Week 1: Worksheet 1

Lecturer: Zhi (George) Lin

Calculus 1, Class 6

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Summary

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2. How TA class works?
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TA Information

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TA Hour:	Fri. 14:00–16:00 or by appointment 化學系館（積學館）B363
Research Interest:	Machine Learning, Stochastic Model, Image Recognition

How TA class works?

Class:	06 餘 0 班
Time:	Every Wed. 17:30–18:20
Location:	普通 204
What will be covered?	Worksheets
	Supplementary materials
	Quiz/Exam review

Worksheet 1: Inverse Function

Definition 1.1

Let $f : \mathbb{R} \to \mathbb{R}$ be a function with domain A and range B. Suppose $g(x)$ is a function whose domain is B such that

$$g(f(x)) = x \quad \text{for all } x \in A.$$

In this case, we say that $g(x)$ is the inverse function of $f(x)$ and denote it by $f^{-1}(x)$.

Theorem 1.2

Let $f(x)$ be a function whose domain is A and whose range is B, and let $f^{-1}(x)$ be its inverse function.

a. The domain of $f^{-1}(x)$ is B and its range is A.
b. The graphs $y = f(x)$ and $y = f^{-1}(x)$ are symmetric along the line $y = x$.

Exercise 1: Inverse Functions

For each of the following functions $f(x)$:

a. write down its inverse function,
b. write down the domain and range of the inverse function,
c. sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same diagram.

What do you observe from (b) and (c)?

i. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 2x$
ii. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 3x - 4$
iii. $f : \mathbb{R}_{\geq 0} \to \mathbb{R}, \quad f(x) = x^2$

Exercise 1: Solutions

(i) $f(x) = 2x$:

$$f^{-1}(x) = \frac{x}{2}, \quad \mathrm{Dom}(f^{-1}) = \mathbb{R},\ \mathrm{Ran}(f^{-1}) = \mathbb{R}.$$

(ii) $f(x) = 3x - 4$:

$$f^{-1}(x) = \frac{x+4}{3}, \quad \mathrm{Dom}(f^{-1}) = \mathbb{R},\ \mathrm{Ran}(f^{-1}) = \mathbb{R}.$$

(iii) $f : \mathbb{R}_{\geq 0} \to \mathbb{R},\ f(x) = x^2$:

$$\mathrm{Ran}(f) = \mathbb{R}_{\geq 0} \ \Rightarrow\ f^{-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad f^{-1}(x) = \sqrt{x}.$$

Observations:

- The *domain* of f^{-1} equals the *range* of f, and the *range* of f^{-1} equals the *domain* of f.
- The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetric about the line $y = x$.
- The domain restriction in (iii) ($\mathbb{R}_{\geq 0}$) is necessary to make f one-to-one so that f^{-1} is a function.

One-to-One Functions and Inverses

Definition 2.1

Let $f : D \to \mathbb{R}$. We say that f is **one-to-one** if for any distinct inputs $x_1, x_2 \in D$, we have

$$f(x_1) \neq f(x_2).$$

Theorem 2.2

The inverse function $f^{-1}(x)$ exists if and only if f is one-to-one.

Exercise 2: One-to-One and Inverses

a. Explain briefly why the above discussion does not contradict your findings in Exercise 1 (iii).
b. Explain briefly why if a function is strictly increasing or strictly decreasing (but not both), then it is one-to-one (and hence its inverse exists).
c. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is neither strictly increasing nor strictly decreasing but is still one-to-one. Graph your example.

Exercise 2: Solutions

(a) In Exercise 1 (iii), $f(x) = x^2$ is not one-to-one on $\mathbb{R}$, but it becomes one-to-one when restricted to $\mathbb{R}_{\geq 0}$. Thus, the inverse $f^{-1}(x) = \sqrt{x}$ exists.

(b) If f is strictly increasing, then $x_1 < x_2 \implies f(x_1) < f(x_2)$. If f is strictly decreasing, then $x_1 < x_2 \implies f(x_1) > f(x_2)$. In either case, distinct inputs give distinct outputs, so f is one-to-one.

(c) Example: $f(x) = 1/x$. This function is not monotonic, but it never takes the same y-value twice, so it is one-to-one. (Graph the function yourselves)

Our goal is to define the inverse functions of trigonometric functions.

Take $f(x) = \sin x$. If we graph it, we encounter a problem:

$$\sin x \quad \text{is not one-to-one on } \mathbb{R},$$

so its inverse does not exist globally.

To define an inverse, we restrict the domain. For example:

$$f : \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] \ \to \ [-1, 1], \quad f(x) = \sin x.$$

On this interval, $\sin x$ is strictly increasing, so it is one-to-one. Hence, the inverse exists!

