Week 11: Integration by Parts

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Calculus 1, Class 6

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Summary

Quiz Reminder

Time:	11/19 (Wed) 17:30-18:20
Quiz Scope:	5.1 - 6.1
Location:	普通103
TA Hour:	Fri. 14-16 化學系館（積學館）B363

Today's Goals

By the end, you should be able to:

- Recognize when to use IBP vs. substitution/other methods.
- Apply one-shot IBP, repeated/tabular IBP, and cyclic IBP (solve for I).
- Derive and use classic identities obtained via IBP.
- Avoid common pitfalls (bad u, sign errors, boundary terms).

Total derivative viewpoint: why "$\times dx$" is OK

Total derivative (1D). For differentiable $f : \mathbb{R} \to \mathbb{R}$,

$$Df(x)[h] = f'(x)\,h \quad \Rightarrow \quad df := Df(x)[dx] = f'(x)\,dx.$$

Leibniz rule. $D(uv) = u\,Dv + v\,Du$. Evaluating at dx gives

$$\boxed{d(uv) = u\,dv + v\,du} \quad \Rightarrow \quad u\,dv = d(uv) - v\,du.$$

Interpret "$\times dx$" as: *evaluate the linear map at the increment* dx, not as cancelling a denominator.

Understanding $Df(x)[h] = f'(x)\,h$

Total derivative: $Df(x)$ is the linear map that gives the change in f for a small input change h:

$$Df(x)[h] = f'(x)\,h.$$

In words: "if x changes by h, f changes by about $f'(x)\,h$."
Setting the input change to the infinitesimal dx gives the differential form:

$$\boxed{df := Df(x)[dx] = f'(x)\,dx.}$$

Thus "$\times\, dx$" means we *apply the total derivative to the increment* dx, not that we cancel a denominator.

When "$\times dx$" is legitimate ($\checkmark$)

- **Convert derivatives to differentials (same variable):** $\frac{d}{dx}(uv) = u\,v' + v\,u' \iff d(uv) = u\,dv + v\,du.$
- **Chain rule (substitution):** $\frac{d}{dx}F(g(x)) = F'(g)g' \iff dF = F'(g)\,dg,\ dg = g'(x)\,dx.$
- **Separation of variables (ODE):** If $\frac{dy}{dx} = F(x)G(y)$, then $\frac{1}{G(y)}\,dy = F(x)\,dx$ (then integrate).
- **Multivariable total differential:** $df = f_x\,dx + f_y\,dy$; along $y = y(x)$, $dy = y'(x)\,dx$.

When not to "$\times dx$" ($\times$)

1. **Not a fraction — $\frac{d}{dx}$ is an operator.**
   - You can multiply by dx *after* differentiation, not before.
   - Wrong: $(d/dx)(fg)\,dx = df \cdot dg$ (this skips the product rule). Correct: $d(fg) = f\,dg + g\,df$.
2. **Don't ignore other variables.**
   - In several variables, $df = f_x\,dx + f_y\,dy + \cdots$.
   - Writing $df = f_x\,dx$ is only valid if $dy = dz = \cdots = 0$ (holding others fixed).
   - Example: $f(x, y) = xy \Rightarrow df = y\,dx + x\,dy$.

IBP: Derivation and The Rule

From the product rule,

$$\frac{d}{dx}(uv) = u\,v' + v\,u' \quad \Longrightarrow \quad u\,dv = d(uv) - v\,du.$$

Integrating both sides,

$$\boxed{\int u\,dv = uv - \int v\,du}.$$

Choice of u and dv: pick u so that du simplifies; pick dv so that v is easy to find.

Choosing u: Heuristics and When *Not* to Use IBP

Heuristics (LIATE/ILATE): Logarithmic $\to$ Inverse trig $\to$ Algebraic $\to$ Trig $\to$ Exponential.

- Use as a *tiebreaker*, not a law. Ask: does differentiating u *simplify*? does integrating dv stay easy?

