Stochastic Process

sup (pun intended)

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3.1 Basics and Existence

The upshot: Brownian paths are Holder continuous. With probability one, Brownian paths are not Lipschitz continuous (and hence not differentiable) at any point.

3.2 Strong Markov Property

Same definition for the stopping time as in martingales. The related σ-field is defined the same as well. We always denote BM by B_t and stopping time by T or S unless otherwise specified.

Example 3.1First hitting time

$$T_a = \inf\{t \geq 0 : B_t = a\}$$

is a stopping time, where a is non-negative

but:

Example 3.2

$$T_a = \sup\{t \geq 0 : B_t = a\}$$

is NOT a stopping time, the intuitive explanation is that it makes use of info after time t by requiring the motion to not hit a after time t.

What's the relation between $\mathcal{F}_T$ and BM?

Proposition 3.1

$$B_s 1_{\{s \leq t\}} \in \mathcal{F}_T$$

♠

A more important construction would be the following. Again, we've seen this type of generalization $t \to T$.

Proposition 3.2

B_T is $\mathcal{F}_T$-measurable, it is defined to be 0 on $\{T = \infty\}$. ♠

Proof Classic approximation:

$$B_T = \lim_{n \to \infty} \sum_{i=-1}^{\infty} 1_{\{\frac{i}{2^n} \leq T < \frac{i+1}{2^n}\}} B_{\frac{i}{2^n}}$$

.

Remark We can have a sequence of stopping times $\{T_n\}$ decreasing monotonously to T, with a similar approximation as above. We'll use this in the proof of SMP and other occasions along with typical theory-building tools such as DCT and MCT.

Now we are ready to state SMP.

Theorem 3.1 (Strong Markov Property)

Let T be a stopping time, assume $P(T < +\infty) > 0$, $\forall t \geq 0$:

$$B_t^{(T)} = 1_{\{T<+\infty\}}(B_{T+t} - B_T)$$

then $\{B_t^{(T)}|t \geq 0\}$ *is a BM independent of* $\mathcal{F}_T$ *(under* $\mathbb{P}(\cdot|T < \infty)$*). Furthermore,*

$$\mathbb{E}[f(B_{T+t})1_{\{T<+\infty\}}|\mathcal{F}_T] = \mathbb{E}[f(x+B_t)]|_{x=B_T} \cdot 1_{\{T<+\infty\}}$$

when T *is essentially finite:*

$$\mathbb{E}[f(B_{T+t})|\mathcal{F}_T] = \mathbb{E}[f(x+B_t)]|_{x=B_T}$$

♡

Proof For the first part, it suffices to prove that the finite-dimensional distribution of $\{B_t^{(T)}|t \geq 0\}$ is the same as the expected BM, so it boils down to computing an expectation. Do that using $T_n \to T$ from the last remark.

Then use monotone class theorem and the expectation identity above to prove that:

$$\mathbb{E}[F(B_T, B_t^{(T)})1_{\{T<+\infty\}}] = \mathbb{E}[1_{\{T<+\infty\}}\mathbb{E}[F(x,B_t)]|_{x=B_T}]$$

which makes SMP obvious.

A lot of theorems involving stopping times are proven by discretizing T first, then taking the limit (usually of the expectations). Equivalently, breaking up the sample space into a countable disjoint union, where T_n is constant on each piece.

Another formulation of the weaker Markov property given in *Durrett* is:

Theorem 3.2

If $s \geq 0$ *and* Y *is bounded and* $\mathcal{C}$*-measurable, then for all* $x \in \mathbb{R}^d$*,*

$$\mathbb{E}_x(Y \circ \theta_s|\mathcal{F}_{s+}) = \mathbb{E}_{B_s}[Y]$$

♡

you have to go to *Durrett* to demystify the notations.

Theorem 3.3 (Reflection Principle)

♡

Next, we study $S_t = \sup_{0 \leq s \leq t} B_s$. It has a surprisingly neat law.

