Stanford University
EE 102A: Signal Processing and Linear Systems I
Professor Joseph M. Kahn

Table of Contents

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Course Overview

Applications of the Course Material

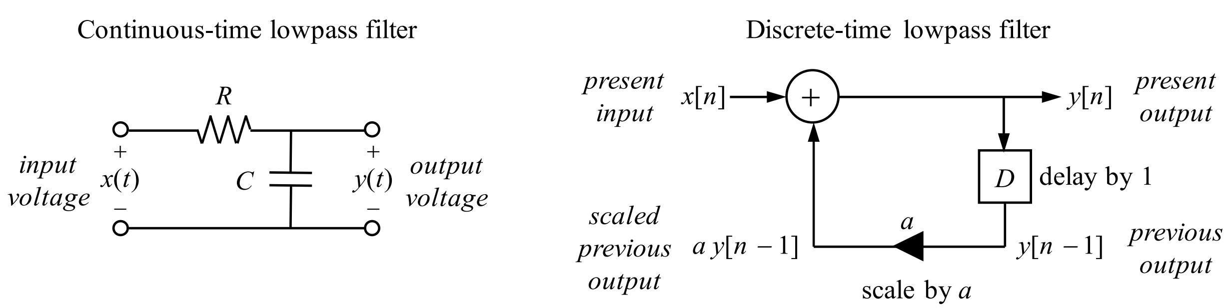
- Electronics: linear circuits and some nonlinear circuits.
- Mechanical systems: automobile suspension, micro-electro-mechanical accelerometer, etc.
- Communication systems: wired, wireless, optical fiber, free-space optical, etc.
- Ranging, positioning and navigation systems: RADAR, LIDAR, global positioning system, etc.
- Imaging systems: digital photography and videography, magnetic resonance imaging, etc.
- Digital signal processing of audio, still images or video: filtering, compression, etc.
- Feedback control systems: thermostat, automobile cruise control, camera autofocus, etc.
- Financial systems: compound interest, securities, etc.

Major Ideas

- Here we briefly glimpse some of the key ideas touched upon in the course.

1. Linear Time-Invariant Systems (Chapter 2)

- Many important systems are *linear* and *time-invariant* (LTI). Two examples are shown below.
  - On the left is a *continuous-time* (CT) lowpass filter. This is an analog circuit comprising a resistor and a capacitor. The input voltage $x(t)$ and output voltage $y(t)$ are functions of a continuous time variable t.
  - On the right is a *discrete-time* (DT) lowpass filter. This is comprised of digital hardware, including a shift register, a multiplier and an adder. The input $x[n]$ and output $y[n]$ are numerical values at a discrete time n that represents integer multiples of a clock interval.



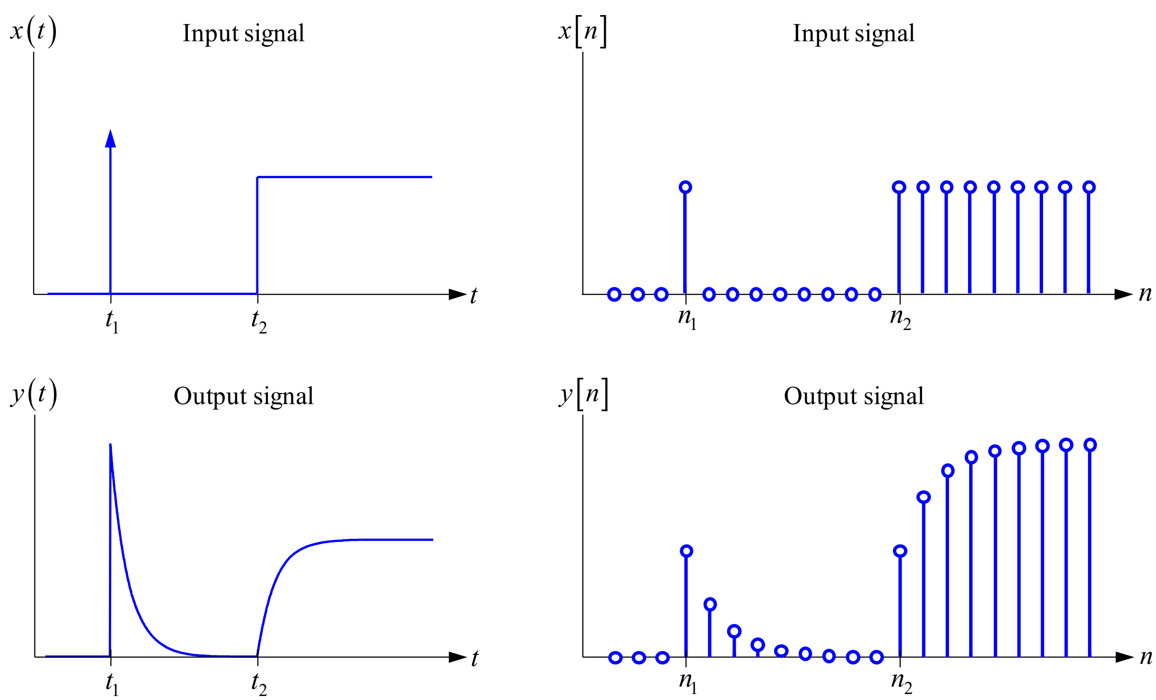
- Like many LTI systems we will study, these systems are described by *linear equations with constant coefficients*. The *differential equation* (for the CT system) and the *difference equation* (for the DT system) are both first-order equations.

Differential equation	Difference equation
$RC\dfrac{dy}{dt}+y(t)=x(t)$	$y[n]-ay[n-1]=x[n]$

- Both these equations represent *implicit* descriptions of the relationship between input x and output y. Given a particular input signal $x(t)$ or $x[n]$, one must solve the equation through some means to obtain an *explicit* expression for the output signal $y(t)$ or $y[n]$. Then, given a different input signal $x(t)$ or $x[n]$, often one must solve the equation again to find the corresponding output signal $y(t)$ or $y[n]$ (depending on the approach used).
- In this course, we will learn to solve a differential or difference equation just once to find an *impulse response* $h(t)$ or $h[n]$ that fully describes the LTI system. (We will extend this concept to include LTI systems not describable by finite-order differential or difference equations.) Then, given any input $x(t)$ or $x[n]$, we can compute the output $y(t)$ or $y[n]$ explicitly without solving the differential or difference equation again. We use a mathematical operation called *convolution*, denoted by "*":

$$y(t)=x(t)*h(t) \qquad\qquad y[n]=x[n]*h[n]$$

The figure below shows exemplary input signals and the resulting output signals for the CT lowpass filter (on the left) and the DT lowpass filter (on the right). In each case, the output signals have been computed using convolution. Although the CT system and the DT system are constructed and described differently, their behavior is similar. In both systems, given a short pulse input signal, the output signal rises abruptly and then decays exponentially. Likewise, in both systems, given a step input signal, the output signal rises and gradually approaches a constant value.



2. Frequency-Domain Analysis (Chapters 3, 4 and 5)

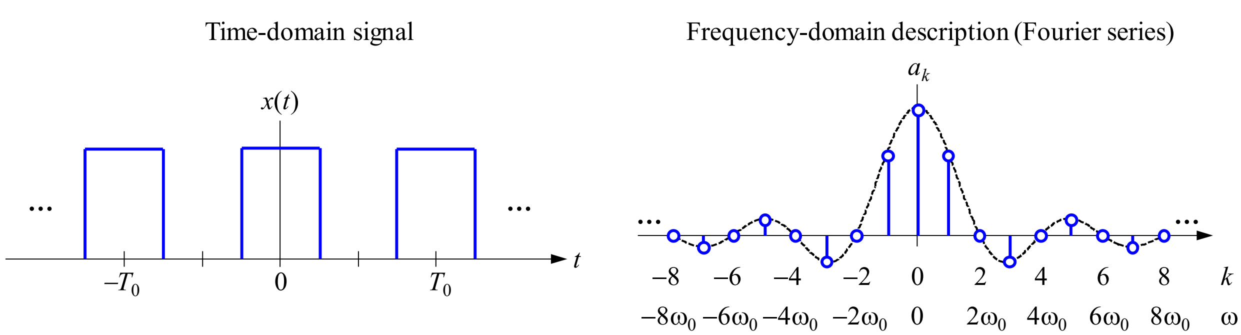
- A set of sinusoids at different frequencies provides a natural basis for representing signals in CT or DT.
    - Many physical systems, including pendula, acoustic tuning forks, radio-frequency oscillators, and lasers, are described by second- or higher-order differential equations, and exhibit oscillatory behavior. Often the oscillations are sinusoidal at one frequency or a small number of frequencies.
    - Human hearing can distinguish sinusoidal acoustic signals at different frequencies in the audio range (Hz to kHz).
- In *frequency-domain analysis*, also known as *Fourier analysis*, given a CT or DT signal, we express it as a sum of sinusoidal time signals. In electrical engineering, instead of sinusoids, we actually use imaginary exponential time signals $e^{j\omega t}$ or $e^{j\Omega n}$, where the real-valued variables ω and Ω denote frequency in CT and DT, respectively. Imaginary exponentials provide several advantages over sinusoids as a basis for Fourier analysis, as explained in Chapter 3.
- We will study two classes of tools for both CT and DT.
    - *Fourier series*: for *periodic* signals.
    - *Fourier transform*: for *aperiodic* (or periodic) signals.
- The figure below illustrates the Fourier series for a periodic CT signal.
    - On the left, we show a periodic time-domain signal $x(t)$, which is an infinite train of rectangular pulses. The pulses repeat with a *period* T_0 and a *fundamental frequency*

$$\omega_0 = \frac{2\pi}{T_0}.$$

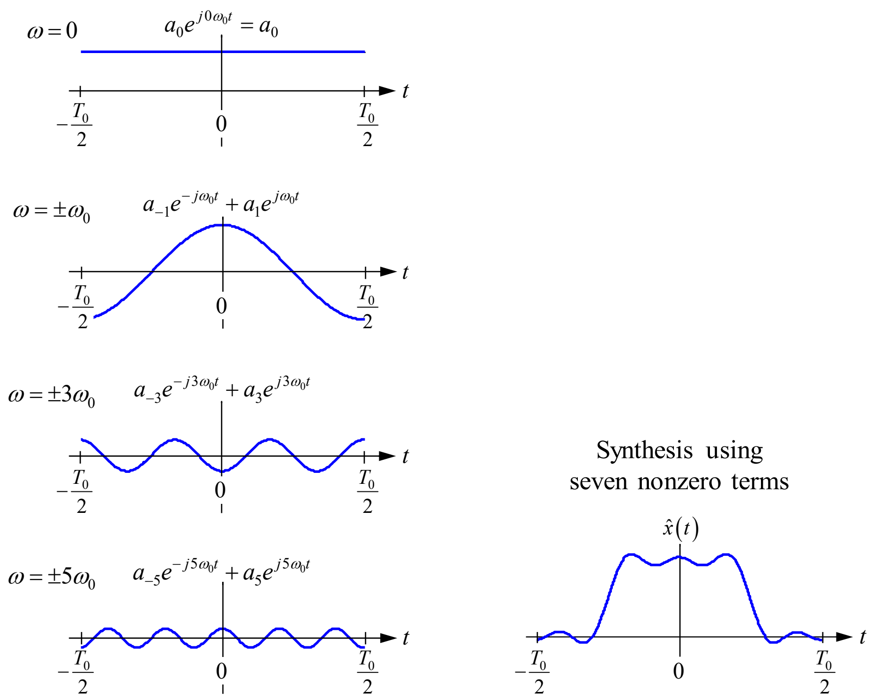
Notice the *inverse relationship between time and frequency*: a shorter period T_0 corresponds to a higher fundamental frequency ω_0 (and vice versa). Stated differently, a signal changing faster in time is described using higher frequencies (and vice versa).

- On the right, we show a frequency-domain description for the periodic signal $x(t)$, which is a sequence of Fourier series coefficients a_k, $-\infty < k < \infty$. These coefficients tell us how to synthesize the periodic signal as a sum of imaginary exponentials whose frequencies are $k\omega_0$, integer multiples of the fundamental frequency:

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad .$$



- The figure below illustrates how the rectangular pulse train is approximated by adding up the seven nonzero terms for $-5 \le k \le 5$. Only one period, $-T_0/2 \le t \le T_0/2$, is shown. The term for $k=0$, given by a_0, represents the average value of the signal. By including terms up to higher $|k|$ in the synthesis, we better approximate the rectangular pulse train and its abrupt transitions.



- Frequency-domain methods provides powerful tools for analyzing LTI systems. Given an LTI system, it is often possible to find a *frequency response* that characterizes its input-output relationship (this is not possible for all LTI systems). The frequency response is denoted by $H(j\omega)$ for CT systems, and by $H\left(e^{j\Omega}\right)$ for DT systems.

- Given an LTI system, if we choose the input signal to be an imaginary exponential at a single frequency ω or Ω, then the output signal will be an imaginary exponential at the identical frequency ω or Ω, simply scaled by the frequency response at frequency ω or Ω:

$$x(t) = e^{j\omega t} \qquad\qquad x[n] = e^{j\Omega n}$$

$$y(t) = H(j\omega)e^{j\omega t} \qquad\qquad y[n] = H\left(e^{j\Omega}\right)e^{j\Omega n}$$

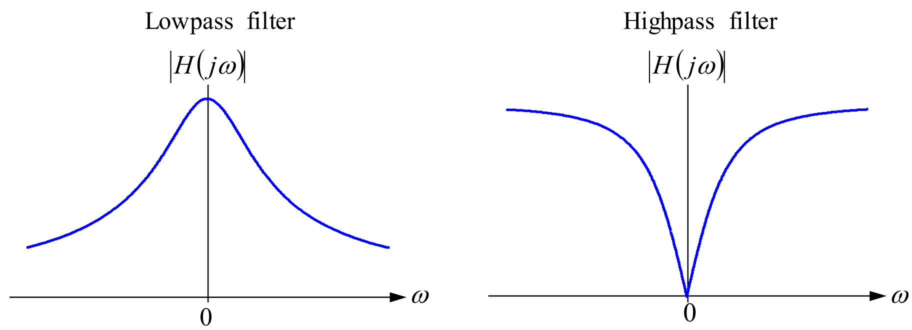
- Suppose an input signal can be expressed as a sum of imaginary exponentials at multiple frequencies using a Fourier series or Fourier transform (this is not possible for all signals). Then we can express the corresponding output signal as the same sum, but with each term scaled by the value of the frequency response at the appropriate frequency. As an example, consider once again a periodic CT input signal expressed as a Fourier series:

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} .$$

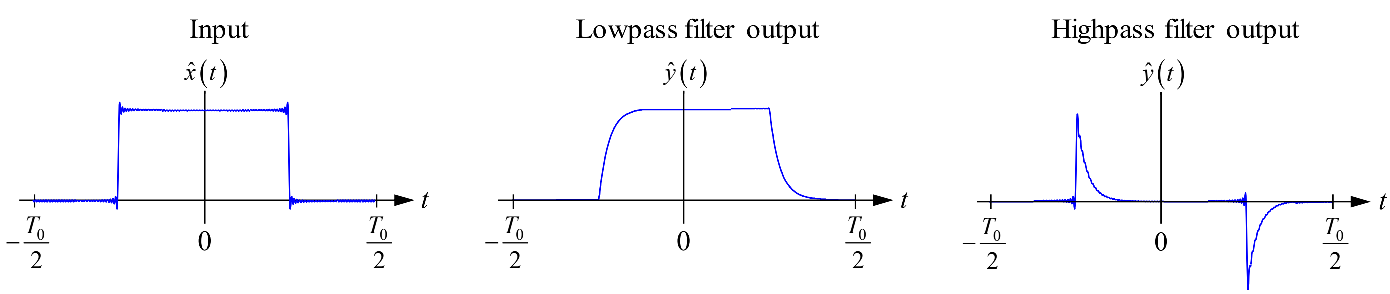
The corresponding output signal can be expressed as a Fourier series:

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t} .$$

- Frequency-domain analysis provides an intuitive understanding of how LTI systems respond to input signals. For example, the figure below compares the magnitudes of the frequency responses for a CT *lowpass filter* (left) and a CT *highpass filter* (right). The first-order lowpass filter is described by the circuit and differential equation on pages iii-iv above. Observe how the lowpass filter passes low frequencies (low $|\omega|$) and attenuates high frequencies (high $|\omega|$), while the highpass filter attenuates low frequencies and passes high frequencies.

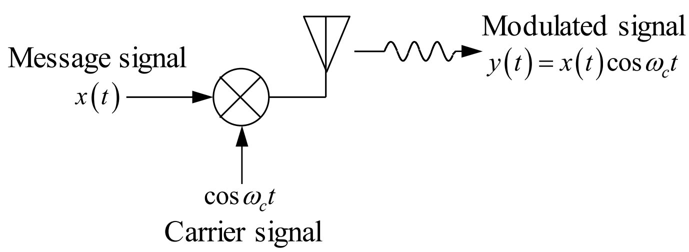


- The figure below illustrates the effect of these filters on a rectangular pulse train. We synthesize each signal by a Fourier series including $-15 \le k \le 15$, a total of 31 terms. The 14 terms for nonzero, even values of k vanish because of a symmetry of the waveform, so there are actually only 17 nonzero terms. The input signal $\hat{x}(t)$ (left) exhibits abrupt upward and downward transitions. The lowpass filter smooths out these transitions, as seen in the output signal $\hat{y}(t)$ (middle). By contrast, the highpass filter emphasizes the abrupt transitions and removes other features of the signal, including its positive average value, leaving an output signal $\hat{y}(t)$ (right) that has a zero average value.

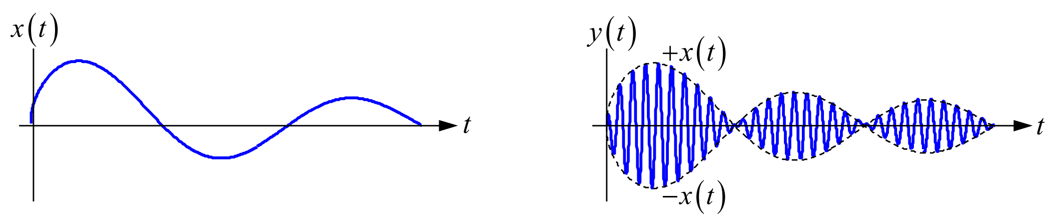


3. Modulation (Chapters 4 and 7)

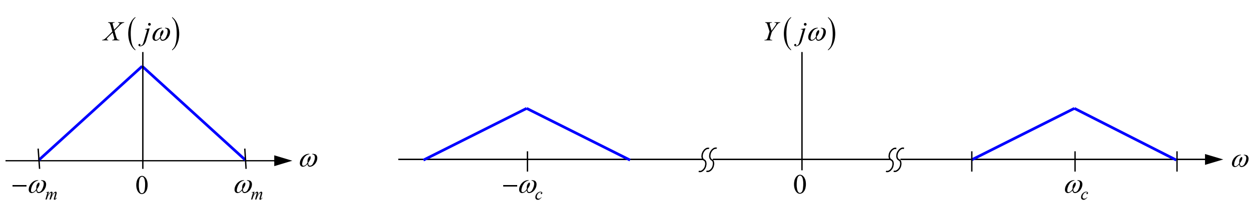
- *Modulation* is a process of embedding an information-bearing *message signal* into another signal called a *carrier signal*, thus creating a *modulated signal*. For example, in the *amplitude modulation* (AM) system shown below, a message signal $x(t)$ is multiplied by a sinusoidal carrier $\cos\omega_c t$, where ω_c is a *carrier frequency*, yielding a modulated signal $y(t)=x(t)\cos\omega_c t$. In AM radio broadcasting, the message $x(t)$ is an audio signal containing frequencies $\omega/2\pi$ from 0 to several kHz, while the carrier frequency $\omega_c/2\pi$ lies between 550 and 1600 kHz. The modulated signal $y(t)$ can propagate as an electromagnetic wave over tens to hundreds of kilometers. (As explained in Chapter 7, a large unmodulated carrier is added in AM broadcasting to enable demodulation using simple receiver hardware.)



- The figure below shows a message signal $x(t)$ and the corresponding modulated signal $y(t)$, which is a sinusoid at frequency ω_c whose envelope is defined by $\pm x(t)$.

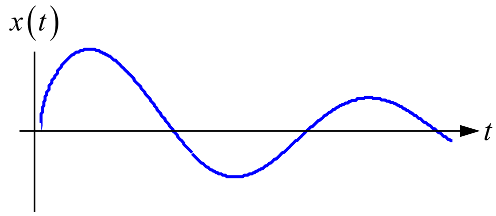


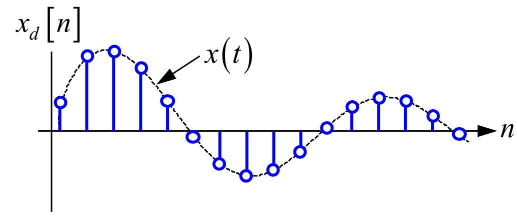
- Frequency-domain tools, especially the CT Fourier transform, provide a simple intuitive understanding of AM and similar techniques. The figure below schematically shows $X(j\omega)$, the CT Fourier transform of a message signal $x(t)$ (not the particular $x(t)$ shown in the figure above). The figure also shows $Y(j\omega)$, the CT Fourier transform of the corresponding message signal $y(t)$. Observe that $X(j\omega)$ is nonzero over a range of low frequencies near $\omega=0$, while $Y(j\omega)$ contains copies of $X(j\omega)$ shifted to high frequencies near $\omega=\pm\omega_c$.



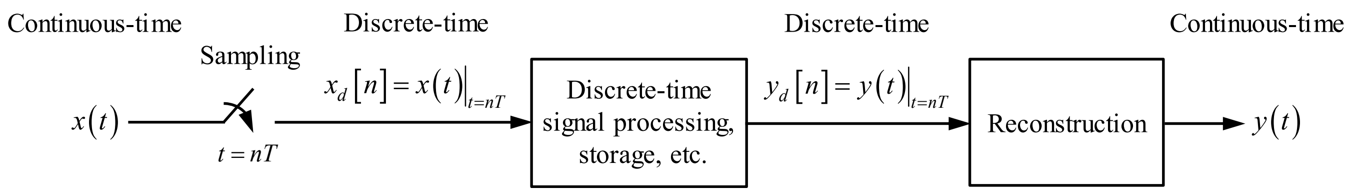
4. Sampling and Reconstruction (Chapter 6)

- As shown here, given a CT signal $x(t)$, we perform *sampling* to obtain a DT signal $x_d[n] = x(t)|_{t=nT}$. The samples are taken at times $t = nT$, corresponding to a *sampling rate* of $1/T$ samples per second.





A system using sampling is shown below. Once we have sampled $x(t)$ and obtained $x_d[n]$ (a sequence of numbers), we can use digital hardware to perform signal processing operations on the sequence, such as filtering or data compression. We can also store a digital representation in a memory or disk, or transmit it through a data network. After performing these operations, we obtain a DT signal $y_d[n]$. Given the DT signal $y_d[n]$, we can perform *reconstruction* to obtain a CT signal $y(t)$.



- When we compare an original CT signal $x(t)$ to its DT samples $x_d[n]$, it appears that $x_d[n]$ contains less information than $x(t)$. Later in the course, we will study sampling and reconstruction and address two questions:

  1. Under what conditions do the samples of a CT signal $x(t)$ taken at rate $1/T$, given by $x_d[n] = x(t)|_{t=nT}$, $-\infty < n < \infty$, represent sufficient information to reconstruct $x(t)$?

  2. Assuming a CT signal $x(t)$ was sampled satisfying the conditions determined in question 1, how can we reconstruct $x(t)$ from the samples $x_d[n] = x(t)|_{t=nT}$, $-\infty < n < \infty$?

Frequency-domain analysis will prove essential in addressing these questions. Surprisingly, we will find that it is possible, in principle, to reconstruct the CT signal $x(t)$ *perfectly*. This perfect reconstruction requires $x(t)$ to be *bandlimited* to a frequency range $|\omega| \le \omega_m$, i.e., its Fourier transform must vanish for $|\omega| > \omega_m$ for some ω_m. Furthermore, it requires that the *sampling frequency* $\omega_s = 2\pi / T$ exceed twice the highest frequency contained in $x(t)$, i.e., $\omega_s > 2\omega_m$. Intuitively, we must sample each sinusoidal frequency component of $x(t)$ *more than twice per cycle* to identify its frequency correctly.

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Chapter 1: Signals and Systems

Major Topics in This Chapter

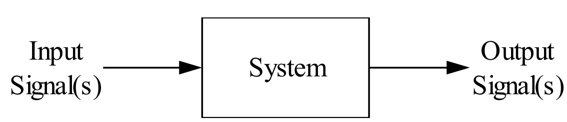
- Signals in continuous or discrete time
  - Classification: continuous- or discrete-time, even or odd, periodic or aperiodic
  - Energy and power
  - Operations on dependent or independent variables
- Elementary signals in continuous or discrete time
  - Exponentials: real, imaginary, complex. Sinusoids: steady, decaying, growing.
  - Singularity function: step, impulse, ramp, etc.
- Systems in continuous or discrete time
  - Representation
  - Examples
  - Properties: stability, memory, invertibility, time-invariance, linearity, causality

Signals

- A *signal* is a function of one or more independent variables, and often represents a variable associated with a physical system.
  - A *one-dimensional signal* depends on one independent variable.
    Example: a voltage on a wire as a function of time.
  - A *multi-dimensional signal* depends on more than one independent variable.
    Example: a still image is a function of two spatial coordinates, so it is a two-dimensional signal.

Systems

- A *system* performs a mapping on one or more signal(s) to produce new signal(s).



- Examples
  - One-dimensional: a speaker converts a voltage on a wire to a sound wave.
  - Two-dimensional: an image processing program converts a blurry image to a sharper image.

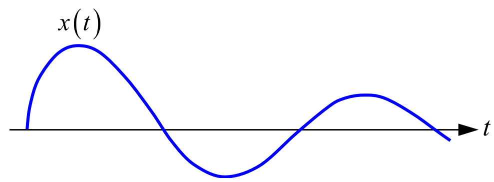
Classification of Signals

- In this course, we mainly discuss one-dimensional signals, where the independent variable is time.

Continuous-Time vs. Discrete-Time Signals

Continuous-Time Signals

- Let a real number t (which has units of seconds) denote *continuous time* (CT). A CT signal $x(t)$ is defined at all instants of time t, at least over a limited domain. A CT signal is sometimes referred to as a *waveform* in order to distinguish it from a discrete-time signal.

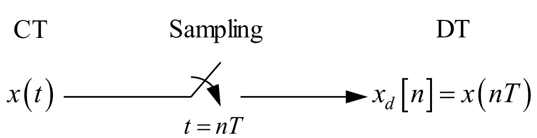


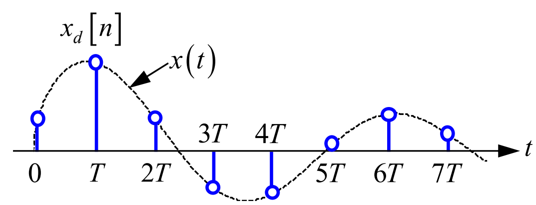
Discrete-Time Signals

- Let an integer n (which is unitless) denote *discrete time* (DT). A DT signal $x[n]$ is defined only at integer values of n. A DT signal is sometimes called a *sequence* in order to distinguish it from a CT signal.

- In some cases, the values of n correspond to *unequally spaced* intervals in CT.
  Example: a DT signal $x[n]$ represents the closing price of a stock.
  Values $\ldots, x[n-1], x[n], x[n+1], \ldots$ may represent values taken on …, Friday, Monday, Tuesday, ….

- In other cases, the values of n correspond to *equally spaced* intervals in CT.
  Example: a DT signal $x_d[n]$ corresponds to samples of a CT signal $x(t)$ taken at integer multiples of a *sampling interval* T. Music signals are often sampled at a sampling rate $1/T = 44.1$ kHz.





Random vs. Deterministic Signals

- *Random signals* are not precisely predictable, and are described by probability theory. For example, noise processes are random signals.

- *Deterministic signals* are completely specified functions of time. Almost all the signals studied in this course are deterministic.

How to Specify Deterministic Signals

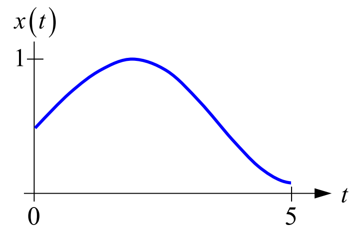
- A deterministic signal may be specified by various methods, depending on the nature of the signal.
  - *Formula*: applicable to CT or DT.
    Examples:

$$x(t) = Ae^{2t} \quad \forall t \qquad\qquad x[n] = 3n \quad \forall n .$$

- *Table*: applicable only to DT, finite domain.
  Example:

n	0	1	2	3
$x[n]$	2.6	1.4	3.7	9.3

- *Graph*: applicable only to CT, limited domain or DT, finite domain.
  Example:



- *Algorithm*: applicable only to DT.
  Example:

$$x[0]=1, \quad x[n+1]=x[n]-6.$$

Even or Odd Signals

- These classifications are equally applicable to CT or DT signals.
- Definitions. They are given here only for DT, but can be extended to CT in an obvious manner.
  - *Even and odd signals*

$$x_e[-n]=x_e[n] \quad \forall n \qquad\qquad x_o[-n]=-x_o[n] \quad \forall n$$

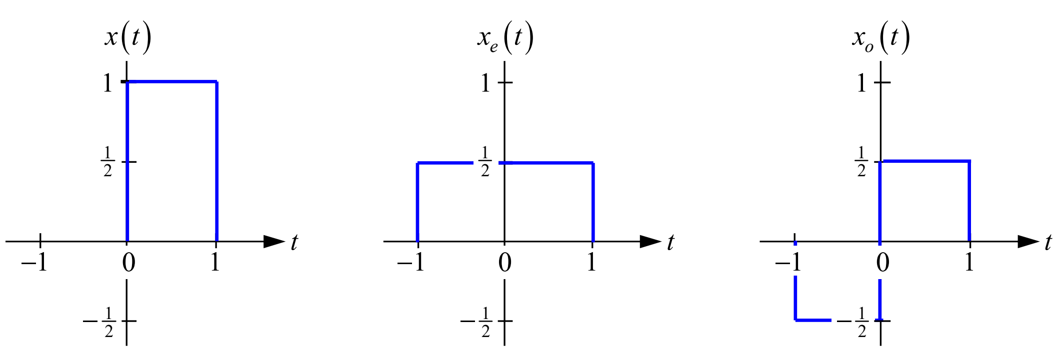
  - *Arbitrary signal as sum of even and odd parts*

$$x[n]=x_e[n]+x_o[n]$$

  Obtaining even and odd parts of an arbitrary signal

$$x_e[n]=\frac{1}{2}\left(x[n]+x[-n]\right) \qquad\qquad x_o[n]=\frac{1}{2}\left(x[n]-x[-n]\right)$$

- Example for CT.



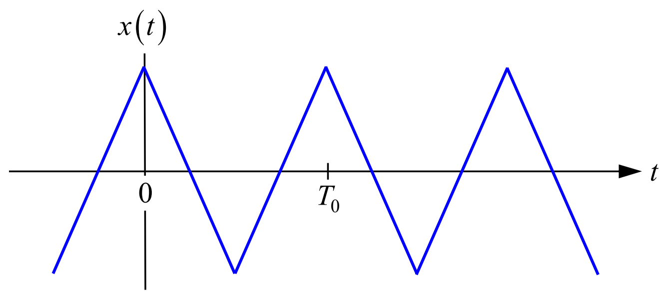
Periodic vs. Aperiodic Signals

Periodic Continuous-Time Signals

- A periodic CT signal satisfies

$$x(t) = x(t + T_0) \quad \forall t$$

for some T_0. The *period* is the smallest positive value of T_0 satisfying the above equation. (In CT, we denote the period by T_0 to avoid confusion with the sampling interval T used later in the course.) Example: a triangle wave with period T_0 is shown.



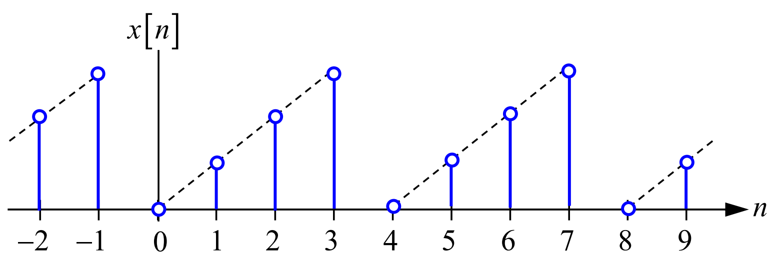
Periodic Discrete-Time Signals

- A periodic DT signal satisfies

$$x[n] = x[n + N] \quad \forall n$$

for some N. The *period* is the smallest positive value of N satisfying the above equation. (In DT, we need not define a special symbol for the period.)

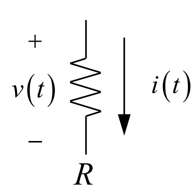
Example: a sawtooth signal with period $N = 4$ is shown.



Energy and Power of Signals

Continuous-Time Signals

- To motivate our general definitions of energy and power, we consider a voltage $v(t)$ across a resistor R inducing a current $i(t)$.



The instantaneous power dissipated is

$$p(t) = v(t)i(t) = i^2(t)R = \frac{v^2(t)}{R},$$

which has units of W. Setting $R = 1\ \Omega$, we obtain simpler expressions for the instantaneous power:

$$p(t) = i^2(t) = v^2(t).$$

The total energy dissipated is

$$E = \int_{-\infty}^{\infty} v^2(t)dt,$$

while the average power dissipated is

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} v^2(t)dt.$$

After we set $R = 1\ \Omega$, these expressions yield values of E and P proportional to, but not generally equal to, the true energy and true average power, which have units of J and W, respectively.

- Given a general complex-valued CT signal $x(t)$, we define the signal *energy* and the *average power* as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt,$$

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt.$$

These should be considered mathematical definitions of signal energy and power and are not generally equal to physical energy and power. Nevertheless, in many examples we will study, where $x(t)$ represents a physical variable such as voltage or mechanical displacement, these expressions yield values of E and P proportional to the physical energy and average power.

- Given a *periodic CT signal*, we can compute the average power by averaging over one period of duration T_0:

$$P = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} |x(t)|^2 dt,$$

where t_1, the start of the integration interval, may be chosen arbitrarily.

Discrete-Time Signals

- Given a general complex-valued signal $x[n]$, we define the signal *energy* and the *average power* as

$$E = \sum_{n=-\infty}^{\infty} \left| x[n] \right|^2$$

$$P = \lim_{N \to \infty} \frac{1}{2N} \sum_{n=-N}^{N} \left| x[n] \right|^2 .$$

- Given a *periodic DT signal*, we can compute the average power by averaging over one period of N samples:

$$P = \frac{1}{N} \sum_{n=n_1}^{n_1+N-1} \left| x[n] \right|^2 ,$$

where n_1, the start of the summation interval, may be chosen arbitrarily.

Classification of Signals

- Many important CT or DT signals can be classified into one of the following two categories.
  - An *energy signal* has finite energy and zero average power:

$$0 \le E < \infty , \quad P = 0 .$$

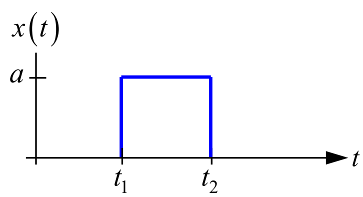
  - A *power signal* has finite average power and infinite energy:

$$0 < P < \infty , \quad E = \infty .$$

  - Some signals cannot be classified as energy signals or power signals, as we may see in homework problems.

Examples

1. A CT energy signal, as shown.



The total energy is

$$E = \int_{t_1}^{t_2} a^2 dt = a^2 \left(t_2 - t_1 \right) ,$$

while the average power is

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{t_1}^{t_2} a^2 dt = 0 .$$

2. A periodic CT power signal:

$$x(t) = a \sin \omega_0 t \ .$$

We use the trigonometric identity $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$. The energy is

$$E = \int_{-\infty}^{\infty} a^2 \sin^2 \omega_0 t dt = \frac{a^2}{2}\left[\underbrace{\int_{-\infty}^{\infty}(1)dt}_{\infty} - \underbrace{\int_{-\infty}^{\infty}\cos 2\omega_0 t dt}_{\text{bounded}}\right] = \infty .$$

To compute the average power, we first use the general definition, which is applicable to either periodic or aperiodic signals:

$$\begin{aligned}
P &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} a^2\sin^2\omega_0 t dt \\
&= \frac{a^2}{2}\lim_{T\to\infty}\frac{1}{2T}\left[\int_{-T}^{T}(1)dt - \int_{-T}^{T}\cos 2\omega_0 t dt\right] \\
&= \frac{a^2}{2}\lim_{T\to\infty}\frac{1}{2T}\left[\underbrace{2T}_{\text{grows large}} - \underbrace{\tfrac{1}{\omega_0}\sin 2\omega_0 T}_{\text{bounded}}\right] \\
&= \frac{a^2}{2}
\end{aligned} .$$

Since the signal is periodic, we may compute the average power by averaging over just one period of duration $T_0 = 2\pi/\omega_0$:

$$P = \frac{\omega_0}{2\pi}\int_{-\pi/\omega_0}^{\pi/\omega_0} a^2\sin^2\omega_0 t dt = \frac{a^2\omega_0}{4\pi}\left[\underbrace{\int_{-\pi/\omega_0}^{\pi/\omega_0}(1)dt}_{\frac{2\pi}{\omega_0}} - \underbrace{\int_{-\pi/\omega_0}^{\pi/\omega_0}\cos 2\omega_0 t dt}_{\frac{1}{\omega_0}\sin\left(2\omega_0\frac{\pi}{\omega_0}\right)=0}\right] = \frac{a^2}{2},$$

which agrees with the first approach.

Operating on Signals to Produce New Signals

Operations on the Dependent Variable

1. *Amplitude Scaling*

$$y(t) = Cx(t) \qquad y[n] = Cx[n].$$

2. *Addition*

$$y(t) = x_1(t) + x_2(t) \qquad y[n] = x_1[n] + x_2[n].$$

3. *Multiplication*

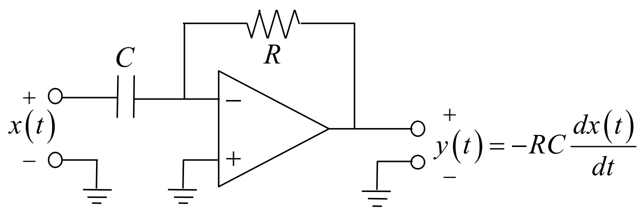
$$y(t) = x_1(t)\cdot x_2(t) \qquad y[n] = x_1[n]\cdot x_2[n].$$

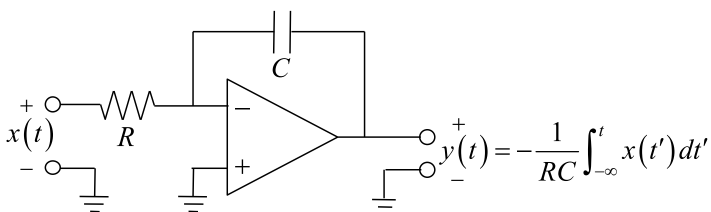
4a. Differentiation and Running Integration, CT Signals Only

$$y(t) = \frac{dx(t)}{dt}$$

$$y(t) = \int_{-\infty}^{t} x(t')dt' \ .$$

- These operations can be implemented on real-valued signals using the circuits shown, assuming the operational amplifiers are ideal (see *EE 102B Course Reader*, Chapter 5).





4b. First Difference and Running Summation (Accumulation), DT Signals Only

$$y[n] = x[n] - x[n-1]$$

$$y[n] = \sum_{k=-\infty}^{n} x[k] \ .$$

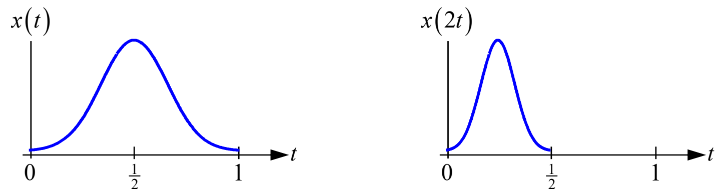
- Strictly speaking, these are not unique operations, but are derivable from addition and scaling. The first difference and running summation of DT signals are somewhat analogous to the differentiation and running integration of CT signals. As we will learn in EE 102B, however, given the samples of a CT signal, the first difference and running summation are not generally the best way to approximate samples of the derivative and running integral of the CT signal.

Operations on the Independent Variable

1a. CT Time Scaling. Consider a real, positive constant $a > 0$. Given a CT signal $x(t)$, the *time-scaled signal* is

$$y(t) = x(at).$$

- Time is *compressed* for $a > 1$. An example is shown for $a = 2$.

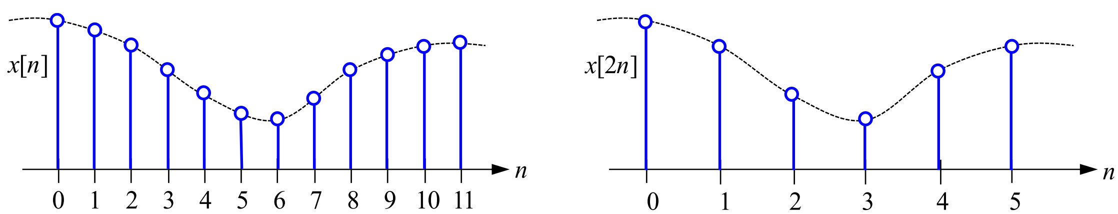


- Time is *expanded* for $0 < a < 1$.

