How to Make Better and/or Cheaper Products Accessible to Buyers through an Optimal Product Testing Mechanism*

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Abstract

In markets where the true qualities of products are not visible to buyers, a product quality testing organization has expertise in finding out and revealing the true qualities to buyers. However, the quality testing organization often has limited testing capacity. In this study, we propose a quality testing mechanism which makes full use of the testing capacity to only test products that maximize consumer surplus. We show that with our proposed mechanism, the unique weak Perfect Bayesian Equilibrium maximizes consumer surplus, and thus our proposed mechanism (weakly) dominates any alternative quality testing mechanism. We also consider a generic benchmark mechanism in which the testing organization randomly selects products to test and reveal their qualities. Our experimental results show that the consumer surplus is significantly higher when the testing organization uses our proposed mechanism than when it uses the benchmark mechanism which randomly tests products.

Keywords: consumer surplus, asymmetric information, product quality, product test, information disclosure, mechanism design, experiment

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1 Introduction

From complex technical products to foodstuffs to consumer products such as toothpaste, buyers are often worse informed about product quality than sellers. This asymmetric infor-

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mation about vertical product quality[1] may decrease consumer surplus (Akerlof, 1970).[2]

Independent product testing organizations such as Consumer Reports (US), Stiftung Warentest (Germany), and Which? (UK) are third-party certifiers who mitigate buyers' informational disadvantage by providing credible information about product quality. They are usually not-for-profit organizations who neither require sellers to pay a fee for the rating service, nor accept advertisements in order to avoid conflicts of interest (see Consumer Reports, Stiftung Warentest (2019), and Which?). Instead, they finance themselves mainly through selling their own publications (International Consumer Research & Testing).[3] As Vollstaedt et al. (2020) note, product testing organizations usually aim to provide a comprehensive rating of *vertical* product quality, i.e., they include ratings for a stroller's weight, how waterproof the raincover is, and the level of toxic substances. However, they do *not* include horizontal quality ratings, e.g., for tasteful colors. Often, these organizations employ their own test buyers to be able to buy products anonymously. In order to obtain a comprehensive quality rating, e.g., good, medium, and poor, product testing organizations weight and add the ratings of all included quality dimensions. Test results are accessible online or in print magazines. Product testing organizations are usually well-known and well-regarded. For instance, 96 % (77 %) of all German consumers know of (strongly trust) Stiftung Warentest (KantarEmnid and Verbraucherzentrale Bundesverband, p. 9). In the US, Consumer Reports has more than 6 million paying members and their website receives an average of 14 million unique visits per month (Consumer Reports). Consequently, information provided by product testing organizations is very close to what Viscusi (1978) proposed in a reply to

[1]As Vollstaedt et al. (2020) note, product quality is a multidimensional construct comprised of horizontal and vertical dimensions. Horizontal quality dimensions are subjective. More precisely, while it may be possible to objectively specify horizontal quality dimensions, consumers differ in their preferences about them (Hotelling, 1929). For example, horizontal dimensions of a stroller's quality include its color. While it is possible to objectively specify a certain color, e.g., by using a spectrophotometer, consumers have different preferences over colors. In contrast, vertical quality dimensions are objectively rateable. To illustrate, vertical dimensions of a stroller's quality include its weight, how waterproof the raincover is, and the level (if any) of toxic substances contained in its materials. Note that these vertical dimensions usually contain search, experience, and credence characteristics (Nelson, 1970; Darby and Karni, 1973). For a stroller, a search characteristic would be its weight since a stroller's weight can be determined before purchasing it. An experience characteristic would be how waterproof the raincover is since this is usually observable only after use. A credence characteristic would be how many toxic substances are contained in the fabric since consumers are usually not able to observe this amount even after having purchased the stroller.

[2]We follow Vollstaedt et al. (2020) in noting that, while most products contain some amount of horizontal and some amount of vertical quality dimensions, the relevance of each one may differ. This paper focuses on products whose vertical quality dimensions are at least as relevant for buyers as its horizontal ones, e.g., toothpaste, strollers, or grills. We do not analyze markets for products whose horizontal quality dimensions are more relevant for buyers than its vertical ones, e.g., fiction movies or books. Note that, while online consumer ratings for such products can be found on websites like amazon.com or imdb.com, independent product testing organizations usually do not test fiction movies or books.

[3]Refer to http://www.international-testing.org/members.html for a list of international product testing organizations.

Akerlof (1970), namely to provide *credible* information to buyers.[4] [5]

While independent product testing organizations offer credible information, they are hampered by limited testing capacities. In particular, when testing a certain product, they select only a sample of all product models which are available for this product.[6] [7] Typically, they select which products to test based on which ones are perceived to be of greatest interest for consumers. Stiftung Warentest, for example, uses current sales numbers to select the bestselling products for testing. Their sample of tested products usually accounts for 2 % to 33 % of all available products (as in the 09/2016 magazine, see GfK SE). By contrast, Consumer Reports and Which? make their testing selections using a combination of sales numbers, price, and other criteria.

A recent paper investigates how different product selection mechanisms influence consumer surplus in the short term, i.e., when quality and price are exogenous (Vollstaedt et al., 2020). They show that, when quality and price have already been set, any current selection mechanism almost always provides suboptimal information for consumers. Instead, they propose a new mechanism which (weakly) dominates any current mechanism. More precisely, under this new mechanism, all products that buyers would have selected under complete information are selected for testing, yielding optimal consumer surplus.

In this paper, we investigate whether our proposed product testing mechanism SELLERS-MAYAPPLY can maximize consumer surplus when sellers endogenously determine the price and quality of their products. We build a theoretical model for a product testing game. In this game, sellers make production, pricing and quality testing application decisions, and then a product testing organization uses our proposed product testing mechanism to determine which seller(s)' products to test and reveal their qualities to buyers. Finally, buyers make purchasing decisions based on the qualities revealed by the product testing organization and all sellers' prices. We prove that in the unique pure-strategy weak Perfect Bayesian Equilibria, all buyers purchase products that maximize their surplus. Therefore, our proposed product testing mechanism (weakly) dominates any other alternative product testing

[4]As Vollstaedt et al. (2020) note, buyers also use other proxies for quality, e.g., online consumer ratings (Rao and Monroe, 1989, and De Langhe et al., 2016). Online consumer ratings are often readily available. However, they are problematic since, first, they usually do not include credence characteristics,e.g., toxic substances in food, cosmetics, or clothing, or under which working conditions a product was manufactured. Second, online consumer ratings often include both vertical and horizontal quality dimensions although the latter are, by definition, not objectively rateable. Third, a considerable amount of fake ratings exists, even among verified purchases (Mayzlin et al., 2014, and Which?, 2018). Interestingly, online consumer ratings correlate poorly with ratings provided by independent product testing organizations (De Langhe et al., 2016, and Köcher and Köcher, 2018).

[5]Some buyers also use price as a proxy for quality. Yet, price seems to be a poor indicator (see Ratchford et al., 1996, and Olbrich and Jansen, 2014 for overviews, as well as Oxenfeldt, 1950, and De Langhe et al., 2016; Diller, 1977, 1988; Yamada and Ackerman, 1984; Bodell et al., 1986; Steenkamp, 1988; Kirchler et al., 2010). We acknowledge that these correlational results are sensitive to the weights of quality dimensions which are, to some degree, arbitrary. Yet, test results published by Consumer Reports show that more than half of all tested products are dominated on *all* quality dimensions (Hjorth-Andersen, 1984).

[6]Usually, several product models belong to one certain type of product, e.g. several smartphone models belong to the product smartphone. However, to improve readibility, we use "product" instead of "product model" hereafter.

[7]Product testing organizations do not only face capacity constraints as to which *product models*, but also as to which *products* to select for testing. This study focuses on the problem of which *product models* to select.

mechanism.

In addition to the SellersMayApply condition in which our proposed mechanism is applied, we also consider a RandomTesting condition in which the product testing organization randomly tests the same number of products and reveals their qualities to buyers. The RandomTesting mechanism is a generic testing mechanism in which sellers cannot affect whether their products will be tested.

We conduct a laboratory experiment to test the effectiveness of our proposed SellersMayApply mechanism. We find that consumer surplus is significantly higher when we use our SellersMayApply mechanism than when we use the RandomTesting mechanism.

This study contributes to the theoretical literature in industrial organization in two important aspects. First, we show that we can incentivize sellers to offer products that maximize consumer surplus through a product testing mechanism in which sellers can influence whether and with what probability their products will be tested. Second, we include a product testing organization as a means to provide credible information for buyers and, most fundamentally, allow for prices which may *not* be positively correlated with quality.

There have been theoretical, empirical and experimental studies that investigate the effectiveness of unraveling and information diclosure (see Dranove and Jin, 2010 and Brendel, 2021 for overviews). Some theoretical studies indicate that full unraveling is usually difficult to achieve due to its requirement for some strong assumptions (e.g., Grossman, 1981, and Milgrom, 1981). However, there have been some empirical studies that find unraveling to an incomplete degree (see, for instance, Mathios, 2000, and Jin and Leslie, 2003, amongst many others), and there is also experimental evidence showing both unraveling to an incompleted degree (see, for example, Benndorf et al., 2015, and Benndorf, 2018) and unraveling to a complete degree when feedback and learning are allowed (see, for example, Forsythe et al., 1989, and Jin et al., 2021). To the best of our knowledge, our study is the first one that investigates whether unraveling increases market efficiency in the long term (i.e., prices and qualities are endogenous) when there are limited information disclosure capacities.

We also contribute to the literature on third-party certifiers by considering independent product testing organizations that are different from other third-party certifiers in several aspects. First, we consider not-for-profit testing organzations which do not charge fees for the purpose of increasing their own profits. These organizations are different from private third-party certifiers such as Moody's and PSA[8], which charge sellers a fee for the rating service (see Dranove and Jin, 2010, List, 2006, and Jin et al., 2010). Because of the not-for-profit feature, independent product testing organizations do not have the incentive to give overgenerous ratings in exchange for future business. Second, due to the not-for-profit property, independent product testing organzations often have limited testing capacities, which are different from private third-party certifiers such as Moody's and PSA and other non-profit third-party certifiers such as USDA organic or Blauer Engel.

The rest of our paper proceeds as follows. Section 2 introduces our theoretical framework and derives our theoretical predictions. Section 3 presents our experimental design and hypotheses. Section 4 reports our experimental results. Section 5 discusses our findings and concludes.

[8]Professional Sports Authenticator (PSA) is one of the largest card grading services world-wide (for more information, see https://www.psacard.com/services/tradingcardgrading).

2 Theory

We start by describing the theoretical framework (subsection 2.1). Subsequently, we present predictions for the theoretical framework.

2.1 Theoretical framework

In this subsection, we establish our theoretical framework.

2.1.1 Product testing game

We consider a market with a non-empty set of rational sellers F with $\emptyset \neq F = \{f_1, \ldots, f_n\}$ $(n \in \mathbb{N})$, and a non-empty set of rational buyers B with $\emptyset \neq B = \{b_1, \ldots, b_s\}$ $(s \in \mathbb{N})$.

Sellers' payoff function Each seller f_i offers products with a certain quality level $q_i \in \{1, 2, 3\}$ and a price $p_i \in \mathbb{R}^+$. We assume each seller can sell as many units of that product as demanded, but all the products she sells must be identical in quality and price. The marginal cost c_i is a function of quality q_i, i.e., $c_i = c(q_i)$. The marginal cost function is assumed to be continuously differentiable, strictly increasing and strictly convex in quality, i.e., $c'\big(q_i\big) > 0$ and $c''\big(q_i\big) > 0$. Since we are not interested in analyzing market entry or exit decisions and since positive fixed costs would thus not influence equilibrium predictions, we assume all sellers' fixed costs equal zero. Each seller f_i's payoff function is

$$\pi_i\big(p_i, q_i\big) = \Big(p_i - c\big(q_i\big)\Big)d_i \tag{1}$$

where d_i represents the demand for f_i's product, i.e., the number of buyers buying seller f_i's product.

Buyers' payoff function Each buyer decides whether, and if so from which seller, to buy at most one product. They are not able to resell. Different buyers have different valuation for the quality of a product. For buyer b_j, with $j \in \{1, \ldots, s\}$, we call θ_j her valuation of quality, with $0 < \theta_j \in \mathbb{R}^+$. Among all buyers, there are two types of buyers with two different θ values: θ_L or θ_H. The numbers of buyers with $\theta_j = \theta_L$ and $\theta_j = \theta_H$ are the same. If the buyer decides to buy a product from f_i, then her payoff function is

$$\pi_j\big(p_i, q_i, \theta_j\big) = \theta_j q_i - p_i \tag{2}$$

$\theta_j q_i$ is a buyer's willingness to pay for q_i. Finally, if a buyer chooses not to purchase a product model, her payoff is zero.

Buyers' preferred product quality when the markup is 0 Buyers with different valuation of quality θ have different preferred product qualities when all products with different qualities have a markup of 0 (in other words, when the prices of all products are equal to their corresponding marginal cost). Specifically, buyers with $\theta = \theta_L$ strictly prefer a quality 2 product, while buyers with $\theta = \theta_H$ strictly prefer a quality 3 product, if all products with different qualities have the same markup.[9] Formally, we have

$$\arg\max_{q \in \{1,2,3\}} \theta_L q - c(q) = 2 \tag{3}$$

$$\arg\max_{q \in \{1,2,3\}} \theta_H q - c(q) = 3 \tag{4}$$

The two equations (3) and (4) have the following implications:

$$2\theta_L - c(2) > \theta_L - c(1) \Leftrightarrow c(2) - c(1) < \theta_L \tag{5}$$

$$2\theta_L - c(2) > 3\theta_L - c(3) \Leftrightarrow c(3) - c(2) > \theta_L \tag{6}$$

$$3\theta_H - c(3) > \theta_L - c(1) \Leftrightarrow c(3) - c(1) < 2\theta_H \tag{7}$$

$$3\theta_H - c(3) > \theta_H - c(2) \Leftrightarrow c(3) - c(2) < \theta_H \tag{8}$$

Product testing organization After all sellers determine the qualities and prices of their products, the prices of all sellers are visible to each buyer. However, the quality of a seller's product is visible to each buyer if and only if the product has been tested by a product testing organization and the organization reveals the quality of the product. The product testing organization can accurately find out the true quality of a product after the test, but the organization has a limited maximum testing capacity $k \in \mathbb{N}$. It selects at most k sellers' products according to a certain product selection mechanism. In this study, we consider an extreme case in which the maximum testing capacity is only 2, which is equal to the number of quality levels preferred by the two types of buyers when all products have a markup of 0. Denote the set of products that are selected by the organization to be tested as K, and denote the set of products whose qualities are revealed to buyers by the organization as K'. There is $K' \subseteq K \subseteq F$. In this study, we consider two product selection mechanisms: our proposed mechanism SELLERSMAYAPPLY and a random selection mechanism RANDOMTESTING. The latter mechanism represents a stylized version of mechanisms in which sellers cannot directly influence whether their products will be tested. We assume both mechanisms are testing capacity-neutral, i.e., they provide the same number of testing slots. We refrain from modeling the testing organization's payoff function since, as mentioned above (section 1), it is a non-profit organization which does not rely on fees for the rating service. Since we model the product testing organization as an algorithm without its own surplus function, we do not call it a player.

2.1.2 Local and global dominance

Before describing the two different product selection mechanisms in detail, we need to establish a set of basic definitions.[10] We begin by defining local and global dominance

[9]Note that, if offered at marginal costs, no buyer would select quality level 1 which corresponds to a "poor" rating. This rating is given when a product is considered unacceptable for all, as when it does not suit its claimed purpose and/or entails unacceptable risks such as high toxic material levels.

[10]These two definitions are adapted from Vollstaedt et al. (2020).

to distinguish if a product is dominated within the whole market when analyzing a world of complete information), or within a certain sub-market like the set of tested products when analyzing worlds of incomplete information with different product model selection mechanisms). Note that we use the terms “seller with (non-) dominated product” and “(non-)dominated seller” equivalently.

Definition 1 (Locally (non-)dominated products)**.** *Let $\emptyset \neq Z \subseteq F$ be a non-empty set of sellers. A seller $f_t \in Z$ offers a locally* **dominated** *product in Z if $\exists f_j \in Z$ with $\Big(\big(p_j \leq p_t\big) \wedge \big(q_j > q_t\big)\Big) \vee \Big(\big(p_j < p_t\big) \wedge \big(q_j \geq q_t\big)\Big)$. A seller $f_t \in Z$ offers a locally* **non-dominated** *product in Z if $\forall f_j \in Z$*

- *if $p_j < p_t$, then $q_j < q_t$,*
- *if $q_j > q_t$, then $p_j > p_t$.*

Essentially, a product is locally dominated in the same set (or market) if at least one seller in this set offers a strictly higher product quality without being more expensive, or a strictly lower price without offering a lower product quality. By comparison, a product model is locally non-dominated in a set if every seller in this set offering a strictly higher product quality also has a strictly higher price, and every seller offering a strictly lower price also offers a strictly lower quality.

We next define a product model vis-à-vis all competitors.

Definition 2 (Globally (non-)dominated product models)**.** *A seller $f_t \in \{f_1, \ldots, f_n\}$ who is locally dominated according to definition 1 with $Z = F$ offers a globally dominated product model. A seller $f_t \in \{f_1, \ldots, f_n\}$ who is locally non-dominated according to definition 1 with $Z = F$ offers a globally non-dominated product model.*

To illustrate definitions 1 and 2, consider the following local market (see figure **??**): $Z = \{f_1, f_2, f_4, f_5, f_6\}$, with
$q_1 = 2$, $p_1 = 5$,
$q_2 = 3$, $p_2 = 10$,
$q_4 = 1$, $p_4 = 11$,
$q_5 = 2$, $p_5 = 9$,
$q_6 = 3$, $p_6 = 28$.
Furthermore, consider the following global market: $F = Z \cup f_3$, with $q_3 = 3$ and $p_3 = 9.5$. While sellers f_1 and f_2 are locally non-dominated in market Z, sellers f_1 and f_3 are globally non-dominated in market F.

2.1.3 The two product testing mechanisms

This section introduces how the product testing organization selects and tests products according to SellersMayApply and Random mechanisms.

Our proposed mechanism SellersMayApply Under our proposed mechanism, sellers are able to influence whether a testing organization will test their product model. The mechanism consists of 4 steps.

- Step 0: After seeing the prices and qualities of all sellers, each seller independently decides whether to apply to have her product tested by the testing organization. If a seller applies, she needs to report the quality of her product to the testing organization (reporting a false quality is allowed). Each applicant pays an application deposit μ to the testing organization.

- Step 1: Among the set of applicants (denoted as F_0), select products which satisfy the following criteria:

  - The reported quality is not 1 (i.e., the reported quality is either 2 or 3).
  - It is locally non-dominated among applicants based on each applicant's reported quality (or updated quality, if available).
  - It has not been tested in the previous iteration (if any).

  Denote the set of these selected products as F_1. The testing organization returns the application deposits $\mu > 0$ to all sellers whose products are in F_1.

