Calculus I – Midterm Review

Problem 1: Limits and Continuity (I)

1. (**Rationalization**) Evaluate $\lim_{x \to 0} \dfrac{\sqrt{1+3x} - \sqrt{1+x}}{x}$.

2. (**Hidden cancellation**) Evaluate $\lim_{x \to 0} \dfrac{\sqrt{x^2+x+1} - \sqrt{x^2+1}}{x}$.

3. (**Trig identity**) Evaluate $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$.

4. (**Squeeze Theorem**) Show that $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$.

5. (**l'Hôpital's Rule**) Compute $\lim_{x \to 0} \dfrac{\ln(1+x) - \sin x}{x^2}$.

Problem 2: Limits and Continuity (II)

6. (**Series / multiple l'Hôpital**) Evaluate $\lim_{x \to 0} \frac{x - \sin x}{x^3}$.
7. (1^∞ **form – log trick**) Evaluate $\lim_{x \to 0} (\cos x)^{1/x^2}$.
8. (**Definition of e**) Evaluate $\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x}$.
9. (**Growth at infinity**) Show that $\lim_{x \to \infty} \frac{x \ln x}{e^x} = 0$.
10. (**Conjugate at infinity**) Evaluate $\lim_{x \to \infty} x(\sqrt{x^2+1} - x)$.

Problem 3: Intermediate Value Theorem (IVT)

11. (**Existence and uniqueness**) Show that the equation $x = \cos x$ has a unique solution in $(0, 1)$.

Problem 4: Mean Value Theorem (MVT)

12. (**MVT bound**) Show that $|\sin b - \sin a| \leq |b - a|$ for all real a, b.

13. (**Specific c value**) Show that there exists $c \in (0, 1)$ such that $e^c = e - 1$.

14. (**Classical inequality**) Show that $\ln(1 + x) \leq x$ for all $x > -1$.

Problem 5: Differentiation Techniques (I)

15. (**Log differentiation**) Compute $\dfrac{d}{dx}(x^x)$ for $x > 0$.

16. (**Chain + product + quotient**) Differentiate $f(x) = \dfrac{x^2 \ln\bigl(1 + x^3\bigr)}{\sin(2x)}$ where defined.

17. (**Log of a trig ratio**) Differentiate $g(x) = \ln\left(\dfrac{\sin(3x)}{x^2 + 1}\right)$ where defined.

18. (**Inverse trig inside a root**) Differentiate $h(x) = \arcsin\bigl(\sqrt{1 - x}\bigr)$ on $(0, 1)$.

19. Differentiate $F(x) = \arctan\left(\dfrac{2x}{1 - x^2}\right)$ where defined.

Problem 6: Linear Approximation (Adavnced)

20. (**Nonlinear Composition**) Let $f(x) = \sqrt{1 + \sin(2x)}$. Use the linearization of f at $a = 0$ to approximate $f(0.1)$. Then, estimate the maximum possible error using the second derivative of f.

21. (**Nested Logarithm–Exponential**) Let $f(x) = \ln\bigl(1 + e^{2x}\bigr)$. (a) Find the linearization $L(x)$ at $a = 0$. (b) Use $L(x)$ to approximate $f(0.2)$. (c) Provide an upper bound on the approximation error using $|R_2(x)| \le \frac{M}{2}|x - a|^2$, where M is the maximum of $|f''(x)|$ on $[0, 0.2]$.

22. (**Multi-step Linearization**) Consider $f(x) = (3x - e^{7x})^{1/5}$. (a) Compute the linearization $L(x)$ at $a = 0$. (b) Use it to approximate $f(0.05)$. (c) Discuss whether this linear approximation overestimates or underestimates the true value, and justify your answer using the sign of $f''(x)$ at $a = 0$.