1b. DT Time Compression. Consider a positive integer $k \geq 1$. Given a DT signal $x[n]$, the *compressed signal* is

$$y[n] = x[kn].$$

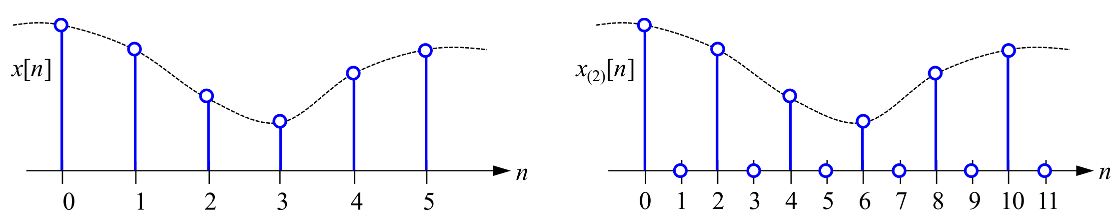
- If $k > 1$, samples of the signal are lost. An example is shown for $k = 2$.



1c. DT Time Expansion. Consider a positive integer $m \geq 1$. Given a DT signal $x[n]$, the *expanded signal* is

$$y[n] = x_{(m)}[n] = \begin{cases} x\left[\dfrac{n}{m}\right] & \dfrac{n}{m} \text{ integer} \\ 0 & \text{otherwise} \end{cases}.$$

- For any positive integer *m*, no samples of the signal are lost. An example is shown for $m = 2$.



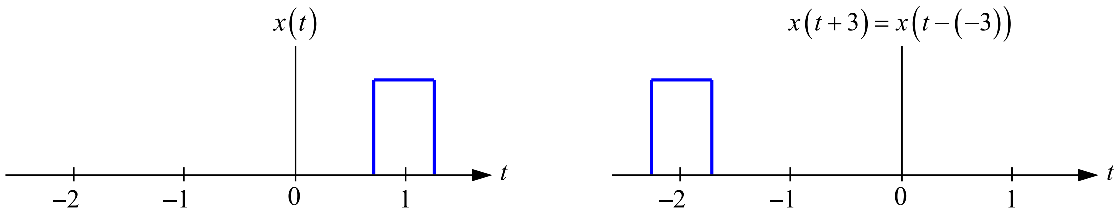
2. *Time Reversal (Reflection)*

$$y(t) = x(-t) \qquad y[n] = x[-n].$$

3a. CT Time Shifting. Consider a real constant t_0. Given a CT signal $x(t)$, the *time-shifted signal* is

$$y(t) = x(t - t_0).$$

- For $t_0 > 0$, the signal is *delayed* (shifted right).
- For $t_0 < 0$, the signal is *advanced* (shifted left). An example is shown with $t_0 = -3$.



3b. DT Time Shifting. Consider an integer n_0. Given a signal $x[n]$, the *time-shifted signal* is

$$y[n]=x[n-n_0].$$

Analogous to the CT case:

- For $n_0>0$, the signal is *delayed* (shifted right).
- For $n_0<0$, the signal is *advanced* (shifted left).

Combining Time Shifting and Time Scaling or Reversal

- We consider only CT signals. Given a signal $x(t)$, we consider

$$y(t)=x\big(a(t-t_0)\big),$$

where t_0 and a are real constants (a is not necessarily positive).

  - For $t_0\neq 0$, the signal is *time-shifted*.
  - For $|a|>1$, the signal is *compressed*, while for $|a|<1$, the signal is *expanded*.
  - For $a<0$, the signal is *time-reversed*.

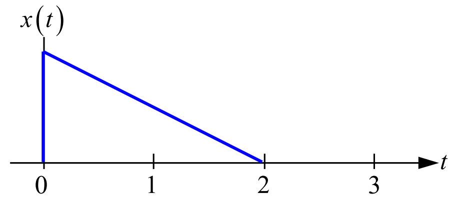
- In order to sketch $y(t)$, we perform three steps:

  1. Express $y(t)$ in the form $y(t)=x\big(a(t-t_0)\big)$.
  2. Form a time-scaled and/or time-reversed signal
  $$v(t)=x(at).$$
  3. Time shift $v(t)$ it in order to obtain the final signal
  $$y(t)=v(t-t_0)=x\big(a(t-t_0)\big).$$

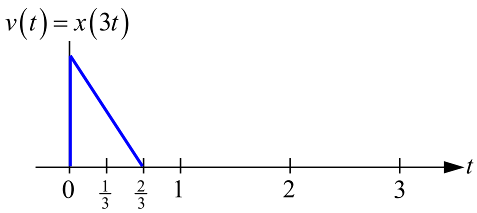
- Two examples are shown below. In the first example, time is shifted and compressed, while in the second example, time is shifted and reversed.

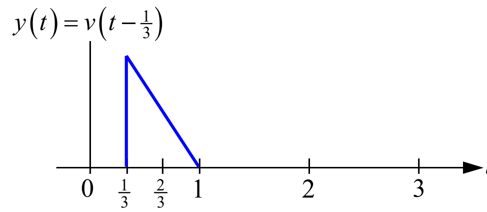
Example: time shifted and compressed

Given



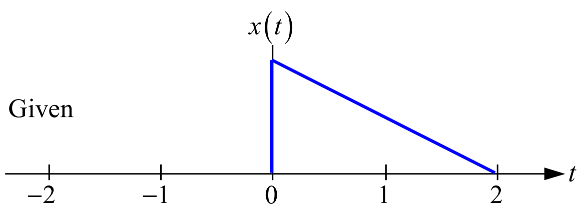
We are asked to sketch $y(t) = x(3t-1)$, which we express as $y(t) = x\left(3\left(t-\frac{1}{3}\right)\right)$.



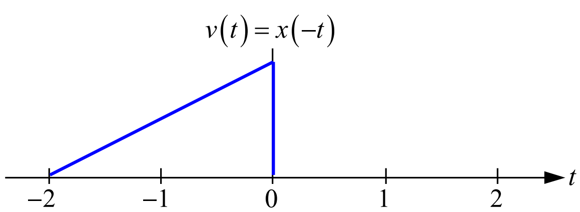


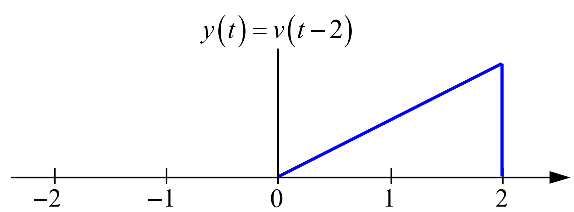
Example: time shifted and reversed

Given



We are asked to sketch $y(t) = x(-t+2)$, which we express as $y(t) = x\left(-\left(t-2\right)\right)$.





Elementary Signals

- We study two families of elementary signals, each in both CT and DT.
  - *Exponentials* (real, imaginary or complex) and *sinusoids* (steady, decaying or growing).
    - These are solutions to common first- or second-order differential or difference equations.
    - The exponential signals are eigenfunctions of linear time-invariant systems.
  - *Singularity functions*: impulse, step, ramp, etc.
    - These are building blocks for other signals.
    - These are important in the analysis of linear time-invariant systems.

Continuous-Time Exponentials and Sinusoids

- CT exponential signals are of the general form

$$x(t) = Ce^{at}, \tag{1}$$

  where a and C are complex constants.

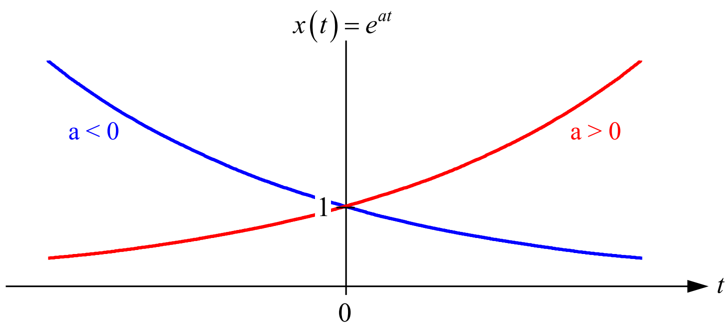
Real Exponentials

- These signals are of the form

$$x(t) = e^{at},$$

  where a is a real constant. We have set the constant C in (1) to unity.

- A CT real exponential is a purely decaying or growing function, depending on the sign of a, as shown.



Imaginary Exponentials

- Starting with the general form of a CT exponential (1), we choose the constants as follows:
  - Let $a = j\omega_0$, where ω_0 is a real constant called the *fundamental frequency* (rad/s).
  - Let $C = e^{j\phi}$, where ϕ is a real constant called the *initial phase* (rad).

- A CT imaginary exponential is of the form

$$\begin{aligned} x(t) &= e^{j\phi}e^{j\omega_0 t} = e^{j(\omega_0 t+\phi)} \\ &= \cos(\omega_0 t+\phi) + j\sin(\omega_0 t+\phi) \end{aligned},$$

  where we use Euler's relation to express its real and imaginary parts in terms of the cosine and sine.

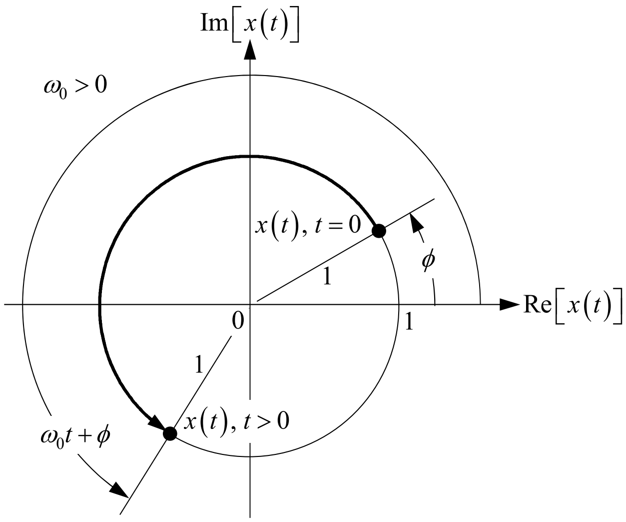
- Two characteristics distinguish a CT imaginary exponential from a DT imaginary exponential:

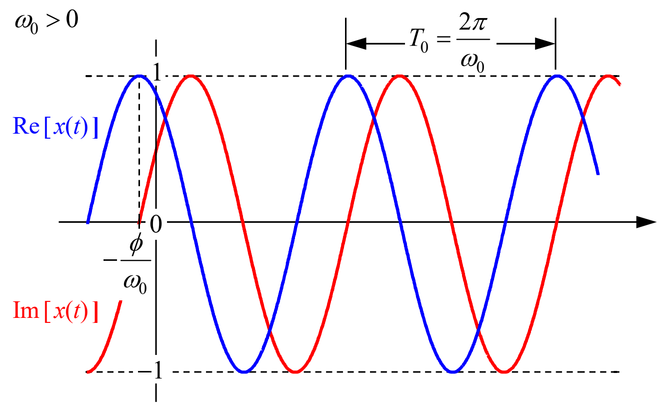
  1. It is periodic for any fundamental frequency ω_0 with period $T_0 = \dfrac{2\pi}{\omega_0}$. For any integer k:

$$e^{j\omega_0(t+T_0)} = e^{j\omega_0\left(t+\frac{2\pi}{\omega_0}\right)} = e^{j\omega_0 t}\underbrace{e^{j2\pi}}_{1} = e^{j\omega_0 t}.$$

  2. The signals for any distinct values of the fundamental frequency ω_0 are distinct.

- As shown below on the left, we can represent a CT imaginary exponential as a point on a unit circle in a complex plane. The figure assumes $\omega_0 > 0$, so the point rotates counterclockwise as time *t* increases.
- As shown below on the right, we can represent a CT imaginary exponential in terms of its real and imaginary signal components, $\cos(\omega_0 t+\phi)$ and $\sin(\omega_0 t+\phi)$.



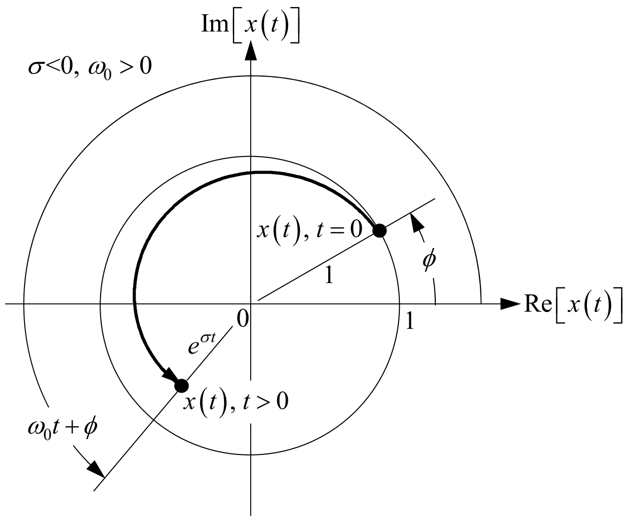


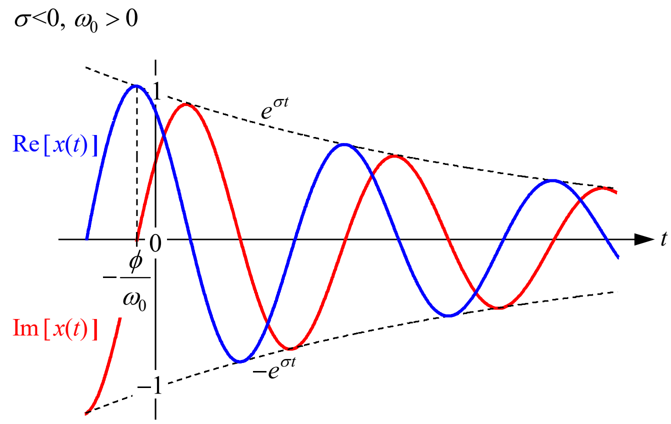
Complex Exponentials

- Now, starting with the general form of a CT exponential (1), we choose the constants as follows:
  - Let $a=\sigma+j\omega_0$, where σ and ω_0 are real constants.
  - Let $C=e^{j\phi}$, where ϕ is a real constant.
- A CT complex exponential is of the form

$$\begin{aligned} x(t) &= e^{j\phi}e^{(\sigma+j\omega_0)t} = e^{\sigma t}e^{j(\omega_0 t+\phi)} \\ &= e^{\sigma t}\left[\cos(\omega_0 t+\phi)+j\sin(\omega_0 t+\phi)\right] \end{aligned}.$$

- As shown below on the left, we can represent a CT complex exponential as a point in a complex plane. As *t* increases, the point traces out a spiral, depending on the values of ω_0 and σ. The figure assumes $\omega_0 > 0$ (so the point rotates counterclockwise) and $\sigma < 0$ (so it spirals in toward the origin).
- As shown below on the right, we can also represent a CT complex exponential in terms of its real and imaginary signal components, $e^{\sigma t}\cos(\omega_0 t+\phi)$ and $e^{\sigma t}\sin(\omega_0 t+\phi)$. These signals oscillate within the envelope defined by $\pm e^{\sigma t}$. We have chosen $\sigma < 0$, so these signal components are decaying (damped) sinusoids. If we had chosen $\sigma > 0$, they would be growing sinusoids.





Discrete-Time Exponentials and Sinusoids

- DT exponential signals can be written in the general form

$$x[n] = Ce^{\beta n}, \tag{2}$$

where β and C are complex constants. Usually we define a complex constant $\alpha = e^{\beta}$, and write a DT exponential signal in a form that looks different but is equivalent to (2):

$$x[n] = C\alpha^n. \tag{2$'$}$$

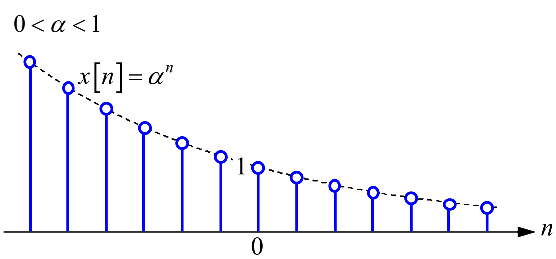
Real Exponentials

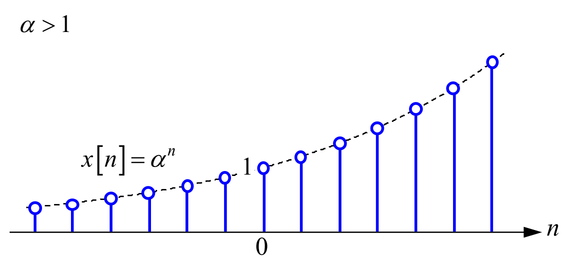
- These signals are of the form

$$x[n] = \alpha^n,$$

where α is a real constant (β need not be real). We have set the constant C in (2′) to unity.

- When $\alpha > 0$, a DT real exponential is a purely decaying function ($0 < \alpha < 1$) or growing function ($\alpha > 1$), as shown.

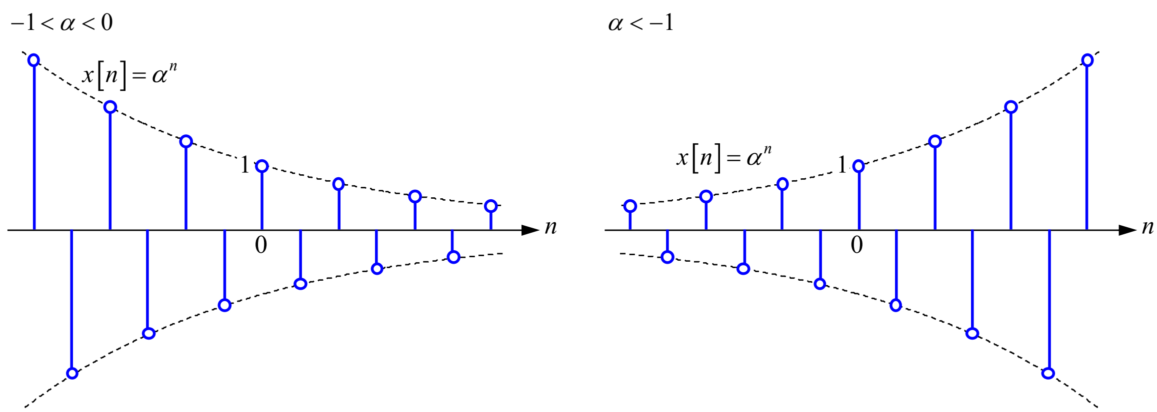




- When $\alpha < 0$, a DT real exponential alternates sign as a function of *n*. We can show this by writing $\alpha = -|\alpha|$ so that

$$x[n] = (-1)^n |\alpha|^n.$$

As *n* increases, the envelope of the signal decays ($-1 < \alpha < 0$) or grows ($\alpha < -1$), as shown.



Imaginary Exponentials

- Starting with the general form of a DT exponential (2) or (2′), we choose the constants as follows:
  - Let $\beta = j\Omega_0$, where Ω_0 is a real constant called the *fundamental frequency* (rad).

    It follows that $\alpha = e^{\beta} = e^{j\Omega_0}$.
  - Let $C = e^{j\phi}$, where ϕ is a real constant, the *initial phase* (rad).
- A DT imaginary exponential is of the form

$$\begin{aligned} x[n] &= e^{j\phi} e^{j\Omega_0 n} = e^{j(\Omega_0 n + \phi)} \\ &= \cos(\Omega_0 n + \phi) + j\sin(\Omega_0 n + \phi) \end{aligned}.$$

- Two characteristics distinguish a DT imaginary exponential from a CT imaginary exponential:

1. It is periodic only if the fundamental frequency Ω_0 is a rational multiple of 2π.
   - Periodicity requires that

$$e^{j\Omega_0(n+N)} = e^{j\Omega_0 n} \quad \forall n \quad \text{for some integer } N > 0.$$

   This can be simplified to

$$e^{j\Omega_0 N} = 1,$$

   which requires that

$$\Omega_0 N = m2\pi \quad \text{for some integer } m,$$

   or

$$\Omega_0 = \frac{m}{N} 2\pi.$$

   - We conclude that a DT imaginary exponential is periodic only if the fundamental frequency Ω_0 is a rational multiple m/N of 2π.
   - The period is the value of N when the ratio m/N is expressed in lowest terms.
2. The signals for distinct values of the fundamental frequency Ω_0 are distinct only if the values of Ω_0 do not differ by an integer multiple of 2π.

- Consider fundamental frequencies differing by an integer k times 2π, Ω_0 and $\Omega_0 + k2\pi$. The signal at frequency $\Omega_0 + k2\pi$ is

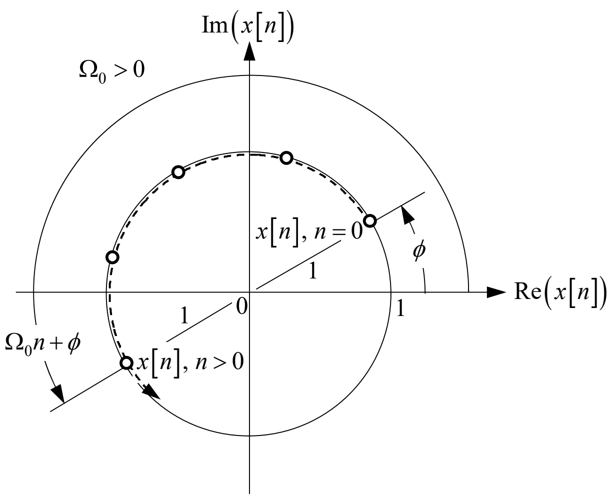
$$e^{j(\Omega_0 + k2\pi)n} = e^{j\Omega_0 n} \underbrace{e^{jk2\pi n}}_{1} = e^{j\Omega_0 n},$$

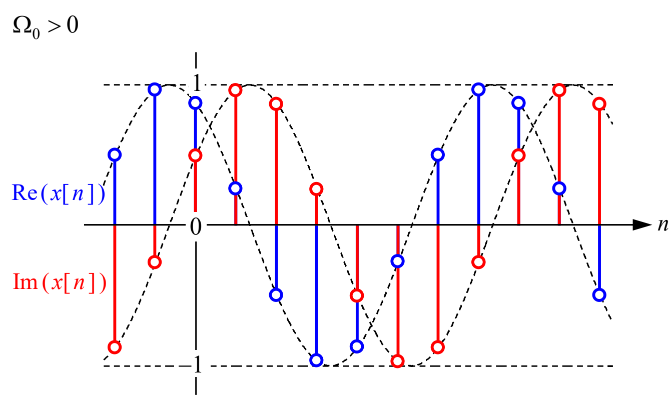
  which is identical to the signal at frequency Ω_0.

- To avoid ambiguity, we often choose the fundamental frequency Ω_0 to lie only within an interval of length 2π, such as $-\pi \le \Omega_0 < \pi$ or $0 \le \Omega_0 < 2\pi$.

- As shown below on the left, we can represent a DT imaginary exponential as a point on a unit-radius circle in a complex plane. The figure assumes $\Omega_0 > 0$, so the point rotates counterclockwise as time n increases. This figure helps illustrate why the signal point will eventually revisit the same sequence of locations, corresponding to a periodic signal, only if Ω_0 is a rational multiple of 2π.

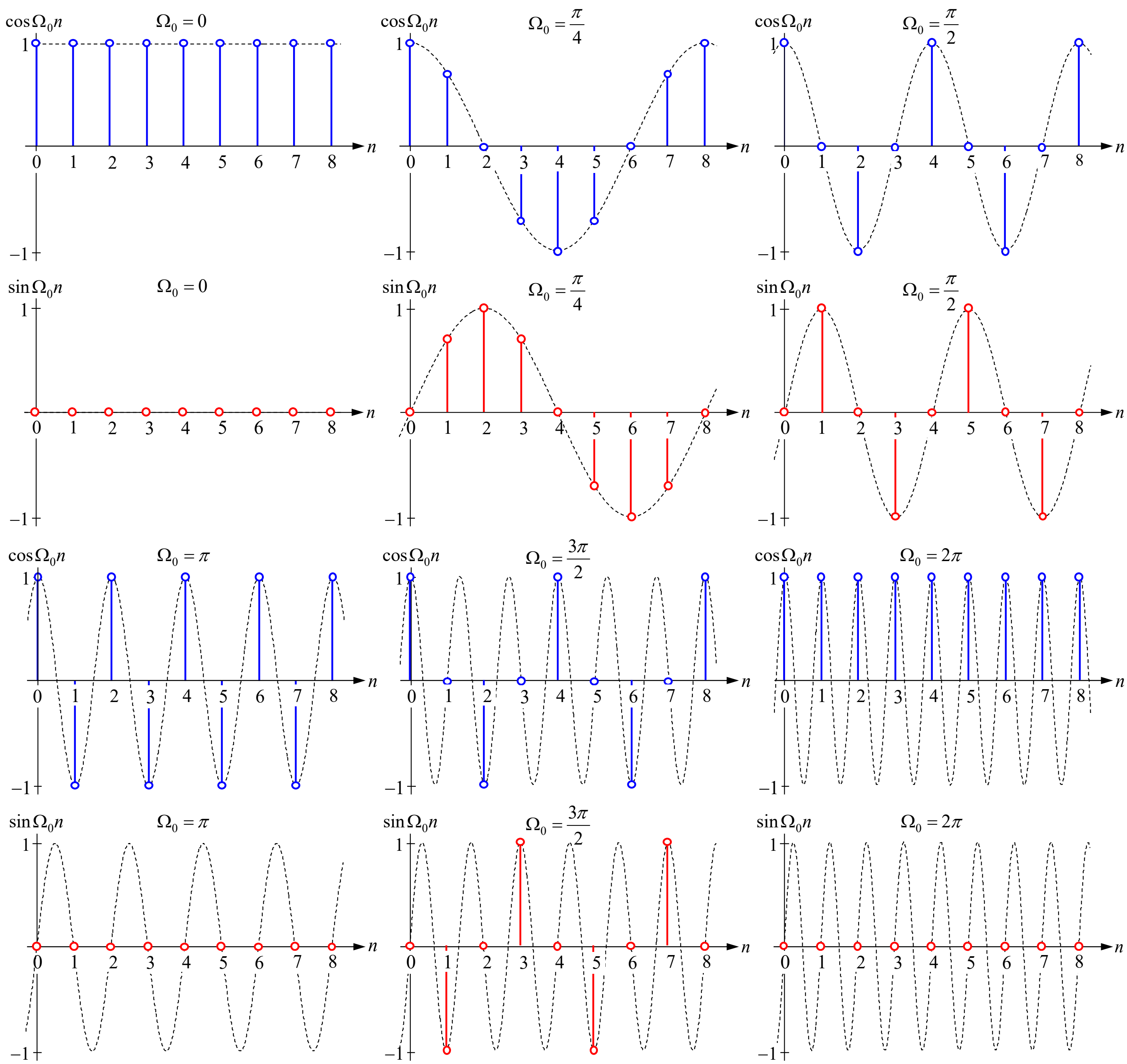
- As shown below on the right, we can represent a DT imaginary exponential in terms of its real and imaginary signal components, $\cos(\Omega_0 n + \phi)$ and $\sin(\Omega_0 n + \phi)$.





- The figure below shows the DT sinusoids $\cos\Omega_0 n$ (in blue) and $\sin\Omega_0 n$ (in red) for several values of Ω_0 between 0 and 2π. The initial phase ϕ has been set to zero. All the values of Ω_0 considered are rational multiples of 2π, so all the DT sinusoids are periodic. The dashed lines show the values the signals would assume if time n were a continuous variable.

- Consider the signals for different values of the fundamental frequency Ω_0. Starting at $\Omega_0 = 0$, as Ω_0 increases toward π, the DT sinusoids oscillate between -1 and $+1$ at an *increasing* rate. Upon reaching $\Omega_0 = \pi$, $\cos\Omega_0 n = \cos\pi n = (-1)^n$, so $\cos\Omega_0 n$ alternates between -1 and $+1$ in successive values of n, which is the *highest rate of oscillation possible*. As Ω_0 increases beyond π, the DT sinusoids oscillate between -1 and $+1$ at a *decreasing* rate. Upon reaching $\Omega_0 = 2\pi$, the DT sinusoids are identical to those for $\Omega_0 = 0$. This illustrates that DT sinusoids whose fundamental frequencies differ by 2π (or a multiple of 2π) are identical.

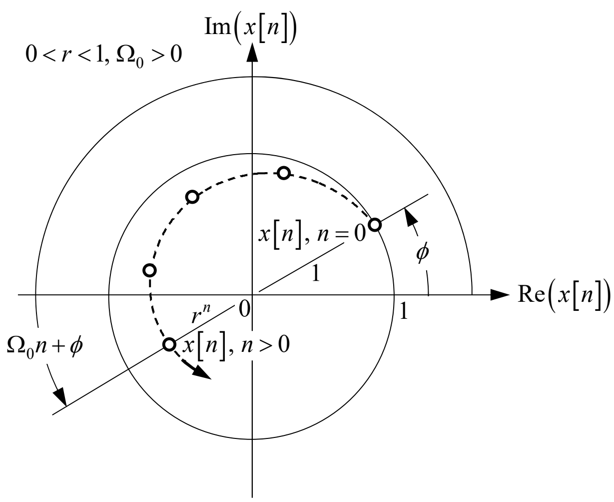


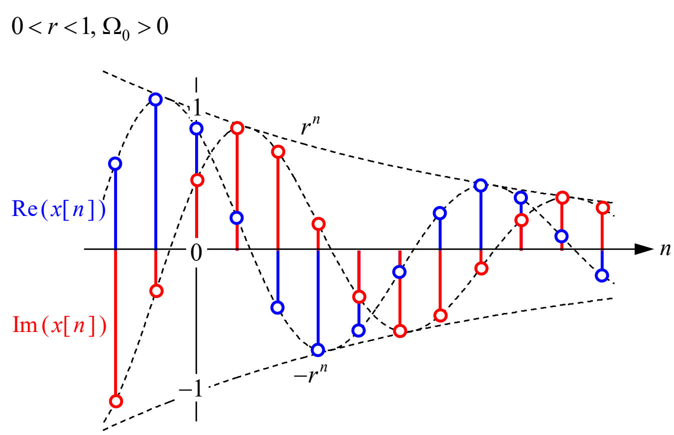
Complex Exponentials

- Now, starting with the general form of the DT exponential (2) or (2′), we choose the constants as follows:
    - Let $\beta = \ln r + j\Omega_0$, where Ω_0 and r are real constants, and $r > 0$. It follows that $\alpha = e^{\beta} = re^{j\Omega_0}$.
    - Let $C = e^{j\phi}$, where ϕ is a real constant.
- A DT complex exponential is of the form

$$\begin{aligned} x[n] &= e^{j\phi}\left(re^{j\Omega_0}\right)^n = r^n e^{j(\Omega_0 n+\phi)} \\ &= r^n\left[\cos\left(\Omega_0 n+\phi\right)+j\sin\left(\Omega_0 n+\phi\right)\right] \end{aligned}.$$

- As shown below on the left, we can represent a DT complex exponential as a point in a complex plane. As n increases, the point traces out a spiral, depending on the values of Ω_0 and r. The figure assumes $\Omega_0 > 0$ (so the point rotates counterclockwise) and $0 < r < 1$ (so it spirals in toward the origin).
- As shown below on the right, we can also represent a DT complex exponential in terms of its real and imaginary signal components, $r^n\cos\left(\Omega_0 n+\phi\right)$ and $r^n\sin\left(\Omega_0 n+\phi\right)$. These signals oscillate within the envelope defined by $\pm r^n$. We have chosen $0 < r < 1$, so the signals decay as n increases. If we had chosen $r > 1$, the signals would grow as n increases.





Continuous-Time Singularity Functions

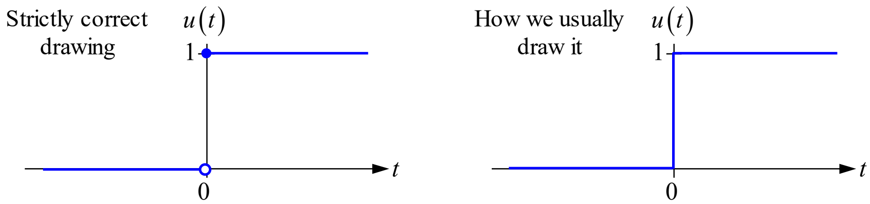
- A CT "singularity function" is so-called because the function, or a derivative of the function, becomes infinite at one or more values of the independent variable t.

Unit Step Function

- The CT unit step function is defined as

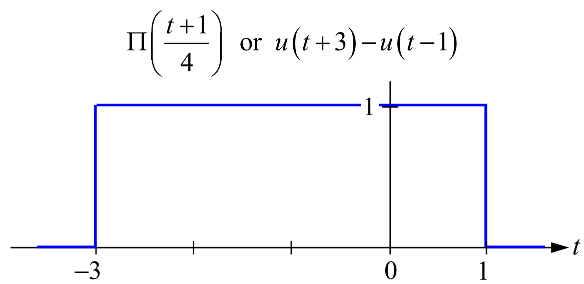
$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}.$$

A strictly correct drawing is shown below on the left. We typically draw it as shown below on the right.



Throughout the course, we will work with CT signals that have discontinuities. If the discontinuity is *finite*, as in a CT step function, we are not concerned with the value the signal assumes precisely at the discontinuity. This is because the modulus of the difference between any two finite-valued CT signals, when integrated over an infinitesimal interval, yields zero. By contrast, it the discontinuity is *infinite*, as in a CT impulse function, we must pay attention to the value of the signal at the discontinuity.

- The CT unit step function is often used as a building block to represent more complicated signals. For example, consider $\Pi\left((t+1)/4\right)$, a rectangular pulse of width 4, centered at $t=-1$ (see the definition of $\Pi(x)$, Appendix, page 290). We can express $\Pi\left((t+1)/4\right)$ as $u(t+3)-u(t-1)$. These signals are shown below.

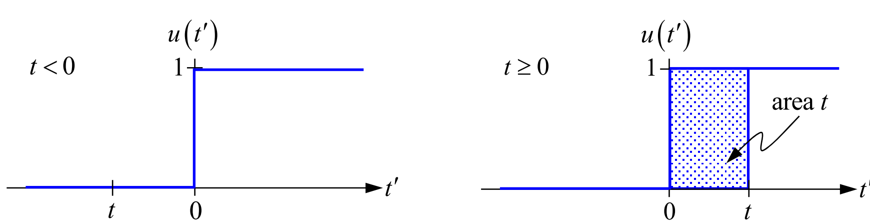


Unit Ramp Function

- The CT unit ramp function is the running integral of the unit step function:

$$r(t)=\int_{-\infty}^{t} u(t')\,dt' .$$

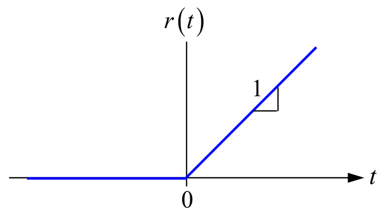
We perform the integration considering the cases $t<0$ and $t\geq 0$, as shown below.



We obtain

$$r(t)=\begin{cases}0 & t<0\\ t & t\geq 0\end{cases}=t\cdot u(t).$$

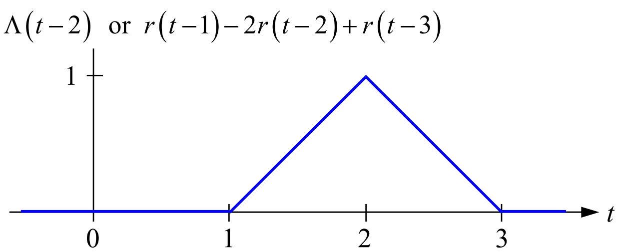
A graph of the unit ramp function is shown below.



- Conversely, the unit step function is the derivative of the unit ramp function:

$$u(t) = \frac{d}{dt} r(t).$$

- The CT unit ramp function may be used to synthesize more complicated signals. For example, consider a unit triangular pulse (which has width 2 at its base) centered at $t = 2$, given by $\Lambda(t-2)$ (see the definition of $\Lambda(x)$, Appendix, page 290). This can be expressed as $r(t-1) - 2r(t-2) + r(t-3)$. The signals are shown below.



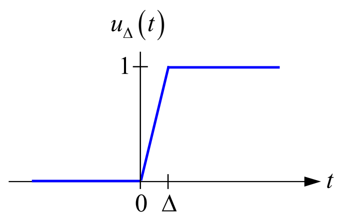
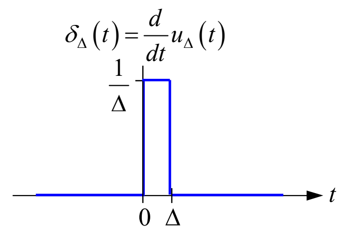
Unit Impulse Function

- The CT unit impulse function is a tool for analyzing systems. In system analysis, the impulse is an idealization of an input signal that is a pulse whose duration is very short in comparison to the response time of the system.

- The unit impulse function $\delta(t)$ is formally defined as the derivative of the unit step function:

$$\delta(t) \stackrel{d}{=} \frac{d}{dt} u(t).$$

Since the unit step function $u(t)$ is discontinuous at $t = 0$, however, it is not straightforward to evaluate its derivative. To solve this problem, we approximate $u(t)$ by a function $u_\Delta(t)$ that is continuous at all t, as shown below on the left. Over the interval $0 < t < \Delta$, the value of $u_\Delta(t)$ ramps linearly from 0 to 1, so $u_\Delta(t)$ has a slope of $1/\Delta$ over the interval. We are going to approximate $\delta(t)$ by a function $\delta_\Delta(t)$, which is the derivative of $u_\Delta(t)$. As shown below on the right, $\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t)$ is a rectangular pulse that has a height $1/\Delta$ over the interval $0 < t < \Delta$. Notice that the area under $\delta_\Delta(t)$ is unity.

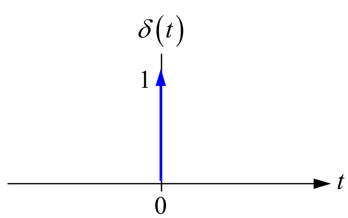
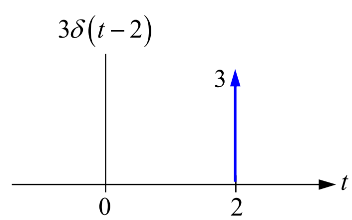
In the limit $\Delta \to 0$, $u_\Delta(t)$ becomes $u(t)$:

$$u(t) = \lim_{\Delta \to 0} u_\Delta(t),$$

so in that limit, $\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t)$ becomes $\delta(t)$:

$$\delta(t) = \lim_{\Delta \to 0} \frac{d}{dt} u_\Delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t).$$

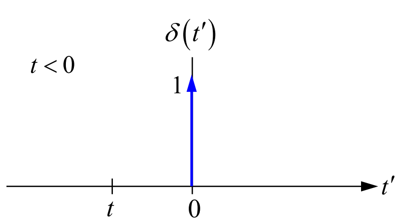
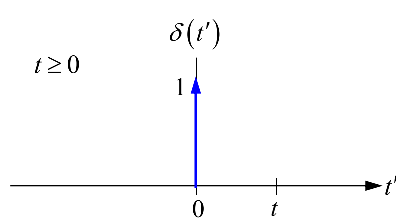
- Based on this argument, we can consider a unit impulse function $\delta(t)$ to be a rectangular pulse $\delta_\Delta(t)$ at $t = 0$ that has *zero width*, *infinite height*, and *unit area*.
- The unit impulse function $\delta(t)$ is drawn as an arrow, and is labeled by the area under the function. In other words, $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$ and $\int_{-\infty}^{\infty} 3\delta(t-2)\, dt = 3$.

- Since the unit impulse is the derivative of the unit step, conversely, the unit step is the running integral of the unit impulse:

$$u(t) = \int_{-\infty}^{t} \delta(t')\, dt'.$$

We perform the integration considering the cases $t < 0$ and $t \geq 0$, as shown below.

We obtain

$$\int_{-\infty}^{t} \delta(t')\, dt' = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} = u(t).$$

Properties of the Unit Impulse Function

1. *Evenness.*

$$\delta(-t)=\delta(t) \ .$$

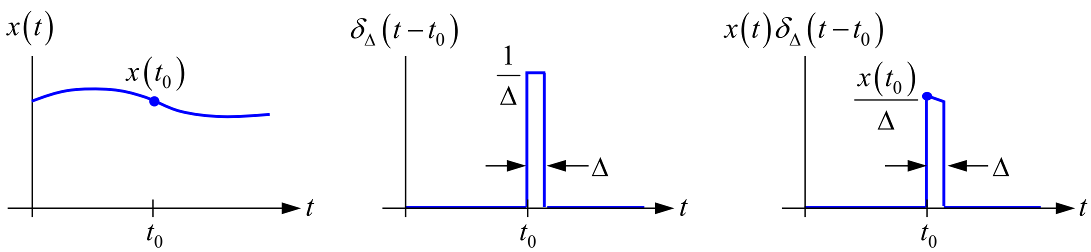
2. *Time Scaling.* Let a be a real constant.

$$\delta(at)=\frac{1}{|a|}\delta(t) \, .$$

3a. *Sampling.* Assume $x(t)$ is continuous at $t=t_0$. Then

$$x(t)\delta(t-t_0)=x(t_0)\delta(t-t_0) \, .$$

- To prove this property, we represent $\delta(t-t_0)$ by $\delta_\Delta(t-t_0)=\frac{d}{dt}u_\Delta(t-t_0)$, as shown below. Then we take the limit $\Delta \to 0$.



3b. *Sampling (Alternate Statement).*

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt=x(t_0) \, .$$

- This form of the sampling property is obtained by integrating sampling property 3a over time.