- Step 2: Among F_1, if there exist identical products (same reported quality and same price), randomly select one product among them. Denote the set of selected products as F_2 (there should be at most 2 products in F_2).

- Step 3:

  - Test all untested products in F_2 and reveal the quality of all products with a true quality statement. Do not reveal the quality of a product with a false quality statement.
  - The seller f_i who is found out to report a false quality, if any, pays a lying fee of $\sigma_i > 0$. To ensure that the lying fee is large enough to deter a false reported quality, we consider a dynamic lying fee which depends on the seller's ex-post revenue and is paid after the transaction is completed. f_i needs to pay a lying fee that is strictly greater than her ex-post revenue. In other words, $\sigma_i = p_i d_i + \underline{\sigma}$, where $\underline{\sigma}$ is a constant strictly greater than 0.
  - If no false quality reporting is detected or if all applicants' products have been tested or all testing capacity has been used up, then finish the algorithm. Otherwise, update F_0 based on tested sellers' true quality and return to Step 1.

The RandomTesting mechanism Under the RandomTesting mechanism, sellers cannot directly influence whether the testing organization will test their products. The testing organization randomly selects $k = 2$ sellers' products from F and reveal their qualities to buyers.

2.1.4 Procedures of the game

The product testing game consists of the following stages:

- Stage 1: n sellers determine quality and price simultaneously.

- Stage 2: The product mechanism (SellersMayApply or RandomTesting) is implemented.

- Stage 3: All buyers see each seller's price as well as the qualities of products revealed by the testing organization. Each buyer decides from whom to purchase a product or buys nothing.

2.2 Theoretical predictions

2.2.1 When each buyer's surplus is maximized

Having established our theoretical framework in the previous subsection, we are now able to analyze the two different versions of the game.

Since we consider a one-shot product testing game, no seller should have the incentive to charge a price lower than her unit cost. Therefore, we know from (3) and (4) that a buyer with $\theta = \theta_L$ will maximize her surplus when she purchases a product with $q = 2$ and $p = c(2)$, while a buyer with $\theta = \theta_H$ will maximize her surplus when she purchases a product with $q = 3$ and $p = c(3)$.

2.2.2 SellersMayApply mechanism

In this subsection, we analyze a world with incomplete information about product quality where a product testing organization uses the SellersMayApply mechanism to select at most $k = 2$ products to test.

Since buyers have incomplete information about product quality, they will form a belief about the expected quality of a product given its price. More precisely, each buyer will have a subjective quality distribution function for each product whose quality is not revealed given the price of the product (hereafter, unrevealed product). We assume that all buyers have the same subjective quality distribution function and this function is common knowledge.

We make the following assumptions about sellers' and buyers' in some tie-breaking or trivial situations. We assume that all these assumptions are also common knowledge among all sellers and buyers.

Assumption 1 (A1) (Rationality). *Every player is rational.*

Assumption 2 (A2) (Zero probability on "unrationalizable" quality levels). *In a buyers' subjective quality distribution for an unrevealed seller, all quality levels which violate A1 or any corollary that can be derived from A1, A2, A3.1, A3.2, A4 and/or A5 will have a 0 probability.*

Assumption 3.1 (A3.1) ("Unraveling quality uncertainty" seller tie-breaking rule). *Given all n sellers' price and quality bundles and all the other $n-1$ sellers' application decisions, if a seller's application decision does not make a difference in her expected quality and her expected payoff, then she chooses to apply to unravel uncertainty about her quality.*

Assumption 3.2 (A3.2) ("Quality-caring" seller tie-breaking rule). *Given all n sellers' price and quality bundles and all the other $(s-1)$ sellers' application decisions, if a seller is indifferent between applying and not applying, she will choose the one that gives him a higher expected quality.*

Assumption 4 (A4) (Not buying from any seller when the maximum expected profit is 0). *When the maximum expected payoff from buying from any seller is 0, then the buyer will choose not buying.*

Assumption 5 (A5) (Non-negative markup). *No seller will set her price to be lower than the marginal cost.*

We make A3.1 based on ambiguity aversion. A3.2 is made based on the assumption that sellers want buyers to believe/observe that their products have a higher quality, even if a product with a higher quality does not increase their monetary payoffs. We assume A5, because sellers usually charge a price lower than the marginal cost for the purpose of predatory pricing. In this study, we do not discuss the possibility of predatory pricing, because we focus on an one-shot interaction.

Our goal is to find all Perfect Bayesian Equilibria in this game, so we use the backward induction method. It is common knowledge that each buyer will maximize her expected payoff based on the subjective quality distribution function for an revealed seller given the seller's price. Since each seller is rational, she should be able to form a correct belief about the buyer's subjective quality distribution function. Then she uses this function to determine whether to apply for quality testing or not in Stage 2, given all n sellers' qualities and prices in Stage 1.

We first show that applying with a false reported quality is a dominated strategy for a seller.

Corollary 0 (C0). *Applying for quality testing with a false reported quality is a dominated strategy for a seller.*

With C0, we only need to consider whether each seller will choose applying with a true reported quality or not applying. We derive the following corollaries about each seller's application decision in Stage 2 given her quality and price in Stage 1, based on the common knowledge about A1, A2, A3.1, A3.2, A4 and A5.

Corollary 1.1 (C1.1). *A seller with $q = 1$ will not apply.*

Corollary 2.1 (C2.1). *A globally non-dominated seller with $q = 3$ must apply with a true reported quality.*

Corollary 2.2 (C2.2). *A globally dominated seller with $q = 3$ will not apply.*

Corollary 3.1 (C3.1). *A globally non-dominated seller with $q = 2$ must apply with a true reported quality.*

Corollary 3.2 (C3.2). *A globally dominated seller with $q = 2$ will not apply.*

The proofs for these corollaries are in Appendix A.

Based on the C1.1, C2.1, C2.2, C3.1, C3.2 and A2, we can derive the buyer's subjective belief about an unrevealed seller's quality distribution:

Corollary 4.1 (C4.1). *The buyer's subjective belief about an unrevealed seller f_t's quality distribution is:*

- ***Case 1:*** *If there are two revealed sellers (i.e., one with $q = 2$, denoted as $f^2_{K'}$ and the other with $q = 3$, denoted as $f^3_{K'}$), and there must be $p^3_{K'} > p^2_{K'}$:*

	$Pr(q = 1)$	$Pr(q = 2)$	$Pr(q = 3)$
if $p_t \geqslant p^3_{K'}$	α_1	β_1	$1 - \alpha_1 - \beta_1$
if $p^2_{K'} \leqslant p_t < p^3_{K'}$	α_2	$1 - \alpha_2$	*0*
if $p_t < p^2_{K'}$	*1*	*0*	*0*

- ***Case 2:*** *If there is only one revealed seller, and she has $q = 2$, denoted as $p^2_{K'}$:*

	$Pr(q = 1)$	$Pr(q = 2)$	$Pr(q = 3)$
if $p_t \geqslant p^2_{K'}$	α_3	$1 - \alpha_3$	*0*
if $p_t < p^2_{K'}$	*1*	*0*	*0*

- ***Case 3:*** *If there is only one revealed seller, and she has $q = 3$, denoted as $p^3_{K'}$:*

	$Pr(q = 1)$	$Pr(q = 2)$	$Pr(q = 3)$
if $p_t \geqslant p^3_{K'}$	α_4	β_2	$1 - \alpha_4 - \beta_2$
if $p_t < p^3_{K'}$	*1*	*0*	*0*

- ***Case 4:*** *If there is no revealed seller, then:*

$Pr(q = 1)$	$Pr(q = 2)$	$Pr(q = 3)$
1	*0*	*0*

where $\alpha_1, \beta_1, (1 - \alpha_1 - \beta_1), \alpha_2, \alpha_3, \alpha_4, \beta_2 \in [0, 1]$

Based on the quality distribution above, we can derive the expected quality of an unrevealed seller's product:

Corollary 4.2 (C4.2). *The buyer's belief about an unrevealed seller f_t's expected quality is:*

- ***C4.2.1*** *If there are two revealed sellers (i.e., one with $q = 2$, denoted as $f^2_{K'}$ and the other with $q = 3$, denoted as $f^3_{K'}$), and there must be $p^3_{K'} > p^2_{K'}$:*

$$E_{f_t \notin K'}(q_t) = \begin{cases} 3 - 2\alpha_1 - \beta_1 & if\ p_t \geqslant p^3_{K'} \\ 2 - \alpha_2 & if\ p^2_{K'} \leqslant p_t < p^3_{K'} \\ 1 & if\ p_t < p^2_{K'} \end{cases}$$

- ***C4.2.2*** *If there is only one revealed seller, and she has $q = 2$, denoted as $p^2_{K'}$:*

$$E_{f_t \notin K'}(q_t) = \begin{cases} 2 - \alpha_3 & if\ p_t \geqslant p^2_{K'} \\ 1 & if\ p_t < p^2_{K'} \end{cases}$$

- ***C4.2.3*** *If there is only one revealed seller, and she has $q = 3$, denoted as $p^3_{K'}$:*

$$E_{f_t \notin K'}(q_t) = \begin{cases} 3 - 2\alpha_4 - \beta_2 & if\ p_t \geqslant p^3_{K'} \\ 1 & if\ p_t < p^3_{K'} \end{cases}$$

- ***C4.2.4*** *If there is no revealed seller, then:*

$$E_{f_t \notin K'}(q_t) = 1$$

where $\alpha_1, \beta_1, (1 - \alpha_1 - \beta_1), \alpha_2, \alpha_3, \alpha_4, \beta_2 \in [0, 1]$.

With all the assumptions and corollaries above, we find that the only pure-strategy profiles to be weak Perfect Bayesian Equilibria should have the following features.

Proposition 1. *In the SellersMayApply condition, the only pure-strategy profiles to be weak Perfect Bayesian Equilibria must have the following features:*

- *γ_2 sellers play $(q = 2, p = c(2), Apply, Report\ q = 2)$, with $\gamma_2 \geqslant 2$;*
- *γ_3 sellers play $(q = 3, p = c(3), Apply, Report\ q = 3)$, with $\gamma_3 \geqslant 2$;*
- *γ_1 sellers play $(q = 1, p = c(2), Not\ Apply)$, with $\gamma_1 \geqslant 1$;*
- *$(n - \gamma_1 - \gamma_2 - \gamma_3)$ sellers play $(q = 1, p = c(3), Not\ Apply)$, with $\gamma_1 + \gamma_2 + \gamma_3 < n$.*
- *Buyers' belief about the quality distribution of an unrevealed seller f_t given her price p_t:*
  - *If there are two revealed sellers (i.e., one with $q = 2$, denoted as $f^2_{K'}$ and the other with $q = 3$, denoted as $f^3_{K'}$), and there must be $p^3_{K'} > p^2_{K'}$:*

	$Pr(q = 1)$	$Pr(q = 2)$	$Pr(q = 3)$
if $p_t \geqslant p^3_{K'}$	$\frac{\gamma_1}{\gamma_1+\gamma_2-1}$	*0*	$1 - \frac{\gamma_1}{\gamma_1+\gamma_2-1}$
if $p^2_{K'} \leqslant p_t < p^3_{K'}$	$\frac{n-\gamma_1-\gamma_2-\gamma_3}{n-\gamma_1-\gamma_2-1}$	$1 - \frac{n-\gamma_1-\gamma_2-\gamma_3}{n-\gamma_1-\gamma_2-1}$	*0*
if $p_t < p^2_{K'}$	*1*	*0*	*0*

  - *If there is only one revealed seller, and she has $q = 2$, denoted as $p^2_{K'}$ $(\alpha_3 > 0)$:*

	$Pr(q = 1)$	$Pr(q = 2)$	$Pr(q = 3)$
if $p_t \geqslant p^2_{K'}$	α_3	$1 - \alpha_3$	*0*
if $p_t < p^2_{K'}$	*1*	*0*	*0*

  - *If there is only one revealed seller, and she has $q = 3$, denoted as $p^3_{K'}$ $(\alpha_4 + \beta_2 > 0)$:*

	$Pr(q=1)$	$Pr(q=2)$	$Pr(q=3)$
if $p_t \geqslant p^3_{K'}$	α_4	β_2	$1-\alpha_4-\beta_2$
if $p_t < p^3_{K'}$	*1*	*0*	*0*

– *If there is no revealed seller, then:*

$Pr(q=1)$	$Pr(q=2)$	$Pr(q=3)$
1	*0*	*0*

where $\alpha_3 > 0$ *and* $\alpha_4 + \beta_2 > 0$.

- *Each buyer with* $\theta = \theta_L$ *will buy a product from the revealed seller with* $q = 2$ *and* $p = c(2)$.
- *Each buyer with* $\theta = \theta_H$ *will buy a product from the revealed seller with* $q = 3$ *and* $p = c(3)$.

Proof. To prove this proposition, we first prove some lemmas (i.e., L0.1 through L7). We prove the lemmas and this proposition in Appendix A. □

Since each buyer with $\theta = \theta_L$ will buy a product with $q = 2$ and $p = c(2)$ and each buyer with $\theta = \theta_H$ will buy a product with $q = 3$ and $p = c(3)$, each buyer is maximized. Therefore, we maximize consumer surplus when using our proposed product testing mechanism, and this mechanism must (weakly) dominate any other alternative mechanism.

2.2.3 RandomTesting mechanism

When analyzing the RandomTesting mechanism, we use all applicable assumptions from the SellersMayApply mechanism. Since each seller cannot directly affect whether and with what probability she will be tested, A3.1 and A3.2 are not applicable in the RandomTesting mechanism. Thus, we have the following assumptions:

Assumption 1 (A1) (Rationality). *Every player is rational.*

Assumption 2 (A2) (Zero probability on "unrationalizable" quality levels). *In a buyers' subjective quality distribution for an unrevealed seller, all quality levels which violate A1 or any corollary that can be derived from A1, A2, A4 and/or A5 will have a 0 probability.*

Assumption 4 (A4) (Not buying from any seller when the maximum expected profit is 0). *When the maximum expected payoff from buying from any seller is 0, then the buyer will choose not buying.*

Assumption 5 (A5) (Non-negative markup). *No seller will set her price to be lower than the marginal cost.*

We prove that with the Randomtesting mechanism, there does not exist any Perfect Bayesian Equilibrium that can achieve the same buyer surplus as the SellersMayApply mechanism or the benchmark CompleteInformation condition.

Proposition 2. *With the* RandomTesting *mechanism, there does not exist any Perfect Bayesian Equilibrium, if any, that can yield the same buyer surplus as the* SellersMayApply *mechanism does.*

Proof. We prove it in Appendix A. □

Therefore, the SellersMayApply mechanism must strictly dominate the RandomTesting mechanism.

3 Experimental design and hypotheses

Based on the product testing game introduced in the previous section, we design a laboratory experiment to test our theoretical predictions and ascertain the extent to which these predictions are observed with human decision makers. We design two experimental conditions: SellersMayApply and RandomTesting according to section 2.

In our experiment, we use a between-subject design, and we conduct three sessions per condition. Each session consists of 20 rounds. At the end of a session, one of the 20 rounds is chosen randomly for payment (2 ECU = US \$ 1). In each session, we include 6 sellers ($n = 6$), 6 buyers ($s = 6$), and one product testing organization. While sellers and buyers are played by actual participants, the product testing organization is simulated by the computer. Player roles, i.e., seller or buyer, are assigned randomly at the beginning of a session and remain constant afterwards. Player IDs, i.e., seller 1, 2 or 3 etc., are re-shuffled, i.e., they are assigned randomly at the beginning of each round. The product testing organization selects at most two products to be tested ($k = 2$).

In each round, sellers choose one of three quality levels, i.e., $q_t \in \{1, 2, 3\}$. As to their cost function, we implement one of the simplest ones fulfilling $c'(q_t) > 0$ and $c''(q_t) > 0$, namely quadratic unit costs, i.e., $c(q_t) = q_t^2$. As to buyers' valuations of quality, θ_L takes the value of 4, and θ_H takes the value of 8, such that θ_L, θ_H , and $c(q_t)$ satisfy (3) and (4). In each round, there are three buyers with θ_L, and three buyers with θ_H. While buyers are neither informed of the cost function nor of the quality distribution, they are, at the beginning of a session, informed that, if all products are offered at marginal costs, quality level 2 (3) would be optimal for buyers with $\theta = 4$ ($\theta = 8$). Buyers learn a product's quality only if the product testing organization has revealed it, or after having purchased a product. If applicable, sellers incur an application deposit μ of 0.1 ECU and a lying fee σ of 10 ECU+revenue if a false quality report is detected.

To avoid bankruptcy, we pay each subject an initial endowment of 38 ECU (16 ECU as a show-up fee + 22 ECU for answering the comprehension questions). Each subject also receives 2 ECU for answering questions about their beliefs regarding the expected quality of untested products in Round 20, i.e., in the last round. We elicit first-order beliefs for buyers and second-order beliefs for sellers. All beliefs are elicited after subjects make their decisions in Round 20, but before they receive feedback on their payoff. Table 1 summarizes the procedures in each experiment session.

We base our hypothesis on our theoretical results from Propositions 1 and 2.

Hypothesis: A buyer's surplus in the SellersMayApply condition is higher than that in the RandomTesting condition.

Table 1: Procedures of the experiment

Experimental instructions Comprehension questions
Product testing game (round 1 to 19): - Decision making - Round feedback Product testing game (round 20): - Decision making - Belief elicitation - Round feedback
Demographic questionnaire Final payoff feedback

Table 2: Number of sellers and buyers per session, and number of sessions and subjects per condition

Condition	Sellers per session	Buyers per session	Sessions	Subjects per condition
SellersMayApply	6	6	3	36
RandomTesting	6	6	3	36
Total			6	72

Our experiment was comprised of 6 sessions (3 per condition) and was conducted in-person between October and November 2022 at the Behavioral Laboratory at the University of Michigan. In total, 72 subjects participated in the experiment. On average, a session lasted 90 minutes, and a subject earned $22.37. More details on the number of subjects are displayed in Table 2. Subjects were invited to participate in the experiment using ORSEE (Greiner, 2015). The experiment was programmed and conducted with zTree (Fischbacher, 2007). Experimental instructions and main decision screens can be found in Appendix B.

4 Results

In this section, we demonstrate our experimental results and test our hypothesis.

As Table 3 shows, we find that a buyer's surplus in the SellersMayApply condition is higher than that in the RandomTesting condition by 2.991 ECUs on average, and this difference is statistically significant ($p < 0.01$). This result supports our hypothesis.

Result: A buyer's surplus in the SellersMayApply condition is significantly higher than that in the RandomTesting condition.

Table 3: Effect of the SellersMayApply mechanism on buyer surplus (Random-effects Linear Regression)

	Buyer's surplus
SellersMayApply	2.991***
	(0.428)
Constant	5.726***
	(0.387)
Observations	720

Note: (1) Standard errors are clustered at the subject level. (2) *** p<0.01, ** p<0.05, * p<0.1.

