Week 6: Quiz 2 Review

Lecturer: Zhi (George) Lin

Calculus 1, Class 6

Oct 8th 2025

Summary

Quiz Reminder

Time:	10/15 17:30-18:20
Quiz Scope:	3.3-4.2
TA Hour:	By appointment
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Mean Value Theorem

Definition 6.1 (Mean Value Theorem, MVT)

Suppose f is **continuous** on the closed interval $[a, b]$ and **differentiable** on the open interval (a, b). Then there exists at least one point $c \in (a, b)$ such that

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Special Case: Rolle's Theorem

If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that

$$f'(c) = 0.$$

Problem 1 (Mean Value Theorem)

(a) Use the **Mean Value Theorem** to prove that for any $0 < a < b$,

$$\frac{\sqrt{b}-\sqrt{a}}{1+b} \leq \tan^{-1}(\sqrt{b}) - \tan^{-1}(\sqrt{a}) \leq \frac{\sqrt{b}-\sqrt{a}}{1+a}.$$

(b) Suppose c is a constant such that the limit

$$L = \lim_{x\to\infty} \frac{\tan^{-1}(\sqrt{x^3+1}) - \tan^{-1}(\sqrt{x^3-1})}{x^c}$$

is non-zero.

Find c and L.

Problem 1(a): Solutions

Let $f(x) = \tan^{-1}(x)$. Since f is differentiable everywhere, we may apply the Mean Value Theorem on $[x, y]$ with $0 < x < y$.

$$\tan^{-1}(y) - \tan^{-1}(x) = \frac{1}{1+c^2}(y-x), \quad \text{for some } x < c < y.$$

Since $x < c < y$, we have

$$\frac{1}{1+y^2} < \frac{1}{1+c^2} < \frac{1}{1+x^2}.$$

Hence,

$$\frac{y-x}{1+y^2} < \tan^{-1}(y) - \tan^{-1}(x) < \frac{y-x}{1+x^2}.$$

Now substitute $x = \sqrt{a}$ and $y = \sqrt{b}$, where $0 < a < b$, to obtain

$$\frac{\sqrt{b}-\sqrt{a}}{1+b} < \tan^{-1}(\sqrt{b}) - \tan^{-1}(\sqrt{a}) < \frac{\sqrt{b}-\sqrt{a}}{1+a}.$$

Problem 1(b): Solutions

Take $a = x^3 - 1$ and $b = x^3 + 1$ in part (a). Then

$$\begin{aligned}\frac{\sqrt{x^3+1} - \sqrt{x^3-1}}{x^3+2} &\le \tan^{-1}(\sqrt{x^3+1}) - \tan^{-1}(\sqrt{x^3-1}) \\ &\le \frac{\sqrt{x^3+1} - \sqrt{x^3-1}}{x^3}.\end{aligned}$$

Rationalizing the numerators, we get

$$\begin{aligned}\frac{2}{(x^3+2)(\sqrt{x^3+1} + \sqrt{x^3-1})} &\le \tan^{-1}(\sqrt{x^3+1}) - \tan^{-1}(\sqrt{x^3-1}) \\ &\le \frac{2}{x^3(\sqrt{x^3+1} + \sqrt{x^3-1})}.\end{aligned}$$

Multiply every term by $x^{4.5}$ (equivalently divide by $x^{-4.5}$):

$$\begin{aligned}\frac{2x^{4.5}}{(x^3+2)(\sqrt{x^3+1} + \sqrt{x^3-1})} &\le \frac{\tan^{-1}(\sqrt{x^3+1}) - \tan^{-1}(\sqrt{x^3-1})}{x^{-4.5}} \\ &\le \frac{2x^{4.5}}{x^3(\sqrt{x^3+1} + \sqrt{x^3-1})}.\end{aligned}$$

Problem 1(b): Solutions

Now compute the limits as $x \to \infty$.

$$\lim_{x\to\infty} \frac{2x^{4.5}}{x^3(\sqrt{x^3+1}+\sqrt{x^3-1})} = \frac{2}{\sqrt{1+\frac{1}{x^3}}+\sqrt{1-\frac{1}{x^3}}} = 1,$$

$$\lim_{x\to\infty} \frac{2x^{4.5}}{(x^3+2)(\sqrt{x^3+1}+\sqrt{x^3-1})} = \frac{2}{(1+\frac{2}{x^3})(\sqrt{1+\frac{1}{x^3}}+\sqrt{1-\frac{1}{x^3}})} = 1.$$

By the Squeeze Theorem,

$$\lim_{x\to\infty} \frac{\tan^{-1}(\sqrt{x^3+1})-\tan^{-1}(\sqrt{x^3-1})}{x^{-4.5}} = 1.$$

Therefore,

$$\boxed{c = -4.5 \quad \text{and} \quad L = 1.}$$

Problem 2 (Linear Approximation)

Consider the function

$$f(x) = 3x - \tan^{-1}(x-1).$$

a. Show that the equation

$$3x - \tan^{-1}(x-1) = 3.01$$

has a unique solution.

b. Let $g(x)$ be the inverse function of f. Find $g(3)$ and $g'(3)$.

c. Apply a linear approximation to g to estimate the solution of $f(x) = 3.01$.

