Stanford University
EE 102A: Signal Processing and Linear Systems I
Summer 2022
Instructor: Ethan M. Liang

Homework 5 Solutions, due Friday, July 29

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CT or DT LTI System Analysis

1. **(8 points)** *Eigenfunctions of CT or DT LTI Systems.* As discussed in lecture, the signals e^{st}, s complex, $-\infty < t < \infty$, are eigenfunctions of all CT LTI systems for which the integral defining $H(s)$ exists. Likewise, the signals z^n, z complex, $-\infty < n < \infty$, are eigenfunctions of all DT LTI systems for which the sum defining $H(z)$ exists.

   We can also show that the e^{st} or z^n are the *only* signals that can be eigenfunctions of all CT or DT LTI systems, respectively. Prove this for the CT case. *Hint*: Identify a simple LTI system for which the e^{st} are the only eigenfunctions. It is a system we studied in lecture on several occasions.

   Solution

   Consider a differentiator defined by

$$H\left[x(t)\right] = y(t) = \frac{dx}{dt}.$$

   The eigenfunction property requires that

$$H\left[x(t)\right] = \frac{dx}{dt} = \lambda x(t),$$

   where the eigenvalue λ is a complex constant. This is a first-order linear constant-coefficient differential equation, which has a unique solution

$$x(t) = e^{\lambda t}.$$

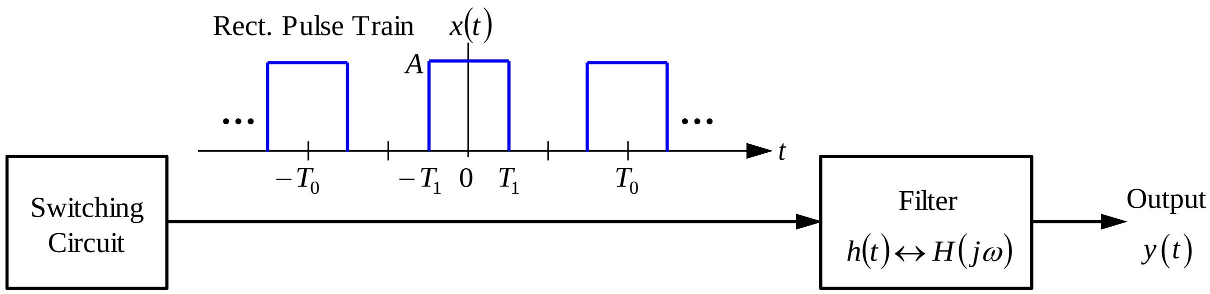
   If we redefine the constant by the substitution $\lambda \to s$, we have

$$x(t) = e^{st}.$$

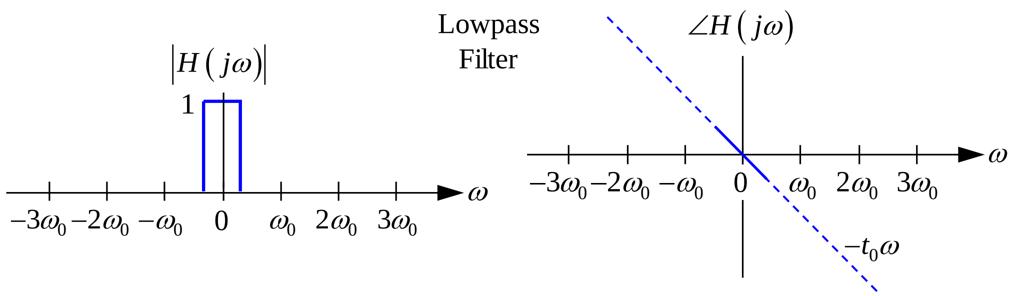
Applications of CT FS and LTI System Analysis

2. **(16 points)** *Switching d.c. power supply or oscillator*. Switching circuits are an efficient way to generate a time-varying voltage from a fixed d.c. supply voltage. They are especially useful in high-power applications. Here, a switching circuit generates a rectangular pulse train $x(t)$ with fixed

amplitude A, fixed fundamental frequency $\omega_0 = 2\pi / T_0$, and variable pulse width $2T_1$. It is convenient to define a duty cycle $\eta = 2T_1 / T_0$. Depending on the design of the filter, the output $y(t)$ can be a d.c. voltage or sinusoid with variable amplitude.



a. **(8 points)** *Variable d.c. supply*. The filter has a lowpass response with the magnitude and phase shown.



Find an expression for $y(t)$ as a function of A, the duty cycle $\eta = 2T_1 / T_0$, and the filter group delay t_0. What value of duty cycle η maximizes $y(t)$, and what is the maximum value? *Hint*: the lowpass filter, described by the frequency response $H(j\omega)$, selects the CT FS component of $x(t)$ at just one value of *k*, corresponding to just one frequency $k\omega_0$. As a result, the CT FS expansion of $y(t)$ comprises just one term.

Solution

For both parts (a) and (b): we know from the Appendix, Table 4 that the FS coefficients of $x(t)$ are

$$
\begin{aligned}
a_k &= \frac{A\omega_0 T_1}{\pi}\operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) \\
&= \frac{A2T_1}{T_0}\operatorname{sinc}\left(k\frac{2T_1}{T_0}\right) \quad , \\
&= A\eta\operatorname{sinc}(k\eta)
\end{aligned}
$$

so

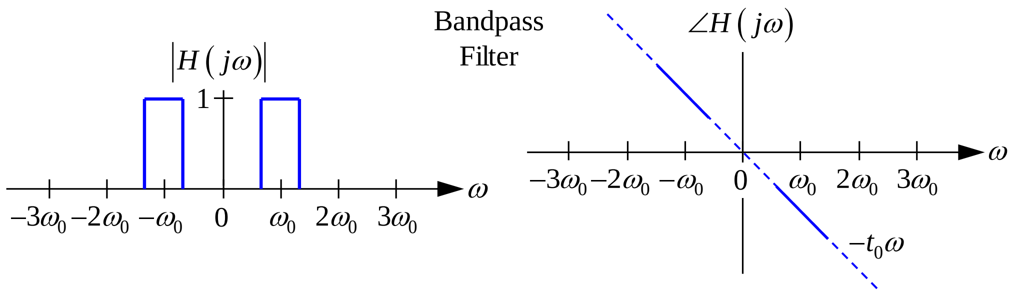
$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

$$= A\eta \sum_{k=-\infty}^{\infty} \operatorname{sinc}(k\eta) \left| H(jk\omega_0) \right| e^{-jk\omega_0 t_0} e^{jk\omega_0 t}$$

$$= A\eta \sum_{k=-\infty}^{\infty} \operatorname{sinc}(k\eta) \left| H(jk\omega_0) \right| e^{jk\omega_0 (t-t_0)}$$

In part (a), $\left| H(jk\omega_0) \right| = \begin{cases} 1 & k=0 \\ 0 & \text{otherwise} \end{cases}$, so $y(t)$ is a d.c. voltage. To maximize $y(t)$, we choose $\eta = 2T_1/T_0 = 1$. Using the fact that $\operatorname{sinc}(0) = 1$, we find $y(t) = A$.

b. **(8 points)** *Variable a.c. supply*. The filter has a bandpass response with the magnitude and phase shown.



Find an expression for $y(t)$ as a function of A, the duty cycle $\eta = 2T_1/T_0$, and the filter group delay t_0. What value of η duty cycle maximizes the peak-to-peak value of $y(t)$, and what is the maximum peak-to-peak value? How does that maximum peak-to-peak value compare to A? *Hint*: the bandpass filter selects the CT FS components of $x(t)$ at two values of *k*, corresponding to two frequencies $k\omega_0$. As a result, the CT FS expansion of $y(t)$ comprises two terms. The bandpass filter imparts phase shifts to these two terms that correspond to a time shift in $y(t)$.

Solution

Here, $\left| H(jk\omega_0) \right| = \begin{cases} 1 & k=\pm 1 \\ 0 & \text{otherwise} \end{cases}$, so $y(t)$ is a sinusoid at frequency ω_0. Using the general expression for $y(t)$ derived above and the fact that $\operatorname{sinc}(-\eta) = \operatorname{sinc}(\eta)$, we have

$$
\begin{aligned}
y(t) &= A\eta \sum_{k=-\infty}^{\infty} \operatorname{sinc}(k\eta) \left|H(jk\omega_0)\right| e^{jk\omega_0(t-t_0)} \\
&= A\eta \operatorname{sinc}(\eta)\left[e^{j\omega_0(t-t_0)} + e^{-j\omega_0(t-t_0)}\right] \\
&= A\eta \frac{\sin(\eta\pi)}{\eta\pi} 2\cos\left(\omega_0(t-t_0)\right) \\
&= \frac{2A}{\pi} \sin(\pi\eta)\cos\left(\omega_0(t-t_0)\right)
\end{aligned}
$$

The peak-to-peak value of $y(t)$ is maximized by choosing $\eta = 1/2$, i.e., a 50% duty cycle. Using the fact that $\sin(\pi/2) = 1$, the peak-to-peak value is $4A/\pi$, which is greater than A.

