Holographic entanglement entropy

Zhe Yu Difei Dong

Sun Yat-san University
School of Physics

January 22, 2024

Background

Entanglement entropy plays a variety of roles in quantum field theory, including the connections between quantum states and gravitation through the holographic principle.

Our report is aimed at providing basic concepts of quantum entanglement and some examples in theoretical physics, especially in conformal field theory.

Here is the content.

Density Operater

Density Operater

$$\rho = \sum_n p_n \left|\varphi_n\right\rangle \left\langle\varphi_n\right| \tag{1}$$

- Density operator is similar to mechanical operator.
- Density operator is a generalization of classical probability.

Density Operater

- **Classic situation**
  - States:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2}$$

  - Density matrix:

$$\rho = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3}$$

  - Probability:

$$p_0 = \langle 0| \rho |0\rangle , \quad p_1 = \langle 1| \rho |1\rangle \tag{4}$$

- **Quantum situation**
  - States:

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{5}$$

  - Density matrix:

$$\rho = \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{6}$$

  - Probability:

$$p_0 = \langle 0| \rho |0\rangle , \quad p_1 = \langle 1| \rho |1\rangle \tag{7}$$

Pure State and Mixed State

1. **Pure State:** A pure state is defined as a density operator that can be expressed as $|\phi\rangle\langle\phi|$ under an appropriate basis choice.

   eg1. $\frac{1}{2}(|1\rangle\langle 1| + |1\rangle\langle 0| + |0\rangle\langle 1| + |0\rangle\langle 0|) = (\frac{|1\rangle+|0\rangle}{\sqrt{2}})(\frac{\langle 1|+\langle 0|}{\sqrt{2}})$

   eg2. $\rho = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} \xrightarrow{\text{Diagonalization}} \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$

2. **Mixed State:** A Mixed state is defined as a density operator that can be expressed as $\sum_i |\phi_i\rangle\langle\phi_i|$ under an appropriate basis choice.

   eg1. $\frac{1}{2}(|1\rangle\langle 1| + |0\rangle\langle 0|) + \frac{1}{4}(|1\rangle\langle 0| + |1\rangle\langle 0|) =$
   $\frac{3}{4}(\frac{|1\rangle+|0\rangle}{\sqrt{2}})(\frac{\langle 1|+\langle 0|}{\sqrt{2}}) + \frac{1}{4}(\frac{|1\rangle-|0\rangle}{\sqrt{2}})(\frac{\langle 1|-\langle 0|}{\sqrt{2}})$

   eg2. $\rho = \begin{bmatrix}\frac{1}{2} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{2}\end{bmatrix} \xrightarrow{\text{Diagonalization}} \begin{bmatrix}\frac{3}{4} & 0\\ 0 & \frac{1}{4}\end{bmatrix}$

3. **Partial Trace:** Let $\{|i\rangle_B, i = 1, 2, \cdots\}$ be an orthonormal basis in H_B, and define the *reduced density matrix* ρ_A of the system A by taking the partial trace over the system B

$$\rho_A \equiv \mathrm{Tr}_B(\rho_{\text{tot}}) \equiv \sum_i {}_B\langle i|\,\rho_{\text{tot}}\,|i\rangle_B\ . \tag{8}$$

Question

What is the amount of information?

We use the function $I(p)$ of p to describe the amount of information. $I(p)$ should satisfy the following properties.

1. The smaller the probability, the greater the information.
2. When an inevitable event occurs, the amount of information is zero.
3. The amount of information brought by different events can be added together.

Translate the above content into an equations.

$$\begin{cases} \dfrac{\mathrm{d}I(p)}{\mathrm{d}p} < 0 \\ I(1) = 0 \\ I(p_1 \cdot p_2) = I(p_1) + I(p_2) \end{cases} \tag{9}$$

Shannon Entropy

From the equation9 above, we can derive the expression for information content.

$$I(p) = -\log(p) \tag{10}$$

If we average the information content according to the probability of events, we can obtain the average information content, which is Shannon entropy.

$$H(x) = -\sum_{i=1}^{n} p(x_i) \log_2 p(x_i) \tag{11}$$

Shannon entropy can serve as a measure of understanding of the future.

Von Neumann Entropy

Von Neumann Entropy is also a generalization of classical probability.