Proposition 3.3

$\forall a \geq 0, b \in (-\infty, a]$,

$$P(S_t \geq a, B_t \leq b) = P(B_t \geq 2a - b)$$

Moreover, the law of S_t *is the same as* $|B_t|$*.*

♠

Proof Draw the path and use reflection principle.

Corollary 3.1

T_a *and* $\frac{a^2}{B_1^2}$ *have the same law/distribution, where* $a \neq 0$*. Moreover,* $T_0 = 0$*.*

♡

Proof Translate from T_a to S_t, to $|B_t|$, to $\sqrt{t}|B_1|$, finally to tB_1^2. We end up with,

$$P(T_a > t) = P(\frac{a^2}{B_1^2} > t)$$

The T_0 case makes use of the fact that the zero set of a BM has no isolated points, thus $T_0 = \inf\{t > 0 : B_t = 0\} = \inf\{t \geq 0 : B_t = 0\} = 0$.

Chapter 4 Continuous Time Martingale

4.1 Basics

We start off by defining a different kind of continuous filtration.

Definition 4.1

$$\mathcal{F}_{s+} = \bigcap_{t>s} \mathcal{F}_t$$

We say that $\mathcal{F}_t$ is right-continuous at time $t \geq 0$ if $\mathcal{F}_{t+} = \mathcal{F}_t$.
$\{\mathcal{F}_t\}$ is said to be complete if all negligible sets (not necessarily measurable in $\mathcal{F}$) are in $\mathcal{F}_0$. ♣

Of course, we can take the completion of a filtration by throwing in all the negligible sets. So the *usual condition* means *completeness* and *right-continuity*.

Definition 3.10 An adapted real-valued process $(X_t)_{t\geq 0}$ such that $X_t \in L^1$ for every $t \geq 0$ is called

- a *martingale* if, for every $0 \leq s < t$, $E[X_t \mid \mathscr{F}_s] = X_s$;
- a *supermartingale* if, for every $0 \leq s < t$, $E[X_t \mid \mathscr{F}_s] \leq X_s$;
- a *submartingale* if, for every $0 \leq s < t$, $E[X_t \mid \mathscr{F}_s] \geq X_s$.

Figure 4.1: Continuous martingale

The most frequently used continuous martingales include: $B_t, B_t^2 - t, exp(\theta B_t - \frac{\theta^2 t}{2})$. The problem is, a random continuous martingale is not always right-continuous! Surely, we seek its modifications below.

Proposition 3.15

(i) (Maximal inequality) *Let $(X_t)_{t\geq 0}$ be a supermartingale with right-continuous sample paths. Then, for every $t > 0$ and every $\lambda > 0$,*

$$\lambda\, P\Big(\sup_{0\leq s\leq t} |X_s| > \lambda\Big) \leq E[|X_0|] + 2E[|X_t|].$$

(ii) (Doob's inequality in L^p) *Let $(X_t)_{t\geq 0}$ be a martingale with right-continuous sample paths. Then, for every $t > 0$ and every $p > 1$,*

$$E\Big[\sup_{0\leq s\leq t} |X_s|^p\Big] \leq \Big(\frac{p}{p-1}\Big)^p E[|X_t|^p].$$

Note that part (ii) of the proposition is useful only if $E[|X_t|^p] < \infty$.

Figure 4.2: Maximal inequality for continuous martingale

Definition 4.2 (up-crossing number)

$$M^f_{ab}(I)$$

♣

Lemma 4.1

$f : \mathbb{Q} \to \mathbb{R}$, assume that $\forall t \in \mathbb{Q}$:

- *f is bounded on $\mathbb{Q} \cap [0, t]$.*
- *$M^f_{ab}(\mathbb{Q} \cap [0, t]) < \infty$, for $\forall a < b$.*

Then both the right and left limits of f exist and both can be achieved by taking limits in the rationals. ♡

Remark We can take $g(t) = f(t+)$ to right-continuify a function like f, serving as a pathway from discrete martingale to continuous martingale.

