 = F^\nabla + [\Phi, \Phi^*]$ implies on $\hat{\Sigma}$ restricted to L.

$$0 = F^{\nabla^L} - \gamma \wedge \gamma^* + \alpha \wedge \alpha^*$$

The energy on 2-fold covering computed

$$\begin{aligned}
\mathcal{E}(f) &= \frac{1}{2}\, 2i \int_{\hat{\Sigma}} tr(\pi^*\Phi \wedge \pi^*\Phi^*) \\
&= i\int_{\hat{\Sigma}} \alpha \wedge \alpha^* + 2i \int_{\hat{\Sigma}} \hat{\omega} \wedge \overline{\hat{\omega}} \\
&\underset{(*)}{=} -i \int_{\hat{\Sigma}} F^{\nabla^L} + i \int_{\hat{\Sigma}} \gamma \wedge \gamma^* + 2i \int_{\hat{\Sigma}} \hat{\omega} \wedge \overline{\hat{\omega}} \\
&= -2\pi \deg L + \underbrace{i \int_{\hat{\Sigma}} \gamma^{(0,1)} \wedge \gamma^*}_{\leq 0} + 2 \int_{\hat{\Sigma}} |\pi^* \det \Phi| \\
&\leq 2\pi(2g-2) + 4 \int_{\Sigma} |\det \Phi|
\end{aligned}$$

$\square$

Uniqueness for solutions to SD-Equation

Lemma: If: (∇^V, Φ^V, h^V) on $V \to \Sigma$, (∇^W, Φ^W, h^W) on $W \to \Sigma$, then the tensor product $(\nabla^V \otimes \nabla^W, \Phi^V \otimes id + id \otimes \Phi^W, h^V \otimes h^W)$ is also a solution of SD-Equation again.

$F^{\nabla^{V \otimes W}} = F^{\nabla^V} \otimes id + id \otimes F^{\nabla^W}$. Also:

$[\Phi^V \otimes id + id \otimes \Phi^W, \Phi^{V*} \otimes id + id \otimes \Phi^{W*}]$

$= [\Phi^V, \Phi^{V*}] \otimes id + id \otimes [\Phi^W, \Phi^{W*}]$

$\therefore$ this is true. #

Def: (Chern-Connection)

For $\begin{array}{c}(V,\bar{\partial},h)\\ \downarrow \\ \Sigma\end{array}$ be a holomorphic vector bund

with hermitian matrix h. Then there exists an unique unitary connection D, with $D'' = \bar{\partial}$

Example: $\bar{\partial} = \bar{\partial}^{0} + A$, then

$$D = d + A - \bar{A}^{t}$$

故我们可以reduce the case to $(\bar{\partial}, \Phi, h)$ if (D, Φ, h) is a solution.

Technical-Lemma:

借此工具，我们可以reformulate the Self Duality Equation.

$(\bar\partial, \bar\Phi, h)$ satisfies the self-duality if and only if: $\exists \lambda_0 \in \mathbb{C}^*$

1° $F^{\nabla^{\lambda_0}} = 0$ (key-point: 从 symmetry 角度去看 "hermitian"

2° $F^{D(\bar\partial + \lambda_0 \bar\Phi^*)} = |\lambda_0|^2 F^{D(\partial + \lambda_0^{-1}\bar\Phi)}$. "skew-hermi"

pf: Notice that: $(\bar\partial, \bar\Phi, h)$ solution

$\Leftrightarrow (\bar\partial, e^{i\varphi}\bar\Phi, h)$ is a solution.

$\therefore$ We can assume $\lambda_0 \in \mathbb{R} > 0$

$\nabla^{\lambda_0} \triangleq \nabla + \lambda_0^{-1}\Phi + \lambda_0 \Phi^*$, then:

$F^{\nabla^{\lambda_0}} = \underline{F^\nabla} + d^\nabla(\lambda_0^{-1}\bar\Phi + \lambda_0 \Phi^*) + \underline{[\Phi, \Phi^*]} = 0$

∇: skew-hermi $\longrightarrow \nabla \cdot \nabla$

hermitian: $[\Phi, \Phi^*]^* = [\Phi, \Phi^*]$ hermitian

$\therefore (r^{-1}\partial\bar\Phi + r\bar\partial\Phi^*)$'s skew hermitian part $= 0$