As we had discussed before, the probability distribution will change as the basis is changed in quantum conditions, so when calculating the information entropy, the summation operation is replaced by the trace operation.[1]

$$S = -\mathsf{Tr}(\rho \log(\rho)) \tag{12}$$

Specifically, the trace here avoids the influence of basis choice on the result.

Clearly, the von Neumann entropy of a pure state is zero.

$$S_{\mathsf{pure}} = -\mathsf{Tr}\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \log\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right) = 0 \tag{13}$$

Von Neumann Entropy

Consider a system C in a pure state ρ_{tot} that is completely divided into two disjoint subsystems A and B.

$$\rho_{\text{tot}} = |\psi\rangle\langle\psi| = (\sum_{i,j} \lambda_{ij} |i\rangle_B \otimes |j\rangle_A)(\sum_{i,j} \lambda_{ij\,A}\langle j| \otimes {}_B\langle i|) \tag{14}$$

Where λ_{ij}^2 is the probability B in state i while A in state j.

$$\rho_A = \text{Tr}_B \rho_{\text{tot}} = \sum_k {}_B\langle k|\rho_{\text{tot}}|k\rangle_B = \sum_k (\sum_j \lambda_{kj} |j\rangle_A)(\sum_j \lambda_{kj\,A}\langle j|) \tag{15}$$

Linear transformation to a new orthonormal basis

$$|j_k\rangle = \frac{\sum_j \lambda_{kj} |j\rangle}{\sqrt{\sum_j \lambda_{kj}^2}} \tag{16}$$

Von Neumann Entropy

$$
\begin{aligned}
\rho_A &= \sum_k \left(\sqrt{\sum_j \lambda_{kj}^2}\,|j_k\rangle\right)\left(\sqrt{\sum_j \lambda_{kj}^2}\,\langle j_k|\right) \\
&= \sum_k \left[(\sum_j \lambda_{kj}^2)\,|j_k\rangle\langle j_k|\right]
\end{aligned} \tag{17}
$$

Here, $\sum_j \lambda_{ij}^2$ is evidently the probability of B being in state k P_k. Note that there may be $|j_n\rangle = |j_m\rangle$, assuming
$|j_{01}\rangle = |j_{02}\rangle = \cdots = |j_{0m_1}\rangle \neq |j_{11}\rangle = \cdots = |j_{1m_2}\rangle \neq \cdots$.

$$
S(A) = -\sum_n P_n \log P_n, \quad P_n = \sum_m P_{nm} \tag{18}
$$

Clearly, the physical meaning of $S(A)$ is how much we can learn about A on average by measuring B.

Von Neumann Entropy

Taking the example of a two-electron spin system.

Non-entangled State:

$|\psi\rangle = \frac{1}{2}(|11\rangle + |10\rangle + |01\rangle + |00\rangle)$,$\rho_{\text{tot}} = |\psi\rangle\langle\psi|$

$\rho_A = \text{Tr}_B \rho_{\text{tot}} = |\psi_{A0}\rangle\langle\psi_{A0}| + |\psi_{A1}\rangle\langle\psi_{A1}|$

$|\psi_{A0}\rangle = {}_B\langle 0|\psi\rangle = \frac{1}{2}(|1\rangle + |0\rangle)$,$|\psi_{A1}\rangle = {}_B\langle 1|\psi\rangle = \frac{1}{2}(|0\rangle + |1\rangle)$

$\rho_A = \frac{1}{2}(|1\rangle + |0\rangle)(\langle 1| + \langle 0|)$

$S(A) = 0$

Bell State:

$|\psi\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$,$\rho_{\text{tot}} = |\psi\rangle\langle\psi|$

$\rho_A = \text{Tr}_B \rho_{\text{tot}} = |\psi_{A0}\rangle\langle\psi_{A0}| + |\psi_{A1}\rangle\langle\psi_{A1}|$

$|\psi_{A0}\rangle = {}_B\langle 0|\psi\rangle = \frac{1}{\sqrt{2}}|1\rangle$,$|\psi_{A1}\rangle = {}_B\langle 1|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle$

$\rho_A = \frac{1}{2}(|1\rangle\langle 1| + |0\rangle\langle 0|)$

$S(A) = -\frac{1}{2}\log(\frac{1}{2}) \times 2 = \log(2)$

Strong Subadditivity

Strong Subadditivity

$$S_{A\cup B\cup C} + S_B \leq S_{A\cup B} + S_{B\cup C}$$
$$S_A + S_C \leq S_{A\cup B} + S_{B\cup C} \tag{19}$$