So we want to control the up-crossing number. We can verify that for almost every $w \in \Omega$, $X(w)$ is a path satisfying the conditions in the last lemma (by using 4.2 in the proof).

Theorem 4.1

Assume that $\mathscr{F}_t$ is right-continuous and complete, let X be a supermartingale s.t. $t \to \mathbb{E}[X_t]$ is right continuous, then X has a right continuous modification $\bar{X}$ which is a $\mathscr{F}_t$-supermartingale. Modification means that $\forall t \geq 0, \mathbb{P}(X_t = \overline{X_t}) = 1$. ♡

Remark if X is a martingale, then $\overline{X}$ is also a martingale.

Actually, the modification is *continue à droite avec des limites à gauche*. The upshot is quite simple: it's safe to assume right continuity.

4.2 Optional Stopping Time

We assume right continuity throughout the rest of this chapter. The leading question in this section is: *When does $\mathbb{E}[X_T] = \mathbb{E}[X_0]$ hold?* We call the limit of a uniformly integrable martingale (*UIM*) Y_t by Y_∞.

Theorem 4.2 (Discrete Doob stopping time)

Let Y_n be a UIM, $M \leq N$ be stopping times, then

$$Y_M = \mathbb{E}[Y_N | \mathscr{F}_M]$$

♡

Proof

$$\mathbb{E}[Y_T] = \sum \mathbb{E}[Y_n | T = n] = \sum \mathbb{E}[\mathbb{E}[Y_\infty | \mathscr{F}_n] 1_{T=n}] = \sum \mathbb{E}[Y_\infty 1_{T=n}]$$

which then evaluates to

$$\mathbb{E}[Y_\infty] = \mathbb{E}[Y_0] = \mathbb{E}[Y_T] \tag{4.1}$$

Now we're ready to verify

$$\mathbb{E}[Y_M 1_A] = \mathbb{E}[Y_N 1_A]$$

Then define $T = M 1_A + N 1^c_A$, verify that T is a stopping time and then use identity 4.1.

Then the main actor in this section.

Theorem 4.3 (Continuous Doob stopping time)

Let X_t be a path-right-continuous UIM, $S \leq T$ be stopping times, then

$$X_S = \mathbb{E}[X_T|\mathscr{F}_S]; X_S, X_T \in L_1$$

In particular,

$$\mathbb{E}[X_\infty] = \mathbb{E}[X_0] = \mathbb{E}[X_T]$$

and

$$X_S = \mathbb{E}[X_\infty|\mathscr{F}_S]$$

Proof Discretize X with $Y_k = X_{\frac{k}{2^n}}$, S and T with their usual discrete counterparts that decrease to them (see the remark of 3.2). We then use thm 4.2 on these objects, so

$$\mathbb{E}[X_{T_n} 1_A] = \mathbb{E}[X_{S_n} 1_A]$$

Now we want L_1 convergence, which is given by a.s. convergence and the following uniform bound

$$sup\mathbb{E}[|X_{S_n}|] \leq \mathbb{E}[|X_\infty|] < \infty$$

because *UIM* ensures that X_∞ is in L_1. Finally, take the limit, proving that $X_{S_n} \to X_S \in L_1$ and $\mathbb{E}[X_T 1_A] = \mathbb{E}[X_S 1_A]$.

Corollary 4.1 (Continuous Doob stopping time without *UIM* condition)

X_t is path-right-continuous, not UIM, but S and T are now bounded. Then

$$X_S = \mathbb{E}[X_T|\mathscr{F}_S]$$

Proof Let a be a common bound, stop X_t with time a, this new martingale is *UIM* and then apply 4.3.

Corollary 4.2

X_t is path-right-continuous and UIM, T is a stopping time. Then

$$X_{t\wedge T} = \mathbb{E}[X_T|\mathscr{F}_t]$$

Namely, $Y_t = X_{t\wedge T}$ is closed by X_T, and UIM is preserved.