CT or DT LTI System Analysis

3. *Sinusoidal steady-state response of real LTI systems*. The property is relevant in both CT and DT.

CT Case: consider an LTI system having real impulse response $h(t)$ for which the frequency response $H(j\omega)$ exists. We input a sinusoid

$$x(t) = \cos(\omega_0 t)$$

and the output is a sinusoid at the same frequency:

$$y(t) = \left|H(j\omega_0)\right| \cos\left(\omega_0 t + \angle H(j\omega_0)\right),$$

which is scaled by the magnitude $\left|H(j\omega_0)\right|$ and phase-shifted by the phase $\angle H(j\omega_0)$. Note that $x(t)$ is not an eigenfunction of the system because $y(t)$ is not a scaled version of $x(t)$, except in the special case $\angle H(j\omega_0) = 0$.

DT Case: consider an LTI system having real impulse response $h[n]$ for which the frequency response $H\left(e^{j\Omega}\right)$ exists. We input a sinusoid

$$x[n] = \cos(\Omega_0 n)$$

and the output is a sinusoid at the same frequency:

$$y[n] = \left|H\left(e^{j\Omega_0}\right)\right| \cos\left(\Omega_0 n + \angle H\left(e^{j\Omega_0}\right)\right),$$

which is scaled by the magnitude $\left|H\left(e^{j\Omega_0}\right)\right|$ and phase-shifted by the phase $\angle H\left(e^{j\Omega_0}\right)$. Note that $x[n]$ is not an eigenfunction of the system because $y[n]$ is not a scaled version of $x[n]$, except in the special case $\angle H\left(e^{j\Omega_0}\right) = 0$.

Prove this property for the CT case. *Hint*: Express $x(t)$ as a sum of scaled imaginary exponentials. Express $H(j\omega)$ and $H(-j\omega)$ in polar form, both in terms of $|H(j\omega)|$ and $\angle H(j\omega)$, considering that $H(-j\omega) = H^*(j\omega)$ since $h(t)$ is real.

Solution We express the input as

$$x(t) = \cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right).$$

Using the eigenfunction property of the imaginary exponentials:

$$y(t) = \frac{1}{2}\left[H(j\omega_0)e^{j\omega_0 t} + H(-j\omega_0)e^{-j\omega_0 t}\right].$$

Since $h(t)$ is real, $H(-j\omega_0) = H^*(j\omega_0)$. Expressing $H(j\omega_0)$ and $H^*(j\omega_0)$ in polar form:

$$\begin{aligned} y(t) &= \frac{1}{2}\left[|H(j\omega_0)|e^{j\angle H(j\omega_0)}e^{j\omega_0 t} + |H(j\omega_0)|e^{-j\angle H(j\omega_0)}e^{-j\omega_0 t}\right] \\ &= |H(j\omega_0)|\cdot\frac{1}{2}\left[e^{j(\omega_0 t + \angle H(j\omega_0))} + e^{-j(\omega_0 t + \angle H(j\omega_0))}\right] \\ &= |H(j\omega_0)|\cos\left(\omega_0 t + \angle H(j\omega_0)\right). \end{aligned}$$

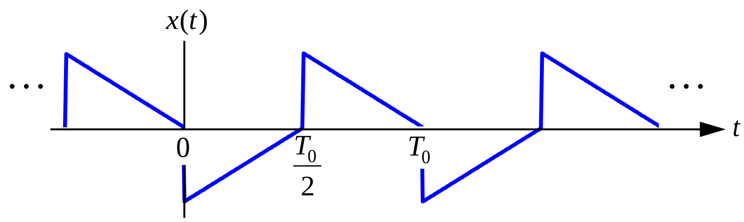
Properties of CT or DT Fourier Series

4. *Half-wave odd symmetry*. This is a property of periodic CT or DT signals in which the values in the second half of each period are the *negative* of the values in the first half of each period. This causes all the FS coefficients for even k to vanish. Intuitively, since the signal is specified fully by its values during half a period, it can be determined by only half as many FS coefficients as a general signal.

   CT Case: a signal having half-wave odd symmetry satisfies

$$x\left(t - \frac{T_0}{2}\right) = -x(t) \qquad \forall t.$$

   An example is shown.



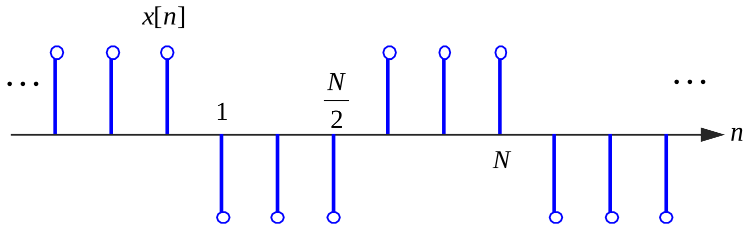
   If $x(t)$ has half-wave odd symmetry, all its even FS coefficients vanish:

$$a_k = 0, \qquad k \text{ even}.$$

   DT Case: half-wave odd symmetry is possible only if the period N is even. It is defined by

$$x\left[n - \frac{N}{2}\right] = -x[n] \qquad \forall n.$$

   An example is shown.



If $x[n]$ has half-wave odd symmetry, all its even FS coefficients vanish:

$$a_k = 0, \qquad k \text{ even.}$$

Prove the CT case: if $x\left(t - \frac{T_0}{2}\right) = -x(t)$, $a_k = 0$, k even. *Hint*: use the time shift property of the FS.

Solution Using the time shift property, the FS coefficients of a signal shifted by half a period are

$$x\left(t - \frac{T_0}{2}\right) = -x(t) \stackrel{FS}{\longleftrightarrow} a_k e^{-jk\frac{\omega_0 T_0}{2}} = -a_k$$

Note that $e^{-jk\frac{\omega_0 T_0}{2}} = e^{-jk\pi} = (-1)^k$ so

$$a_k(-1)^k = -a_k$$

or

$$a_k\left[1 + (-1)^k\right] = 0.$$

Note that

$$\left[1 + (-1)^k\right] = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}.$$

Thus we have:

$$\begin{cases} 2a_k = 0 & k \text{ even} \\ 0a_k = 0 & k \text{ odd} \end{cases},$$

which implies that

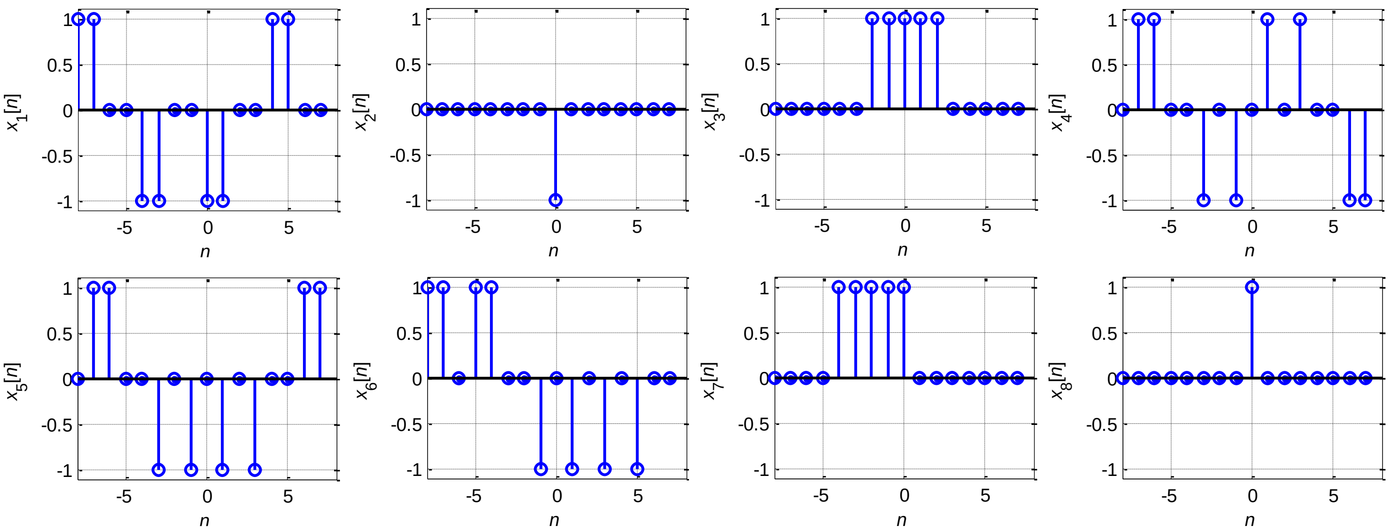
$$\begin{cases} a_k = 0 & k \text{ even} \\ a_k \text{ unrestricted} & k \text{ odd} \end{cases}.$$

