

Week 11: Worksheet 2

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Calculus 1, Class 6

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Summary

1 Quiz Reminder

2 Arc Length

3 Average Value and Center of Mass

Quiz Reminder

Time: 12/03 (Wed) 17:30-18:20
Quiz Scope: 7.1 - 7.5, WS1-2
Location: 普通103
TA Hour: Fri. 14-16
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Arc Length Formula

The concepts of Riemann sums and integration both aim to measure total accumulation. The length of a curve can be viewed as the total length of many short segments. Any *smooth* curve that is short enough resembles a line segment, whose length is approximately $\sqrt{(dx)^2 + (dy)^2}$. Thus, integrating this expression gives the arc length formula.

Arc Length Formula for Graphs. Consider the graph $y = f(x)$. If f' is continuous on the interval $[a, b]$, then the length of $y = f(x)$ for $a \leq x \leq b$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2} dx.$$

A related application is the **arc length function**, defined by

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2} dt.$$

The arc length function helps determine the exact point you would arrive at after traveling a fixed distance along the curve.

Exercise 1

Find the exact length of the curve

$$y = \frac{x^3}{3} + \frac{1}{4x}, \quad 1 \leq x \leq 2.$$

Exercise 1: Solution

$$\frac{dy}{dx} = x^2 - \frac{1}{4x^2},$$

Compute the arc length:

$$\begin{aligned} L &= \int_1^2 \sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2} dx. \\ &= \int_1^2 \sqrt{x^4 + \frac{1}{16x^4} + \frac{1}{2}} dx \\ &= \int_1^2 \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2} dx = \int_1^2 \left(x^2 + \frac{1}{4x^2}\right) dx. \\ &= \left(\frac{x^3}{3} - \frac{1}{4x}\right) \bigg|_{x=1}^{x=2} = \frac{59}{24}. \end{aligned}$$

Exercise 2

Find the exact length of the curve

$$y = e^x, \quad 0 \leq x \leq 1.$$

Exercise 2: Solution

Since

$$\frac{dy}{dx} = e^x,$$

we compute

$$L = \int_0^1 \sqrt{1 + e^{2x}} \, dx.$$

Let $u = e^x$, so $dx = \frac{1}{u} du$. Then

$$L = \int_1^e \frac{\sqrt{1 + u^2}}{u} \, du.$$

Next let $u = \tan \theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, so $du = \sec^2 \theta \, d\theta$. Then

$$L = \int_{\pi/4}^{\arctan e} \frac{\sec \theta}{\tan \theta} \sec^2 \theta \, d\theta = \int_{\pi/4}^{\arctan e} \frac{1}{\cos^2 \theta \sin \theta} \, d\theta.$$

Write the integrand as

$$\frac{1}{\cos^2 \theta \sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta \sin \theta} = \frac{\sin \theta}{\cos^2 \theta} + \csc \theta.$$

Exercise 2: Solution

Thus

$$L = \frac{1}{\cos \theta} \Big|_{\theta=\pi/4}^{\theta=\arctan e} - \ln |\csc \theta + \cot \theta| \Big|_{\theta=\pi/4}^{\theta=\arctan e}.$$

Rewriting back in terms of $u = \tan \theta$,

$$L = \sqrt{1+u^2} \Big|_{u=1}^{u=e} - \ln \left| \frac{\sqrt{1+u^2}+1}{u} \right| \Big|_{u=1}^{u=e}.$$

Therefore

$$L = \sqrt{1+e^2} - \sqrt{2} - \ln \left(\frac{1+\sqrt{1+e^2}}{e} \right) + \ln(\sqrt{2}+1).$$

Exercise 3

Consider the curve

$$y = \frac{2}{3}\sqrt{x^3}, \quad 1 \leq x \leq 4.$$

- a) Find the exact length.
- b) Find the length function $s(x)$. Does it have an inverse?
- c) Find an interval for x where the exact length of the curve is equal to 1.

