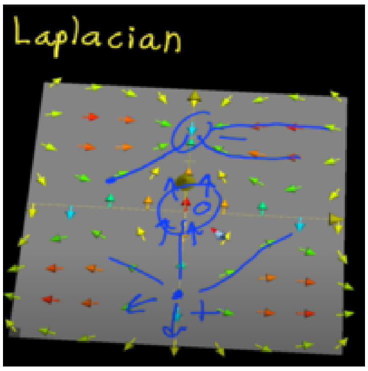
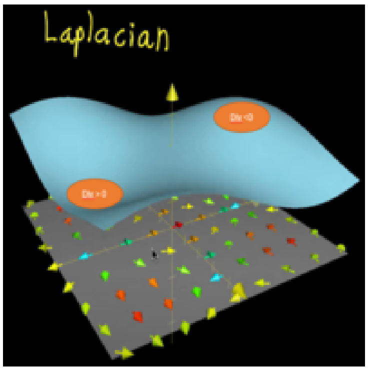
Intuition of the math behind Laplacian & Laplacian Matrix

Overview

Laplacian of a function, f is defined as:

$$\Delta f = div(grad(f)) = \nabla \cdot \nabla f$$

Vector field	Topology
	

Referring left image, **positive** divergence diverges **out**, whereas **negative** divergence **converges** to a point.
Referring right image, a **positive** divergence appears as **valley** in topology, whereas **negative** divergence appears as **mountaintop**.

Laplacian Graph

Laplacian kinda measures how **"smooth"** the function is over its domain.
On graphs, a smooth function:

> connected vertices -> changes slightly
> unconnected vertices -> changes significantly

Therefore, representation in mathematical way:

$$\sum_{u,v} w_{uv}(f(u) - f(v))^2$$

> u, v are the vertices
> w is the weight of the edge between node u and v

More formally:.

$$\frac{1}{2}\sum_{u,v} w_{uv}(f(u)-f(v))^2 = f^T L f$$

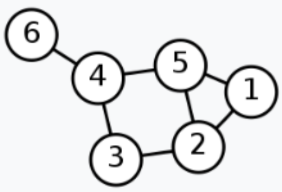
- **Minimizing** the equation's *left* part = minimizing **distance** between **connected**(neighbouring) nodes
- **Minimizing** the equation's *right* part = eigenvectors the the Laplacian （Refer PCA note）

Laplacian Matrix

Given L as Laplacian Matrix, D as Degree Matrix, W as adjacency matrix

$$L = D - W$$

Example:

Labelled graph	Degree matrix	Adjacency matrix	Laplacian matrix
	$\begin{pmatrix} 2&0&0&0&0&0\\ 0&3&0&0&0&0\\ 0&0&2&0&0&0\\ 0&0&0&3&0&0\\ 0&0&0&0&3&0\\ 0&0&0&0&0&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&0&0&1&0\\ 1&0&1&0&1&0\\ 0&1&0&1&0&0\\ 0&0&1&0&1&1\\ 1&1&0&1&0&0\\ 0&0&0&1&0&0 \end{pmatrix}$	$\begin{pmatrix} 2&-1&0&0&-1&0\\ -1&3&-1&0&-1&0\\ 0&-1&2&-1&0&0\\ 0&0&-1&3&-1&-1\\ -1&-1&0&-1&3&0\\ 0&0&0&-1&0&1 \end{pmatrix}$

As show in table, in Laplacian Matrix,

> diagonal value = degree of a node
> sum of number of "-1" in row/column = degree of the node

Reference

- Quora-Laplacian Matrix Intuition
- Youtube-Khan Academy
- 2.3.2 Laplacian Introduction

Author

Author: Lee Zhicheng
Date: 30/04/2020