Reading note of ”1906.08616 Covariant phase space with boundaries”

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Abstract: This reading note is mainly based on Daniel Harlow and Jieqiang Wu’s paper Covariant phase space with boundaries and Daniel’s two talks on IAS and QGI Virtual Seminar.

Contents

1 Notation

Notation of p-forms:

$$\begin{aligned}(\omega\wedge\sigma)_{\mu_1\ldots\mu_p\nu_1\ldots\nu_q} &= \frac{(p+q)!}{p!q!}\omega_{[\mu_1\ldots\mu_p}\sigma_{\nu_1\ldots\nu_q]}\\ (d\omega)_{\mu_0\ldots\mu_p} &= (p+1)\partial_{[\mu_0}\omega_{\mu_1\ldots\mu_p]}\\ (\star\omega)_{\mu_1\ldots\mu_{d-p}} &= \frac{1}{p!}\epsilon^{\nu_1\ldots\nu_p}{}_{\mu_1\ldots\mu_{d-p}}\omega_{\nu_1\ldots\nu_p}.\end{aligned} \tag{1.1}$$

Cartan's magic formula

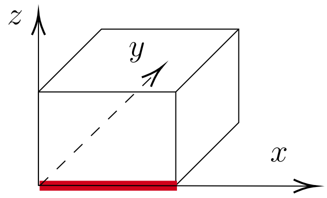
$$\pounds_X\omega = X\cdot d\omega + d(X\cdot\omega) \tag{1.2}$$

$X\cdot$ means i_X. We denote the volume form of M by ϵ, and the volume form of its boundary is given by

$$\epsilon = n\wedge\epsilon_{\partial M}.$$

There is a very important detail, that the signature of $\epsilon_{\partial M}$ depends on how you construct it. For example, we want to get the volume form of an edge of a cube (red line). The volume form of a cube is given by $dx\wedge dy\wedge dz$, you can view it as the boundary of xy plane

$$\begin{aligned}dx\wedge dy\wedge dz &= dz\wedge\epsilon_{xy}\\ \epsilon_{xy} &= dy\wedge\epsilon_{red},\\ \epsilon_{red} &= -dx\end{aligned}$$



or boundary of xz plane

$$\begin{aligned} dx \wedge dy \wedge dz &= dy \wedge \epsilon_{xz} \\ \epsilon_{xz} &= dz \wedge \epsilon_{red}. \\ \epsilon_{red} &= dx \end{aligned}$$

Stokes' theorem

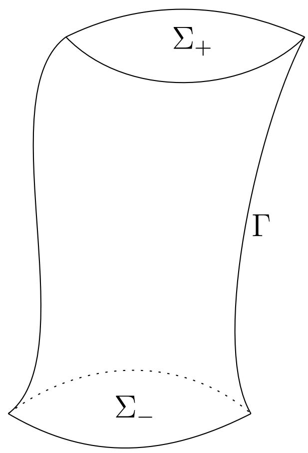
$$\int_M d\omega = \int_{\partial M} \omega \tag{1.3}$$

In this note, we consider two distinct manifolds: spacetime and phase space, the latter typically being infinite-dimensional. d denotes the exterior derivative on spacetime, δ denotes the exterior derivative on configuration space (and also its pullback to pre-phase space and phase space).

2 Covariant phase space

2.1 Basic ideas

The basic idea of covariant space is to view the phase space as a collection of EoM which obey the spatial boundary condition but not future/past boundary condition. In other words, the future past boundaries are not fixed, and the spatial boundary condition is part of the definition of the theory.



Things will become much easier if there's no redundant freedoms in the theory. We view the phase space as an abstract manifold $\mathcal{P}$, endowed with a closed non-degenerate two-form Ω called the **symplectic form**. A manifold equipped with such a form is called **symplectic manifold**. We view the symplectic form as a map:

$$\begin{matrix} X_f = \Omega^{-1}(\cdot, \delta f) \\ \delta f = \Omega(\cdot, X_f) \end{matrix} \quad or \quad \begin{matrix} X_f = \Omega^{-1}(\delta f) \\ \delta f = \Omega(X_f) \end{matrix}.$$

The Hamiltonian is a scalar field H on phase space, we can then define a vector field

$$X_H(f) \equiv \Omega^{-1}(\delta f, \delta H), \tag{2.1}$$

and $\delta H = -X_H \cdot \Omega$.