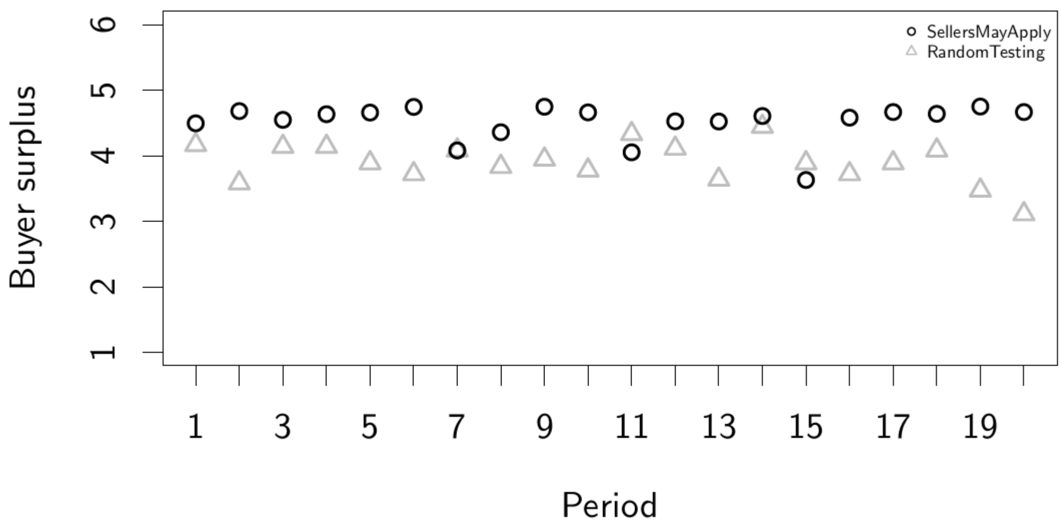

Figure 1: Buyer surplus over time

Figure 1 demonstrates the average buyer surplus in each round in two conditions. We see that the average buyer surplus in the SellersMayApply condition is higher than that in the RandomTesting condition in most rounds.

To better understand how the SellersMayApply mechanism improves buyer surplus relative to the RandomTesting mechanism, we plot all 6 sellers' products in the last round (i.e., Round 20) of each session in Figure 2. The three graphs on the first row demonstrate the distribution of six sellers' products in each session of the SellersMayApply condition, while the the three graphs on the second row shows the distribution of six sellers' products in each session of the RandomTesting condition. The three dashed horizontal lines on each graph indicate the unit costs of products with qualities 1, 2 and 3.

From Figure 2, We can see that there are mainly two reasons why the SellersMayApply mechanism improves buyer surplus. First, in all three SellersMayApply sessions, there are always sellers who offer products that can close to $(q = 2, p = 4)$ and $(q = 3, p = 9)$, which maximize $\theta = \theta_L$ and $\theta = \theta_H$ buyers' surplus respectively. In the three RandomTesting sessions, sellers' provision of products tend to deviate more from the optimal quality-

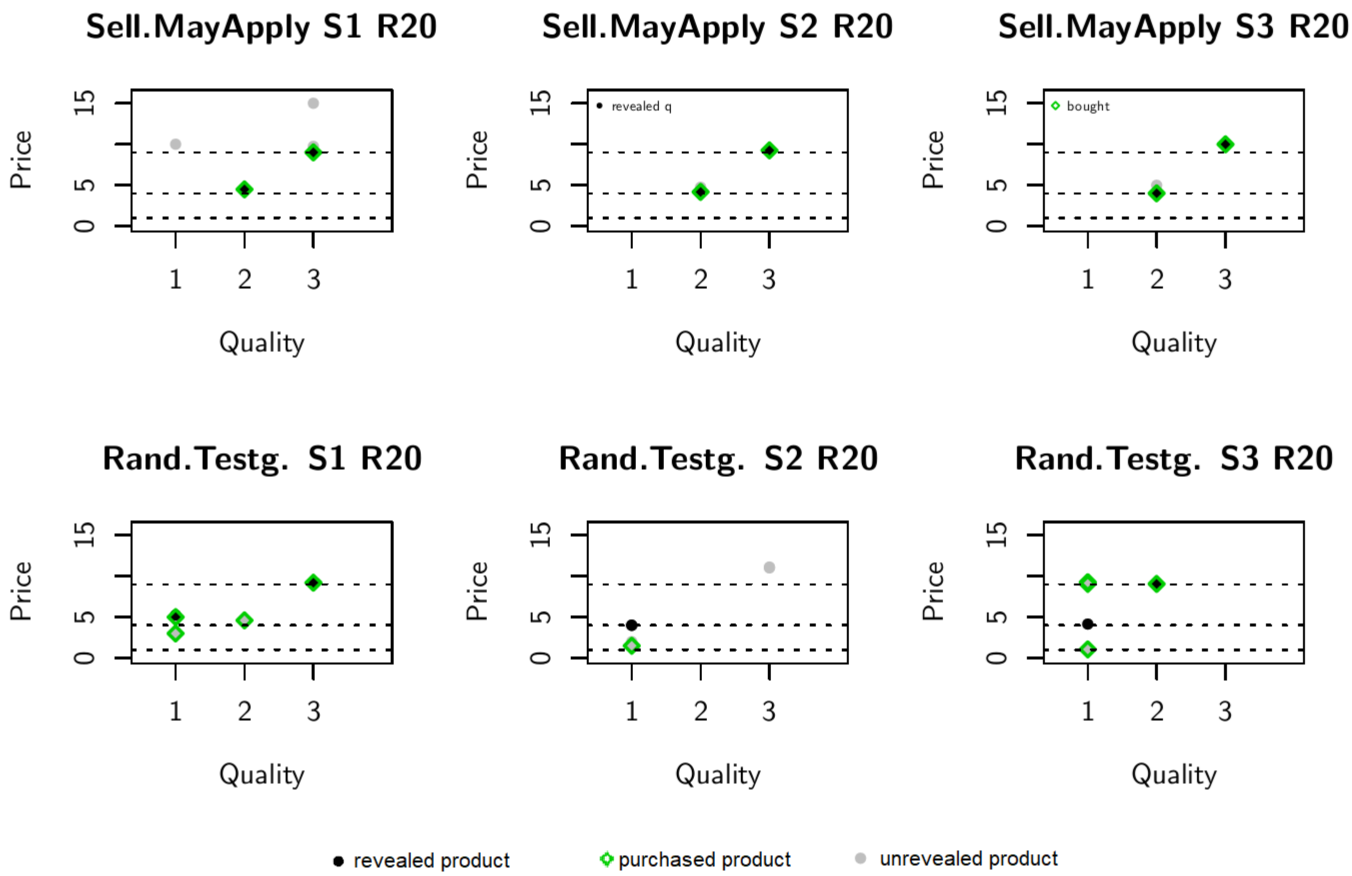

Figure 2: Product distribution in the last round of each session

price bundles. Second, the product testing organization is always able to reveal globally non-dominated products to buyers when using the SellersMayApply mechanism. Knowing how the SellersMayApply mechanism select and test products, buyers always buy products revealed by the testing organization. In the RandomTesting condition, which two products are tested is random, so buyers sometimes choose to purchase an unrevealed product. Many of the revealed products yield low surplus to buyers.

5 Discussion and conclusion

In this paper, we discuss markets in which the vertical dimension of product characteristic, i.e., quality, is not visible to consumers unless it is tested by an independent product testing organization. We investigate whether the product testing organization can make full use of its limited testing capacity to test and reveal the qualities of products that maximize consumer surplus through our proposed product testing mechanism SellersMayApply. We prove that in the product testing game where the product testing organization uses the SellersMayApply mechanism, the unique weak Perfect Bayesian Equilibria maximize consumer surplus, so that the SellersMayApply mechanism theoretically (weakly) dominates any other alternative product testing mechanism. We also discuss a generic testing mechanism, RandomTesting, which randomly test products within the testing capacity of the product testing organization. We show that under the RandomTesting mechanism, there does not exist a (weak) perfect Bayesian Equilibrium that maximizes consumer surplus.

The results from our laboratory experiment supports our prediction that consumer surplus is significantly higher when the SellersMayApply mechanism is used than when the RandomTesting mechanism is used.

Our theoretical and experimental results demonstrate that our proposed product testing mechanism increases consumer surplus through two channels. First, the mechanism incentivizes enough sellers to produce products with qualities that are preferred by consumers and with prices close to or equal to the unit cost. Second, the mechanism enables sellers to influence the product testing outcomes and ensures that only the qualities of globally non-dominated products will be revealed to consumers. Overall, our study shows that we can improve consumer surplus in a market with information asymmetry through a product testing mechanism which only tests and reveals the qualities of a small fraction of products on the market.

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A Appendix: Proofs of corollaries, lemmas and propositions

Corollary 0 (C0). *Applying for quality testing with a false reported quality is a dominated strategy for a seller.*

Proof. Let's consider the perspective of one seller f_i with a true quality q_i. Holding all the other $n-1$ sellers' qualities, prices and application decisions constant. Let's discuss two cases.

- **Case 1:** f_i would not be selected into F_1 if she applies with a false reported quality $\tilde{q}_i$ (with $\tilde{q}_i \neq q_i$).
  - **Case 1.1:** If f_i applies, there is 0% chance that she will be selected to be tested, so she will pay the application deposit μ which will not be returned to her.
  - **Case 1.2:** If f_i does not apply, her expected quality will be the same as that in Case 1.1, because holding all n sellers' price and quality bundles and all the other $n-1$ sellers' application decisions constant, her decision of not applying should not make any difference in terms of the outcome buyers can see (i.e., the revealed qualities and which sellers are tested and which sellers are not) compared with Case 1.1. However, she does not pay the application deposit.
  - Therefore, not applying strictly dominates applying with a false reported quality for f_i.
- **Case 2:** f_i would be selected into F_1 and would be tested with a probability of 1 if she applies with a false reported quality $\tilde{q}_i$ (with $\tilde{q}_i \neq q_i$).
  - **Case 2.1:** If f_i applies, f_i needs to pay σ. Since σ is greater than her ex-post revenue, f_i's payoff must be strictly negative.
  - **Case 2.2:** If f_i does not apply, then with A5, her expected payoff must be non-negative.
  - Therefore, not applying strictly dominates applying with a false reported quality for f_i.
- **Case 3:** f_i would selected into F_1 and would be tested with a probability of λ ($\lambda \in (0, 1)$) if she applies with a false reported quality $\tilde{q}_i$ (with $\tilde{q}_i \neq q_i$).
  - **Case 3.1:** If f_i applies:
    * **Case 3.1.1:** (With probability λ) If f_i is tested. f_i needs to pay σ. Since σ is greater than her ex-post revenue, f_i's payoff must be strictly negative. Denote this negative (expected) payoff as $\pi^* < 0$.
    * **Case 3.1.2:** (With probability $1-\lambda$) If f_i is not tested but is in the F_1, f_i will be returned the application deposit μ. There could be one or multiple testing outcomes (i.e., which seller(s)' is (are) tested and revealed), denoted as $o_1, ..., o_\xi$. Denote f_i's ex-post payoff in o_t as π_{o_t} ($t = 1, ..., \xi$). Denote the

probability of the occurrence of o_t as λ_{o_t}. With A5, we know that $\pi_{o_t} \geqslant 0$ $(t = 1, ..., \xi)$. Therefore, f_i's expected payoff is $\sum_{t=1}^{\xi} \pi_{o_t}\lambda_{o_t} \geqslant 0$ (with $\sum_{t=1}^{\xi} \lambda_{o_t} = 1 - \lambda$).

    * Therefore, f_i's expected payoff when applying is $\lambda\pi^* + \sum_{t=1}^{\xi} \pi_{o_t}\lambda_{o_t}$.

- **Case 3.2:** If f_i does not apply:
    * **Case 3.2.1:** When f_i does not apply, each of the possible outcomes in Case 2.1.2, o_t $(t = 1, ..., \xi)$ will still happen. Denote the probability of the occurrence of o_t in Case 3.2.1 as $\lambda_{o_t}^{'}$. Since f_i does not apply, the testing slot will be equally or less congested, so there must be $\lambda_(o_t)^{'} \geqslant \lambda_{o_t}$ (with $(t = 1, ..., \xi)$). f_i's expected payoff is $\sum_{t=1}^{\xi} \pi_{o_t}\lambda_{o_t}^{'} \geqslant 0$, and there must be $\sum_{t=1}^{\xi} \pi_{o_t}\lambda_{o_t}^{'} \geqslant \sum_{t=1}^{\xi} \pi_{o_t}\lambda_{o_t}$.
    * **Case 3.2.2:** In addition to Case 3.2.1, it is also possible (with a probability $1 - \sum_{t=1}^{\xi} \lambda_{o_t}^{'}$) that with f_i's quitting, there are new sellers selected into the F_1 compared with Case 3.2.1. In other words, there could be new testing outcome(s) that would not happen in Case 3.2.1. With A5, we know that f_i's ex-post payoff in each of these new testing outcomes (if any) must be non-negative. Therefore, f_i's (expected) payoff, denoted as π^{**}, must be non-negative (i.e., $\pi^{**} \geqslant 0$).
    * Therefore, f_i's expected payoff when not applying is $\sum_{t=1}^{\xi} \pi_{o_t}\lambda_{o_t}^{'} + (1 - \sum_{t=1}^{\xi} \lambda_{o_t}^{'})\pi^{**}$.

- With $\sum_{t=1}^{\xi} \pi_{o_t}\lambda_{o_t}^{'} \geqslant \sum_{t=1}^{\xi} \pi_{o_t}\lambda_{o_t}$, $\pi^* < 0$ and $\pi^{**} \geqslant 0$, we can conclude that $\lambda\pi^* + \sum_{t=1}^{\xi} \pi_{o_t}\lambda_{o_t} < \sum_{t=1}^{\xi} \pi_{o_t}\lambda_{o_t}^{'} + (1 - \sum_{t=1}^{\xi} \lambda_{o_t}^{'})\pi^{**}$. Thus, not applying strictly dominates applying with a false reported quality for f_i.

□

Corollary 1.1 (C1.1). *A seller with $q = 1$ will not apply.*

Proof. When there exists sellers with $q = 1$. Denote one of them as f_1. Consider f_1's best response without loss of generality.

- **Case 1:** If f_1 applies, she will not be selected to be tested, so she will pay the application deposit σ which will not be returned to him.

- **Case 2:** If f_1 does not apply, her expected quality will be the same as that in Case 1, because holding all n sellers' price and quality bundles and all the other 5 sellers' application decisions constant, her decision of not applying should not make any difference in terms of the outcome buyers can see (i.e., the revealed qualities and which sellers are tested and which sellers are not) compared with Case 1. However, she does not pay the application deposit σ.

- Therefore, not applying is a best response for f_1. According to A1, f_1 will not apply, and it is common knowledge.

- If there are multiple sellers with $q = 1$, each of them will not apply either, using the same reasoning as we used for f_1.

□

Corollary 2.1 (C2.1). *A globally non-dominated seller with $q = 3$ must apply.*

Proof. Let's discuss different cases.

- **Case 1:** Suppose there is only one globally non-dominated sellers with $q = 3$. Denote him as f_{ND_3}. Consider f_{ND_3}'s best response.
  - **Case 1.1:** If f_{ND_3} applies, then she must be selected to be tested, so her quality is revealed to be 3.
  - **Case 1.2:** If f_{ND_3} does not apply, then her expected quality can be 3 or smaller than 3.
    - **Case 1.2.1:** If her expected quality is 3, then according to A3.1, she will choose to apply to unravel uncertainty about her quality (her expected payoff from applying and not applying should also be the same since her application deposit must be returned to him).
    - **Case 1.2.2:** If her expected quality is smaller than 3, then applying will bring him a high expected quality. If this higher expected quality brings up her expected profit (which in turns brings up her expected profit), then applying is a best response. If this higher expected quality does not change her expected profit, applying is still a (weakly) best response, and according to A3.2, she chooses applying.
  - Therefore, f_{ND_3} will apply, and it is common knowledge.

- **Case 2:** Suppose there are $n (n \geqslant 2)$ identical globally non-dominated sellers with $q = 3$. Denote one globally non-dominated seller with $q = 3$ as $f_{ND_3^1}$ and the other globally non-dominated sellers with $q = 3$ as $f_{ND_3^2}, ..., f_{ND_3^n}$. Consider $f_{ND_3^1}$'s best response without loss of generality.
  - **Case 2.1:** Suppose there are α sellers among $f_{ND_3^2}, ..., f_{ND_3^n}$ who apply ($\alpha > 0$):
    - **Case 2.1.1:** If $f_{ND_3^1}$ applies, then:
      - **Case 2.1.1.1:** With $\frac{1}{\alpha+1}$ probability, $f_{ND_3^1}$ will be randomly selected to be tested, and then her quality is revealed to be 3.
      - **Case 2.1.1.2:** With $\frac{\alpha}{\alpha+1}$ probability, $f_{ND_3^1}$ will not be randomly selected, and then her expected quality must satisfy $E(q) \leqslant 3$. Her application deposit σ will be returned to him, because she is also globally non-dominated with $q = 3$.
    - **Case 2.1.2:** If $f_{ND_3^1}$ does not apply, then there must be one seller among $f_{ND_3^2}, ..., f_{ND_3^n}$ who is selected to be tested, and $f_{ND_3^1}$'s expected quality will be the same as that in Case 2.1.1.2. This is because holding all n sellers' price and quality bundles and all the other 5 sellers' application decisions constant, her decision of not applying should not make any difference in terms of the outcome buyers can see (i.e., the revealed qualities and which sellers are tested and which sellers are not) compared with Case 2.1.1.2.

    - * Therefore, $f_{ND_3^1}$'s expected quality from applying must be higher than or equal to that from not applying. Using the same reasoning as we do in Case 1.2, $f_{ND_3^1}$ will apply.
  - **Case 2.2:** Suppose there is no seller among $f_{ND_3^2}, ..., f_{ND_3^n}$ who applies:
    - * **Case 2.2.1:** If $f_{ND_3^1}$ applies, then her quality is revealed to be 3.
    - * **Case 2.2.2:** If $f_{ND_3^1}$ does not apply, then her expected quality must satisfy $E(q) \leqslant 3$.
    - * Therefore, using the same reasoning as we do in Case 1.2, $f_{ND_3^1}$ will apply.
  - Based on the conclusions from Case 2.1 and Case 2.2, $f_{ND_3^1}$ will apply, and it is common knowledge, according to A1.
  - Since $f_{ND_3^1}, ..., f_{ND_3^n}$ are identical sellers, they face the same situation, so all globally non-dominated sellers with $q = 3$ will apply.

□

Corollary 2.2 (C2.2). *A globally dominated seller with $q = 3$ will not apply.*

Proof. Denote the globally dominated seller with $q = 3$ as f_{D_3}. Consider f_{D_3}'s best response. Let's discuss different cases.

- **Case 1:** If she applies, since she knows that globally non-dominated sellers with $q = 3$ must apply according to C2.1 and C2.2, she will not be selected to be tested. She needs to pay the application deposit σ which will not be returned to him.
- **Case 2:** If she does not apply, then her expected quality would be the same as that in Case 1, because holding all n sellers' price and quality bundles and all the other 5 sellers' application decisions constant, her decision of not applying should not make any difference in terms of the outcome buyers can see (i.e., the revealed qualities and which sellers are tested and which sellers are not) compared with Case 1. However, she does not pay the application deposit σ.
- Therefore, not applying is a best response for f_{D_3}. According to A1, f_{D_3} will not apply, and it is common knowledge.
- If there are multiple globally dominated sellers with $q = 3$, each of them will not apply either, using the same reasoning as we used for f_{D_3}.