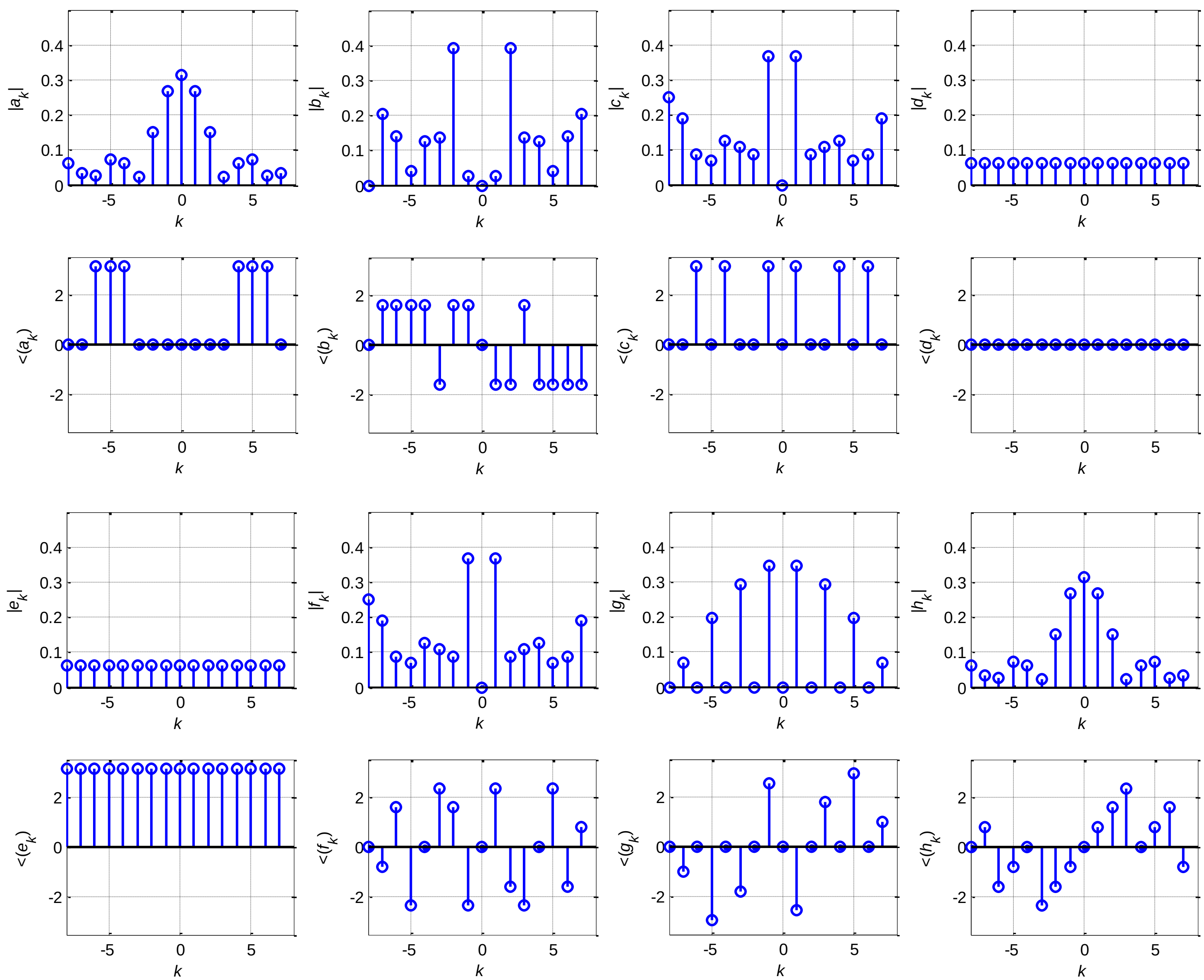
Discrete-Time Fourier Series

5. *Discrete-time Fourier series*. Consider the eight real signals $x_1[n], \ldots, x_8[n]$ and the eight DTFS coefficients $a_k, \ldots, h_k$. Each is periodic with period N = 16. Values are shown over one period, for $-8 \le n \le 7$ or $-8 \le k \le 7$. Match each signal to its DTFS coefficients. Provide a table like the following, filling the appropriate choices from $a_k, \ldots, h_k$ in the second column. Provide a brief justification based on recognizing the DTFS coefficients of familiar signals or based on symmetry (even/odd, half-wave odd), average value, slope of the phase of the DTFS coefficients, etc.

Signal	DTFS	Explanation

1		
2		
3		
4		
5		
6		
7		
8		


$x_1[n]$
$x_2[n]$
$x_3[n]$
$x_4[n]$
$x_5[n]$
$x_6[n]$
$x_7[n]$
$x_8[n]$
n


$|a_k|$
$|b_k|$
$|c_k|$
$|d_k|$
$<(a_k)$
$<(b_k)$
$<(c_k)$
$<(d_k)$
$|e_k|$
$|f_k|$
$|g_k|$
$|h_k|$
$<(e_k)$
$<(f_k)$
$<(g_k)$
$<(h_k)$
k

Solution

Signal	DTFS	Explanation
1	g_k	$x_1[n]$ is the only signal having half-wave odd symmetry, and g_k are the only DTFS coefficients that are zero for all even k.
2	e_k	$x_2[n]$ is a negative-valued periodic impulse train centered at the origin $n = 0$. As it is an impulse train, all the DTFS coefficients have equal value $-1/N$, i.e., they have magnitude $1/N$ and phase $\pm\pi$.
3	a_k	$x_3[n]$ is a rectangular pulse train of pulse width $2N_1+1 = 5$ centered at the origin $n = 0$. It is real and even, so its DTFS coefficients are real and even, and have phases of 0 or $\pm\pi$. Its average value of 5/16 is equal to the DTFS coefficient at $k = 0$.
4	b_k	$x_4[n]$ is a real, odd signal, so its DTFS coefficients are imaginary and odd in k. The DTFS coefficients thus have even magnitude and odd phase, with phase values of $\pm\pi/2$ (except where the magnitude vanishes, in which case, the phase is irrelevant).
5	c_k	$x_5[n]$ is a real, even signal, so its DTFS coefficients are real and even in k. The DTFS coefficients thus have even magnitude and odd phase, with phase values of 0 or $\pm\pi$.
6	f_k	$x_6[n]$ is a right-shifted (delayed) version of the real, even signal $x_5[n]$. Its DTFS coefficients have the same magnitude as those of $x_5[n]$, i.e., the c_k, but have a different phase that has a negative slope, corresponding to the right shift.
7	h_k	$x_7[n]$ is a rectangular pulse train of pulse width $2N_1+1 = 5$ shifted to the left of the origin $n = 0$. Its average value of 5/16 is equal to the DTFS coefficient at $k = 0$. Its DTFS coefficients have the same magnitude as those of signal $x_3[n]$, but their phases have a positive slope, corresponding to the left shift.
8	d_k	$x_8[n]$ is a positive-valued periodic impulse train centered at the origin $n = 0$. As it is an impulse train, all the DTFS coefficients have equal value $1/N$, i.e., they have magnitude $1/N$ and zero phase.

6. *Orthogonality of DT FS basis signals*. In the DT FS, we express an N-periodic signal $x[n]$ in terms of N orthogonal basis signals

$$\phi_k[n] = e^{jk\left(\frac{2\pi}{N}\right)n}, \qquad k = \langle N \rangle ,$$

where k runs over N consecutive values. Consider the inner product between two sequences $\phi_k[n]$ and $\phi_l[n]$ over N consecutive values of n:

$$S = \sum_{n=\langle N \rangle} \phi_k[n]\phi_l^*[n] = \sum_{n=0}^{N-1} e^{j(k-l)\left(\frac{2\pi}{N}\right)n}$$

For concreteness, we choose $0 \le n \le N-1$, $0 \le k \le N-1$ and $0 \le l \le N-1$. When $k = l$, we have

$$S = \sum_{n=0}^{N-1} (1) = N.$$

Considering $k \neq l$, by summing a geometric series, show that $S = 0$.

