Week 8: Midterm Review

Lecturer: Zhi (George) Lin

Calculus 1, Class 6

Oct 22th 2025

Summary

Midterm Reminder

Time:	11/01 09:00-11:30
Midterm Scope:	2.1-2.8, 3.1-3.6, 3.10, 4.1-4.5, 4.7, 4.9, WS 1-3
Location:	Gongtong Lecture Building 205 共同205
TA Hour:	By appointment
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Limits: Core Techniques

Algebraic & analytic tools

- Factor/cancel; common denominators; conjugates (rationalization).
- Trig identities and standard limits:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}.$$

- **Squeeze Theorem**: if $g \le f \le h$ and $\lim g = \lim h = L$, then $\lim f = L$.
- **l'Hôpital's Rule**: for $0/0$, ∞/∞ (with differentiability).
- Non-quotient indeterminate forms: $0 \cdot \infty$, 1^∞, 0^0, ∞^0. Use logs to reduce to $0/0$ or ∞/∞ and apply l'Hôpital.

Problems 1–5 (Limits and Techniques I)

1. (**Rationalization**) Evaluate $\lim_{x\to 0} \dfrac{\sqrt{1+3x}-\sqrt{1+x}}{x}$.

2. (**Hidden cancellation**) Evaluate $\lim_{x\to 0} \dfrac{\sqrt{x^2+x+1}-\sqrt{x^2+1}}{x}$.

3. (**Trig identity**) Evaluate $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$.

4. (**Squeeze Theorem**) Show that $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$.

5. (**l'Hôpital's Rule**) Compute $\lim_{x\to 0} \dfrac{\ln(1+x)-\sin x}{x^2}$.

Solutions (Problems 1–3)

1. Multiply by conjugate:

$$\lim_{x\to 0} \frac{\sqrt{1+3x}-\sqrt{1+x}}{x} = \lim_{x\to 0} \frac{2}{\sqrt{1+3x}+\sqrt{1+x}} = 1.$$

2. Let $A = x^2 + x + 1,\ B = x^2 + 1.$

$$\lim_{x\to 0} \frac{\sqrt{A}-\sqrt{B}}{x} = \lim_{x\to 0} \frac{A-B}{x(\sqrt{A}+\sqrt{B})} = \lim_{x\to 0} \frac{1}{\sqrt{A}+\sqrt{B}} = \frac{1}{2}.$$

3. Use $1-\cos x = 2\sin^2(x/2)$:

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{1}{2}\left(\frac{\sin(x/2)}{x/2}\right)^2 = \frac{1}{2}.$$

Solutions (Problems 4–5)

4. Since $-1 \le \sin(1/x) \le 1$,

$$-|x| \le x\sin(1/x) \le |x| \Rightarrow \lim_{x\to 0} x\sin(1/x) = 0.$$

5. 0/0 form — apply l'Hôpital twice:

$$\lim_{x\to 0} \frac{\ln(1+x) - \sin x}{x^2} = \lim_{x\to 0} \frac{\frac{1}{1+x} - \cos x}{2x} = \lim_{x\to 0} \frac{-\frac{1}{(1+x)^2} + \sin x}{2} = -\tfrac{1}{2}.$$

Problems 6–10 (Limits and Techniques II)

6. (**Series or multiple l'Hôpital**) Evaluate $\lim_{x \to 0} \dfrac{x - \sin x}{x^3}$.

7. (1^∞ form — **log trick**) Evaluate $\lim_{x \to 0} (\cos x)^{1/x^2}$.

8. (**Definition of e**) Evaluate $\lim_{x \to 0} \dfrac{(1+x)^{1/x} - e}{x}$.

9. (**Growth at infinity**) Show that $\lim_{x \to \infty} \dfrac{x \ln x}{e^x} = 0$.

10. (**Conjugate at infinity**) Evaluate $\lim_{x \to \infty} x(\sqrt{x^2+1} - x)$.

Solutions (Problems 6–7)

6. Series: $\sin x = x - \frac{x^3}{6} + O(x^5) \Rightarrow \lim_{x\to\infty} \frac{x - \sin x}{x^3} = \frac{1}{6}$.
Don't use Taylor's series to calculate limits in midterm.