[1] In the current scenario, there are two subsystems X and Y. Their union, denoted as $X \cup Y$, also forms a subsystem, with the reduced density matrix of $X \cup Y$ being represented as

$$\rho_{X\cup Y} = \sum_{ij} P_{ij}(|x_i\rangle \otimes |y_j\rangle)(\langle x_i| \otimes \langle y_j|), P_{ij} = P_{XY}(x_i, y_j) \tag{20}$$

At the same time, we can also derive the reduced density matrix for subsystem Y based on this.

$$\rho_Y = \sum_j (\sum_i P_{ij}) |y_j\rangle \langle y_j| , \sum_i P_{ij} = P_Y(y_j) \tag{21}$$

Strong Subadditivity

$$
\begin{aligned}
S_{X\cup Y} - S_Y &= \sum_{i,j} P_{XY}(x_i, y_j)\log(P_{XY}(x_i, y_j)) - \sum_j P_Y(y_j)\log(P_Y(y_j)) \\
&= \sum_j \left[\sum_i P_{XY}(x_i, y_j)\log(P_{XY}(x_i, y_j)) - P_Y(y_j)\log(P_Y(y_j))\right] \\
&= \sum_{i,j}\left[P_{XY}(x_i, y_j)\left(\log\frac{P_{XY}(x_i, y_j)}{P_Y(y_j)}\right)\right] = \sum_j P_Y(y_j) S_{X|y_i}
\end{aligned} \tag{22}
$$

We can observe that the physical significance of $S_{X|Y} = \sum\limits_j P_Y(y_j) S_{X|y_i}$ is the average amount of information about the overall system that can still be obtained through measurements on subsystem X, after repeated measurements on subsystem Y. Therefore, as Y becomes larger, A should become smaller. Notice that $B \cup C \geq B$.

$$
S_{A|B\cup C} \leq S_{A|B} \Rightarrow S_{A\cup B\cup C} - S_{B\cup C} \leq S_{A\cup B} - S_B \tag{23}
$$

Limitations

The von Neumann entropy still has the following limitations[1]:

1. It can only measure the entanglement between the quantum subsystem and the overall system.
2. The overall system needs to be in a pure state.
3. It is computationally challenging.

For the first issue, we can use mutual information and relative entropy, while the latter two problems can be addressed using Rényi entropy.

Mutual Information

$$I(A, B) = S_A + S_B - S_{A \cup B} \tag{24}$$

Relative Entropy

$$S(A||B) = \mathsf{Tr}[\rho_A(\log \rho_A - \log \rho_B)] \tag{25}$$

Rényi Entropy

$$S_n(A) = \frac{1}{1-n} \log[\mathsf{Tr}_A(\rho_A^n)] \tag{26}$$

Rényi Entropy vs Von Neumann Entropy

Von Neumann Entropy is the limitation of Rényi Entropy

$$S(A) = \lim_{n \to 1} S_n(A) \tag{27}$$

Under unitary transformations, the trace of a matrix remains invariant. Therefore, it is convenient for us to diagonalize matrix A.

$$\begin{aligned} S_n(A) &= \frac{1}{1-n} \log[\mathsf{Tr}(\rho_A^n)] = \frac{1}{1-n} \log[\mathsf{Tr}(\begin{bmatrix} p_1^n & & & \\ & p_2^n & & \\ & & \ddots & \\ & & & p_N^n \end{bmatrix})] \\ &= \frac{1}{1-n} \log \sum_{i=1}^{n} p_i^n \end{aligned} \tag{28}$$

$$\lim_{n \to 1} S_n(A) = \frac{\partial(\log \sum\limits_{i=1}^{N} p_i^n)}{\partial n} \Big/ \left.\frac{\partial(1-n)}{\partial n}\right|_{n=1} = -\left.\frac{\sum\limits_{i=1}^{N} p_i^n \log p_i}{\sum\limits_{i=1}^{N} p_i^n}\right|_{n=1} = S(A) \tag{29}$$

Density operater in QFT

At any time t, all field configurations form a complete basis of Hilbert space.