3c. *Sifting.*

$$x(t)=\int_{-\infty}^{\infty} x(t')\delta(t-t')\,dt' \ .$$

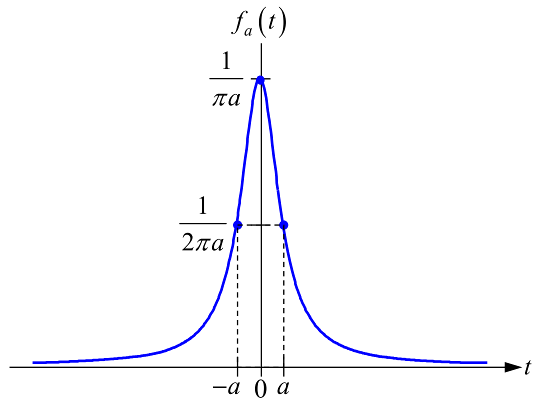
- From a strictly mathematical standpoint, the sifting property is equivalent to the sampling property 3b, and is obtained by using evenness property 1 and changing variables.
- The sifting property is used in Chapter 2 to analyze linear time-invariant systems, where its interpretation is entirely different from that of the sampling property.

Alternate Representation of the Unit Impulse Function

- Thus far, we have represented the unit impulse $\delta(t)$ as rectangular pulse $\delta_\Delta(t)$ in the limit of zero width, infinite height and unit area. It is often useful to represent $\delta(t)$ using a smooth (differentiable) function that is peaked near $t=0$, in the limit that the function has zero width and infinite height, while maintaining unit area. As an example, consider the function

$$f_a(t)=\frac{a}{\pi}\frac{1}{a^2+t^2},$$

which is sometimes called the *Lorentzian line shape function*. It is plotted here.



- The real parameter a governs the height and width of $f_a(t)$. It is easy to show that $f_a(t)$ has:
  - Height $f_a(0)=\frac{1}{a\pi}$.
  - Full width at half-height $2a$, since $\frac{f_a(\pm a)}{f_a(0)}=\frac{1}{2}$.
  - Unit area $\int_{-\infty}^{\infty} f_a(t)\,dt=1$ for any value of a.
- Thus we can represent the unit impulse $\delta(t)$ as $f_a(t)$ in the limit that a becomes small:

$$\delta(t)=\lim_{a\to 0} f_a(t).$$

This representation of the unit impulse arises in deriving some CT Fourier transforms in Chapter 4.

Unit Doublet Function

- The *unit doublet function* is the derivative of the unit impulse function:

$$\delta'(t)=\frac{d\delta(t)}{dt}.$$

We will discuss the unit doublet in analyzing some linear time-invariant systems in Chapter 2.

Discrete-Time Singularity Functions

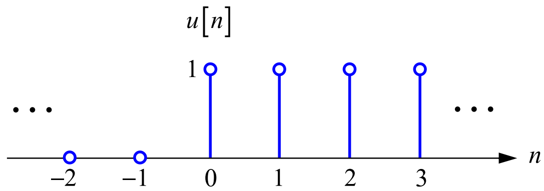
- The DT "singularity functions" are not actually singular. They are so-called because of their similarity to the CT singularity functions.

Unit Step Function

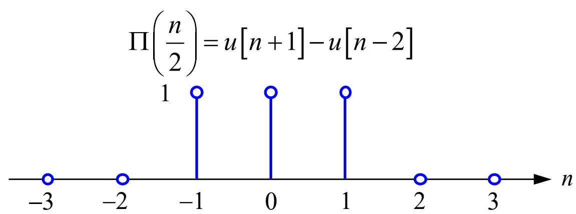
- The DT unit step function is defined as

$$u[n]=\begin{cases}0 & n<0\\ 1 & n\geq 0\end{cases},$$

and is shown below.



- The DT step function is often used as a building block to construct other DT signals. For example, consider $\Pi(n/2)$, which is a rectangular pulse of three nonzero samples, centered at the origin (see the definition of $\Pi(x)$, Appendix, page 290). It can be expressed as $u[n+1]-u[n-2]$. The two signals, as shown below, are identical.

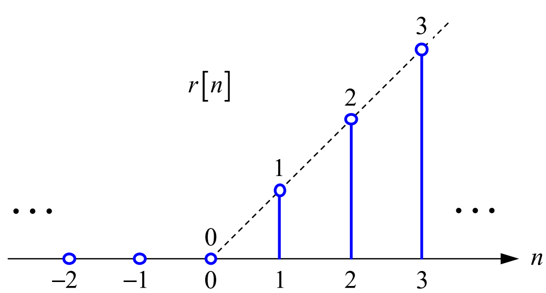


Unit Ramp Function

- The DT unit ramp function is defined as

$$r[n]=\begin{cases}0 & n<0\\ n & n\geq 0\end{cases}=n\cdot u[n],$$

and is shown below.



- It is related to the DT unit step function through running summation and first difference as follows:

$$r[n]=\sum_{k=-\infty}^{n-1}u[k]$$

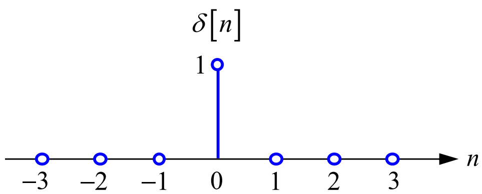
$$u[n]=r[n+1]-r[n].$$

Unit Impulse Function

- The DT unit impulse function is defined as

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases},$$

and is shown below.



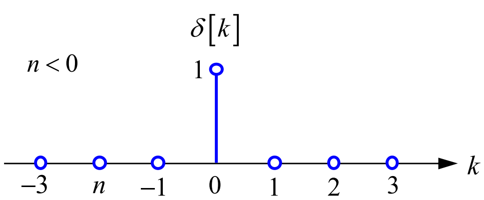
- The unit impulse function $\delta[n]$ is the first difference of the unit step function $u[n]$:

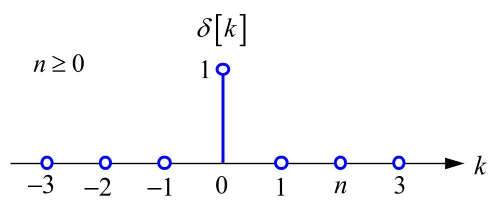
$$\delta[n] = u[n] - u[n-1].$$

- Conversely, the unit step function $u[n]$ is the running summation of the unit impulse function $\delta[n]$:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k].$$

To perform the summation, we consider the cases $n < 0$ and $n \geq 0$, as shown below.





We obtain

$$\sum_{k=-\infty}^{n} \delta[k] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases} = u[n].$$

Properties of the Unit Impulse Function

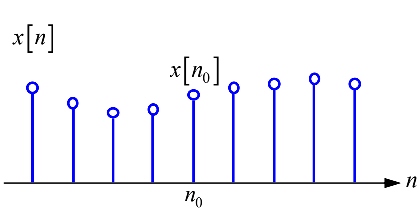
1. Evenness.

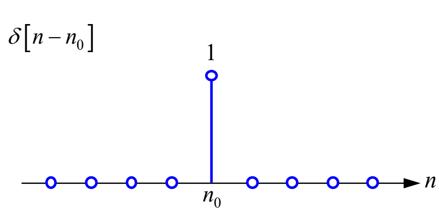
$$\delta[-n] = \delta[n] .$$

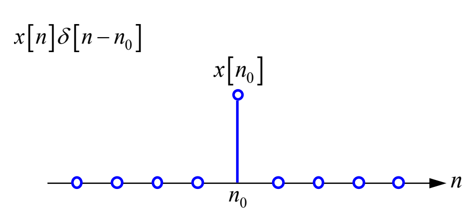
2a. Sampling. Given a signal $x[n]$,

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0].$$

- This property follows from the fact that $\delta[n-n_0]$ is nonzero only for $n = n_0$, as illustrated below.







2b. Sampling (Alternate Statement).

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n-n_0] = x[n_0].$$

- This form of the sampling property is obtained by summing sampling property 2a over time.

2c. Sifting.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

- The sifting property is mathematically equivalent to the sampling property 2b. It is obtained from property 2b by using evenness property 1 and changing variables.
- The sifting property is used in Chapter 2 in the analysis of linear time-invariant systems.

Representing Systems

Block Diagram

- A CT or DT system H, having input x and output y, can be represented as a block diagram, as shown.

$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t) \qquad x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

Symbolic

- A CT or DT system H, having input x and output y, can be represented with the following notation:

$$y(t) = H\{x(t)\} \qquad\qquad y[n] = H\{x[n]\}.$$

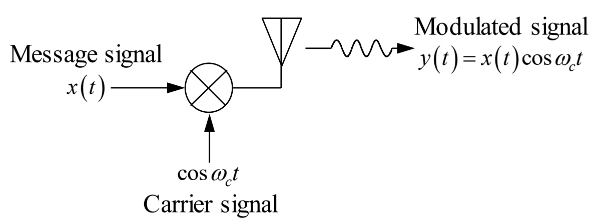
We may read such an expression as "the system H acts on input x to yield output y".

Systems Examples

- In this section, we present several examples of CT and DT systems. In the following section, these examples will be used to illustrate various properties of systems.

1. CT Amplitude Modulation.

- *Amplitude modulation* (AM) is used to shift a signal from low frequencies to high frequencies, often so it can be transmitted as an electromagnetic wave. (A particular method, double-sideband amplitude modulation with suppressed carrier, is shown here. Several AM methods are described in Chapter 7.)

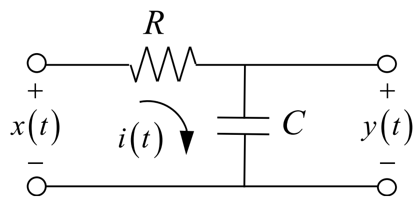


In this form of AM, an information-bearing *message signal* $x(t)$ is multiplied by a sinusoidal carrier $\cos\omega_c t$, where ω_c is a *carrier frequency*, yielding a *modulated signal*

$$y(t) = x(t)\cos\omega_c t \ . \tag{3}$$

2. CT First-Order Lowpass Filter.

- A CT first-order lowpass filter, shown below, may be used intentionally to smooth out unwanted noise or high-frequency components appearing in CT signals. In circuits, resistance, inductance and capacitance are unavoidable, causing lowpass filtering to occur even when it is not desired, potentially attenuating or distorting high-frequency or high-speed signals.



- Using elementary circuit analysis, we can relate the input voltage $x(t)$ and output voltage $y(t)$ to the current $i(t)$:

$$x(t) = i(t)R + y(t)$$

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} i(t')dt' \ .$$

By differentiating the second equation with respect to t, solving for $i(t)$, and substituting in the first equation, we obtain a first-order differential equation relating $x(t)$ and $y(t)$:

$$\tau\frac{dy}{dt} + y(t) = x(t) \ , \tag{4}$$

where $\tau = RC$. This is a *first-order* equation because the highest derivative of the output $y(t)$ appearing is the first derivative.

- We will assume that the system is *at initial rest*. In general, an initial rest condition means that the *output is zero before any input is applied*. Here we assume that the input $x(t)$ is specified explicitly for all time starting at $t = -\infty$, so the initial rest condition is expressed as

$$y(t) = 0 \ \text{ at } \ t = -\infty . \tag{5}$$

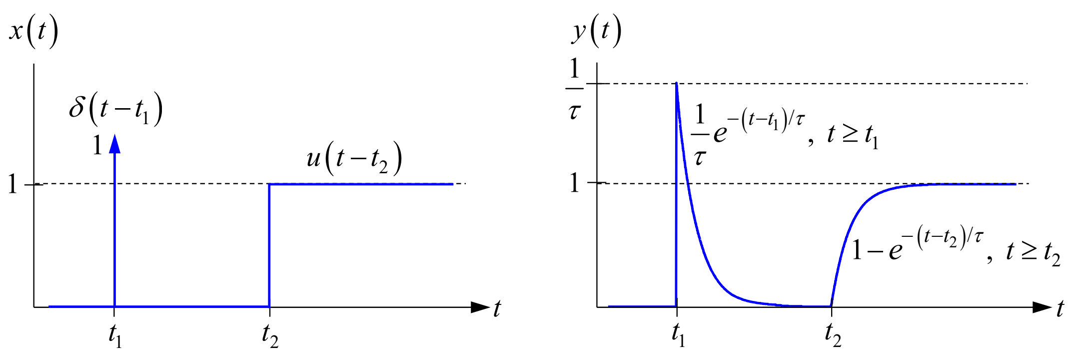
The unique solution to differential equation (4) satisfying initial condition (5) is

$$y(t) = \frac{1}{\tau}\int_{-\infty}^{t} e^{-\frac{t-t'}{\tau}} x(t')dt' \ . \tag{6}$$

Some comments about the solution (6):

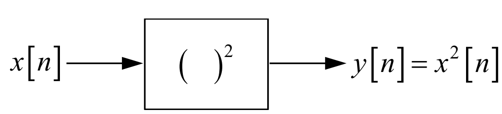
- The output $y(t)$ at time t is a weighted sum of inputs $x(t')$ at all past and present times $t' \le t$, with weighting factor $(1/\tau)e^{-\frac{t-t'}{\tau}}$. This weighting factor gives more weight to recent inputs (small $t-t'$) and less weight to older inputs (large $t-t'$).
- Expression (6) is in the form of a *convolution integral* between the input signal $x(t)$ and an impulse response $h(t)$ describing the system. This will be explained fully in Chapter 2.
- You should verify that (6) solves differential equation (4) and satisfies the initial rest condition (5). In doing this, it is helpful to write (6) as $y(t) = (1/\tau)\exp(-t/\tau)\int_{-\infty}^{t}\exp(t'/\tau)x(t')dt'$, a product of two functions of t. Differentiate it with respect to t using the product rule of differentiation.
- Some CT or DT systems in EE 102A are specified by linear, constant-coefficient differential or difference equations. You will not be asked to solve these equations. You will be given their solutions and may be asked to verify them. Methods for solving linear constant-coefficient equations efficiently using Laplace transforms or Z transforms are taught in EE 102B.

- A representative input signal $x(t)$ and the resulting output signal $y(t)$ are shown below. These will be explained fully in Chapter 2.



3. DT Squarer.

- A DT squaring system is shown.

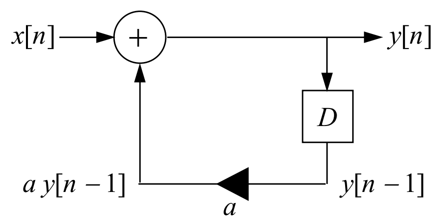


Given an input signal $x[n]$, the output signal is

$$y[n] = x^2[n]. \tag{7}$$

4. DT First-Order Lowpass Filter.

- A DT first-order system is shown below. Assuming a is real and $0 \le a \le 1$, it is a lowpass filter, which may be used to smooth out unwanted noise or high-frequency components of DT signals. Using other values of a, this system is a highpass filter or can model accrual of interest in a bank account or a loan.



- In the diagram above, the circle containing "+" denotes addition, the box containing "*D*" denotes a delay by one time step, and the triangle labeled by "*a*" denotes multiplication by a constant *a*. For now, we assume *a* is real and $0 \le a \le 1$. The input $x[n]$ and output $y[n]$ are related by a difference equation

$$y[n] = x[n] + ay[n-1]. \tag{8}$$

Equation (8) is called a *first-order* difference equation because at time *n*, the output $y[n]$ is dependent on past outputs back to $y[n-1]$.

- We will assume that the input $x[n]$ is specified explicitly for all time starting at $n = -\infty$. As for the CT lowpass filter, we assume the system is at initial rest, so the output is zero before any input is applied:

$$y[n] = 0 \text{ at } n = -\infty. \tag{9}$$

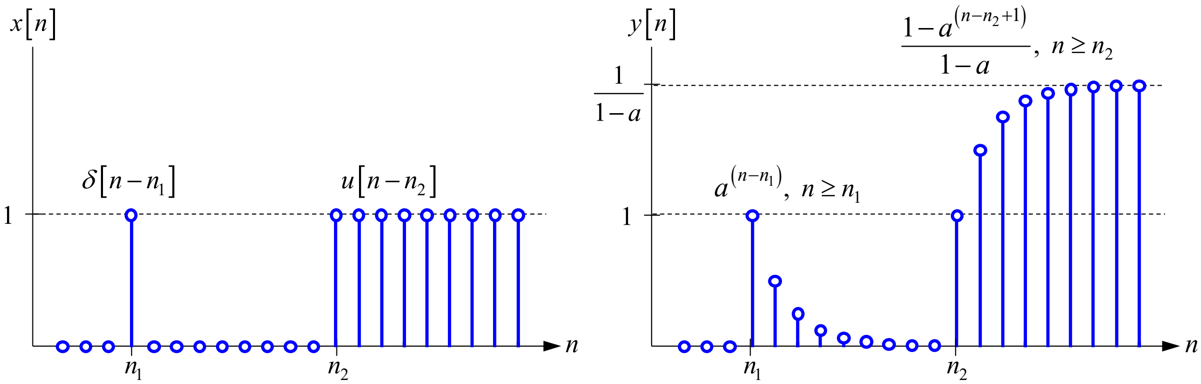
The unique solution to difference equation (8) satisfying initial condition (9) is

$$y[n] = \sum_{k=-\infty}^{n} a^{n-k} x[k]. \tag{10}$$

Some comments about the solution (10):

  - The output $y[n]$ at time n is a weighted sum of inputs $x[k]$ at all past and present times $k \le n$, with weighting factor a^{n-k}. Under the assumption that $0 \le a \le 1$, this gives more weight to recent inputs (small $n-k$) and less weight to older inputs (large $n-k$).
  - Expression (10) is in the form of a *convolution sum* between the input signal $x[n]$ and an impulse response $h[n]$ describing the system. This will be explained fully in Chapter 2.
  - You should verify that (10) solves difference equation (8) and satisfies the initial rest condition (9).

- A representative input signal $x[n]$ and the resulting output signal $y[n]$ are shown below. These will be explained fully in Chapter 2.



- The difference equation (8) can, with appropriate choice of the real constant a, describe a variety of relevant DT systems.

System	**Value of *a***	**Stable**
Highpass filter	$-1 < a < 0$	Yes
Lowpass filter	$0 < a < 1$	Yes
Running summation (accumulation)	$a = 1$	No
Compound interest	$1 < a < \infty$ (typically)	No

In compound interest applications, $a > 1$ provided the interest rate is positive. The property of *stability* is discussed shortly.

Properties of Systems

- The system properties presented here are:
  - Stability
  - Memory
  - Invertibility
  - Time invariance
  - Linearity
  - Causality
- Unless noted otherwise, the definition given for a property is relevant for both CT and DT systems. If it is given for only one of the two types of systems, extension to the other type is considered obvious.

Stability

- A system is *bounded-input, bounded-output stable* (BIBO stable) if and only if every bounded input induces a bounded output.
- *DT Definition*. A DT system with input $x[n]$ and output $y[n]$ is BIBO stable if and only if

$$\left|x[n]\right| \le M_x < \infty \ \ \forall n \ \ \Rightarrow \ \ \left|y[n]\right| \le M_y < \infty \ \ \forall n ,$$

where M_x and M_y are positive real constants. The definition for a CT system is entirely analogous.

1. *CT Amplitude Modulation.* Assume the input $x(t)$ is bounded:

$$\left|x(t)\right| \le M_x < \infty \ \ \forall t .$$

Using the input-output relation (3), the magnitude of the output $y(t)$ satisfies

$$\left|y(t)\right| = \left|x(t)\right|\left|\cos\omega_c t\right| \le M_x < \infty \ \ \forall t ,$$

and the system is *stable*.

2. *CT First-Order Lowpass Filter.* Assume the input $x(t)$ is bounded:

$$\left|x(t)\right| \le M_x < \infty \ \ \forall t .$$

In the input-output relation (6), the integrand is $x(t')$ times a positive factor, so the magnitude of the output $y(t)$ satisfies

$$\left|y(t)\right| \le \frac{1}{\tau}\int_{-\infty}^{t} e^{-\frac{t-t'}{\tau}} M_x dt' = M_x < \infty \ \ \forall t ,$$

and the system is *stable*.

3. *DT Squarer.* Assume the input $x[n]$ is bounded:

$$\left|x[n]\right| \le M_x < \infty \ \ \forall n .$$

Using the input-output relation (7), the magnitude of the output $y[n]$ satisfies

$$\left|y[n]\right| \le M_x^2 < \infty \ \ \forall n ,$$

and the system is *stable*.

4. *DT First-Order Lowpass Filter.* Assume the input $x[n]$ is bounded:

$$\left|x[n]\right| \le M_x < \infty \ \ \forall n .$$

In the input-output relation (10), the summand is $x[k]$ times a positive factor, so the magnitude of the output $y[n]$ satisfies

$$\left|y[n]\right| \le \sum_{k=-\infty}^{n} a^{n-k} M_x .$$

Changing the variable of summation to $m = n - k$, this becomes

$$\left|y[n]\right| \le M_x \sum_{m=\infty}^{0} a^m = M_x \sum_{m=0}^{\infty} a^m = \frac{M_x}{1-a} < \infty \; .$$

We have used the assumption that $0 < a < 1$ to sum the geometric series $\sum_{m=0}^{\infty} a^m = 1/(1-a)$. The system is *stable*. The system is unstable for some other choices of *a*, as we will see.

Memory

- A system is *memoryless* if, at any given time, the value of the output depends only on the present value of the input, and not on past or future values of the input. If the output depends on past or future values of the input, the system has *memory*.

1. *CT Amplitude Modulation.* In the input-output relation

$$y(t) = x(t)\cos\omega_c t \; , \tag{3}$$

the output $y(t)$ depends only on the present input $x(t)$. The system is *memoryless*.

2. *CT First-Order Lowpass Filter.* In the input-output relation

$$y(t) = \frac{1}{\tau}\int_{-\infty}^{t} e^{-\frac{t-t'}{\tau}} x(t')dt' \; , \tag{6}$$

the output $y(t)$ depends on past and present values of the input, $x(t')$, $t' \le t$. The system has *memory*.

3. *DT Squarer.* In the input-output relation

$$y[n] = x^2[n] , \tag{7}$$

the output $y[n]$ depends only on the present input $x[n]$. The system is *memoryless*.

4. *DT First-Order Lowpass Filter.* In the input-output relation

$$y[n] = \sum_{k=-\infty}^{n} a^{n-k} x[k] , \tag{10}$$

the output $y[n]$ depends on past and present values of the input, $x[k]$, $k \le n$. The system has *memory*.

Invertibility

- A system is *invertible* if the input can always be recovered from the output.
- *DT Definition*. A DT system H with input $x[n]$ and output $y[n] = H\{x[n]\}$ is invertible if and only if there exists a stable *inverse system* H^{-1} such that

$$x[n] = H^{-1}\{y[n]\} \quad \forall x[n], y[n] .$$

The system and inverse system satisfy $H^{-1}H = I$, where I is an *identity operator*. The definition for a CT system is entirely analogous.

1. *CT Amplitude Modulation.* Recall the input-output relation

$$y(t) = x(t)\cos\omega_c t \,. \tag{3}$$

The system is *not invertible*. For example, if the input $x(t)$ has impulses that coincide with zero crossings of the sinusoidal carrier $\cos\omega_c t$, these impulses do not appear in the output $y(t)$, and $x(t)$ cannot be recovered from $y(t)$. Although the system is not strictly invertible, if we place suitable restrictions on $x(t)$, we can guarantee that it is recoverable from $y(t)$. If $x(t)$ is suitably bandlimited, then we can recover $x(t)$ from $y(t)$ using a system called a *demodulator* (see Chapter 7). An input signal $x(t)$ including impulses is not bandlimited.

2. *CT First-Order Lowpass Filter.* The system is *not invertible*. A CT lowpass filter attenuates high frequencies. Its inverse system would need to have infinite gain at high frequencies, and would be unstable. (As explained in Chapter 3, the CT lowpass filter has a frequency response whose magnitude is proportional to $|\omega|^{-1}$ for large $|\omega|$. The inverse system would need to have a magnitude response proportional to $|\omega|$ for large $|\omega|$, becoming infinite at high frequencies.) If we place suitable restrictions on $x(t)$, we can always recover it from $y(t)$. If $x(t)$ is bandlimited to any finite bandwidth, we can recover it from $y(t)$ using a stable inverse system.

3. *DT Squarer.* The system is *not invertible*. For example, if $x[n]$ is real, then given $y[n] = x^2[n]$, we cannot determine the sign of $x[n]$.

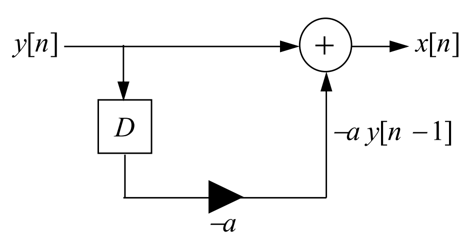
4. *DT First-Order Lowpass Filter.* The system is *invertible*. Recall the input $x[n]$ and output $y[n]$ are related by the difference equation

$$y[n] = x[n] + ay[n-1]. \tag{8}$$

We can solve this for a difference equation allowing us to obtain $x[n]$ from $y[n]$:

$$x[n] = y[n] - ay[n-1].$$

The inverse system, which realizes this difference equation, is shown.



This inverse system is stable for any $|a| < \infty$ (this is explained in Chapter 2).

Time Invariance

- A system is *time-invariant* if any shift of the input signal leads only to an identical shift of the output signal.
- *CT Definition*. Consider a system *H*. If

$$H\{x(t)\} = y(t),$$

then H is time-invariant if

$$H\{x(t-t_0)\} = y(t-t_0)$$

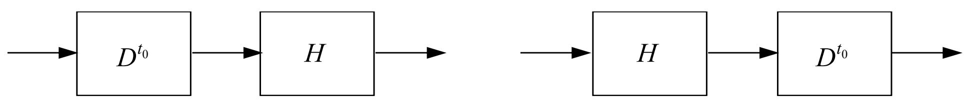
for any $x(t)$ and any t_0. Equivalently, we can define a *time-shift operator* D^{t_0}:

$$D^{t_0}\{x(t)\} = x(t-t_0).$$

A system *H* is time-invariant if

$$D^{t_0}\{H\{x(t)\}\} = H\{D^{t_0}\{x(t)\}\}$$

for any $x(t)$ and any t_0. In other words, the system H is time-invariant if the following two block diagrams are equivalent.



- *DT Definition*. Consider a system *H*. If

$$H\{x[n]\} = y[n],$$

then H is time-invariant if

$$H\{x[n-n_0]\} = y[n-n_0]$$

for any $x[n]$ and any n_0. Equivalently, we can define a *time-shift operator* D^{n_0}:

$$D^{n_0}\{x[n]\} = x[n-n_0].$$

A system *H* is time-invariant if

$$D^{n_0}\{H\{x[n]\}\} = H\{D^{n_0}\{x[n]\}\}$$

for any $x[n]$ and any n_0.

- In each of the following examples, H denotes the system under consideration.

1. *CT Amplitude Modulation.* The input-output relation (3) states

$$H\{x(t)\} = y(t) = x(t)\cos\omega_c t.$$

Time-shifting the input, the output is

$$H\{x(t-t_0)\} = x(t-t_0)\cos\omega_c t,$$

while modulating the input and then time-shifting the output yields

$$y(t-t_0) = x(t-t_0)\cos\omega_c(t-t_0).$$

These are different, and the system is *not time-invariant*.

2. *CT First-Order Lowpass Filter.* The input-output relation (6) states

$$H\{x(t)\} = y(t) = \frac{1}{\tau}\int_{-\infty}^{t} e^{-\frac{t-t'}{\tau}} x(t')dt'.$$

Time-shifting the input, the output is

$$H\{x(t-t_0)\} = \frac{1}{\tau}\int_{-\infty}^{t} e^{-\frac{t-t'}{\tau}} x(t'-t_0)dt'.$$

Changing the variable of integration to $t'' = t' - t_0$, so that $t' = t'' + t_0$, yields

$$H\{x(t-t_0)\} = \frac{1}{\tau}\int_{-\infty}^{t-t_0} e^{-\frac{t-t_0-t''}{\tau}} x(t'')dt''.$$

Filtering the input and then time-shifting the output yields

$$y(t-t_0) = \frac{1}{\tau}\int_{-\infty}^{t-t_0} e^{-\frac{t-t_0-t'}{\tau}} x(t')dt'$$

The two previous expressions are equivalent. The system is *time-invariant*. Note, however, that if the system were not assumed to be at initial rest, it would not be time invariant, as shown in a homework problem.

3. *DT Squarer.* The input-output relation (7) states

$$H\{x[n]\} = y[n] = x^2[n]$$

Time-shifting the input, the output is

$$H\{x[n-n_0]\} = x^2[n-n_0].$$

Squaring the input and then time-shifting the output yields

$$y[n-n_0] = x^2[n-n_0].$$

These are equivalent, and the system is *time-invariant*.

4. *DT First-Order Lowpass Filter.* The input-output relation (10) states

$$H\{x[n]\} = y[n] = \sum_{k=-\infty}^{n} a^{n-k} x[k]$$

Time-shifting the input, the output is

$$H\{x[n-n_0]\} = \sum_{k=-\infty}^{n} a^{n-k} x[k-n_0].$$

Changing the variable of summation to $m = k - n_0$, so that $k = m + n_0$, yields

$$H\{x[n-n_0]\} = \sum_{m=-\infty}^{n-n_0} a^{n-n_0-m} x[m].$$

Filtering the input and then time-shifting the output yields

$$y[n-n_0] = \sum_{k=-\infty}^{n-n_0} a^{n-n_0-k} x[k].$$

The two previous expressions are equivalent. The system is *time-invariant*. Like the CT lowpass filter, if the system were not assumed to be at initial rest, it would not be time invariant.

Linearity

- A system is *linear* if, given an input that is a weighted sum of several signals, the output is the weighted sum of the responses of the system to each of the signals.
- *DT Definition*. A DT system H is *linear* if

$$H\left\{\sum_{i=1}^{N} a_i x_i[n]\right\} = \sum_{i=1}^{N} a_i H\{x_i[n]\}$$

for any constants a_i and signals $x_i[n]$, $i = 1, \ldots, N$. The definition for a CT system is analogous.

1. *CT Amplitude Modulation.* It is easy to verify that the system is *linear*.
2. *CT First-Order Lowpass Filter.* It is easy to verify that the system is *linear*. If the system were not assumed to be at initial rest, however, it would not be linear (as shown in a homework problem).
3. *DT Squarer.* The system is *nonlinear*, owing to the squaring operation.
4. *DT First-Order Lowpass Filter.* It is easy to verify that the system is *linear*. Like the CT lowpass filter, if the system were not assumed to be at initial rest, it would not be linear.

Causality

- A system is *causal* if, at any given time, the value of the output depends only on the present and past values of the input, and not on future values of the input.
- *CT Definition*. A CT system with input $x(t)$ and output $y(t)$ is *causal* if $y(t)$ depends only upon $x(t-t')$, $t' \geq 0$. The definition for a DT system is analogous.

1. *CT Amplitude Modulation.* The system is *causal*.
2. *CT First-Order Lowpass Filter.* The system is *causal*.
3. *DT Squarer.* The system is *causal*.
4. *DT First-Order Lowpass Filter.* The system is *causal*.

Stanford University
EE 102A: Signal Processing and Linear Systems I
Professor Joseph M. Kahn

Chapter 2: Linear Time-Invariant Systems

Linear Time-Invariant Systems

- Many important CT or DT systems are linear or may be modeled as approximately linear for small changes in the input that induce small changes in the output. Many important CT or DT systems are time-invariant, or may be modeled as approximately time-invariant over relevant time scales.

Tools for Analyzing Linear Time-Invariant Systems

- Powerful tools exist for analyzing and implementing CT and DT linear time-invariant (LTI) systems.
- The *impulse response* is a function of time that captures all relevant properties of an LTI system. Once we know the impulse response of the system, then given any input signal, we can use a linear operation called *convolution* to find an explicit expression for the output signal. (*Applicable to all LTI systems*.)
- The *frequency response* is a function of frequency that, when it exists, captures all relevant properties of an LTI system. Once we know the frequency response of the system, then given any input signal that can be expressed as a sum of imaginary exponential time signals, we can use simple *multiplication* to find an expression for the output signal. (*Applicable to stable LTI systems and some unstable ones*. *Generalized to broader class of systems using complex exponential time signals and transfer function*.)
- *Linear, constant-coefficient differential equations* (in CT) or *difference equations* (in DT) describe many LTI systems. These equations can be solved using systematic procedures to find the impulse response, the frequency response, or the output in response to an input signal.
(*Applicable to finite-order causal LTI systems.*)
- *Systematic realization*: many LTI systems can be realized using simple operations, including multiplication, addition and time-shifting (in DT) or integration (in CT).
(*Applicable to finite-order causal LTI systems.*)

Major Topics in This Chapter (studied for both CT and DT unless noted otherwise)

- Impulse response and convolution for LTI systems
  - Determining the impulse response for a given system
  - Evaluating the convolution sum (in DT) or the convolution integral (in CT)
- Properties of convolution and of LTI systems
  - Distributive, associative, commutative
- Properties of an impulse response corresponding to properties of the LTI system it describes
  - Real, memoryless, causal, stable, invertible
- Systems described by finite-order, linear constant-coefficient differential equations (in CT) or difference equations (in DT)

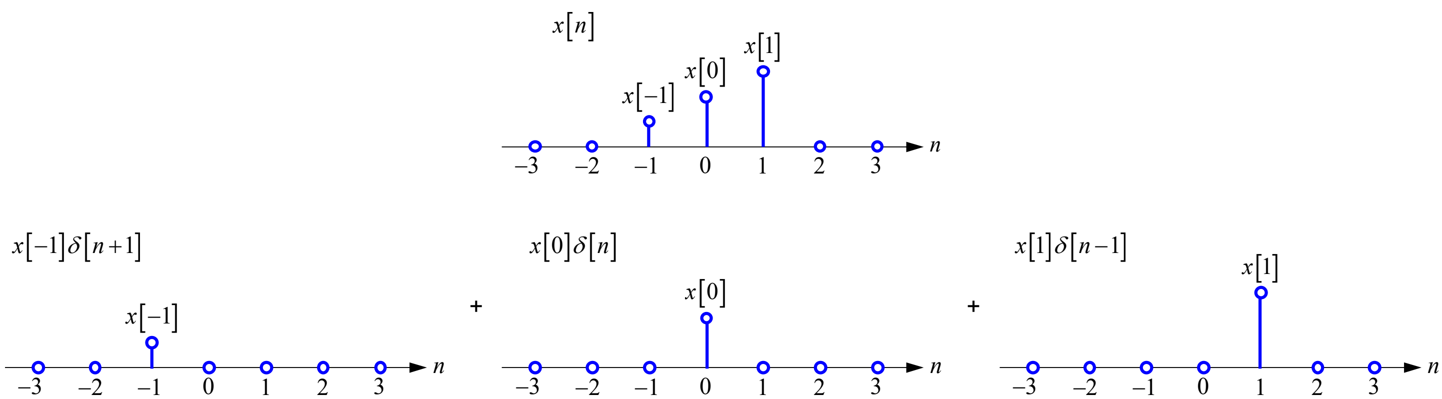
Impulse Response and Convolution Sum for Discrete-Time Linear Time-Invariant Systems

- Any DT signal $x[n]$ can be represented as a sum of scaled, shifted impulses:

$$
\begin{aligned}
x[n] &= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \\
&= \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots
\end{aligned} \tag{1}
$$

  - Expression (1) is the *sifting property* of the DT impulse function (see Chapter 1, page 26).
  - In interpreting (1), bear in mind that a DT signal is a function of the time variable n. On the left-hand side of (1), $x[n]$ is a *signal*. On the right-hand side of (1), the $\delta[n-k]$ are *signals*, while the $x[k]$ are *coefficients* that scale the signals.

- This figure shows an example of (1). A short-duration signal, with three nonzero samples, is represented by a sum of three scaled, shifted impulses.



- Consider a DT LTI system H. Let the signal $x[n]$ be input to the system. We would like to compute the output signal $y[n]$, so we let the system act on the input, i.e., $y[n] = H\{x[n]\}$. Representing the input by (1), the output is

$$
y[n] = H\left\{ \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \right\}.
$$

Since the system H is linear, and since it acts on signals (functions of time n) but not on the coefficients scaling the signals, the output is

$$
y[n] = \sum_{k=-\infty}^{\infty} x[k] H\{\delta[n-k]\}. \tag{2}
$$

- As in Chapter 1, page 34, we define a *time-shift operator* D^k, which time shifts a signal by k:

$$
D^k\{z[n]\} = z[n-k].
$$

The signals appearing in each term of (2) can be represented as

$$\begin{aligned} H\{\delta[n-k]\} &= H\{D^k\{\delta[n]\}\} \\ &= D^k\{H\{\delta[n]\}\} \end{aligned}, \tag{3}$$

where we exploit the time invariance of *H* to interchange the operators *H* and D^k.

- Let us define the *impulse response* $h[n]$ of the LTI system *H*, which is the output of the system when the input is a unit impulse $\delta[n]$:

$$h[n] \stackrel{d}{=} H\{\delta[n]\}. \tag{4}$$

Using definition (4) and the time-shift operator, we can rewrite (3) as

$$\begin{aligned} H\{\delta[n-k]\} &= D^k\{h[n]\} \\ &= h[n-k] \end{aligned}.$$

Substituting this into (2), we can express the system output as

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &\stackrel{d}{=} x[n]*h[n] \end{aligned}. \tag{5}$$

Expression (5) is called a *convolution sum*. It defines the mathematical operation of convolution between DT signals $x[n]$ and $h[n]$, denoted by the symbol "*", which yields a DT signal $y[n]$.

- We find another form of the convolution sum by changing the summation variable in (5) to $l = n-k$:

$$\begin{aligned} y[n] &= \sum_{l=-\infty}^{\infty} x[n-l]h[l] \\ &\stackrel{d}{=} h[n]*x[n] \end{aligned}. \tag{5'}$$

Formula (5') defines the convolution between DT signals $h[n]$ and $x[n]$, which yields a DT signal $y[n]$. Comparing (5) and (5'), we see that $y[n] = x[n]*h[n] = h[n]*x[n]$, i.e., *convolution is commutative*.

- In the convolution defined by (5) or (5'), values of the input signal $x[n]$ are redistributed in time, in a way that depends on the impulse response $h[n]$, to yield values of the output signal $y[n]$.

- Expressions (5) and (5') are *extremely important* in the analysis of DT LTI systems. Given a DT LTI system *H*, suppose we know its impulse response $h[n]$. Then, given any input signal $x[n]$, we can compute the resulting output signal $y[n]$ using (5) or (5'). In solving any problem, we may choose whichever form, (5) or (5'), we find easiest to evaluate.

Step Response of a Discrete-Time Linear Time-Invariant Systems

- The *step response* $s[n]$ of a DT LTI system H is the output obtained when the input is a unit step function $u[n]$:

$$s[n] \stackrel{d}{=} H\{u[n]\}.$$

Using (5'), the step response is given by a convolution

$$\begin{aligned} s[n] &= h[n] * u[n] \\ &= \sum_{k=-\infty}^{\infty} h[k]u[n-k] \end{aligned}. \tag{6}$$

When a DT unit step function, which can only assume values of 0 or 1, appears in a summation like (6), it restricts the summation to a limited domain. Note that

$$u[n-k] = \begin{cases} 1 & n-k \geq 0 \\ 0 & n-k < 0 \end{cases} = \begin{cases} 1 & k \leq n \\ 0 & k > n \end{cases},$$

so (6) can be rewritten without a unit step function as

$$s[n] = \sum_{k=-\infty}^{n} h[k]. \tag{7}$$

According to (7), the step response $s[n]$ is the *running sum* of the impulse response $h[n]$.

- Conversely, using (7), we can evaluate the first difference of the step response:

$$\begin{aligned} s[n] - s[n-1] &= \sum_{k=-\infty}^{n} h[k] - \sum_{k=-\infty}^{n-1} h[k] \\ &= h[n] \end{aligned}. \tag{8}$$

As shown in (8), the impulse response $h[n]$ is the *first difference* of the step response $s[n]$.

- Given the step response $s[n]$ of a DT LTI system H, we can always use (8) to obtain the impulse response $h[n]$. Hence, the step response $s[n]$, like the impulse response $h[n]$, completely characterizes the input-output relationship of the system. As we will see, there are reasons we sometimes study the step response $s[n]$ of a DT LTI system instead of the impulse response $h[n]$:
  - The step response gives insight into key system properties, such as *rise time* or *overshoot/undershoot*. We study these properties for exemplary systems later in the course.
  - We may experimentally characterize the input-output relationship of a system by inputting one or more particular $x[n]$ and measuring the resulting output(s) $y[n]$. If the device generating the $x[n]$ has a limited peak amplitude, it can be easier to measure the step response than the impulse response.

Computing the Impulse Response of a Discrete-Time Linear Time-Invariant System

- Assume we have a description of the input-output relation of a DT LTI system H in some form, and we would like to obtain an expression for the impulse response $h[n]$. We only need to do this once. Then, given any input signal $x[n]$, we can predict the output signal $y[n]$.

Methods

- Here is a list of several methods we will use. The list is not exhaustive, as other methods exist.
  1. *Direct substitution* using $x[n]=\delta[n]$ and $y[n]=h[n]$. This is useful if the system is specified in terms of an explicit time-domain input-output relationship.
  2. *Solution of difference equation*. This is useful if the system is specified in terms of a linear, constant-coefficient difference equation.
  3. *Inverse Fourier transform*. This is useful if the system is specified in terms of a frequency response $H\left(e^{j\Omega}\right)$.

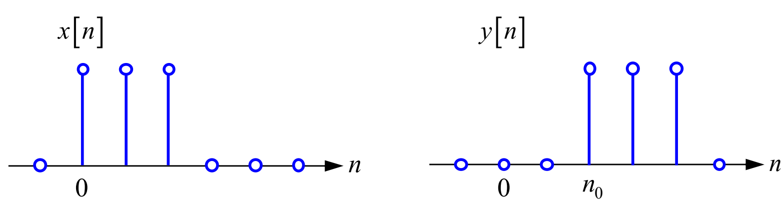
Examples

- Here, we study examples of methods 1 and 2. We will study method 3 after developing the DT Fourier transform in Chapter 5.