Often better than IBP:

- Pure rational functions $\Rightarrow$ partial fractions.
- Obvious chain rule $\Rightarrow$ substitution.
- Trig powers $\Rightarrow$ trig identities first.

Cyclic case: for $\int e^x \sin x \, dx$ or $\int e^x \cos x \, dx$, use IBP twice and solve for I.

Warm-ups

Compute using one-shot IBP:

$$\int xe^x\,dx = e^x(x-1) + C,$$

$$\int x\sin x\,dx = -x\cos x + \sin x + C,$$

$$\int \ln x\,dx = x\ln x - x + C,$$

$$\int \arctan x\,dx = x\arctan x - \tfrac{1}{2}\ln(1+x^2) + C.$$

Emphasize the choice $u = \ln x, \arctan x$ (they simplify upon differentiation).

Repeated (Tabular) IBP: When and How

When: polynomial $\times$ (exp or sin/cos) — the polynomial "dies" under differentiation.

D–I table idea:

Differentiate	Integrate
x^n	e^{ax}, $\sin bx$, $\cos bx$

Alternate signs; stop when the derivative hits 0.

Example (Tabular): $\int x^3 e^x \, dx$

$$\int x^3 e^x \, dx = e^x \big(x^3 - 3x^2 + 6x - 6 \big) + C.$$

(Derivative chain of x^3: x^3, $3x^2$, $6x$, 6, 0; integrate e^x repeatedly; alternate signs.)

Cyclic IBP: Solve for the Integral

Let $I = \int e^x \cos x \, dx$. One IBP gives an integral with $\sin x$; doing IBP again returns to I. Collect terms:

$$2I = e^x\big(\sin x + \cos x\big) \quad \Rightarrow \quad \boxed{I = \tfrac{e^x}{2}\big(\sin x + \cos x\big) + C}.$$

Same idea works for $\int e^x \sin x \, dx$.

Cyclic IBP: Detailed Solution for $\int e^x \cos x \, dx$

Let

$$I = \int e^x \cos x \, dx.$$

First IBP. Choose $u = e^x$, $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$. Hence

$$I = \int u \, dv = uv - \int v \, du = e^x \sin x - \int e^x \sin x \, dx.$$

Set

$$J = \int e^x \sin x \, dx,$$

so the first step gives $I = e^x \sin x - J$.

Second IBP (for J). Again take $u = e^x$, $dv = \sin x \, dx$. Then $du = e^x \, dx$, $v = -\cos x$. Thus

$$J = \int u \, dv = uv - \int v \, du = -e^x \cos x + \int e^x \cos x \, dx = -e^x \cos x + I.$$

Cyclic IBP: Detailed Solution for $\int e^x \cos x \, dx$

Close the cycle and solve for I.

$$I = e^x \sin x - J = e^x \sin x - (-e^x \cos x + I) = e^x(\sin x + \cos x) - I.$$

Therefore

$$2I = e^x(\sin x + \cos x) \quad \Longrightarrow \quad \boxed{I = \frac{e^x}{2}\big(\sin x + \cos x\big) + C}.$$

Classics via IBP

(A) Logarithm:

$$\int \ln x \, dx = x \ln x - x + C.$$

(B) Inverse tangent:

$$\int \arctan x \, dx = x \arctan x - \tfrac{1}{2} \ln(1 + x^2) + C.$$

(C) General log moment:

$$\boxed{\int x^n \ln x \, dx = \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + C}.$$

Derivation: $\int x^n \ln x \, dx \quad (\text{for } n \neq -1,\ x > 0)$

Use integration by parts with

$$u = \ln x \quad \Rightarrow \quad du = \frac{1}{x}\,dx, \qquad dv = x^n\,dx \quad \Rightarrow \quad v = \frac{x^{n+1}}{n+1} \ \ (n \neq -1).$$

Then

$$\int x^n \ln x\,dx = uv - \int v\,du = \frac{x^{n+1}}{n+1}\ln x \ - \ \int \frac{x^{n+1}}{n+1} \cdot \frac{1}{x}\,dx.$$