Solution The inner product can be written

$$S = \sum_{n=0}^{N-1} \left(e^{j(k-l)\left(\frac{2\pi}{N}\right)} \right)^n .$$

Using the partial sum of a geometric series

$$\sum_{n=0}^{N-1} z^n = \frac{1-z^N}{1-z} \qquad z \neq 1,$$

the inner product is

$$S = \frac{1-e^{j(k-l)\left(\frac{2\pi}{N}\right)N}}{1-e^{j(k-l)\left(\frac{2\pi}{N}\right)}} = \frac{1-e^{j(k-l)2\pi}}{1-e^{j(k-l)\left(\frac{2\pi}{N}\right)}}.$$

Note that

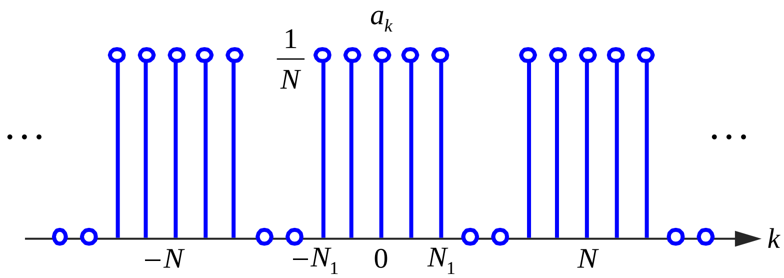
$$e^{j(k-l)2\pi} = 1$$

$$e^{j(k-l)\left(\frac{2\pi}{N}\right)} \neq 1,$$

since $1 \leq |k-l| \leq N-1$, which follows from assuming $0 \leq k \leq N-1$, $0 \leq l \leq N-1$ and $k \neq l$. So $S = 0$.

Properties of Discrete-Time Fourier Series

7. *Duality and Parseval's Identity for DT FS.* Consider a periodic DT signal $x[n]$ of period *N*. Its DT FS coefficients are defined in the following figure.



This looks like a periodic rectangular pulse train in the frequency domain. If you were to synthesize the periodic signal $x[n]$ using these a_k, using mathematics similar to the derivation of the DT FS of the rectangular pulse train, you would find that the periodic signal is

$$x[n] = \frac{1}{N} \frac{\sin\left(2\pi n\left(N_1 + \frac{1}{2}\right)/N\right)}{\sin(\pi n/N)}.$$

Calculate the average power of $x[n]$, P_x. *Hint*: use Parseval's Identity.

Solution Using Parseval's Identity for power (see *EE 102A Course Reader*, Appendix, Table 2), we have

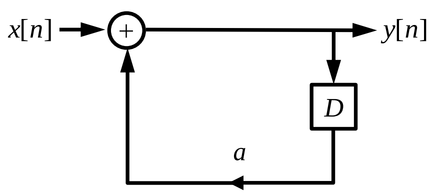
$$P_x = \frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$
$$= \frac{2N_1+1}{N^2}.$$

DT LTI System Analysis

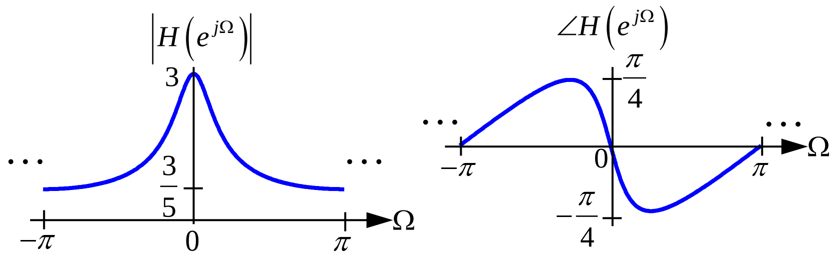
8. *Improving DT infinite impulse response lowpass filter*. In lecture we discussed a first-order DT system described by a difference equation

$$y[n] - ay[n-1] = x[n].$$

and realized as shown.



Here we consider $0 < a < 1$, so it describes a lowpass filter. Because this IIR filter uses just one shift register with feedback, if we choose a close to 1, it can realize a lowpass filter with a low cutoff frequency using very little hardware. For example, here is a plot of its frequency response magnitude and phase for $a = 2/3$.

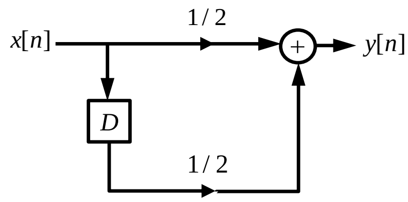


This filter has two obvious deficiencies, however. First, its magnitude response does not go to zero at $\Omega = \pm\pi$. Second, its phase response is not a linear function of frequency, so it causes phase distortion. Here we address the first deficiency.

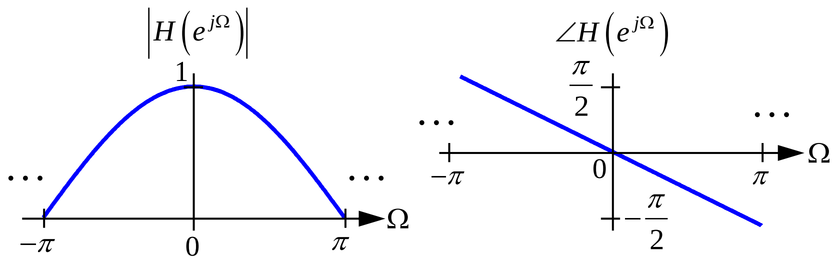
A two-sample moving average filter is described by a difference equation

$$y[n] = \frac{1}{2}\left(x[n] + x[n-1]\right)$$

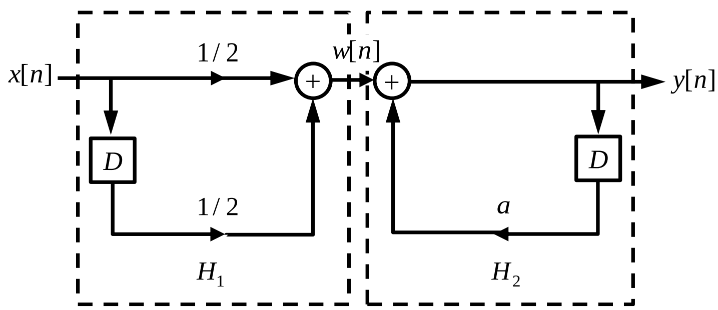
and is realized as shown



Its magnitude response goes to zero at $\Omega = \pm\pi$, as shown. Since its phase is a linear function of Ω, it causes no phase distortion.



But since this FIR filter does not use feedback, achieving a low cutoff frequency would require many more shift registers. We can cascade the two systems as shown.



a. What is the difference equation relating $x[n]$ to $y[n]$? *Hint*: express the intermediate signal $w[n]$ in terms of $x[n]$, express the output $y[n]$ in terms of $w[n]$, and substitute the first relation into the second.

Solution The difference equation relating $x[n]$ to $w[n]$ is

$$w[n] = \frac{1}{2}\left(x[n] + x[n-1]\right).$$

The difference equation relating $w[n]$ to $y[n]$ is

$$y[n] - ay[n-1] = w[n].$$

Therefore, the difference equation relating $x[n]$ to $y[n]$ is

$$y[n] - ay[n-1] = \frac{1}{2}\left(x[n] + x[n-1]\right).$$

b. What is the impulse response $h[n]$ of the overall system with input $x[n]$ and output $y[n]$? *Hint*: find $h_1[n]$ such that $w[n] = x[n] * h_1[n]$ and $h_2[n]$ such that $y[n] = w[n] * h_2[n]$. Then $h[n] = h_1[n] * h_2[n]$.

Solution We have

$$h_1[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]$$

and

$$h_2[n] = a^n u[n].$$

Therefore, the impulse response $h[n]$ of the overall system is

$$\begin{aligned} h[n] &= h_1[n] * h_2[n] \\ &= \left(\frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]\right) * a^n u[n]. \\ &= \frac{1}{2}a^n u[n] + \frac{1}{2}a^{n-1}u[n-1] \end{aligned}$$

c. Give an expression for the frequency response $H\left(e^{j\Omega}\right)$ of the overall system. *Hint*: use the difference equation you found in part (a). You can express $H\left(e^{j\Omega}\right)$ in terms of the frequency responses $H_1\left(e^{j\Omega}\right)$ and $H_2\left(e^{j\Omega}\right)$ of the two constituent systems.