□

Corollary 3.1 (C3.1). *A globally non-dominated seller with $q = 2$ must apply.*

Proof. Let's discuss different cases.

- **Case 1:** If there also exist sellers with $q = 3$:
  - **Case 1.1:** Suppose there is only one globally non-dominated sellers with $q = 2$. Denote her as f_{ND_2}. Since she is globally non-dominated, we must have $p_{ND_2} < p_{ND_3}$. Consider f_{ND_2}'s best response.

* **Case 1.1.1:** If f_{ND_2} applies, then she must be selected to be tested, so her quality is revealed to be 2.
* **Case 1.1.2:** If f_{ND_2} does not apply, since $p_{ND_2} < p_{ND_3}$ buyers will conclude that f_{ND_2} cannot have $q = 3$ based on C2.2 (which would make f_{ND_3} a globally dominated seller and would have a conflict with f_{ND_3} applying). Therefore, based on A2 and C2.2, f_{ND_2}'s expected quality must satisfy $1 \leqslant E(q) \leqslant 2$.
  * **Case 1.1.2.1:** If f_{ND_2}'s expected quality is 2, then f_{ND_2} will apply according to A3.1 to unravel uncertainty about her quality (her expected payoff from applying and not applying should also be the same since her application deposit must be returned to her).
  * **Case 1.1.2.2:** If f_{ND_2}'s expected quality is smaller than 2, then applying will bring her a high expected quality. If this higher expected quality brings up her expected demand (which in turns brings up her expected profit), then applying is a best response. If this higher expected quality does not change her expected profit, applying is still a (weakly) best response, and according to A3.2, she chooses applying.
* Therefore, f_{ND_2} will apply according to A1.

- **Case 1.2:** Suppose there are $n (n \geqslant 2)$ identical globally non-dominated sellers with $q = 2$ $(2 \leqslant n \leqslant 5)$. Denote one globally non-dominated seller with $q = 2$ as $f_{ND_2^1}$ and the other globally non-dominated sellers with $q = 2$ as $f_{ND_2^2}, ..., f_{ND_2^n}$. Since they are globally non-dominated, we must have $p_{ND_2^i} < p_{ND_3}$ $(i = 1, ..., n)$. Consider $f_{ND_2^1}$'s best response without loss of generality.
  * **Case 1.2.1:** Suppose there are α sellers among $f_{ND_2^2}, ..., f_{ND_2^n}$ who apply $(\alpha > 0)$. Using the a similar reasoning as that in C2.1 Case 2.1, it can be proved that $f_{ND_2^1}$ will apply.
  * **Case 1.2.2:** Suppose there is no seller among $f_{ND_2^2}, ..., f_{ND_2^n}$ who applies. Using the a similar reasoning as that in C2.1 Case 2.2, it can be proved that $f_{ND_2^1}$ will apply.
- Based on the conclusions from Case 1.2.1 and Case 1.2.2, $f_{ND_2^1}$ must apply according to A1.
- Since $f_{ND_2^1}, ..., f_{ND_2^n}$ are identical sellers, they face the same situation, so all globally non-dominated sellers with $q = 2$ will apply.

• **Case 2:** If there do not exist sellers with $q = 3$:

- **Case 2.1:** Suppose there is only one globally non-dominated sellers with $q = 2$. Denote her as f_{ND_2}. Consider f_{ND_2}'s best response.
  * **Case 2.1.1:** If f_{ND_2} applies, then she must be selected to be tested, so her quality is revealed to be 2.
  * **Case 2.1.2:** If f_{ND_2} does not apply, then since there is no $q = 3$ seller, it is impossible for buyers to see a revealed $q = 3$ product. Buyers will conclude that f_{ND_2} cannot have $q = 3$ based on C2.1. Therefore, based on A2 and

C2.2, f_{ND_2}'s expected quality must satisfy $1 \leqslant E(q) \leqslant 2$. Using the same reasoning as that in Case 1.1.2, it can be proved that f_{ND_2} will apply.

- **Case 2.2:** Suppose there are $n (n \geqslant 2)$ identical globally non-dominated sellers with $q = 2$ $(2 \leqslant n \leqslant s)$. Using a similar reasoning as that in Case 1.2, it can be proved that every globally non-dominated sellers with $q = 2$ will apply.

□

Corollary 3.2 (C3.2). *A globally dominated seller with $q = 2$ will not apply.*

Proof. Denote (one) of the globally dominated seller(s) with $q = 2$ as f_{D_2}. Let's discuss different cases.

- **Case 1:** If there also exist sellers with $q = 3$. In this case, there are two possible reasons that f_{D_2} is globally dominated:
  - **Case 1.1:** If there exists a globally non-dominated seller f_{ND_2} such that $p_{ND_2} < p_{D_2} < p_{ND_3}$.
    - **Case 1.1.1:** If she applies, she knows that f_{ND_2} must apply and she would not be selected. She needs to pay the application deposit which will not be returned to him.
    - **Case 1.1.2:** If she does not apply, then her expected quality would be the same as that in Case 1.1.1, because holding all n sellers' price and quality bundles and all the other 5 sellers' application decisions constant, her decision of not applying should not make any difference in terms of the outcome buyers can see compared with Case 1.1.1. However, she does not pay the application deposit.
  - **Case 1.2:** If $p_{D_2} > p_{ND_3}$.
    - **Case 1.2.1:** If she applies, she knows that f_{D_2}, who globally dominate him, must apply and thus she would not be selected. She needs to pay the application deposit which will not be returned to him.
    - **Case 1.2.2:** If she does not apply, then using the same reasoning as that in Case 1.1.2, it can be proved that her expected quality would be the same as that in Case 1.2.1, but she does not pay the application deposit.
  - Therefore, not applying is always a best response for f_{D_2}. According to A1, f_{D_2} will not apply.
  - If there are multiple globally dominated sellers with $q = 2$, each of them will not apply either, using the same reasoning as we used for f_{D_2}.
- **Case 2:** If there do not exist sellers with $q = 3$, then using the same reasoning as that in Case 1.1, it can be proved that f_{D_2} will not apply. If there are multiple globally dominated sellers with $q = 2$, each of them will not apply either.

□

Lemma 0.1 (L0.1). *If there exist sellers with $q = 2$ and $q = 3$ and (one of) the seller with the lowest price among $q = 2$, denoted as $f_{l_2^1}$, and (one of) the seller with the lowest price among $q = 3$, denoted as $f_{l_3^1}$, satisfy $p_{l_2^1} < p_{l_3^1}$, then:*

- ***L0.1.1:*** *The maximum expected profit a buyer with $\theta = \theta_L$ can earn from a seller with $q = 1$ is $max\{\theta_L - c(1), 2\theta_L - c(2) - \theta_L\alpha_2, 3\theta_L - c(3) - 2\theta_L\alpha_1 - \theta_L\beta_1\} \leqslant 2\theta_L - c(2)$.*
- ***L0.1.2:*** *The maximum expected profit a buyer with $\theta = \theta_H$ can earn from a seller with $q = 1$ is $max\{\theta_H - c(1), 2\theta_H - c(2) - \theta_H\alpha_2, 3\theta_H - c(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\} \leqslant 3\theta_H - c(3)$.*
- ***L0.1.3:*** *The maximum expected profit a buyer with $\theta = \theta_H$ can earn from a seller with $q = 2$ is $max\{2\theta_H - c(2), 3\theta_H - c(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\} \leqslant 3\theta_H - c(3)$.*
- ***L0.1.4:*** *The maximum expected profit a buyer with $\theta = \theta_L$ can earn from a seller with $q = 3$ is $3\theta_L - c(3) < 2\theta_L - c(2)$.*

Proof. Both $f_{l_2^1}$ and $f_{l_3^1}$ are globally non-dominated, and re-denote them as $f_{ND_2^1}$ and $f_{ND_3^1}$ (so there should be $p_{ND_2^1} < p_{ND_3^1}$. According to C2.1 and C3.1, both $f_{ND_2^1}$ and $f_{ND_3^1}$ must apply.

- **Proof for L0.1.1:** Let's discuss different cases:
  - **Case L0.1.1.1:** If a seller with $q = 1$, denoted as f_1, has $p_1 < p_{ND_2^1}$, then f_1's expected quality is 1 according to C4.2.1. Since $p_1 \geqslant c(1)$, a $\theta = \theta_L$ buyer's maximum expected profit from buying from f_1 is $\theta_L - c(1)$.
  - **Case L0.1.1.2:** If a seller with $q = 1$, denoted as f_1, has $p_{ND_2^1} \leqslant p_1 < p_{ND_3^1}$, then f_1's expected quality is $2 - \alpha_2 \in [1, 2]$ according to C4.2.1. Since $p_{ND_2^1} \geqslant c(2)$, a $\theta = \theta_L$ buyer's maximum expected profit from buying from f_1 is $\theta_L(2-\alpha_2)-c(2) = 2\theta_L - c(2) - \theta_L\alpha_2$.
  - **Case L0.1.1.3:** If a seller with $q = 1$, denoted as f_1, has $p_1 \geqslant p_{ND_3^1}$, then f_1's expected quality is $3 - 2\alpha_1 - \beta_1 \in [1, 3]$ according to C4.2.1. Since $p_{ND_3^1} \geqslant c(3)$, a $\theta = \theta_L$ buyer's maximum expected profit from buying from f_1 is $\theta_L(3 - 2\alpha_1 - \beta_1) - c(3) = 3\theta_L - c(3) - 2\theta_L\alpha_1 - \theta_L\beta_1$.
  - The inequality holds because according to (5), $c(2) - c(1) < \theta_L \Rightarrow \theta_L - c(1) < 2\theta_L - c(2)$, and according to (6), $c(3) - c(2) > \theta_L \Rightarrow 3\theta_L - c(3) < 2\theta_L - c(2)$.
- **Proof for L0.1.2:** Let's discuss different cases:
  - **Case L0.1.2.1:** If a seller with $q = 1$, denoted as f_1, has $p_1 < p_{ND_2^1}$, then f_1's expected quality is 1 according to C4.2.1. Since $p_1 \geqslant c(1)$, a $\theta = \theta_H$ buyer's maximum expected profit from buying from f_1 is $\theta_H - c(1)$.
  - **Case L0.1.2.1:** If a seller with $q = 1$, denoted as f_1, has $p_{ND_2^1} \leqslant p_1 < p_{ND_3^1}$, then f_1's expected quality is $2 - \alpha_2 \in [1, 2]$ according to C4.2.1. Since $p_{ND_2^1} \geqslant c(2)$, a $\theta = \theta_H$ buyer's maximum expected profit from buying from f_1 is $\theta_H(2 - \alpha_2) - c(2) = 2\theta_H - c(2) - \theta_H\alpha_2$.

- **Case L0.1.2.3:** If a seller with $q = 1$, denoted as f_1, has $p_1 \geqslant p_{ND_3^1}$, then f_1's expected quality is $3 - 2\alpha_1 - \beta_1 \in [1, 3]$ according to C4.2.1. Since $p_{ND_3^1} \geqslant c(3)$, a $\theta = \theta_H$ buyer's maximum expected profit from buying from f_1 is $\theta_H(3 - 2\alpha_1 - \beta_1) - c(3) = 3\theta_H - c(3) - 2\theta_H\alpha_1 - \theta_H\beta_1$.
- The inequality holds because according to (8), $c(3) - c(2) < \theta_H \Rightarrow 2\theta_H - c(2) < 3\theta_H - c(3)$, and according to (6), $c(3) - c(2) < \theta_H \Rightarrow 2\theta_L - c(2) < 3\theta_L - c(3)$.

- **Proof for L0.1.3:** Let's discuss different cases:
  - **Case L0.1.3.1:** If an untested seller with $q = 2$, denoted as f_2, has $p_{ND_2^1} \leqslant p_2 < p_{ND_3^1}$, a $\theta = \theta_H$ buyer's expected quality is $2 - \alpha_2 \in [1, 2]$ according to C4.2.1. Since $p_{ND_2^1} \geqslant c(2)$, a $\theta = \theta_H$ buyer's maximum expected profit from buying from f_2 is $\theta_H(2 - \alpha_2) - c(2) = 2\theta_H - c(2) - \theta_H\alpha_2$. Then a $\theta = \theta_H$ buyer's maximum expected profit from any $q = 2$ sellers (globally non-dominated or dominated) is $max\{\theta_H \times 2 - c(2), 2\theta_H - c(2) - \theta_H\alpha_2\} = 2\theta_H - c(2)$.
  - **Case L0.1.3.2:** If an untested seller with $q = 2$, denoted as f_2, has $p_2 \geqslant p_{ND_3^1}$, then f_2's expected quality is $3 - 2\alpha_1 - \beta_1$ according to C4.2.1. Since $p_{ND_3^1} \geqslant c(3)$, a $\theta = \theta_H$ buyer's maximum expected profit from buying from f_2 is $\theta_H(3 - 2\alpha_1 - \beta_1) - c(3) = 3\theta_H - c(3) - 2\theta_H\alpha_1 - \theta_H\beta_1$. Then a $\theta = \theta_H$ buyer's maximum expected profit from buying from any $q = 2$ sellers (globally non-dominated or dominated) is is $max\{\theta_H \times 2 - c(2), 3\theta_H - c(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\} = max\{2\theta_H - c(2), 3\theta_H - c(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\}$.
  - The inequality holds because according to (8), $c(3) - c(2) < \theta_H \Rightarrow 2\theta_H - c(2) < 3\theta_H - c(3)$.

- **Proof for L0.1.4:** Let's discuss different cases:
  - Denote (one of) the globally dominated seller(s) with $q = 3$ as $f_{ND_3^1}$. According to C2.1, $f_{ND_3^1}$ must apply. According to C4.2.1, any untested seller with $q = 3$ must have an expected quality of $3 - 2\alpha_1 - \beta_1 \in [1, 3]$. On the other hand, all sellers with $q = 3$ must have $p \geqslant c(3)$. Therefore, a $\theta = \theta_L$ buyer's maximum expected profit from buying from any seller with $q = 3$ is $max\{3\theta_L - c(3), \theta_L(3 - 2\alpha_1 - \beta_1) - c(3)\} = 3\theta_L - c(3)$. The inequality holds because according to (6), $c(3) - c(2) > 4 \Rightarrow 3\theta_L - c(3) < 2\theta_L - c(2)$.

□

Lemma 0.2 (L0.2). *If there exist sellers with $q = 2$ and $q = 3$ and (one of) the seller with the lowest price among $q = 2$, denoted as $f_{l_2^1}$, and (one of) the seller with the lowest price among $q = 3$, denoted as $f_{l_3^1}$, satisfy $p_{l_2^1} \geqslant p_{l_3^1}$, then:*

- ***L0.2.1:*** *The maximum expected profit a buyer with $\theta = \theta_H$ can earn from a seller with $q = 1$ is $max\{\theta_H - c(1), 3\theta_H - c(3) - 2\theta_H\alpha_4 - \theta_H\beta_2\} \leqslant 3\theta_H - c(3)$.*

- ***L0.2.2:*** *The maximum expected profit a buyer with $\theta = \theta_H$ can earn from a seller with $q = 2$ is $3\theta_H - c(3) - 2\theta_H\alpha_4 - \theta_H\beta_2$.*

Proof. $f_{l_2^1}$ is globally dominated and $f_{l_3^1}$ is globally non-dominated, and re-denote $f_{l_3^1}$ as $f_{ND_3^1}$ (so there should be $p_{l_2^1} \geqslant p_{ND_3^1}$). According to C2.1 and C3.2, $f_{l_2^1}$ will not apply (and none of other $q = 2$ sellers, if any, will apply) and $f_{ND_3^1}$ will apply.

- **Proof for L0.2.1:** Let's discuss different cases:
  - **Case L0.2.1.1:** If a seller with $q = 1$, denoted as f_1, has $p_1 < p_{ND_3^1}$, then f_1's expected quality is 1 according to C4.2.3. Since $p_{ND_3^1} \geqslant c(1)$, a $\theta = \theta_H$ buyer's maximum expected profit from buying from f_1 is $\theta_H - c(1)$.
  - **Case L0.2.1.2:** If a seller with $q = 1$, denoted as f_1, has $p_1 \geqslant p_{ND_3^1}$, then f_1's expected quality is $3 - 2\alpha_1 - \beta_1 \in [1, 3]$ according to C4.2.3. Since $p_{ND_3^1} \geqslant c(3)$, a $\theta = \theta_H$ buyer's maximum expected profit from buying from f_1 is $\theta_H(3 - 2\alpha_4 - \beta_2) = 3\theta_H - c(3) - 2\theta_H\alpha_4 - \theta_H\beta_2$.
  - The inequality holds because according to (7), $c(3) - c(1) < 2\theta_H \Rightarrow \theta_H - c(1) < 3\theta_H - c(3)$.
- **Proof for L0.2.2:** According to C4.2.3, an untested seller with $q = 2$, denoted as f_2, must have $p_2 \geqslant p_{ND_3^1}$. According to C4.2.3, f_2's expected quality is $3 - 2\alpha_1 - \beta_1 \in [1, 3]$. Since $p_{ND_3^1} \geqslant c(3)$, a $\theta = \theta_H$ buyer's maximum expected profit from buying from f_2 is $\theta_H(3 - 2\alpha_4 - \beta_2) = 3\theta_H - c(3) - 2\theta_H\alpha_4 - \theta_H\beta_2$.

□

Lemma 0.3 (L0.3). *If there exist sellers with $q = 2$ but not any seller with $q = 3$, denote (one of) the globally non-dominated seller(s) with $q = 2$ as $f_{ND_2^1}$, then the maximum expected profit a buyer with $\theta = \theta_L$ can earn from a seller with $q = 1$ is $max\{\theta_L - c(1), 2\theta_L - c(2) - \theta_L\alpha_3\} \leqslant 2\theta_L - c(2)$.*

Proof. Let's discuss different cases:

- **Case L0.3.1:** If a seller with $q = 1$, denoted as f_1, has $p_1 < p_{ND_2^1}$. According to C4.2.2, f_1's expected quality is 1. Since $p_1 \geqslant c(1)$, a $\theta = \theta_L$ buyer's maximum expected profit from buying from f_1 is $\theta_L - c(1)$.
- **Case L0.3.2:** If a seller with $q = 1$, denoted as f_1, has $p_1 \geqslant p_{ND_2^1}$. According to C4.2.2, f_1's expected quality is $2 - \alpha_3 \in [1, 2]$. Since $p_{ND_2^1} \geqslant c(2)$, a $\theta = \theta_L$ buyer's maximum expected profit from buying from f_1 is $\theta_L(2 - \alpha_3) - c(2) = 2\theta_L - c(2) - \theta_L\alpha_3$.
- This inequality holds because according to (5), $c(2) - c(1) < \theta_L \Rightarrow \theta_L - c(1) < 2\theta_L - c(2)$.

