Week 3: Worksheet 2

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Calculus 1, Class 6

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Summary

Quiz Reminder

Time:	09/24 17:30-18:20
Quiz Scope:	2.1-3.2, Worksheet 1-2
TA Hour:	Fri. 14:00–16:00 or by appointment
	化學系館（積學館）B363

Definition of Limits

Definition 3.1

Let $f : \mathbb{R} \to \mathbb{R}$ and let $a, L \in \mathbb{R}$. We say that

$$\lim_{x \to a} f(x) = L$$

if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

Exercise 1

Consider

$$f(x) = \begin{cases} \dfrac{2x^2 - x - 1}{x - 1}, & x \neq 1, \\ 0, & x = 1. \end{cases}$$

a. Simplify $f(x)$ for $x \neq 1$. $\quad f(x) = ?$
b. From (a), the limit
$$L = \lim_{x \to 1} f(x) = ?$$
c. Starting from the goal inequality $|f(x) - L| < \varepsilon$ for various $\varepsilon > 0$, derive an *equivalent inequality* for $|x - 1|$. Hence, find a suitable δ in each case.
d. What happens at $x = 1$? Obviously, $|f(1) - L| = |0 - L| > 0.1$. Does this violate the statement $\lim\limits_{x \to 1} f(x) = L$?

Excercise 1(a)(b): Solutions

(a) Simplify the numerator:

$$2x^2 - x - 1 = (2x+1)(x-1).$$

Thus, for $x \neq 1$,

$$f(x) = \frac{2x^2 - x - 1}{x - 1} = \frac{(2x+1)(x-1)}{x-1} = 2x + 1.$$

$$\boxed{f(x) = 2x + 1 \quad (x \neq 1)}$$

(b) From part (a),

$$L = \lim_{x \to 1} f(x) = \lim_{x \to 1} (2x + 1) = 2(1) + 1 = 3.$$

$$\boxed{L = 3}$$

Excercise 1(c)

By the definition of limits, we are claiming that:

We can make $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to 1 but not equal to 1.

To justify the above claim, we need to show that, for each (small) positive number ε, there is a positive number δ (which may depend on ε) such that $0 < |x - 1| < \delta$ **implies** $|f(x) - L| < \varepsilon$.

Exercise 1(c): Solutions

1. ($\varepsilon = 0.1$)

   $$|f(x)-L| < 0.1 \iff |2(x+1)-3| < 0.1 \iff |x-1| < 0.05.$$

   Therefore, we can pick $\delta = 0.05$, which satisfies

   $$\boxed{0 < |x-1| < 0.05 \implies |f(x)-L| < 0.1}$$

2. ($\varepsilon = 0.01$)

   $$|f(x)-L| < 0.01 \iff 2|x-1| < 0.01 \iff |x-1| < 0.005.$$

   Therefore, we can pick $\delta = 0.005$, which satisfies

   $$\boxed{0 < |x-1| < 0.005 \implies |f(x)-L| < 0.01}$$

Exercise 1(c): Solutions

iii. (General case)

$$|f(x) - L| < \varepsilon \quad \Longleftrightarrow \quad 2|x-1| < \varepsilon \quad \Longleftrightarrow \quad |x-1| < \frac{\varepsilon}{2}.$$

Therefore, we can pick $\delta = \frac{\varepsilon}{2}$, which satisfies

$$\boxed{0 < |x-1| < \frac{\varepsilon}{2} \quad \Longrightarrow \quad |f(x) - L| < \varepsilon}$$

Excercise 1(d): Solutions

- (d) What happens at $x = 1$? Obviously, $|f(1) - L| = |0 - L| > 0.1$. Does this violate the statement $\lim_{x \to 1} f(x) = L$?

No. Recall the definition of limit,

$$\ldots \text{ such that } 0 < |x - a| < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

That is, we don't care about the case $|x - 1| = 0$ in the definition of limit. Thus, it does not violate the statement $\lim_{x \to 1} f(x) = L$.