Solution The frequency response H_1 of the two-sample moving average filter is

$$H_1\left(e^{j\Omega}\right) = \frac{1}{2}\left(1 + e^{-j\Omega}\right).$$

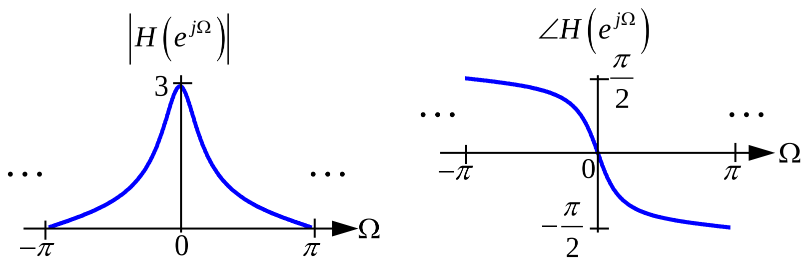
The frequency response H_2 of the lowpass filter is

$$H_2\left(e^{j\Omega}\right) = \frac{1}{1 - ae^{-j\Omega}}.$$

The overall frequency response $H\left(e^{j\Omega}\right)$ of the cascaded system is

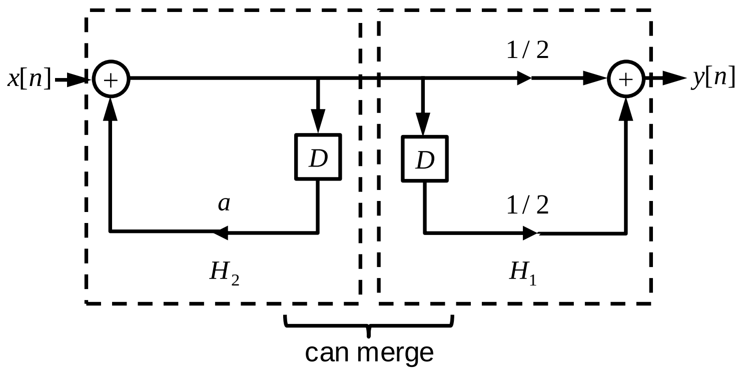
$$\begin{aligned} H\left(e^{j\Omega}\right) &= H_1\left(e^{j\Omega}\right) \cdot H_2\left(e^{j\Omega}\right) \\ &= \frac{1}{2}\left(\frac{1 + e^{-j\Omega}}{1 - ae^{-j\Omega}}\right). \end{aligned}$$

The overall magnitude response is $\left|H\left(e^{j\Omega}\right)\right| = \left|H_1\left(e^{j\Omega}\right)\right| \cdot \left|H_2\left(e^{j\Omega}\right)\right|$ and the overall phase response is $\angle H\left(e^{j\Omega}\right) = \angle H_1\left(e^{j\Omega}\right) + \angle H_2\left(e^{j\Omega}\right)$.

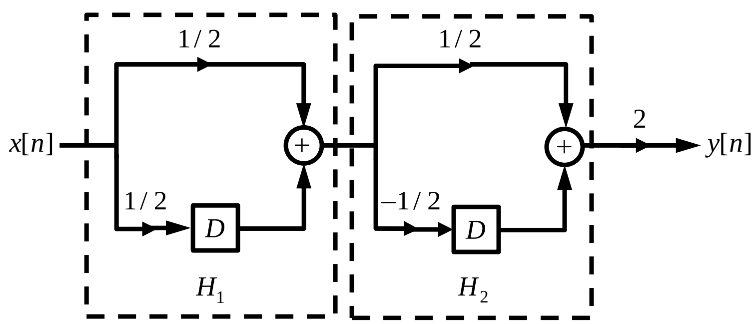


Note that the cascaded system is an improved lowpass IIR filter because its magnitude response goes to zero at $\Omega = \pm\pi$.

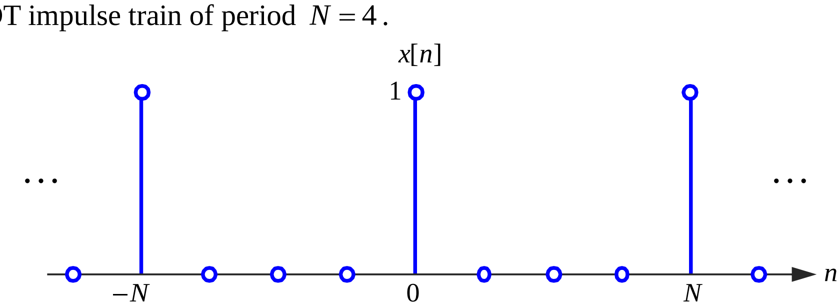
Note: The constituent systems H_1 and H_2 are LTI, and therefore commute. By reversing their order and merging the two shift registers, one can realize the overall system using one shift register.



9. *DT finite impulse response filter; filtering periodic DT signals.* We cascade a two-sample moving average filter H_1, an edge detector H_2, and a constant gain of 2:



a. Consider a DT impulse train of period $N = 4$.



Find its DT FS coefficients a_k and express $x[n]$ using the synthesis equation as

$$x[n] = \sum_{k=-1}^{2} a_k e^{jk\Omega_0 n}$$

with $\Omega_0 = \pi / 2$.

Solution We can find the DT FS coefficients a_k using the analysis equation:

$$\begin{aligned} a_k &= \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\Omega_0 n} \\ &= \frac{1}{N} x[0] e^{-jk\Omega_0 \cdot 0} \\ &= \frac{1}{4} \qquad k = -1,\, 0,\, 1,\, 2 \end{aligned}$$

Therefore, we can express $x[n]$ using the synthesis equation as

$$x[n] = \sum_{k=-1}^{2} \frac{1}{4} e^{jk\frac{\pi}{2}n}.$$

b. The impulse train $x[n]$ is input to the system considered in parts (a)-(d) above. Find a purely real expression for the output $y[n]$. *Hint*: express $y[n]$ using the synthesis equation with DT FS coefficients $a_k H\left(e^{jk\Omega_0}\right)$, where $H\left(e^{jk\Omega_0}\right) = je^{-j\Omega} \sin\left(\Omega\right)\Big|_{\Omega = k\frac{\pi}{2}}$, $k = -1,\, 0,\, 1,\, 2$.

Solution Following the hint, if we input $x[n] = \sum_{k=-1}^{2} \frac{1}{4} e^{jk\frac{\pi}{2}n}$ into the system, the output is

$$y[n] = \sum_{k=-1}^{2} a_k H\left(e^{jk\Omega_0}\right) e^{jk\Omega_0 n}.$$

We examine the values of $H\left(e^{jk\Omega_0}\right) = je^{-j\Omega} \sin\left(\Omega\right)\Big|_{\Omega = k\frac{\pi}{2}}$ one-by-one for each frequency.

$k = -1$: $H\left(e^{-j\Omega_0}\right) = je^{j\frac{\pi}{2}} \sin\left(-\frac{\pi}{2}\right) = j \cdot j \cdot -1 = 1$.

$k = 0$: $H\left(e^{j0\Omega_0}\right) = je^{-j0} \sin\left(0\right) = j \cdot 1 \cdot 0 = 0$.

$k = 1$: $H\left(e^{j\Omega_0}\right) = je^{-j\frac{\pi}{2}} \sin\left(\frac{\pi}{2}\right) = j \cdot -j \cdot 1 = 1$.

$k = 2$: $H\left(e^{j2\Omega_0}\right) = je^{-j\pi} \sin\left(\pi\right) = j \cdot -1 \cdot 0 = 0$.

Only the terms for $k = -1, 1$ are nonzero at the output. Combining them, the output is

$$
\begin{aligned}
y[n] &= \sum_{k=\pm 1} a_k H\left(e^{jk\Omega_0}\right) e^{jk\Omega_0 n} \\
&= \frac{1}{4}\left(1 \cdot e^{-j\frac{\pi}{2}n} + 1 \cdot e^{j\frac{\pi}{2}n}\right) \\
&= \frac{1}{2}\left(\frac{e^{-j\frac{\pi}{2}n} + e^{j\frac{\pi}{2}n}}{2}\right) \\
&= \frac{1}{2}\cos\left(\frac{\pi}{2}n\right)
\end{aligned}
.
$$

The system is a bandpass filter. We input an impulse train and obtain a cosine at the output.

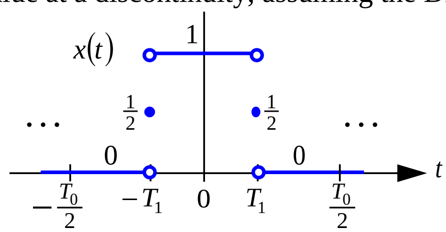
Laboratory 4

In lecture, we have been discussing the use of Fourier series to represent periodic continuous-time signals. Here, we will see Fourier series in action. In order to maximize the learning-to-work ratio, we provide all the scripts and some of the functions required, and ask you to implement four key functions called by the scripts. However, we strongly recommend that you go through the provided MATLAB codes line-by-line and make sure you understand them fully. They are posted in **`Homework_4_MATLAB_Files.zip`** in the Homework Assignments folder on the Canvas website.

Task 1: Reconstruction Error in Fourier Series

In this task, you will write a function to compute the Fourier series coefficients a_k for a rectangular pulse train. Then you will reconstruct the rectangular pulse train using a Fourier series and examine differences between the ideal rectangular pulse train and the Fourier series reconstruction.

We provide a script and a function you may use. The function, **`x_ideal_rectpulsetrain.m`**, creates an ideal pulse train to which you will be comparing your Fourier series reconstruction. This ideal pulse train has values of 1 inside each pulse, 0 outside each pulse, and ½ exactly at the boundaries, since a Fourier series converges to the average value at a discontinuity, assuming the Dirichlet conditions are satisfied.