7. Let $L = \lim_{x\to 0}(\cos x)^{1/x^2}$. Then

$$\ln L = \lim_{x\to 0} \frac{\ln|\cos x|}{x^2} = \lim_{x\to 0} \frac{-\tan x}{2x} = -\tfrac{1}{2}, \quad L = e^{-1/2}.$$

8. Define $h(x) = \frac{\ln(1+x)}{x}$ and $k(x) = h(x) - 1$. Then

$$\lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x} = \lim_{x\to 0} \frac{e^{h(x)} - e}{x} = e \cdot \lim_{x\to 0} \frac{e^{h(x)-1} - 1}{x}$$
$$= e \cdot \lim_{x\to 0} \left(\frac{e^{k(x)} - 1}{k(x)}\right)\left(\frac{k(x)}{x}\right).$$

Using the definition of e (equivalently $\lim_{u\to 0} \frac{e^u - 1}{u} = 1$),

$$\lim_{x\to 0} \frac{e^{k(x)} - 1}{k(x)} = 1,$$

Apply l'Hôpital's Rule (twice) to the last limit:

$$\lim_{x\to 0} \frac{\ln(1+x) - x}{x^2} = \lim_{x\to 0} \frac{\frac{1}{1+x} - 1}{2x} = \lim_{x\to 0} \frac{-\frac{1}{(1+x)^2}}{2} = -\frac{1}{2}.$$

Therefore,

$$\lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x} = e \cdot 1 \cdot \left(-\frac{1}{2}\right) = -\frac{e}{2}.$$

Solutions (Problems 9–10)

9. Repeated l'Hôpital:

$$\lim_{x\to\infty} \frac{x \ln x}{e^x} = \lim_{x\to\infty} \frac{\ln x + 1}{e^x} \lim_{x\to\infty} \frac{1/x}{e^x} = 0.$$

Thus, $\displaystyle\lim_{x\to\infty} \frac{x \ln x}{e^x} = 0$.

10. Multiply by conjugate:

$$\lim_{x\to 0} x(\sqrt{x^2+1} - x) = \lim_{x\to 0} \frac{x((x^2+1) - x^2)}{\sqrt{x^2+1} + x} = \lim_{x\to 0} \frac{x}{\sqrt{x^2+1} + x} = \frac{1}{2}.$$

Limits: Quick Reference

Indeterminate Form	**Typical Fix**
$0/0,\ \infty/\infty$	Apply l'Hôpital, simplify, rationalize
$0 \cdot \infty$	Rewrite as a quotient ($0/0$ or ∞/∞)
$1^\infty,\ 0^0,\ \infty^0$	Take ln, use l'Hôpital, exponentiate back
Oscillation near 0	Apply Squeeze Theorem ($x \sin(1/x) \to 0$)

Intermediate Value Theorem (IVT)

Definition 8.1 (Intermediate Value Theorem, IVT)

Suppose f is **continuous** on the closed interval $[a, b]$. Let N be any number between $f(a)$ and $f(b)$, that is,

$$f(a) < N < f(b) \quad \text{or} \quad f(b) < N < f(a).$$

Then there exists at least one point $c \in (a, b)$ such that

$$f(c) = N.$$

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Geometric Interpretation

The continuous curve $y = f(x)$ must cross every horizontal line $y = N$ between $y = f(a)$ and $y = f(b)$.

Mean Value Theorem

Definition 8.2 (Mean Value Theorem, MVT)

Suppose f is **continuous** on the closed interval $[a, b]$ and **differentiable** on the open interval (a, b). Then there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

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The tangent at some point c is **parallel** to the secant through $(a, f(a))$ and $(b, f(b))$.

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Special Case: Rolle's Theorem

If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that

$$f'(c) = 0.$$

Problems 11–14 (IVT and MVT Applications)

11. (**IVT + Monotonicity**) Show that the equation $x = \cos x$ has a unique solution in $(0, 1)$.

12. (**MVT Bound**) Show that $|\sin b - \sin a| \leq |b - a|$ for all real a, b.

13. (**Locating a specific c**) Show there exists $c \in (0, 1)$ with $e^c = e - 1$.

14. (**Classical MVT Inequality**) Show that $\ln(1 + x) \leq x$ for all $x > -1$.

Solutions (Problems 11–12)

Problem 11 (IVT + Monotonicity) Let $f(x) = x - \cos x$. Then f is continuous, $f(0) = -1 < 0$, and $f(1) = 1 - \cos 1 > 0$. By the **Intermediate Value Theorem**, a root exists in $(0, 1)$. Moreover, $f'(x) = 1 + \sin x > 0$ on $(0, 1)$, so f is strictly increasing, hence the root is **unique**.