□

Lemma 0.4 (L0.4). *If there exist sellers with $q = 3$ but not any seller with $q = 2$, denote (one of) the globally non-dominated seller(s) with $q = 3$ as $f_{ND_3^1}$, then*

- ***L0.4.1:*** *The maximum expected profit a buyer with $\theta = \theta_L$ can earn from a seller with $q = 1$ is $max\{\theta_L - c(1), 3\theta_L - c(3) - 2\theta_L\alpha_4 - \theta_L\beta_2\} < 2\theta_L - c(2)$.*

- ***L0.4.2:*** *The maximum expected profit a buyer with $\theta = \theta_H$ can earn from a seller with $q = 1$ is $max\{\theta_H - c(1), 3\theta_H - c(3) - 2\theta_H\alpha_4 - \theta_H\beta_2\} \leqslant 3\theta_H - c(3)$.*

Proof. According to C2.1, $f_{ND_3^1}$ must apply.

- **Proof for L0.4.1:** Let's discuss different cases:
  - **Case L0.4.1.1:** If a seller with $q = 1$, denoted as f_1, has $p_1 < p_{ND_3^1}$. According to C4.2.3, f_1's expected quality is 1. Since $p_1 \geqslant c(1)$, a $\theta = \theta_L$ buyer's maximum expected profit from buying from f_1 is $\theta_L - c(1)$.
  - **Case L0.4.1.2:** If a seller with $q = 1$, denoted as f_1, has $p_1 \geqslant p_{ND_3^1}$. According to C4.2.3, f_1's expected quality is $3 - 2\alpha_1 - \beta_1 \in [1, 3]$. Since $p_{ND_3^1} \geqslant c(3)$, a $\theta = \theta_L$ buyer's maximum expected profit from buying from f_1 is $\theta_L(3 - 2\alpha_1 - \beta_1) - c(3) = 3\theta_L - c(3) - 2\theta_L\alpha_4 - \theta_L\beta_2$.
  - The inequality holds because according to (5), $c(2) - c(1) < \theta_L \Rightarrow \theta_L - c(1) < 2\theta_L - c(2)$, and according to (6), $c(3) - c(2) > \theta_L \Rightarrow 3\theta_L - c(3) < 2\theta_L - c(2)$.
- **Proof for L0.4.2:** Let's discuss different cases:
  - **Case L0.4.2.1:** If a seller with $q = 1$, denoted as f_1, has $p_1 < p_{ND_3^1}$. According to C4.2.3, f_1's expected quality is 1. Since $p_1 \geqslant c(1)$, a $\theta = \theta_H$ buyer's maximum expected profit from buying from f_1 is $\theta_H - c(1)$.
  - **Case L0.4.2.2:** If a seller with $q = 1$, denoted as f_1, has $p_1 \geqslant p_{ND_3^1}$. According to C4.2.3, f_1's expected quality is $3 - 2\alpha_1 - \beta_1 \in [1, 3]$. Since $p_{ND_3^1} \geqslant c(3)$, a $\theta = \theta_H$ buyer's maximum expected profit from buying from f_1 is $\theta_H(3 - 2\alpha_1 - \beta_1) - c(3) = 3\theta_H - c(3) - 2\theta_H\alpha_4 - \theta_H\beta_2$.
  - The inequality holds because according to (7), $c(3) - c(1) < 2\theta_H \Rightarrow \theta_H - c(1) < 3\theta_H - c(3)$.

□

Lemma 1 (L1). *If in a strategy profile, no seller applies, then this strategy profile cannot be a PBE.*

Proof. If no seller applies, then according to C1.1, C2.1 and C3.1, all sellers must have $q = 1$, and every (untested) seller's expected quality is also 1 (according to C4.2.4), so only the seller with the lowest price will have a positive demand, if any seller has a positive demand (if all sellers have zero demand, then the only reason is that their prices are all too high, so obviously any seller would have the incentive to deviate by lowering her price).

- **Case 1:** If all sellers' prices are higher than $c(1)$.
  - **Case 1.1:** If not all sellers' prices are the same. Denote the seller with the highest price as f_h and the seller with the lowest price as f_l. f_h must have a 0 demand and thus a 0 expected profit, and she must have the incentive to deviate to $(1,\ p_l - \epsilon,\ \textit{Not Apply})$, where ϵ can be any number which satisfies $\epsilon < p_l - c(1)$, and then she would have a positive expected demand and thus a positive expected profit.

- If all sellers' prices are the same. Then every seller must have an expected demand of 1. Then any seller must have the incentive to reduce their price by ϵ, where ϵ is small enough, so that her expected demand becomes s, but her markup is only reduced by a little, so her expected profit is still increased.

- **Case 2:** If at least one seller's price is $c(1)$. Then the seller(s) with $p = c(1)$ will have an expected profit of 0. Suppose the lowest price among all the other sellers is $p = c(1) + \tau$ ($\tau \geq 0$). The seller(s) with $p = 1$ must have an incentive to deviate to (2, $c(2)+\epsilon$, *Apply*), where ϵ is small enough such that all $\theta = \theta_L$ will strictly prefer this deviated seller (i.e., $2\theta_L - (c\,(2) + \epsilon) > \theta_L - (c\,(1) + \tau) \iff \epsilon < \tau + c\,(1) - c\,(2) + \theta_L$. According to (5), $c\,(2) - c\,(1) < \theta_L \Rightarrow\ c\,(1) - c\,(2) + \theta_L > 0$, so there must exist $\epsilon > 0$ that is small enough) to get a positive expected profit.

□

Lemma 2 (L2). *If in a strategy profile, there do not exist sellers with $q = 3$, then this strategy profile cannot be a PBE, when* $\begin{cases}\alpha_3 > 0\\ \alpha_1 + \beta_1 > 0\end{cases}$.

Proof. According to C1.1, C3.1 and C3.2, if there exist sellers with $q = 2$, the globally non-dominated one(s) must apply, and all other sellers (including sellers with $q = 1$, if any, and globally dominated sellers with $q = 2$, if any) will not apply. Denote (one of) the globally non-dominated seller(s) with $q = 2$ as $f_{ND^1_2}$. If there exist sellers with $q = 1$, then according to C1.1, no seller with $q = 1$ will apply. Denote the seller with the lowest price among $q = 1$ as f_l. We then prove the following lemmas when there does not exist any seller with $q = 3$:

- **Lemma 2.1 (L2.1):** Suppose there does not exist any seller with $q = 3$, and there exist seller(s) with $q = 2$. Any seller with $p > p_{ND^1_2}$, denoted as f_h, must have a 0 expected demand and thus a 0 expected profit.
  - Proof: This other seller must have an expected quality of $2-\alpha_3 \in [1, 2]$ according to C4.2.2. $f_{ND^1_2}$ must be strictly preferred to f_h by all $\theta = \theta_L$ and $\theta = \theta_H$ buyers (because $2\theta_L - p_{ND^1_2} > \theta_L(2 - \alpha_3) - p_h$ and $2\theta_H - p_{ND^1_2} > \theta_H(2 - \alpha_3) - p_h$), so she must have a 0 expected demand.

- **Lemma 2.2 (L2.2):** Suppose there does not exist any seller with $q = 3$, and there exist seller(s) with $q = 2$. If there exist at least two sellers with $q = 1$, then if any seller with $q = 1$, denoted as f_i, has $p_i > p_l$, then f_i must have a 0 expected profit.
  - Proof:
    * **Case L2.2.1:** If $p_i < p_{ND^1_2}$, then there must be $p_l < p_{ND^1_2}$, and thus both f_l and f_i's expected qualities are 1 according to C4.2.2, but $p_i > p_l$, so f_l must be strictly preferred to f_i, so f_i must have a 0 expected demand.
    * **Case L2.2.2:** If $p_i \geqslant p_{ND^1_2}$, then f_i has an expected quality of $2-\alpha_3$ according to C4.2.2. When $\alpha_3 > 0$, we have $2 - \alpha_3 < 2$, and then $f_{ND^1_2}$ must be strictly preferred to f_i by all $\theta = \theta_L$ and $\theta = \theta_H$ buyers (because $2\theta_L - p_{ND^1_2} > \theta_L(2 - \alpha_3) - p_i$ and $2\theta_H - p_{ND^1_2} > \theta_H(2 - \alpha_3) - p_i$), so f_i must have a 0 expected demand.

- **Lemma 2.3 (L2.3):** Suppose there does not exist any seller with $q = 3$, and there exist seller(s) with $q = 2$. If any seller with $q = 2$ has $p = c(2)$, then this strategy profile cannot be a PBE.
  - Proof: This seller, denoted as f_2, must have a 0 expected profit, because she has a 0 markup. Then she must have the incentive to deviate to $(3,\ c(3)+\epsilon,\ Apply)$, where ϵ is small enough, so that all $\theta = \theta_H$ buyers will strictly prefer her product (because when $\alpha_1 + \beta_1 > 0$, there must be $2\alpha_1 + \beta_1 > 0$, and then $\theta = \theta_H$ buyers' expect profit is at most $max\{\theta_H - c(1), 2\theta_H - c(2) - \theta_H\alpha_2, 3\theta_H - c(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\} < 3\theta_H - c(3)$ from any $q = 1$ seller according to L0.1.2, and at most $max\{2\theta_H - c(2), 3\theta_H - c(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\} < 3\theta_H - c(3)$ from any $q = 2$ seller according to L0.1.3, both of which are smaller than that from f_2, which is $3\theta_H - (c(3) + \epsilon) = 3\theta_H - c(3) - \epsilon > max\{\theta_H - c(1), 2\theta_H - c(2) - \theta_H\alpha_2, 2\theta_H - c(2), 3\theta_H - c(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\}$, then she would have a positive expected profit.
- **Lemma 2.4 (L2.4):** Suppose there does not exist any seller with $q = 3$, and there exist seller(s) with $q = 2$. If any seller with $q = 1$ has $p = c(1)$, then this strategy profile cannot be a PBE.
  - Proof: This seller with $q = 1$ and $p = c(1)$, denoted as f_1, must have an expected profit of 0, regardless of whether she has any expected demand, because she has a 0 markup. Then she must have the incentive to deviate to $(3,\ c(3)+\epsilon,\ Apply)$, where ϵ is small enough, so that all $\theta = \theta_H$ buyers will strictly prefer her product (because when $\alpha_1 + \beta_1 > 0$, there must be $2\alpha_1 + \beta_1 > 0$, and then $\theta = \theta_H$ buyers' expect profit is at most $max\{\theta_H - c(1), 2\theta_H - c(2) - \theta_H\alpha_2, 3\theta_H - c(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\} < 3\theta_H - c(3)$ from any $q = 1$ seller according to L0.1.2, and at most $max\{2\theta_H - c(2), 3\theta_H - c(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\}$ from any $q = 2$ seller according to L0.1.3, both of which are smaller than that from f_2, which is $3\theta_H - (c(3) + \epsilon) = 3\theta_H - c(3) - \epsilon > max\{\theta_H - c(1), 2\theta_H - c(2) - \theta_H\alpha_2, 2\theta_H - c(2), 3\theta_H - c(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\}$, then she would have a positive expected profit.

Then we begin to discuss different cases.

- **Case 1:** If there exist sellers with $q = 1$ and $q = 2$.
  - **Case 1.1:** If $p_{ND_2^1} > c(2)$:
    - **Case 1.1.1:** If $f_{ND_2^1}$ has a positive expected demand and f_l, the seller with the lowest price among $q = 1$ sellers, has a 0 expected demand, then f_l must have the incentive to deviate to be identical to $f_{ND_2^1}$'s action so that she would have a positive expected demand (shared with $f_{ND_2^1}$) and thus a positive expected profit.
    - **Case 1.1.2:** If $f_{ND_2^1}$ has a 0 expected demand and f_l has a positive expected demand.
      - **Case 1.1.2.1:** If $p_l = c(1)$, then according to L2.4, it cannot be a PBE.
      - **Case 1.1.2.2:** If $p_l > c(1)$, denote the seller with the lowest price among all other $q = 2$ sellers, if any, as f_2 (and there must be $p_2 \geqslant p_{ND_2^1} > c(2)$).

Then $f_{ND_2^1}$ must have the incentive to deviate to $(2, c(2)+\epsilon, Apply)$, where ϵ is small enough, to guarantee a positive expected demand (because $\theta = \theta_L$ buyers' expected profit from the deviated $f_{ND_2^1}$ is $2\theta_L - (c\,(2) + \epsilon) = 2\theta_L - c(2) - \epsilon$. When $\alpha_3 > 0$, this is larger than that from a $q = 1$ seller, which is at most $max\{\theta_L - c(1), 2\theta_L - c(2) - \theta_L\alpha_3\} < 2\theta_L - c(2)$ (this inequality holds because according to (5), $c\,(2) - c\,(1) < \theta_L \Longrightarrow \theta_L - c\,(1) < 2\theta_L - c(2)$) based on L0.3, and larger than that from any other q=2 seller, which is at most $2\theta_L - p_2 < 2\theta_L - c(2)$).

* **Case 1.1.3:** If $f_{ND_2^1}$ and f_l both have positive expected demand.
  - **Case 1.1.3.1:** If $p_l = c(1)$, then according to L2.4, it cannot be a PBE.
  - **Case 1.1.3.2:** If $p_l > c(1)$:
    - **Case 1.1.3.2.1:** If there is only 1 seller with $q = 1$ and $n - 1$ sellers with $q = 2$.
      - **Case 1.1.3.2.1.1:** If at least one other seller with $q = 2$, denoted as $f_{ND_2^2}, \ldots,\ f_{ND_2^k}$ $(2 \leqslant k \leqslant s - 1)$, has the same price as $f_{ND_2^1}$ (i.e., these sellers are identical to $f_{ND_2^1}$), then $f_{ND_2^1}, f_{ND_2^2}, \ldots, f_{ND_2^k}$ must share demand. Any of them will have the incentive to decrease her price by ϵ and still apply, where $\epsilon > 0$ is small enough, so that she would not need to share demand with other identical sellers (because only she alone will be tested after the deviation), and her expected profit would increase, despite of a very small decrease in the markup.
      - **Case 1.1.3.2.1.2:** If $f_{ND_2^1}$ is the unique globally non-dominated seller with $q = 2$, then according to L2.1, other sellers with $q = 2$ must have an expected demand of 0 and thus an expected profit of 0, so any of these sellers must have an incentive to deviate to be identical to $f_{ND_2^1}$'s action so that she would have a positive expected demand and thus a positive expected profit.
    - **Case 1.1.3.2.2:** If there are x seller with $q = 2$ and $(s-x)$ sellers with $q = 1$. $(1 \leqslant x \leqslant s - 2)$:
      - **Case 1.1.3.2.2.1:** If at least one other seller with $q = 1$, denoted as $f_{l_2}, \ldots, f_{l_k}$ $(2 \leqslant k \leqslant s-1)$, has the same price as f_l (i.e., these sellers also have the lowest price and are identical to f_l), then $f_l, f_{l_2}, \ldots, f_{l_k}$ must share demand. Notice that in this case, it is impossible that $p_l = p_{ND_2^1}$ (because that would make $f_{ND_2^1}$ strictly preferred to f_l by all $\theta = \theta_L$ and $\theta = \theta_H$ buyers and thus make f_l have a 0 expected demand, which would contradict with the condition in Case 1.1.3). Therefore, any of $f_l, f_{l_2}, \ldots, f_{l_k}$ will have the incentive to decrease her price by ϵ and still not apply, where $\epsilon > 0$ is small enough, so that she would not need to share demand with other identical sellers, and her expected profit would increase, despite of a very small decrease in the markup.
      - **Case 1.1.3.2.2.2:** If f_l is the unique seller with the lowest price among sellers with $q = 1$, then other sellers with $q = 1$, whose price

is higher than f_l, must have an expected demand of 0 and thus an expected profit of 0 according to L2.2, so any of these sellers must have an incentive to deviate to be identical to f_l's action so that she would have a positive demand and thus a positive expected profit (due to a positive markup).

    * **Case 1.1.4:** If $f_{ND_2^1}$ and f_l both have 0 expected demand (and thus both of them have a 0 expected profit), then according to L2.1 and L2.2, all buyers will choose not to buy from any seller (and thus all sellers have a 0 expected profit). The only possibility for this case is that for any buyer, the expected profit from buying from any seller is non-positive. Then any seller would have the incentive to deviate to, for example, $(2,\ c(2)+\epsilon, Apply)$, where $\epsilon < 2\theta_H - c(2)$, so that buyers with $\theta = \theta_H$ would buy from her and thus she would have a positive expected profit.

  - **Case 1.2:** If $p_{ND_2^1} = c(2)$, then according to L2.3, we know that it cannot be a PBE.

- **Case 2:** All sellers have $q = 1$. In this case, according to C1.1, no seller will apply. Then according to L1, it cannot be a PBE.

- **Case 3:** All sellers have $q = 2$. In this case, according to C3.1, $f_{ND_2^1}$ must apply.

  - **Case 3.1:** If$p_{ND_2^1} = c(2)$, then according to L2.3, we know that it cannot be a PBE.

  - **Case 3.2:** If $c(2) < p_{ND_2^1} < 2\theta_H$, then $p_{ND_2^1}$ must have a positive expected demand and thus a positive expected profit.

    * **Case 3.2.1:** If at least one other seller with $q = 2$, denoted as $f_{ND_2^2}, \ldots,\ f_{ND_2^k}$ $(2 \leqslant k \leqslant s-1)$, has the same price as $f_{ND_2^1}$ (i.e., these sellers are identical to $f_{ND_2^1}$), then $f_{ND_2^1}, f_{ND_2^2}, \ldots, f_{ND_2^k}$ must share demand. Any of them will have the incentive to decrease her price by ϵ and still apply, where $\epsilon > 0$ is small enough, so that she would not need to share demand with other identical sellers (because only she alone will be tested after the deviation), and her expected profit would increase, despite of a very small decrease in the markup.

    * **Case 3.2.2:** If $f_{ND_2^1}$ is the unique globally non-dominated seller with $q = 2$, then according to L2.1, other sellers with $q = 2$ must have an expected demand of 0 and thus an expected profit of 0, so any of these sellers must have an incentive to deviate to be identical to $f_{ND_2^1}$'s action so that she would have a positive demand and thus a positive expected profit.

  - **Case 3.3:** If $p_{ND_2^1} \geqslant 2\theta_H$, then $f_{ND_2^1}$ must have a 0 expected demand and thus a 0 expected profit. According to L2.1, any other seller must have a 0 expected profit too. Then any seller must have the incentive to deviate to, for example, $(2,\ c(2)+\epsilon, Apply)$, where $\epsilon < 2\theta_H - c(2)$, so that all $\theta = \theta_H$ buyers will strictly prefer her product, and then she would have a positive expected profit.

We have discussed all cases and do not find any PBE.

□

Lemma 3 (L3). *If in a strategy profile, there do not exist sellers with $q = 2$, then this strategy profile cannot be a PBE when* $\begin{cases} \alpha_4 + \beta_2 > 0 \\ \alpha_2 > 0 \\ \alpha_1 + \beta_1 > 0 \end{cases}$.

