Week 10: Worksheet 1

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Calculus 1, Class 6

Nov 5th 2025

Summary

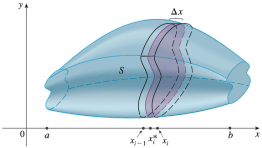
Quiz Reminder

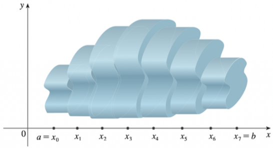
Time:	11/12 (Wed) 17:30-18:20
Quiz Scope:	5.1 - 6.1
Location:	普通103
TA Hour:	Fri. 14-16
	化學系館（積學館）B363

Volumes by Cylindrical Shells

Definition of Volume as a Riemann Sum.

For a *cylinder* (of an arbitrary base), its volume equals to $V = Ah$ where A is its base area and h is its height. But how do we compute the volume of a *general solid* S?





To do this, we place S in a coordinate system. Suppose that the x-coordinates of S is contained in the interval $[a, b]$ (see the left figure). Then we cut S by a family of parallel planes, $P_{x_i} : x = x_i$, where $a = x_1 < x_2 < ... < x_n = b$ are constants (see the right figure). Thus we slice S into many small slabs which are parts of S between two adjacent planes P_{x_i} and $P_{x_{i+1}}$, $i = 1, 2, \dots, n-1$. Each slab can be approximated by a cylinder with height $x_{i+1} - x_i = \Delta x_i$ and base $S \cap P_{x_i}$. If we know the area of the cross-section $S \cap P_x$ for all $x \in [a, b]$ which is denoted by $A(x)$, then the approximating cylinders have volumes $A(x_i)\Delta x_i$, $i = 1, 2, \dots, n-1$. Adding volumes of these cylinders, we can approximate the volume of S by

$$\sum_{i=1}^{n-1} A(x_i)\Delta x_i$$

which is a finite Riemann sum of $A(x)$ over the interval $[a, b]$. When Δx_i tends to 0, the slices become thinner and thus the approximation becomes better. Hence, we will *define the volume of S as the limit of these Riemann sums :*

$$V(S) = \int_a^b A(x)\,dx$$

In conclusion, the volume of S is the *definite integral of the cross-sectional area sliced by a family of parallel planes.*

Exercise 1(a): Volume of a square pyramid

First we derive the volume of the pyramid whose base is a square of area A and whose height is h. Follow the following steps. (Hint: Textbook p. 454)
Base area A, height h.

(i) Place the pyramid in a coordinate system. To simplify the computation, we choose the origin to be ____________ of the pyramid and the x-axis to be ____________ of the pyramid.

(ii) Let P_{x_0} be the plane $x = x_0$ for some $0 < x_0 < h$. The shape of cross section $S \bigcap P_{x_0}$ is a ____________ whose area equals ____________.

(iii) Write the volume of S as firstly a Riemann sum and then a definite integral of $A(x) =$ area of $S \bigcap P_x$. Hence, compute the volume.

Exercise 1(a): Solutions

(i) Put the *origin* at the apex; take the x-*axis* along the altitude (perpendicular to the base).

(ii) $S \cap \{x = x_0\}$ is a *square*. By similarity, linear scale $= x_0/h$, so

$$A(x_0) = \left(\frac{x_0}{h}\right)^2 A = \frac{A}{h^2} x_0^2.$$

(iii) We divide the interval $[0, h]$ into n equal parts with $\Delta x = \frac{h}{n}$ and choose sample points x_i in each sub-interval. The i-th slice is approximately a thin slab of volume $A(x_i)\,\Delta x$ where $A(x_i) = \dfrac{A}{h^2} x_i^2$.

$$V_n = \sum_{i=1}^{n} A(x_i)\,\Delta x = \sum_{i=1}^{n} \frac{A}{h^2} x_i^2\,\Delta x.$$

Taking the limit as $\Delta x \to 0$ (i.e. $n \to \infty$), we obtain the definite integral:

$$V = \lim_{n\to\infty} V_n = \int_0^h \frac{A}{h^2} x^2\,dx = \frac{A}{h^2} \cdot \frac{h^3}{3} = \boxed{\tfrac{1}{3} Ah.}$$

Exercise 1(b): Volume of a circular cone

Imitate the procedure in (a) and find the volume of a circular cone whose base is a disc of area A and whose height is h.

