Week 2: Limit

Lecturer: Zhi (George) Lin

Calculus 1, Class 6

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Summary

TA Information

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TA Hour:	Fri. 14:00–16:00 or by appointment 化學系館（積學館）B363
Research Interest:	Machine Learning, Stochastic Model, Image Recognition

How TA class works?

Class:	06 餘 0 班
Time:	Every Wed. 17:30–18:20
Location:	普通 204
What will be covered?	Worksheets
	Supplementary materials
	Quiz/Exam review

Problem 1: Limit Tricks

- (a) Evaluate $\lim_{x\to 0}\frac{\sin x}{x}$.
- (b) Evaluate $\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x}$.
- (c) Evaluate $\lim_{x\to 0^+}(e^x-1)^{\frac{1}{\ln x}}$.
- (d) Evaluate $\lim_{x\to 0}(1+|\sin x|)^{\frac{1}{x}}$.
- (e) Evaluate $\lim_{x\to 0}(\frac{2^x+3^x+5^x}{3})^{\frac{1}{x}}$.

Problem 1(a): Solutions

(a) Evaluate $\lim_{x \to 0} \frac{\sin x}{x}$.

Method: Squeeze Theorem. For $x > 0$, from geometry we have

$$\cos x \leq \frac{\sin x}{x} \leq 1.$$

By symmetry, the same inequalities hold for $x < 0$.

Taking the limit as $x \to 0$, both bounding functions converge to 1:

$$\lim_{x \to 0} \cos x = 1, \qquad \lim_{x \to 0} 1 = 1.$$

Therefore, by the squeeze theorem:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Problem 1(b): Solutions

(b) Evaluate $\lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x}$.

Since $|\sin \frac{1}{x}| \leq 1$,

$$\lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x} \leq |\frac{x^2}{\sin x}|$$

Hence,

$$\lim_{x\to 0} |\frac{x^2}{\sin x}| = \lim_{x\to 0} |\frac{x}{\frac{\sin x}{x}}| = \frac{0}{1} = 0$$

By Squeeze Theorem,

$$\lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x} = 0.$$

Problem 1(c): Solutions

(c) Evaluate $\lim_{x\to 0^+}(e^x-1)^{\frac{1}{\ln x}}$. Set $y=(e^x-1)^{1/\ln x}$. Take logarithms:

$$\ln y = \frac{\ln(e^x-1)}{\ln x}.$$

By l'Hôpital rule,

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln\left(e^x-1\right)}{\ln x} = \lim_{x\to 0^+}\frac{e^x/(e^x-1)}{1/x}$$
$$= \lim_{x\to 0^+}\frac{xe^x}{e^x-1}$$

By l'Hôpital rule again,

$$\lim_{x\to 0^+}\frac{xe^x}{e^x-1} = \lim_{x\to 0^+}\frac{e^x+xe^x}{e^x} = \frac{1+0}{1} = 1$$

Therefore $y\to e^1=e$, i.e.

$$\lim_{x\to 0^+}(e^x-1)^{\frac{1}{\ln x}} = e.$$

Problem 1(d): Solutions

(d) Evaluate $\lim_{x\to 0}(1+|\sin x|)^{\frac{1}{x}}$. Consider one-sided limits.
For $x \to 0^+$:

$$\lim_{x\to 0^+}(((1+|\sin x|)^{1/x})) = \lim_{x\to 0^+}\exp\left(\frac{1}{x}\ln(1+|\sin x|)\right),$$

By l'Hôpital rule,

$$= \lim_{x\to 0^+}\left((1+|\sin x|)^{1/x}\right) = \lim_{x\to 0^+}\exp\left(\frac{\cos x}{1+|\sin x|}\right) = e^1 = e,$$

hence $\lim_{x\to 0^+}(1+|\sin x|)^{1/x} = e$.
For $x \to 0^-$: Similarly, $\lim_{x\to 0^-}(1+|\sin x|)^{1/x} = e^{-1}$.
Since the left and right limits differ, the two-sided limit does not exist.