Problem 2(a): Solutions

(a) The function f is continuous and differentiable.

$$f(0) = \frac{\pi}{4}, \quad f(1) = 3, \quad f(2) = 6 - \frac{\pi}{4}.$$

By the **Intermediate Value Theorem**,

$$f(1) < 3.01 < f(2),$$

so there exists at least one solution.
If there were two solutions, then by **Rolle's Theorem**, $f'(c) = 0$ for some c. However,

$$f'(x) = 3 - \frac{1}{1 + (x-1)^2} \geq 2,$$

which is never zero. Therefore, there is exactly **one unique solution**.

Problem 2(b)(c): Solutions

(b) For the inverse function $g(x)$,

$$g(3) = 1, \qquad g'(3) = \frac{1}{f'(1)} = \frac{1}{2}.$$

(c) Using linear approximation:

$$g(3.01) \approx 1 + \frac{1}{2}(3.01 - 3) = 1.005.$$

Problem 3 (Differentiation Skills)

For each of the following relations between x and y, find an expression for $\dfrac{dy}{dx}$ in terms of x and y.

a. $y = (3x - e^{7x})^{\cos(5x)}$.
b. $y = \dfrac{\sin(3x + x^2)}{(6 - x^4)^3}$.

Problem 3(a): Solutions

Take the natural logarithm of both sides:

$$\ln y = \ln\left[(3x - e^{7x})^{\cos(5x)}\right] = \cos(5x)\,\ln|3x - e^{7x}|.$$

Differentiate both sides with respect to x:

$$\frac{y'}{y} = -5\sin(5x)\ln|3x - e^{7x}| + \cos(5x)\frac{3 - 7e^{7x}}{3x - e^{7x}}.$$

Solve for y' and substitute back y:

$$\begin{aligned} y' &= y\left[-5\sin(5x)\ln|3x - e^{7x}| + \cos(5x)\frac{3 - 7e^{7x}}{3x - e^{7x}}\right] \\ &= \boxed{(3x - e^{7x})^{\cos(5x)}\left[-5\sin(5x)\ln|3x - e^{7x}| + \cos(5x)\frac{3 - 7e^{7x}}{3x - e^{7x}}\right].} \end{aligned}$$

Problem 3(b): Solutions

Take the natural logarithm of both sides:

$$\ln y = \ln\left[\frac{\sin(3x+x^2)}{(6-x^4)^3}\right] = \ln|\sin(3x+x^2)| - 3\ln|6-x^4|.$$

Differentiate both sides with respect to x:

$$\frac{y'}{y} = \frac{(3+2x)\cos(3x+x^2)}{\sin(3x+x^2)} + 12\frac{x^3}{6-x^4}.$$

Simplify:

$$\frac{y'}{y} = (3+2x)\cot(3x+x^2) + \frac{12x^3}{6-x^4}.$$

Substitute back y:

$$y' = y\left[(3+2x)\cot(3x+x^2) + \frac{12x^3}{6-x^4}\right].$$

$$\boxed{y' = \frac{\sin(3x+x^2)}{(6-x^4)^3}\left[(3+2x)\cot(3x+x^2) + \frac{12x^3}{6-x^4}\right].}$$

Problem 4 (Implicit Differentiation)

Find the highest and the lowest points of the curve given by

$$x^2 + xy + 2y^2 = 28.$$

1. (a) Find an expression for $\dfrac{dy}{dx}$ in terms of x and y.
2. (b) At the highest and the lowest points, what is $\dfrac{dy}{dx}$? What equation can we obtain from $\dfrac{dy}{dx}$ and part (a)?
3. (c) Using parts (a) and (b), find the coordinates of the highest and lowest points.

Problem 4(a)(b): Solutions

1. (a) Differentiate both sides with respect to x:

$$2x + y + x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0.$$

Rearrange and solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \boxed{\frac{-2x - y}{x + 4y}}.$$

2. (b) At the highest and lowest points, the slope is horizontal:

$$\frac{dy}{dx} = 0.$$

Substitute into part (a):

$$2x + y = 0.$$

Problem 4(c): Solutions

- Substitute $y = -2x$ into the curve equation:

$$x^2 + x(-2x) + 2(-2x)^2 = 28.$$

Simplify:

$$x^2 - 2x^2 + 8x^2 = 28 \Rightarrow 7x^2 = 28 \Rightarrow x = \pm 2.$$

Then $y = -2x$ gives $y = \mp 4$.

$$\boxed{\text{Highest point: } (-2, 4)}, \qquad \boxed{\text{Lowest point: } (2, -4)}.$$