$r^{-1}\partial\bar\Phi + r\bar\partial\Phi^* - r^{-1}\bar\partial\Phi^* - r\partial\bar\Phi = 0$

$\rightsquigarrow \partial\bar\Phi + \bar\partial\Phi^* = 0$

$$F^{D(\bar\partial+\lambda_0\Phi^*)} = F^\nabla - r(\bar\partial\Phi - \partial\Phi^*) - r^2[\Phi,\Phi^*]$$

$$F^{D(\partial+\lambda_0^{-1}\Phi)} = F^\nabla + r^{-1}(\bar\partial\Phi - \bar\partial\Phi^*) - r^{-2}[\Phi,\Phi^*]$$

$$F^\nabla - r(\bar\partial\Phi - \partial\Phi^*) - r^2[\Phi,\Phi^*] = r^2F^\nabla + r(\bar\partial\Phi - \partial\Phi^*) - [\Phi,\Phi^*]$$

$$\underline{(r^2-1)F^\nabla} + 2r(\bar\partial\Phi - \partial\Phi^*) + \underline{(r^2-1)[\Phi,\Phi^*]} = 0.$$

this is hermitian

$$\bar\partial\Phi - \partial\Phi^* = 0.$$

仍是分别看 hermitian 与 skew-hermitian

$$\Rightarrow \bar\partial\Phi = 0 \ ; \ \partial\Phi^* = 0$$

#

Thm: If $\nabla^{\lambda_0} = \nabla + \lambda_0^{-1}\Phi + \lambda_0\Phi^*$ is reducible, then (∇, Φ, h) is reducible, i.e. it is direct sum of SD solution ~~on~~ lower rank vector bundle.

Idea: To show $L \hookrightarrow (V, \nabla)$ is parallel, one point is to show the inclusion $i \in L^* V$ is parallel, i.e. $(\nabla^L)^* \otimes \nabla\, i = 0$

(As we show before about the parallel endmorphism bundle:

For $s \in \{\text{parallel Endmorphism}\}$, $e_1 \in L$,

$$(\nabla^L)^* \otimes \nabla s = 0 \Leftrightarrow \nabla s(e_1) = s(\nabla^L e_1)$$

check: Let $s = e^i \otimes v_j$, then:

$$(\nabla^L)^* \otimes \nabla (e^i \otimes v_j) = (\nabla^L)^* e^i \otimes v_j + e^i \otimes \nabla v_j$$

$$= 0 \rightarrow s(\nabla^L e_1) = \nabla s(e_1)$$

Now come to our problem:

先看 Line-Bundle 上的 Self-Duality Equation:

$$\begin{tikzcd} \mathbb{C} \arrow[d] \\ L \arrow[d] \\ \Sigma \end{tikzcd}$$

As $\mathrm{End}_0(L) = \mathbb{C}$ is Abelian:

The Self-Duality Equation is just

(∇, Φ, h), $\begin{cases} F^{\nabla} = 0 \\ \bar\partial^{\nabla}\Phi = 0 \end{cases}$，因此，the family of

flat connections are $\nabla^{\lambda} = d + \underset{H^0(\Sigma, K_\Sigma)}{\beta} - \bar\beta + \lambda^{-1}\omega + \lambda\bar\omega \quad \omega \to H^0(\Sigma, K_\Sigma)$

故：给定 $\begin{matrix}\mathbb{C} \\ \downarrow \\ L \\ \downarrow \\ \Sigma\end{matrix}$ 上 $\forall$ 一个 flat-connection ∇, $\lambda_0 \in \mathbb{C}^*$

一定∃ solution of SD $(\hat\nabla, \hat\Phi, h)$, s.t.

$\hat\nabla^{\lambda_0} = \nabla$. 此即解二元一次方程组 $\nabla = d + \alpha - \bar\alpha$

$= d + \underline{\beta} - \bar\beta + \underline{\lambda_0^{-1}\Phi} + \lambda_0\bar\Phi$ $\quad \begin{cases} \beta + \lambda_0^{-1}\Phi = \alpha \\ -\bar\beta + \lambda_0\bar\Phi = -\bar\alpha \end{cases} \Rightarrow \checkmark$

As ∇^{λ_0} is reducible, Let $E \hookrightarrow V$ be such parallel-line-bundle, we'll show: This is also for ∇.