(i) Place the cone in a coordinate system. To simplify the computation, we choose the origin to be ____________ of the cone and the x-axis to be ____________ of the cone.

(ii) Let P_{x_0} be the plane $x = x_0$ for some $0 < x_0 < h$. The shape of cross section $S \bigcap P_{x_0}$ is a ____________ whose area equals ____________.

(iii) Write the volume of S as firstly a Riemann sum and then a definite integral of $A(x) =$ area of $S \bigcap P_x$. Hence, compute the volume.

Exercise 1(b): Solutions

(i) Put the *origin* at the apex; take the *x-axis* along the altitude (perpendicular to the base).

(ii) $S \cap \{x = x_0\}$ is a *circle*. Its radius scales proportionally to x_0/h, so

$$A(x_0) = \pi\left(\frac{R}{h}x_0\right)^2 = \frac{A}{h^2}x_0^2, \quad \text{since } A = \pi R^2.$$

(iii) Divide the interval $[0, h]$ into n equal parts with $\Delta x = \frac{h}{n}$ and pick sample points x_i. Each thin slice has approximate volume $A(x_i)\,\Delta x = \frac{A}{h^2}x_i^2\Delta x$. Hence the Riemann sum is

$$V_n = \sum_{i=1}^{n} A(x_i)\,\Delta x = \sum_{i=1}^{n} \frac{A}{h^2}x_i^2\,\Delta x.$$

Taking the limit as $\Delta x \to 0$ gives

$$V = \lim_{n\to\infty} V_n = \int_0^h \frac{A}{h^2}x^2\,dx = \frac{A}{h^2}\cdot\frac{h^3}{3} = \boxed{\tfrac{1}{3}Ah.}$$

Exercise 1(c): Volume of a general cone

Actually, a pyramid or a circular cone are just special kinds of "cones." Suppose that B is a region on a plane P and O is a point not on P. A cone with base B and apex O consists of all points on line segments that join the apex O to points of B.
If the area of the base is A and the distance from the apex O to the plane P (the height of the cone) is h, can you derive the volume of the cone in terms of A and h?

(i) Place the cone in a coordinate system. Choose the origin to be ____________ of the cone and the x-axis to be ____________ of the cone.

(ii) Let P_{x_0} be the plane $x = x_0$ for some $0 < x_0 < h$. The shape of cross section $S \cap P_{x_0}$ is a ____________ whose area equals ____________.

(iii) Write the volume of S first as a Riemann sum and then as a definite integral of $A(x) =$ area of $S \cap P_x$. Hence, compute the volume.

Exercise 1(c): Solutions

(i) Put the *origin* at the apex O and take the x-*axis* along the altitude (perpendicular to the base plane P).

(ii) $S \cap \{x = x_0\}$ is a region similar to the base B. Linear dimensions scale as x_0/h, so

$$A(x_0) = \left(\frac{x_0}{h}\right)^2 A = \frac{A}{h^2} x_0^2.$$

(iii) Divide the interval $[0, h]$ into n equal parts of width $\Delta x = \frac{h}{n}$ and choose sample points x_i. Each thin slice has approximate volume $A(x_i)\,\Delta x = \frac{A}{h^2} x_i^2 \Delta x$. Thus

$$V_n = \sum_{i=1}^{n} A(x_i)\,\Delta x = \sum_{i=1}^{n} \frac{A}{h^2} x_i^2\,\Delta x.$$