1. *Direct substitution*.
   - We are given a system D^{n_0} that time-shifts the input signal by n_0:

$$\begin{aligned} y[n] &= D^{n_0}\{x[n]\} \\ &= x[n-n_0] \end{aligned}.$$

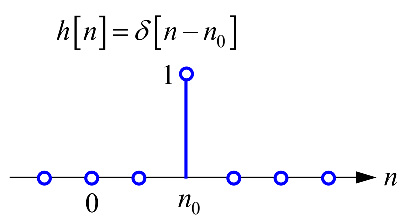
An example of input and output signals is shown.



   - Choosing $x[n]=\delta[n]$ as the input, we obtain the impulse response as the resulting output:

$$\begin{aligned} y[n] &= h[n] \\ &= \delta[n-n_0] \end{aligned}.$$

This impulse response is shown.



- We often find it useful to represent time shifting by n_0 as a convolution with $\delta[n-n_0]$:

$$\begin{aligned} D^{n_0}\{x[n]\} &= x[n-n_0] \\ &= x[n]*\delta[n-n_0] \end{aligned}. \tag{9}$$

2. *Solution of difference equation.*

- We are given a *first-order system* described by a constant-coefficient linear difference equation

$$y[n]=x[n]+ay[n-1], \tag{10}$$

where a is a real constant. Assume we know that the system is at initial rest, i.e., $y[n]=0$ until $x[n]$ first becomes nonzero. Then the system is LTI, as shown in Chapter 1, pages 34-36.

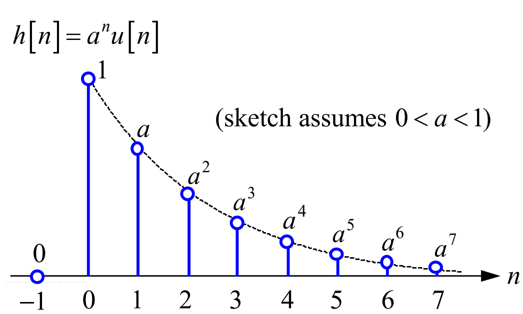
- To compute the impulse response, we assume an input $x[n]=\delta[n]$ and solve (10) for the output $y[n]=h[n]$. The input first becomes nonzero at $n=0$, so the initial rest condition corresponds to $y[-1]=0$. An efficient way to solve for the output is to use the Z transform, as taught in EE 102B. Here we solve (10) by iteratively substituting for $x[n]$ and $y[n]$ in (10). We create a table of inputs and outputs as a function of time *n*, starting at $n=-1$. At each *n*, knowing $x[n]$ and $y[n-1]$, we use (10) to find $y[n]$. We can extend the procedure to arbitrary *n*.

n	-1	0	1	2	$\cdots$	n
$x[n]=\delta[n]$	0	1	0	0	$\cdots$	0
$y[n]=h[n]$	0	1	a	a^2	$\cdots$	a^n

We have found the impulse response to be

$$h[n]=\begin{cases} 0 & n<0 \\ a^n & n\geq 0 \end{cases} = a^n u[n]. \tag{11}$$

This impulse response is shown here, assuming $0<a<1$ (lowpass filter).



- As stated in Chapter 1, the difference equation (10) can describe a variety of relevant DT systems with appropriate choice of the real constant *a*.

System	Value of *a*
Highpass filter	$-1 < a < 0$
Lowpass filter	$0 < a < 1$
Running summation (accumulation)	$a = 1$
Compound interest	$1 < a < \infty$ (typically)

Evaluating Discrete-Time Convolution Sums

- Here we discuss how to evaluate a convolution sum in the form (5) or (5'):

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{5}$$

$$= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \tag{5'}$$

The two forms are equivalent. In solving a problem, we can choose the form we find easiest to evaluate.

Starting and Ending Times

- This table summarizes how the starting time (first nonzero sample), ending time (last nonzero sample), and length (spanning the first to last nonzero samples) of $y[n]$ relate to those of $x[n]$ and $h[n]$. Knowing the starting and ending times can simplify evaluation of some convolutions, and can help you check your results. As we see in the table:
  - The starting time of $y[n]$ is the *sum of the starting times* of $x[n]$ and $h[n]$.
  - The ending time of $y[n]$ is the *sum of the ending times* of $x[n]$ and $h[n]$.
  - The length of $y[n]$ is *one less than the sum of the lengths* of $x[n]$ and $h[n]$.
- These formulas are applicable when the starting or ending times are infinite. For example, given any nonzero $x[n]$, if $h[n]$ ends at $n = \infty$ ($n_{2h} = \infty$), then $y[n]$ ends at $n = \infty$ ($n_{2y} = \infty$).

	$x[n]$	$h[n]$	$y[n]$
First nonzero sample	n_{1x}	n_{1h}	$n_{1y} = n_{1x} + n_{1h}$
Last nonzero sample	n_{2x}	n_{2h}	$n_{2y} = n_{2x} + n_{2h}$
Length (spanning first to last nonzero samples)	$L_x = n_{2x} - n_{1x} + 1$	$L_h = n_{2h} - n_{1h} + 1$	$L_y = L_x + L_h - 1$

Methods

- Here is a list of several methods for evaluating convolution sums. The list is not exhaustive, as other methods exist. Also, given a pair of signals to be convolved, more than one technique may be applicable. Before computing a convolution, think about which technique is likely to be easiest.
    1. *Add up scaled, shifted copies* of $x[n]$ (or $h[n]$). This method is applicable if $h[n]$ (or $x[n]$) has finite length.
    2. *Flip and drag.* This method is always applicable, but it is not necessarily the easiest method.
    3. *Symbolic.* This method is applicable if both $x[n]$ and $h[n]$ are specified as mathematical formulas.
    4. *Numerical.* This method is applicable if both $x[n]$ and $h[n]$ can be represented as finite-length numerical vectors **x** and **h**. In MATLAB, we use the command `y = conv(x,h)`.

Discrete-Time Convolution Methods and Illustrative Examples

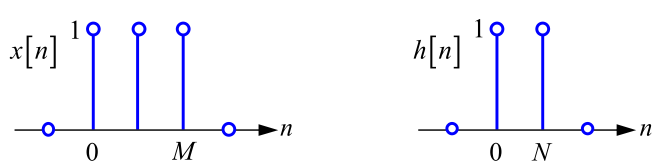
1. *Adding up scaled, shifted copies*. This method is applicable if at least one of the signals has finite length. For concreteness, we consider the case that $h[n]$ has finite length. (It is straightforward to address the case that $x[n]$ has finite length by interchanging $h[n]$ and $x[n]$ in the description given here. If both $h[n]$ and $x[n]$ have finite length, then treat the shorter signal as we treat $h[n]$ here.) Suppose $h[n]$ has first and last nonzero samples n_{1h} and n_{2h} and length $L_h = n_{2h} - n_{1h} + 1$. The convolution sum (5') can be written

$$y[n] = \sum_{k=n_{1h}}^{n_{2h}} h[k]x[n-k]. \tag{12}$$

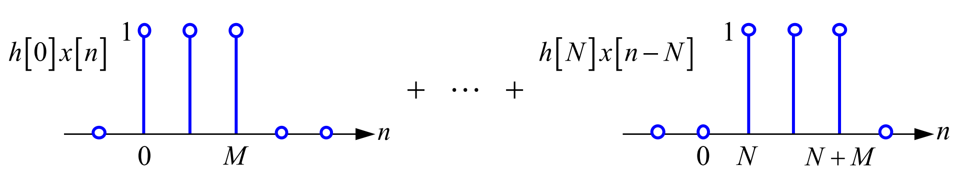
This is a sum of L_h shifted copies $x[n-k]$, each scaled by a coefficient $h[k]$.

Example: convolution of two rectangular pulses

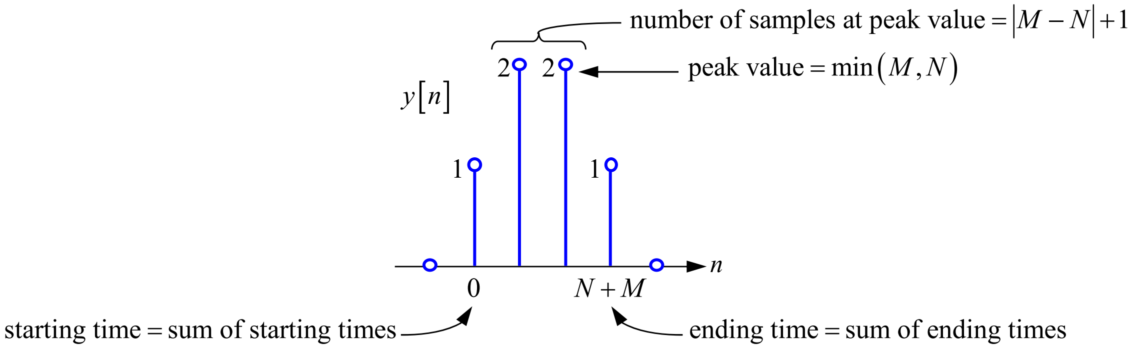
- Signals $x[n]$ and $h[n]$ are unit-amplitude rectangular pulses of length $M+1$ and $N+1$.



- Using (12), we obtain $y[n]$ by adding up the $N+1$ shifted copies $x[n],\ldots,x[n-N]$, each scaled by $h[0] = \cdots = h[N] = 1$, as shown.



- The resulting convolution $y[n]$ is shown. This example shows that when we convolve two rectangular pulses, we obtain a *trapezoid*, assuming the pulses have unequal lengths ($M \neq N$). In the special case of equal lengths ($M = N$), we obtain a *triangle*. This is a very useful example.



2. *Flip and drag*. This method is applicable to any signals, and follows directly from the general convolution sum (5) or (5'). For concreteness, we focus on computing the sum in form (5'). (It is straightforward to compute the sum in form (5) by interchanging $h[n]$ and $x[n]$ in the description given here.) In order to compute the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k], \tag{5'}$$

for $-\infty < n < \infty$, we perform the following steps:

1. Plot $h[k]$ vs. k.

Start with n negative and large.

2. Plot $x[n-k]$ vs. k. (In order to do this, reflect or *flip* $x[k]$ to obtain $x[-k]$, then shift or *drag* $x[-k]$ to obtain $x[n-k]$. When n is negative, shift it to the left.)

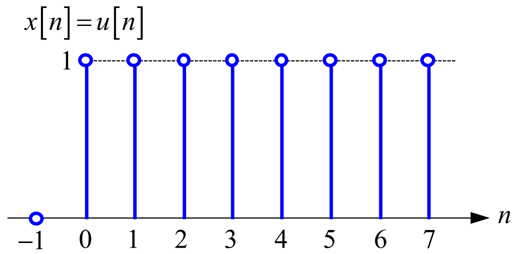
3. At a given value of n, compute the value of the convolution sum $y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$. In order to do this, multiply $h[k]$ by $x[n-k]$ at each value of k, then sum the product $x[n-k]h[k]$ over all k.

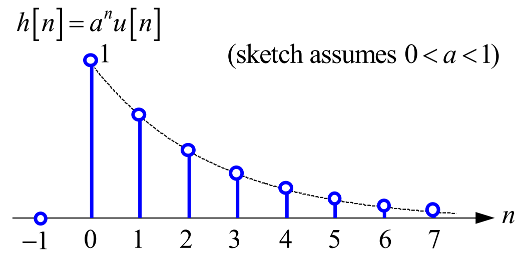
Increase n by 1 and return to step 2.

Example: step response of first-order system

- In this example, we compute the step response of a first-order system. The input signal is $x[n] = u[n]$, and the impulse response (11) is $h[n] = a^n u[n]$, where a is real, assuming $a \neq 1$. The output is $y[n] = s[n]$. In working out the example, to the extent possible, we will denote these signals as $x[n]$, $h[n]$ and $y[n]$ to best illustrate the general technique.

- The input signal $x[n]$ and impulse response $h[n]$ to be convolved are shown below.

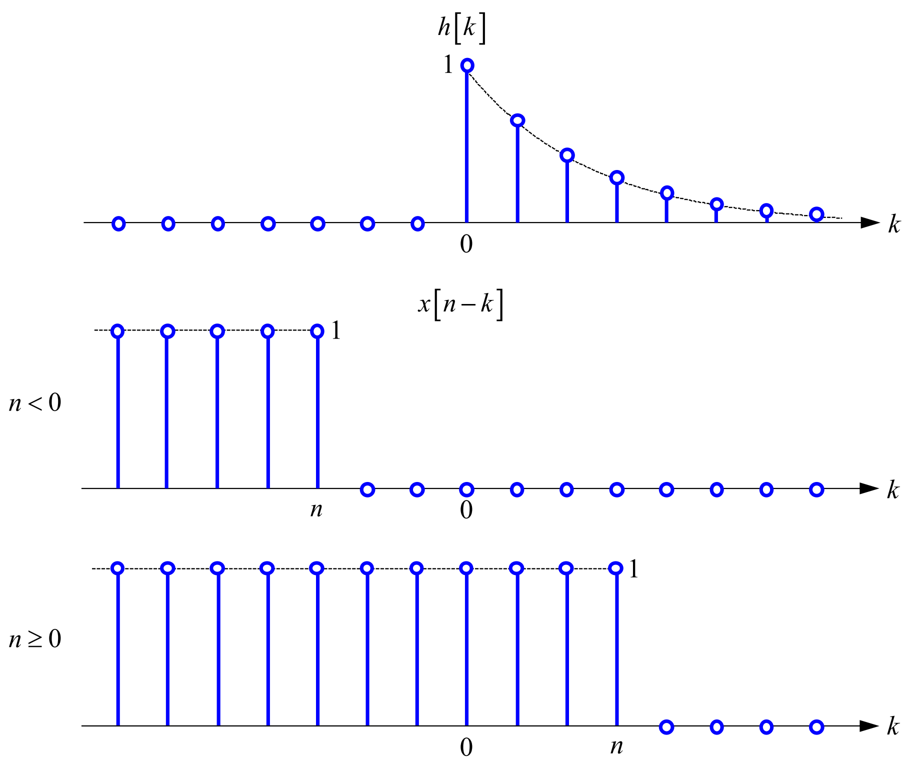




- In order to compute the convolution sum

$$y[n]=\sum_{k=-\infty}^{\infty} x[n-k]h[k] \tag{5'}$$

using the flip and drag method, we plot $h[k]$ and $x[n-k]$ vs. k below. We show $x[n-k]$ for the cases $n<0$ and $n\geq 0$.



- Considering these two cases:
  - For $n<0$, multiplying $h[k]$ by $x[n-k]$, we find

$$x[n-k]h[k]=0 \quad \forall k,$$

    so the summation (5') yields $y[n]=0$.

  - For $n\geq 0$, multiplying $h[k]$ by $x[n-k]$, we find

$$x[n-k]h[k]=\begin{cases}0 & k<0\\ a^k & 0\le k\le n\\ 0 & k>n\end{cases}.$$

Performing the summation (5'):

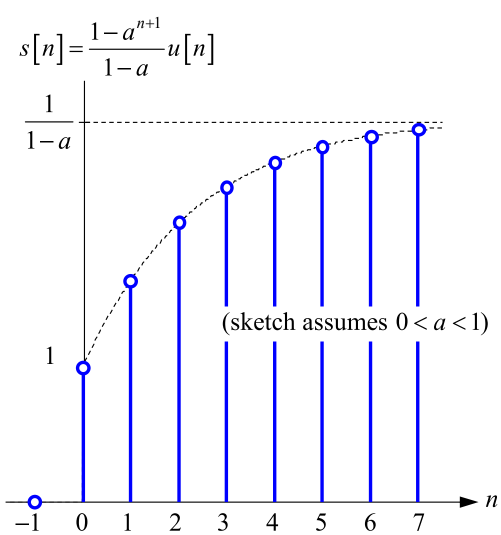
$$\begin{aligned}y[n]&=\sum_{k=-\infty}^{\infty}x[n-k]h[k]\\ &=\sum_{k=0}^{n}a^k\\ &=\frac{1-a^{n+1}}{1-a}\end{aligned}.$$

We have used the finite summation of a geometric series valid for $a\neq 1$ (see Appendix, page 287).

- Combining the two cases $n<0$ and $n\ge 0$, the step response of the first-order system, assuming $a\neq 1$, is

$$y[n]=s[n]=\begin{cases}0 & n<0\\ \dfrac{1-a^{n+1}}{1-a} & n\ge 0\end{cases}=\frac{1-a^{n+1}}{1-a}u[n]. \tag{13}$$

The step response (13) is shown below.



3. *Symbolic*. Symbolic convolution is applicable when both the input signal $x[n]$ and the impulse response $h[n]$ are specified as mathematical formulas. Instead of trying to explain it in general terms, we will provide two examples.

Example: step response of first-order system

- In this example, we compute the step response of a first-order system with impulse response $h[n] = a^n u[n]$, where a is real, assuming $a \neq 1$. We computed this using the flip and drag method in the preceding example.
- Earlier in this chapter, we showed that for any DT LTI system, the step response $s[n]$ is the running sum of the impulse response $h[n]$:

$$s[n] = \sum_{k=-\infty}^{n} h[k]. \tag{7}$$

For the particular impulse response $h[n] = a^n u[n]$, the step response is the running sum

$$s[n] = \sum_{k=-\infty}^{n} a^k u[k].$$

Using the fact that $u[k] = 0$, $k < 0$, when $n < 0$, the sum is zero. When $n \geq 0$, the step response becomes

$$s[n] = \sum_{k=0}^{n} a^k .$$

This is the finite sum of a geometric series, $\sum_{k=0}^{n} a^k = \left(1 - a^{n+1}\right)/\left(1 - a\right)$, $a \neq 1$ (see Appendix, page 287). The step response can be written as

$$s[n] = \begin{cases} 0 & n < 0 \\ \dfrac{1 - a^{n+1}}{1 - a} & n \geq 0 \end{cases} = \frac{1 - a^{n+1}}{1 - a} u[n]. \tag{13}$$

This agrees with the result obtained using the flip and drag method in the previous example.

Example: output of first-order system given arbitrary input

- In this example, given an arbitrary input signal $x[n]$, we compute the output from a first-order system with impulse response $h[n] = a^n u[n]$, a real. Substituting this expression for $h[n]$ in the general expression (5) for the convolution sum, the output is

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k] h[n-k] \\ &= \sum_{k=-\infty}^{\infty} x[k] a^{n-k} u[n-k] \end{aligned}.$$

Using the fact that

$$u[n-k] = \begin{cases} 0 & n-k < 0 \\ 1 & n-k \geq 0 \end{cases} = \begin{cases} 0 & k > n \\ 1 & k \leq n \end{cases},$$

The output is given by

$$y[n] = \sum_{k=-\infty}^{n} x[k] a^{n-k} .$$

- This agrees with expression (10) given in Chapter 1, page 29. As noted there, at time n, the output $y[n]$ is a weighted sum of inputs $x[k]$ at all past and present times $k \le n$, with weighting factor a^{n-k}. If we further assume $0 \le a \le 1$, which corresponds to a lowpass filter, the output gives more weight to recent inputs (small $n-k$) and less weight to older inputs (large $n-k$).

Impulse Response and Convolution Integral for Continuous-Time Linear Time-Invariant Systems

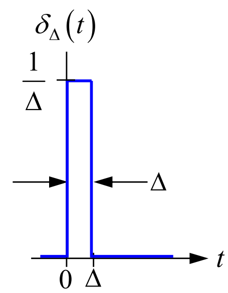
- Any sufficiently smooth CT signal $x(t)$ can be represented as a weighted sum of shifted impulses:

$$x(t) = \int_{-\infty}^{\infty} x(t')\delta(t-t')dt' . \tag{14}$$

Expression (14) is called the *sifting property* of the CT impulse function.

Proof

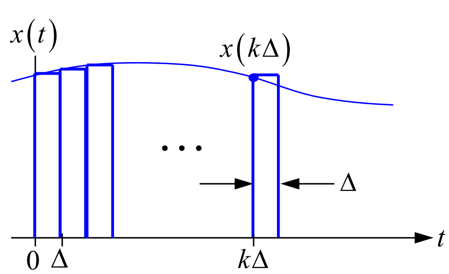
As in Chapter 1, we approximate the CT impulse function $\delta(t)$ as $\delta_\Delta(t)$, a rectangular pulse of width Δ and height $1/\Delta$, as shown.



Note that $\Delta \cdot \delta_\Delta(t)$ is a rectangular pulse of width Δ and unit height. Given a smooth signal $x(t)$, we approximate it as a weighted sum of shifted rectangular pulses:

$$x(t) \approx \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta . \tag{15}$$

An example is shown below.



Now we consider the limit $\Delta \to 0$. In this limit:

$$\delta_\Delta(t) \to \delta(t)$$

$$k\Delta \to t'$$

$$\Delta \to dt'$$

and the sum (15) approaches a Riemann sum representation of the integral

$$x(t) = \int_{-\infty}^{\infty} x(t')\delta(t-t')dt' . \tag{14}$$

QED

- Now we consider a CT LTI system *H*. Let the signal $x(t)$ be input to the system. In order to compute the output signal $y(t)$, we let the system act on the input, i.e., $y(t) = H\{x(t)\}$. Representing the input signal by (14), the output is

$$y(t) = H\left\{\int_{-\infty}^{\infty} x(t')\delta(t-t')dt'\right\}.$$

Since the system *H* is linear, and since it acts on signals (functions of time *t*) but not on the coefficients scaling the signals, the output is

$$y(t) = \int_{-\infty}^{\infty} x(t')H\{\delta(t-t')\}dt' . \tag{16}$$

- As in Chapter 1, page 34, we define a *time-shift operator* $D^{t'}$, which time shifts a signal by t':

$$D^{t'}\{z(t)\} = z(t-t').$$

This allows us to represent the signal appearing in the integrand of (16) as

$$\begin{aligned} H\{\delta(t-t')\} &= H\{D^{t'}\{\delta(t)\}\} \\ &= D^{t'}\{H\{\delta(t)\}\} \end{aligned}. \tag{17}$$

We have exploited the time invariance of *H* to interchange the operators *H* and $D^{t'}$.

- Let us define the *impulse response* $h(t)$ of the LTI system *H*, which is the output of the system when the input is a unit impulse $\delta(t)$:

$$h(t) \stackrel{d}{=} H\{\delta(t)\}. \tag{18}$$

Using definition (18) and the time-shift operator, we can rewrite (17) as

$$\begin{aligned} H\{\delta(t-t')\} &= D^{t'}\{H\{\delta(t)\}\} \\ &= h(t-t') \end{aligned}.$$

Substituting this into (16), we can express the system output as

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(t')h(t-t')dt' \\
&\stackrel{d}{=} x(t)*h(t)
\end{aligned}
. \qquad (19)
$$

Expression (19) is called a *convolution integral*. It defines the mathematical operation of convolution between CT signals $x(t)$ and $h(t)$, denoted by the symbol "*", which yields a signal $y(t)$.

- We obtain another form of the convolution integral by changing the integration variable in (19) to $\mu = t - t'$:

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(t-\mu)h(\mu)d\mu \\
&\stackrel{d}{=} h(t)*x(t)
\end{aligned}
. \qquad (19')
$$

Expression (19') defines the convolution between CT signals $h(t)$ and $x(t)$, which yields a CT signal $y(t)$. Comparing (19) and (19'), we see that $y(t) = x(t)*h(t) = h(t)*x(t)$, i.e., *convolution is commutative*.

- In the convolution defined by (19) or (19'), values of the input signal $x(t)$ are redistributed in time, in a way that depends on the impulse response $h(t)$, to yield the output signal $y(t)$.

- The convolution integrals (19) and (19') are *very useful* in analyzing CT LTI systems. Given a CT LTI system *H*, suppose we know its impulse response $h(t)$. Given any input signal $x(t)$, we can compute the resulting output signal $y(t)$ using (19) or (19'). We are free to choose whichever form, (19) or (19'), we find easiest to evaluate in solving a given problem.

Step Response of a Continuous-Time Linear Time-Invariant Systems

- The *step response* $s(t)$ of a CT LTI system *H* is the output that results when the input is a unit step function $u(t)$:

$$
s(t) \stackrel{d}{=} H\{u(t)\}.
$$

Making use of (19'), the step response can be computed as a convolution integral

$$
\begin{aligned}
s(t) &= h(t)*u(t) \\
&= \int_{-\infty}^{\infty} h(t')u(t-t')dt'
\end{aligned}
. \qquad (20)
$$

When a CT unit step function, which can only assume values of 0 or 1, appears in an integral such as (20), it restricts the region of integration but does not otherwise change the integrand. Observe that

$$u(t-t') = \begin{cases} 1 & t-t' \geq 0 \\ 0 & t-t' < 0 \end{cases} = \begin{cases} 1 & t' \leq t \\ 0 & t' > t \end{cases}.$$

Hence, (20) can be rewritten without a unit step function as

$$s(t) = \int_{-\infty}^{t} h(t')dt' . \tag{21}$$

As shown by (21), the step response $s(t)$ is the *running integral* of the impulse response $h(t)$.

- Conversely, using (21), we can evaluate the derivative of the step response, which is

$$\frac{ds(t)}{dt} = h(t). \tag{22}$$

As shown by (22), the impulse response $h(t)$ is the *derivative* of the step response $s(t)$.

- If we know the step response $s(t)$ of a CT LTI system *H*, we can obtain the impulse response $h(t)$ using (22). In other words, like the impulse response $h(t)$, the step response $s(t)$ fully characterizes the input-output relation of the system *H*. There are reasons we sometimes prefer to study the step response $s(t)$ instead of the impulse response $h(t)$ for CT LTI systems, as for DT LTI systems:
  - The step response gives insight into important system properties, such as *rise time* or *overshoot/undershoot*. We will study the rise time and overshoot of some important systems later in the course.
  - In practice, we may measure the input-output relationship of a system by applying one or more input signal(s) $x(t)$ and measuring the resulting output signal(s) $y(t)$. It can be easier to measure the step response than the impulse response, particularly if the device generating the $x(t)$ has a limited peak amplitude.

Computing the Impulse Response of a Continuous-Time Linear Time-Invariant System

- Suppose we have a specification of the input-output relation of a CT LTI system *H*, and we want to obtain an expression for the impulse response $h(t)$. We need only compute $h(t)$ once. Then, given any input signal $x(t)$, we can compute the output signal $y(t)$.

Methods

- Here is a list of several methods we will employ. Other methods exist, so the list is not exhaustive.
  1. *Direct substitution* using $x(t) = \delta(t)$ and $y(t) = h(t)$. This is applicable if we have an explicit input-output relation in the time domain.
  2. *Solution of differential equation*. This is applicable if the system is specified in terms of a linear, constant-coefficient differential equation.

3. *Inverse Fourier transform*. This is applicable if the system is specified in terms of a frequency response $H(j\omega)$.

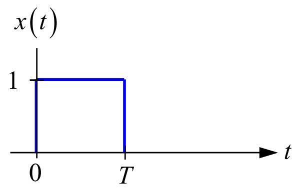
Examples

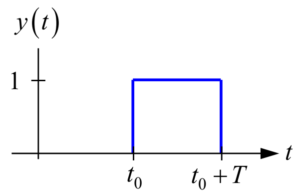
- Here, we study examples of methods 1 and 2. The examples are analogous to those studied for DT LTI systems above. We will study method 3 after developing the CT Fourier transform in Chapter 4.

1. *Direct substitution*.
   - We consider a system D^{t_0} that time-shifts the input signal by t_0:

$$\begin{aligned} y(t) &= D^{t_0}\{x(t)\} \\ &= x(t-t_0) \end{aligned}.$$

Exemplary input and output signals are shown below.

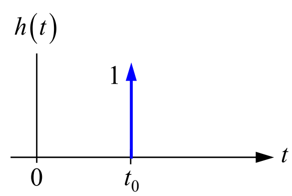




   - Choosing the input to be $x(t) = \delta(t)$, the resulting output is the impulse response:

$$\begin{aligned} y(t) &= h(t) \\ &= \delta(t-t_0) \end{aligned}.$$

This impulse response is shown.



   - We often represent time shifting by t_0 as a convolution with $\delta(t-t_0)$:

$$\begin{aligned} D^{t_0}\{x(t)\} &= x(t-t_0) \\ &= x(t) * \delta(t-t_0) \end{aligned}. \tag{23}$$

2. *Solution of differential equation*.
   - We are given a *first-order lowpass filter* described by a constant-coefficient linear differential equation

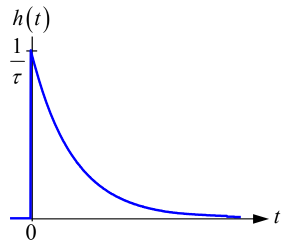
$$\tau \frac{dy}{dt} + y(t) = x(t) \ , \tag{24}$$

where τ is real and $\tau > 0$. Assume we know the system is at initial rest, i.e., $y(t) = 0$ until $x(t)$ first becomes nonzero. Then the system is LTI, as shown in Chapter 1, pages 34-36.

- In order to compute the impulse response, we assume an input $x(t)=\delta(t)$ and solve (24) for the output $y(t)=h(t)$. This input first becomes nonzero at $t=0$, so the initial rest condition corresponds to $y(t)=0$, $t<0$. We can solve the differential equation easily using the Laplace transform, as taught in EE 102B. Here, we guess the solution and verify that it satisfies the differential equation and the initial conditions. We guess the impulse response to be

$$h(t)=\frac{1}{\tau}e^{-\frac{t}{\tau}}u(t), \tag{25}$$

which is shown here.



- We can verify that (25) satisfies (24) with $x(t)=\delta(t)$ and $y(t)=h(t)$, i.e.,

$$\tau\frac{dh}{dt}+h(t)=\delta(t). \tag{26}$$

Differentiating the product of two functions in (25) and using $\frac{du(t)}{dt}=\delta(t)$, we find

$$\begin{aligned}\frac{dh}{dt}&=-\frac{1}{\tau^2}e^{-\frac{t}{\tau}}u(t)+\frac{1}{\tau}e^{-\frac{t}{\tau}}\delta(t)\\&=-\frac{1}{\tau^2}e^{-\frac{t}{\tau}}u(t)+\frac{1}{\tau}\delta(t)\end{aligned}.$$

We have simplified the second term using the CT impulse sampling property, i.e., $e^{-\frac{t}{\tau}}\delta(t)=e^{-\frac{0}{\tau}}\delta(t)=\delta(t)$. Substituting this expression for $\frac{dh}{dt}$ and (25) for $h(t)$ into (26), we verify that (26) is satisfied. Likewise, we can verify that (25) satisfies the initial rest condition $h(t)=0$, $t<0$.

Evaluating Continuous-Time Convolution Integrals

- In this section, we discuss how to evaluate a convolution integral in the form (19) or (19'):

$$y(t)=\int_{-\infty}^{\infty}x(t')h(t-t')dt' \tag{19}$$

$$=\int_{-\infty}^{\infty}x(t-t')h(t')dt' \tag{19'}$$

The two forms are equivalent, so in solving a problem, we may use whichever form we find easiest to evaluate.

Starting and Ending Times

- This table shows how the starting time (first nonzero value), ending time (last nonzero value), and duration (interval between starting and ending times) of $y(t)$ are related to those of $x(t)$ and $h(t)$. Knowing the starting and ending times can simplify the evaluation of some convolutions, and can help you check your results. We can make some key observations about the table:
  - The starting time of $y(t)$ is the *sum of the starting times* of $x(t)$ and $h(t)$.
  - The ending time of $y(t)$ is the *sum of the ending times* of $x(t)$ and $h(t)$.
  - The duration of $y(t)$ is the *sum of the durations* of $x(t)$ and $h(t)$.
- The formulas are applicable when the starting or ending times are infinite. As an example, given any nonzero $x(t)$, if $h(t)$ ends at $t=\infty$ ($t_{2h}=\infty$), then $y(t)$ ends at $t=\infty$ ($t_{2y}=\infty$).

	$x(t)$	$h(t)$	$y(t)$
First nonzero value	t_{1x}	t_{1h}	$t_{1y}=t_{1x}+t_{1h}$
Last nonzero value	t_{2x}	t_{2h}	$t_{2y}=t_{2x}+t_{2h}$
Duration (spanning first to last nonzero values)	$T_x=t_{2x}-t_{1x}$	$T_h=t_{2h}-t_{1h}$	$T_y=T_x+T_h$

Methods

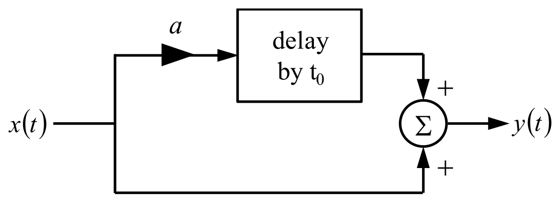
- Here are several methods for evaluating convolution integrals. Other methods exist, so the list is not exhaustive. Given any two signals to be convolved, it may be possible to use more than one technique. Hence, before evaluating a convolution, you should think about which technique is easiest to use.
  1. *Add up scaled, shifted copies* of $x(t)$ (or $h(t)$). This method is applicable if $h(t)$ (or $x(t)$) is comprised only of one or more scaled, shifted impulses.
  2. *Flip and drag.* This method is always applicable. It is not necessarily the easiest method to use, however.
  3. *Symbolic.* This method is applicable if both $x(t)$ and $h(t)$ are given as mathematical formulas.
  4. *Numerical.* CT convolution can be approximated by numerical computation, like any integration of a function of a continuous variable. We will study the numerical approximation of CT convolution in homework MATLAB exercises.

Continuous-Time Convolution Methods and Illustrative Examples

1. *Adding up scaled, shifted copies*. This method is applicable when either $x(t)$ or $h(t)$ comprises only a sum of scaled, shifted impulses. It is a trivial consequence of (23) and the linearity of convolution. We provide an example only for the sake of completeness.

Example: delay-and-add system

- A *delay-and-add* system describes propagation of signals along two paths with different delays. In the case of acoustic waves, this can cause echoes. In the case of electromagnetic waves, this is often called *multipath propagation*, and can cause signal distortion and fluctuation of received signal power. A block diagram representation of the system is shown.



The impulse response is

$$h(t) = \delta(t) + a\delta(t - t_0).$$

where a and t_0 are real and $t_0 \geq 0$. Given any input signal $x(t)$, using (23) and the linearity of convolution, the output is the sum of $x(t)$ and a scaled, delayed version of it:

$$y(t) = x(t) + ax(t - t_0).$$

2. *Flip and drag*. This convolution method follows directly from the general convolution integral (19) or (19'), and is applicable to any signals. For the sake of definiteness, we focus on computing the integral in the form (19'). (To compute the integral in form (19), we simply interchange $x(t)$ and $h(t)$ in the description given here.) In order to compute the convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(t - t')h(t')dt' \tag{19'}$$

for $-\infty < t < \infty$, we perform the following steps:

1. Plot $h(t')$ vs. t'.

Start with t negative and large.

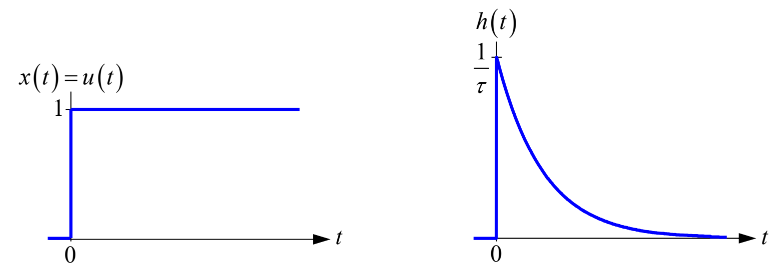
2. Plot $x(t - t')$ vs. t'. (In order to do this, reflect or *flip* $x(t')$ to obtain $x(-t')$, then shift or *drag* $x(-t')$ to obtain $x(t - t')$. When t is negative, shift it to the left.)

3. At a given value of t, compute the value of the convolution integral $y(t) = \int_{-\infty}^{\infty} x(t - t')h(t')dt'$. In order to do this, multiply $h(t')$ by $x(t - t')$ at each value of t', then integrate the product $x(t - t')h(t')$ over all t'.

Increase t and return to step 2.

Example: step response of first-order lowpass filter

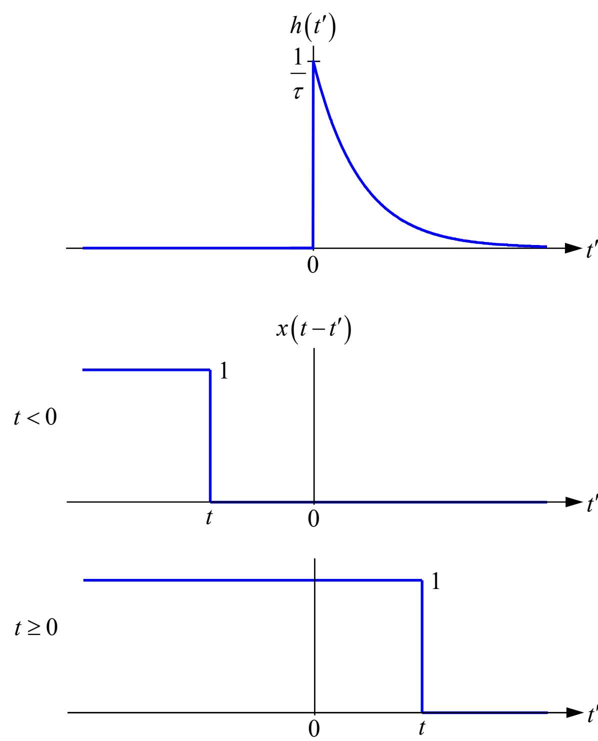
- Here, we compute the step response of a first-order lowpass filter. The input is $x(t) = u(t)$, and the impulse response (25) is $h(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t)$, where τ is real and $\tau > 0$. The output is $y(t) = s(t)$. Throughout this example, we will denote these signals by $x(t)$, $h(t)$ and $y(t)$ to illustrate the general technique.
- The input signal $x(t)$ and impulse response $h(t)$ we wish to convolve are shown below.



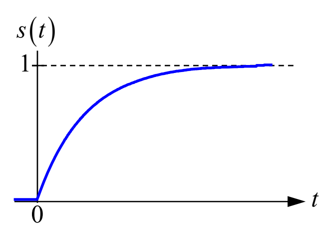
- To compute the convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(t - t') h(t') dt' , \tag{19'}$$

using the flip and drag method, we plot $h(t')$ and $x(t - t')$ vs. t', as shown below. We show $x(t - t')$ for the two cases $t < 0$ and $t \geq 0$.



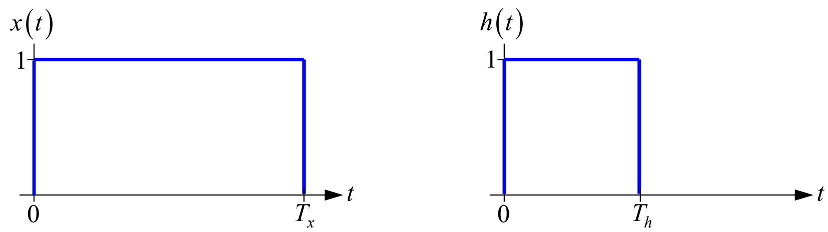
- Considering these two cases:
  - For $t < 0$, multiplying $h(t')$ by $x(t-t')$, we find
    $$x(t-t')h(t') = 0 \quad \forall t',$$
    so the integral (19') yields $y(t) = 0$.
  - For $t \geq 0$, multiplying $h(t')$ by $x(t-t')$, we find
    $$x(t-t')h(t') = \begin{cases} 0 & t' < 0 \\ \frac{1}{\tau} e^{-\frac{t'}{\tau}} & 0 \leq t' \leq t \\ 0 & t' > t \end{cases}.$$
    Performing the integration (19'):
    $$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(t-t')h(t')dt' \\ &= \frac{1}{\tau}\int_0^t e^{-\frac{t'}{\tau}} dt' \\ &= 1 - e^{-\frac{t}{\tau}} \end{aligned}.$$
- Combining the two cases $t < 0$ and $t \geq 0$, the step response of the first-order lowpass filter is
  $$y(t) = s(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-\frac{t}{\tau}} & t \geq 0 \end{cases} = \left(1 - e^{-\frac{t}{\tau}}\right)u(t). \qquad (27)$$
  The step response (27) is shown below.



3. *Symbolic*. Symbolic convolution is applicable when both the input signal $x(t)$ and the impulse response $h(t)$ are described by mathematical formulas. We will study three examples of symbolic convolution.

Example: convolution of two rectangular pulses

- The signals $x(t)$ and $h(t)$ are unit-amplitude rectangular pulses of duration T_x and T_h, respectively, as shown.