There exists solution $(\hat\nabla, \hat\Phi, \hat h)$ on E, such that: $\hat\nabla^{\lambda_0} = \nabla_E^{\lambda_0}$, Considering the bundle: $E^* \otimes V$ with connection $(\tilde{\nabla}^{\lambda})^* \otimes \nabla^{\lambda}$: then the ① inclusion $i: E \hookrightarrow V$ is parallel for $\lambda = \lambda_0$.

notice that:

$((\tilde{\nabla}^{\lambda})^* \otimes \nabla^{\lambda}, \hat{\Phi} \otimes id + id \otimes \Phi, h_1 \otimes h_2)$ is also a solution of SD-Equation,

Denote: $\mathcal{D}^{\lambda} = (\tilde{\nabla}^{\lambda})^* \otimes \nabla^{\lambda}$, and

$D_1 = D((\mathcal{D}^{\lambda_0})^{0,1})$, $D_2 = D((\mathcal{D}^{\lambda_0})^{1,0})$ be the associated Chern-Connection.

$\therefore \bar{\partial}^{D_1} i = 0 \quad \partial^{D_2} i = 0$, Let:

$L = \mathrm{span}(i) \subset E^* \otimes V$ be its subbundle.

$D_1 : L \oplus L^{\perp} \longrightarrow L \oplus L^{\perp}$

$\begin{pmatrix} \nabla^{L} & \gamma \\ -\gamma^* & \nabla^{L^{\perp}} \end{pmatrix}$ $\gamma \leftarrow (0,1)$ — form, $F^{D_1}|_L = F^{\nabla^L} - \gamma \wedge \gamma^*$

$D_2 : L \oplus L^{\perp} \longrightarrow L \oplus L^{\perp}$

$$\begin{pmatrix} \nabla^L & \underset{(1,0)\text{-form}}{\beta} \\ -\beta^* & \nabla^{L^\perp} \end{pmatrix} \qquad F^{D_3}\big|_L = F^{\nabla^L} - \beta\wedge\beta^*$$

As $i \in \Gamma^* V$ is parallel $\rightarrow$ $F^{\nabla^L} = 0$.

By above Lemma:

$$F^{D_1}\big|_L = |\lambda_0|^2 F^{D_2}\big|_L$$

$\therefore$ $F^{D_1}\big|_L = 0 = F^{D_2}\big|_L$ $\quad\therefore$ $\gamma = \beta = 0$

$\therefore$ i is parallel w.r.t. D^1, D^2

i.e. $$D^1 = \hat{\nabla}\otimes\nabla + \lambda_0^{-1}(\hat{\Phi}^*\otimes id + id\otimes\Phi^*) - \lambda_0^{-1}(\hat{\Phi}\otimes id + id\otimes\Phi)$$

Take $e_1 \in E$, then:

$$\nabla(ie_1) + \cancel{\lambda_0^{-1}e_1} + \lambda_0^{-1}\Phi^*(e_1) - \lambda_0^{-1}e_1 - \lambda_0^{-1}\Phi(e_1)$$

$$= i(\hat{\nabla}e_1) + \lambda_0^{-1}\hat{\Phi}^*(e_1) + \lambda_0^{-1}e_1 - \lambda_0^{-1}\hat{\Phi}(e_1) - \cancel{\lambda_0^{-1}e_1}$$

$\therefore$ $\nabla + \lambda_0^{-1}\Phi^* - \lambda_0^{-1}\Phi \in L$.

$\nabla + \lambda_0\Phi - \lambda_0\Phi^* \in L$

$\nabla + \lambda_0^{-1}\Phi + \lambda_0\Phi^* \in L$

Thus:

∇, Φ, Φ^* preserves L.

$\therefore V = E \oplus E^{\perp}$ gives the reduction.