$$I = \int [D\phi]_t \left|\phi\right\rangle \left\langle\phi\right| \tag{30}$$

For thermal state, $\rho = \frac{1}{Z}e^{-\beta H}$, replace β with it. Here, we have extended the time analysis to the imaginary axis. So, path integral is also useful in the imaginary time.

$$\begin{aligned} Tr(e^{-\beta H}) &= \int [D\phi]_{t=0} \left\langle\phi\right| e^{-iH(-i\beta)} \left|\phi\right\rangle \\ &= \int d\phi_{t=0} \int [D\phi]_{\phi(t=0)=\phi(t=-i\beta)} \exp\left(i \int_{t=0}^{t=-i\beta} \mathcal{L} dx\right) \\ &= Z \end{aligned}$$

Density operater in QFT

Define

$$\tau = it$$
$$\mathcal{L}_E(\phi) = -\mathcal{L}(\phi)$$

Then a microcanonical ensemble can be described by a Euclidean field theory

$$Tr(e^{-\beta H}) = \int [D\phi]_{\phi(\tau=0)=\phi(\tau=\beta)} \exp\left(-\int_{\tau=0}^{\tau=\beta} \mathcal{L}_E dx\right) \tag{31}$$

Density operater in QFT

And we can give density operator as

$$\rho = \frac{1}{Z}\int [D\phi]^{\phi(\tau=\beta)=?}_{\phi(\tau=0)=?} \exp\left(-\int_{\tau=0}^{\tau=\beta} \mathcal{L}_E dx\right) \tag{32}$$

Because of the same boundary condition, we can sew the boundary together and change time into a circle S^1.

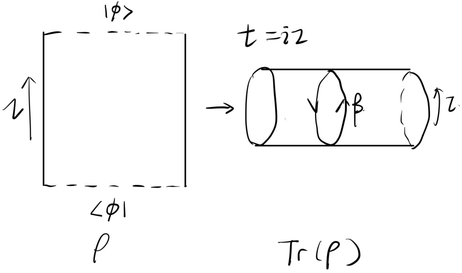


Figure: density operater with finite temperature $1/\beta$

Reduced density operater in QFT

When $\beta \to \infty$, it is a zero-temperature QFT. For simpilicity, the following example we are going to consider are all zero-temperature QFT. Vacuum state is given by

$$|0\rangle = \int [D\phi]_{\phi(t=-\infty)=\phi_0}^{\phi(t=0)=?} (-\int_{\tau=-\infty}^{\tau=0} \mathcal{L}_E dx) \tag{33}$$

Reduced density matrix indicates fixing part of the boundary.

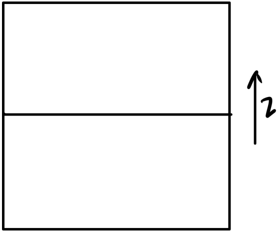


Figure: Fix the boundary completely
$Z = Tr(|0\rangle \langle 0|) = \langle 0|0\rangle$

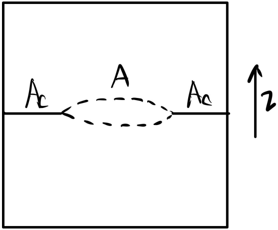


Figure: Fix part of the boundary
$\rho_A = Tr_{A_c}(|0\rangle \langle 0|)$

Renyi entropy and replica Trick

Replica trick allows us to calculate renyi entropy first which is much easier to do, and let n tend to 1 to get von neunman entropy. Notice that

$$Tr\rho_A^3 = \sum_{i,j,k} \langle i|\rho_A|j\rangle \langle j|\rho_A|k\rangle \langle k|\rho_A|i\rangle \tag{34}$$

This indicates that we can sew the un-fixed boundary together and then calculate path integral on a larger manifold just like the Riemann sheet.