Proof

$$\mathbb{E}[X_{t\wedge T} 1_A] = \mathbb{E}[X_t 1_{A\cap\{t\leq T\}}] + \mathbb{E}[X_T 1_{A\cap\{T\leq t\}}]$$

$$\mathbb{E}[X_T 1_A] = \mathbb{E}[X_T 1_{A\cap\{t\leq T\}}] + \mathbb{E}[X_T 1_{A\cap\{T\leq t\}}]$$

it suffices to show

$$\mathbb{E}[X_t 1_{A\cap\{t\leq T\}}] = \mathbb{E}[X_T 1_{A\cap\{t\leq T\}}] \tag{4.2}$$

notice that $A\cap\{t\leq T\} \in \mathscr{F}_t \cap \mathscr{F}_T = \mathscr{F}_{t\wedge T}$ and by thm 4.3, we have

$$X_{t\wedge T} = \mathbb{E}[X_T|\mathscr{F}_{t\wedge T}]$$

which proves 4.2.

We have an arsenal now, so let's shoot some birds.

Example 4.1 Let B_t be a BM starting from 0, set $a < 0 < b$, T_a and T_b be first hitting time resp. Lastly, let $T = T_a \wedge T_b$. We're interested in:

- $\mathbb{P}(T_a < T_b)$
- $\mathbb{E}[T], \mathbb{E}[T_a], \mathbb{E}[T_b]$

$B_{t\wedge T}$ is bounded, thus by DCT

$$\mathbb{E}[B_T] = \mathbb{E}[B_0] = 0 = a\mathbb{P}(T_a < T_b) + b\mathbb{P}(T_a > T_b)$$

we can obtain $\mathbb{P}(T_a < T_b)$ along with

$$\mathbb{P}(T_a < T_b) + \mathbb{P}(T_a > T_b) = 1$$

Same thing for the second part, but use $B_t^2 - t$ this time and stop it with T.

$$\mathbb{E}[T] = \lim_{t\to\infty} \mathbb{E}[B_{t\wedge T}^2] = \mathbb{E}[B_T^2] = -ab$$

Maybe a bit surprisingly, $\mathbb{E}[T_a] = \infty$, proven by taking $b \to \infty$ in $\mathbb{E}[T]$.

We finish this section a stopping time theorem for supermartingales.

Theorem 4.4

Let X_t be a nonnegative supermartingale, $S \leq T$, then

$$X_S \geq \mathbb{E}[X_T|\mathscr{F}_S]; X_S, X_T \in L_1$$

♡

Proof

$$sup\mathbb{E}[|X_t|] = sup\mathbb{E}[X_t] \leq \mathbb{E}[X_0] < \infty$$

i.e. uniformly bounded in L_1, so X_t converges $X_\infty \in L_1$ a.s.

Step 1: assume that $S \leq T \leq a$, then use the same old S_n and T_n, since $sup\mathbb{E}[|X_{S_n}|] < \infty$, we have convergence in L_1 and take the limit on the discrete version $\mathbb{E}[|X_{S_n}|] \geq \mathbb{E}[|X_{T_n}|]$, which produces

$$\mathbb{E}[|X_S|] \geq \mathbb{E}[|X_T|]$$

Step 2: without boundedness, by Fatou's lemma

$$\mathbb{E}[|X_S|] \leq \mathbb{E}[\mathit{liminf}|X_{S\wedge a}|] \leq \mathit{liminf}\mathbb{E}[|X_{S\wedge a}|] \leq \mathbb{E}[|X_0|]$$

So $X_S, X_T \in L_1$. With a similar T as in 4.2, we have

$$\mathbb{E}[|X_{S\wedge a}|1_A] \geq \mathbb{E}[|X_{T\wedge a}|1_A]$$

which then produces

$$\mathbb{E}[|X_S|1_{A\cap\{S\leq a\}}] \geq \mathbb{E}[|X_{T\wedge a}|1_{A\cap\{S\leq a\}}]$$

treat LHS with DCT and RHS with Fatou's lemma, then it's done.