You are asked to write a function **`a_rectpulsetrain.m`**, which is called from the main script and returns the Fourier series coefficients a_k for the rectangular pulse train of period $T_0 = 2\pi/\omega_0$ and pulse width $2T_1$. This function will accept three parameters: **`k`**, the coefficient index; **`omega0`**, the fundamental frequency; and **`T1`**, the pulse half-width. The script calls it as **`a_rectpulsetrain(k,omega0,T1)`**.

The FS reconstruction including imaginary exponentials up to $\pm K\omega_0$ is

$$\hat{x}_K(t) = \sum_{k=-K}^{K} a_k e^{jk\omega_0 t} \ .$$

In Task 1, we will study differences between the ideal rectangular pulse train $x(t)$ and its reconstruction $\hat{x}_K(t)$. We quantify the differences by an integrated absolute-square error between $x(t)$ and $\hat{x}(t)$, given by

$$\varepsilon_K = \int_{T_0} \left| x(t) - \hat{x}_K(t) \right|^2 dt \ , \tag{1}$$

where the integral runs over any period, such as $-T_0/2 \le t < T_0/2$. Ideally, $x(t)$ and $\hat{x}_K(t)$ are continuous-time signals. As in Laboratory 3, to represent them in MATLAB, we must discretize time with interval Δt, represented by the MATLAB variable **`deltat`**, and describe the signals by their discrete values $x(n\Delta t)$ and $\hat{x}_K(n\Delta t)$, represented by the MATLAB vectors **`x`** and **`xhatK`**. In Laboratory 3, we approximated a convolution integral by a Riemann sum. In Task 1, we will approximate the error integral (1) by a sum

$$\varepsilon_K = \lim_{\Delta t \to 0} \sum_n \left| x(n\Delta t) - \hat{x}_K(n\Delta t) \right|^2 \Delta t \ , \tag{2}$$

where the values of n correspond to one period of these signals. In MATLAB, we compute the summation (2) by **`epsK = sum(abs(x - xhatK).^2)*deltat`**, choosing **`deltat`** to be suitably small.

In Task 1, in reconstructing a signal using imaginary exponentials with frequencies up to $\pm K\omega_0$, we will choose **`deltat = T0/K/128`**. With this choice, the time increment Δt corresponds to 1/128 of a cycle of a sinusoid at frequency $K\omega_0$. This is necessary to model the reconstruction error accurately for small K.

After writing the function **`a_rectpulsetrain.m`**, run the script for **`K = 4, 16, 64, 256`**. Turn in the function and all four plots. Comment briefly on any trends you notice. *Do not worry if the legends* $x(t)$ *and* $\hat{x}_K(t)$ *do not print properly. You will not lose any credit.*

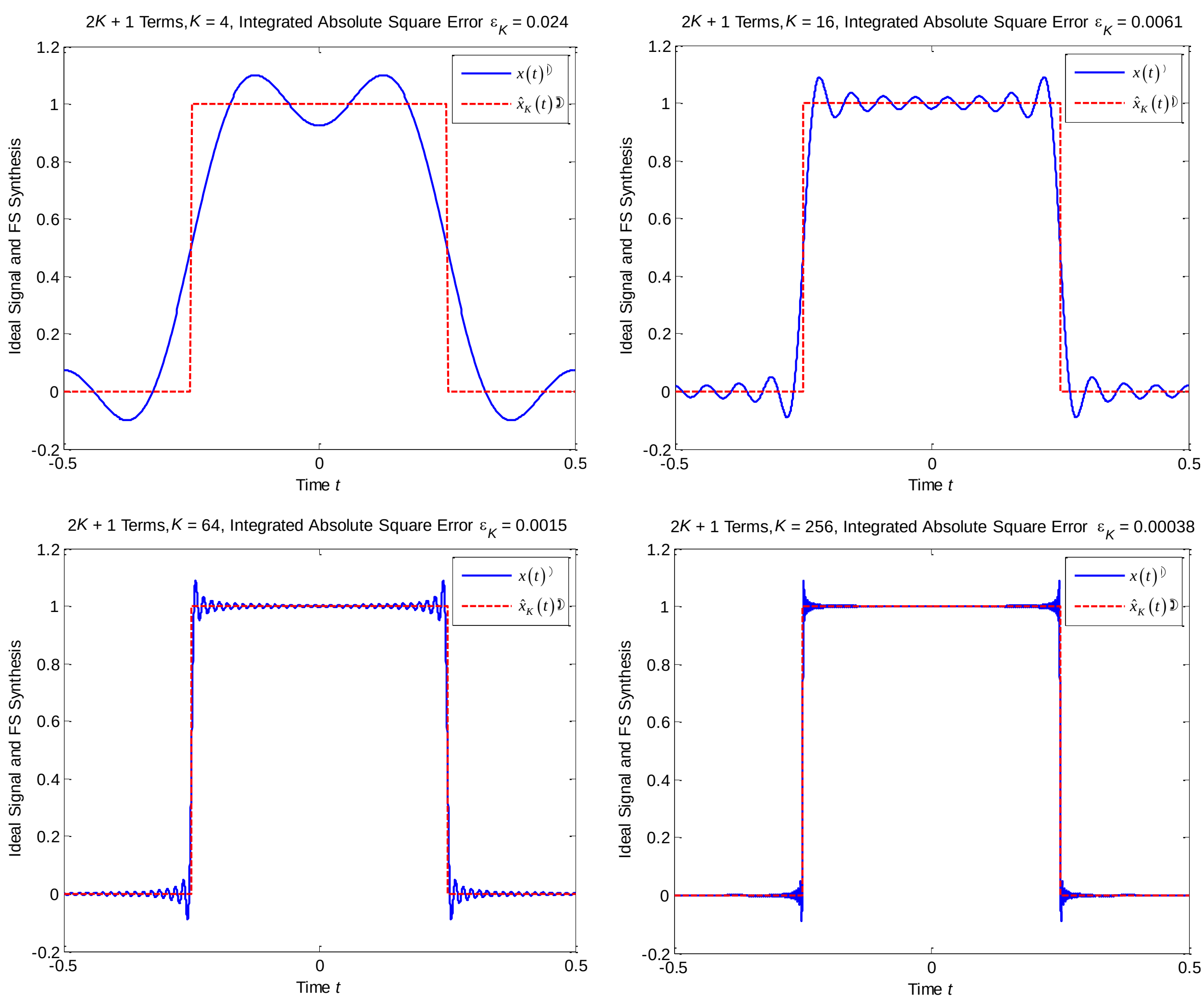
```
%% Task 1: reconstruction error

clear all
lw = 1.5;                % line width for plots

% Signal and FS parameters
T0 = 1;                  % period
omega0 = 2*pi/T0;        % fundamental frequency
T1 = T0/4;               % pulse half-width, T1 < T0/2 to avoid overlap
K = 256;                 % FS reconstruction sums from -K to +K
% Try K = 4, 16, 64, 256
deltat = T0/K/128;       % discretization interval chosen to give 128
                         % samples per cycle of highest frequency
components
t1 = -T0/2; t2 = T0/2;
t = t1:deltat:t2;        % time vector for all signals

% x(t), ideal rectangular pulse train to be synthesized
x = x_ideal_rectpulsetrain(t,T0,T1);

% xhatK(t), Fourier series synthesis using terms from -K to K
xhatK = zeros(size(t));
for k = -K:K;
    ak = a_rectpulsetrain(k,omega0,T1);
    xhatK = xhatK + ak*exp(j*k*omega0*t);
end

% epsK, integrated squared error between x(t) and xhatK(t)
epsK = sum(abs(x - xhatK).^2)*deltat;

% display results
figure(1)
plot(t,real(xhatK),'b-',t,x,'r--')
l=get(gca,'children'); set(l,'linewidth',lw)
xlabel('Time \itt'); ylabel('Ideal Signal and FS Synthesis');
leg = legend({'$x(t)$';'$\hat{x}_{K}(t)$'});
set(leg,'Interpreter','latex');
title(['2\itK\rm + 1 Terms, \itK\rm = ' num2str(K,4) ...
    ', Integrated Absolute Square Error \epsilon_{\itK} = '
num2str(epsK,2)]);
```

```
function x = x_ideal_rectpulsetrain(t,T0,T1)
% Returns ideal rectangular pulse train, with value of 1 inside each
% pulse, 0 outside the pulse, and exactly 1/2 at the boundaries.
x = double(abs(mod(t-T0/2,T0)-T0/2)<T1) + ...
    1/2*double(abs(mod(t-T0/2,T0)-T0/2)==T1);
end
```

(7 points) Task 1 Solution

```
function ak = a_rectpulsetrain(k,omega0,T1);
% Fourier series coeffcients for rectangular pulse train
ak = omega0*T1/pi*sinc(k*omega0*T1/pi);
end
```

A comment that larger values of K correspond to a more faithful representation of $x(t)$, except for the Gibbs phenomenon, is sufficient. You may observe that $\varepsilon_K \propto 1/K$. Also, as noted in lecture, as K increases, the peak error does not diminish, but the error tends to become confined to a narrower time interval.