Problem 12 (MVT Bound) Apply the **Mean Value Theorem** to $f(x) = \sin x$ on $[a, b]$:

$$\sin b - \sin a = \cos(c)(b - a) \quad \text{for some } c \in (a, b).$$

Since $|\cos c| \leq 1$, it follows that

$$|\sin b - \sin a| \leq |b - a|.$$

Solutions (Problems 13–14)

Problem 13 (Locating a specific c) Apply the **Mean Value Theorem** to $f(x) = e^x$ on $[0, 1]$:

$$e^1 - e^0 = e - 1 = f'(c)(1 - 0) = e^c.$$

Hence $e^c = e - 1$, and $c = \ln(e - 1) \in (0, 1)$.

Problem 14 (Classical MVT Inequality) Let $f(t) = \ln(1 + t)$. For $x > 0$, apply MVT on $[0, x]$:

$$\ln(1 + x) = \frac{1}{1 + c}\, x, \quad c \in (0, x).$$

Since $\frac{1}{1+c} \leq 1$, we get $\ln(1 + x) \leq x$.
For $x < 0$, use $[x, 0]$; then $\frac{1}{1+c} \geq 1$ and $x < 0$, so again $\ln(1 + x) \leq x$. **Thus,** $\boxed{\ln(1 + x) \leq x \text{ for all } x > -1.}$

Problems 15–19 (Differentiation Techniques I)

15. (**Log differentiation**) Compute $\dfrac{d}{dx}\big(x^x\big)$ for $x > 0$.

16. (**Chain + product + quotient**) Differentiate $f(x) = \dfrac{x^2 \ln(1 + x^3)}{\sin(2x)}$ where defined.

17. (**Log of a trig ratio**) Differentiate $g(x) = \ln\left(\dfrac{\sin(3x)}{x^2 + 1}\right)$ where defined.

18. (**Inverse trig inside a root**) Differentiate $h(x) = \arcsin\big(\sqrt{1 - x}\,\big)$ on $(0, 1)$.

19. (**Disguised double-angle**) Differentiate $F(x) = \arctan\!\left(\dfrac{2x}{1 - x^2}\right)$ where defined.

Solutions (Problems 15–19)

15. Let $y = x^x$. Then $\ln y = x \ln |x| \Rightarrow \frac{y'}{y} = \ln |x| + 1$. Hence $y' = x^x(\ln x + 1)$.

Solutions (Problems 15–19)

15. Let $y = x^x$. Then $\ln y = x \ln |x| \Rightarrow \frac{y'}{y} = \ln |x| + 1$. Hence $y' = x^x(\ln x + 1)$.

16. Let $u = x^2 \ln(1 + x^3)$, $v = \sin(2x)$. $u' = 2x \ln(1 + x^3) + x^2 \frac{3x^2}{1+x^3}$,
$v' = 2\cos(2x)$.

$$f'(x) = \frac{u'v - uv'}{v^2} = \frac{\left(2x \ln(1 + x^3) + \frac{3x^4}{1+x^3}\right)\sin(2x) - x^2 \ln(1 + x^3)\, 2\cos(2x)}{\sin^2(2x)}.$$

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17. Use $\ln \frac{A}{B} = \ln A - \ln B$: $g'(x) = \frac{3\cos(3x)}{\sin(3x)} - \frac{2x}{x^2+1} = 3 \cot(3x) - \frac{2x}{x^2+1}$.

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18. Let $u = \sqrt{1 - x} \Rightarrow u' = -\frac{1}{2\sqrt{1-x}}$. Then $h'(x) = \frac{u'}{\sqrt{1-u^2}} = -\frac{1}{2\sqrt{x(1-x)}}$.

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19. Let $u = \frac{2x}{1-x^2} \Rightarrow u' = \frac{2(1+x^2)}{(1-x^2)^2}$. Also $1 + u^2 = \frac{(1+x^2)^2}{(1-x^2)^2}$.

$$F'(x) = \frac{u'}{1 + u^2} = \frac{2(1 + x^2)}{(1 - x^2)^2} \cdot \frac{(1 - x^2)^2}{(1 + x^2)^2} = \frac{2}{1 + x^2}.$$

Problems 20–22 (Linear Approximation — Advanced)

20. (**Nonlinear Composition**) Let $f(x) = \sqrt{1 + \sin(2x)}$. Use the linearization of f at $a = 0$ to approximate $f(0.1)$. Then, estimate the maximum possible error using the second derivative of f.

21. (**Nested Logarithm–Exponential**) Let $f(x) = \ln(1 + e^{2x})$. (a) Find the linearization $L(x)$ at $a = 0$. (b) Use $L(x)$ to approximate $f(0.2)$. (c) Provide an upper bound on the approximation error using $|R_1(x)| \leq \dfrac{M}{2}\,|x - a|^2$, where $M = \max_{[0,0.2]} |f''(x)|$.