Chapter 5 Continuous Markov Processes

We discuss Continuous Markov Processes in this chapter.

5.1 Basics

Let (E, ϵ) be a measure space, which will be the value space, here ϵ is a σ-algebra.

Definition 5.1 (Markov process)

An E-valued Markov process satisfies:

- $X_t \in \mathscr{F}_t, \forall t \geq 0$.
- *for any bounded measurable function f on E, $s < t$*

$$\mathbb{E}[f(X_t)|\mathscr{F}_s] = \mathbb{E}[f(X_t)|\sigma(X_s)]$$

♣

One can replace the filtration in the definition with $\sigma(X_s, s \leq t)$, that's the more usual definition.

Definition 5.2 (transition kernel)

If Q is a mapping $E \times \epsilon \to [0, 1]$ s.t.

- *fix any $x \in E$, $Q(x, \cdot)$ is a probability on (E, ϵ).*
- *fix any $A \in \epsilon$, $Q(\cdot, A)$ is a ϵ-measurable function .*

then Q is a transition kernel. (from E to ϵ)

♣

Definition 5.3 (transition semi-group)

A collection of transition kernels Q_t s.t.

- *$Q_0(x, \cdot)$ is the dirac measure δ_x.*
- *$Q_{t+s}(x, A) = \int_E Q_t(y, A)Q_s(x, dy)$. (CK)*
- *$(t, x) \to Q_t(x, A)$ is measurable for any fixed A.*

is called a transition semi-group.

♣

Remark how to make sense of (CK)? Define a bounded linear op Q_t on $B_b(E)$:

$$Q_t(f)(x) = \int_E f(y)Q_t(x, \cdot)$$

then (CK) essentially says: $Q_{s+t}f = Q_t(Q_s f)$, which is also where the name *semi-group* comes from.

Now we're interested in processes that have these *transition semi-groups*.

Definition 5.4 (Markov process with transition semi-group)

the additional condition is $\forall f \in B_b(E)$

$$\mathbb{E}[f(X_{s+t})|\mathscr{F}_t] = \mathbb{Q}_s(f)(X_t)$$

♣

One should verify that this condition implies the second point in the original definition. Moreover, for any time-homogeneous Markov process X, we can construct $Q_t(x, A) = \mathbb{E}[1_A(X_{s+t})|X_s = x]$ (need to show it's independent in s), then X is a Markov process with transition semi-group.

Next, we compute the finite dimensional distribution of X_t. $0 \leq t_0 < t_1 < \cdots < t_p$, consider

$(X_{t_0}, \cdots, X_{t_p})$ and suppose the law of X_0 is γ, then we claim

$$\mathbb{P}(X_{t_0} \in A_0, \cdots, X_{t_p} \in A_p) = \int \gamma(dx_0) \int_{A_0} Q_{t_0}(x_0, dx_1) \cdots \int_{A_p} Q_{t_p - t_{p-1}}(x_p, dx_{p-1}) \tag{5.1}$$

read this chain of integrals from right to left. Note that by definition $Q_t(X_0, A) = Q_t 1_A(X)$, so $Q_t(X_0, A) = \mathbb{E}[1_A(X_t)|\sigma(X_0)]$, which implies the $p = 0$ case, then do induction to prove 5.1.

Remark Suppose Q_t is a transition semi-group, then 5.1 is a sufficient and necessary condition for X_t to be a Markov process with Q_t.

Example 5.1 Let

$$Q_t(x, dy) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x-y|^2}{2t}} dy$$

be a Gaussian measure with mean x and standard deviation t. Then *B.M.* is a $\mathbb{R}$-valued Markov process with Q_t, this can be proven by checking its finite dimensional distribution and the last remark.

We can consider the converse.

Aim: given a transition semi-group, find a probability triplet and an E-valued sto process X_t s.t. X_t is a Markov process with Q_t.