Exercise 2

Our goal is to show, by using the precise definition, that

$$\lim_{x \to 0} \sqrt[3]{x} = 0.$$

a. Write down the precise definition of $\lim_{x \to 0} \sqrt[3]{x} = 0$.
b. We now attempt to verify the precise definition.
   i. Starting from the 'goal inequality' $|\sqrt[3]{x} - 0| < \varepsilon$, derive an equivalent inequality for $|x - 0|$.
   ii. Hence, for each $\varepsilon > 0$, find δ and complete the proof of $\lim_{x \to 0} \sqrt[3]{x} = 0$.

Exercise 2(a): Solutions

By the definition of limit:
For all $\varepsilon > 0$, there is $\delta > 0$ such that

$$0 < |x - 0| < \delta \quad \text{implies} \quad |\sqrt[3]{x} - 0| < \varepsilon.$$

$$\boxed{\text{This is the precise definition of } \lim_{x\to 0} \sqrt[3]{x} = 0.}$$

Exercise 2(b): Solutions

(i) Start from the goal inequality:

$$|\sqrt[3]{x} - 0| < \varepsilon.$$

This is equivalent to

$$|\sqrt[3]{x}| < \varepsilon.$$

Cubing both sides (since cube is monotone increasing):

$$|x| < \varepsilon^3.$$

(ii) From part (i), the condition becomes:

$$|x| < \varepsilon^3.$$

Therefore, for each $\varepsilon > 0$, we can choose

$$\delta = \varepsilon^3.$$

Then, whenever $0 < |x| < \delta$, we have

$$|\sqrt[3]{x} - 0| < \varepsilon.$$

Exercise 3

By imitating the precise definition of a limit, write down precise definitions of one-sided limits.

1. $\lim\limits_{x \to a^+} f(x) = L$
2. $\lim\limits_{x \to a^-} f(x) = L$

Exercise 3: Solutions

1. $\lim_{x \to a^+} f(x) = L$ means: For every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$0 < x - a < \delta \;\Rightarrow\; |f(x) - L| < \varepsilon.$$
2. $\lim_{x \to a^-} f(x) = L$ means: For every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$0 < a - x < \delta \;\Rightarrow\; |f(x) - L| < \varepsilon.$$

Definition of Infinite Limits

Definition 3.2

Let $f : \mathbb{R} \to \mathbb{R}$ and let $a \in \mathbb{R}$. We say that

$$\lim_{x \to a} f(x) = +\infty$$

if, for every $M > 0$, there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Longrightarrow \quad f(x) > M.$$

Similarly,

$$\lim_{x \to a} f(x) = -\infty$$

if, for every $N < 0$, there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Longrightarrow \quad f(x) < N.$$

Exercise 4

Imitating the above definition, write down precise definitions of other limits regarding infinity:

1. $\lim_{x \to a^-} f(x) = -\infty$
2. $\lim_{x \to \infty} f(x) = L$
3. $\lim_{x \to -\infty} f(x) = \infty$

Exercise 4: Solutions

1. $\lim_{x \to a^-} f(x) = -\infty$ means: For every $N < 0$, there exists a $\delta > 0$ such that
$$0 < a - x < \delta \implies f(x) < N.$$
2. $\lim_{x \to \infty} f(x) = L$ means: For every $\varepsilon > 0$, there exists $M > 0$ such that
$$x > M \implies |f(x) - L| < \varepsilon.$$
3. $\lim_{x \to -\infty} f(x) = \infty$ means: For every $K > 0$, there exists $M < 0$ such that
$$x < M \implies f(x) > K.$$

Exercise 5

Our goal is to show, by using the precise definition, that

$$\lim_{x \to 0^+} \ln x = -\infty.$$

a. Write down the precise definition of ' $\lim_{x \to 0^+} \ln x = -\infty$'.
b. We now attempt to verify the precise definition.
   i. Firstly, starting from the 'goal inequality' $\ln x < N$, derive an inequality for $x - 0$.
   ii. For a given $N < 0$, find δ and complete the proof of ' $\lim_{x \to 0^+} \ln x = -\infty$' by using the precise definition.

Exercise 5: Solutions

(a) Precise definition: $\lim_{x\to 0^+} \ln x = -\infty$ means: For every $N < 0$, there exists $\delta > 0$ such that

$$0 < x < \delta \implies \ln x < N.$$

(b) Verification:

(i) Starting from $\ln x < N$, we exponentiate to obtain

$$x < e^N.$$

So the condition becomes $0 < x < e^N$.