Task 2: Filtering Rectangular Pulse Train in Frequency Domain

Tasks 2 and 3 will use the function **a_rectpulsetrain.m** you wrote for Task 1, so please complete Task 1 first.

In Homework 3 and Laboratory 3, we filtered a rectangular pulse in the *time domain*, by convolution of the signal with the impulse response of an LTI system, such as a lowpass filter. Here, in Task 2, we learn how to filter a rectangular pulse train in the *frequency domain*, by multiplication of the Fourier series coefficients by the frequency response of an LTI system. We will study three different LTI systems: a time shift, a differentiator, and a first-order lowpass filter.

Instead of filtering a single rectangular pulse, we filter a periodic pulse train input signal $x(t)$ because we can represent it in the frequency domain by its Fourier series coefficients a_k. Given any LTI system, we determine its frequency response $H(j\omega)$. The output signal $y(t)$ is periodic, so we represent it by its Fourier series coefficients b_k. To find these coefficients, we multiply each of the input signal Fourier series coefficients a_k by the frequency response at frequency $\omega = k\omega_0$, to obtain $b_k = a_k H(jk\omega_0)$. Finally, we synthesize the output signal using

$$\hat{y}_K(t) = \sum_{k=-K}^{K} b_k e^{jk\omega_0 t} = \sum_{k=-K}^{K} a_k H(jk\omega_0) e^{jk\omega_0 t} . \qquad (3)$$

We provide the main script for Task 2 along with a function that returns the frequency response of a time shift filter, **Hshift.m**. You are asked to write functions **Hdiff.m** and **Hfolpf.m**, which return the frequency responses of the differentiator and the first-order low pass filter, respectively. We have derived the frequency responses of all three systems in lecture and include them below.

As stated already, you will need to use the function **a_rectpulsetrain.m** that you wrote in Task 1.

Time Shift

$$H_{\text{shift}}(j\omega) = e^{-j\omega t_0}.$$

This frequency response is implemented in the function **Hshift.m**. It accepts two input parameters, the frequency **omega** and the time shift **t0**. It is called as **Hshift(k*omega0,t0)** in the main script.

Differentiator

$$H_{\text{diff}}(j\omega) = j\omega$$

Write a function **Hdiff.m** to return this frequency response. This function accepts a single input parameter **omega**. It is called as **Hdiff(k*omega0)** in the main script.

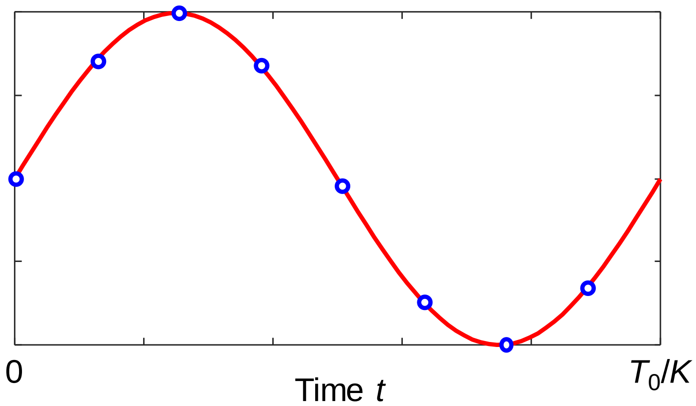
First-order Lowpass Filter

$$H_{\text{flopf}}(j\omega) = \frac{1}{1 + j\omega\tau}$$

Write the function **Hfolpf.m** to return this frequency response. This function will accept two parameters **omega** and **tau**. It is called as **Hfolpf(k*omega0,tau)** in the main script.

In Tasks 2 and 3, we are not studying reconstruction error. We will always use $K = 256$. So we will simply write the LTI system output (3) as $y(t)$.

In Tasks 2 and 3, we only need to achieve a reasonable visual rendering of the input and output signals. We will choose a time discretization interval **`deltat = T0/K/8`**. With this choice, the time increment Δt corresponds to 1/8 of a cycle of a sinusoid at frequency $K\omega_0$, the highest frequency used in our Fourier series synthesis. One cycle of this sinusoid and the eight discrete samples are shown in this figure.



After writing the functions **`Hdiff.m`** and **`Hfolpf.m`**, run the main script. Turn in the two functions and the plot obtained.

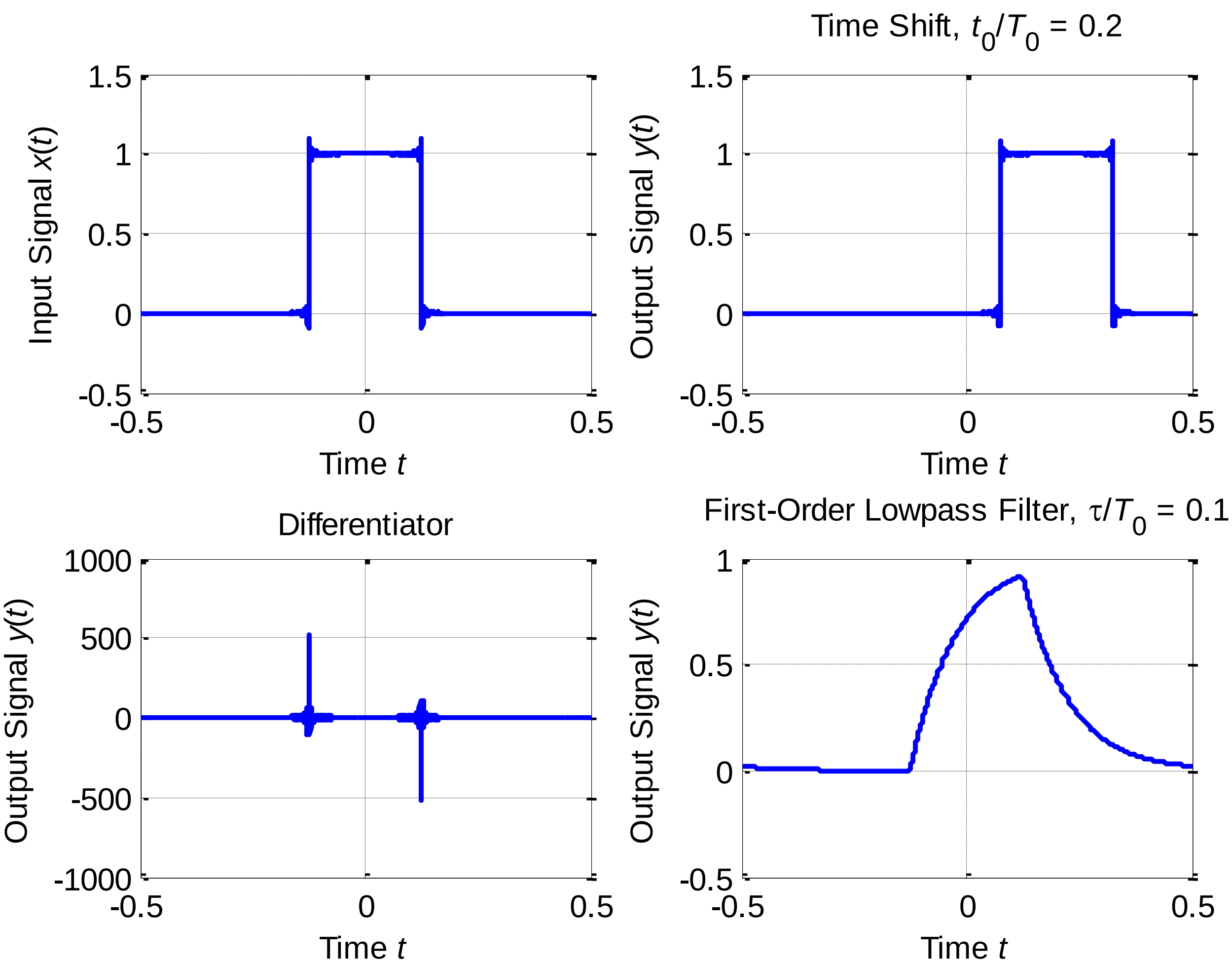
```
function H = Hshift(omega,t0)
% Time shift
H = exp(-sqrt(-1)*omega*t0);
end
```

```
%% Task 2: filtering of pulse train by time shift, differentiator or
%% first-order lowpass filter

clear all
lw = 1.5;                % line width for plots

% Signal and FS parameters
T0 = 1;                  % period
omega0 = 2*pi/T0;        % fundamental frequency
T1 = T0/8;               % pulse half-width, require T1 < T0/2 to avoid
overlap
K = 256;                 % FS reconstruction sums from -K to +K
deltat = T0/K/8;         % discretization interval chosen to give 8
                         % samples per cycle of highest freq. comps.
t1 = -T0/2; t2 = T0/2;
t = t1:deltat:t2;        % time vector for all signals

% Time shift parameters
t0 = 0.2*T0;
```

```
% First-order lowpass filter parameters
tau = 0.1*T0;              % time constant

% y(t), Fourier series synthesis of output using terms from -K to K
x = zeros(size(t));
yshift = zeros(size(t));
ydiff = zeros(size(t));
yfo = zeros(size(t));
for k = -K:K;
    ak = a_rectpulsetrain(k,omega0,T1);
    x = x + ak*exp(j*k*omega0*t);
    yshift = yshift + ak*Hshift(k*omega0,t0)*exp(j*k*omega0*t);
    ydiff = ydiff + ak*Hdiff(k*omega0)*exp(j*k*omega0*t);
    yfo = yfo + ak*Hfolpf(k*omega0,tau)*exp(j*k*omega0*t);
end

% display results
figure(2)
subplot(221)
plot(t,real(x)); grid
l=get(gca,'children'); set(l,'linewidth',lw)
xlabel('Time \itt'); ylabel('Input Signal \itx\rm(\itt\rm)');

subplot(222)
plot(t,real(yshift)); grid
l=get(gca,'children'); set(l,'linewidth',lw)
xlabel('Time \itt'); ylabel('Output Signal \ity\rm(\itt\rm)');
title(['Time Shift, \itt\rm_{0}/\itT\rm_{0} = ' num2str(t0/T0,3)]);

subplot(223)
plot(t,real(ydiff)); grid
l=get(gca,'children'); set(l,'linewidth',lw)
xlabel('Time \itt'); ylabel('Output Signal \ity\rm(\itt\rm)');
title('Differentiator');

subplot(224)
plot(t,real(yfo)); grid
l=get(gca,'children'); set(l,'linewidth',lw)
xlabel('Time \itt'); ylabel('Output Signal \ity\rm(\itt\rm)');
title(['First-Order Lowpass Filter, \tau/\itT\rm_{0} = '
num2str(tau/T0,3)]);
```