#

换言之: 找3个与 ∇, Φ, Φ^* 有关的 connection on $L^* \otimes V$ parallel the inclusion ✓

Thm: (The gauge is generic condition w.r.t solutions of SD)

If $\nabla_1^\lambda = \nabla_1 + \lambda^{-1}\Phi_1 + \lambda\Phi_1^*$ are the ★

$\nabla_2^\lambda = \nabla_2 + \lambda^{-1}\Phi_2 + \lambda\Phi_2^*$

associated family of solutions of

self-Duality Equations such that:

$\exists\, \lambda_0 \in \mathbb{C}^*$. $\nabla_1^{\lambda_0} \sim \nabla_2^{\lambda_0}$ (gauge-equiva)

one $\lambda_0 \in \mathbb{C}^*$ 即包含了所有关于solution之信息。

then:

$(\nabla_1, \Phi_1, h_1) \sim (\nabla_2, \Phi_2, h_2)$

pf: Considering the parallel-bundle

$\{ s \in Hom((V_1, \nabla_1), (V_2, \nabla_2)) \mid s \text{ is parallel} \}$

想法是类似的：设 $\nabla_2^{\lambda_0} = \nabla_1^{\lambda_0} \cdot g$

$\therefore g \in \{Hom((V_1, V_2)) \mid g \text{ is parallel}\}$

考虑：$V_1^* \otimes V_2$，$(\nabla_1^{\lambda})^* \otimes \nabla_2^{\lambda}$

$\nabla_2(ge_1) + \cancel{\lambda_0^{-1}e_1} + \lambda_0^{-1}\Phi_2^*(e_1) - \cancel{\lambda_0^{-1}e_1} - \lambda_0^{-1}\Phi_2(e_1)$

$= g(\nabla_1 e_1) + \cancel{\lambda_0^{-1}\Phi_1^*(e_1)} + \lambda_0^{-1}e_1 - \lambda_0^{-1}\Phi_1(e_1) - \cancel{\lambda_0^{-1}e_1}$

类似考虑 F^{D_1}，F^{D_2} $\therefore F^{D_1}\big|_{span\{g\}}$

$= F^{D_2}\big|_{span\{g\}} = 0$ $\therefore \nabla_1 = \nabla_2 \cdot g$

#

☆ 这个定理事实上告诉我们：

the map: $(\nabla, \Phi, h) \longmapsto \nabla^{\lambda_0}$ is an

injective-map. (up to gauge)

Thm:

If (∇, Φ, h) is a solution of SD-Equation, then $\forall$ holo Line subbundle $L \hookrightarrow V$ with $\deg L = 0$, (∇, Φ, h) is direct sum of 2 Lower-rank solutions.

pf: $V = L \oplus L^{\perp}$, $\nabla = \begin{pmatrix} \nabla^{L} & -\alpha^{*} \\ \underset{\substack{\downarrow \\ (1,0)}}{\alpha} & \nabla^{L^{\perp}} \end{pmatrix}$, $\Phi = \begin{pmatrix} \omega & \beta \\ 0 & -\omega \end{pmatrix}$

$\omega \in H^{0}(\Sigma, K_{\Sigma})$

$$F^{\nabla} + [\Phi, \Phi^{*}] = 0 \rightarrow F^{\nabla^{L}} - \alpha^{*} \wedge \alpha + \beta + \beta^{*} = 0$$

$$\therefore \deg L = \frac{i}{2\pi} \int_{\Sigma} F^{\nabla^{L}} = \underbrace{\frac{i}{2\pi} \int_{\Sigma} \alpha^{*} \wedge \alpha - \beta \wedge \beta^{*}}_{\geq 0} = 0$$

$$\therefore \alpha = \beta = 0$$

#

Def: A Higgs Pair $(\bar{\partial}, \Phi)$ is called stable if $\forall E \hookrightarrow V$ holomorphic has negative degree.

Thm:

If $(\nabla^1, \bar{\Phi}^1, h^1)$, $(\nabla^2, \bar{\Phi}^2, h^2)$ are two solutions of SD-Equation s.t. $(\bar{\partial}^{\nabla^1}, \bar{\Phi}^1) \sim (\bar{\partial}^{\nabla^2}, \bar{\Phi}_2)$ gauge equivalent, then:

$$\nabla^1 \cong \nabla^2. \text{ (up to gauge-equiva).}$$

Lemma:

Stable Higgs Pairs don't admit non-trivial automorphism.

pf: Let: $(\bar{\partial}, \bar{\Phi}).g = (\bar{\partial}, \bar{\Phi})$

$\therefore g \in H^0(V^* V)$, if g is not the trivial element: $\exists p \in \Sigma$, s.t. g_p has eigenvalue λ. Let $\hat{g} = g - \lambda E \in \Gamma(\mathrm{End}\, E)$, then it cannot be isomorphism.

$\therefore g = \lambda E$.

#.

pf: now we come to our main thm.