Problem 1(e): Solutions

(e) Evaluate $\lim_{x \to 0} \left(\dfrac{2^x + 3^x + 5^x}{3} \right)^{1/x}$.

Let

$$L = \lim_{x \to 0} \left(\frac{2^x+3^x+5^x}{3} \right)^{1/x}.$$

Take logarithms:

$$\ln L = \lim_{x \to 0} \frac{\ln\left(\frac{2^x+3^x+5^x}{3} \right)}{x}.$$

As $x \to 0$, the inside tends to 1, so the numerator tends to $\ln 1 = 0$. Thus the limit is of the indeterminate form $\frac{0}{0}$, so we may apply **L'Hôpital's Rule**:

$$\ln L = \lim_{x \to 0} \frac{\frac{d}{dx} \ln\left(\frac{2^x+3^x+5^x}{3} \right)}{1}.$$

Problem 1(e): Solutions

Differentiate:

$$\frac{d}{dx}\ln\left(\tfrac{2^x+3^x+5^x}{3}\right) = \frac{2^x\ln 2 + 3^x\ln 3 + 5^x\ln 5}{2^x+3^x+5^x}.$$

Taking the limit as $x \to 0$:

$$\ln L = \frac{\ln 2 + \ln 3 + \ln 5}{3}.$$

Therefore,

$$L = \exp\left(\tfrac{\ln 2+\ln 3+\ln 5}{3}\right) = (2\cdot 3\cdot 5)^{1/3} = 30^{1/3}.$$

Problem 2: Limit Tricks

Let α and β be two constants. Suppose

$$\lim_{x\to-\infty}\left(\sqrt{x^2+3x+2}-\alpha x-\beta\right)=0.$$

Find α and β.

Problem 2: Solutions

If

$$\lim_{x \to -\infty} \left(\sqrt{x^2 + 3x + 2} - \alpha x - \beta \right) = 0,$$

then

$$\lim_{x \to -\infty} \frac{\sqrt{x^2 + 3x + 2} - \alpha x - \beta}{x} = 0.$$

That is,

$$\lim_{x \to -\infty} \left(-\sqrt{1 + \frac{3}{x} + \frac{2}{x^2}} - \alpha - \frac{\beta}{x} \right) = 0.$$

Hence,

$$\alpha = \lim_{x \to -\infty} \left(-\sqrt{1 + \frac{3}{x} + \frac{2}{x^2}} - \frac{\beta}{x} \right) = -1.$$

Problem 2: Solutions

Next, compute β:

$$\beta = \lim_{x \to -\infty} \left(\sqrt{x^2 + 3x + 2} + x\right).$$

Rationalizing,

$$\beta = \lim_{x \to -\infty} \frac{(x^2 + 3x + 2) - x^2}{\sqrt{x^2 + 3x + 2} - x} = \lim_{x \to -\infty} \frac{2x + 3}{\sqrt{x^2 + 3x + 2} - x}.$$

Divide numerator and denominator by x:

$$\beta = \lim_{x \to -\infty} \frac{3 + \frac{2}{x}}{-\sqrt{1 + \frac{3}{x} + \frac{2}{x^2}} - 1}.$$

Hence,

$$\beta = \frac{3}{-2} = -\tfrac{3}{2}.$$

Problem 3: Continuity and Differentiability

Let

$$f(x) = \begin{cases} (1+x)^{\frac{1}{x}}, & x \neq 0,\ x > -1, \\ a, & x = 0. \end{cases}$$

a. Find the value of a such that $f(x)$ is continuous at $x = 0$.
b. Find $\lim_{x \to \infty} f(x)$.
c. Compute $f'(x)$, for $x \neq 0$.
d. Is $f(x)$ differentiable at $x = 0$? If $f(x)$ is differentiable at $x = 0$, then find $f'(0)$.