- We express each signal as a sum of scaled, shifted step functions:

$$x(t) = u(t) - u(t - T_x) \quad \text{and} \quad h(t) = u(t) - u(t - T_h),$$

and represent the convolution as

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \left[u(t) - u(t - T_x)\right] * \left[u(t) - u(t - T_h)\right] \end{aligned}. \tag{28}$$

- Recall that the unit ramp function is the running integral of the unit step function (Chapter 1, page 19):

$$r(t) = \int_{-\infty}^{t} u(t')\,dt'. \tag{29}$$

Using (29), we can rewrite (28) as

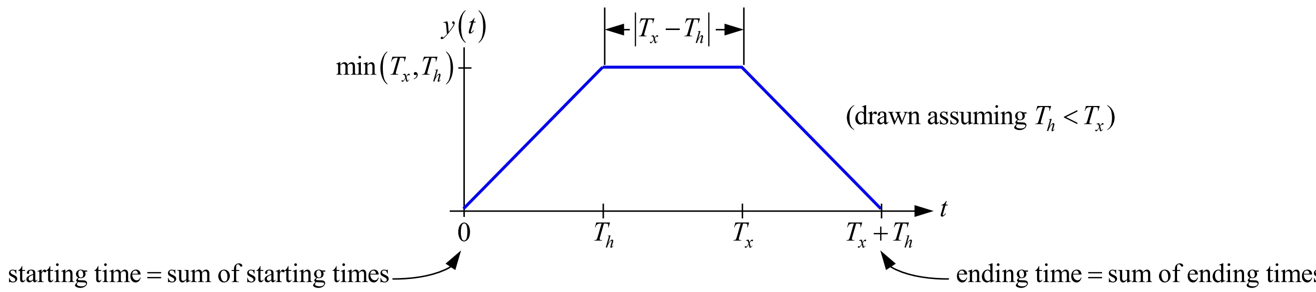
$$y(t) = r(t) - r(t - T_x) - r(t - T_h) + r\left(t - (T_x + T_h)\right). \tag{30}$$

- It is worth explaining some details of how we evaluated (28) to obtain (30). For example, the third term on the right-hand side of (30) arises from a convolution between two of the terms in (28). Omitting the minus sign here, it is

$$\begin{aligned} u(t) * u(t - T_h) &= u(t) * \left[u(t) * \delta(t - T_h)\right] \\ &= \left[u(t) * u(t)\right] * \delta(t - T_h) \\ &= r(t) * \delta(t - T_h) \\ &= r(t - T_h) \end{aligned}.$$

In the first line, we have represented time shifting as convolution with a shifted impulse function, (23). In the second line, we have used the associative property of convolution, which is discussed shortly. In the third line, we have used (29). In the fourth line, we have used (23) once again.

- The convolution $y(t)$ is shown below. This example demonstrates that when we convolve two rectangular pulses, we obtain a *trapezoid*, assuming the pulses have unequal durations ($T_x \neq T_h$). In the special case of equal-duration pulses ($T_x = T_h$), we obtain a *triangle*. This example is extremely useful.



Example: step response of first-order lowpass filter

- In this example, we compute the step response of a first-order lowpass filter, by symbolic convolution of an input $x(t) = u(t)$ with the impulse response (25), given by $h(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t)$, where τ is real and $\tau > 0$. We computed this using the flip and drag method on pages 57-58.
- Recall that the step response $s(t)$ is the running integral of the impulse response $h(t)$:

$$s(t) = \int_{-\infty}^{t} h(t')dt' . \tag{21}$$

For the particular impulse response $h(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t)$, the step response is the running integral

$$s(t) = \frac{1}{\tau} \int_{-\infty}^{t} e^{-\frac{t'}{\tau}} u(t')dt' .$$

Using the fact that $u(t') = 0$, $t' < 0$, when $t < 0$, the integral yields zero. When $t \geq 0$, the step response becomes

$$s(t) = \frac{1}{\tau} \int_{0}^{t} e^{-\frac{t'}{\tau}} dt' .$$

This integral yields $1 - e^{-\frac{t}{\tau}}$. We can write the step response as

$$s(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-\frac{t}{\tau}} & t \geq 0 \end{cases} = \left(1 - e^{-\frac{t}{\tau}}\right) u(t), \tag{27}$$

which agrees with the result obtained using the flip and drag method. This step response is plotted on page 58.

Example: output of first-order lowpass filter given arbitrary input

- In this example, given an arbitrary input signal $x(t)$, we compute the output from a first-order lowpass filter with impulse response $h(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t)$, where τ is real and $\tau > 0$. Substituting this particular $h(t)$ in the general expression (19) for the convolution integral, the output is

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(t')h(t-t')dt' \\
&= \frac{1}{\tau}\int_{-\infty}^{\infty} x(t')e^{-\frac{t-t'}{\tau}}u(t-t')dt'
\end{aligned}.
$$

Using the fact that

$$
u(t-t') = \begin{cases} 0 & t-t'<0 \\ 1 & t-t' \geq 0 \end{cases} = \begin{cases} 0 & t'>t \\ 1 & t' \leq t \end{cases},
$$

The output is given by

$$
y(t) = \frac{1}{\tau}\int_{-\infty}^{t} x(t')e^{-\frac{t-t'}{\tau}}dt'.
$$

- This agrees with expression (6) given in Chapter 1, page 27. At time *t*, the output $y(t)$ is a weighted sum of inputs $x(t')$ at all past and present times $t' \leq t$, with weighting factor $(1/\tau)e^{-\frac{t-t'}{\tau}}$. This gives more weight to recent inputs (small $t-t'$) and less weight to older inputs (large $t-t'$), behavior consistent with a lowpass filter.

Properties of Convolution and of Linear Time-Invariant Systems

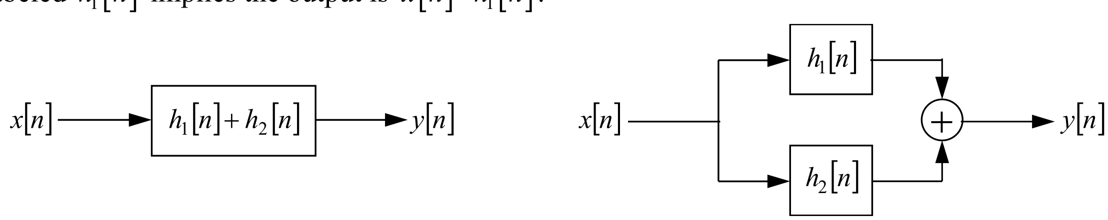
- The mathematical operations of DT and CT convolution satisfy the *distributive*, *associative* and *commutative* properties stated here. These properties can be derived using the definition of DT convolution as a sum (5) or (5'), or the definition of CT convolution as an integral (19) or (19').
- Likewise, all DT and CT LTI systems satisfy the distributive, associative and commutative properties. This follows from the fact that the operation of any LTI system can be represented as a convolution with an appropriately chosen impulse response $h[n]$ in DT, or $h(t)$ in CT.
- These properties are stated here only for DT. Their extension to CT should be obvious.

Distributive Property

- Convolution is a *distributive* operation:

$$
x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n].
$$

This corresponds to the distributive property of LTI systems shown below. Throughout this course, we will use the notation implied by these block diagrams. For example, passing a signal $x[n]$ into a block labeled $h_1[n]$ implies the output is $x[n]*h_1[n]$.



Associative Property

- Convolution is an *associative* operation:

$$\left(x[n]*h_1[n]\right)*h_2[n]=x[n]*\left(h_1[n]*h_2[n]\right).$$

This corresponds to the associative property of LTI systems shown below.

$$x[n] \longrightarrow \boxed{h_1[n]} \longrightarrow \boxed{h_2[n]} \longrightarrow y[n] \qquad\qquad x[n] \longrightarrow \boxed{h_1[n]*h_2[n]} \longrightarrow y[n]$$

Commutative Property

- Convolution is a *commutative* operation:

$$h_1[n]*h_2[n]=h_2[n]*h_1[n].$$

This corresponds to the commutative property of LTI systems shown below.

$$x[n] \longrightarrow \boxed{h_1[n]} \longrightarrow \boxed{h_2[n]} \longrightarrow y[n] \qquad\qquad x[n] \longrightarrow \boxed{h_2[n]} \longrightarrow \boxed{h_1[n]} \longrightarrow y[n]$$

Properties of Impulse Responses and Corresponding LTI Systems

- In this section, we consider LTI systems with various properties – real, memoryless, causal, stable or invertible – and study how these properties are reflected in the system impulse response.

Real Systems

- If a DT or CT LTI system maps all real input signals to real output signals, then its impulse response, $h[n]$ or $h(t)$, must be a real-valued function.

Memoryless Systems

- Recall that a system is memoryless if, at any given time, the value of the output depends only on the present value of the input, and not on past or future values of the input. If the output depends on past or future values of the input, the system has memory.
- The impulse response of a memoryless DT LTI system must be of the form

$$h[n]=C\cdot\delta[n].$$

Likewise, the impulse response of a memoryless CT LTI system must be of the form

$$h(t)=C\cdot\delta(t).$$

In both cases, C is an arbitrary constant.

Causal Systems

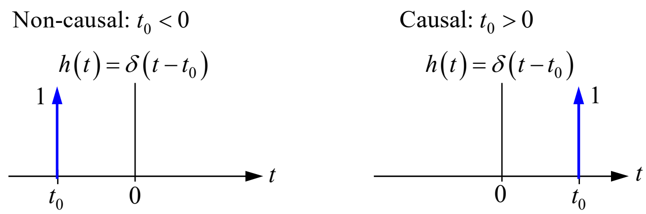
- Recall that a system is causal if, at any given time, the value of the output depends only on the present and past values of the input, and not on future values of the input.
- The impulse response of a causal DT LTI system must satisfy

$$h[n]=0\ ,\ n<0\ .$$

Similarly, the impulse response of a causal CT LTI system must satisfy

$$h(t)=0\ ,\ t<0\ .$$

- As an example, we compare the impulse responses of non-causal and causal CT time-shifting systems.



- Here we justify the condition for the CT case. Given an LTI system with impulse response $h(t)$ and input $x(t)$, the output is given by

$$y(t)=\int_{-\infty}^{\infty} x(t-t')h(t')dt'\ . \tag{19'}$$

Consider a general LTI system that is not necessarily causal. Expression (19') states that at time *t*, the output $y(t)$ incorporates the effects of inputs $x(t-t')$ at past, present and future times $t-t'$, $-\infty<t'<\infty$. In order to satisfy causality, $y(t)$ should depend only on $x(t-t')$ at past and present times $t-t'$, $0\le t'<\infty$. In order for the impulse response to enforce causality, it must exclude the influence of inputs $x(t-t')$ at future times $t-t'$, $-\infty<t'<0$, which requires that it satisfy the condition $h(t')=0$, $t'<0$. The explanation for the DT case, which uses (5'), is entirely analogous.

Stable Systems

- Recall that a system is bounded-input, bounded-output stable (BIBO stable) if and only if every bounded input induces a bounded output.
- A DT LTI system is stable if and only if its impulse response is *absolutely summable*:

$$\sum_{n=-\infty}^{\infty}\left|h[n]\right|<\infty\ . \tag{31}$$

Similarly, a CT LTI system is stable if and only if its impulse response is *absolutely integrable*:

$$\int_{-\infty}^{\infty}\left|h(t)\right|dt<\infty\ . \tag{32}$$

- We justify CT condition (32) here and prove that for an LTI system H with input $x(t)$, impulse response $h(t)$ and output $y(t)$, and for positive real constants M_x and M_y, the BIBO stability condition

$$\left|x(t)\right|\le M_x<\infty\ \forall t\ \Rightarrow\ \left|H\{x(t)\}\right|=\left|y(t)\right|\le M_y<\infty\ \forall t$$

is satisfied if and only if

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty .$$

Proof

- Show that $\int_{-\infty}^{\infty} |h(t)| dt < \infty$ is sufficient. The magnitude of the output can be expressed as

$$\begin{aligned} |y(t)| &= |x(t) * h(t)| \\ &= \left| \int_{-\infty}^{\infty} h(t') x(t-t') dt' \right| \\ &\le \int_{-\infty}^{\infty} |h(t') x(t-t')| dt' \quad \text{(Riemann sum representation of integral and } |a+b| \le |a| + |b|\text{)} \\ &= \int_{-\infty}^{\infty} |h(t')| |x(t-t')| dt' \quad (|ab| = |a| \cdot |b|) \end{aligned}$$

  Assume that $|x(t)| \le M_x < \infty \ \forall t$. Then

$$|y(t)| \le M_x \int_{-\infty}^{\infty} |h(t')| dt' .$$

  We conclude that $|y(t)| \le M_y < \infty \ \forall t$ if $\int_{-\infty}^{\infty} |h(t)| dt < \infty$.

- Show that $\int_{-\infty}^{\infty} |h(t)| dt < \infty$ is necessary. Given an impulse response $h(t)$, we choose an input

$$x(t) = \begin{cases} \dfrac{h^*(-t)}{|h(-t)|} & h(-t) \ne 0 \\ 0 & h(-t) = 0 \end{cases} .$$

  The resulting output is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(t') x(t-t') dt' \\ &= \int_{-\infty}^{\infty} \frac{h(t') h^*(t'-t)}{|h(t'-t)|} dt' \end{aligned} .$$

  At $t = 0$, the output is

$$\begin{aligned} y(0) &= \int_{-\infty}^{\infty} \frac{h(t') h^*(t')}{|h(t')|} dt' \\ &= \int_{-\infty}^{\infty} |h(t')| dt' \end{aligned} .$$

  Satisfying $|y(t)| \le M_y < \infty \ \forall t$ requires $\int_{-\infty}^{\infty} |h(t)| dt < \infty$.

QED

The proof for the DT case is entirely analogous.

Invertible Systems

- Recall that a system is invertible if the input can always be recovered from the output.
- A DT LTI system *H* with impulse response $h[n]$ is invertible if and only if there exists a stable inverse system H^{-1} with impulse response $h^{-1}[n]$ such that

$$h^{-1}[n]*h[n]=\delta[n].$$

- Similarly, a CT LTI system *H* with impulse response $h(t)$ is invertible if and only if there exists a stable inverse system H^{-1} with impulse response $h^{-1}(t)$ such that

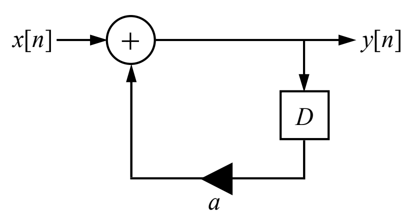
$$h^{-1}(t)*h(t)=\delta(t).$$

Example: Discrete-Time First-Order System and its Inverse

- A DT first-order system *H* is described by the constant-coefficient linear difference equation

$$y[n]=x[n]+ay[n-1], \tag{10}$$

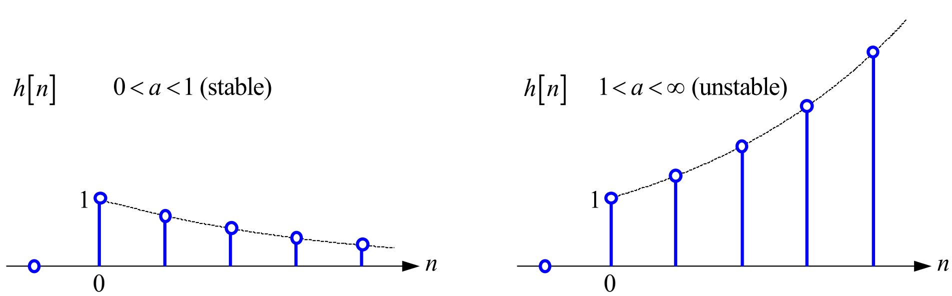
and can be realized by the block diagram shown. The constant *a* is assumed real.



We found earlier that it has an impulse response

$$h[n]=a^n u[n]. \tag{11}$$

The impulse response (11) is shown below for positive *a*, including $0<a<1$ (on the left) and $1<a<\infty$ (on the right).

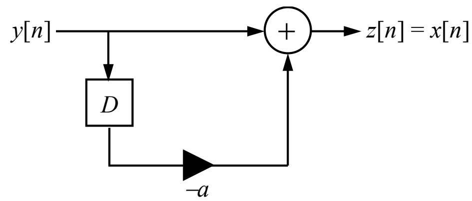


The system is stable only for $|a|<1$, in which case, the impulse response (11) is absolutely summable, $\sum_{n=-\infty}^{\infty}|h[n]|<\infty$, and thus satisfies condition (31) for stability.

- The inverse of the first-order system, denoted by H^{-1}, with input $y[n]$ and output $z[n]$, is described by a constant-coefficient linear difference equation

$$z[n]=y[n]-ay[n-1]=x[n], \tag{33}$$

and can be realized by the block diagram shown.



It is easy to show by direct substitution, with input $y[n]=\delta[n]$ and output $z[n]=h^{-1}[n]$, that the inverse system has an impulse response

$$h^{-1}[n]=\delta[n]-a\delta[n-1]. \tag{34}$$

- We can verify through two different methods that this system is, indeed, the inverse of the first-order system. First, we can substitute expression (10) for $y[n]$ into (33), obtaining

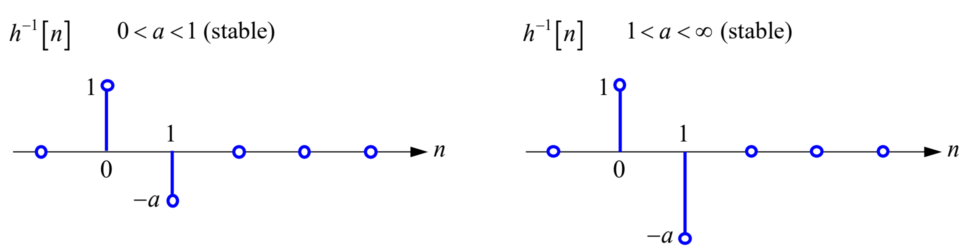
$$z[n]=x[n]+ay[n-1]-ay[n-1]=x[n],$$

showing that the inverse system allows us to recover $x[n]$. Second, we can convolve the impulse responses of the first-order system and its inverse, (11) and (34):

$$\begin{aligned} h[n]*h^{-1}[n] &= a^n u[n]*\{\delta[n]-a\delta[n-1]\} \\ &= a^n\{u[n]-u[n-1]\} \\ &= a^n\delta[n] \\ &= \delta[n] \end{aligned},$$

verifying that we obtain $\delta[n]$. We have used (9) in the second line, and have used the sampling property of the DT impulse function (Chapter 1, page 25) in the last line.

- The inverse system impulse response (34) is shown below for positive a, including $0<a<1$ (on the left) and $1<a<\infty$ (on the right).



Unlike the first-order system, which is stable only for $|a|<1$, the inverse system is stable for any $|a|<\infty$, in which case, the impulse response (34) is absolutely summable, $\sum_{n=-\infty}^{\infty}\left|h^{-1}[n]\right|<\infty$, and satisfies condition (31) for stability.

Continuous-Time Systems Described by Linear, Constant-Coefficient Differential Equations

- A general linear, constant-coefficient differential equation, describing a CT system with input $x(t)$ and output $y(t)$, is of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} . \tag{35}$$

  - The constants a_k, $k=0,\ldots,N$ and b_k, $k=0,\ldots,M$ are real-valued in systems that map real inputs to real outputs.
  - The right-hand side of (35) is a linear combination of the input $x(t)$ and its first *M* derivatives.
  - The left-hand side of (35) is a linear combination of the output $y(t)$ and its first *N* derivatives. The parameter *N*, denoting the highest derivative of $y(t)$ in (35), is called the *order* of the equation and of the system. Typically, *N* corresponds to the number of independent energy storage elements in the system. In a circuit, *N* typically equals the number of inductors and capacitors. In a mechanical system, *N* typically equals the number of elements storing potential and kinetic energies.
  - By construction, (35) describes a causal system.

General Case $N \neq 0$

- In this general case, (35) provides an *implicit* description for the output $y(t)$ given an input $x(t)$. Suppose we specify:
  - *N* initial conditions $y(t_0),\ldots,y^{(N-1)}(t_0)$, values of the output and its $N-1$ derivatives at time t_0.
  - An input $x(t)$, $t>t_0$.

  Then we can solve for the output $y(t)$, $t>t_0$ *uniquely*.

- If we assume the system is *at initial rest* (the output is zero until the input becomes nonzero)

$$x(t)=0,\ t<t_0 \ \Rightarrow\ y(t)=\cdots=y^{(N-1)}(t)=0,\ t<t_0,$$

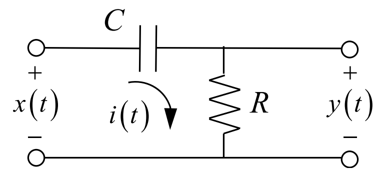
  then the linear, constant-coefficient differential equation (35) describes a *causal, LTI system*. We can calculate its impulse response $h(t)$ by solving (35) assuming input, output and initial conditions

$$x(t)=\delta(t) \qquad y(t)=h(t)$$

$$y(t)=\cdots=y^{(N-1)}(t)=0,\ t<0 .$$

Example: First-Order Highpass Filter

- A first-order *highpass filter*, shown below, may be used to remove the d.c. component or low-frequency components from signals.



We can relate the input voltage $x(t)$ and output voltage $y(t)$ to the current $i(t)$:

$$x(t) - y(t) = \frac{1}{C}\int_{-\infty}^{t} i(t')dt'$$

$$y(t) = i(t)R$$

Using the second equation to express $i(t)$ in terms of $y(t)$, substituting into the first equation, and differentiating, we obtain a first-order differential equation

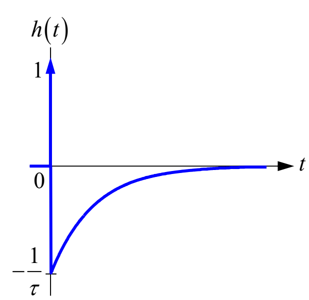
$$\frac{dy}{dt} + \frac{1}{\tau}y(t) = \frac{dx}{dt} \ , \tag{36}$$

where $\tau = RC$.

- We can compute the impulse response of the highpass filter by solving (36), assuming an input $x(t) = \delta(t)$, an output $y(t) = h(t)$, and an initial condition $y(t) = 0$, $t < 0$. We obtain

$$h(t) = \delta(t) - \frac{1}{\tau}e^{-\frac{t}{\tau}}u(t).$$

You will be asked in a homework problem to verify that this impulse response satisfies (36) and the initial condition stated. The impulse response $h(t)$ is shown here.



Special Case $N = 0$

- In this special case, the differential equation (35) provides an *explicit* description for the output $y(t)$ given an input $x(t)$. We can immediately solve (35) to obtain an expression for the output:

$$y(t) = \frac{1}{a_0}\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \tag{37}$$

The system described by (37) is LTI. We can obtain its impulse response by directly substituting $x(t)=\delta(t)$ and $y(t)=h(t)$ into (37):

$$h(t)=\frac{1}{a_0}\sum_{k=0}^{M} b_k \frac{d^k \delta(t)}{dt^k}. \tag{38}$$

In this special case, the impulse response is a scaled sum of a CT impulse function and its derivatives.

Example: Differentiator

- The CT differentiator, which was discussed in Chapter 1, page 8, has an input-output relation

$$y(t)=\frac{dx(t)}{dt}.$$

This input-output relation is already in the form (37). Using (38), the impulse response of the differentiator is

$$h(t)=\frac{d\delta(t)}{dt}. \tag{39}$$

The differentiator's impulse response is the derivative of the unit impulse, which we now discuss.

Unit Doublet Function

- The *unit doublet function* is the derivative of the CT unit impulse function:

$$\delta'(t)=\frac{d\delta(t)}{dt}.$$

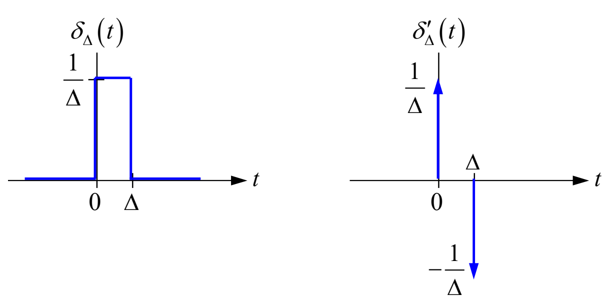
In order to understand the doublet, recall that the unit impulse function can be represented as the limiting case of a very narrow, tall rectangular pulse that has unit area (Chapter 1, page 21):

$$\delta(t)=\lim_{\Delta\to 0}\delta_\Delta(t).$$

Differentiating this representation of the unit impulse yields

$$\begin{aligned}\delta'(t)&=\lim_{\Delta\to 0}\delta'_\Delta(t)\\ &=\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\delta(t)-\delta(t-\Delta)\right]\end{aligned}. \tag{40}$$

The rectangular pulse $\delta_\Delta(t)$ and its derivative $\delta'_\Delta(t)$ are shown, assuming nonzero Δ.



We can think of $\delta'_\Delta(t)$ as a pair of unit impulses, scaled by $1/\Delta$ and $-1/\Delta$, and offset in time by Δ, in the limit that Δ tends to zero.

- We can use this description of the unit doublet to verify that this function is the impulse response of a differentiator, as stated by (39). Consider a signal $x(t)$ that is differentiable near $t=0$. Convolving it with the differentiator impulse response (39) and representing the doublet using (40) yields

$$\begin{aligned} x(t)*\delta'(t) &= \lim_{\Delta\to 0} x(t)*\delta'_\Delta(t) \\ &= \lim_{\Delta\to 0} x(t)*\left[\frac{1}{\Delta}\delta(t)-\frac{1}{\Delta}\delta(t-\Delta)\right] \\ &= \lim_{\Delta\to 0}\frac{x(t)-x(t-\Delta)}{\Delta} \\ &= x'(t) \end{aligned}.$$

In the next-to-last line, we have used (23). In the last line, we have used the fact that in the limit, a finite difference becomes a derivative.

Block Diagram Realizations

- Any CT system described by a finite-order, linear, constant-coefficient differential equation can be realized using differentiators (or integrators), scale factors and adders. Such realizations were used historically in analog computers to solve differential equations before the advent of digital computers. Such realizations are still used in feedback control systems. See *EE 102B Course Reader*, Chapter 6.

Example: First-Order Equation

- Consider a first-order constant-coefficient equation

$$\frac{dy}{dt}+ay(t)=bx(t) \ . \tag{41}$$

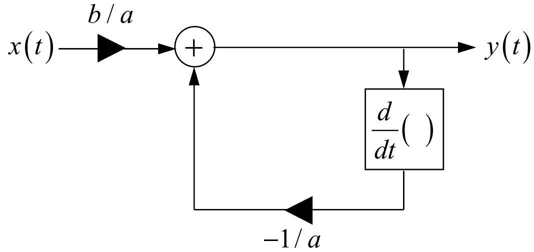
The first-order lowpass filter equation (24) is an example of (41) with $a=b=\dfrac{1}{\tau}$.

Differentiator-Based Realization

- We rewrite (41) as

$$y(t)=\frac{b}{a}x(t)-\frac{1}{a}\frac{dy}{dt},$$

which is realized by the system shown.

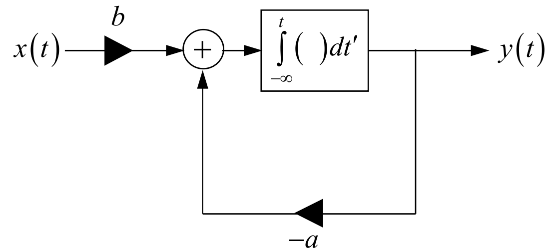


Integrator-Based Realization

- We rearrange (41) as $\frac{dy}{dt} = bx(t) - ay(t)$ and integrate from time $-\infty$ to time *t*, obtaining

$$y(t) = \int_{-\infty}^{t} \left[bx(t') - ay(t') \right] dt' .$$

This is realized by the system shown.



Discrete-Time Systems Described by Linear, Constant-Coefficient Difference Equations

- A general linear, constant-coefficient difference equation, describing a DT system with input $x[n]$ and output $y[n]$, is of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \tag{42}$$

  - The constants a_k, $k = 0,\ldots,N$ and b_k, $k = 0,\ldots,M$ are real-valued in systems that map real inputs to real outputs.
  - The right-hand side of (42) is a linear combination of the present and *M* past values of the input $x[n]$.
  - The left-hand side of (42) is a linear combination of the present and *N* past values of the output $y[n]$. The parameter *N*, which specifies the oldest output values that contribute to the present output $y[n]$ in (42), is called the *order* of the equation and of the system.
  - By construction, (42) describes a causal system (provided $a_0 \neq 0$).
  - We can draw the block diagram of a realization of the general DT system (42) using shift registers (discrete-time delays), scale factors and adders. Such realizations are used extensively in the implementation of digital signal processing. The general realization is presented in the *EE 102B Course Reader*, Chapter 7. Throughout EE 102A, we will study block diagrams of specific DT systems that can be described by the general form (42).

General Case $N \neq 0$

- In this general case, the difference equation (42) describes a system in which *N* past output values fed back into the system and contribute to the present output $y[n]$. We say that the system is *recursive*.
- The difference equation (42) provides an *implicit* description of the output $y[n]$ given an input $x[n]$.

Suppose we specify:

- N initial conditions $y[n_0-1],\ldots,y[n_0-N]$, the N most recent past output values at time n_0.
- An input $x[n]$, $n \geq n_0$.

Then we can solve for the output $y[n]$, $n \geq n_0$ *uniquely*.

- If we assume the system is *at initial rest* (the output is zero until the input becomes nonzero)

$$x[n]=0,\ n<n_0 \quad \Rightarrow \quad y[n]=0,\ n<n_0,$$

then the linear, constant-coefficient difference equation (42) describes a *causal, LTI system*. We can calculate its impulse response $h[n]$ by solving (42) assuming input, output and initial conditions

$$x[n]=\delta[n] \qquad y[n]=h[n]$$

$$y[n]=0,\ n<0.$$

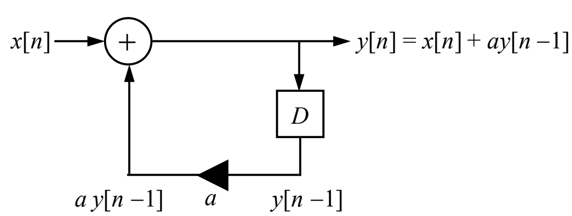
Such a DT LTI system is called an *infinite impulse response* (IIR) system, because the impulse response $h[n]$ never becomes identically zero for any finite time $n>0$. Intuitively, the impulse response has an infinite duration because N past values of the output are fed back and contribute to the present output $y[n]$.

Example: First-Order System

- Recall once again the first-order system described by a constant-coefficient linear difference equation

$$y[n]=x[n]+ay[n-1], \tag{10}$$

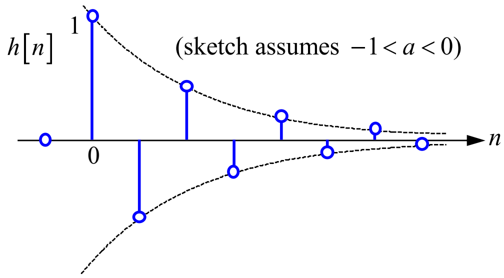
where a is a real constant. Difference equation (10) is an instance of the general form (42) with $N = 1$. The first-order system can be realized by the block diagram below, which shows that the past output $y[n-1]$ is fed back and contributes to the present output $y[n]$.



As a result of this feedback or recursion, the first-order system impulse response

$$h[n]=a^n u[n] \tag{11}$$

has infinite duration. The impulse response (11) is shown below for a negative value of a, $-1<a<0$, corresponding to a DT *highpass filter*. (Recall that positive a, $0<a<1$, describes a DT lowpass filter.) Because these values satisfy $|a|<1$, impulse response (11) is absolutely summable, $\sum_{n=-\infty}^{\infty}|h[n]|<\infty$, corresponding to a stable system.



Special Case $N = 0$

- In this special case, the difference equation (42) provides an *explicit* description for the output $y[n]$ given an input $x[n]$. We can immediately solve (42) to obtain an expression for the output:

$$y[n] = \frac{1}{a_0}\sum_{k=0}^{M} b_k x[n-k]. \tag{43}$$

  The system described by (43) is LTI. We can obtain its impulse response by directly substituting $x[n] = \delta[n]$ and $y[n] = h[n]$ into (43):

$$h[n] = \frac{1}{a_0}\sum_{k=0}^{M} b_k \delta[n-k]. \tag{44}$$

  In this special case, the impulse response is a scaled sum of delayed DT impulse functions.

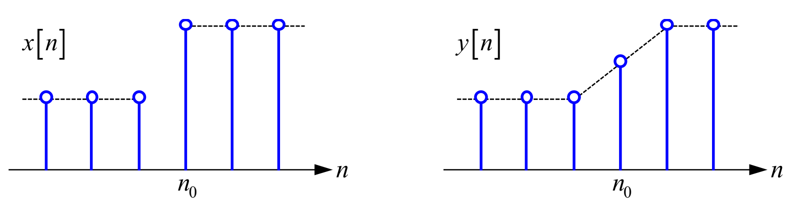
- Such a DT LTI system is called a *finite impulse response* (FIR) system, because the impulse response $h[n]$ becomes identically zero for $n > M$, and thus has finite length. FIR DT filters are used extensively in practical digital signal processing. We study two simple examples of FIR filters here, and will study more examples in Chapters 3, 5 and 6 below.

Example: Two-Point Moving Average (Lowpass Filter)

- A two-point *moving average filter* is described by an input-output relation

$$y[n] = \frac{1}{2}\left(x[n] + x[n-1]\right). \tag{45}$$

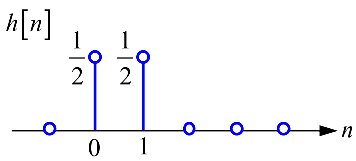
  Expression (45) is an instance of (43) with $M = 1$. As the term "moving average" implies, at time *n*, the output $y[n]$ is an average of the present and preceding input values, $x[n]$ and $x[n-1]$. Moving average filters, a type of lowpass filter, are often used to smooth out fluctuations or noise appearing in signals or data. Two-dimensional versions of such filters may be used in smoothing images. An example of an input signal and the corresponding output signal are shown.

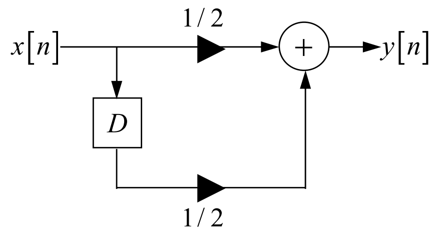


- The impulse response is found by substituting $x[n]=\delta[n]$ and $y[n]=h[n]$ into (45):

$$h[n]=\frac{1}{2}\left(\delta[n]+\delta[n-1]\right). \tag{46}$$

The impulse response is plotted below (on the left). A block diagram realizing (45) and (46) is shown (on the right).



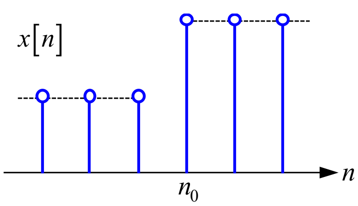


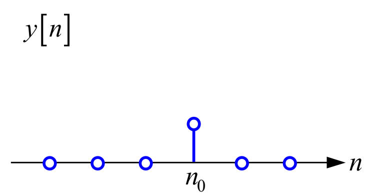
Example: Edge Detector (Highpass Filter)

- An *edge detector filter* is described by an input-output relation

$$y[n]=\frac{1}{2}\left(x[n]-x[n-1]\right), \tag{47}$$

and is another instance of (43) with $M=1$. At time n, the output $y[n]$ is half the difference between the present and preceding input value, $x[n]$ and $x[n-1]$. Edge detectors are a type of highpass filter used to accentuate differences appearing in signals or data. Two-dimensional versions of edge detectors may be used to accentuate edges in images. An input signal and the resulting output signal are shown below.

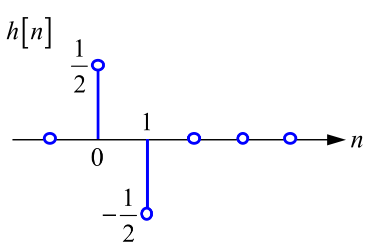


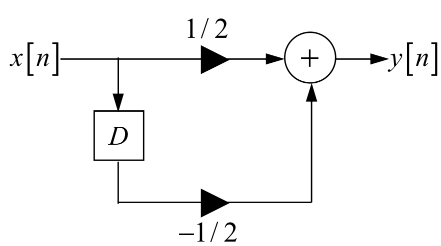


- The impulse response, found by substituting $x[n]=\delta[n]$ and $y[n]=h[n]$ into (47), is

$$h[n]=\frac{1}{2}\left(\delta[n]-\delta[n-1]\right). \tag{48}$$

The impulse response is plotted below (on the left). A block diagram realization of (47) and (48) is shown (on the right).





Stanford University
EE 102A: Signal Processing and Linear Systems I
Professor Joseph M. Kahn

Chapter 3: Fourier Series

Motivations

- Many CT or DT signals may be expressed as a discrete sum or a continuous integral of imaginary or complex exponential signals at different frequencies.
- Expressing signals in terms of imaginary or complex exponentials simplifies LTI system analysis.
- This table summarizes the methods for so expressing signals that are used in EE 102A and 102B.

Method	Principal Signal Type	Expressed in Terms of	Introduced in
CT Fourier series	Periodic power, CT	Imaginary exponentials $e^{jk\omega_0 t}$, ω_0 real, k integer	EE 102A
DT Fourier series	Periodic power, DT	Imaginary exponentials $e^{jk\Omega_0 n}$, Ω_0 real, k integer	
CT Fourier transform	Aperiodic energy, CT	Imaginary exponentials $e^{j\omega t}$, ω real and continuous	
DT Fourier transform	Aperiodic energy, DT	Imaginary exponentials $e^{j\Omega n}$, Ω real and continuous	
Laplace transform	Aperiodic, CT	Complex exponentials e^{st}, s complex and continuous	EE 102B
Z Transform	Aperiodic, DT	Complex exponentials z^n, z complex and continuous	

Major Topics in This Chapter (studied for both CT and DT unless noted otherwise)

- Complex exponentials as eigenfunctions of LTI systems. Transfer function of LTI system.
  Imaginary exponentials as eigenfunctions of LTI systems. Frequency response of LTI system.
- Fourier series
  - Trigonometric vs. exponential. Synthesis and analysis. Application to periodic or aperiodic signals.
- Fourier series properties
  - Linearity, time-shift, multiplication, time reversal, conjugation. Parseval's identity.
- Response of LTI systems to periodic inputs
  - Computing the frequency response.
  - CT system examples: first-order lowpass and highpass filters.
  - DT system examples. Recursive: first-order. Non-recursive: moving average, edge detector.
- Fourier series representations of different signal types: periodic, time-limited or general aperiodic.

Eigenfunctions of Continuous-Time Linear Time-Invariant Systems

General Case: Complex Exponentials

- Consider a CT LTI system *H* that has an impulse response $h(t)$. Given a general input signal $x(t)$, if we wish to predict the output $y(t) = H\{x(t)\}$, we perform a *convolution* between $x(t)$ and $h(t)$.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) * x(t)$$

- The *eigenfunctions* of CT LTI systems are *complex exponential* time signals

$$e^{st}, \ s \text{ complex}, \ -\infty < t < \infty .$$

If we input one of these signals to an LTI system *H*, the output is the same signal e^{st}, multiplied by an *eigenvalue* denoted by $H(s)$, as shown.

$$x(t) = e^{st} \longrightarrow \boxed{h(t) \leftrightarrow H(s)} \longrightarrow y(t) = H(s) \cdot e^{st}$$

We refer to the variable *s* as *complex frequency*, and refer to $H(s)$ as the *transfer function* of the LTI system *H*. Assuming we know the transfer function $H(s)$ as a function of *s*, then for an input e^{st}, we can predict the output by using *multiplication*, and do not need to use convolution.

- To show this is true, we input $x(t) = e^{st}$ to the system and compute the output $y(t)$ using convolution:

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau \\ &= e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \\ &= e^{st} \cdot H(s) \end{aligned} \quad . \tag{1}$$

- In (1), we have defined the system transfer function as

$$H(s) \stackrel{d}{=} \int_{-\infty}^{\infty} h(t) e^{-st} dt . \tag{2}$$

Given an impulse response $h(t)$, we compute the integral (2) to obtain the transfer function $H(s)$. The integral (2) defines $H(s)$ as the *bilateral Laplace transform* of the impulse response $h(t)$.

- The Laplace transform integral (2) converges for a large class of impulse responses $h(t)$, including some corresponding to unstable systems (see *EE 102B Course Reader*, Chapter 5).

Special Case: Imaginary Exponentials

- In (1) and (2) above, s is assumed complex-valued. Now we consider the special case that s is purely imaginary, $s = j\omega$, where ω is real. We are considering a subset of the complex exponential signals, namely, the *imaginary exponential* signals

$$e^{j\omega t},\ \omega \text{ real},\ -\infty < t < \infty .$$

These, too, are eigenfunctions of LTI systems. If we input an imaginary exponential $e^{j\omega t}$ to an LTI system, the output is the same signal, multiplied by an eigenvalue $H(j\omega)$, as shown. We refer to the variable ω simply as *frequency*, and refer to $H(j\omega)$ as the *frequency response* of the LTI system *H*.

$$x(t) = e^{j\omega t} \longrightarrow \boxed{h(t) \leftrightarrow H(j\omega)} \longrightarrow y(t) = H(j\omega)\cdot e^{j\omega t}$$

As in the general case above, if we know $H(j\omega)$ as a function of ω, then for an imaginary exponential input signal $e^{j\omega t}$, we can predict the system output by using multiplication, and need not use convolution. To prove this, we simply use an input $x(t) = e^{j\omega t}$ in (1). Then (1) yields an output

$$y(t) = e^{j\omega t}\cdot H(j\omega). \tag{3}$$

- The frequency response appearing in (3) is defined by the following expression:

$$H(j\omega) \stackrel{d}{=} H(s)\Big|_{s=j\omega} = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt . \tag{4}$$

As in the general case above, given an impulse response $h(t)$, we evaluate the integral (4) to obtain the frequency response $H(j\omega)$. The integral (4) defines $H(j\omega)$ as the *CT Fourier transform* of the impulse response $h(t)$. We will study the CT Fourier transform extensively in Chapter 4.

- The Fourier transform integral (4) converges for many impulse responses $h(t)$, but does not converge in some cases (notably, some important unstable systems) for which the Laplace transform (2) does converge. This motivates us to use the Laplace transform in studying feedback control and other applications involving potentially unstable systems (see *EE 102B Course Reader*, Chapters 5-6).