The integral curves of X_H in $\mathcal{P}$ give the time evolution of the system.

$$\dot{f} \equiv X_H(f) = \{f, H\} \tag{2.2}$$

where the *Poisson bracket* is defined as

$$\{f, g\} \equiv \Omega^{-1}(\delta f, \delta g) = \Omega(X_g, X_f). \tag{2.3}$$

So, why the 2-form on phase is required to be non-degenerate and closed? Since,

- Non-degeneracy is obvious, because Ω has an inverse.
- A closed form ensures that the Poisson bracket is preserved under time evolution.

$$\begin{aligned} \pounds_{X_H}\Omega &= X_H \cdot \delta\Omega + \delta(X_H \cdot \Omega) \\ &= X_H \cdot \delta\Omega + \delta(-\delta H) \\ &= X_H \cdot \delta\Omega \end{aligned}$$

2.2 Redundancy

But things are not that easy, because the system contains redundant degrees of freedom just like diffeomorphism. When we consider a covariant phase space, it means that the choice of coordinates is not fixed; in other words, it entails considering all possible coordinate systems. Therefore, two field configurations that might appear different can actually be connected through a diffeomorphism. Thus, there is redundancy in the phase space formed by simply considering all solutions of the EoM, which is denoted as pre-phase space $\tilde{\mathcal{P}}$. Let me summarize them as follows:

- $\mathcal{P}$(phase space): All physical field configurations which obeys spatial boundary condition and EoM.
- $\tilde{\mathcal{P}}$(pre-phase space): Solutions of EoM which obeys spatial boundary condition.
- $\mathcal{C}$(configuration space): Field configurations satisfying spatial boundary condition.

$$\mathcal{P} \subset \tilde{\mathcal{P}} \subset \mathcal{C}$$

Importantly, the symplectic form $\tilde{\Omega}$ of pre-phase space is **degenerate**, and the zero modes of $\tilde{\Omega}$ form a Lie algebra. **Zero modes** refer to some non-zero vectors $\tilde{X}, \tilde{Y}$ that satisfy $\tilde{\Omega}\left(\tilde{X}, \tilde{Y}\right) = 0$. Moreover, the commutator of zero modes is also a zero mode, implying that they form a Lie algebra. After exponentiating the Lie algebra, we obtain a group $\tilde{G}$(you can call it gauge group if you like). The quotient space of pre-phase space $\tilde{\mathcal{P}}$ with respect to $\tilde{G}$ is $\mathcal{P}$.

$$\mathcal{P} \equiv \frac{\tilde{\mathcal{P}}}{\tilde{G}},\ \Omega \equiv \frac{\tilde{\Omega}}{\tilde{G}}.$$

2.3 Lagrangian

Because we do not impose boundary conditions on future and past. The stationarity of the action is to require that the variation of action is zero up to some fields localized at future and past.

$$\delta S = \int_{\Sigma_+} \Psi - \int_{\Sigma_-} \Psi \tag{2.4}$$

The variation of Lagrangian is conveniently denoted as

$$\delta L = E_a \delta\phi^a + d\Theta, \tag{2.5}$$

Where E_a is the EoM, Θ is a local functional of the dynamical/background fields and their derivatives.