(7 points) Task 2 Solution:

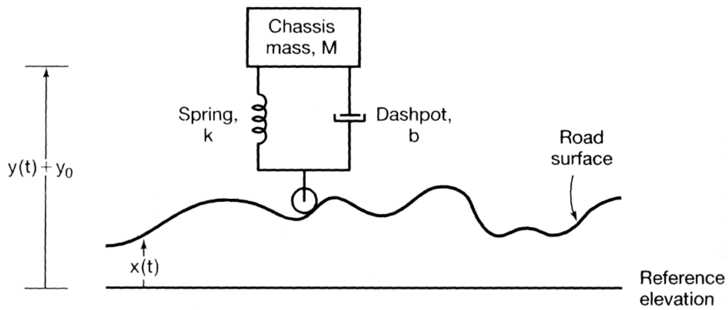
```
function H = Hdiff(omega)
% Differentiator
H = sqrt(-1)*omega;
end

function H = Hfolpf(omega,tau)
% Frequency response of first-order lowpass filter
H = 1./(1 + sqrt(-1)*omega*tau);
end
```



Task 3: Automobile Suspension

In this task, we will use Fourier series to model the response of an automobile driving at constant speed over a periodic series of speed bumps, represented by a rectangular pulse train. We will compare the responses of three different suspension types, roughly representative of sports, standard and luxury cars.

The key to this task is deriving the frequency response describing an automobile suspension (see OWN Section 6.7.1 for more details). As shown in this diagram, the input $x(t)$ represents the road elevation, while the output $y(t)$ represents the vehicle elevation.



The vehicle elevation $y(t)$ and road elevation $x(t)$ are related by a second-order constant-coefficient linear differential equation

$$m\frac{d^2y(t)}{dt} + b\frac{dy(t)}{dt} + ky(t) = kx(t) + b\frac{dx(t)}{dt}, \tag{4}$$

where m is the mass of the chassis and k and b are the spring and shock absorber constants, respectively. Knowing that the frequency response of this system exists, we can derive the frequency response using the method presented in lecture. Let the input and output be $x(t) = e^{j\omega t}$ and $y(t) = H(j\omega)e^{j\omega t}$. We substitute these into (4):

$$H(j\omega)[m(j\omega)^2e^{j\omega t} + b(j\omega)e^{j\omega t} + ke^{j\omega t}] = ke^{j\omega t} + b(j\omega)e^{j\omega t}.$$

Canceling the factors of $e^{j\omega t}$, and solving for $H(j\omega)$ yields

$$H(j\omega) = \frac{k + b(j\omega)}{M(j\omega)^2 + b(j\omega) + k}\,. \tag{5a}$$

For convenience, we make the parameter substitutions $\omega_n = \sqrt{k/M}$ and $2\zeta\omega\_n = b/M$, obtaining a canonical form of the equation:

$$H(j\omega) = \frac{\omega_n^2 + 2\zeta\omega_n(j\omega)}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}. \tag{5b}$$

The natural frequency ω_n is the resonant frequency at which the system would oscillate in the absence of damping. Choosing ω_n large (stiff suspension) makes the car more agile, while choosing it small (soft suspension) makes the car more sluggish but gives a smoother ride. The damping constant ζ is a measure of how strongly the oscillations are damped by the shock absorber. If $\zeta < 1$ (underdamped), the car is agile, but it will oscillate up and down after going over a bump. If $\zeta > 1$ (overdamped), the response of the car becomes sluggish. A good compromise is to choose $\zeta = 1$ (critically damped), the value we consider in this task. Many second-order systems, such as dynamic feedback control systems, are chosen to have a critically damped response.

In this task, you are given a main script. You are asked to write a function **`Hautosusp.m`**, which accepts input parameters **`omega`**, **`omegan`** and **`zeta`**, and returns the frequency response of the automobile suspension (5b). The main script call it as **`Hautosusp(k*omega0,omegan,zeta)`**.

You are asked to run the main script and turn in plots of $x(t)$ and $y(t)$ for the following values of the natural frequency. Comment briefly on the differences between the plots and provide an intuitive interpretation of these differences.

```
omegan = 10*omega0;       % stiff (sports car)
omegan = 1*omega0;        % moderate (standard car)
omegan = 0.1*omega0;      % soft (luxury car)

%% Task 3: automobile rolling over periodic speed bumps

clear all
lw = 1.5;                 % line width for plots

T0 = 1;                   % period
omega0 = 2*pi/T0;         % fundamental frequency
T1 = T0/32;               % pulse half-width, T1 < T0/2 to avoid overlap
K = 256;                  % FS reconstruction sums from -K to +K
deltat = T0/K/8;          % discretization interval chosen to give 8
                          % samples per cycle of highest freq. comps.
t1 = -3*T0/2; t2 = 3*T0/2;
t = t1:deltat:t2;         % time vector for all signals

% Automobile suspension system
omegan = 1*omega0;        % natural frequency
zeta = 1;                 % damping constant

% y(t), Fourier series synthesis of output using terms from -K to K
x = zeros(size(t));
yautosusp = zeros(size(t));
for k = -K:K;
    ak = a_rectpulsetrain(k,omega0,T1);
    x = x + ak*exp(j*k*omega0*t);
    yautosusp = yautosusp +
ak*Hautosusp(k*omega0,omegan,zeta)*exp(j*k*omega0*t);
end

% display results
figure(3)
subplot(211)
plot(t,real(x)); grid
l=get(gca,'children'); set(l,'linewidth',lw)
xlabel('Time \itt'); ylabel('Road Elevation \itx\rm(\itt\rm)');

subplot(212)
plot(t,real(yautosusp)); grid
l=get(gca,'children'); set(l,'linewidth',lw)
xlabel('Time \itt'); ylabel('Automobile Elevation \ity\rm(\itt\rm)');
title(['Auto Suspension, \omega_{\itn}/\omega_{0} = '
num2str(omegan/omega0,3),...
   ', \zeta = ' num2str(zeta,3)]);
```

(7 points) Task 3 Solution

```
function H = Hautosusp(omega,omegan,zeta)
% Automobile suspension, OWN Section 6.7.1
H = (2*zeta*omegan*j*omega + omegan^2)./...
    ((j*omega).^2 + 2*zeta*omegan*j*omega + omegan^2);
end
```

An acceptable answer includes comments that a smaller value of the ratio ω_n/ω_0, corresponding to a softer suspension, reduces the peak excursion of the automobile elevation, although it extends the response over a longer period of time.