MATH-249 Project Proposal

Topic: Birkhoff Polytopes and its Ehrhart polynomial

Guiding questions, scope, and background:

This is most likely to be a survey paper but it's good to have guiding problems and sub-topics. The first two are related and the third is independent and a minor topic compared to the first two.

- This is the main question, it's been conjectured that *Birkhoff polytopes show Ehrhart positivity*. Based on the study of the first nine Birkhoff polytopes, all roots seem to have negative parts and follow a pattern based on experimental data from *COEFFICIENTS AND ROOTS OF EHRHART POLYNOMIALS*. Thus it's promising to attack this conjecture by studying the roots of the Ehrhart polynomial of Birkhoff polytopes. Which current results can we apply to find the distribution of the roots? Can we find or produce more data, conjecture about the distribution, and try to prove it?
- A wide-open problem concerns the volumes of these polytopes, in The Ehrhart polynomial of the Birkhoff polytope, the authors gave a complex analysis way of computing the Ehrhart polynomials, but it becomes almost computationally impossible once $n \geq 11$, can we find a formula for the volume of Birkhoff polytope that's *fast* as an algorithm? It could be precise or asymptotic. Coming up with something new can be hard, can we modify the current algorithm or formula to make it more efficient?
- Study Birkhoff-von Neumann decomposition of a doubly stochastic matrix, such as the upper and lower bounds for the number of permutation matrices that take part in the decomposition of a given doubly stochastic matrix, and runtime analysis. This part should be purely reading one or two papers and summarizing results.

Weekly Timeline:

- 4.3 - 4.9: read papers on Ehrhart theory to understand the current state of the main question and more generally, summarize techniques and facts we have about the roots of Ehrhart polynomial, such as norm bounds and theorems that can be used to determine the root distribution. Make a list of tools, theorems, or approaches that may be useful in attacking the main question.
- 4.10 - 4.16: read papers on the volume of Birkhoff polytopes and its Ehrhart polynomial, again, summarize, and make notes.

- 4.17 - 4.23: Use everything learned from reading to tackle the first two, by the end of this week, I should have understood the difficulties that can arise and had several ideas that may work such as guesses on the distribution of the roots.
- 4.24 - 4.30: keep the work from last week, this time review some of the tools we learned from class and see if any one of them can be useful. Again, read papers on the third topic and summarize.
- 5.1 - 5.7: Write a summary of the progress, which could be a survey of my understanding of the subject or a paper on the progress I make. By the start of May, I should have known the background, significance, and state-of-art of all three topics/questions mentioned before.

Sources:

There are sources more than these I found, I'll include a more comprehensive list in the final write-up paper.

1. *A Generating Function for all Semi-Magic Squares and the Volume of the Birkhoff Polytope*
2. *The Ehrhart polynomial of the Birkhoff polytope*
3. *RESTRICTED BIRKHOFF POLYTOPES AND EHRHART PERIOD COLLAPSE*
4. *EHRHART POLYNOMIALS, SIMPLICIAL POLYTOPES, MAGIC SQUARES AND A CONJECTURE OF STANLEY*
5. *COEFFICIENTS AND ROOTS OF EHRHART POLYNOMIALS*
6. Notes on Birkhoff–von Neumann decomposition of doubly stochastic matrices

Responsibilities:

I am working alone on this project and will do my best to stick to the timeline.