Proof. According to C1.1, C2.1 and C2.2, if there exist sellers with $q = 3$, the globally non-dominated one(s) must apply, and all other sellers (including sellers with $q = 1$, if any, and globally dominated sellers with $q = 3$, if any) will not apply. Denote (one of) the globally non-dominated seller(s) with $q = 3$ as $f_{ND_3^1}$. If there exist sellers with $q = 1$, then according to C1.1, no seller with $q = 1$ will apply. Denote the seller with the lowest price among $q = 1$ as f_l. We then prove the following lemmas when there does not exist any seller with $q = 2$:

- **Lemma 3.1 (L3.1):** Suppose that in a strategy profile, there do not exist sellers with $q = 2$, and there exist sellers with $q = 3$. Any seller with $p > p_{ND_3^1}$, denoted as f_h, must have a 0 expected demand and thus a 0 expected profit.
  - Proof: f_h must have an expected quality of $3 - 2\alpha_4 - \beta_2$ according to C4.2.3. $f_{ND_3^1}$ must be strictly preferred to f_h by all $\theta = \theta_L$ and $\theta = \theta_H$ buyers (because $3\theta_L - p_{ND_3^1} > \theta_L(3 - 2\alpha_4 - \beta_2) - p_h$ and $3\theta_H - p_{ND_3^1} > \theta_H(3 - 2\alpha_4 - \beta_2) - p_h$), so f_h must have a 0 expected demand.
- **Lemma 3.2 (L3.2):** Suppose that in a strategy profile, there do not exist sellers with $q = 2$, and there exist sellers with $q = 3$. If there exist at least two sellers with $q = 1$, then if any seller with $q = 1$, denoted as f_i, has $p_i > p_l$, then f_i must have a 0 expected profit.
  - **Case L3.2.1:** If $p_i < p_{ND_3^1}$, then there must be $p_l < p_{ND_3^1}$, and thus both f_l and f_i's expected qualities are 1 according to C4.2.3, but $p_i > p_l$, so f_l must be strictly preferred to f_i, so f_i must have a 0 expected demand.
  - **Case L3.2.2:** If $p_i \geqslant p_{ND_3^1}$, then f_i has an expected quality of $3 - 2\alpha_4 - \beta_2$ according to C4.2.3. Then when $a_4 + \beta_2 > 0$, there must be $2\alpha_4 + \beta_2 > 0$, so $f_{ND_3^1}$ must be strictly preferred to f_i by all $\theta = \theta_L$ and $\theta = theta_H$ buyers (because $3\theta_L - p_{ND_3^1} > \theta_L(3 - 2\alpha_4 - \beta_2) - p_i$ and $3\theta_H - p_{ND_3^1} > \theta_H(3 - 2\alpha_4 - \beta_2) - p_i$), so f_i must have a 0 expected demand.
- **Lemma 3.3 (L3.3):** Suppose that in a strategy profile, there do not exist sellers with $q = 2$, and there exist sellers with $q = 3$. If any seller with $q = 3$ has $p = c(3)$, then this strategy profile cannot be a PBE.
  - Proof: This seller, denoted as f_3, must have a 0 expected profit, because she has a 0 markup. Then she must have the incentive to deviate to $(2, c(2) + \epsilon, Apply)$, where ϵ is small enough, so that all $\theta = \theta_L$ buyers will strictly prefer her product (because $\begin{cases} \alpha_1 + \beta_1 > 0 \\ \alpha_2 > 0 \end{cases}$, there must be $\begin{cases} 2\theta_L\alpha_1 + \theta_L\beta_1 > 0 \\ \alpha_2 > 0 \end{cases}$, and then $\theta = \theta_L$

buyers' expect profit is at most $max\{\theta_L - c(1), 2\theta_L - c(2) - \theta_L\alpha_2, 3\theta_L - c(3) - 2\theta_L\alpha_1 - \theta_L\beta_1\} < 2\theta_L - c(2)$ from any $q = 1$ seller according to L0.1.1, and at most $3\theta_L - c(3) < 2\theta_L - c(2)$ from any $q = 3$ seller according to L0.1.4, both of which are smaller than that from the deviated f_3, which is $2\theta_L - (c(2) + \epsilon) = 2\theta_L - c(2) - \epsilon > max\{\theta_L - c(1), 2\theta_L - c(2) - \theta_L\alpha_2, 3\theta_L - c(3) - 2\theta_L\alpha_1 - \theta_L\beta_1, 3\theta_L - c(3)\}$), and then she would have a positive expected profit.

- **Lemma 3.4 (L3.4):** Suppose that in a strategy profile, there do not exist sellers with $q = 2$, and there exist sellers with $q = 3$. If any seller with $q = 1$ has $p = c(1)$, then this strategy profile cannot be a PBE.
  - Proof: This seller, denoted as f_1, must have a 0 expected profit, because she has a 0 markup. Then she must have the incentive to deviate to $(2, c(2) + \epsilon, Apply)$, where ϵ is small enough, so that all $\theta = \theta_L$ buyers will strictly prefer her product (because when $\alpha_1 + \beta_1 > 0$, there must be $2\alpha_1 + \beta_1 > 0 \Rightarrow 3\theta_L - c(3) - 2\theta_L\alpha_1 - \theta_L\beta_1 < 3\theta_L - c(3)$, and $\alpha_2 > 0$, and then $\theta = \theta_L$ buyers' expect profit is at most $max\{\theta_L - c(1), 2\theta_L - c(2) - \theta_L\alpha_2, 3\theta_L - c(3) - 2\theta_L\alpha_1 - \theta_L\beta_1\} < 2\theta_L - c(2)$ from any $q = 1$ seller according to L0.1.1, and at most $3\theta_L - c(3) < 2\theta_L - c(2)$ from any $q = 3$ seller according to L0.1.4, both of which are smaller than that from the deviated f_3, which is $2\theta_L - (c(2) + \epsilon) = 2\theta_L - c(2) - \epsilon > max\{3\theta_L - c(3), \theta_L - c(1), 3\theta_L - c(3) - 2\theta_L\alpha_4\}$), and then she would have a positive expected profit.

Then we discuss different cases.

- **Case 1:** If there exist sellers with $q = 1$ and $q = 3$.
  - **Case 1.1:** If $p_{ND_3^1} > c(3)$:
    - **Case 1.1.1:** If $f_{ND_3^1}$ has a positive expected demand and f_l, the seller with the lowest price among $q = 1$ sellers, has a 0 expected demand, then f_l must have the incentive to deviate to be identical to $f_{ND_3^1}$'s action so that she would have a positive expected demand (shared with $f_{ND_3^1}$) and thus a positive expected profit.
    - **Case 1.1.2:** If $f_{ND_3^1}$ has a 0 expected demand and f_l has a positive expected demand.
      - **Case 1.1.2.1:** If $p_l = c(1)$, then according to L3.4, it cannot be a PBE.
      - **Case 1.1.2.2:** If $p_l > c(1)$, denote the seller with the lowest price among all other $q = 3$ sellers, if any, as f_3 (and there must be $p_3 \geqslant p_{ND_3'} > c(3)$). Then $f_{ND_3^1}$ must have the incentive to deviate to $(3, c(3)+\epsilon, Apply)$, where ϵ is small enough, to guarantee a positive expected demand (because $\theta = \theta_H$ buyers' expected profit from the deviated $f_{ND_3^1}$ is $3\theta_H - (c(3) + \epsilon) = 3\theta_H - c(3) - \epsilon$. When $\alpha_4 + \beta_2 > 0$, there is $2\theta_H\alpha_4 + \theta_H\beta_2 > 0$, so this is larger than that from any $q = 1$ seller, which is at most $max\{\theta_H - c(1),\ 3\theta_H - c(3) - 2\theta_H\alpha_4 - 2\theta_H\beta_2\} < 3\theta_H - c(3)$ based on L0.4.2, and larger than that from any other $q = 3$ seller, which is at most $3\theta_H - p_3 < 3\theta_H - c(3)$).

- * **Case 1.1.3:** If $f_{ND_3^1}$ and f_l both have positive expected demand. In this case, using the same reasoning as that in Lemma 2 Case 1.1.3, it can be proved that the strategy profile cannot be a PBE.
  * **Case 1.1.4:** If $f_{ND_3^1}$ and f_l both have 0 expected demand (and thus both of them have a 0 expected profit), then according to L3.1 and L3.2, all buyers will choose not to buy from any seller (and thus all sellers have a 0 expected profit). The only possibility for this case is that for any buyer, the expected profit from buying from any seller is non-positive. Then any seller would have the incentive to deviate to, for example, $(2,\ c(2)+\epsilon, Apply)$, where $\epsilon < 2\theta_H - c\,(2)$, so that buyers with $\theta = \theta_H$ would buy from her and thus she would have a positive expected profit.
- **Case 1.2:** If $p_{ND_3^1} = c(3)$, then according to L3.3, we know that it cannot be a PBE.

- **Case 2:** All sellers have $q = 1$. In this case, according to C1.1, no seller will apply. Then according to L1, it cannot be a PBE.

- **Case 3:** All sellers have $q = 3$. In this case, according to C2.1, $f_{ND_3^1}$ must apply and any seller with a price higher than $p_{ND_3^1}$ must have an expected quality of $3 - 2\alpha_4 - \beta_2$ according to C4.2.3.
  - **Case 3.1:** If $p_{ND_3^1} = c(3)$, then according to L3.3, we know that it cannot be a PBE.
  - **Case 3.2:** If $c\,(3) < p_{ND_3^1} < 3\theta_H$, then $p_{ND_3^1}$ must have a positive expected demand and thus a positive expected profit.
    * **Case 3.2.1:** If at least one other seller with $q = 3$, denoted as $f_{ND_3^2}, \ldots, f_{ND_3^k}$ $(2 \leqslant k \leqslant s-1)$, has the same price as $f_{ND_3^1}$ (i.e., these sellers are identical to $f_{ND_3^1}$), then $f_{ND_3^1}, f_{ND_3^2}, \ldots, f_{ND_3^k}$ must share demand. Any of them will have the incentive to decrease her price by ϵ and apply, where $\epsilon > 0$ is small enough, so that she would not need to share demand with other identical sellers (because only she alone will be tested after the deviation), and her expected profit would increase, despite of a very small decrease in the markup.
    * **Case 3.2.2:** If $f_{ND_3^1}$ is the unique globally non-dominated seller with $q = 3$, then according to L3.1, other sellers with $q = 3$ must have an expected demand of 0 and thus an expected profit of 0, so any of these sellers must have an incentive to deviate to be identical to $f_{ND_3^1}$'s action so that she would have a positive demand and thus a positive expected profit.
  - **Case 3.3:** If $p_{ND_3^1} \geqslant$, then $f_{ND_3^1}$ must have a 0 expected demand and thus a 0 expected profit. According to C3.1, any other seller must have a 0 expected profit too. Then any seller must have the incentive to deviate to, for example, $(3,\ c(3)+\epsilon, Apply)$, where $\epsilon < 3\theta_H - c(3)$, so that all $\theta = \theta_H$ buyers will strictly prefer her product, and then she would have a positive expected profit.

We have discussed all cases and do not find any PBE.

□

Lemma 4 (L4). *If in a strategy profile, there exist sellers with $q = 2$ and $q = 3$, and (one of) the seller(s) with the lowest price among $q = 2$ sellers, denoted as $f_{l_2^1}$, has $p_{l_2^1} > c(2)$, then this strategy profile cannot be a PBE when* $\begin{cases} \alpha_4 + \beta_2 > 0 \\ \alpha_2 > 0 \\ \alpha_1 + \beta_1 > 0 \end{cases}$.

Proof. We first prove the following lemma when there exist sellers with $q = 2$ and $q = 3$:

- **Lemma 4.1 (L4.1):** Suppose there exist sellers with $q = 2$ and $q = 3$. Any seller with $p > p_{l_2^1}$, denoted as f_h, must have a 0 expected demand and thus a 0 expected profit.
  - Proof: Denote (one of) the globally non-dominated seller(s) with $q = 3$ as $f_{ND_3^1}$.
    * **Case L4.1.1:** If $p_{l_2^1} < p_{ND_3^1}$, then $f_{l_2^1}$ is globally non-dominated and thus must apply according to C3.1, then:
      · **Case L4.1.1.1:** If $p_{l_2^1} < p_h < p_{ND_3^1}$, then f_h must have an expected quality of $2 - \alpha_2$ according to C4.2.1, so $f_{l_2^1}$ must be strictly preferred to f_h by all $\theta = \theta_L$ and $\theta = \theta_H$ buyers (because when $a_2 > 0$, $2\theta_L - p_{l_2^1} > \theta_L(2 - \alpha_2) - p_h$ and $2\theta_H - p_{l_2^1} > \theta_H(2 - \alpha_2) - p_h$), so f_h must have a 0 expected demand.
      · **Case L4.1.1.2:** If $p_h \geqslant p_{ND_3^1}$, then f_h must have an expected quality of $3 - 2\alpha_1 - \beta_1$ according to C4.2.1, so $f_{l_2^1}$ must be strictly preferred to $f_{ND_3^1}$ by all $\theta = \theta_L$ and $\theta = \theta_H$ buyers (because when $\alpha_1 + \beta_1 > 0$, there must be $2\alpha_1 + \beta_1 > 0$, and then there is $3 - 2\alpha_1 - \beta_1 < 3$, and thus $3\theta_L - p_{ND_3^1} > \theta_L(3 - 2\alpha_1 - \beta_1) - p_h$ and $3\theta_H - p_{ND_3^1} > \theta_H(3 - 2\alpha_1 - \beta_1) - p_h$), so f_h must have a 0 expected demand.
    * **Case L4.1.2:** If $p_{l_2^1} \geqslant p_{ND_3^1}$, then $f_{l_2^1}$ is globally dominated and thus will not apply according to C3.2 (and none of other sellers with $q = 2$ will apply), then both $f_{l_2^1}$ and f_h must have an expected quality of $3 - 2\alpha_4 - \beta_2$ according to C4.2.3. When $\alpha_4 + \beta_2 > 0$, there must be $2\alpha_4 + \beta_2 > 0$, so there is $3 - 2\alpha_4 - \beta_2 < 3$, and thus $f_{ND_3^1}$ must be strictly preferred to f_h by all $\theta = \theta_L$ and $\theta = \theta_H$ buyers (because $3\theta_L - p_{l_2^1} > \theta_L(3 - 2\alpha_4 - \beta_2) - p_h$ and $3\theta_H - p_{l_2^1} > \theta_H(3 - 2\alpha_4 - \beta_2) - p_h$), so f_h must have a 0 expected demand.

Now let's discuss the following cases:

- **Case 1:** If $f_{l_2^1}$ has a positive demand, then she must have a positive expected profit (due to a positive markup). In this case, there must be $p_{l_2^1} < p_{ND_3^1}$ and $f_{l_2^1}$ is globally non-dominated and must apply according to C3.1, because otherwise she would have a 0 expected demand (if $p_{l_2^1} \geqslant p_{ND_3^1}$, then $f_{l_2^1}$ will not apply according to C3.2 and would have an expected quality of $3 - 2\alpha_4$ according to C4.2.3. When $\alpha_4 + \beta_2 > 0$, $3 - 2\alpha_4 - \beta_2 < 3$, and then $f_{ND_3^1}$ would be strictly preferred to $f_{l_2^1}$ by all $\theta = \theta_L$ and $\theta = \theta_H$ buyers, because $3\theta_L - p_{ND_3^1} > \theta_L(3 - 2\alpha_4 - \beta_2) - p_{l_2^1}$ and $3\theta_H - p_{ND_3^1} > \theta_H(3 - 2\alpha_4 - \beta_2) - p_{l_2^1}$).

- **Case 1.1:** If at least one other seller with $q = 2$, denoted as $f_{l_2^2}, \ldots, f_{l_2^k}$ $(2 \leqslant k \leqslant s-1)$, has the same price as $f_{l_2^1}$ (i.e., these sellers are identical to $f_{ND_2^1}$), then $f_{l_2^1}, f_{l_2^2}, \ldots, f_{l_2^k}$ must all apply (according to C3.1) and thus share demand. Any of them will have the incentive to decrease her price by ϵ and still apply, where $\epsilon > 0$ is small enough, so that she would not need to share demand with other identical sellers (because only she alone will be tested after the deviation), and her expected profit would increase, despite of a very small decrease in the markup.
- **Case 1.2:** If $f_{l_2^1}$ is the unique seller with the lowest price among sellers with $q = 2$, then according to L4.1, other sellers with $q = 2$ must have an expected demand of 0 and thus an expected profit of 0, so any of these sellers must have an incentive to deviate to be identical to $f_{l_2^1}$'s action so that she would have a positive demand and thus a positive expected profit.

- **Case 2:** If $f_{l_2^1}$ has a 0 expected demand, then she must have a 0 expected profit, then according to L4.1, other sellers with $q = 2$ must have an expected demand of 0 and thus an expected profit of 0 (denote the seller with the lowest price among all other $q = 2$ sellers, if any, as f_2, and there must be $p_2 \geqslant p_{l_2^1} > c(2)$). The only possibility for this case is that $p_{l_2^1}$ is too high. $f_{l_2^1}$ must have the incentive to deviate to $(2, c(2) + \epsilon, Apply)$, where ϵ is small enough, to guarantee that all $\theta = \theta_L$ buyers would strictly prefer her (because when $\alpha_1 + \beta_1 > 0$, there must be $2\alpha_1 + \beta_1 > 0 \Rightarrow -2\theta_L\alpha_1 - \theta_L\beta_1 < 0$, and then there is $2\theta_L - c\,(2) - 2\theta_L\alpha_1 - \theta_L\beta_1 < 2\theta_L - c(2)$, and since $\alpha_2 > 0$, there is $2\theta_L - c\,(2) - \alpha_2 < 2\theta_L - c(2)$. Thus, the maximum expected profit a $\theta = \theta_L$ buyer can get from a $q = 1$, according to L0.1.1, is $max\{\theta_L - c\,(1)\,, 2\theta_L - c\,(2) - \theta_L\alpha_2, 3\theta_L - c(3) - 2\theta_L\alpha_1 - \theta_L\beta_1\} < 2\theta_L - c(2)$, and the maximum expected profit a $\theta = \theta_L$ buyer can get from a $q = 3$, according to L0.1.4, is $3\theta_L - c\,(3) < 2\theta_L - c(2)$. The maximum expected profit a $\theta = \theta_L$ buyer can get from any other seller with $q = 2$ who must have a price no less than the deviated $p_{l_2^1}$ is $2\theta_L - p_2 < 2\theta_L - c(2)$. Then a $\theta = \theta_L$ buyer's expected profit from the deviated $f_{l_2^1}$ would be $2\theta_L - (c\,(2) + \epsilon) = 2\theta_L - c(2) - \epsilon > max\{\theta_L - c\,(1)\,, 2\theta_L - c\,(2) - \theta_L\alpha_2, 3\theta_L - c(3) - 2\theta_L\alpha_1 - \theta_L\beta_1, 3\theta_L - c\,(3)\,, 2\theta_L - p_2\}$).
- We have discussed all cases and do not find any PBE.