Any hermitian-matrix can be transformed to another by $SL(2,\mathbb{C}) \rhd \mathcal{H}$

$(B, A) \longmapsto BAB^*$

In this case: $(\bar{\partial}_2 . g,\ g^{-1}\Phi_2 g,\ g^* h_2 = h_1)$

W.L.O.G. we can assue $h_1 = h_2$

Now if g is a gauging between $\bar{\partial}^{\nabla_1}, \bar{\partial}^{\nabla_2}$ with same hermitian metric h, then it is also for ∇^1, ∇^2.

So here the question is :

(∇^1, Φ^1, h^1) solution of SD

(D^2, Φ^2, h^1) satisfies :

$(\bar\partial^{\nabla^1}, \Phi^1) \underset{g}{\sim} (\bar\partial^{\nabla^2}, \Phi^2)$ which are two stable Higgs-Pairs, then g is unitary, hence:

(∇^2, Φ^2, h^1) is also a solution.

As: $\bar\partial^{\nabla_2} = \bar\partial^{\nabla_1} \cdot g$, then: $\bar\partial^{\nabla_2} \cdot g^* = \bar\partial^{\nabla_1}$

Computation: $\partial^{\nabla_2} = \partial^{\nabla_1} g$

$$\therefore h(\bar\partial^{\nabla_2} g^* s_1, s_2) = h((g^{-1})^* \bar\partial^{\nabla_2} g^* s_1, s_2)$$

$$= h(\bar\partial^{\nabla_2}(g^* s_1), g^{-1} s_2) = h(g^* s_1, \partial^{\nabla_2}(g^{-1} s_2))$$

$$\| $$

$$h(\bar\partial^{\nabla_1} s_1, s_2) = h(s_1, \partial^{\nabla_2} g^{-1} s_2)$$

$$= h(s_1, g\partial^{\nabla_2}(g^{-1} s_2)) = h(g^* s_1, \partial^{\nabla_2}(g^{-1} s_2))$$

$$\therefore \bar\partial^{\nabla_1} \underset{g^* g}{\overset{g g^*}{\rightleftarrows}} \bar\partial^{\nabla_1} \quad \therefore \begin{array}{l} g g^* = \lambda E \\ g^* g = \lambda E \end{array} \quad \therefore \lambda = 1$$

$\therefore g$ unitary

Big Picture

$$\nabla^\lambda = \nabla + \lambda^{-1}\Phi + \lambda\Phi^*$$

one ∇^{λ_0} contains everything about (∇, Φ, h)

$\left\{ \text{A Family of Flat Connections} \right\}$

the gauge between 2 family is a generic condition.

$\left\{ \begin{array}{l} \rho\text{-equiva} \\ \text{harmonic} \\ f : \tilde{\Sigma} \to \mathbb{H}^3 \end{array} \right\}$

irreducible $\left\{ \text{Self-Duality Equation} \right\}$

(∇, Φ, h)

$$\begin{cases} \bar{\partial}^\nabla \Phi = 0 \\ F^\nabla + [\Phi, \Phi^*] = 0 \end{cases}$$

$$\mathcal{E}(f) \leq (2g-2)2\pi + 4\int_\Sigma |\det|$$

Now we want to show a big Correspondence :

$$\begin{tikzcd}
 & \mathcal{M}_{SD}^{irr} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\mathcal{M}_{Dol}^{S} & & \mathcal{M}_{dR}^{irr}
\end{tikzcd}$$

We have given to map

injective map: $\mathcal{M}_{SD}^{irr} \longrightarrow \mathcal{M}_{dR}^{irr}$

$[\nabla, \Phi, h] \longmapsto \nabla^{\lambda_0} \quad (\lambda_0 \in \mathbb{C}^*)$

$\mathcal{M}_{SD}^{irr} \longrightarrow \mathcal{M}_{Dol}^{S}$

$[\nabla, \Phi, h] \longmapsto (\bar{\partial}^{\nabla}, \Phi)$

Hyperkahler Manifold.

见 Appendix.

We hope to construct this structure on our Moduli – Space.

(Delign, Simpson): Twistor – Approach.