Simplify the remaining integral:

$$\int \frac{x^{n+1}}{n+1} \cdot \frac{1}{x}\,dx = \frac{1}{n+1}\int x^n\,dx = \frac{1}{n+1} \cdot \frac{x^{n+1}}{n+1}.$$

Therefore,

$$\int x^n \ln x\,dx = \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C = \boxed{\frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + C}.$$

Special case $n = -1$: $\displaystyle\int \frac{\ln x}{x}\,dx = \tfrac{1}{2}(\ln x)^2 + C.$

Exponential–Trig (Laplace-type) Integrals ($a > 0$)

Two IBPs (or integrate $e^{-ax}\cos bx$ directly) yield

$$\boxed{\int_0^\infty e^{-ax}\cos(bx)\,dx = \frac{a}{a^2+b^2}, \qquad \int_0^\infty e^{-ax}\sin(bx)\,dx = \frac{b}{a^2+b^2}}.$$

Always justify boundary terms: $e^{-ax}\cos(bx) \to 0$ as $x \to \infty$ when $a > 0$.

Derivation by Cyclic IBP ($a > 0$)

Define

$$C := \int_0^\infty e^{-ax} \cos(bx)\, dx, \qquad S := \int_0^\infty e^{-ax} \sin(bx)\, dx, \qquad (a > 0,\ b \in \mathbb{R}).$$

Step 1: IBP for C. Let $u = e^{-ax}$, $dv = \cos(bx)\, dx$. Then $du = -ae^{-ax}\, dx$, $v = \frac{1}{b} \sin(bx)$.

$$C = \left[e^{-ax} \cdot \tfrac{1}{b} \sin(bx)\right]_0^\infty - \int_0^\infty \tfrac{1}{b} \sin(bx) \left(-ae^{-ax}\right) dx = \frac{a}{b}\, S.$$

(Boundary term vanishes since $e^{-ax} \to 0$ as $x \to \infty$ and $\sin(0) = 0$.)

Derivation by Cyclic IBP ($a > 0$)

Step 2: IBP for S. Let $u = e^{-ax}$, $dv = \sin(bx)\,dx$. Then $du = -ae^{-ax}\,dx$, $v = -\frac{1}{b}\cos(bx)$.

$$S = \left[e^{-ax} \cdot \left(-\tfrac{1}{b}\cos(bx)\right)\right]_0^\infty - \int_0^\infty \left(-\tfrac{1}{b}\cos(bx)\right)\left(-ae^{-ax}\right)dx$$
$$= \frac{1}{b} - \frac{a}{b}\,C.$$

(Boundary term: at ∞ it is 0; at 0 it is $-\frac{1}{b}\cos 0 = -\frac{1}{b}$; subtracting yields $+\frac{1}{b}$.)

Derivation by Cyclic IBP ($a > 0$)

Step 3: Solve the linear system. From Step 1, $C = \frac{a}{b}S$. Substitute into Step 2:

$$S = \frac{1}{b} - \frac{a}{b} \cdot \frac{a}{b} S \quad \Longrightarrow \quad S\left(1 + \frac{a^2}{b^2}\right) = \frac{1}{b} \quad \Longrightarrow \quad \boxed{S = \frac{b}{a^2 + b^2}}.$$

Then

$$\boxed{C = \frac{a}{b}\,S = \frac{a}{a^2 + b^2}}.$$

Checks. If $b = 0$: $C = \int_0^\infty e^{-ax}\,dx = \frac{1}{a}$ and $S = 0$. The formulas give $\frac{a}{a^2+0} = \frac{1}{a}$ and $\frac{0}{a^2+0} = 0$, consistent.
Boundary justification: For $a > 0$, e^{-ax} dominates any bounded trig factor, so $e^{-ax}\cos(bx) \to 0$, $e^{-ax}\sin(bx) \to 0$ as $x \to \infty$.