22. (**Multi-step Linearization**) Consider $f(x) = (3x - e^{7x})^{1/5}$. (a) Compute the linearization $L(x)$ at $a = 0$. (b) Use it to approximate $f(0.05)$. (c) Decide whether this linear approximation overestimates or underestimates the true value near $a = 0$, justifying via the sign of $f''(0)$.

Solutions (Problems 20–21)

20. $f(x) = \sqrt{1+\sin(2x)}$. $f(0) = 1$.

$f'(x) = \dfrac{\cos(2x)}{\sqrt{1+\sin(2x)}} \Rightarrow f'(0) = 1$. $\Rightarrow$ Linearization at $a = 0$:

$L(x) = 1 + x$. Approximation: $f(0.1) \approx L(0.1) = 1.1$.

Error bound via second derivative: $f'(x) = g(x)h(x)$ with $g = \cos(2x)$, $h = (1+\sin 2x)^{-1/2}$. $g' = -2\sin(2x)$, $\quad h' = -\cos(2x)(1+\sin 2x)^{-3/2}$.

$$f''(x) = \frac{-2\sin(2x)(1+\sin 2x) - \cos^2(2x)}{(1+\sin 2x)^{3/2}}.$$

On $[0, 0.1]$: $|\sin(2x)| \le \sin(0.2)$, $|\cos(2x)| \le 1$. Hence

$$|f''(x)| \le \frac{2|\sin(2x)|(1+|\sin(2x)|) + \cos^2(2x)}{(1-|\sin(2x)|)^{3/2}} \le \frac{4\sin(0.2)+1}{(1-\sin(0.2))^{3/2}} \approx 2.5.$$

Thus $|R_1(0.1)| \le \dfrac{2.5}{2}(0.1)^2 \approx 0.0125$.

21. $f(x) = \ln(1 + e^{2x})$. (a) $f(0) = \ln 2$.
$f'(x) = \dfrac{2e^{2x}}{1 + e^{2x}} = \dfrac{2}{1 + e^{-2x}} \Rightarrow f'(0) = 1. \Rightarrow L(x) = \ln 2 + x.$
(b) $f(0.2) \approx L(0.2) = \ln 2 + 0.2.$
(c) $f''(x) = 4\,\sigma(2x)\big(1 - \sigma(2x)\big)$ with $\sigma(t) = \dfrac{e^t}{1 + e^t}$. Since $0 \le \sigma \le 1$,
$\sigma(1 - \sigma) \le \frac{1}{4} \Rightarrow f''(x) \le 1$ on any interval.
Hence $M \le 1$ on $[0, 0.2]$ and $|R_1(0.2)| \le \dfrac{1}{2}(0.2)^2 = 0.02.$

Solutions (Problem 22)

22. $f(x) = (3x - e^{7x})^{1/5}$. Let $g(x) = 3x - e^{7x}$. (a) $f'(x) = \frac{1}{5}g(x)^{-4/5}g'(x)$. At $x = 0$: $g(0) = -1$, $g'(x) = 3 - 7e^{7x} \Rightarrow g'(0) = -4$. $(-1)^{-4/5} = 1$ (principal real fifth root). Thus $f(0) = (-1)^{1/5} = -1$ and $f'(0) = \frac{1}{5} \cdot 1 \cdot (-4) = -\frac{4}{5}$.

$$\Rightarrow \quad L(x) = f(0) + f'(0)(x - 0) = -1 - \frac{4}{5}x.$$

(b) $f(0.05) \approx L(0.05) = -1 - 0.8(0.05) = -1.04$.
(c) Concavity at 0: $f''(x) = \frac{1}{5}\left(-\frac{4}{5}\right)g(x)^{-9/5}\big(g'(x)\big)^2 + \frac{1}{5}g(x)^{-4/5}g''(x)$. At 0: $g''(x) = -49e^{7x} \Rightarrow g''(0) = -49$, and $(-1)^{-9/5} = -1$. Hence

$$f''(0) = \frac{1}{5}\left(-\frac{4}{5}\right)(-1) \cdot 16 + \frac{1}{5} \cdot 1 \cdot (-49) = -\frac{181}{25} \approx -7.24 < 0.$$

So f is **concave down** at 0; the tangent line lies above the curve nearby. Therefore the linear approximation $L(0.05) = -1.04$ **overestimates** the true value (i.e., $f(0.05) < -1.04$).