Prop: Let (M, g, I, J, K) be hyperkahler, then: $xI+yJ+zK$ is an integrable cpx structure $\Leftrightarrow$ $x^2+y^2+z^2=1$ $(x, y, z \in \mathbb{R})$

Def: For a hyperkahler mfd (M, g, I, J, K) the twistor Space P is given by $M \times \mathbb{C}P^1$ with almost complex structure at (p, x, y, z) by $I_{(p,x,y,z)} := (xI_p + yJ_p + zK_p,\ I_{(x,y,z)})$ on $T_pM \oplus T_{(x,y,z)}S^2$

Rk: The surjection of S^2 and $\mathbb{C}P^1$ is given by:

$$\underset{\mathbb{C}P^1}{\lambda} \longmapsto \frac{1}{1+\lambda\bar{\lambda}}(1-\lambda\bar{\lambda},\ \lambda+\bar{\lambda},\ i(\lambda-\bar{\lambda}))$$

$P \approx M\times\mathbb{C}P^1$ (smooth意义下是一样的)

Thm:

(P, I) is a complex mfd with holo-fibration

$\pi: P \longrightarrow \mathbb{C}P^1$! (此复结构十分有趣)

For $\forall p\in P$

We got a section of $\begin{array}{c} P \\ \downarrow \pi \\ \mathbb{C}P^1 \end{array}$, $s: \mathbb{C}P^1 \longrightarrow P$

which is given by: $s_p(\lambda) = (p, \lambda)$

Prop: the section $s_p(\lambda)\in P$ is holomorphic.

(此即section $s: \mathbb{C}P^1 \to P$ is a holomorphic map)

Def: The normal-bundle for $s: \mathbb{CP}^1 \longrightarrow \mathbb{P}$ is defined by: $\mathcal{N}_s = s^*T\mathbb{P}/TS$

$$\begin{tikzcd} P = M \times S^2 \arrow[d, bend right, "\pi"'] \\ \mathbb{CP}^1 = S^2 \arrow[u, "s_p"'] \end{tikzcd} \qquad T_{s_p(\lambda)}P = \operatorname{Im} d_{s_p(\lambda)} \oplus \ker d_{s_p(\lambda)}\pi.$$

(Smooth 意义下: $\mathcal{N}_{s_p(\lambda)} \cong T_pM$).

Prop: $\mathcal{N}_s \cong \mathcal{O}(1)^{\oplus 2m}$, where $\dim_{\mathbb{C}} M = 2m$

$\dim_{\mathbb{R}} M = 4m$.

pf: by construction: $\dim \mathcal{N}_s = \dim_{\mathbb{C}} TM = 2m$

$T_{s_p(\lambda)}P = T_pM \oplus \underline{T_\lambda\mathbb{CP}^1} = T_pM \oplus \mathcal{O}(-2)$

Identify: $T_pM = \mathbb{H}^* \otimes T_pM \otimes \mathbb{H} = \operatorname{Hom}(\mathbb{H}, T_pM) \otimes \mathbb{H}$

记作 $E \otimes \mathbb{H}$, where $E = \operatorname{Hom}(\mathbb{H}, T_pM)$

Identify $\mathbb{H} = \mathbb{C}^2$, $I = \begin{pmatrix} i & \\ & -i \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $K = \begin{pmatrix} & i \\ i & \end{pmatrix}$

$\mathbb{CP}^1 \cong S^2$

$\lambda \longmapsto \frac{1}{1+\lambda\bar{\lambda}}(1-\lambda\bar{\lambda}, \lambda+\bar{\lambda}, i(\lambda-\bar{\lambda}))$.

(取 holomorphic-basis, then 考虑 transition-function)

Lemma: There exists a isomorphism:

$$T_pM \xrightarrow{\varphi} \operatorname{Hom}_{\mathbb{H}}(\mathbb{H}, T_pM) \otimes \mathbb{H}$$

s.t. $\varphi\hat{I} = I\varphi$, $\varphi\hat{J} = J\varphi$, $\varphi\tilde{K} = K\varphi$

$\left(T_pM, \frac{1-\lambda\bar\lambda}{1+\lambda\bar\lambda} I_p + \frac{\lambda+\bar\lambda}{1+\lambda\bar\lambda} J_p + \frac{i(\lambda-\bar\lambda)}{1+\lambda\bar\lambda} K_p\right)$

$\mathrm{Hom}_{\mathbb{H}}(\mathbb{H}, T_pM) = \mathbb{H}^{\dim_{\mathbb{H}} T_pM = m}$.