Gamma Function: Explicit Definition

Definition (Euler integral of the second kind). For $\Re(s) > 0$,

$$\boxed{\Gamma(s) \;:=\; \int_0^\infty x^{\,s-1}\, e^{-x}\, dx}$$

This integral converges because x^{s-1} is integrable near 0 when $\Re(s) > 0$, and e^{-x} decays exponentially as $x \to \infty$.

Basic values.

$$\Gamma(1) = \int_0^\infty e^{-x}\, dx = 1, \qquad \Gamma\!\left(\tfrac{1}{2}\right) = \int_0^\infty x^{-1/2} e^{-x}\, dx = \sqrt{\pi}.$$

Connection to Factorials and Your I_n

For $n \in \mathbb{N} \cup \{0\}$, set $I_n = \int_0^\infty x^n e^{-x}\, dx$ (from the previous slide). Then with $s = n+1$,

$$I_n = \int_0^\infty x^{(n+1)-1} e^{-x}\, dx = \Gamma(n+1).$$

IBP recursion: For $n \geq 1$,

$$I_n = n\, I_{n-1} \quad \Longleftrightarrow \quad \Gamma(n+1) = n\, \Gamma(n).$$

Since $\Gamma(1) = 1$, induction yields the **factorial identity**

$$\boxed{\Gamma(n+1) = n! \quad (n = 0, 1, 2, \dots)}.$$

(Optional) Extension Beyond Integers

For $s > 0$ define $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\,dx$. The same IBP yields the functional equation

$$\Gamma(s+1) = s\,\Gamma(s), \qquad \Gamma(1) = 1.$$

Thus $\Gamma(n+1) = n!$ for integers $n \geq 0$, and Γ extends the factorial to nonintegers (e.g. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$).

(Optional) Quick Table of Explicit Values

$$\Gamma(1) = 1, \quad \Gamma(2) = 1!, \quad \Gamma(3) = 2!, \quad \Gamma(4) = 3!, \ \ldots$$

$$\Gamma\Big(\tfrac{1}{2}\Big) = \sqrt{\pi}, \quad \Gamma\Big(\tfrac{3}{2}\Big) = \tfrac{1}{2}\sqrt{\pi}, \quad \Gamma\Big(\tfrac{5}{2}\Big) = \tfrac{3}{4}\sqrt{\pi}, \quad \Gamma\Big(\tfrac{7}{2}\Big) = \tfrac{15}{8}\sqrt{\pi}.$$

Pattern for half-integers: for $k \in \mathbb{N}$,

$$\Gamma\Big(k + \tfrac{1}{2}\Big) = \frac{(2k-1)!!}{2^{\,k}}\,\sqrt{\pi} = \frac{(2k)!}{4^{\,k}\,k!}\,\sqrt{\pi}.$$

Common Pitfalls

- Choosing u that *does not* simplify upon differentiation.
- Forgetting the minus: $\int u\,dv = uv - \int v\,du$.
- Swapping u and dv mid-calculation.
- In cyclic IBP, forgetting to collect terms and solve for the original I.

Identity Toolbox (Handy Results)

- $\displaystyle\int \ln x \, dx = x \ln x - x + C$
- $\displaystyle\int \arctan x \, dx = x \arctan x - \tfrac{1}{2} \ln(1 + x^2) + C$
- $\displaystyle\int x^n \ln x \, dx = \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + C$
- $\displaystyle\int_0^1 x^m \ln x \, dx = -\frac{1}{(m+1)^2} \; (m > -1)$
- $\displaystyle I_n = \int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} I_{n-2}$
- $\displaystyle\int_0^\infty e^{-ax} \cos(bx) \, dx = \frac{a}{a^2 + b^2}, \int_0^\infty e^{-ax} \sin(bx) \, dx = \frac{b}{a^2 + b^2} \; (a > 0)$
- $\displaystyle\int_0^\infty x^n e^{-x} \, dx = n! \quad \text{(Gamma at integers)}$