Problem 3(a): Solutions

(a) We compute

$$\lim_{x\to 0}(1+x)^{1/x} = \exp\Big(\lim_{x\to 0}\frac{\ln(1+x)}{x}\Big).$$

Since $\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$, the limit is e. Hence to make f continuous at 0 we must take

$$\boxed{a = e.}$$

Problem 3(b): Solutions

(b) Write

$$\ln f(x) = \frac{\ln(1+x)}{x}.$$

As $x \to \infty$, this is an indeterminate form $\frac{\infty}{\infty}$, so we apply L'Hôpital's rule:

$$\lim_{x\to\infty} \frac{\ln(1+x)}{x} = \lim_{x\to\infty} \frac{\frac{1}{1+x}}{1} = \lim_{x\to\infty} \frac{1}{1+x} = 0.$$

Therefore,

$$\lim_{x\to\infty} f(x) = \exp\left(\lim_{x\to\infty} \frac{\ln(1+x)}{x}\right) = e^0 = \boxed{1}.$$

Problem 3(c): Solutions

- We have

$$f(x) = (1+x)^{1/x} = e^{\frac{1}{x}\ln(1+x)}.$$

Let

$$u = \frac{1}{x}\ln(1+x), \quad \text{so that} \quad f(x) = e^u.$$

By the chain rule,

$$\frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx}. \tag{2}$$

Since

$$\frac{df}{du} = \frac{d}{du}\left(e^u\right) = e^u = e^{\frac{1}{x}\ln(1+x)}, \tag{3}$$

we only need $\frac{du}{dx}$.

Problem 3(c): Solutions

$$u = \frac{\ln(1+x)}{x} = x^{-1}\ln(1+x).$$

Differentiate using the product rule:

$$\frac{du}{dx} = \frac{d(x^{-1})}{dx}\ \ln(1+x) + x^{-1}\cdot\frac{1}{1+x}. \tag{4}$$

That is,

$$\frac{du}{dx} = -x^{-2}\ln(1+x) + \frac{1}{x(1+x)}.$$

Therefore,

$$\frac{df}{dx} = e^{\frac{1}{x}\ln(1+x)}\left(-\frac{\ln(1+x)}{x^2} + \frac{1}{x(1+x)}\right).$$

So,

$$\boxed{f'(x) = (1+x)^{1/x}\left(\frac{1}{x(1+x)} - \frac{\ln(1+x)}{x^2}\right).}$$

Problem 3(d): Solutions

(d) We compute

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x},$$

which is an indeterminate form $\frac{0}{0}$.
Apply L'Hôpital's rule and use the derivative from part (c):

$$\frac{d}{dx}(1+x)^{1/x} = (1+x)^{1/x}\left(\frac{1}{x(1+x)} - \frac{\ln(1+x)}{x^2}\right).$$

Thus

$$\begin{aligned} f'(0) &= \lim_{x\to 0}(1+x)^{1/x}\left(\frac{1}{x(1+x)} - \frac{\ln(1+x)}{x^2}\right) \\ &= e \cdot \lim_{x\to 0}\left(\frac{1}{x(1+x)} - \frac{\ln(1+x)}{x^2}\right). \end{aligned}$$

Problem 3(d): Solutions

Combine the two terms into a single fraction:

$$\frac{1}{x(1+x)} - \frac{\ln(1+x)}{x^2} = \frac{x-(1+x)\ln(1+x)}{x^2(1+x)}.$$

The latter is again 0/0, so apply L'Hôpital's rule.
Differentiate numerator and denominator:

$$\frac{d}{dx}\big(x-(1+x)\ln(1+x)\big) = 1 - \big(\ln(1+x)+1\big) = -\ln(1+x),$$

$$\frac{d}{dx}\big(x^2(1+x)\big) = 3x^2+2x.$$

Hence

$$\lim_{x\to 0}\frac{x-(1+x)\ln(1+x)}{x^2(1+x)} = \lim_{x\to 0}\frac{-\ln(1+x)}{3x^2+2x},$$

again an indeterminate form 0/0.