Application to LTI System Analysis

- Consider a CT LTI system *H*, and assume we know its transfer function $H(s)$. Suppose we are given an input signal that is expressed as a linear combination of complex exponential signals at *K* distinct values of *s*:

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + \cdots + a_K e^{s_K t} . \tag{5}$$

Using the linearity of the system and the eigenfunction property (1), we can compute the output signal as

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + \cdots + a_K H(s_K) e^{s_K t} . \tag{6}$$

Notice that we have computed each term in the output (6) by using only multiplication, not convolution. While we have considered the case of general complex exponential input signals, it is easy to apply (5) and (6) to the special case of imaginary exponential input signals.

Example

- An LTI system *H* with input $x(t)$ and output $y(t)$ has an input-output relation

$$y(t) = \frac{dx(t)}{dt} + x(t-1). \tag{7}$$

We are given an input signal

$$x(t) = e^{jt} + e^{2t} ,$$

which is in the form (5). Using the input-output relation (7), we can compute the output signal:

$$\begin{aligned} y(t) &= je^{jt} + e^{j(t-1)} + 2e^{2t} + e^{2(t-1)} \\ &= \left(j + e^{-j}\right) e^{jt} + \left(2 + e^{-2}\right) e^{2t} \\ &= H(j) e^{jt} + H(2) e^{2t} \end{aligned} . \tag{8}$$

The output (8) is consistent with the general form (6). In EE 102B, we will learn that the input-output relation (7) corresponds to a transfer function $H(s) = s + e^{-s}$. Knowing this transfer function, we can use (6) to write down the output $y(t)$ without performing the computations in the first line of (8).

Types of Fourier Series Representations

- We are given a periodic CT signal that satisfies

$$x(t) = x(t + T_0) \quad \forall t , \tag{9}$$

where T_0 is the *period* and the *fundamental frequency* is

$$\omega_0 = \frac{2\pi}{T_0} . \tag{10}$$

- We can represent the periodic signal $x(t)$ as linear combination of sinusoidal or imaginary exponential basis signals, each at a frequency $k\omega_0$, which is an integer *k* times the fundamental frequency ω_0. Such a representation is called a *Fourier series* (FS). As shown in the table below, there are several ways to construct a FS representation. For now, we use the symbol $\hat{x}(t)$ to denote the FS representation of a signal $x(t)$ because, as we will see, $\hat{x}(t)$ is not necessarily identical to $x(t)$.

Fourier Series Synthesis	Frequencies	Pros	Cons
$\hat{x}(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$	Positive	• Real coefficients for all real $x(t)$.	• Harder algebra. • Not eigenfunctions: cannot simply use multiplication.
$\hat{x}(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k \cos k\omega_0 t - C_k \sin k\omega_0 t]$	Positive		
$\hat{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	Positive and negative	• Easier algebra. • Eigenfunctions: use multiplication.	• Complex coefficients for many real $x(t)$.

- The *trigonometric FS* representations in the first two rows use only positive frequencies and use purely real coefficients in representing real signals. The sine and cosine basis signals, however, are not eigenfunctions of LTI systems, so if a FS representation $\hat{x}(t)$ is input to an LTI system, the output $y(t)$ cannot be computed simply by using multiplication, as in (6).
- The *exponential FS* representation is shown in the third row. Algebraic manipulation of imaginary exponentials is far easier than that of trigonometric functions. The imaginary exponential basis signals $e^{jk\omega_0 t}$ are eigenfunctions of LTI systems, so if an exponential FS representation $\hat{x}(t)$ is input to an LTI system, simple multiplication may be used to compute the output $y(t)$, as in (6). In EE 102A and 102B, we use only the exponential FS for CT signals.
- The considerations governing representation of DT signals in terms of FS are entirely analogous, and we will use only the exponential FS for DT signals.

Continuous-Time Fourier Series

- We are given a periodic CT signal $x(t)$ with period T_0 and fundamental frequency $\omega_0 = 2\pi / T_0$. We will represent $x(t)$ as an exponential continuous-time Fourier series (CTFS) in the form

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} . \tag{11}$$

We refer to (11) as a *synthesis equation*, as it shows how the periodic signal is synthesized as a linear combination of imaginary exponential at frequencies $k\omega_0$, $-\infty < k < \infty$, which are integer multiples of the fundamental frequency ω_0. The a_k, $-\infty < k < \infty$, are the *CTFS coefficients* for the signal $x(t)$.

- We can verify that the CTFS synthesis (11) is periodic in time with period T_0:

$$\begin{aligned} \hat{x}(t+T_0) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (t+T_0)} \\ &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \underbrace{e^{jk2\pi}}_{=1} . \\ &= \hat{x}(t) \end{aligned} \tag{12}$$

In the second line, we have used (10) to write $\omega_0 T_0 = 2\pi$.

- Now we must address the question of how to perform *analysis*. Given a periodic signal $x(t)$, how can we determine the CTFS coefficients a_k, $-\infty < k < \infty$?

Obtaining the Fourier Series by Inspection

- In some simple cases, the CTFS coefficients can be found by inspection. For example, consider

$$x(t) = 1 + \sin 2\pi t + \cos 3\pi t .$$

This signal is periodic with period $T_0 = 2$. We can express it as a linear combination of imaginary exponentials with frequencies

$$k\omega_0 = k\frac{2\pi}{T_0} = k\pi .$$

The representation is

$$x(t) = 1\left(e^{j0}\right) + \frac{1}{2j}\left(e^{j2\pi t} - e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j3\pi t} + e^{-j3\pi t}\right).$$

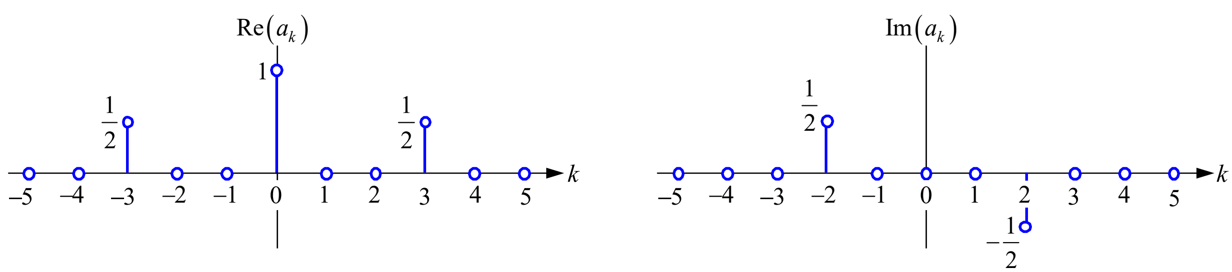
Comparing this representation to the synthesis equation (11), we have found the CTFS coefficients as

$$a_0 = 1$$

$$a_{-2} = -a_2 = \frac{1}{2} j$$

$$a_{-3} = a_3 = \frac{1}{2},$$

while all other CTFS coefficients are zero. These CTFS coefficients are plotted here.



Imaginary Exponential Basis Signals

- In deriving a general analysis method, we use a set of *imaginary exponential basis signals*, which are defined as

$$\phi_k(t) = e^{jk\omega_0 t}, \quad -\infty < k < \infty . \tag{13}$$

Each basis signal is periodic, satisfying

$$\phi_k(t + T_0) = \phi_k(t) \quad \forall t ,$$

which can be proven in the same way (12) was proven.

- We would like to show that the basis signals form an *orthogonal set*, so we compute the following integral. (You may recognize it as an *inner product* between $\phi_k(t)$ and $\phi_m(t)$ from your study of linear algebra.)

$$\int_{T_0} \phi_k(t)\phi_m^*(t)\,dt = \int_{T_0} e^{j(k-m)\omega_0 t}\,dt \ . \tag{14}$$

In (14), and throughout our study of CTFS, we use the following notation to denote an integral over an interval of length T_0 starting at an arbitrary time t_1:

$$\int_{T_0} (\ )\,dt \quad \text{means} \quad \int_{t_1}^{t_1+T_0} (\ )\,dt\,,\ t_1 \text{ arbitrary.}$$

We will use Euler's relation (see Appendix, page 288) to express the imaginary exponential in (14) in terms of sinusoids. Evaluating (14) for $k \neq m$, we find

$$\int_{T_0} e^{j(k-m)\omega_0 t}\,dt = \int_{T_0} \cos\big((k-m)\,\omega_0 t\big)dt + j\int_{T_0} \sin\big((k-m)\,\omega_0 t\big)dt = 0\,,$$

which vanishes because each integral on the right-hand side includes an integer number of cycles of a sinusoid. Evaluating (14) for $k = m$ yields

$$\int_{T_0} e^{j(k-m)\omega_0 t}\,dt = \int_{T_0} (1)\,dt = T_0\,.$$

In summary, the basis signals form an *orthogonal set* with pairwise integrals (inner products) given by

$$\int_{T_0} \phi_k(t)\phi_m^*(t)\,dt = \int_{T_0} e^{j(k-m)\omega_0 t}\,dt = \begin{cases} 0 & k \neq m \\ T_0 & k = m \end{cases}. \tag{15}$$

Analysis Equation

- We are given a periodic signal $x(t)$ with period T_0 and fundamental frequency $\omega_0 = 2\pi / T_0$. We will represent $x(t)$ by an exponential CTFS or, equivalently, approximate it as a linear combination of imaginary exponential basis signals (a restatement of synthesis equation (11)):

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \ . \tag{11'}$$

- In order to determine the CTFS coefficients a_k, $-\infty < k < \infty$, that appear in (11′), we assume for the moment that the approximation is identical to the original signal

$$\hat{x}(t) = x(t)\,. \tag{16}$$

We compute an inner product integral between both sides of (16) and imaginary exponential basis signal $\phi_m(t) = e^{jm\omega_0 t}$:

$$\int_{T_0} x(t) e^{-jm\omega_o t}\,dt \ = \int_{T_0} \hat{x}(t) e^{-jm\omega_o t}\,dt \ . \tag{17}$$

Now we use expression (11) and substitute it for $\hat{x}(t)$ on the right-hand side of (17):

$$
\begin{aligned}
\int_{T_0} x(t) e^{-jm\omega_o t} dt &= \int_{T_0} \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jm\omega_o t} dt \\
&= \sum_{k=-\infty}^{\infty} a_k \underbrace{\int_{T_0} e^{j(k-m)\omega_0 t} dt}_{=\begin{cases} 0 & k \neq m \\ T_0 & k=m \end{cases}} \quad . \\
&= T_0 a_m
\end{aligned}
\tag{18}
$$

In the second line, we have interchanged the order of summation and integration, and have evaluated the integral using (15). In the third line, we have evaluated the sum, finding that only the term for $k = m$ is nonzero. Rearranging (18) yields

$$
a_m = \frac{1}{T_0} \int_{T_0} x(t) e^{-jm\omega_o t} dt \, . \tag{19}
$$

Expression (19) is the *analysis equation* we sought. Given a periodic signal $x(t)$, it tells us how to obtain the CTFS coefficients a_m, $-\infty < m < \infty$, to represent the signal using synthesis equation (11).

Summary of Continuous-Time Fourier Series

- The synthesis and analysis equations are

$$
\hat{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \text{(synthesis)} \tag{11}
$$

$$
a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_o t} dt \qquad \text{(analysis)} \tag{19}
$$

- We often denote a CT signal $x(t)$ and its FS coefficients a_k as a *CTFS pair*:

$$
x(t) \overset{FS}{\leftrightarrow} a_k \, . \tag{20}
$$

- The CTFS coefficients for some important periodic CT signals are given in Table 4, Appendix.

Convergence of Continuous-Time Fourier Series

- Suppose a periodic signal $x(t)$, with period T_0, is used in (19) to compute CTFS coefficients a_k, and these coefficients are used in (11) to synthesize a periodic CTFS approximation $\hat{x}(t)$.

- It can be shown that if $x(t)$ has finite power

$$
\frac{1}{T_0} \int_{T_0} |x(t)|^2 dt < \infty \, ,
$$

the power in the difference between $x(t)$ and $\hat{x}(t)$ vanishes:

$$
\frac{1}{T_0} \int_{T_0} |x(t) - \hat{x}(t)|^2 dt = 0 \, .
$$

This does not imply that $\hat{x}(t) = x(t)$ at all t. In fact, $\hat{x}(t)$ differs from $x(t)$ near any values of t where $x(t)$ has discontinuities. Near these values of t, $\hat{x}(t)$ exhibits ripples called the *Gibbs phenomenon*.

- It can be shown that $\hat{x}(t) = x(t)$ except near values of t where $x(t)$ has discontinuities if $x(t)$ satisfies the *Dirichlet conditions*:
  - $x(t)$ is absolutely integrable over any period:

$$\frac{1}{T_0}\int_{T_0} |x(t)|dt < \infty .$$

  - $x(t)$ has a finite number of local maxima and minima in each period.
  - $x(t)$ has a finite number of discontinuities in each period.
  - Any discontinuities of $x(t)$ are finite.
- Later in this chapter, we will often use the symbol $x(t)$ to represent both the original signal used in the analysis (19) and the signal formed by the synthesis (11), unless it is necessary to draw a distinction between them.
- For further discussion on the convergence of CTFS, see *OWN*, Section 3.4.

Fourier Series of a Rectangular Pulse Train

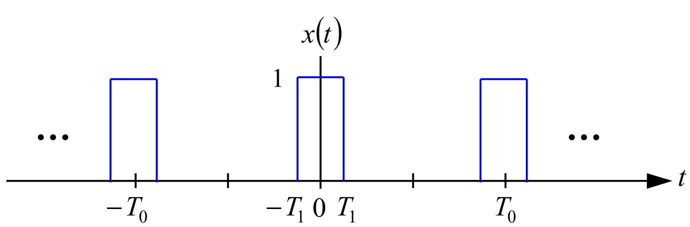
- As an example of applying the analysis and synthesis equations we have derived, we consider a periodic rectangular pulse train described by the formula

$$x(t) = \begin{cases} 1 & |t| \leq T_1 \\ 0 & T_1 < |t| \leq \frac{T_0}{2} \end{cases}, \quad x(t+T_0) = x(t) .$$

  It can, alternatively, be expressed as a sum of shifted rectangular pulse functions (defined on page 290):

$$x(t) = \sum_{l=-\infty}^{\infty} \Pi\left(\frac{t - lT_0}{2T_1}\right).$$

  This pulse train is shown below. The period is T_0 and the width of each pulse is $2T_1$.



- We compute the CTFS coefficients using the analysis equation (19). Given the symmetry of $x(t)$, it is natural to choose a symmetric integration interval:

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T_0} \int_{-T_1}^{T_1} e^{-jk\omega_0 t} dt . \quad (21)$$

Evaluating (21) for $k \neq 0$, we have

$$a_k = \frac{1}{-T_0 jk\omega_0} e^{-jk\omega_0 t} \Big|_{-T_1}^{T_1} = \frac{2\sin(k\omega_0 T_1)}{T_0 k\omega_0} = \frac{\omega_0 T_1}{\pi} \frac{\sin\left(\pi \frac{k\omega_0 T_1}{\pi}\right)}{\pi \frac{k\omega_0 T_1}{\pi}} . \quad (22)$$

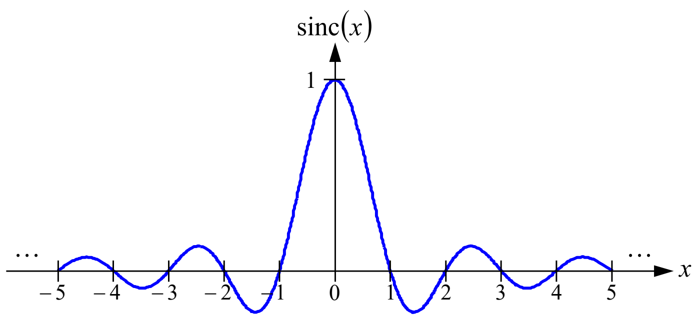
For $k = 0$, we cannot apply (22), which would involve division by zero. We return to (21), set $e^{-jk\omega_0 t} = 1$, and evaluate the integral to obtain

$$a_0 = \frac{1}{T_0} \int_{-T_1}^{T_1} (1) dt = \frac{2T_1}{T_0} = \frac{\omega_0 T_1}{\pi} . \quad (23)$$

- Refer to the Appendix, page 290, for a discussion of the sinc function, which is defined as

$$\operatorname{sinc}(x) \stackrel{d}{=} \frac{\sin(\pi x)}{\pi x} .$$

A plot of the sinc function is shown here.

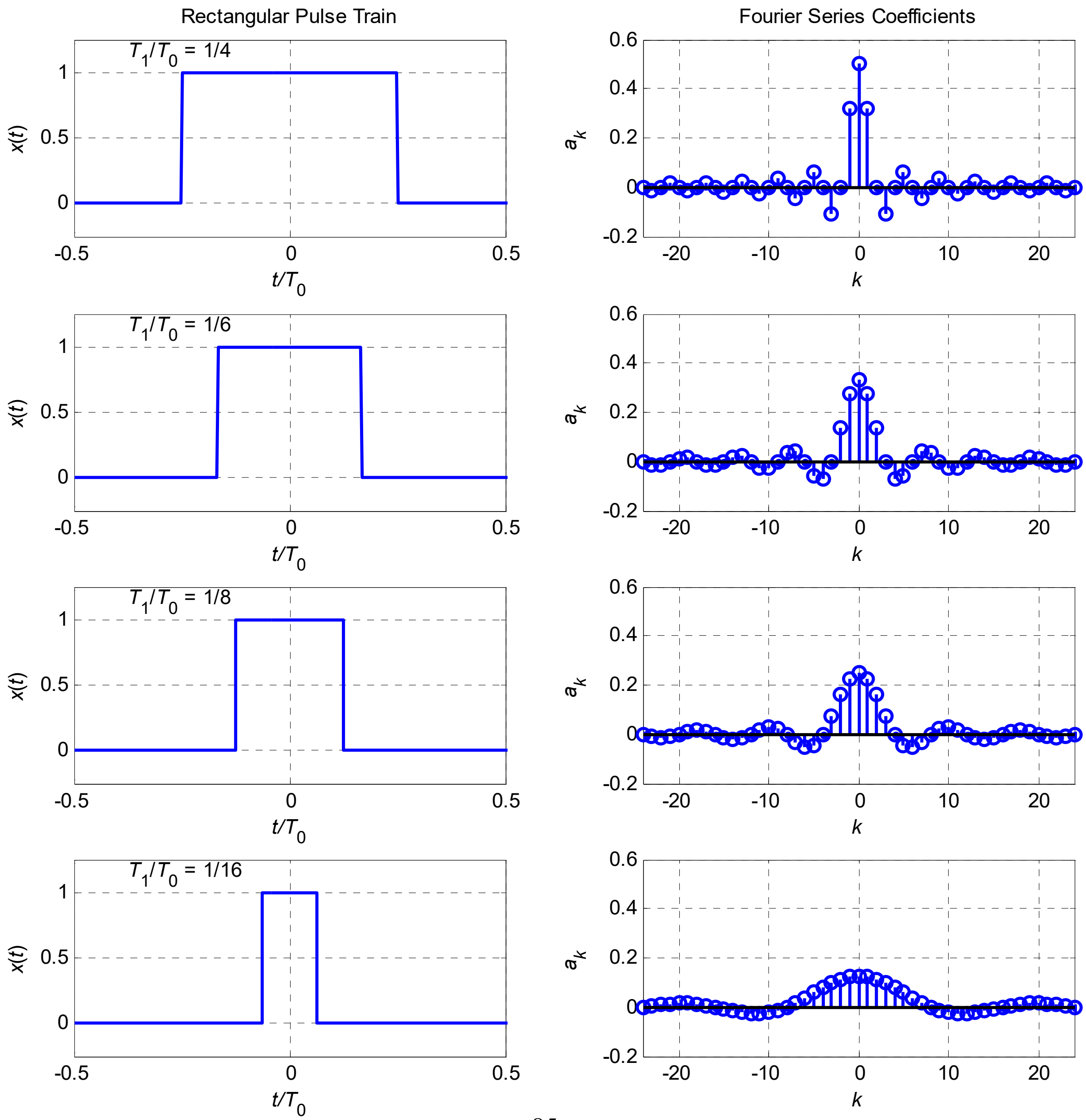


Note that as the argument x approaches zero, $\operatorname{sinc}(x)$ approaches a limiting value of 1.

- Using the sinc function, the CTFS coefficients of the pulse train can be expressed for all values of k as

$$a_k = \frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right), \quad -\infty < k < \infty . \quad (24)$$

- The figure below shows one period ($-T_0/2 \le t \le T_0/2$) of the periodic pulse train $x(t)$ and the corresponding Fourier series coefficients a_k for various values of the ratio T_1/T_0. We can observe that:
  - Because $x(t)$ is real and even in t, the a_k are real and even in k (see the CTFS properties below).
  - In this figure, we fix the period T_0, thus fixing the fundamental frequency $\omega_0 = 2\pi / T_0$, and we vary the ratio T_1/T_0, thus varying the width of each pulse. As we decrease T_1, each pulse becomes narrower. The spectrum of CTFS coefficients a_k at frequencies $k\omega_0$, $-\infty < k < \infty$, spreads out, occupying a wider range of frequencies. This is an example of *the inverse relationship between time and frequency*, a basic principle in Fourier series and Fourier transforms. A narrower pulse, representing a signal changing faster in time, is described using higher frequencies. Conversely, a wider pulse, representing a signal changing more slowly in time, is described using lower frequencies.



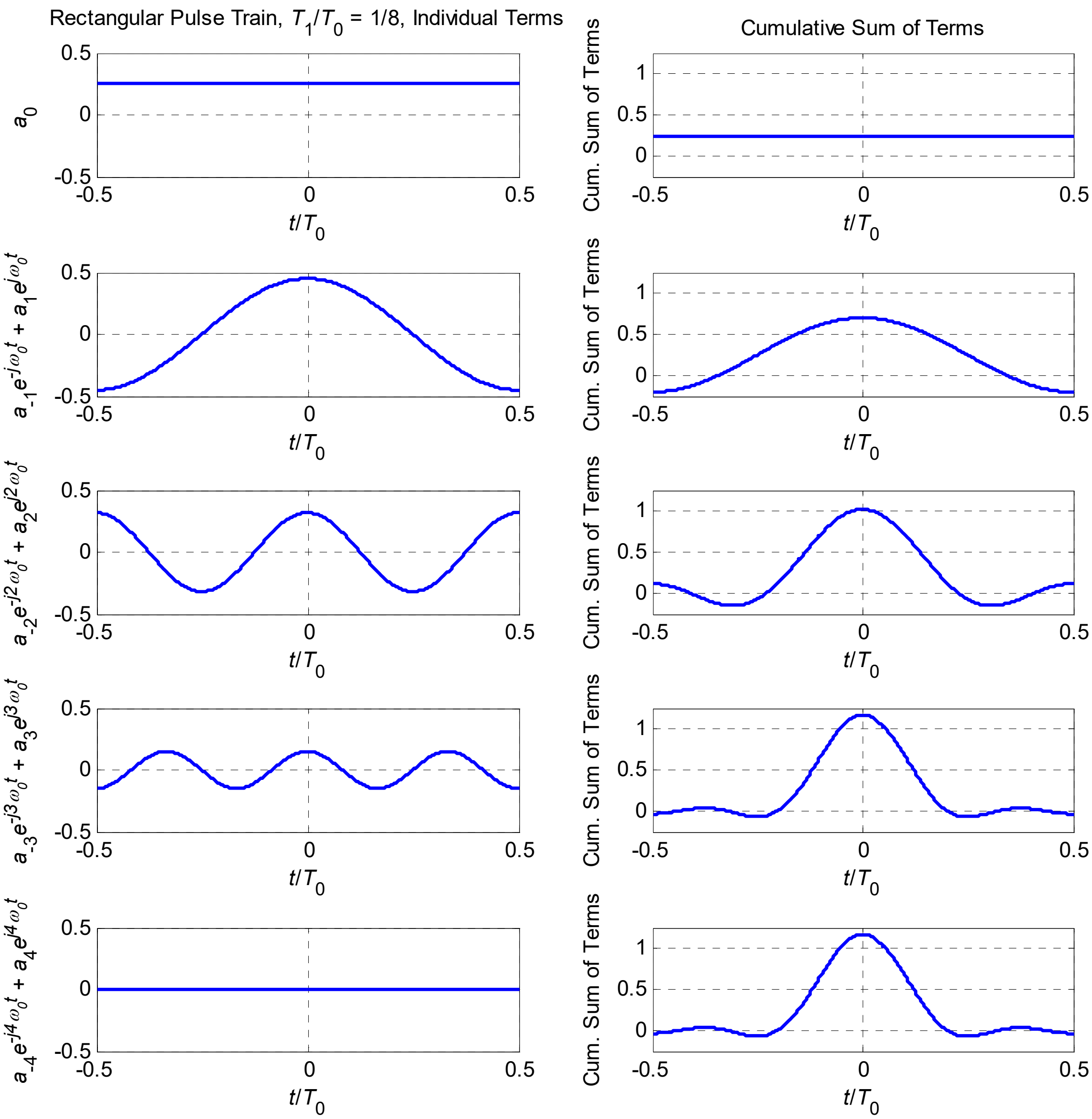
- Now we study synthesis of the pulse train using the CTFS coefficients (24) in the synthesis equation (11). In the following two figures, the left column shows the contribution to the synthesis (11) from the term(s) at $\pm k$:

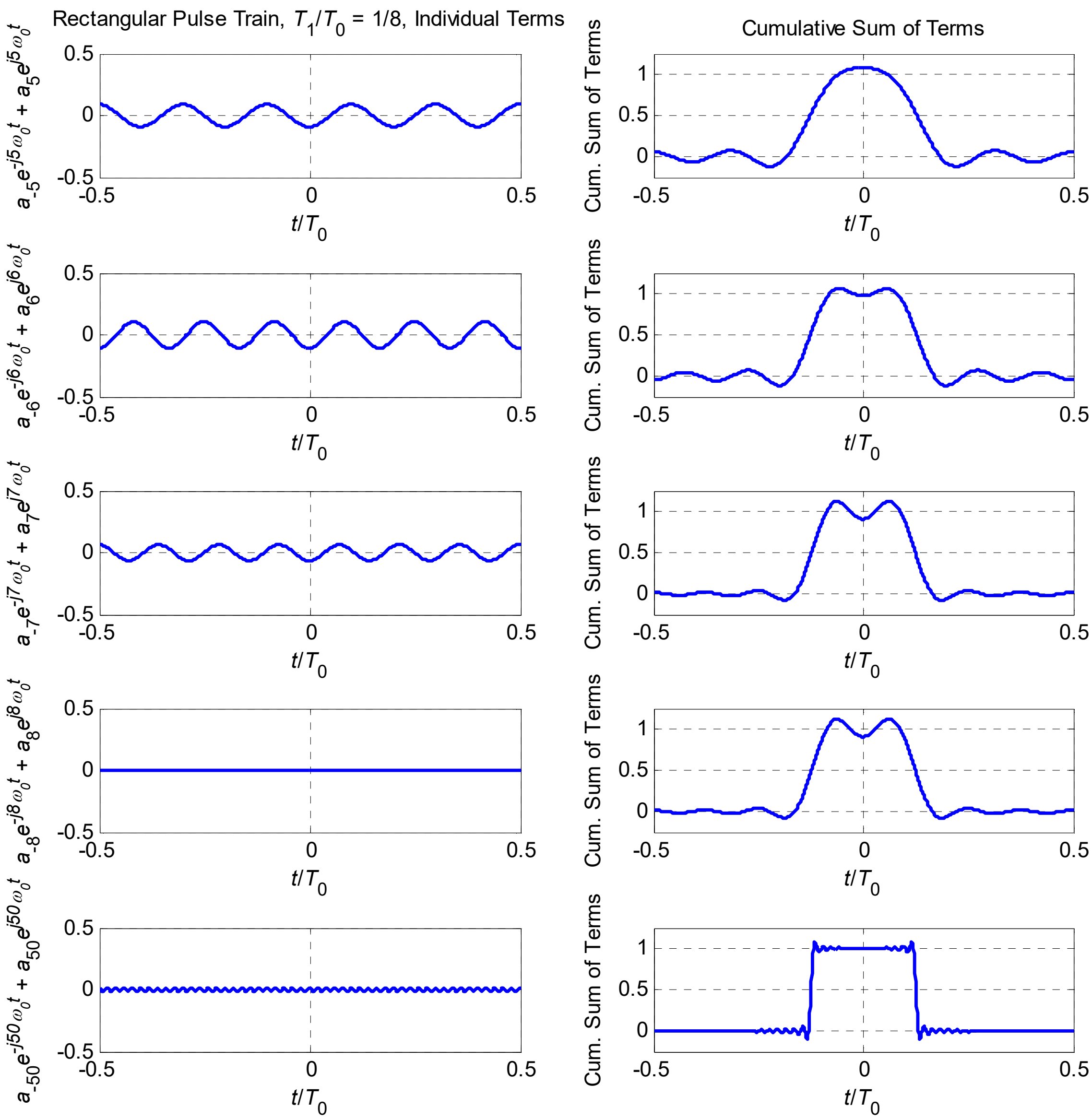
$$\begin{cases} a_0 & k=0 \\ a_{-k}e^{-jk\omega_0 t}+a_k e^{jk\omega_0 t} & k\neq 0 \end{cases}. \tag{25}$$

Contribution (25) is a real constant or cosine, because the a_k are conjugate-symmetric, $a_{-k}=a_k^*$, since $x(t)$ is real (refer to the explanation on pages 90-91). The right column shows a synthesis using a finite number of terms up to $\pm K$ in (11), which is

$$\hat{x}_K(t)=\sum_{k=-K}^{K} a_k e^{jk\omega_0 t}. \tag{26}$$

All plots consider $T_1/T_0=1/8$ and show the waveforms over one period, $-T_0/2\le t\le T_0/2$.





- We can make the following observations about these plots.
  - As we increase K and include more terms in the partial synthesis $\hat{x}_K(t)$, given by (26), we observe that $\hat{x}_K(t)$ better approximates $x(t)$. In particular, the transitions at the leading and trailing edges of the pulses become more abrupt.
  - We observe ripple in $\hat{x}_K(t)$ near the discontinuities (leading and trailing pulse edges) in $x(t)$. This is a manifestation of the Gibbs phenomenon. As we increase K, the ripple becomes confined to a narrower time interval, but its peak amplitude does not diminish.

Properties of the Continuous-Time Fourier Series

- These properties are useful for:
  - Computing the CTFS coefficients for new signals, with minimal effort, by using CTFS coefficients that we already know for other signals.
  - Checking the CTFS coefficients that we compute for new signals.
- For a complete list of CTFS properties, see Table 1 in the Appendix. We discuss only a few of the properties here.
- We assume periodic signals having a common period T_0 and a common fundamental frequency $\omega_0 = 2\pi / T_0$. We consider one or two signals and their CTFS coefficients:

$$x(t) \overset{FS}{\leftrightarrow} a_k \quad \text{and} \quad y(t) \overset{FS}{\leftrightarrow} b_k .$$

Linearity

- A linear combination of $x(t)$ and $y(t)$ is periodic with the same period T_0. It has CTFS coefficients given by the corresponding linear combination of the CTFS coefficients a_k and b_k:

$$Ax(t) + By(t) \overset{FS}{\leftrightarrow} Aa_k + Bb_k .$$

Time Shift

- A signal time-shifted by t_0 has CTFS coefficients multiplied by a complex-valued factor $e^{-jk\omega_0 t_0}$:

$$x(t - t_0) \overset{FS}{\leftrightarrow} e^{-jk\omega_0 t_0} a_k . \tag{27}$$

  The magnitude and phase of $e^{-jk\omega_0 t_0} a_k$ are related to those of a_k by

$$\begin{cases} \left| e^{-jk\omega_0 t_0} a_k \right| = |a_k| \\ \angle\left(e^{-jk\omega_0 t_0} a_k \right) = \angle a_k - k\omega_0 t_0 \end{cases} . \tag{27'}$$

  Time-shifting a signal by t_0 affects its CTFS coefficients by

  - Leaving the magnitude unchanged.
  - Adding a phase shift proportional to the negative of the time shift, $-t_0$, which varies linearly with frequency $k\omega_0$.

  Proof

  We define a time-shifted signal and its CTFS coefficients:

$$x(t - t_0) \overset{FS}{\leftrightarrow} b_k .$$

We compute its CTFS coefficients using analysis equation (19):

$$b_k = \frac{1}{T_0} \int_{T_0} x(t - t_0) e^{-jk\omega_o t} dt .$$

We change the integration variable to $\tau = t - t_0$. Although the integration limits change, we can denote the integration interval using the same notation:

$$\begin{aligned} b_k &= \frac{1}{T_0} \int_{T_0} x(\tau) e^{-jk\omega_o(\tau + t_0)} d\tau \\ &= e^{-jk\omega_o t_0} \frac{1}{T_0} \int_{T_0} x(\tau) e^{-jk\omega_o \tau} d\tau . \\ &= e^{-jk\omega_o t_0} a_k \end{aligned}$$

We have used the analysis equation (19) to substitute a_k in the third line.

QED

Multiplication

- The product of $x(t)$ and $y(t)$ is periodic with the same period T_0, and has CTFS coefficients that are the convolution between the sequences of CTFS coefficients a_k and b_k:

$$x(t) y(t) \overset{FS}{\leftrightarrow} \sum_{l=-\infty}^{\infty} a_l b_{k-l} .$$

Time Reversal

- Reversal in the time domain corresponds to reversal in the frequency domain:

$$x(-t) \overset{FS}{\leftrightarrow} a_{-k} . \tag{28}$$

- As a consequence, if a signal is even in time, its CTFS coefficients are even in frequency:

$$x(-t) = x(t) \overset{FS}{\leftrightarrow} a_{-k} = a_k ,$$

and if a signal is odd in time, its CTFS coefficients are odd in frequency:

$$x(-t) = -x(t) \overset{FS}{\leftrightarrow} a_{-k} = -a_k .$$

Conjugation

- Complex conjugation of a time signal corresponds to frequency reversal and complex conjugation of its CTFS coefficients:

$$x^*(t) \overset{FS}{\leftrightarrow} a_{-k}^* . \tag{29}$$

Conjugate Symmetry for Real Signal

- A real signal $x(t)$ is equal to its complex conjugate $x^*(t)$. This, in combination with (29), implies:

$$x(t) = x^*(t) \overset{FS}{\leftrightarrow} a_k = a_{-k}^* . \tag{30}$$

  If a signal is real, its CTFS coefficients at positive frequency equal the complex conjugates of its CTFS coefficients at negative frequency. This property of the CTFS coefficients is called *conjugate symmetry*.

- We can restate the conjugate symmetry property (30) in two alternate ways. First, if a signal is real, the magnitudes of its CTFS coefficients are even in frequency, while the phases of its CTFS coefficients are odd in frequency:

$$x(t) = x^*(t) \overset{FS}{\leftrightarrow} \begin{cases} |a_k| = |a_{-k}| \\ \angle a_k = -\angle a_{-k} \end{cases} . \tag{30a}$$

  Second, if a signal is real, the real parts of the CTFS coefficients are even in frequency, and the imaginary parts of its CTFS coefficients are odd in frequency:

$$x(t) = x^*(t) \overset{FS}{\leftrightarrow} \begin{cases} \mathrm{Re}(a_k) = \mathrm{Re}(a_{-k}) \\ \mathrm{Im}(a_k) = -\mathrm{Im}(a_{-k}) \end{cases} . \tag{30b}$$

Real, Even or Real, Odd Signals

- Combining the time reversal and conjugation properties (28) and (29), we find that

$$x(t) \text{ real and even in } t \overset{FS}{\leftrightarrow} a_k \text{ real and even in } k \tag{31}$$

  and

$$x(t) \text{ real and odd in } t \overset{FS}{\leftrightarrow} a_k \text{ imaginary and odd in } k . \tag{32}$$

Fourier Series Synthesis of a Real Signal

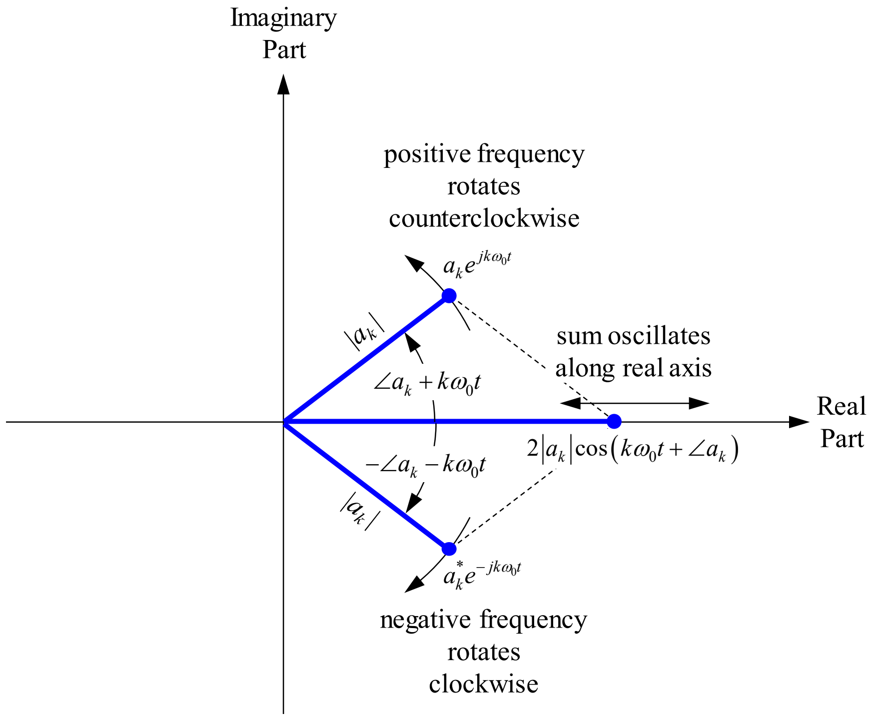
- Here, we further explain the roles of positive and negative frequencies and conjugate-symmetric CTFS coefficients in the synthesis of a real, periodic signal (as on pages 86-87). Consider the synthesis

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} . \tag{11}$$

  Assuming the signal is real, the CTFS coefficients a_k must satisfy (30). The term in (11) at zero frequency satisfies $a_0 = a_{-0}^*$, so it is a real constant equal to the average value of the signal. Consider any pair of terms in (11) at positive and negative frequencies $\pm k\omega_0$, $k \neq 0$. Their CTFS coefficients must satisfy $a_k = a_{-k}^*$ (or $a_{-k} = a_k^*$), so they add up to yield a real cosine signal at frequency $k\omega_0$:

$$\begin{aligned} a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t} &= a_k e^{jk\omega_0 t} + \left(a_k e^{jk\omega_0 t}\right)^* \\ &= 2|a_k| \cos\left(\angle a_k + k\omega_0 t\right) \end{aligned} . \tag{33}$$

- These two terms in (33) are illustrated in the figure below. The positive-frequency term $a_k e^{jk\omega_0 t}$ corresponds to a vector rotating *counterclockwise*, with magnitude $|a_k|$ and phase $\angle a_k + k\omega_0 t$. The negative-frequency term $a_{-k} e^{-jk\omega_0 t} = a_k^* e^{-jk\omega_0 t} = \left(a_k e^{jk\omega_0 t}\right)^*$ corresponds to a vector rotating *clockwise*, with magnitude $|a_k|$ and phase $-\angle a_k - k\omega_0 t$. The sum of these two vectors is always a real cosine, which oscillates along the real axis.



Inner Products

This section may be skipped in a first reading of this chapter.

- The *dot product* is an important tool in the study of ordinary vectors, which are *N*-tuples with real- or complex-valued entries. The dot product is a function that maps a pair of vectors, **x** and **y**, to a scalar quantity denoted by $\mathbf{x} \cdot \mathbf{y}$. In the table below, the first column reviews key properties of the dot product.

  Given a vector **x**, the *norm* or *length* of **x** is $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$, which is the square root of the dot product of **x** with itself. Given nonzero vectors **x** and **y**, if their dot product is zero, $\mathbf{x} \cdot \mathbf{y} = 0$, the two vectors are *orthogonal*, i.e., they point along *perpendicular* directions. Finally, the magnitude of the dot product between the two vectors cannot exceed the product of their norms, $|\mathbf{x} \cdot \mathbf{y}| \le \|\mathbf{x}\| \|\mathbf{y}\|$, with equality only when the two vectors are *parallel*, i.e., when one vector is a scalar multiple of the other.

- The *inner product* is a generalization of the dot product to more general types of vectors. The inner product is a function that maps a pair of vectors, *x* and *y*, to a scalar quantity denoted by $\langle x, y \rangle$. The second column of the table below presents key properties of the inner product. In EE 102A and 102B, *x* and *y* may denote CT signals $x(t)$ and $y(t)$ or DT signals $x[n]$ and $y[n]$. The definition of the inner product $\langle x,y\rangle$ depends on whether the signals are CT or DT, and whether they are periodic or aperiodic. Given a signal *x*, the *norm* of *x* is $\|x\|=\left(\langle x,x\rangle\right)^{1/2}$, which is the square root of the inner product of *x* with itself. The square of a signal's norm equals the signal's energy computed over one period (if the signal is periodic) or over all time (if it is aperiodic). Given nonzero signals *x* and *y*, if their inner product is zero, $\langle x,y\rangle=0$, the two signals are *orthogonal*. Lastly, the magnitude of the inner product between the two signals is less than or equal to the product of their norms, $\left|\langle x,y\rangle\right|\le\|x\|\|y\|$, with equality only when one signal is a scalar multiple of the other.