Construction: define

$$\Omega = \{f | f : \mathbb{R}^+ \to E\}$$

let X_t be the coordinate process $X_t(w) = w(t)$, with $\mathscr{F}_t = \sigma(X_s, s \le t)$. Given some discrete time points $\{t_0, \cdots, t_p\}$ and a distribution γ on (E, ϵ), let $\mu_\gamma^I(A_0 \times \cdots \times A_p)$ be the RHS of 5.1, where μ_γ^I is a measure on (E^I, ϵ). We then verify the consistency condition on μ_γ^I, and get a probability measure on $(E^{R^+}, \mathscr{F}_\infty) = (\Omega, \mathscr{F}_\infty)$.

If X is a fixed $x \in E$, denote $X_t = X_t^x$. Recall that

$$Q_t f(x) = \int_E f(y) Q_t(x, dy)$$

The transition kernels are related to X_t via finite distribution 5.1. Since the law of X_t^x equals $Q_t(x, \cdot)$, one can check

$$Q_t f(x) = \mathbb{E}[f(X_t^x)]$$

Then we switch to discuss some related quantities from semi-group/functional analysis theory.

Definition 5.5 (Resolvant)

Let $\lambda > 0$, define a linear operator $R_\lambda : B_b(E) \to B_b(E)$ with

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda t} Q_t f(x) dx$$

♣

A few simple facts on the resolvant:

- R_λ is bounded, $||R_\lambda|| \le \frac{1}{\lambda}$.
- if $0 \le f \le 1$, $0 \le \lambda R_\lambda \le 1$.
- If $\lambda, \mu > 0$, we have this identity $R_\lambda f - R_\mu f + (\lambda - \mu) R_\lambda R_\mu f = 0$.

The following lemma makes use of resolvant and connect continuous Markov process (M.P.) with martingales, which will later be used to understand the *paths* of a M.P..

Lemma 5.1

Let X_t be a M.P. with Q_t, $h \in B_b(E)$, $h \ge 0$, $\lambda > 0$, then $Y_t = e^{-\lambda t} R_\lambda h(X_t)$ is a supermartingale. ♡

Proof

$$\begin{aligned}
\mathbb{E}[Y_{s+t}|\mathscr{F}_s] &= e^{-(s+t)\lambda}\mathbb{E}[R_\lambda h(X_{t+s})|\mathscr{F}_s] \\
&= e^{-(s+t)\lambda}Q_t R_\lambda h(X_s) \\
&= e^{-(s+t)\lambda}Q_t \int_0^\infty e^{-\lambda s}Q_s h(x)ds \\
&= e^{-(s+t)\lambda}\int_0^\infty e^{-\lambda s}Q_{s+t}h(x)ds \\
&= e^{-(s+t)\lambda}\int_t^\infty e^{-\lambda\nu+\lambda t}Q_\nu h(x)\nu \\
&\le e^{-(s+t)\lambda}\int_0^\infty e^{-\lambda\nu+\lambda t}Q_\nu h(x)\nu \\
&= e^{-(s+t)\lambda}e^{\lambda t}R_\lambda h(x) \\
&= e^{-s\lambda}R_\lambda h(x)
\end{aligned}$$

We'd love for $R_\lambda h(x)$ to continuous, so we can pass around continuity later. This motivates the following definition.

Definition 5.6 (Feller's semi-group)

Let E be a locally compact metric space, let $f \in \mathcal{C}_0(E)$, i.e. f vanishes at infinity. We say that Q_t is a Feller semi-group if

- $\forall f \in \mathcal{C}_0(E)$, *we have* $Q_t f \in \mathcal{C}_0(E), \forall t \geq 0$.
- $\forall f \in \mathcal{C}_0(E)$, $||Q_t f - f|| \to 0$ *as* $t \to 0$.

