Calculus I – Quiz 2 Review

Problem 1: Mean Value Theorem

(a) Use the **Mean Value Theorem** to prove that for any $0 < a < b$,

$$\frac{\sqrt{b} - \sqrt{a}}{1 + b} \leq \tan^{-1}(\sqrt{b}) - \tan^{-1}(\sqrt{a}) \leq \frac{\sqrt{b} - \sqrt{a}}{1 + a}.$$

(b) Suppose c is a constant such that the limit

$$L = \lim_{x \to \infty} \frac{\tan^{-1}(\sqrt{x^3 + 1}) - \tan^{-1}(\sqrt{x^3 - 1})}{x^c}$$

is non-zero.

Find c and L.

Problem 2: Linear Approximation

Consider the function

$$f(x) = 3x - \tan^{-1}(x-1).$$

(a) Show that the equation

$$3x - \tan^{-1}(x-1) = 3.01$$

has a unique solution.

(b) Let $g(x)$ be the inverse function of f. Find $g(3)$ and $g'(3)$.

(c) Apply a linear approximation to g to estimate the solution of $f(x) = 3.01$.

Problem 3: Differentiation Skills

For each of the following relations between x and y, find an expression for $\dfrac{dy}{dx}$ in terms of x and y.

(a) $y = (3x - e^{7x})^{\cos(5x)}$.

(b) $y = \dfrac{\sin\left(3x + x^2\right)}{(6 - x^4)^3}$.

Problem 4: Implicit Differentiation

Find the highest and the lowest points of the curve given by

$$x^2 + xy + 2y^2 = 28.$$

(a) Find an expression for $\dfrac{dy}{dx}$ in terms of x and y.

(b) At the highest and the lowest points, what is $\dfrac{dy}{dx}$? What equation can we obtain from $\dfrac{dy}{dx}$ and part (a)?

(c) Using parts (a) and (b), find the coordinates of the highest and lowest points.