In this case: if $v \in T_pM$ is an i-eigenvalue vector: $\hat{I}v = iv$, then: $I\varphi v = \varphi \hat{I} v = i\varphi v$

故：先看 I 的 i-eigen space.

$$\begin{tikzcd} T_pM \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & \mathrm{Hom}_{\mathbb{H}}(\mathbb{H}, T_pM) \otimes \mathbb{H} \arrow[d, "\pi"] \\ & M \end{tikzcd} \quad (*)$$

$(T_pM, x_0 I + y_0 J + z_0 K) \cong (\mathrm{Hom}_{\mathbb{H}}(\mathbb{H}, T_pM) \otimes \mathbb{H}, x_0\hat{I} + y_0\hat{J} + z_0\hat{K})$

the complex-structure of $(*)$ is given by: $I_\lambda^{\oplus 2m} = \frac{1}{1+\lambda\bar\lambda}\begin{pmatrix} i(1-\lambda\bar\lambda) & 2\lambda \\ -2\bar\lambda & -i(1-\lambda\bar\lambda) \end{pmatrix}$

the i-eigenvalue space $= \mathrm{span}\left\{\begin{pmatrix} 1 \\ i\lambda \end{pmatrix}, \begin{pmatrix} -\frac{i}{\lambda} \\ 1 \end{pmatrix}\right\}$

$\therefore$ the transition function $\hat\lambda = \frac{1}{\lambda} \rightsquigarrow \mathcal{O}(1)^{\oplus 2}$.

$\therefore TM \cong \left(\mathcal{O}(1)^2\right)^{\oplus T_p^{\mathbb{H}} M} \cong \mathcal{O}(1)^{\oplus 2m}$

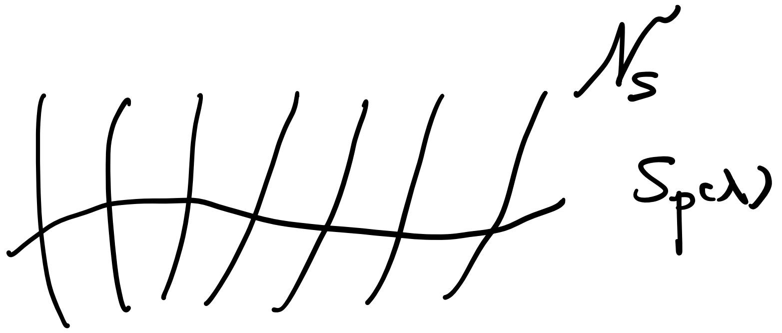
For $P = M \times S^2$, where M is a hyperkahler manifold with (M, g, I, J, K), the complex structure at (m, x_0, y_0, z_0) is given by:

$$(x_0 I + y_0 J + z_0 K, \; J_{(x_0, y_0, z_0)} \mathbb{C}P^1).$$

Def: Fix one $p \in M$, the sections $S_p(\lambda) = (p, \lambda)$ are called the twistor lines in P.

Kodaira-Spencer Deformation Theory of Twistor Lines in P. (Deformation of complex submanifold)

We try to deform the $S_p(\lambda) \subset P$ along the normal bundle N_S.



$$H^2_{\bar{\partial}}(\mathbb{C}P^1, T_S)$$

$$= H^1_{\bar{\partial}}(\mathbb{C}P^1, N_S).$$

So the Obstruction:

$$H^1_{\bar\partial}(\mathbb{C}P^1, \mathcal{O}(1)^{\oplus 2m}) = \bigoplus_{2m} H^1_{\bar\partial}(\mathbb{C}P^1, \mathcal{O}(1))$$

by Serre-Duality: $H^1(\mathbb{C}P^1, \mathcal{O}(1)) \cong H^0(K\mathcal{O}(-1))$
$= 0.$

$\therefore$ We have no obstructions $\Rightarrow$ We can deform S by $\forall X \in H^0(\Sigma, N_S)$ which is a $4m$ real-mfd.

Prop: $\exists$ a involution $\mathcal{J}: P \to P$, such that:

① $d\mathcal{J}$ is anti-complex-linear which covers $\lambda \mapsto \bar\lambda^{-1}$ on $\mathbb{C}P^1$

② the twistor-line are real:

$$\mathcal{J}(S(-\bar\lambda^{-1})) = S(\lambda), \ \forall \lambda \in \mathbb{C}P^1$$