♣

Example 5.2 the transition semi-group of a B.M. is a Feller semi-group. Again, let

$$Q_t(x, dy) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{|x-y|^2}{2t}}dy$$

then $Q_t f$ is a convolution

$$Q_t f(x) = \int \frac{1}{\sqrt{2\pi t}}e^{-\frac{|y|^2}{2t}}f(x-y)dy$$

since $\frac{1}{\sqrt{2\pi t}}e^{-\frac{|y|^2}{2t}}$ is concentrated around zero, the convolution can be decomposed into deux parts as per usual. To see why the second point holds, notice that the measure $\frac{1}{\sqrt{2\pi t}}e^{-\frac{|x-y|^2}{2t}}dy \to \delta_x$ as $t \to \infty$. More concretely,

$$Q_t f(x) - f(x) = \int \frac{1}{\sqrt{2\pi t}}e^{-\frac{|y|^2}{2t}}(f(x-y) - f(x))dy$$

since $Q_t f \in \mathcal{C}_0(E)$, $\sup_t \sup_{|x|>R_0}|Q_t f(x)| \leq \epsilon$, for sone R_0. It suffices to prove

$$\lim_{t\to 0}\sup_{|x|\leq R_0}|Q_t f(x) - f(x)| = 0$$

then it's basic

$$\begin{aligned}
|Q_t f(x) - f(x)| &= |\int \frac{1}{\sqrt{2\pi t}}e^{-\frac{|y|^2}{2t}}(f(x-y) - f(x))dy| \\
&= |\int_{[-\delta,\delta]} \frac{1}{\sqrt{2\pi t}}e^{-\frac{|y|^2}{2t}}(f(x-y) - f(x))dy + \int_{|y|\geq\delta} \frac{1}{\sqrt{2\pi t}}e^{-\frac{|y|^2}{2t}}(f(x-y) - f(x))dy| \\
&\leq \epsilon + 2||f||\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}}e^{-\frac{|y|^2}{2t}}dy = \epsilon + Ce^{-\frac{\delta^2}{4t}}
\end{aligned}$$

which tends to zero.

Recall that Q_t is a compression operator so the map $t \to Q_t f$ given $f \in \mathcal{C}_0(E)$ is uniformly continuous.

Especially $\int_0^T e^{-\lambda t} Q_t f(x) dx = P_T f \in \mathcal{C}_0(E)$, $P_T f \to R_\lambda f$ in sup norm, thus by completeness, $R_\lambda f \in \mathcal{C}_0(E)$, $R_\lambda : \mathcal{C}_0(E) \to \mathcal{C}_0(E)$.

So far, we haven't established the continuity of R_λ, but at least it's mapping spaces correctly.

Proposition 5.1

Let $D = \{R_\lambda f : f \in \mathcal{C}_0(E)\}$, then

- *D is independent of λ,*
- *D is dense.*

Proof use this identity $R_\lambda f - R_\mu f + (\lambda - \mu) R_\lambda R_\mu f = 0$ for the first point, and $||Q_t f - f|| \to 0$ for the second point.

I don't know why but we gotta introduce a different thingy here.

Definition 5.7

$$D(L) = \{f \in \mathcal{C}_0(E) : \lim_{t \to 0} \frac{Q_t f - f}{t} \text{exists in } \mathcal{C}_0(E)\}$$

and $Lf = \lim_{t \to 0} \frac{Q_t f - f}{t}$ defined on $D(L)$.

It's easy to see that Q_t maps $D(L)$ to itself, and $LQ_t f = Q_t L f$, namely $\frac{dQ_t f}{dt} = Q_t(Lf)$.

Proposition 5.2

$$Q_t f - f = \int_0^t Q_s(Lf) ds$$

Proof $\frac{dQ_t f}{dt} = Q_t(Lf)$.

Proposition 5.3

- *$R_\lambda f \in D(L)$, and $D(L)$ is dense in $\mathcal{C}_0$.*
- *$R_\lambda(\lambda - L) f = f$ and $(\lambda - L) R_\lambda f = f$. Notice they are defined on different subspace, so it doesn't mean R_λ and $\lambda - L$ are commutative, but we do denote $R_\lambda = (\lambda - L)^{-1}$.*

Proof omitted.