We denote the inverse by

$$f^{-1}(x) = \sin^{-1}(x),$$

called the **arcsine function**. (Some textbooks also write $\arcsin(x)$.)

Exercise 3: Inverse Sine

a. What are the domain and the range of $f^{-1}(x) = \sin^{-1}(x)$?
b. Write down the following values:

$$\sin^{-1}(1), \quad \sin^{-1}\left(\tfrac{1}{2}\right), \quad \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right),$$

$$\sin^{-1}\big(\sin(-\tfrac{\pi}{8})\big), \quad \sin^{-1}\big(\sin(\tfrac{3\pi}{2})\big).$$

c. Sketch the graph $y = \sin^{-1}(x)$.

Exercise 3: Solutions

(a)

$$\mathrm{Dom}(\sin^{-1}) = [-1, 1], \qquad \mathrm{Ran}(\sin^{-1}) = \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right].$$

(b)

$$\sin^{-1}(1) = \tfrac{\pi}{2}, \quad \sin^{-1}\left(\tfrac{1}{2}\right) = \tfrac{\pi}{6}, \quad \sin^{-1}\left(-\tfrac{\sqrt{3}}{2}\right) = -\tfrac{\pi}{3},$$

$$\sin^{-1}\big(\sin(-\tfrac{\pi}{8})\big) = -\tfrac{\pi}{8}, \quad \sin^{-1}\big(\sin(\tfrac{3\pi}{2})\big) = -\tfrac{\pi}{2}.$$

(c) The graph $y = \sin^{-1}(x)$ is the reflection of $y = \sin(x)$ (restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$) across the line $y = x$. It passes through $(-1, -\frac{\pi}{2})$, $(0, 0)$, $(1, \frac{\pi}{2})$.

Exercise 4: Inverse Tangent

a. Sketch the graph $y = \tan^{-1}(x)$.
b. From (a), write down $\lim_{x\to\infty} \tan^{-1} x$ and $\lim_{x\to-\infty} \tan^{-1} x$.
c. Look up the relevant section in the textbook (Stewart): write down the domains and ranges of the 'inverse' and 'inverse reciprocal' trig functions listed on the right-hand side of the table.

Exercise 4: Solutions

(a) Graph of $y = \tan^{-1} x$ (Draw it yourselves):

- $y = \tan^{-1} x$ is odd, continuous, strictly increasing, passes through $(0, 0)$.
- Horizontal asymptotes at $y = \pm\frac{\pi}{2}$.
- No vertical asymptotes (domain all real numbers).

(b) Limits:

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}, \qquad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}.$$

(c) Domains and ranges (usual conventions):

Function	Domain	Range
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	$\mathbb{R}$	$(0, \pi)$
$\sec^{-1} x$	$\{x : \lvert x\rvert \geq 1\}$	$[0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$
$\csc^{-1} x$	$\{x : \lvert x\rvert \geq 1\}$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$

Exercise 5: Simplifying Compositions of Inverse Trig Functions

Consider the function $f(x) = \tan(\sin^{-1}(x))$.

a. Let $\theta = \sin^{-1}(x)$. For $x > 0$, draw a right triangle with θ being one of the angles such that $\sin(\theta) = x$.
b. Hence, guess a simplified expression of $f(x)$.
c. Using a similar method, simplify the expression

$$\cos\left(\tan^{-1}\frac{1}{\sqrt{x^2-1}}\right).$$

Hence, evaluate

$$\lim_{x\to 1^+}\frac{1}{\sqrt{x^2-1}}\cos\left(\tan^{-1}\frac{1}{\sqrt{x^2-1}}\right).$$

Exercise 5: Solutions

(a) Triangle visualization: Let $\theta = \sin^{-1}(x)$, $x > 0$. Construct a right triangle:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{1} \implies \text{opposite} = x,\ \text{hypotenuse} = 1.$$

Then the adjacent side is $\sqrt{1-x^2}$.

(b) Simplifying $f(x)$:

$$\tan(\sin^{-1}(x)) = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{\sqrt{1-x^2}}.$$

(c) Similar method: Let $\phi = \tan^{-1} \frac{1}{\sqrt{x^2-1}}$. Then a right triangle with $\tan\phi = \frac{1}{\sqrt{x^2-1}}$ has opposite 1, adjacent $\sqrt{x^2-1}$, hypotenuse $\sqrt{1+(x^2-1)} = x$. Thus:

$$\cos\left(\tan^{-1}\frac{1}{\sqrt{x^2-1}}\right) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{x^2-1}}{x}.$$

Limit:

$$\lim_{x\to1^+} \frac{1}{\sqrt{x^2-1}} \cdot \frac{\sqrt{x^2-1}}{x} = \lim_{x\to1^+} \frac{1}{x} = 1.$$