The variation of the action is thus

$$\delta S = \int_M E_a \delta\phi^a + \int_{\partial M} (\delta\ell + \Theta), \tag{2.6}$$

where l denotes the boundary terms for example the Gibbs-Hawking-York term. We believe the stationarity doesn't depend on spacial boundary term, so

$$(\Theta + \delta\ell)|_\Gamma = dC. \tag{2.7}$$

Therefore,

$$\begin{aligned}
\delta S &= \int_M E_a \delta\phi^a + \int_{\Sigma_+ - \Sigma_-} (\Theta + \delta\ell) + \int_\Gamma (\Theta + \delta\ell) \\
&= \int_M E_a \delta\phi^a + \int_{\Sigma_+ - \Sigma_-} (\Theta + \delta\ell) + \int_{\partial\Gamma} C \\
&= \int_M E_a \delta\phi^a + \int_{\Sigma_+ - \Sigma_-} (\Theta + \delta\ell - dC),
\end{aligned}$$

You can view C as the boundary of Γ or boundary of Σ, as we showed before, a minus sign will come up. So,

$$\Psi = \Theta + \delta\ell - dC. \tag{2.8}$$

In addition, you can regard δ as the exterior derivative for differential forms on configuration space $\mathcal{C}$ and on its pull-backs.

Using this interpretation, we can now introduce the **pre-symplectic current:**

$$\omega \equiv \delta\Psi|_{\tilde{\mathcal{P}}} = \delta(\Theta - dC)|_{\tilde{\mathcal{P}}}. \tag{2.9}$$

Here we have used $\delta^2 = 0$. Now we write the properties of ω:

- $\omega|_\Gamma = \delta(\Theta + \delta\ell - dC)|_{\tilde{\mathcal{P}},\Gamma} = 0$
- ω is closed as a $(d-1)$-form on spacetime:

$$d\omega = d\delta(\Theta - dC) = \delta d\Theta = \delta\left(\delta L - E_a \delta\phi^a\right) = -\delta E_a \wedge \delta\phi^a = 0.$$

Finally we define the pre-symplectic form on $\tilde{\mathcal{P}}$ as

$$\tilde{\Omega} \equiv \int_\Sigma \omega \tag{2.10}$$

$\tilde{\Omega}$ is independent of the choice of Σ, because for two different cauchy slice Σ, Σ'

$$\begin{aligned}\int_\Sigma \omega - \int_{\Sigma'} \omega &= \int_\Sigma \omega - \int_{\Sigma'} \omega + \int_\Gamma \omega \\ &= \int_M d\omega = 0.\end{aligned}$$

Let us briefly summarize,

	Sparetime space	Configuration space $\tilde{C}$
L	d	0
ℓ	$d-1$	0
Θ	$d-1$	1
C	$d-2$	1
w	$d-1$	2

2.4 Covariant lagrangians

We define a vector field on configuration space

$$X_\xi \equiv \int d^dx \pounds_\xi \phi^a(x) \frac{\delta}{\delta\phi^a},$$

in terms of which we have

$$\delta_\xi \phi^a(x) = \pounds_{X_\xi} \phi^a(x) = X_\xi \cdot \delta\phi^a(x),$$

More generally we define

$$\delta_\xi \sigma \equiv \pounds_{X_\xi} \sigma.$$

We now introduce a key definition: a configuration-space tensor σ which is also a spacetime tensor locally constructed out of the dynamical and background fields is covariant under the infinitesimal diffeomorphism generated by a vector field ξ^μ if

$$\delta_\xi \sigma = \pounds_\xi \sigma,$$

where we emphasize that $\pounds_\xi$ is the spacetime Lie derivative. In order to have symmetry, we want L, ℓ, Θ, C to all be covariant in the sense that they obey this equation. They transform as tensors even if you only transform the dynamical field and not the background field. We can write the variation of the action by an infinitesimal diffeomorphism under which L is covariant as

$$\begin{aligned}\delta_\xi S &= \int_M \delta_\xi L + \int_{\partial M} \delta_\xi \ell \\ &= \int_{\partial M} (\xi \cdot L + \delta_\xi \ell)\end{aligned}$$

Here we have used $\delta_\xi L = \pounds_\xi L = d(\xi \cdot L) + \xi \cdot dL = d(\xi \cdot L)$. Here we have used $\xi \cdot dL = 0$, since L is a closed form. Note that $\delta_\xi S$ is pure boundary term.