	Dot Product	**Inner Product**
Applicable To	Vectors $\mathbf{x}=\{x_1,\ldots,x_N\}$ and $\mathbf{y}=\{y_1,\ldots,y_N\}$ Generally complex.	CT signals $x(t)$ and $y(t)$ or DT signals $x[n]$ and $y[n]$ Generally complex.
Denoted By	$\mathbf{x}\cdot\mathbf{y}$	$\langle x,y\rangle$
Definition	$\mathbf{x}\cdot\mathbf{y}=\sum_{i=1}^{N}x_i y_i^*$ Generally complex.	CT periodic: Chapter 3 (34) DT periodic: Chapter 3 (81) CT aperiodic: Chapter 4 (36) DT aperiodic: Chapter 5 (42) Generally complex.
Norm	$\|\mathbf{x}\|=\left(\mathbf{x}\cdot\mathbf{x}\right)^{1/2}$ Always real. $\|\mathbf{x}\|$ is length of **x**.	$\|x\|=\left(\langle x,x\rangle\right)^{1/2}$ Always real. $\|x\|^2$ is energy of *x* $\begin{cases}\text{over one period} & x\text{ periodic}\\ \text{over all time} & x\text{ aperiodic}\end{cases}.$
Orthogonality	Assume **x**, **y** nonzero. If $\mathbf{x}\cdot\mathbf{y}=0$, **x** and **y** are orthogonal, i.e., one is perpendicular to the other.	Assume *x*, *y* nonzero. If $\langle x,y\rangle=0$, *x* and *y* are orthogonal.
Cauchy-Schwarz Inequality	Assume **x**, **y** nonzero. $\left|\mathbf{x}\cdot\mathbf{y}\right|\le\|\mathbf{x}\|\|\mathbf{y}\|$ Equality only if one is a multiple of the other, i.e., **x** and **y** are parallel.	Assume *x*, *y* nonzero. $\left|\langle x,y\rangle\right|\le\|x\|\|y\|$ Equality only if one is a multiple of the other.

- The inner products between signals are important in many applications. For example, in digital communications, where the goal is to convey information bits, we may use two different CT signals, $x(t)$ and $y(t)$, to encode the bits 0 and 1. The inner product between the two signals is a measure of how easily we can distinguish one signal from the other in the presence of noise, and determines the probability of mistaking one signal for the other. The signals are most easily distinguished if one signal is the negative of the other, $y(t) = -x(t)$, in which case, $\langle x(t), y(t) \rangle = -\|x\|^2$. As another example from digital communications, we often want to transmit several different signals simultaneously through a shared communication medium in order to maximize the rate at which information is conveyed. This technique is called *multiplexing* (see page 286). It is desirable for a set of multiplexed signals to be *mutually orthogonal* so they do not interfere with each other. In other words, the inner product between any pair of signals should be zero.

Parseval's Identity

- *Parseval's Identity* (also known as *Parseval's Theorem*) is a property of Fourier representations, and exists in different forms for periodic or aperiodic signals in CT or DT. It allows us to compute the inner product between two signals, or the power or energy of one signal, either in the time domain or the frequency domain. Given any particular signal(s), the calculation of an inner product (or power or energy) is often found to be easier in one domain or the other, as we will see in examples throughout EE 102A and 102B.
- Parseval's identity for the CTFS allows us to compute the inner product between two periodic CT signals, or the power of one periodic CT signal, either in the time domain or in the frequency domain.
- As above, we assume periodic signals having a common period T_0 and a common fundamental frequency $\omega_0 = 2\pi / T_0$. We consider one or two signals and their CTFS coefficients:

$$x(t) \overset{FS}{\leftrightarrow} a_k \quad \text{and} \quad y(t) \overset{FS}{\leftrightarrow} b_k \, .$$

Inner Product Between Signals

- The general form of Parseval's identity, for an *inner product between two periodic CT signals*, states that

$$\langle x(t), y(t) \rangle = \int_{T_0} x(t) y^*(t)\, dt = T_0 \sum_{k=-\infty}^{\infty} a_k b_k^* \, . \tag{34}$$

The middle expression in (34) is an inner product between the CT signals $x(t)$ and $y(t)$ computed in the time domain. Since both are periodic signals, the integration is performed over only a single period, of duration T_0. The rightmost expression in (34) is an inner product between the corresponding sequences of CTFS coefficients, a_k and b_k^*, which are infinite-length discrete vectors.

Example: Orthogonality of Even and Odd Square Waves

- In a homework problem, you will study the CTFS coefficients for an *even square wave* $y(t) \overset{FS}{\leftrightarrow} b_k$ and an *odd square wave* $z(t) \overset{FS}{\leftrightarrow} c_k$. By using symmetry properties (31) and (32) and using (34), you will show that they are orthogonal in both the time and the frequency domains:

$$\int_{T_0} \underbrace{y(t)}_{\substack{\text{even} \\ \text{in } t}} \underbrace{z^*(t)}_{\substack{\text{odd} \\ \text{in } t}} dt = T_0 \sum_{k=-\infty}^{\infty} \underbrace{b_k}_{\substack{\text{even} \\ \text{in } k}} \underbrace{c_k^*}_{\substack{\text{odd} \\ \text{in } k}} = 0 .$$

Signal Power

- By considering the special case of (34) with $x(t) = y(t)$ and $a_k = b_k$, we obtain an expression for the *power of a periodic CT signal*:

$$P = \frac{1}{T_0} \int_{T_0} \left| x(t) \right|^2 dt = \sum_{k=-\infty}^{\infty} \left| a_k \right|^2 . \tag{35}$$

  We recognize the middle expression in (35) as the power of the periodic signal $x(t)$, which is the energy of $x(t)$ in one period divided by the period. Expression (35) shows that we can, alternatively, compute the power of $x(t)$ by summing the squared magnitudes of its CTFS coefficients a_k over $-\infty < k < \infty$, as in the rightmost expression.

- Based on (35), we can identify $\left| a_k \right|^2$ as the *power density spectrum* of the periodic signal $x(t)$, as $\left| a_k \right|^2$ indicates the power contained in the component of the signal at frequency $k\omega_0$. We can interpret the rightmost expression in (35) as the sum of the powers at all frequencies $k\omega_0$, $-\infty < k < \infty$.

Proof

- In order to prove the general form (34) for the inner product, we start with the integral in the middle expression in (34). We represent $y^*(t)$ by the complex conjugate of the CTFS synthesis of $y(t)$:

$$y^*(t) = \left(\sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} \right)^* ,$$

  and substitute this into the integral to obtain

$$\int_{T_0} x(t) y^*(t) dt = \int_{T_0} x(t) \left(\sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} \right)^* dt .$$

  Interchanging the order of summation and integration yields

$$\int_{T_0} x(t) y^*(t) dt = \sum_{k=-\infty}^{\infty} \left(\int_{T_0} x(t) e^{-jk\omega_0 t} dt \right) b_k^* = T_0 \sum_{k=-\infty}^{\infty} a_k b_k^* .$$

  We have used the analysis equation (19) to substitute $T_0 a_k$ for the integral in parentheses.

QED

Response of Continuous-Time Linear Time-Invariant Systems to Periodic Inputs

- Suppose we have an LTI system whose impulse response is $h(t)$, and assume the integral defining the system frequency response

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \tag{4}$$

converges. As expressed by (3), the imaginary exponential signals

$$e^{j\omega t}, \ \omega \text{ real}, \ -\infty < t < \infty .$$

are eigenfunctions of the system, with eigenvalues given by $H(j\omega)$, as shown here.

$$x(t) = e^{j\omega t} \longrightarrow \boxed{h(t) \leftrightarrow H(j\omega)} \longrightarrow y(t) = H(j\omega) \cdot e^{j\omega t}$$

- Now suppose we input a signal $x(t)$, which is periodic with period $T_0 = 2\pi / \omega_0$ and can be expressed by a CTFS with coefficients a_k, $-\infty < k < \infty$:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} . \tag{11}$$

The output $y(t)$, like the input $x(t)$, is periodic with period T_0. Using linearity of the system and the eigenfunction property (3) of the imaginary exponentials, the output $y(t)$ can be expressed by a CTFS

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t} , \tag{36}$$

which has CTFS coefficients

$$b_k = a_k H(jk\omega_0), \ -\infty < k < \infty . \tag{37}$$

These are the CTFS coefficients of the input $x(t)$, scaled by values of the frequency response $H(j\omega)$ evaluated at $\omega = k\omega_0$. This is summarized in the figure below.

$$x(t) \overset{FS}{\leftrightarrow} a_k \longrightarrow \boxed{h(t) \leftrightarrow H(j\omega)} \longrightarrow y(t) \overset{FS}{\leftrightarrow} b_k = a_k H(jk\omega_0)$$

We can rewrite (37) to relate the magnitudes and phases of the input and output CTFS coefficients:

$$\begin{cases} |b_k| = |a_k| |H(jk\omega_0)| \\ \angle b_k = \angle a_k + \angle H(jk\omega_0) \end{cases} . \tag{37'}$$

Frequency Response of Continuous-Time Linear Time-Invariant Systems

- The frequency response $H(j\omega)$ of a CT LTI system is defined by (4) as the Fourier transform of the impulse response $h(t)$, and by (3) as the eigenvalue for an imaginary exponential input signal $e^{j\omega t}$. In this section, we study several important aspects of the frequency response.

Frequency Response of a Real System

- Consider an LTI system whose impulse response $h(t)$ is real, and thus satisfies

$$h(t) = h^*(t). \tag{38}$$

The frequency response $H(j\omega)$ at frequency ω is given by (4). To compute the frequency response at frequency $-\omega$, we evaluate (4) with the substitution $\omega \to -\omega$, and use (38):

$$\begin{aligned} H(-j\omega) &= \int_{-\infty}^{\infty} h(t) e^{j\omega t} dt \\ &= \int_{-\infty}^{\infty} h^*(t) e^{j\omega t} dt \\ &= \left(\int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \right)^* \\ &= H^*(j\omega) \end{aligned}. \tag{39}$$

- We can restate our finding succinctly as

$$h(t) = h^*(t) \leftrightarrow H(-j\omega) = H^*(j\omega). \tag{40}$$

If the impulse response is real, the frequency response at negative frequency equals the complex conjugate of the frequency response at positive frequency. This property of the frequency response is called *conjugate symmetry*. We can restate (40) in two alternate ways. First, if the impulse response is real, the magnitude of the frequency response is even in frequency, while the phase of the frequency response is odd in frequency:

$$h(t) = h^*(t) \leftrightarrow \begin{cases} |H(-j\omega)| = |H(j\omega)| \\ \angle H(-j\omega) = -\angle H(j\omega) \end{cases}. \tag{40a}$$

Second, if the impulse response is real, the real part of the frequency response is even in frequency, while the imaginary part of the frequency response is odd in frequency:

$$h(t) = h^*(t) \leftrightarrow \begin{cases} \mathrm{Re}\left[H(-j\omega)\right] = \mathrm{Re}\left[H(j\omega)\right] \\ \mathrm{Im}\left[H(-j\omega)\right] = -\mathrm{Im}\left[H(j\omega)\right] \end{cases}. \tag{40b}$$

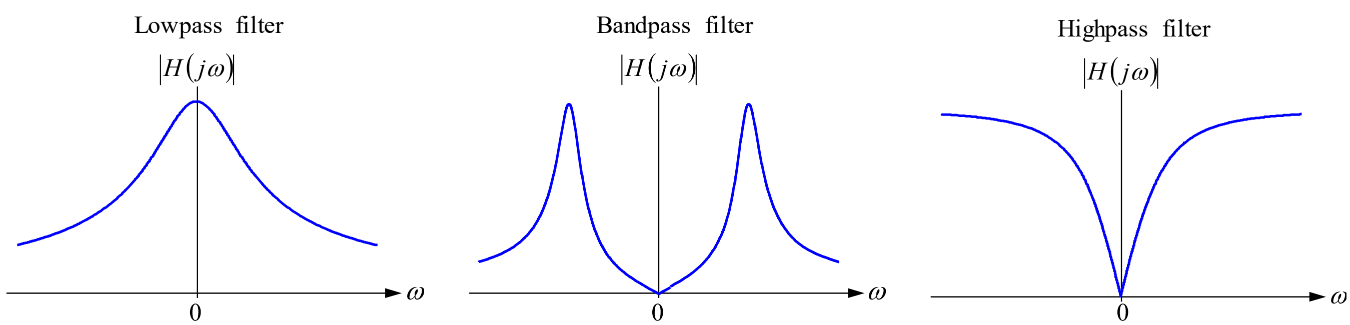
- Suppose we input a real, periodic signal $x(t)$ to the system. Its CTFS coefficients a_k have conjugate symmetry, $a_k = a_{-k}^*$ by (30). We obtain a periodic output $y(t)$ with CTFS coefficients $b_k = a_k H(jk\omega_0)$, as shown by (37). The CTFS coefficients of the output $y(t)$ satisfy

$$
\begin{aligned}
b_k &= a_k H(jk\omega_0) \\
&= a_{-k}^* H^*(-jk\omega_0), \\
&= b_{-k}^*
\end{aligned}
$$

where we have used (40) in the second line. Since the output CTFS coefficients are conjugate symmetric, $b_k = b_{-k}^*$, the output $y(t)$ is real, as expected.

Types of Linear Distortion and Filters

- The *magnitude response* $|H(j\omega)|$ determines the scaling of different frequency components appearing at the output of an LTI system. We sometimes refer to $|H(j\omega)|$ as the *amplitude response*.
  - *Magnitude distortion* (also called *amplitude distortion*) occurs if $|H(j\omega)|$ is frequency-dependent.
  - Filters with a frequency-dependent magnitude response $|H(j\omega)|$ are often classified as *lowpass*, *bandpass*, or *highpass*, as illustrated here by typical examples.



- The *phase response* $\angle H(j\omega)$ determines the phase shifts of different frequency components appearing at the output of an LTI system.
  - If the phase is a linear function of frequency

$$
\angle H(j\omega) = -t_0\omega ,
$$

  then all frequency components are subject to an equal time shift t_0.

  - Given any phase response $\angle H(j\omega)$, we define the *group delay* as $-\dfrac{d\angle H(j\omega)}{d\omega}$. The group delay describes how different frequency components are time-shifted.
  - *Phase distortion* occurs when $\angle H(j\omega)$ is not a linear function of frequency or, equivalently, when the group delay $-d\angle H(j\omega)/d\omega$ is frequency-dependent.

- A *distortionless system* may scale and time-shift signals, but causes no magnitude or phase distortion. Using (23) from Chapter 2, page 53, its impulse response is of the form

$$
h(t) = C\,\delta(t - t_0), \tag{41}
$$

where C is a constant. We can compute the corresponding frequency response using (41) in (4), which is the CT Fourier transform of the impulse response:

$$\begin{aligned} H(j\omega) &= \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} C\,\delta(t-t_0) e^{-j\omega t} dt \ . \\ &= Ce^{-j\omega t_0} \end{aligned} \tag{42}$$

We have evaluated the integral in (42) using the sampling property of the CT impulse function. If we apply frequency response (42) to filtering a periodic signal using (37), we obtain CTFS coefficients that are consistent with the CTFS time-shift property (27). A distortionless system is illustrated below.

$$x(t) \longrightarrow \boxed{h(t) = C\,\delta(t-t_0) \leftrightarrow H(j\omega) = C\,e^{-j\omega t_0}} \longrightarrow y(t) = C\,x(t-t_0)$$

The output is a scaled and time-shifted version of the input: $y(t) = C\,x(t-t_0)$.

Methods for Evaluating the Frequency Response

- Here we describe two methods for computing the frequency response of a given LTI system. The list is not exhaustive, as other methods exist.

1. *Fourier transform of impulse response*. Suppose an LTI system is specified in terms of an *impulse response*. We can evaluate the CT Fourier transform integral

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \tag{4}$$

to find an expression for the frequency response. For example, we used this procedure just above to find the frequency response of a distortionless system (42) from its impulse response (41). The integral (4) represents the CT Fourier transform of the impulse response $h(t)$. We will not be able to evaluate the integral (4) for more complicated impulse responses, however, until we study the CT Fourier transform in Chapter 4.

2. *Substitution in differential equation*. Suppose an LTI system is specified by a *linear, constant-coefficient differential equation* of the form (35) in Chapter 2, page 67. If we know that the integral (4) converges, then we can compute the frequency response directly from the differential equation without using the integral (4) by the following simple procedure:

   1. Substitute the following input and output signals in the differential equation:

   $$x(t) = e^{j\omega t} \quad \text{and} \quad y(t) = H(j\omega)e^{j\omega t} .$$

   Evaluate any time derivatives using

$$\frac{d}{dt}e^{j\omega t} = j\omega e^{j\omega t}.$$

2. Cancel all factors of $e^{j\omega t}$ and solve for $H(j\omega)$.

Once we study the CT Fourier transform in Chapter 4, we will understand when the integral (4) converges and therefore this method is valid. Until then, we will apply this method only to carefully chosen examples.

Examples

- Here we apply Method 2 to three examples.
- For details on computing and plotting the magnitudes and phases of the first-order lowpass and highpass filters, see Appendix, pages 298-299.

1. Differentiator

- This is described by a differential equation

$$y(t) = \frac{dx}{dt}.$$

Substituting for $x(t)$ and $y(t)$ and evaluating the derivative:

$$H(j\omega)e^{j\omega t} = j\omega e^{j\omega t}.$$

Cancelling factors of $e^{j\omega t}$ and solving for the frequency response:

$$H(j\omega) = j\omega.$$

2. First-Order Lowpass Filter

- This is described by a differential equation

$$\tau\frac{dy}{dt} + y(t) = x(t).$$

Substituting for $x(t)$ and $y(t)$ and evaluating the derivative:

$$j\omega\tau H(j\omega)e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t}.$$

Cancelling factors of $e^{j\omega t}$ and solving for the frequency response:

$$H(j\omega) = \frac{1}{1+j\omega\tau}.$$

- In computing the magnitude and phase, we use the *reciprocal property* (see Appendix, page 289). Given a complex-valued $z = |z|e^{j\angle z}$, its reciprocal is

$$\frac{1}{z} = \frac{1}{|z|e^{j\angle z}} = \frac{1}{|z|}e^{-j\angle z}.$$

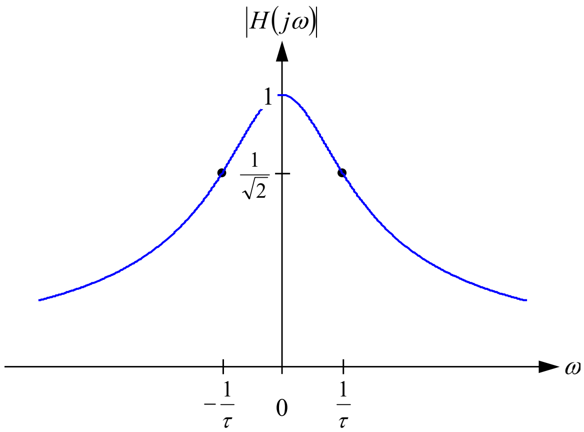
The magnitude and phase of $1/z$ are related to those of z as

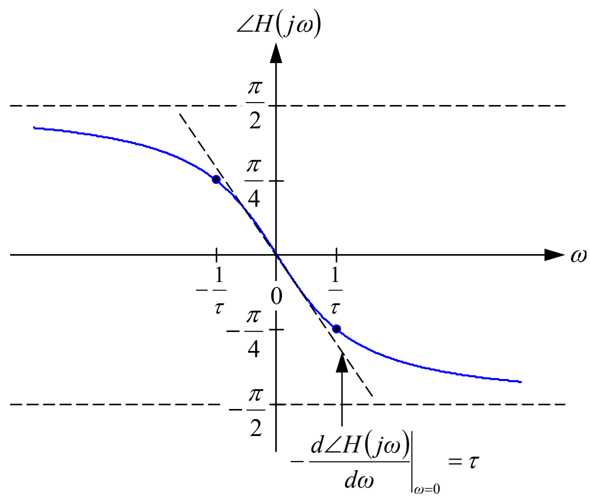
$$\left|\frac{1}{z}\right| = \frac{1}{|z|} \quad \text{and} \quad \angle\left(\frac{1}{z}\right) = -\angle z .$$

- Using the reciprocal property with $z = 1 + j\omega\tau$, we write the magnitude and phase responses of the lowpass filter as

$$\left|H(j\omega)\right| = \frac{1}{\left|1 + j\tau\omega\right|} = \frac{1}{\sqrt{1 + (\tau\omega)^2}} \quad \text{and} \quad \angle H(j\omega) = -\angle(1 + j\tau\omega) = -\tan^{-1}(\tau\omega).$$

These are plotted below. This filter causes both magnitude distortion and phase distortion.





- Magnitude and phase values at some key frequencies are as follows:

$$\left|H(j\omega)\right| = \begin{cases} 1 & \omega = 0 \\ \frac{1}{\sqrt{2}} & \omega = \pm\frac{1}{\tau} \\ 0 & \omega \to \pm\infty \end{cases} \quad \text{and} \quad \angle H(j\omega) = \begin{cases} 0 & \omega = 0 \\ \mp\frac{\pi}{4} & \omega = \pm\frac{1}{\tau} \\ \mp\frac{\pi}{2} & \omega \to \pm\infty \end{cases} .$$

A first-order lowpass filter is often characterized by a *cutoff frequency* at $\omega = 1/\tau$, at which $\left|H(j\omega)\right|^2$ has half the value it has for $\omega = 0$, corresponding to a decrease of $10\log_{10}(1/2) \approx -3$ dB. The nominal passband is often considered to be the frequency range $|\omega| \le 1/\tau$.

- The group delay is obtained by differentiating the phase:

$$-\frac{d\angle H(j\omega)}{d\omega} = \frac{\tau}{1 + (\tau\omega)^2} .$$

Near ω = 0, where the magnitude is largest, the phase $\angle H(j\omega)$ has a slope –τ, corresponding to a group delay $-d\angle H(j\omega)/d\omega = \tau$, as indicated on the plot above.

3. *First-Order Highpass Filter*

- This is described by a differential equation

$$\frac{dy}{dt}+\frac{1}{\tau}y(t)=\frac{dx}{dt}.$$

Substituting for $x(t)$ and $y(t)$ and evaluating the derivatives:

$$j\omega H(j\omega)e^{j\omega t}+\frac{1}{\tau}H(j\omega)e^{j\omega t}=j\omega e^{j\omega t}.$$

Cancelling factors of $e^{j\omega t}$ and solving for the frequency response:

$$H(j\omega)=\frac{j\omega\tau}{1+j\omega\tau}.$$

- In computing the magnitude and phase, we use the *quotient property* (see Appendix, page 289). Given complex-valued $z_1=|z_1|e^{j\angle z_1}$ and $z_2=|z_2|e^{j\angle z_2}$, their quotient is

$$\frac{z_1}{z_2}=\frac{|z_1|e^{j\angle z_1}}{|z_2|e^{j\angle z_2}}=\frac{|z_1|}{|z_2|}e^{j(\angle z_1-\angle z_2)}.$$

The magnitude and phase of z_1/z_2 are related to those of z_1 and z_2 as

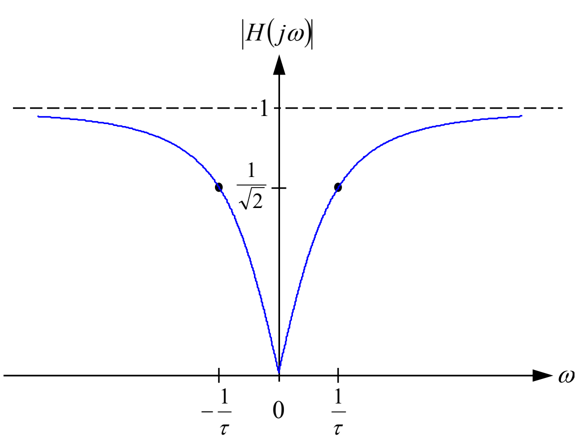
$$\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}\quad\text{and}\quad\angle\left(\frac{z_1}{z_2}\right)=\angle z_1-\angle z_2.$$

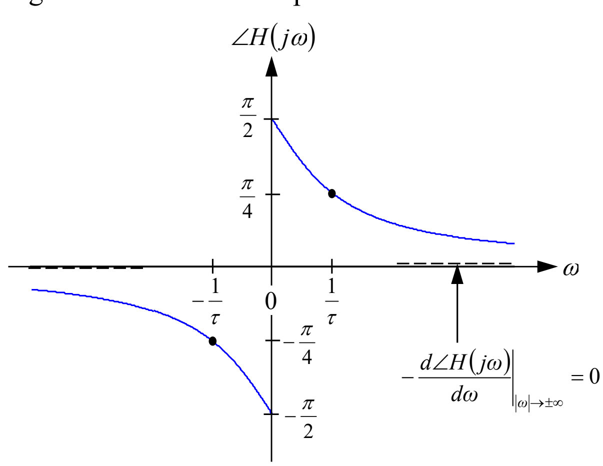
- Using the quotient property with $z_1=j\omega\tau$ and $z_2=1+j\omega\tau$, we write the magnitude and phase responses of the highpass filter as

$$|H(j\omega)|=\frac{|j\tau\omega|}{|1+j\tau\omega|}=\frac{\tau|\omega|}{\sqrt{1+(\tau\omega)^2}}$$

$$\angle H(j\omega)=\angle j\tau\omega-\angle(1+j\tau\omega)=\frac{\pi}{2}\operatorname{sgn}\omega-\tan^{-1}(\tau\omega).$$

These are plotted below. This filter causes both magnitude distortion and phase distortion.





- Magnitude and phase values at some key frequencies are as follows:

$$\left|H(j\omega)\right| = \begin{cases} 0 & \omega = 0 \\ \dfrac{1}{\sqrt{2}} & \omega = \pm\dfrac{1}{\tau} \\ 1 & \omega \to \pm\infty \end{cases} \quad \text{and} \quad \angle H(j\omega) = \begin{cases} \pm\dfrac{\pi}{2} & \omega = 0 \\ \pm\dfrac{\pi}{4} & \omega = \pm\dfrac{1}{\tau} \\ 0 & \omega \to \pm\infty \end{cases}.$$

  A first-order highpass filter is characterized by a *cutoff frequency* at $\omega = 1/\tau$, at which $\left|H(j\omega)\right|^2$ has half the value it has for $|\omega| \to \infty$. The nominal passband may be considered to be the range $|\omega| \geq 1/\tau$.

- The group delay is computed by differentiating the phase:

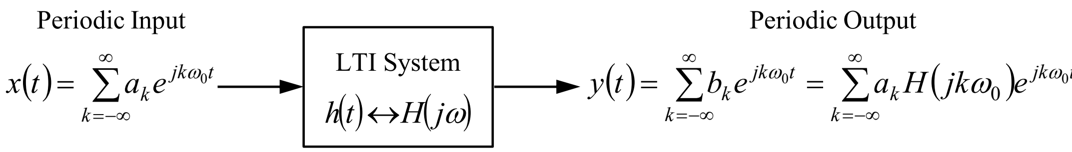
$$-\frac{d\angle H(j\omega)}{d\omega} = \frac{\tau}{1+(\tau\omega)^2}, \ \omega \neq 0.$$

  As $\omega \to \pm\infty$, where the magnitude response is largest, the phase $\angle H(j\omega)$ has a zero slope, corresponding to a group delay $-d\angle H(j\omega)/d\omega = 0$, as indicated on the plot above.

Examples of Filtering Periodic Continuous-Time Signals by Linear Time-Invariant Systems

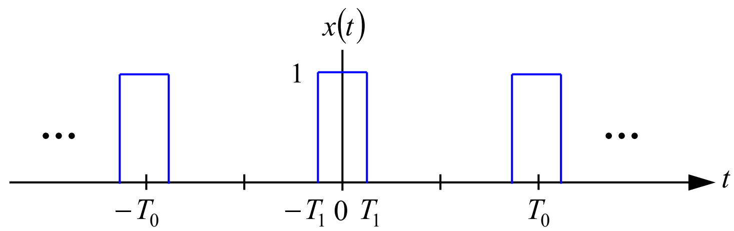
Method of Analysis

- We use the analysis presented on page 95 above, which is summarized in the figure below.



Input Signal

- The input signal $x(t)$ is a rectangular pulse train with period $T_0 = 2\pi/\omega_0$ and pulse width $2T_1$, as shown below.



Its CTFS coefficients are given by

$$a_k = \frac{\omega_0 T_1}{\pi}\operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right). \tag{24}$$

We choose $\frac{T_1}{T_0}=\frac{1}{8}$, so the CTFS coefficients become

$$a_k = \frac{1}{4}\operatorname{sinc}\left(\frac{k}{4}\right),$$

and the CTFS representation of the input becomes

$$x(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t}=\sum_{k=-\infty}^{\infty}\frac{1}{4}\operatorname{sinc}\left(\frac{k}{4}\right)e^{jk\omega_0 t}.$$

- Because $x(t)$ is real and even in t, its CTFS coefficients a_k are real and even in k (see (31)). Since the a_k are purely real, their phases can only be an integer multiple of π, and are typically chosen as

$$\angle a_k = \begin{cases} 0 & a_k > 0 \\ \pm\pi & a_k < 0 \end{cases}.$$

In the figures below, when $a_k < 0$, we make the specific choices $\angle a_k = +\pi$ for $k < 0$ and $\angle a_k = -\pi$ for $k > 0$ so the phase appears with the odd symmetry expected, but this is not necessary. (See Appendix, pages 300-301, for further explanation.)

Linear Time-Invariant Systems

- We consider the first-order lowpass and first-order highpass filters, whose frequency responses were analyzed on pages 99-102 above.
- For each filter, we consider two choices of the time constant τ, which determines the filter cutoff frequency at $\omega = 1/\tau$:
  - High cutoff frequency: $\tau / T_0 = 0.03$.
  - Low cutoff frequency: $\tau / T_0 = 0.3$.

Output Signal

- Given an LTI system with frequency response $H(j\omega)$, using (36), the CTFS representation of the output is

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{1}{4}\text{sinc}\left(\frac{k}{4}\right) H(jk\omega_0) e^{jk\omega_0 t} .$$

  Recall that the output CTFS coefficients b_k are given by the input CTFS coefficients a_k, scaled by values of the frequency response $H(j\omega)$ evaluated at $\omega = k\omega_0$:

$$b_k = a_k H(jk\omega_0) . \tag{37}$$

  As a result, the magnitudes and phases of the input and output CTFS coefficients are related by

$$\begin{cases} |b_k| = |a_k| |H(jk\omega_0)| \\ \angle b_k = \angle a_k + \angle H(jk\omega_0) \end{cases} . \tag{37'}$$

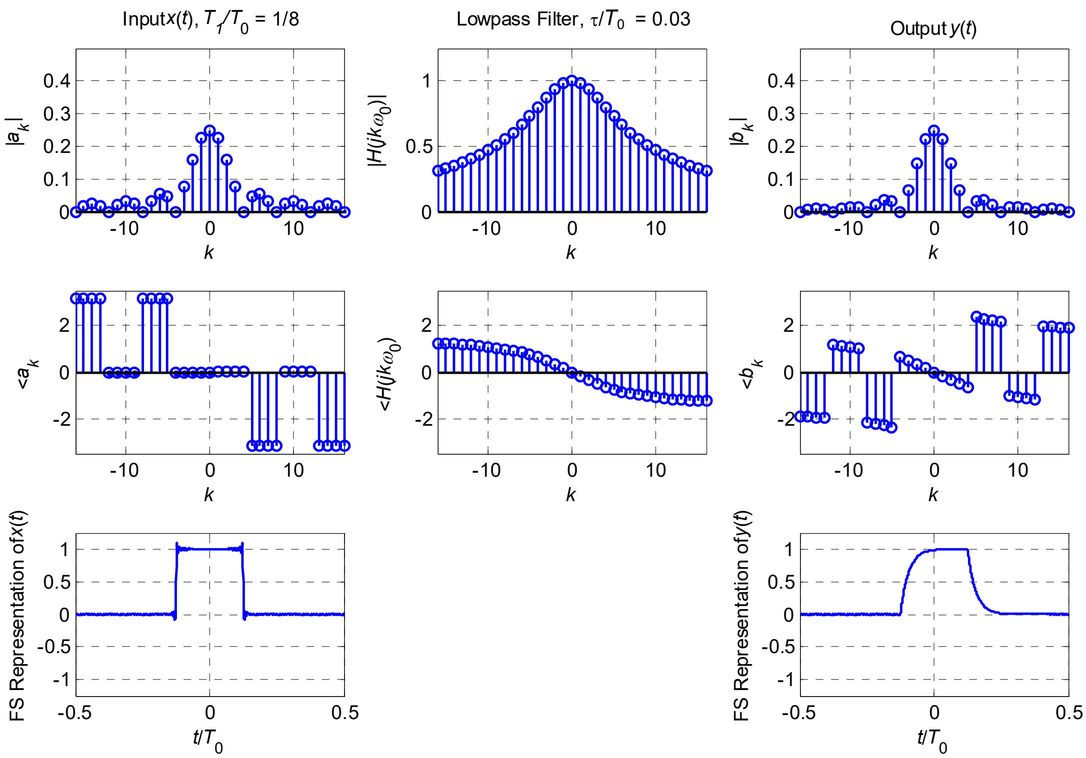
  In each figure below, the relationship (37') should be evident in the first row (which shows $|a_k|$, $|H(jk\omega_0)|$ and $|b_k|$) and in the second row (which shows $\angle a_k$, $\angle H(jk\omega_0)$ and $\angle b_k$).

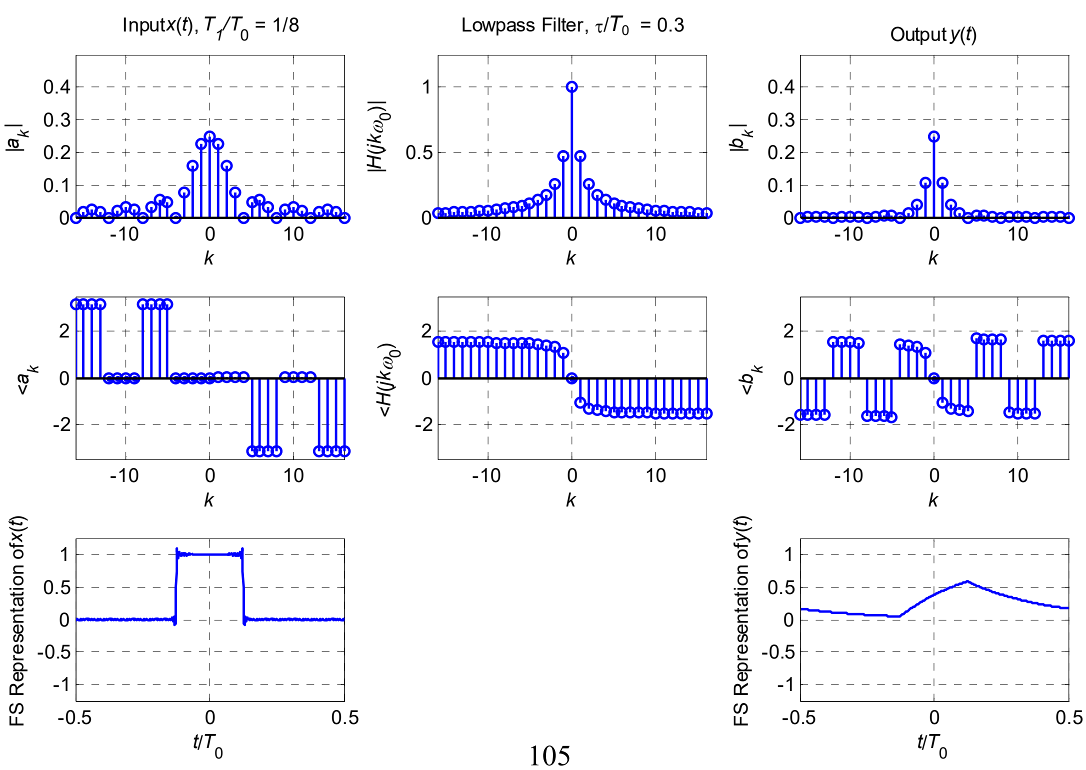
Generating Plots

- The plots of $x(t)$ and $y(t)$ shown are computed using CTFS terms for $-128 \le k \le 128$, a total of 257 terms.

Filtering by First-Order Lowpass Filter

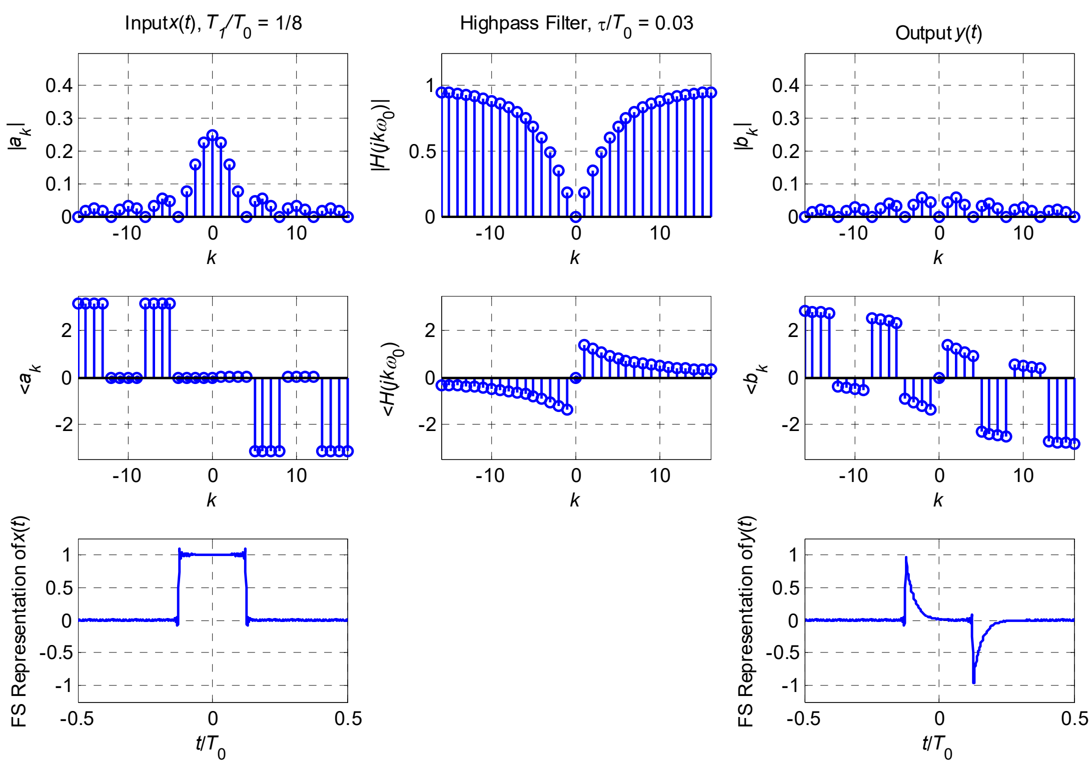
- The d.c. level (average value) is preserved, since $H(j0)=1$.
- As τ increases, the rise and fall times increase.
- The centroid of each pulse, determined mainly by low-frequency components, is delayed noticeably, and is consistent with $-d\angle H(j\omega)/d\omega\big|_{\omega=0}=\tau$.

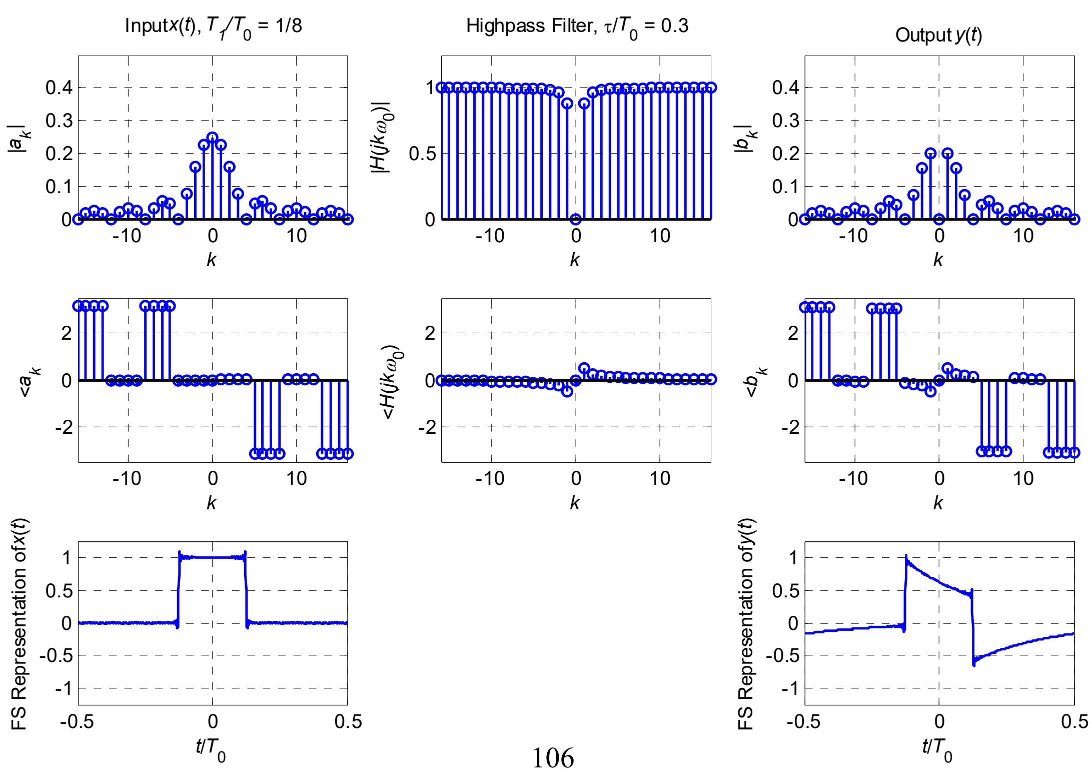




Filtering by First-Order Highpass Filter

- The d.c. level (average value) is removed, since $H(j0) = 0$.
- At small τ, only the leading and trailing edges of each pulse remain. At larger τ, more of each pulse remains, but the baseline (formerly the zero level) wanders up and down.
- The leading and trailing edges, determined mainly by high-frequency components, are delayed little, consistent with $-d\angle H(j\omega)/d\omega\big|_{\omega\to\pm\infty} = 0$.