□

Lemma 5 (L5). *If in a strategy profile, there exist sellers with $q = 2$ and $q = 3$, and the globally non-dominated seller(s) with $q = 3$, denoted as $f_{ND_3^1}$, has $p_{ND_3^1} > c(3)$, then this strategy profile cannot be a PBE when* $\begin{cases} \alpha_4 + \beta_2 > 0 \\ \alpha_1 + \beta_1 > 0 \end{cases}$.

Proof. We first prove the following lemma when there exist sellers with $q = 2$ and $q = 3$:

- **Lemma 5.1 (L5.1):** Any seller with $p > p_{ND_3^1}$, denoted as f_h, must have a 0 expected demand and thus a 0 expected profit.
  - Proof: Denote (one of) the globally non-dominated seller(s) with $q = 3$ as $f_{ND_3^1}$. According to C2.1, $f_{ND_3^1}$ must apply. According to C4.2.1 and C4.2.3, any

untested seller with $p_h > p_{ND_3^1}$ must have an expected quality of $3-2\alpha_1-\beta_1$ or $3-2\alpha_4-\beta_2$. $f_{ND_3^1}$ must be strictly preferred to f_h by all $\theta=\theta_L$ and $\theta=\theta_H$ buyers (because $3\theta_L-p_{ND_3^1} > \theta_L(3-2\alpha_1-\beta_1)-p_h$ and $3\theta_H-p_{ND_3^1} > \theta_H(3-2\alpha_1-\beta_1)-p_h$ and $3\theta_L-p_{ND_3^1} > \theta_L(3-2\alpha_4-\beta_2)-p_h$ and $3\theta_H-p_{ND_3^1} > \theta_H(3-2\alpha_4-\beta_2)-p_h$), so f_h must have a 0 expected demand.

Now let's discuss the following cases:

- **Case 1:** If $f_{ND_3^1}$ has a positive expected demand, then she must have a positive expected profit (due to a positive markup).
  - **Case 1.1:** If at least one other seller with $q=3$, denoted as $f_{ND_3^2},\ldots,f_{ND_3^k}$ $(2\leqslant k\leqslant s-1)$, has the same price as $f_{ND_3^1}$ (i.e., these sellers are identical to $f_{ND_3^1}$), then $f_{ND_3^1}, f_{ND_3^2},\ldots,f_{ND_3^k}$ must all apply share demand. Any of them will have the incentive to decrease her price by ϵ, where $\epsilon>0$ is small enough, so that she would not need to share demand with other identical sellers (because only she alone will be tested after the deviation), and her expected profit would increase, despite of a very small decrease in the markup.
  - **Case 1.2:** If $f_{ND_3^1}$ is the unique globally non-dominated seller with $q=3$, then according to L5.1, other sellers with $q=3$ must have an expected demand of 0 and thus an expected profit of 0, so any of these sellers must have an incentive to deviate to be identical to $f_{ND_3^1}$'s action so that she would have a positive demand and thus a positive expected profit.
- **Case 2:** If $f_{ND_3^1}$ has a 0 expected demand, then she must have a 0 expected profit, then according to L5.1, other sellers with $q=3$ must have an expected demand of 0 and thus an expected profit of 0. The only possibility for this case is that $p_{ND_3^1}$ is too high. $f_{ND_3^1}$ must have the incentive to deviate to $(3, c(3)+\epsilon, Apply)$, where ϵ is small enough, to guarantee that all $\theta=\theta_H$ buyers would strictly or weakly prefer her (because when $\begin{cases}\alpha_4+\beta_2>0\\ \alpha_1+\beta_1>0\end{cases}$ there must be $\begin{cases}2\theta_H\alpha_4+\theta_H\beta_2>0\\ 2\theta_H\alpha_1+\theta_H\beta_1>0\end{cases}$, and then according to L0.1.2 and L0.2.2, the maximum expected profit a $\theta=\theta_H$ buyer can get from a $q=1$ seller is $max\{\theta_H-c\,(1)\,,\ 2\theta_H-c\,(2)-\theta_H\alpha_2, 3\theta_H-c(3)-2\theta_H\alpha_1-\theta_H\beta_1\} < 3\theta_H-c(3)$, and according to L0.1.3 and L0.2.3, the maximum expected profit a $\theta=\theta_H$ buyer can get from a $q=2$ seller is $max\{2\theta_H-c\,(2)\,, 3\theta_H-c(3)-2\theta_H\alpha_1-\theta_H\beta_1\} < 3\theta_H-c(3)$ or $3\theta_H-c\,(3)-2\theta_H\alpha_4-\theta_H\beta_2 < 3\theta_H-c(3)$. Then a $\theta=\theta_H$ buyer's expected profit from the deviated $f_{ND_3^1}$ would be $3\theta_H-(c\,(3)+\epsilon)=3\theta_H-c(3)-\epsilon > max\{2\theta_H-c(1), 2\theta_H-c(2)-\theta_H\alpha_2, 3\theta_H-c(3)-2\theta_H\alpha_1-\theta_H\beta_1, 3\theta_H-c(3)-2\theta_H\alpha_4-\theta_H\beta_2, 2\theta_H-c(2)\}$).
- We have discussed all cases and do not find any PBE.

□

Lemma 6 (L6). *Suppose* $\begin{cases}\alpha_1+\beta_1>0\\ \alpha_2>0\\ \alpha_3>0\\ \alpha_4+\beta_2>0\end{cases}$. *If in a strategy profile, there is only one globally* *non-dominated seller with $q = 2$ who has $p = c(2)$ (denoted as f_{ND_2}) and only one globally non-dominated seller with $q = 3$ who has $p = c(3)$ (denoted as f_{ND_3}) (i.e., all other sellers with $q = 2$, if any, have $p > c(2)$ and all other sellers with $q = 3$, if any, have $p > c(3)$), then this strategy profile cannot be a PBE.*

Proof. Let's discuss different cases.

- **Case 1:** If there exist globally dominated sellers with $q = 2$. Denote the globally dominated seller with $q = 2$ who has the second lowest price among all $q = 2$ sellers as $f_{D_2^l}$. $f_{D_2^l}$ will not apply according to C3.2. We know that f_{ND_2} must have a 0 expected profit due to her 0 markup.
  - **Case 1.1:** If $p_{ND_2} < p_{D_2^l} < p_{ND_3}$, then according to C4.2.1 $f_{D_2^l}$'s expected quality is $2 - \alpha_2$. Then f_{ND_2} must have the incentive to deviate to $(2, c(2) + \epsilon, Apply)$,

$$\text{where} \begin{cases} 2\theta_L - (c(2) + \epsilon) > \theta_L(2 - \alpha_2) - p_{D_2^l} \\ c(2) + \epsilon < p_{D_2^l} \\ 2\theta_L - (c(2) + \epsilon) > max\{\theta_L - c(1), 2\theta_L - c\,(2) - \theta_L\alpha_2, 3\theta_L - c\,(3) - 2\theta_L\alpha_1 - \theta_L\beta_1\} \\ 2\theta_L - (c(2) + \epsilon) > 3\theta_L - c(3) \end{cases} \Leftrightarrow$$

$$\begin{cases} \epsilon < p_{D_2^l} + \theta_L\alpha_2 - c(2) \\ \epsilon < p_{D_2^l} - c(2) \\ \epsilon < 2\theta_L - c(2) - max\{\theta_L - c(1), 2\theta_L - c\,(2) - \theta_L\alpha_2, 3\theta_L - c\,(3) - 2\theta_L\alpha_1 - \theta_L\beta_1\} \\ \epsilon < c(3) - c(2) - \theta_L \end{cases}$$

  (The second inequality means that the deviated f_{ND_2}'s price should still be lower than $p_{D_2^l}$ so that the deviated f_{ND_2} is still globally non-dominated. The third inequality means that a $\theta = \theta_L$ buyer's expected profit from the deviated f_{ND_2} should be higher than the maximum expected profit from a $q = 1$ seller, which is $max\{\theta_L - c\,(1)\,, 2\theta_L - c\,(2) - \theta_L\alpha_2, 3\theta_L - c\,(3) - 2\theta_L\alpha_1 - \theta_L\beta_1\} < 2\theta_L - c\,(2)$ according to L0.1.1. The fourth inequality means that a $\theta = \theta_L$ buyer's expected profit from the deviated f_{ND_2} should be higher than the maximum expected profit from a $q = 3$ seller, which is $3\theta_L - c\,(3) < 2\theta_L - c(2)$ according to L0.1.4), so that all $\theta = 4$ buyers will strictly prefer the deviated f_{ND_2}.
  - **Case 1.2:** If $p_{D_2^l} \geqslant p_{ND_3}$, then according to C4.2.1, $f_{D_2^l}$'s expected quality is $3 - 2\alpha_1 - \beta_1$. Then f_{ND_2} must have the incentive to deviate to $(2, c(2) + \epsilon, Apply)$,

$$\text{where} \begin{cases} 2\theta_L - (c(2) + \epsilon) > \theta_L(3 - 2\alpha_1 - \beta_1) - p_{D_2^l} \\ c(2) + \epsilon < p_{ND_3} = c(3) \\ 2\theta_L - (c(2) + \epsilon) > max\{\theta_L - c(1), 2\theta_L - c(2) - \theta_L\alpha_2, 3\theta_L - c(3) - 2\theta_L\alpha_1 - \theta_L\beta_1\} \\ 2\theta_L - (c(2) + \epsilon) > 3\theta_L - c(3) \end{cases} \Leftrightarrow$$

$$\begin{cases} \epsilon < p_{D_2^l} - c(2) + 2\theta_L - \theta_L(3 - 2\alpha_1 - \beta_1) \\ \epsilon < c(3) - c(2) \\ \epsilon < 2\theta_L - c(2) - max\{\theta_L - c(1), 2\theta_L - c(2) - \theta_L\alpha_2, 3\theta_L - c(3) - 2\theta_L - \theta_L\beta_1\} \\ \epsilon < c(3) - c(2) - \theta_L \end{cases}$$

  (The second inequality means that the deviated f_{ND_2}'s price should still be lower than p_{ND_3} so that the deviated f_{ND_2} is still globally non-dominated. The third inequality means that a $\theta = \theta_L$ buyer's expected profit from the deviated f_{ND_2} should be higher than the maximum expected profit from a $q = 1$ seller, which is $max\{\theta_L - c\,(1)\,, 2\theta_L - c\,(2) - \theta_L\alpha_2, 3\theta_L - c\,(3) - 2\theta_L\alpha_1 - \theta_L\beta_1\} < 2\theta_L - c(2)$ according to L0.1.1. The fourth inequality means that a $\theta = \theta_L$ buyer's expected profit from the deviated f_{ND_2} should be higher than the maximum expected profit from a $q = 3$ seller, which is $3\theta_L - c\,(3) < 2\theta_L - c(2)$ according to L0.1.4), so that all $\theta = \theta_L$ buyers will strictly prefer the deviated f_{ND_2}.

- **Case 2:** If there exist globally dominated sellers with $q = 3$. Denote the globally dominated seller with $q = 3$ who has the second lowest price among all $q = 3$ sellers as $f_{D_3^l}$. $f_{D_3^l}$ will not apply according to C2.2 and has an expected quality of $3 - 2\alpha_1 - \beta_1$ according to C4.2.1. We know that f_{ND_3} must have a 0 expected profit due to her 0 markup. Then f_{ND_3} must have the incentive to deviate to $(3, c(3) + \epsilon, Apply)$, where
$$\begin{cases} 3\theta_H - (c(3) + \epsilon) > \theta_H(3 - 2\alpha_1 - \beta_1) - p_{D_3^l} \\ c(3) + \epsilon < p_{D_3^l} \\ 3\theta_H - (c(3) + \epsilon) > max\{\theta_H - c\,(1)\,, 2\theta_H - c\,(2) - \theta_H\alpha_2, 3\theta_H - c\,(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\} \\ 3\theta_H - (c(3) + \epsilon) > max\{2\theta_H - c\,(2)\,, 3\theta_H - c\,(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\} \end{cases} \Leftrightarrow$$
$$\begin{cases} \epsilon > p_{D_3^l} - c(3) + 2\theta_H\alpha_1 + \theta_H\beta_1 \\ \epsilon < p_{D_3^l} - c(3) \\ \epsilon < 3\theta_H - c(3) - max\{\theta_H - c\,(1)\,, 2\theta_H - c\,(2) - \theta_H\alpha_2, 3\theta_H - c\,(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\} \\ \epsilon < 3\theta_H - c(3) - max\{2\theta_H - c\,(2)\,, 3\theta_H - c\,(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\} \end{cases}$$
(The second inequality means that the deviated f_{ND_3}'s price should still be lower than $p_{D_3^l}$ so that the deviated f_{ND_3} is still globally non-dominated. The third inequality means that a $\theta = \theta_H$ buyer's expected profit from the deviated f_{ND_3} should be higher than the maximum expected profit from a $q = 1$ seller, which is $\$max\{\theta_H - c\,(1)\,, 2\theta_H - c\,(2) - \theta_H\alpha_2, 3\theta_H - c\,(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\} < 3\theta_H - c(3)$ according to L0.1.2. The fourth inequality means that a $\theta = \theta_H$ buyer's expected profit from the deviated f_{ND_3} should be higher than the maximum expected profit from a $q = 2$ seller, which is $max\{2\theta_H - c\,(2)\,, 3\theta_H - c\,(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\} < 3\theta_H - c(3)$ according to L0.1.3), so that all $\theta = \theta_H$ buyers will strictly prefer the deviated f_{ND_3}.

- **Case 3:** If there does not exist any globally dominated sellers with $q = 2$ or $q = 3$ (so that all the other $n - 2$ sellers have $q = 1$). We know that:

  - **Case 3.1:** f_{ND_2} must have a 0 expected profit due to her 0 markup. Then f_{ND_2} must have the incentive to deviate to $(2, c(2) + \epsilon, Apply)$, where
  $$\begin{cases} c(2) + \epsilon < p_{ND_3} = c(3) \\ 2\theta_L - (c(2) + \epsilon) > max\{\theta_L - c(1), 2\theta_L - c(2) - 4\alpha_2, 3\theta_L - c(3) - 2\theta_L - \theta_L\beta_1\} \\ 2\theta_L - (c(2) + \epsilon) > 2\theta_L - c(3) \end{cases} \Leftrightarrow$$
  $$\begin{cases} \epsilon < c(3) - c(2) \\ \epsilon < 2\theta_L - c(2) - max\{\theta_L - c(1), 2\theta_L - c(2) - 4\alpha_2, 3\theta_L - c(3) - 2\theta_L - \theta_L\beta_1\} \\ \epsilon < c(3) - c(2) \end{cases}$$

(The first inequality means that the deviated f_{ND_2}'s price should still be lower than p_{ND_3} so that the deviated f_{ND_2} is still globally non-dominated. The second inequality means that a $\theta = \theta_L$ buyer's expected profit from the deviated f_{ND_2} should be higher than the maximum expected profit from a $q = 1$ seller, which is $max\{\theta_L - c\,(1)\,, 2\theta_L - c\,(2) - \theta_L\alpha_2, 3\theta_L - c\,(3) - 2\theta_L\alpha_1 - \theta_L\beta_1\} < 2\theta_L - c(2)$. according to L0.1.1. The third inequality means that a $\theta = \theta_L$ buyer's expected profit from the deviated f_{ND_2} should be higher than the expected profit from f_{ND_3}, which is $2\theta_L - c\,(3) = 2\theta_L - c(3)$), so that all $\theta = \theta_L$ buyers will strictly prefer the deviated f_{ND_2}.

- **Case 3.2:** f_{ND_3} must have the incentive to deviate to $(3, c(3) + \epsilon, Apply)$, where
$$\begin{cases} 3\theta_H - (c(3) + \epsilon) > max\{\theta_H - c\,(1)\,, 2\theta_H - c\,(2) - \theta_H\alpha_2, 3\theta_H - c\,(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\} \\ 3\theta_H - (c(3) + \epsilon) > 2\theta_H - c(2) \end{cases} \Leftrightarrow$$
$$\begin{cases} \epsilon < 3\theta_H - c(3) - max\{\theta_H - c\,(1)\,, 2\theta_H - c\,(2) - \theta_H\alpha_2, 3\theta_H - c\,(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\} \\ \epsilon < \theta_H - c(3) + c(2) \end{cases}$$
(The first inequality means that a $\theta = \theta_H$ buyer's expected profit from the deviated f_{ND_3} should be higher than the maximum expected profit from a $q = 1$ seller, which is $max\{\theta_H - c\,(1)\,, 2\theta_H - c\,(2) - \theta_H\alpha_2, 3\theta_H - c\,(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\} < 3\theta_H - c(3)$ according to L0.1.2. The second inequality means that a $\theta = \theta_H$ buyer's expected profit from the deviated f_{ND_3} should be higher than the expected profit from f_{ND_2}, which is $2\theta_H - c\,(2)$), so that all $\theta = \theta_H$ buyers will strictly prefer the deviated f_{ND_3}.

We have discussed all cases and do not find any PBE.

□

Lemma 7 (L7). *Suppose* $\begin{cases} \alpha_1 + \beta_1 > 0 \\ \alpha_2 > 0 \\ \alpha_3 > 0 \\ \alpha_4 + \beta_2 > 0 \end{cases}$. *If in a strategy profile, there are at least two globally non-dominated sellers with $q = 2$ who have $p = c(2)$, both of which apply, and at least two globally non-dominated sellers with $q = 3$ who have $p = c(3)$, both of which apply, then no seller or buyer has an incentive to deviate.*

Proof. In this strategy profile, the testing organization will randomly select one globally non-dominated seller with $q = 2$ and $p = c(2)$, denoted as $f_{ND_2^{K'}}$, and one globally non-dominated seller with $q = 3$ and $p = c(3)$, denoted as $f_{ND_3^{K'}}$. We now prove the following lemmas in this type of strategy profiles:

- **Lemma 7.1 (L7.1):** All buyers with $\theta = \theta_L$ must buy and only buy $f_{ND_2^{K'}}$.
  - Proof: This is because $f_{ND_2^{K'}}$ maximizes a $\theta = \theta_L$ buyer's expected profit (The expected profit from $f_{ND_2^{K'}}$ is $2\theta_L - c\,(2)$. Any other seller with $q = 2$, whose price must be no lower than $p_{ND_2^{K'}}$, will either have an expected quality of $2 - \alpha_2$ (when $p_{ND_2^{K'}} \leqslant p < p_{ND_3^{K'}}$) or $3 - 2\alpha_1 - \beta_1$ (when $p \geqslant p_{ND_3^{K'}}$) according to

C4.1, and thus the maximum expected profit from this other seller with $q = 2$ is $max\{\theta_L(2-\alpha_2) - c(2), \theta_L(3-2\alpha_1-\beta_1) - c(3)\} < 2\theta_L - c(2)$ (this inequality holds because according to (6), $c\left(3\right) - c\left(2\right) > \theta_L \Rightarrow 3\theta_L - c\left(3\right) < 2\theta_L - c(2) \Rightarrow \theta_L(3-2\alpha_1-\beta_1)-c(3) < 2\theta_L - c(2)$). According to L0.1.1, the maximum expected profit from any seller with $q = 1$ is $max\{\theta_L - c\left(1\right), 2\theta_L - c\left(2\right) - \theta_L\alpha_2, 3\theta_L - c\left(3\right) - 2\theta_L\alpha_1 - \theta_L\beta_1\} < 2\theta_L - c(2)$. According to L0.1.4, the maximum expected profit from any seller with $q = 3$ is $3\theta_L - c\left(3\right) < 2\theta_L - c(2)$).