Exercise 3(a)(b): Solution

(a) Since

$$\frac{dy}{dx} = \sqrt{x},$$

we have

$$\begin{aligned} L &= \int_1^4 \sqrt{1+x} \, dx = \int_2^5 \sqrt{u} \, du \quad (\text{Let } u = 1+x). \\ &= \left. \frac{2}{3} u^{3/2} \right|_2^5 = \frac{10}{3} \sqrt{5} - \frac{4}{3} \sqrt{2}. \end{aligned}$$

(b) The length function is

$$\begin{aligned} s(x) &= \int_1^x \sqrt{1+t} \, dt = \int_2^{x+1} \sqrt{u} \, du \quad (\text{Let } u = 1+t). \\ &= \left. \frac{2}{3} u^{3/2} \right|_2^{x+1} = \frac{2}{3} (x+1)^{3/2} - \frac{4\sqrt{2}}{3}. \end{aligned}$$

Exercise 3(b)(c): Solution

From

$$s(x) = \frac{2}{3}(x+1)^{3/2} - \frac{4\sqrt{2}}{3},$$

we note that

$$s'(x) = \sqrt{1+x} > 0 \quad (x \in [1, 4]).$$

Hence $s(x)$ is strictly increasing and has an inverse:

$$x(s) = \left(\frac{3}{2}s + 2\sqrt{2} \right)^{2/3} - 1.$$

(c) For a length of 1,

$$x(1) = \left(\frac{3}{2} + 2\sqrt{2} \right)^{2/3} - 1 \approx 1.656.$$

Thus, the exact length of the curve on

$$\left[1, \left(\frac{3}{2} + 2\sqrt{2} \right)^{2/3} - 1 \right]$$

equals to 1.

Average Value and Center of Mass

Average Value of a Function. For a continuous function f on the interval $[a, b]$, the average value is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Center of Mass. For an object with density function $\rho(x)$, the center of mass is the *average position* of all the mass. For a one-dimensional rod covering $[a, b]$ with density $\rho(x)$:

$$m = \int_a^b \rho(x) dx \quad (\text{total mass})$$

and the position of the center of mass is

$$\bar{x} = \frac{1}{m} \int_a^b x \rho(x) dx.$$

Exercise 4

- a Find the average value of $f(x) = 25 - x^2$ on the interval $[0, 2]$.
- b Find all values of c in the interval $[0, 2]$ such that $f(c)$ is equal to the average value.

Exercise 4: Solution

(a)

$$f_{\text{avg}} = \frac{1}{2-0} \int_0^2 (25 - x^2) dx = \frac{1}{2} \left(25x - \frac{x^3}{3} \right) \bigg|_0^2 = \frac{71}{3}.$$

(b) Solve

$$f(c) = 25 - c^2 = \frac{71}{3}.$$

$$c^2 = 25 - \frac{71}{3} = \frac{75 - 71}{3} = \frac{4}{3}, \quad c = \sqrt{\frac{4}{3}} \quad (\text{the other solution } -\sqrt{\frac{4}{3}} \notin [0, 2]).$$

Exercise 5

Find the mass and the center of mass of a thin steel pipe with density function

$$\rho(x) = \frac{1}{1+x^2}$$

over the interval $[0, \sqrt{3}]$.

Exercise 5: Solution

Mass.

$$m = \int_0^{\sqrt{3}} \frac{1}{1+x^2} dx = \arctan(\sqrt{3}) - \arctan(0) = \frac{\pi}{3}.$$

Center of mass.

$$\bar{x} = \frac{1}{m} \int_0^{\sqrt{3}} \frac{x}{1+x^2} dx.$$

Let $u = 1 + x^2$, so $du = 2x dx$, hence

$$\bar{x} = \frac{3}{\pi} \int_1^4 \frac{1}{2u} du = \frac{3}{2\pi} \ln |u| \Big|_1^4 = \frac{3}{\pi} \ln 2 \approx 0.6619.$$

Exercise 6: Mean Value Theorem for Integrals

Prove the Mean Value Theorem for Integrals: if f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that

$$\frac{1}{b-a} \int_a^b f(x) dx = f(c).$$

Interpret this result geometrically.

Exercise 6: Solution

If $f = \text{const.}$ on $[a, b]$, then the statement is trivially true.

If f is not constant, then by the Extreme Value Theorem there exist $m, M \in [a, b]$ such that

$$f(m) \leq f(x) \leq f(M), \quad \forall x \in [a, b].$$

Multiplying $(b - a)$,

$$f(m)(b - a) \leq \int_a^b f(x) dx \leq f(M)(b - a),$$

which implies

$$f(m) \leq \frac{1}{b - a} \int_a^b f(x) dx \leq f(M).$$

Since f is continuous on $[a, b]$, the Intermediate Value Theorem ensures that there exists c between m and M such that

$$f(c) = \frac{1}{b - a} \int_a^b f(x) dx.$$