(ii) For a given $N < 0$, set $\delta = e^N > 0$. Then for all x with $0 < x < \delta$, we have

$$\ln x < N.$$

Hence, by the precise definition,

$$\boxed{\lim_{x\to 0^+} \ln x = -\infty.}$$

Exercise 6

Consider a function f which is defined for $x > 0$.

a. Prove that if $f(x) > 0$ for all $x > 0$ and the limit $\lim_{x \to \infty} f(x) = L$ exists, then $L \geq 0$. (Hint. Argue by contradiction.)
b. Deduce that if $f(x) > g(x)$ for all $x > 0$ and the limits $\lim_{x \to \infty} f(x) = L$ and $\lim_{x \to \infty} g(x) = M$ exist, then $L \geq M$.

Exercise 6: Solutions

(a) Suppose $L < 0$. Since $\lim_{x\to\infty} f(x) = L$, there exists $M > 0$ such that

$$x > M \implies |f(x) - L| < \varepsilon = -L.$$

Then, we obtain

$$|f(x) - L| < -L \implies f(x) < 0.$$

This contradicts to $f(x) > 0$ for all $x > 0$. Thus, $L \geq 0$

(b) Let $h(x) = f(x) - g(x)$. Since $f(x) > g(x)$ for all $x > 0$, $h(x) > 0$ for all $x > 0$. Also,

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} [f(x) - g(x)] = L - M \quad \text{exists (by 7(a))}$$

By 6(a), $L - M \geq 0 \implies L \geq M$

Exercise 7

Consider a function $f(x)$ which is defined around $x = a$. Prove the following statements.

a. Prove that if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ then $\lim_{x \to a} [f(x) + g(x)] = L + M$.
b. Prove that if $\lim_{x \to a} f(x) = 0$ and $|g(x)| < M$ for all x where $M > 0$ is a constant then $\lim_{x \to a} f(x)g(x) = 0$
c. Prove that if $\lim_{x \to a} f(x) = L > 0$ and $\lim_{x \to a} g(x) = \infty$ then $\lim_{x \to a} f(x)g(x) = \infty$

Exercise 7(a): Solutions

(a) Since $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, for every $\varepsilon > 0$ there exist $\delta_1, \delta_2 > 0$ such that

$$|f(x) - L| < \tfrac{\varepsilon}{2}, \quad |g(x) - M| < \tfrac{\varepsilon}{2}$$

whenever $0 < |x - a| < \delta = \min(\delta_1, \delta_2)$. Then

$$|(f(x) + g(x)) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \varepsilon.$$

Hence, $\lim_{x\to a}[f(x) + g(x)] = L + M$.

Exercise 7(b): Solutions

- (b) Given $\lim_{x \to a} f(x) = 0$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x)| < \frac{\varepsilon}{M} \quad \text{whenever } 0 < |x - a| < \delta.$$

Since $|g(x)| < M$, we have

$$|f(x)g(x)| \leq |f(x)| \cdot |g(x)| < \tfrac{\varepsilon}{M} \cdot M = \varepsilon.$$

Thus $\lim_{x \to a} f(x)g(x) = 0$.

Exercise 7(c): Solutions

- (c) Suppose $\lim_{x \to a} f(x) = L > 0$.

  Then for $\varepsilon = L/2 > 0$, there exists $\delta_1 > 0$ such that $|f(x) - L| < L/2 \Rightarrow f(x) > \frac{L}{2}$ when $0 < |x - a| < \delta_1$.

  Since $\lim_{x \to a} g(x) = \infty$, for any $N > 0$, there exists $\delta_2 > 0$ such that $g(x) > \frac{2N}{L}$ whenever $0 < |x - a| < \delta_2$.

  Thus, for $\delta = \min(\delta_1, \delta_2)$ we have

$$f(x)g(x) > \frac{L}{2} \cdot \frac{2N}{L} = N.$$

  Since N is arbitrary, $\lim_{x \to a} f(x)g(x) = \infty$.