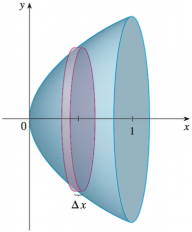
Taking the limit as $\Delta x \to 0$ gives the definite integral

$$V = \lim_{n\to\infty} V_n = \int_0^h \frac{A}{h^2} x^2\,dx = \frac{A}{h^2} \cdot \frac{h^3}{3} = \boxed{\tfrac{1}{3} Ah.}$$

Therefore, the volume of any cone—regardless of the shape of its base—is $V = \dfrac{1}{3} Ah.$

Volumes of Solids of Revolution.

We will proceed to study the volume of a special class of solids. If we revolve a plane region about a line, we obtain a **solid of revolution**. To compute the volume of a solid of revolution, we often cut the solid with planes perpendicular to the axis of revolution. Then the cross sections are just disks or annular rings. Integrating the cross section areas, we can easily obtain the volume.



Exercise 2(a): Volume of a solid of revolution

Let D be the region under $y = \sqrt{1 - x^2}$, above the x axis, between lines $x = 0$ and $x = \frac{1}{2}$. Sketch the region D.

Exercise 2(b)(i)(ii): Volume of a solid of revolution

Now we rotate the region D (under $y = \sqrt{1 - x^2}$, above the x-axis, between $x = 0$ and $x = \frac{1}{2}$) about the x-axis and obtain a solid S. Find the volume of S.

(i) Let P_{x_0} be the plane $x = x_0$ where $0 \leq x_0 \leq \frac{1}{2}$. The cross section $S \cap P_{x_0}$ is a ____________ of radius ____________ and area ____________.

(ii) Hence, express the volume of S as, firstly, a Riemann sum and then a definite integral. Use it to find the volume of S.

Exercise 2(b)(i)(ii): Solutions

(i) For each $x_0 \in [0, \frac{1}{2}]$, the cross section $S \cap \{x = x_0\}$ is a *disk* of radius $r(x_0) = \sqrt{1 - x_0^2}$ and area

$$A(x_0) = \pi r(x_0)^2 = \pi(1 - x_0^2).$$

(ii) Divide the interval $[0, \frac{1}{2}]$ into n subintervals of width $\Delta x = \frac{1}{2n}$, and let x_i be sample points. Each thin slice has approximate volume $A(x_i)\,\Delta x = \pi(1 - x_i^2)\Delta x$. The total volume is approximated by the Riemann sum

$$V_n = \sum_{i=1}^{n} A(x_i)\,\Delta x = \sum_{i=1}^{n} \pi(1 - x_i^2)\,\Delta x.$$

Taking the limit as $\Delta x \to 0$ gives

$$V = \lim_{n\to\infty} V_n = \int_0^{1/2} \pi(1 - x^2)\,dx = \pi\left[x - \frac{x^3}{3}\right]_0^{1/2} = \boxed{\frac{11\pi}{24}}.$$

Exercise 2(iii)(iv): Volume of a ball and a spherical cap

(iii) Indeed S is part of a solid ball of radius 1. Can you compute, in general, the volume of a ball with radius r?

(iv) How about the volume of a cap of a ball

$$\{(x,y,z) \mid x^2+y^2+z^2 \leq R^2,\ x \geq r\} \quad \text{where } 0 < r < R?$$

Exercise 2(iii)(iv): Solutions

(iii) Volume of a ball: Consider the sphere $x^2 + y^2 + z^2 = r^2$. At position x, the cross section is a disk of radius $\sqrt{r^2 - x^2}$ and area

$$A(x) = \pi(r^2 - x^2).$$

By the definition of volume as a Riemann sum,

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi(r^2 - x_i^2)\,\Delta x = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = \pi\left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} = \boxed{\tfrac{4}{3}\pi r^3.}$$