Problem 3(d): Solutions

Apply L'Hôpital's rule once more:
Differentiate numerator and denominator:

$$\frac{d}{dx}\big(-\ln(1+x)\big) = -\frac{1}{1+x}, \qquad \frac{d}{dx}(3x^2+2x) = 6x+2.$$

Thus

$$\lim_{x\to 0}\frac{-\ln(1+x)}{3x^2+2x} = \lim_{x\to 0}\frac{-\frac{1}{1+x}}{6x+2} = \frac{-1}{(1+0)\cdot 2} = -\tfrac{1}{2}.$$

Putting everything together:

$$f'(0) = e\cdot\Big(-\tfrac{1}{2}\Big) = -\tfrac{e}{2}.$$

Therefore f is differentiable at 0 and

$$\boxed{f'(0) = -\tfrac{e}{2}.}$$

Problem 4: Derivatives

Consider the function $f(x) = xe^{\frac{1}{x}}$ for $x \neq 0$.

a. Find $\lim\limits_{x \to 0^+} f(x)$ and $\lim\limits_{x \to 0^-} f(x)$.
b. Find all the vertical asymptotes of $y = f(x)$.
c. Find the slant asymptote(s) of $y = f(x)$.
d. Find $f'(x)$. Write down the interval(s) of increase and interval(s) of decrease of $y = f(x)$.
e. Find $f''(x)$. Write down the interval(s) on which $y = f(x)$ is concave upward and the interval(s) on which $y = f(x)$ is concave downward.

Problem 4(a): Solutions

a. We consider
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} xe^{\frac{1}{x}}.$$
Let $y = \frac{1}{x}$, so as $x \to 0^+$, $y \to +\infty$. Then
$$\lim_{x\to 0^+} xe^{1/x} = \lim_{y\to +\infty} \frac{e^y}{y}.$$
Since $\dfrac{e^y}{y} \to \infty$, we conclude
$$\lim_{x\to 0^+} f(x) = \infty.$$
Similarly,
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} xe^{1/x} = \left(\lim_{x\to 0^-} x\right)\left(\lim_{x\to 0^-} e^{1/x}\right) = 0 \cdot 0 = 0.$$
Hence $x = 0$ is a vertical asymptote.

Problem 4(b): Solutions

- Since $\lim_{x\to 0^+} f(x) = \infty$, we confirm $x = 0$ is a vertical asymptote. For $a \neq 0$, $f(x) = a \cdot \frac{1}{e^{1/a}}$ is finite, so $x = a$ is not a vertical asymptote. Therefore $x = 0$ is the only vertical asymptote.

Problem 4(c): Solutions

- First compute the slope:

$$\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}e^{1/x}=e^0=1.$$

Then compute the intercept:

$$\lim_{x\to\pm\infty}\big(f(x)-x\big)=\lim_{x\to\pm\infty}\big(x(e^{1/x}-1)\big).$$

Let $y=\frac{1}{x}\to 0^+$:

$$\lim_{x\to\pm\infty}x(e^{1/x}-1)=\lim_{y\to 0^+}\frac{e^y-1}{y}=1.$$

Hence the slant asymptote is $y=x+1$.

Problem 4(d): Solutions

(d) We compute

$$f'(x) = e^{1/x} - \frac{1}{x}e^{1/x} = e^{1/x}\frac{x-1}{x}.$$

Thus: - For $x < 0$, $f'(x) > 0$. - For $0 < x < 1$, $f'(x) < 0$. - For $x > 1$, $f'(x) > 0$.
Hence $f(x)$ is increasing on $(-\infty, 0) \cup (1, \infty)$ and decreasing on $(0, 1)$.

Problem 4(e): Solutions

- (e) We compute

$$f''(x) = -\tfrac{1}{x^2}e^{1/x} + \tfrac{1}{x^2}e^{1/x} + \tfrac{1}{x^3}e^{1/x} = \frac{1}{x^3}e^{1/x}(x-1).$$

Thus: - For $x > 0$, $f''(x) > 0$ if $x > 1$, and $f''(x) < 0$ if $0 < x < 1$.
- For $x < 0$, $f''(x) < 0$.
Therefore f is concave upward on $(1, \infty)$ and concave downward on $(-\infty, 0) \cup (0, 1)$.