2.5 Diffeomorphism charges

We now turn to construct the Hamiltonian H_ξ that generates the evolition in phase space corresponding to diffeomorphisms generated by any vector field ξ^μ which respects the boundary conditions and under which L, ℓ, and C are covariant. First, we are going to find function H_ξ on pre-phase space obeying

$$\delta H_\xi = -X_\xi \cdot \tilde{\Omega}. \tag{2.11}$$

Let us start doing some calculation. In order to compute the RHS of $\delta H_\xi = -X_\xi \cdot \tilde{\Omega}$, it is useful to introduce the Noether current

$$J_\xi \equiv X_\xi \cdot \Theta - \xi \cdot L. \tag{2.12}$$

If L is covariant under ξ then J_ξ is closed as a spacetime form:

$$\begin{aligned}dJ_\xi &= d(X_\xi \cdot \Theta) - d(\xi \cdot L) \\ &= X_\xi \cdot (\delta L - E_a \delta\phi^a) - \pounds_\xi L \\ &= \delta_\xi L - \pounds_\xi L - E_a \pounds_\xi \phi^a \\ &= 0.\end{aligned}$$

Here we have used that $d(X_\xi \cdot \Theta) = X_\xi \cdot d\Theta$. We then have the following calculation:

$$\begin{aligned}-X_\xi \cdot \omega &= -X_\xi \cdot \delta(\Theta - dC) \\ &= \delta(X_\xi \cdot (\Theta - dC)) - \pounds_{X_\xi}(\Theta - dC)\end{aligned}$$

Substitute $X_\xi \cdot \Theta = J_\xi + \xi \cdot L$ into the equation above

$$\begin{aligned}
-X_\xi \cdot \omega &= \delta(J_\xi + \xi \cdot L - X_\xi \cdot dC) - \pounds_{X_\xi}(\Theta - dC) \\
&= \delta J_\xi + \xi \cdot \delta L - \delta(X_\xi \cdot dC) - \pounds_\xi \Theta + \pounds_{X_\xi} dC \\
&= \delta J_\xi + \xi \cdot \delta L - \pounds_\xi \Theta + d(-\delta(X_\xi \cdot C) + \delta_\xi C) \\
&= \delta J_\xi + \xi \cdot (E_a \delta\phi^a + d\Theta) - \pounds_\xi \Theta + d(-\delta(X_\xi \cdot C) + \delta_\xi C) \\
&= \delta J_\xi + d(\xi \cdot \Theta - \delta(X_\xi \cdot C) + \delta_\xi C)
\end{aligned} \tag{2.13}$$

Where we have substituted EoM. After integrating over a Cauchy slice Σ, we obtain

$$\begin{aligned}
-X_\xi \cdot \tilde{\Omega} &= \int_\Sigma \delta J_\xi + \int_{\partial\Sigma} (\pounds_\xi C - \delta(X_\xi \cdot C) - \xi \cdot \Theta) \\
&= \int_\Sigma \delta J_\xi + \int_{\partial\Sigma} (\xi \cdot dC - \delta(X_\xi \cdot C) - \xi \cdot (dC - \delta\ell)) \\
&= \delta \left(\int_\Sigma J_\xi + \int_{\partial\Sigma} (\xi \cdot \ell - X_\xi \cdot C) \right).
\end{aligned} \tag{2.14}$$

Compared with $\delta H_\xi = -X_\xi \cdot \tilde{\Omega}$, we see that we have obtained that

$$H_\xi \equiv \int_\Sigma J_\xi + \int_{\partial\Sigma} (\xi \cdot \ell - X_\xi \cdot C) + \text{ constant} \tag{2.15}$$

If we express the Noether current as

$$J_\xi = dQ_\xi,$$

We obtain the following expression, true only in **generally-covariant theories**:

$$H_\xi \equiv \int_{\partial\Sigma} (Q_\xi + \xi \cdot \ell - X_\xi \cdot C) + \text{ constant} . \tag{2.16}$$

3 Examples