(iv) Volume of a spherical cap: For a sphere of radius R, cut by the plane $x = r$ $(0 < r < R)$, each cross section at x has area $A(x) = \pi(R^2 - x^2)$. Hence

$$V = \int_{r}^{R} \pi(R^2 - x^2)\,dx = \pi\left[R^2 x - \frac{x^3}{3}\right]_{r}^{R} = \boxed{\tfrac{\pi}{3}(2R^3 - 3R^2 r + r^3)}.$$

If $h = R - r$ is the cap height, equivalently

$$\boxed{V = \frac{\pi h^2(3R - h)}{3}}.$$

Exercise 3: Torus from $x^2 + (y-2)^2 \leq 1$

Let $D = \{(x, y) \mid x^2 + (y-2)^2 \leq 1\}$.

(a) Sketch the region D.

(b) By rotating D about the x-axis we obtain a donut-shaped solid S.

  (i) Let P_{x_0} be the plane $x = x_0$, $-1 \leq x_0 \leq 1$. The cross section $S \cap P_{x_0}$ is an annular ring of outer radius ______________, inner radius ______________, and area ____________________.

  (ii) Hence, express the volume of S as, firstly, a Riemann sum and then a definite integral. Use it to find the volume of S.

Exercise 3: Solutions

(a) D is the disk of radius 1 centered at $(0, 2)$ (a unit circle shifted up by 2).

(b)(i) At $x = x_0$, the vertical chord of D runs from $y = 2 - \sqrt{1 - x_0^2}$ to $y = 2 + \sqrt{1 - x_0^2}$.
Outer radius $R_{\text{out}} = 2 + \sqrt{1 - x_0^2}$, inner radius $R_{\text{in}} = 2 - \sqrt{1 - x_0^2}$.
Area of the annulus:

$$A(x_0) = \pi\left(R_{\text{out}}^2 - R_{\text{in}}^2\right) = 8\pi\sqrt{1 - x_0^2}.$$

(b)(ii) Divide $[-1, 1]$ into n subintervals of width $\Delta x = \frac{2}{n}$, choose samples x_i.
Riemann sum:

$$V_n = \sum_{i=1}^{n} A(x_i)\,\Delta x = \sum_{i=1}^{n} 8\pi\sqrt{1 - x_i^2}\,\Delta x.$$

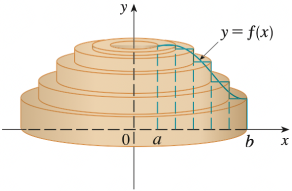
Taking the limit,

$$V = \lim_{n\to\infty} V_n = \int_{-1}^{1} 8\pi\sqrt{1 - x^2}\,dx = 8\pi \cdot \frac{\pi}{2} = \boxed{4\pi^2}.$$

(Agrees with the torus formula $V = 2\pi^2 R r^2$ for $R = 2,\ r = 1$.)

Volumes by Cylindrical Shells

There is another way to compute the volume of a solid of revolution which is called the *Method of Cylindrical Shells*. Consider a solid S obtained by rotating about the y-axis the plane region D bounded by $y = f(x) \geq 0$, $y = 0$, $x = a$, and $x = b$, where $0 \leq a < b$.



To compute its volume, we first divide D by lines $x = x_i$, where $a = x_0 < x_1 < \cdots < x_n = b$. Then the part of D between $x = x_{i-1}$ and $x = x_i$ is approximated by a rectangle $[x_{i-1}, x_i] \times [0, f(\bar{x_i})]$ where $\bar{x_i} = \dfrac{x_{i-1} + x_i}{2}$. Rotating the rectangle about the y-axis, we obtain a thin cylindrical shell with outer radius x_i, inner radius x_{i-1}, and height $f(\bar{x_i})$. Hence this cylindrical shell has volume $\pi(x_i^2 - x_{i-1}^2)f(\bar{x_i})$ which is

$$2\pi\bar{x_i}f(\bar{x_i})\Delta x_i \text{ where } \Delta x_i = x_i - x_{i-1}.$$

By adding up volumes of these cylindrical shells and taking limit, we obtain the volume of S : [a]

$$V(S) = \int_a^b 2\pi x f(x)\, dx.$$

Exercise 4: Volumes by the Cylindrical Shell Method

Let D be the region under $y = f(x) = 3x^2 - x^3$, above the x-axis, from $x = 0$ to $x = 3$.