- **Lemma 7.2 (L7.2):** All buyers with $\theta = \theta_H$ must buy and only buy $f_{ND_3^{K'}}$.
  - Proof: This is because $f_{ND_3^{K'}}$ maximizes a $\theta = \theta_H$ buyer's expected profit (The expected profit from $f_{ND_3^{K'}}$ is $3\theta_H - c\left(3\right) = 3\theta_H - c(3)$. Any other seller with $q = 3$, whose price must be no lower than $p_{ND_3^{K'}}$, will have an expected quality of $3 - 2\alpha_1 - \beta_1$ according to C4.1, and thus the maximum expected profit from this other seller with $q = 3$ is $max\{\theta_H - c\left(1\right), 2\theta_H - c\left(2\right) - \theta_H\alpha_2, 3\theta_H - c\left(3\right) - 2\theta_H\alpha_1 - \theta_H\beta_1\} < 3\theta_H - c(3)$. According to L0.1.2, the maximum expected profit from any seller with $q = 1$ is $max\{\theta_H - c\left(1\right), 2\theta_H - c\left(2\right) - \theta_H\alpha_2, 3\theta_H - c\left(3\right) - 2\theta_H\alpha_1 - \theta_H\beta_1\} < 3\theta_H - c(3)$. According to L0.1.3, the maximum expected profit from any seller with $q = 2$ is $max\{2\theta_H - c\left(2\right), 3\theta_H - c(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\} < 3\theta_H - c(3)$).

Now let's consider whether each seller has an incentive to deviate:

1. For the two (or more than two) globally non-dominated sellers with $q = 2$ and $p = c(2)$, their expected profits are both 0. Neither of them has an incentive to deviate unilaterally, because:
   - **Case 1:** If one of them deviates to $q = 2$ and $p > c(2)$, then according to C3.2, this deviated seller will not apply.
     - **Case 1.1:** If $p < f_{ND_3^{K'}} = c(3)$, then according to C4.1, she would have an expected quality of $2 - \alpha_2$, but all $\theta = \theta_L$ buyers would still strictly prefer $f_{ND_2^{K'}}$ (because $2\theta_L - c(2) > \theta_L(2-\alpha_2) - p$ where $c(2) < p < c(3)$), and all $\theta = \theta_H$ buyers would still strictly prefer $f_{ND_3^{K'}}$ (because $3\theta_H - c(3) > \theta_H(2-\alpha_2) - p$ where $c(2) < p < c(3)$ so that according to (8), $c\left(3\right) - c\left(2\right) > \theta_H \Rightarrow 3\theta_H - c\left(3\right) > 2\theta_H - c(2)$). Therefore, her expected profit after deviation is still 0.
     - **Case 1.2:** If $p \geq f_{ND_3^{K'}} = c(3)$, then according to C4.1, she would have an expected quality of $3 - 2\alpha_1 - \beta_1$, but all $\theta = \theta_L$ buyers would still strictly prefer $f_{ND_2^{K'}}$ (because $2\theta_L - c(2) > \theta_L(3-2\alpha_1-\beta_1) - p$ where $p \geqslant c(3)$ so that according to (6), $c\left(3\right) - c\left(2\right) > \theta_L \Rightarrow 2\theta_L - c\left(2\right) > 3\theta_L - c(3)$), and all $\theta = \theta_H$ buyers would still strictly prefer $f_{ND_3^{K'}}$ (because $3\theta_H - c(3) > \theta_H(3-2\alpha_1-\beta_1) - p$ where $p \geqslant c(3)$). Therefore, her expected profit after deviation would still be 0.
   - **Case 2:** If one of them deviates to $q = 1$ and $p \geqslant c(1)$, then according to L0.1.1, the maximum expected profit a buyer with $\theta = \theta_L$ can earn from a seller with $q = 1$ is $max\{\theta_L - c\left(1\right), 2\theta_L - c\left(2\right) - \theta_L\alpha_2, 3\theta_L - c\left(3\right) - 2\theta_L\alpha_1 - \theta_L\beta_1\} < 2\theta_L - c(2)$,

which is smaller than the expected profit from $f_{ND_2^{K'}}$ (which is $2\theta_L - c(2)$), and according to L0.1.2, the maximum expected profit a buyer with $\theta = \theta_H$ can earn from a seller with $q = 1$ is $max\{\theta_H - c(1), 2\theta_H - c(2) - \theta_H\alpha_2, 3\theta_H - c(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\} < 3\theta_H - c(3)$, which is smaller than the expected profit from $f_{ND_3^{K'}}$ (which is $3\theta_H - c(3)$). Therefore, her expected demand and expected profit after deviation would still be 0.

- **Case 3:** If one of them deviates to $q = 3$ and $p \geqslant c(3)$:
  - **Case 3.1:** If her deviating price is $p = c(3)$, then she would be identical to the globally non-dominated sellers with $q = 3$ and $p = c(3)$, and thus she would still have an expected profit of 0.
  - **Case 3.2:** If her deviating price $p > c(3)$, then according to **??**C2.2]C2.2, she will not apply. Then according to **??**C4.1]C4.1, she would have an expected quality of $3 - 2\alpha_1 - \beta_1$, but all $\theta = \theta_L$ buyers would still strictly prefer $f_{ND_2^{K'}}$ (because $2\theta_L - c(2) > \theta_L(3 - 2\alpha_1 - \beta_1) - p$ where $p > c(3)$ so that according to (6), $c(3) - c(2) > \theta_L \Rightarrow 2\theta_L - c(2) > 3\theta_L - c(3)$), and all $\theta = \theta_H$ buyers would still strictly prefer $f_{ND_3^{K'}}$ (because $3\theta_H - c(3) > \theta_H(3 - 2\alpha_1 - \beta_1) - p$ where $p > c(3)$). Therefore, her expected profit after deviation would still be 0.
- Therefore, none of the globally non-dominated sellers with $q = 2$ and $p = 4$ has the incentive to deviate.

2. For the globally non-dominated sellers with $q = 3$ and $p = c(3)$, their expected profits are both 0. Neither of them has an incentive to deviate unilaterally, because:

   - **Case 1:** If one of them deviates to $q = 3$ and $p > c(3)$, then according to C3.2, this deviated seller will not apply. Then according to C4.1, she would have an expected quality of $3 - 2\alpha_1 - \beta_1$, but all $\theta = \theta_L$ buyers would still strictly prefer $f_{ND_2^{K'}}$ (because $2\theta_L - c(2) > \theta_L(3 - 2\alpha_1 - \beta_1) - p$ where $p > c(3)$ so that according to (6), $c(3) - c(2) > \theta_L \Rightarrow 2\theta_L - c(2) > 3\theta_L - c(3)$), and all $\theta = \theta_H$ buyers would still strictly prefer $f_{ND_3^{K'}}$ (because $3\theta_H - c(3) > \theta_H(3 - 2\alpha_1 - \beta_1) - p$ where $p > c(3)$). Therefore, her expected profit after deviation is still 0.
   - **Case 2:** If one of them deviates to $q = 1$ and $p \geqslant c(1)$, then according to L0.1.1, the maximum expected profit a buyer with $\theta = \theta_L$ can earn from a seller with $q = 1$ is $max\{\theta_L - c(1), 2\theta_L - c(2) - \theta_L\alpha_2, 3\theta_L - c(3) - 2\theta_L\alpha_1 - \theta_L\beta_1\} < 2\theta_L - c(2)$, which is smaller than the expected profit from $f_{ND_2^{K'}}$ (which is $2\theta_L - c(2)$), and according to L0.1.2, the maximum expected profit a buyer with $\theta = \theta_H$ can earn from a seller with $q = 1$ is $max\{\theta_H - c(1), 2\theta_H - c(2) - \theta_H\alpha_2, 3\theta_H - c(3) - 2\theta_H\alpha_1 - \theta_H\beta_1\} < 3\theta_H - c(3)$, which is smaller than the expected profit from $f_{ND_3^{K'}}$ (which is $3\theta_H - c(3)$). Therefore, her expected profit after deviation would still be 0.
   - **Case 3:** If one of them deviates to $q = 2$ and $p \geqslant c(2)$:
     - **Case 3.1:** If her deviating price $p = c(2)$, then she would be identical to the two globally non-dominated sellers with $q = 2$ and $p = c(2)$, and thus she would still have an expected profit of 0.

- **Case 3.2:** If her deviating price $p > c(2)$, then according to C2.2, she will not apply.
  * **Case 3.2.1:** If $c(2) < p < f_{ND_3^{K'}} = c(3)$, Then according to C4.1, she would have an expected quality of $2 - \alpha_2$, but all $\theta = \theta_L$ buyers would still strictly prefer $f_{ND_2^{K'}}$ (because $2\theta_L - c(2) > \theta_L(2 - \alpha_2) - p$ where $c(2) < p < c(3)$), and all $\theta = \theta_H$ buyers would still strictly prefer $f_{ND_3^{K'}}$ (because $3\theta_H - c(3) > \theta_H(2 - \alpha_2) - p$ where $c\,(2) < p < c(3)$). Therefore, her expected profit after deviation would still be 0.
  * **Case 3.2.2:** If $p \geqslant f_{ND_3^{K'}} = c(3)$, Then according to C4.1, she would have an expected quality of $3 - 2\alpha_1 - \beta_1$, but all $\theta = \theta_L$ buyers would still strictly prefer $f_{ND_2^{K'}}$ (because $2\theta_L - c(2) > \theta_L(3 - 2\alpha_1 - \beta_1) - p$ where $p \geqslant c(3)$ so that according to (6), $c\,(3) - c\,(2) > \theta_L \Rightarrow 2\theta_L - c\,(2) > 3\theta_L - c(3)$), and all $\theta = \theta_H$ buyers would still strictly prefer $f_{ND_3^{K'}}$ (because $3\theta_H - c(3) > \theta_H(3 - 2\alpha_1 - \beta_1) - p$ where $p \geqslant c(3)$). Therefore, her expected profit after deviation would still be 0.
- Therefore, none of the globally non-dominated sellers with $q = 3$ and $p = c(3)$ has the incentive to deviate.

3. For any other seller who does not have $(q = 2, p = c(2))$ or $(q = 3, p = c(3))$:

   - **Case 1:** If she also has $q = 2$ and $p = c(2)$, or $q = 3$ and $p = c(3)$, then she will also have a 0 expected profit. Using the same reasoning as the first two globally non-dominated sellers with $q = 2$ and the first two globally non-dominated sellers with $q = 3$, she would not have the incentive to deviate.
   - **Case 2:** If she has $q = 1$, or $q = 2$ and $p > c(2)$, or $q = 3$ and $p > c(3)$, then according to L7.1 and L7.2, she must have an expected demand of 0 and thus an expected profit of 0. If she deviates to any bundle other than $(2, c(2), \textit{Apply}, \textit{Report } q = 2)$ and $(3, c(3), \textit{Apply}, \textit{Report } q = 3)$, she will still have a 0 expected demand according to L7.1 and L7.2, and thus still an expected profit of 0. If she deviates to $(2, c(2), \textit{Apply}, \textit{Report } q = 2)$ or $(3, c(3), \textit{Apply}, \textit{Report } q = 3)$, she will have a 0 markup and thus still have an expected profit of 0. Therefore, she would not have the incentive to deviate.

We have considered all sellers' incentives, and no seller has an incentive to deviate. □

Proposition 1. *In the SellersMayApply condition, the only pure-strategy profiles to be weak Perfect Bayesian Equilibria must have the following features:*

- γ_2 *sellers play* $(q = 2, p = c(2), \textit{Apply}, \textit{Report } q = 2)$*, with* $\gamma_2 \geqslant 2$*;*
- γ_3 *sellers play* $(q = 3, p = c(3), \textit{Apply}, \textit{Report } q = 3)$*, with* $\gamma_3 \geqslant 2$*;*
- γ_1 *sellers play* $(q = 1, p = c(2), \textit{Not Apply})$*, with* $\gamma_1 \geqslant 1$*;*
- $(n - \gamma_1 - \gamma_2 - \gamma_3)$ *sellers play* $(q = 1, p = c(3), \textit{Not Apply})$*, with* $\gamma_1 + \gamma_2 + \gamma_3 < n$*.*

- *Buyers' belief about the quality distribution of an unrevealed seller f_t given her price p_t:*

  – *If there are two revealed sellers (i.e., one with $q = 2$, denoted as $f^2_{K'}$ and the other with $q = 3$, denoted as $f^3_{K'}$), and there must be $p^3_{K'} > p^2_{K'}$:*

	$Pr(q = 1)$	$Pr(q = 2)$	$Pr(q = 3)$
if $p_t \geqslant p^3_{K'}$	$\frac{\gamma_1}{\gamma_1+\gamma_2-1}$	*0*	$1 - \frac{\gamma_1}{\gamma_1+\gamma_2-1}$
if $p^2_{K'} \leqslant p_t < p^3_{K'}$	$\frac{n-\gamma_1-\gamma_2-\gamma_3}{n-\gamma_1-\gamma_2-1}$	$1 - \frac{n-\gamma_1-\gamma_2-\gamma_3}{n-\gamma_1-\gamma_2-1}$	*0*
if $p_t < p^2_{K'}$	*1*	*0*	*0*

  – *If there is only one revealed seller, and she has $q = 2$, denoted as $p^2_{K'}$ ($\alpha_3 > 0$):*

	$Pr(q = 1)$	$Pr(q = 2)$	$Pr(q = 3)$
if $p_t \geqslant p^2_{K'}$	α_3	$1 - \alpha_3$	*0*
if $p_t < p^2_{K'}$	*1*	*0*	*0*

  – *If there is only one revealed seller, and she has $q = 3$, denoted as $p^3_{K'}$ ($\alpha_4+\beta_2 > 0$):*

	$Pr(q = 1)$	$Pr(q = 2)$	$Pr(q = 3)$
if $p_t \geqslant p^3_{K'}$	α_4	β_2	$1 - \alpha_4 - \beta_2$
if $p_t < p^3_{K'}$	*1*	*0*	*0*

  – *If there is no revealed seller, then:*

$Pr(q = 1)$	$Pr(q = 2)$	$Pr(q = 3)$
1	*0*	*0*

  where $\alpha_3 > 0$ and $\alpha_4 + \beta_2 > 0$.

- *Each buyer with $\theta = \theta_L$ will buy a product from the revealed seller with $q = 2$ and $p = c(2)$.*

- *Each buyer with $\theta = \theta_H$ will buy a product from the revealed seller with $q = 3$ and $p = c(3)$.*

Proof. According to L1, L2, L3, L4, L5, L6 and L7, we know that the only possible pure-strategy profiles in which no seller would have the incentive to deviate is that there are at least two sellers with ($q = 2, p = c(2), Apply, Report\ q = 2$) and at least two sellers with ($q = 3, p = c(3), Apply, Report\ q = 3$). In order to find Perfect Bayesian Equilibria, we need to make sure that buyers' belief about the distribution of each unrevealed seller's quality given the price is consistent with the actual quality distribution of unrevealed sellers in the equilibrium.

In L7, we require that $\alpha_1 + \beta_1 > 0$ and $\alpha_2 > 0$. This means that in the equilibrium, in addition to at least two sellers with ($q = 2, p = c(2), Apply, Report\ q = 2$) and at least two sellers with ($q = 3, p = c(3), Apply, Report\ q = 3$), there must exist at least one seller with ($q = 1, p = c(2), Not\ Apply$) and at least one seller with ($q = 1, p = c(3), Not\ Apply$), and there cannot exist any other seller with other strategy. Therefore, if there are γ_1 seller(s) with ($q = 1, p = c(2), Not\ Apply$) and ($n - \gamma_1 - \gamma_2 - \gamma_3$) sellers with ($q = 1, p = c(3), Not\ Apply$), then buyers' belief about the quality distribution of an unrevealed seller f_t give her price p_t when there are two revealed sellers (i.e., one with $q = 2$ and the other with $q = 3$) must satisfy:

	$Pr(q=1)$	$Pr(q=2)$	$Pr(q=3)$
if $p_t \geqslant p^3_{K'}$	$\frac{\gamma_1}{\gamma_1+\gamma_2-1}$	0	$1-\frac{\gamma_1}{\gamma_1+\gamma_2-1}$
if $p^2_{K'} \leqslant p_t < p^3_{K'}$	$\frac{n-\gamma_1-\gamma_2-\gamma_3}{n-\gamma_1-\gamma_2-1}$	$1-\frac{n-\gamma_1-\gamma_2-\gamma_3}{n-\gamma_1-\gamma_2-1}$	0
if $p_t < p^2_{K'}$	1	0	0

Buyers can form any arbitrary belief for all off-path situations. Therefore, the only requirements we need in the situation in which there is only one revealed seller with $q = 2$ and in the situation in which there is only one revealed seller with $q = 3$ are that $\alpha_3 > 0$ and $\alpha_4 + \beta_2 > 0$.

From L7.1 and L7.2, we know that each $\theta = \theta_L$ buyer must strictly prefer the product from the revealed seller with $(q = 2, p = c(2))$, and each $\theta = \theta_H$ buyer must strictly prefer the product from the revealed seller with $(q = 3, p = c(3))$. □

Proposition 2. *With the* RandomTesting *mechanism, there does not exist any weak Perfect Bayesian Equilibrium, if any, that can yield the same buyer surplus as the* Sellers-MayApply *mechanism does.*

Proof. Suppose there existed such a strategy profile which is a weak PBE. The only possibility for a strategy profile to yield the same buyer surplus as the SellersMayApply mechanism does would be that all $\theta = \theta_L$ buyers strictly prefer any seller with $(q = 2, p = c(2))$ in all testing scenarios, while all $\theta = \theta_H$ buyers strictly prefer any seller with $(q = 3, p = c(3))$. On the other hand, since 2 out of n sellers are tested in each testing scenario, there does not exist any strategy profile in which there are always one seller with $(q = 2, p = c(2))$ and one seller with $(q = 3, p = c(3))$ being tested in all testing scenarios. Therefore, there must exist at least one seller with $(q = 2, p = c(2))$ or $(q = 3, p = c(3))$, denoted (one of) them as f_0, who has a positive demand in at least one testing scenario in which she is not tested. Since these sellers have a zero markup, they must all have a zero expected profit.

However, since any f_0 with $(q = 2, p = c(2))$ have a positive demand when not being tested, they must have the incentive to deviate to $(q = 1, p = c(2))$, because after deviation they would have a positive markup and would still have the same demand in all testing scenarios in which they are not tested (because only changing the price will not change the outcome buyers can see in these testing scenarios), which would result in a positive expected profit. For the same reason, any f_0 with $(q = 3, p = c(3))$ must have the incentive to deviate to $(q = 1, p = c(3))$ or $(q = 2, p = c(3))$.

□