Figure: Replica sheet with 3 copy. Resource:[2]

Calculation through twist field

Denote the partition function on n-sheet Riemann surface by $Tr(\rho_A^n) = \frac{1}{Z^n} Z_n[A]$. The difference between fields living on one sheet and fields living on n-sheet Riemann surface is that the latter has connection conditions between different Riemann surfaces. To describe the difference, it is useful to associate a local primary field to any branch point which is called twist field by Cardy[3].

$$Z_n[A] = \langle \mathcal{T}_n(u,0)\tilde{\mathcal{T}}_n(v,0)\rangle \tag{35}$$

The two-point function is completely determined by the conformal weight of the twist field d_n.

$$\langle \mathcal{T}_n(u,0)\tilde{\mathcal{T}}_n(v,0)\rangle \propto \frac{1}{|u-v|^{2d_n}} \tag{36}$$

More generally, the expected value of any operator is given by

$$\langle O(x,\tau)\rangle = \frac{\langle \mathcal{T}_n(u,0)O(x,\tau)\tilde{\mathcal{T}}_n(v,0)\rangle}{\langle \mathcal{T}_n(u,0)\tilde{\mathcal{T}}_n(v,0)\rangle} \tag{37}$$

Determination of the conformal weight of twist field

By comparing the expected value of the energy and momentum tensor of the field obtained by two different calculation methods, the conformal weight of the field can be completely determined.

$$d_n = \frac{c}{12}(n - \frac{1}{n}) \tag{38}$$

Thus, renyi entropy $S_A^{(n)}$ can be given by

$$S_A^{(n)} = \frac{1}{1-n} \log \frac{Z_n[A]}{Z^n} = \frac{c}{6}(1 + \frac{1}{n}) \ln \frac{l}{a} + c' \tag{39}$$

where $l = |u - v|$ and a is the UV cut-off factor. c is the central charge of a CFT, and $c' << $ the first term.

AdS/CFT correspondence

In theoretical physics, the AdS/CFT correspondence is a conjectured relationship between two kinds of physical theories.

AdS	$\leftrightarrow$	CFT
Quantum gravity in asymptotic $AdS_{d+1} \times M$ space-time.		Any Conformal field theory on the $R \times S^{d-1}$ spacetime(AdS boundary).

The most famous example of the AdS/CFT correspondence proposed by Maldacena in 1997[4] states that type IIB string theory on the product space $AdS_5 \times S^5$ is dynamically equivalent to $N = 4$ supersymmetric Yang–Mills theory on the four-dimensional boundary.

RT surface

Based on the considerations of AdS/CFT, the entanglement entropy in CFT should be able to find a correspondence in the dual theory. Ryu and Takayanagi's remarkable work in 2006[5] showed that it corresponds to a minimal surface in AdS space-time.

$$S(A) = \frac{Area(\gamma)}{4G} \tag{40}$$

where S(A) is von neumann entropy, γ is called RT surface and G is the gravitational constant. This result was improved in [6] to make it suitable for non-static AdS space-time.

In 2013, Maldacena et al.[7] improved the RT again and added quantum corrections.

$$S(A) = \frac{Area(\gamma)}{4G} + S_{field}^{R_A} \tag{41}$$

and this result was improved again in 2014[8].

RT surface: AdS_3/CFT_2

The first term, $\frac{Area(\gamma)}{4G}$, is considered in AdS_3/CFT_2, as shown below:

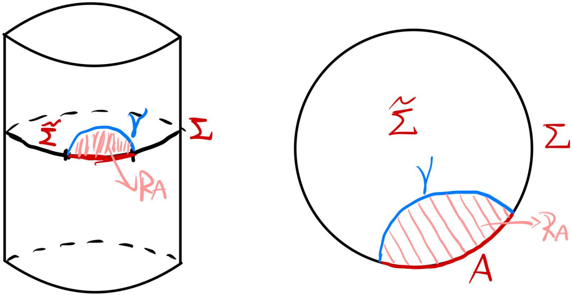


Figure: RT surface in AdS_3

Using the variational method, it is easy to obtain the equation of a minimal surface and then find the area.

RT surface: AdS_3/CFT_2

Using Poincare coordinates for AdS_3,

$$ds^2 = \frac{l_{AdS}^2}{z^2}(-dt^2 + dx^2 + dz^2), boundary\ z = 0$$

$$ds^2 = \frac{l_{AdS}^2}{z^2}(dx^2 + dz^2), \qquad fixed \quad t$$

The area can be shown as:

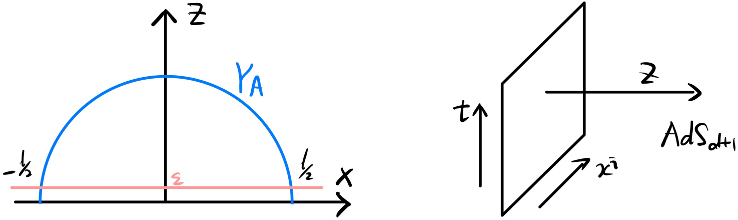


Figure: minimal Surface in Poincare coordinates

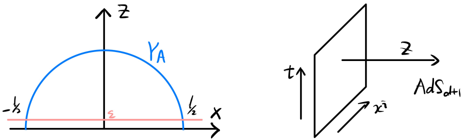


Figure: minimal Surface in Poincare coordinates

The length is given as:

$$length = \int \frac{l_{AdS}}{z}\sqrt{dx^2 + dz^2} = \int \frac{l_{AdS}}{z}\sqrt{1 + z'^2}dx \tag{42}$$

Upon substitution into the Lagrange equation, we obtain the following:

$$x^2 + z^2 = (l/2)^2 \tag{43}$$

So, the length is given by

$$length = 2l \ln \frac{l}{\epsilon} \tag{44}$$

RT surface: AdS_3/CFT_2

According to RT formula, the entanglement entropy is

$$S = \frac{length = 2l}{4G} = \frac{l}{2G} \ln \frac{l}{\epsilon} \tag{45}$$

The map between gravity parameters and CFT parameters in AdS_3/CFT_2 is

$$c = \frac{3l}{2G_N} \tag{46}$$

thus

$$S = \frac{c}{3} \ln \frac{l}{\epsilon} \tag{47}$$

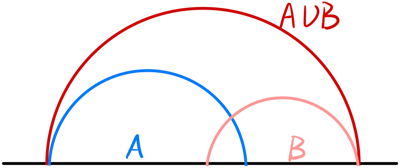
The holographic method gives the same results as the CFT method,

Properties of entanglement entropy : an intuitive proof

Recall the properties of entanglement entropy: *subadditivity* and *strong subadditivity*.

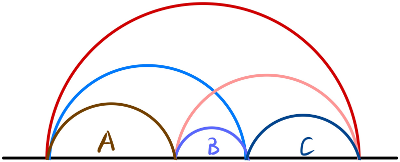
subadditivity

$$S_{A\cup B} \leq S_A + S_B$$



strong subadditivity

$$S_{A\cup B\cup C} + S_B \leq S_{A\cup B} + S_{B\cup C}$$
$$S_A + S_C \leq S_{A\cup B} + S_{B\cup C}$$



Summary

In this report, we introduce the basic concepts of quantum entanglement and calculate the simplest example of entanglement entropy in CFT using two different methods, demonstrating the interesting nature of the holographic principle.

The study of entanglement entropy greatly promoted the solution of the black hole information paradox. The RT formula was subsequently developed into the quantum extreme surface, which in turn led to the establishment of the island rule.

However, entanglement entropy is so complex that there are very few systems for which we can calculate it analytically. There are still many problems waiting for us to solve.

Reference I

- T. Nishioka, *Entanglement entropy: Holography and renormalization group*, *Reviews of Modern Physics* **90** (2018) .
- P. Calabrese and J. Cardy, *Entanglement entropy and conformal field theory*, *Journal of Physics A: Mathematical and Theoretical* **42** (2009) 504005.
- J.L. Cardy, O.A. Castro-Alvaredo and B. Doyon, *Form factors of branch-point twist fields in quantum integrable models and entanglement entropy*, *Journal of Statistical Physics* **130** (2007) 129.
- J. Maldacena, *The large-n limit of superconformal field theories and supergravity*, *International journal of theoretical physics* **38** (1999) 1113.
- S. Ryu and T. Takayanagi, *Holographic derivation of entanglement entropy from the anti–de sitter space/conformal field theory correspondence*, *Physical Review Letters* **96** (2006) .
- V.E. Hubeny, M. Rangamani and T. Takayanagi, *A covariant holographic entanglement entropy proposal*, *Journal of High Energy Physics* **2007** (2007) 062.
- A. Lewkowycz and J. Maldacena, *Generalized gravitational entropy*, *Journal of High Energy Physics* **2013** (2013) .
- N. Engelhardt and A.C. Wall, *Quantum extremal surfaces: holographic entanglement entropy beyond the classical regime*, *Journal of High Energy Physics* **2015** (2015) .

Thank you!