(a) By using the cylindrical shell method, find the volume of the solid obtained by rotating D about the y-axis.

(b) Find the volume of the solid obtained by rotating D about the line $x = -3$.

Hint. Modify your work in (a): change the outer and inner radii of a cylindrical shell.

Exercise 4(a): Solutions

(a) When D is rotated about the y-axis, a typical vertical strip at x produces a thin *cylindrical shell* of radius x, height $f(x) = 3x^2 - x^3$, and thickness Δx. The approximate volume is $2\pi x f(x)\,\Delta x$. Summing over all shells gives the Riemann sum

$$V_n = \sum_{i=1}^{n} 2\pi x_i f(x_i)\,\Delta x = \sum_{i=1}^{n} 2\pi x_i (3x_i^2 - x_i^3)\,\Delta x.$$

Taking the limit, we obtain

$$V = \lim_{n\to\infty} V_n = \int_0^3 2\pi x(3x^2 - x^3)\,dx = 2\pi \int_0^3 (3x^3 - x^4)\,dx.$$

Evaluating:

$$V = 2\pi \left[\frac{3}{4}x^4 - \frac{1}{5}x^5\right]_0^3 = 2\pi\left(\frac{243}{4} - \frac{243}{5}\right) = \boxed{\frac{243\pi}{10}}.$$

Exercise 4(b): Solutions

(b) For rotation about the line $x = -3$, the shell radius becomes $(x+3)$ while the height remains $f(x)$. Hence

$$V = \int_0^3 2\pi(x+3)\, f(x)\, dx = 2\pi \int_0^3 (x+3)(3x^2 - x^3)\, dx = 2\pi \int_0^3 (9x^2 - x^4)\, dx.$$

Compute:

$$V = 2\pi \left[3x^3 - \frac{x^5}{5} \right]_0^3 = \boxed{\tfrac{324\pi}{5}}.$$

Exercise 4(c): Cylindrical shells with two curves

Let $\tilde{D}$ be the region under $y = f(x) = 3x^2 - x^3$, above $y = g(x) = -\sqrt{9 - x^2}$, from $x = 0$ to $x = 3$. Find the volume of the solid obtained by rotating $\tilde{D}$ about the y-axis.
Hint. Express the height of a cylindrical shell in x for each $0 < x < 3$.

Exercise 4(c): Solutions

For rotation about the y-axis, a vertical strip at x forms a shell of radius x, height

$$\text{height} = f(x) - g(x) = 3x^2 - x^3 + \sqrt{9 - x^2},$$

and thickness Δx.

Riemann sum. The i-th shell has volume $2\pi x_i \left[f(x_i) - g(x_i)\right] \Delta x$, so

$$V_n = \sum_{i=1}^{n} 2\pi x_i \big(3x_i^2 - x_i^3 + \sqrt{9 - x_i^2}\big) \, \Delta x.$$

Integral. Taking the limit,

$$\begin{aligned} V = \lim_{n\to\infty} V_n &= \int_0^3 2\pi x \big(3x^2 - x^3 + \sqrt{9 - x^2}\big) \, dx \\ &= 2\pi \left[\int_0^3 (3x^3 - x^4) \, dx + \int_0^3 x\sqrt{9 - x^2} \, dx \right]. \end{aligned}$$

Hence

$$V = 2\pi \left(\frac{243}{20} + 9 \right) = \boxed{\frac{423\pi}{10}}.$$