Comment on Method of Analysis

- We have analyzed these examples using CTFS. Alternatively, we could analyze them using convolution methods from Chapter 2. In that case, we would represent the periodic rectangular pulse train input $x(t)$ as an infinite sum of scaled and shifted step functions, and would represent the periodic output $y(t)$ as a corresponding sum of scaled and shifted step responses. The periodic outputs $y(t)$ we obtain here can be understood using this approach. The FS method we have chosen to use offers important advantages, however. (a) It naturally takes account of the overlap between all the scaled, shifted step responses. (b) It can be applied to *any* periodic input signal $x(t)$ with finite power, even if that signal cannot be represented easily in terms of simple functions, such as step functions.

First-Order and Higher-Order Systems

- The simple first-order filters considered here are easy to analyze and understand. But they are not suitable for many applications, since they have only one energy storage element, and are described by only a single parameter τ that governs both their time-domain response and frequency response. Important time-domain properties include impulse response, step response, rise time and overshoot. Important frequency-domain properties include the abruptness of the transition from the passband to the stopband (the frequency range over which the magnitude $\left|H(j\omega)\right|$ changes from large to small), the group delay $-d\angle H(j\omega)/d\omega$, and variations of the magnitude and group delay within the passband.

- If we introduce more energy storage elements, such as inductors and capacitors in electrical circuits, or kinetic and potential energy in mechanical systems, we obtain systems described by *higher-order differential equations*. Such *higher-order systems* offer more flexibility in their characteristics. For example, they can achieve a sharper passband-stopband transition, and offer more control over tradeoffs between time-domain response and frequency response.

- When we study the CT Fourier transform in Chapter 4, we will learn more about the time- and frequency-domain properties of systems, and will study second-order continuous-time systems. These topics are addressed in more depth using the Laplace transform in EE 102B.

Eigenfunctions of Discrete-Time Linear Time-Invariant Systems

- We have discussed the concepts of eigenfunctions, FS and frequency response for CT, and have applied them to analyzing the response of CT LTI systems to periodic inputs. Now we will present analogous concepts for DT. While much of the development is similar to CT, there are important differences, as we point out below.

General Case: Complex Exponentials

- Consider a DT LTI system *H*, which has an impulse response $h[n]$. If we are given a general input signal $x[n]$ and wish to predict the output $y[n]=H\{x[n]\}$, we must perform a *convolution* between $x[n]$ and $h[n]$, as shown.

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[n] * x[n]$$

- The *eigenfunctions* of DT LTI systems are *complex exponential* time signals

$$z^n, \ z \text{ complex}, \ -\infty < n < \infty .$$

If we input one of these signals to an LTI system *H*, the output is the same signal z^n, multiplied by an *eigenvalue* denoted by $H(z)$, as shown below.

$$x[n] = z^n \longrightarrow \boxed{h[n] \leftrightarrow H(z)} \longrightarrow y[n] = H(z) \cdot z^n$$

We refer to the variable *z* as *complex frequency*, and call $H(z)$ the *transfer function* of the LTI system *H*. If we know the transfer function $H(z)$ as a function of *z*, then for an input z^n, we can predict the output by using *multiplication*, and need not use convolution.

- To prove that the z^n are eigenfunctions, we input $x[n] = z^n$ to the system and compute the output $y[n]$ using the convolution sum

$$\begin{aligned} y[n] &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} h[k] x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k] z^{n-k} \\ &= z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k} \\ &= z^n \cdot H(z) \end{aligned} \qquad . \tag{43}$$

- We have defined the transfer function in (43) as

$$H(z) \overset{d}{=} \sum_{n=-\infty}^{\infty} h[n] z^{-n} . \tag{44}$$

If we are given an impulse response $h[n]$, we compute the sum (44) to obtain the transfer function $H(z)$. The sum (44) defines $H(z)$ as the *bilateral Z transform* of the impulse response $h[n]$.

- The Z transform sum (44) converges for a many impulse responses $h[n]$, including some describing unstable systems (see *EE 102B Course Reader*, Chapter 7).

Special Case: Imaginary Exponentials

- In (43) and (44) above, we have assumed that z takes on a general complex value. Now we consider the special case in which $z = e^{j\Omega}$, where Ω is real. We are considering a subset of the complex exponential signals, which are the *imaginary exponential* signals

$$e^{j\Omega n}, \ \Omega \text{ real}, \ -\infty < n < \infty .$$

These, too, are eigenfunctions of DT LTI systems. If we input an imaginary exponential $e^{j\Omega n}$ to an LTI system, the output is the same signal, multiplied by an eigenvalue $H\left(e^{j\Omega}\right)$, as shown below. We refer to Ω as *frequency*, and refer to $H\left(e^{j\Omega}\right)$ as the *frequency response* of the LTI system *H*.

$$x[n]=e^{j\Omega n} \longrightarrow \boxed{h[n] \leftrightarrow H\left(e^{j\Omega}\right)} \longrightarrow y[n]=H\left(e^{j\Omega}\right)\cdot e^{j\Omega n}$$

Just as in the general case above, if we know $H\left(e^{j\Omega}\right)$ as a function of Ω, then given an imaginary exponential input signal $e^{j\Omega n}$, we can compute the system output by using *multiplication*, and do not need to use convolution. In order to show this, we use an input $x[n]=e^{j\Omega n}$ in (43). We obtain an output

$$y[n]=e^{j\Omega n}\cdot H\left(e^{j\Omega}\right). \tag{45}$$

- The frequency response appearing in (45) is defined by the following expression:

$$H\left(e^{j\Omega}\right) \stackrel{d}{=} H(z)\Big|_{z=e^{j\Omega}} = \sum_{n=-\infty}^{\infty} h[n]e^{-jn\Omega} . \tag{46}$$

As in the general case above, if we are given an impulse response $h[n]$, we evaluate the sum (46) to find the frequency response $H\left(e^{j\Omega}\right)$. The sum (46) defines $H\left(e^{j\Omega}\right)$ as the *DT Fourier transform* of $h[n]$. We discuss the DT Fourier transform extensively in Chapter 5.

- The DT Fourier transform sum (46) converges for many impulse responses $h[n]$, but does not converge in all the cases for which the Z transform (44) converges (for example, for some important unstable systems). This provides motivation for studying the Z transform in EE 102B (see *EE 102B Course Reader*, Chapter 7).

Application to LTI System Analysis

- Consider a DT LTI system *H*. Assume we know its transfer function $H(z)$, and we are given an input signal expressed as a linear combination of complex exponential signals at *K* distinct values of z:

$$x[n]=a_1z_1^n+a_2z_2^n+\cdots+a_Kz_K^n . \tag{47}$$

Using the linearity of the system and the eigenfunction property (43), we can compute the output signal as

$$y[n] = a_1 H(z_1) z_1^n + a_2 H(z_2) z_2^n + \cdots + a_K H(z_K) z_K^n . \quad (48)$$

We have computed each term in the output (48) by using multiplication, without using convolution. Although we have considered the general case of complex exponential input signals, we can easily apply (47) and (48) in the special case of imaginary exponential signals.

Example

- We are given an LTI system *H* with input $x[n]$ and output $y[n]$ that has an input-output relation

$$y[n] = x[n] - x[n-2] . \quad (49)$$

We are given an input signal

$$x[n] = e^{j\frac{\pi}{4}n} + 2^n ,$$

which is in the form of (47). We can compute the output signal by using the input-output relation (49):

$$\begin{aligned} y[n] &= e^{j\frac{\pi}{4}n} - e^{j\frac{\pi}{4}(n-2)} + 2^n - 2^{(n-2)} \\ &= \left(1 - e^{-j\frac{\pi}{2}}\right) e^{j\frac{\pi}{4}n} + \left(1 - 2^{-2}\right) 2^n . \\ &= H\left(e^{j\frac{\pi}{4}}\right) e^{j\frac{\pi}{4}n} + H(2) 2^n \end{aligned} \quad (50)$$

We confirm that the output signal (50) is consistent with the general form (48). We will learn in EE 102B that the input-output relation (49) corresponds to a transfer function $H(z) = 1 - z^{-2}$. Once we know this transfer function, we can use (48) to obtain the output $y[n]$, and do not need to do the computations shown in the first line of (50).

Discrete-Time Fourier Series

- We are given a periodic DT signal that satisfies

$$x[n] = x[n+N] \quad \forall n , \quad (51)$$

where N is the *period* and the *fundamental frequency* is

$$\Omega_0 = \frac{2\pi}{N} . \quad (52)$$

- We will represent the periodic signal $x[n]$ as a *discrete-time Fourier series* (DTFS). This is a linear combination of imaginary exponential basis sequences, each at a frequency $k\Omega_0$, which is an integer *k* times the fundamental frequency $\Omega_0 = 2\pi / N$. We will use only exponential FS for DT signals, for the same reasons we used only exponential FS for CT signals (see pages 78-79 above).

- We will show that for some simple periodic DT signals, it is possible to determine the FS coefficients by inspection, as in CT. The development will be clearer, however, if we first explain the properties of the basis sequences and derive the synthesis and analysis equations.

Imaginary Exponential Basis Sequences

- The *imaginary exponential basis sequences* are of the form

$$\phi_k[n] = e^{jk\left(\frac{2\pi}{N}\right)n},\ k = 0,\ \pm 1,\ \pm 2, \ldots . \tag{53}$$

- Each of these sequences is periodic with period N:

$$\phi_k[n+N] = e^{jk\left(\frac{2\pi}{N}\right)(n+N)} = e^{jk\left(\frac{2\pi}{N}\right)n}\underbrace{e^{jk2\pi}}_{=1} = \phi_k[n] \quad \forall n . \tag{54}$$

- Recall that any two DT imaginary exponential sequences are identical if their frequencies differ by an integer multiple of 2π (Chapter 1, pages 15-16). As a consequence, $\phi_k[n]$ and $\phi_{k+N}[n]$ are identical:

$$\phi_{k+N}[n] = e^{j(k+N)\left(\frac{2\pi}{N}\right)n} = e^{jk\left(\frac{2\pi}{N}\right)n}\underbrace{e^{j2\pi n}}_{=1} = \phi_k[n] \quad \forall n . \tag{55}$$

We only need to consider the N distinct sequences

$$\phi_k[n],\ k = \langle N \rangle . \tag{56}$$

In (56), and throughout our study of the DTFS, we use the following notation to denote any N consecutive values of a discrete variable k or n:

$$k = \langle N \rangle \quad \text{means} \quad k_1 + 1 \le k \le k_1 + N, \quad k_1 \text{ arbitrary}$$

$$n = \langle N \rangle \quad \text{means} \quad n_1 + 1 \le n \le n_1 + N, \quad n_1 \text{ arbitrary} .$$

- We would like to show that the basis sequences form an orthogonal set over any time interval of length N, so we compute the following summation, which is an inner product between $\phi_k[n]$ and $\phi_l[n]$:

$$\sum_{n=\langle N \rangle} \phi_k[n]\phi_l^*[n] = \sum_{n=\langle N \rangle} e^{j(k-l)\left(\frac{2\pi}{N}\right)n} . \tag{57}$$

We will use Euler's relation (see Appendix, page 288) to express the imaginary exponentials in (57) in terms of sinusoids. Evaluating (57) for $k \ne l$, we find

$$\sum_{n=\langle N \rangle}\left[\cos\left((k-l)\left(\frac{2\pi}{N}\right)n\right) + j\sin\left((k-l)\left(\frac{2\pi}{N}\right)n\right)\right] = 0 ,$$

since each of the two sums (real and imaginary) is over an integer number of cycles, and contains positive and negative contributions that cancel precisely. To evaluate (57) for $k = l$, we set the summand to unity, obtaining

$$\sum_{n=\langle N\rangle}(1)=N\ .$$

In summary, the basis sequences form an orthogonal set, with pairwise inner product sums given by

$$\sum_{n=\langle N\rangle}\phi_k[n]\phi_l^*[n]=\sum_{n=\langle N\rangle}e^{j(k-l)\left(\frac{2\pi}{N}\right)n}=\begin{cases}0 & k\neq l\\ N & k=l\end{cases}. \tag{58}$$

Synthesis Equation

- We are given a periodic DT signal $x[n]$ with period N and fundamental frequency $\Omega_0=2\pi/N$. We will represent $x[n]$ by an exponential DTFS, which is a linear combination of N distinct, mutually orthogonal imaginary exponential basis sequences:

$$\hat{x}[n]=\sum_{k=\langle N\rangle}a_k\phi_k[n]=\sum_{k=\langle N\rangle}a_ke^{jk\left(\frac{2\pi}{N}\right)n}\ . \tag{59}$$

  Expression (59) is the *synthesis equation* for the DTFS. The coefficients a_k, $k=\langle N\rangle$, in (59) are the *DTFS coefficients* for the DT signal $x[n]$. The DTFS synthesis (59) is a finite summation of terms at N different frequencies. By contrast, the CTFS synthesis (11) is, in general, the summation of terms at an infinite number of different frequencies. This represents a major difference between the CTFS and the DTFS.

- We can verify that the DTFS synthesis (59) describes a DT signal that is *periodic in time n* with period N, as desired:

$$\hat{x}[n+N]=\sum_{k=\langle N\rangle}a_ke^{jk\left(\frac{2\pi}{N}\right)(n+N)}=\sum_{k=\langle N\rangle}a_ke^{jk\left(\frac{2\pi}{N}\right)n}\underbrace{e^{jk2\pi}}_{=1}=\hat{x}[n]. \tag{60}$$

- A periodic DT signal $x[n]$ is defined entirely by its values over any N consecutive time samples, $n=\langle N\rangle$, which implies that it can be represented *exactly* using a linear combination of N orthogonal sequences. In other words, the DTFS synthesis is *identical* to the original signal at all times n:

$$\hat{x}[n]=x[n]. \tag{61}$$

  By contrast, in CT, we saw that the CTFS representation $\hat{x}(t)$ is not necessarily identical to the original signal $x(t)$ at all t. In particular, they differ near discontinuities of $x(t)$ (see pages 82-83 above). This represents another major difference between the CTFS and the DTFS.

Analysis Equation

- Now we derive an analysis equation to obtain the DTFS coefficients a_k representing a periodic DT signal $x[n]$. Starting with (61), we compute an inner product sum between that equation and the imaginary exponential basis signal $\phi_l[n]=e^{jl\left(\frac{2\pi}{N}\right)n}$:

$$\sum_{n=\langle N\rangle} x[n] e^{-jl\left(\frac{2\pi}{N}\right)n} = \sum_{n=\langle N\rangle} \hat{x}[n] e^{-jl\left(\frac{2\pi}{N}\right)n} . \tag{62}$$

Now we substitute synthesis equation (59) for $\hat{x}[n]$ on the right-hand side of (62):

$$\begin{aligned} \sum_{n=\langle N\rangle} x[n] e^{-jl\left(\frac{2\pi}{N}\right)n} &= \sum_{n=\langle N\rangle} \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} e^{-jl\left(\frac{2\pi}{N}\right)n} \\ &= \sum_{k=\langle N\rangle} a_k \underbrace{\sum_{n=\langle N\rangle} e^{j(k-l)\left(\frac{2\pi}{N}\right)n}}_{=\begin{cases} 0 & k\neq l \\ N & k=l \end{cases}} \\ &= N a_l \end{aligned} . \tag{63}$$

In the second line, we have interchanged the order of the summations, and have evaluated the sum over n using (58). In the third line, we have evaluated the sum over k, finding that only the term for $k=l$ is nonzero. Rearranging (63), we obtain

$$a_l = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jl\left(\frac{2\pi}{N}\right)n} . \tag{64}$$

Equation (64) is the *analysis equation* for the DTFS. Given a periodic signal $x[n]$, it tells us how to obtain the DTFS coefficients a_k, $k=\langle N\rangle$, to represent the signal using synthesis equation (59).

- Observe that the DTFS coefficients given by analysis equation (64) are *periodic in frequency k* with period N:

$$a_{k+N} = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-j(k+N)\left(\frac{2\pi}{N}\right)n} = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n} \underbrace{e^{-jk2\pi n}}_{=1} = a_k \quad \forall k . \tag{65}$$

This periodicity of the DTFS coefficients is equivalent to the fact that the synthesis (59) only needs to use terms at N different frequencies. By contrast, the CTFS coefficients are not generally periodic in frequency k, and the CTFS synthesis requires, in general, an infinite number of terms. We cited this above as a major difference between the CTFS and the DTFS.

Summary of Discrete-Time Fourier Series

- The synthesis and analysis equations are

$$\hat{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \qquad \text{(synthesis)} \tag{59}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n} \qquad \text{(analysis)} \tag{64}$$

- We often denote a signal $x[n]$ and its DTFS coefficients a_k as a *DTFS pair*:

$$x[n] \overset{FS}{\leftrightarrow} a_k . \tag{66}$$

- The DTFS coefficients for some important periodic DT signals are given in Table 6, Appendix.

Choice of Summation Interval

- In the synthesis and analysis equations (59) and (64), we can sum over any N consecutive values of k or n, denoted by $k = \langle N \rangle$ or $n = \langle N \rangle$. In some cases, it is natural to sum over an asymmetric interval, such as $0 \le k \le N-1$ or $0 \le n \le N-1$. In many cases, it is desirable to sum over an interval that is approximately symmetric about $k = 0$ or $n = 0$. In such cases, we can choose the interval as shown in the following table.

N	Minimum n or k	Maximum n or k
Odd	$-\frac{N-1}{2}$	$\frac{N-1}{2}$
Even	$-\frac{N}{2}+1$	$\frac{N}{2}$
	or	
	$-\frac{N}{2}$	$\frac{N}{2}-1$

Obtaining the Fourier Series by Inspection

- The DTFS coefficients of some simple periodic DT signals can be found by inspection. As an example, we consider

$$x[n] = \cos\left(\frac{3\pi}{5}n\right) - \sin\left(\frac{4\pi}{5}n\right).$$

This signal is periodic with period $N = 10$, so we can express it as a linear combination of imaginary exponentials with frequencies

$$k\Omega_0 = k\frac{2\pi}{N} = k\frac{\pi}{5}.$$

We obtain

$$x[n] = \frac{1}{2}\left[e^{j3\left(\frac{\pi}{5}\right)n} + e^{-j3\left(\frac{\pi}{5}\right)n}\right] - \frac{1}{2j}\left[e^{j4\left(\frac{\pi}{5}\right)n} - e^{-j4\left(\frac{\pi}{5}\right)n}\right].$$

If we compare this to the synthesis equation (59), we obtain the DTFS coefficients

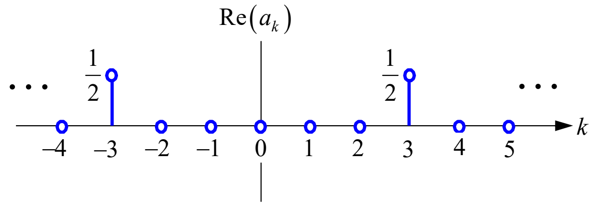
$$a_{-3} = a_3 = \frac{1}{2}$$

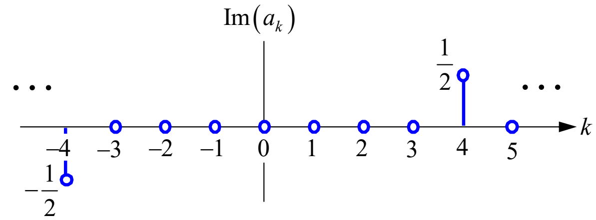
$$-a_{-4} = a_4 = \frac{1}{2}j,$$

while all other a_k are zero, $-4 \le k \le 5$. By (65), the FS coefficients are periodic with period $N = 10$:

$$a_{k+10} = a_k \quad \forall k \quad .$$

These DTFS coefficients are plotted here over one period, $-4 \le k \le 5$.





Fourier Series of a Rectangular Pulse Train

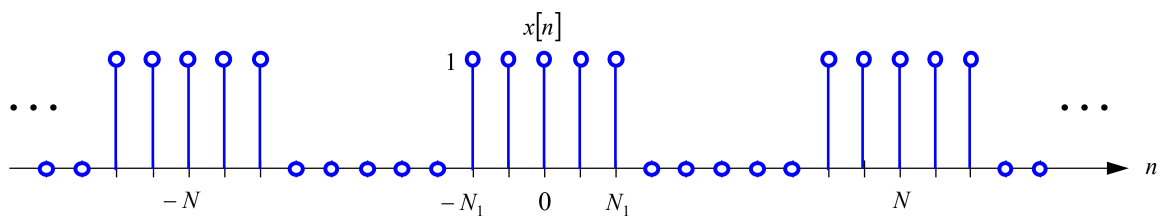
- As an example of using the analysis and synthesis equations we have derived, we consider a periodic rectangular pulse train. It is described by the formula

$$x[n] = \begin{cases} 1 & |n| \le N_1 \\ 0 & N_1 < |n| \le \dfrac{N}{2} \end{cases}, \qquad x[n+N] = x[n].$$

Alternatively, it can be expressed as a series of shifted rectangular pulses (see page 290) as:

$$x[n] = \sum_{l=-\infty}^{\infty} \Pi\left(\frac{n-lN}{2N_1}\right).$$

The pulse train is shown below. The period is N and the width of each pulse is $2N_1+1$.



- We will compute the DTFS coefficients using the analysis equation (64). As $x[n]$ is symmetric, it is convenient to choose the summation interval, of length N, centered at the origin so it includes the range $-N_1 \le n \le N_1$. The analysis equation can be expressed as

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1}(1)e^{-jk\left(\frac{2\pi}{N}\right)n}. \tag{67}$$

We change the summation variable to $m = n + N_1$, so (67) becomes

$$a_k = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)(m-N_1)} = \frac{1}{N}e^{jk\left(\frac{2\pi}{N}\right)N_1}\sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)m}. \tag{68}$$

The summation in (68) is a sum of the first $2N_1+1$ terms of a geometric series. Evaluating it for $k \ne 0, \pm N, \pm 2N, \ldots$ yields

$$a_k = \frac{1}{N} e^{jk\left(\frac{2\pi}{N}\right)N_1} \left(\frac{1 - e^{-jk(2N_1+1)\left(\frac{2\pi}{N}\right)}}{1 - e^{-jk\left(\frac{2\pi}{N}\right)}} \right), \ k \neq 0, \pm N, \pm 2N, \ldots$$

Factoring out $e^{-jk\left(\frac{2\pi}{2N}\right)}$ from both the numerator and denominator yields

$$a_k = \frac{1}{N} \frac{e^{-jk\left(\frac{2\pi}{2N}\right)} \left[e^{jk\left(N_1+\frac{1}{2}\right)\left(\frac{2\pi}{N}\right)} - e^{-jk\left(N_1+\frac{1}{2}\right)\left(\frac{2\pi}{N}\right)} \right]}{e^{-jk\left(\frac{2\pi}{2N}\right)} \left[e^{jk\left(\frac{2\pi}{2N}\right)} - e^{-jk\left(\frac{2\pi}{2N}\right)} \right]},$$

which can be simplified to

$$a_k = \frac{1}{N} \frac{\sin\left(2\pi k\left(N_1 + 1/2\right)/N\right)}{\sin\left(\pi k / N\right)} \quad k \neq 0, \pm N, \pm 2N, \ldots \tag{69}$$

For $k = 0, \pm N, \pm 2N, \ldots$, we cannot use (69), as that would entail division by zero. We return to (67) and sum it setting $e^{-jk\left(\frac{2\pi}{N}\right)n} = 1$, which yields

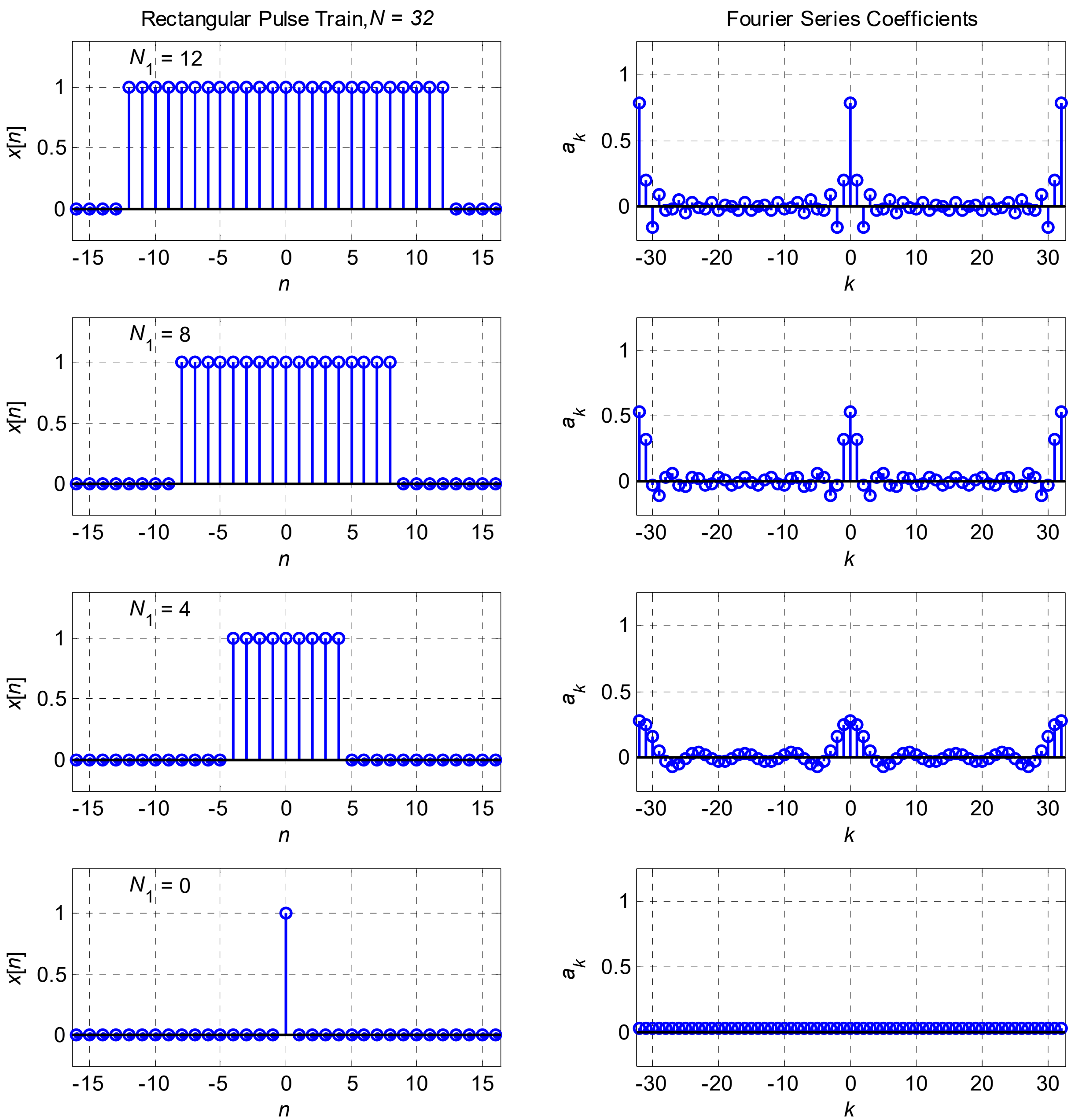
$$a_k = \frac{2N_1 + 1}{N}, \ k = 0, \pm N, \pm 2N, \ldots \tag{70}$$

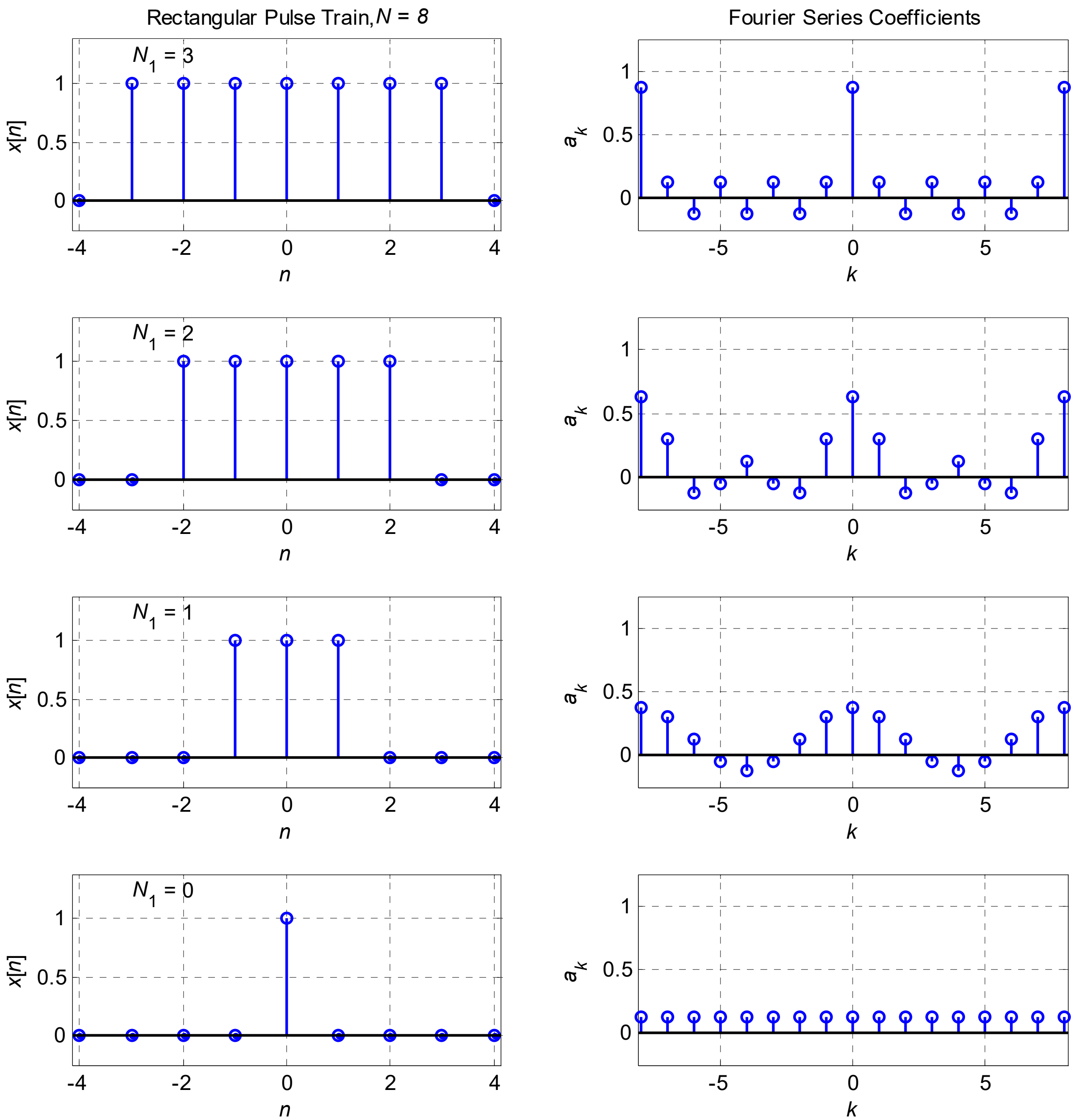
Note that (69) converges to (70) in the limit $k \to 0, \pm N, \pm 2N, \ldots$, which can be shown using L'Hôpital's rule. Hence, we will express the DTFS coefficients for all values of k as

$$a_k = \frac{1}{N} \frac{\sin\left(2\pi k\left(N_1 + 1/2\right)/N\right)}{\sin\left(\pi k / N\right)} \quad \forall k. \tag{71}$$

- The preceding discussion reminds us of the sinc function used to express the CTFS coefficients for the CT rectangular pulse train (see pages 83-84 above). The function in (71) is the DT counterpart of the sinc function, and we learn more about it in our study of DT Fourier transforms (see Chapter 5). While the sinc function is aperiodic, and has just one peak, at $k = 0$, the function in (71) is periodic, and has peaks at $k = 0, \pm N, \pm 2N, \ldots$ In both functions, the peaks occur at values of k where the denominator vanishes.

- The two figures below show the rectangular pulse train $x[n]$ (over one period) and the corresponding DTFS coefficients a_k (over two periods) for $N = 32$ and $N = 8$, for various values of N_1. We observe:
  - Because $x[n]$ is real and even in n, the a_k are real and even in k (see the DTFS properties below).
  - Given a value of N, the $x[n]$ and a_k are periodic with period N. Fixing the period N (and thus the fundamental frequency $\Omega_0 = 2\pi / N$), as we decrease N_1 and the pulses become narrower, the spectrum at frequencies $k\Omega_0$, $k = \langle N \rangle$, described by a_k, spreads out in frequency. In the limiting case $N_1 = 0$, corresponding to a pulse width $2N_1 + 1 = 1$, $x[n]$ becomes a periodic impulse train, and the formula (71) for the DTFS coefficients yields $a_k = 1/N$, $\forall k$.





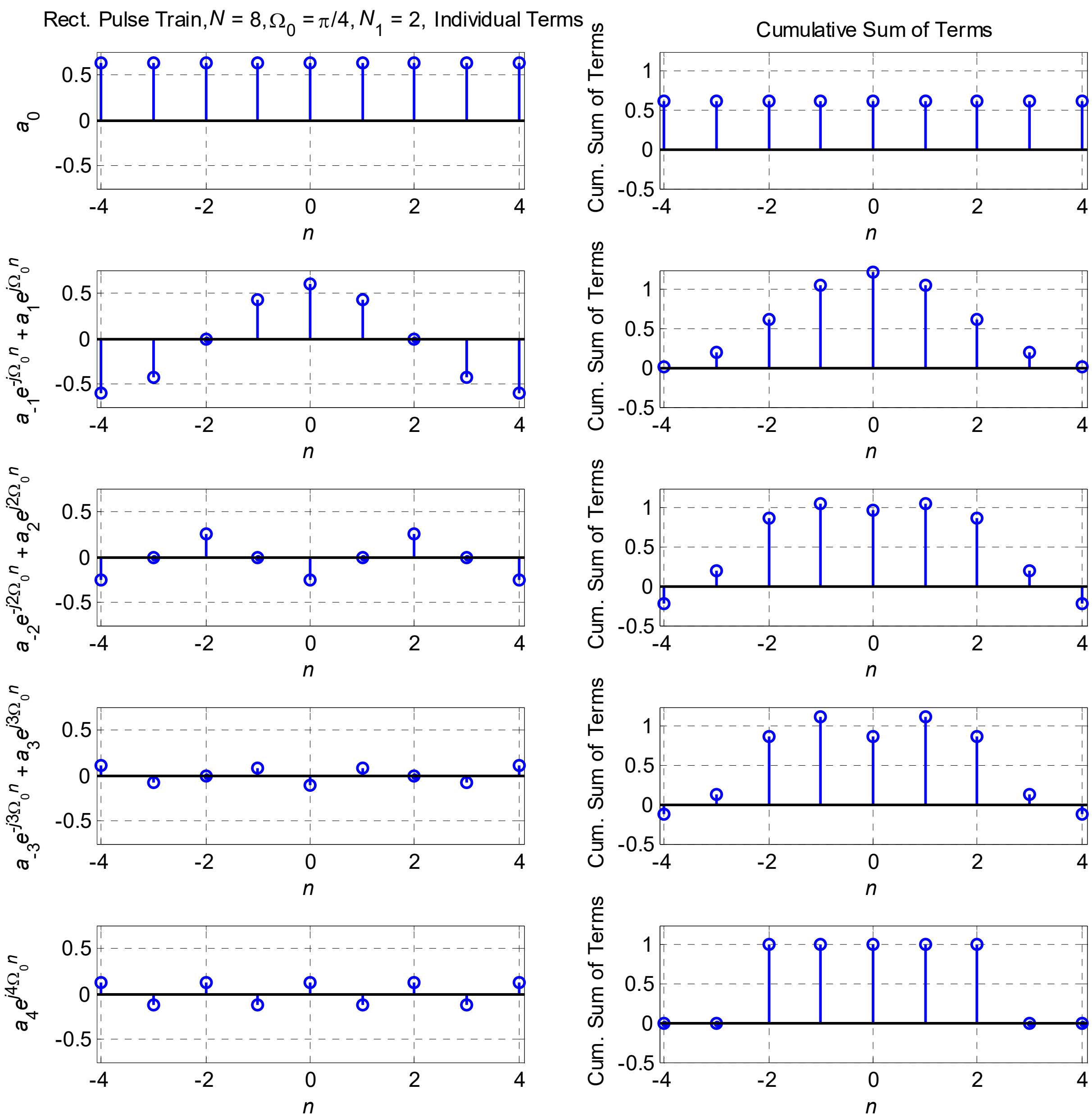
- Now we examine synthesis of the pulse train by using the DTFS coefficients (71) in the synthesis equation (59). We consider $N = 8$ and $N_1 = 2$, so the pulse width is $2N_1 + 1 = 5$. In the figure below, the left column shows the contribution to the synthesis (59) from the terms at $\pm k$, which describe frequencies $\pm k\Omega_0 = \pm k(2\pi / N) = \pm k(\pi / 4)$. We choose to enumerate the N consecutive values of k for reconstruction as $k = \langle N \rangle = -3, \ldots, 4$, so these contributions to the synthesis (59) are

$$\begin{cases} a_0 & k = 0 \\ a_{-k}e^{-jk\Omega_0 n} + a_k e^{jk\Omega_0 n} & k = 1,\, 2,\, 3\,. \\ a_4 e^{j4\Omega_0 n} & k = 4 \end{cases} \tag{72}$$

Each of these contributions is real, because the a_k are conjugate-symmetric, $a_{-k} = a_k^*$, since $x[n]$ is real (see the explanation on pages 121-122). To show that the contribution for $k = 4$ is real, we use also the periodicity of the a_k to note that $a_4 = a_{-4} = a_4^*$, and observe that $e^{j4\Omega_0 n} = e^{j\pi n} = (-1)^n$ is real. In the figure below, the right column shows a synthesis using the cumulative sum of all the terms shown in the left column, which is

$$\hat{x}_K[n] = \sum_{k \le K} a_k e^{jk\Omega_0 n} . \tag{73}$$

We observe that as we include more terms, the partial synthesis $\hat{x}_K[n]$ better approximates $\hat{x}[n]$. Once we include N terms in the synthesis, $\hat{x}_K[n] = x[n]$ *exactly*, as expected from (61).



Properties of the Discrete-Time Fourier Series

- The properties of the DTFS, just like those of the CTFS, are useful for computing DTFS coefficients for new signals by using DTFS coefficients known for other signals, and for checking the DTFS coefficients that we compute for new signals.
- A complete list of DTFS properties is given in Table 2 in the Appendix. We discuss only a subset of the properties here.
- We assume periodic signals having a common period N and a common fundamental frequency $\Omega_0 = 2\pi / N$. We consider one or two signals and their DTFS coefficients:

$$x[n] \overset{FS}{\leftrightarrow} a_k \quad \text{and} \quad y[n] \overset{FS}{\leftrightarrow} b_k .$$

- Many properties of the DTFS are similar to CTFS properties, and we discuss those first. Then we present key DTFS properties that are different from CTFS properties.

Properties Similar to Continuous-Time Fourier Series

Linearity

- A linear combination of $x[n]$ and $y[n]$ is periodic with the same period N, and has DTFS coefficients given by the same linear combination of DTFS coefficients a_k and b_k:

$$Ax[n] + By[n] \overset{FS}{\leftrightarrow} Aa_k + Bb_k .$$

Time Shift

- A signal time-shifted by n_0 has DTFS coefficients multiplied by a complex-valued factor $e^{-jk\Omega_0 n_0}$:

$$x[n-n_0] \overset{FS}{\leftrightarrow} e^{-jk\Omega_0 n_0} a_k . \tag{74}$$

  The magnitude and phase of $e^{-jk\Omega_0 n_0} a_k$ related to those of a_k as follows:

$$\begin{cases} \left| e^{-jk\Omega_0 n_0} a_k \right| = |a_k| \\ \angle\left(e^{-jk\Omega_0 n_0} a_k \right) = \angle a_k - k\Omega_0 n_0 \end{cases} . \tag{74'}$$

  Time-shifting a signal by n_0 affects the DTFS coefficients by leaving the magnitude unchanged, and adding a phase shift proportional to the negative of the time shift, $-n_0$, which varies linearly with frequency $k\Omega_0$.

Time Reversal

- Time-domain reversal corresponds to frequency-domain reversal:

$$x[-n] \overset{FS}{\leftrightarrow} a_{-k} . \tag{75}$$

As a consequence, an even signal has even DTFS coefficients:

$$x[-n] = x[n] \overset{FS}{\leftrightarrow} a_{-k} = a_k \ ,$$

while an odd signal has odd DTFS coefficients:

$$x[-n] = -x[n] \overset{FS}{\leftrightarrow} a_{-k} = -a_k \ .$$

Conjugation

- Complex conjugation of a time signal corresponds to frequency reversal and complex conjugation of its DTFS coefficients:

$$x^*[n] \overset{FS}{\leftrightarrow} a_{-k}^* \ . \tag{76}$$

Conjugate Symmetry for Real Signal

- The DTFS coefficients of a real signal have conjugate symmetry:

$$x[n] = x^*[n] \overset{FS}{\leftrightarrow} a_k = a_{-k}^* \ . \tag{77}$$

Stated alternatively, the DTFS coefficients have even magnitudes and odd phases:

$$x[n] = x^*[n] \overset{FS}{\leftrightarrow} \begin{cases} |a_k| = |a_{-k}| \\ \angle a_k = -\angle a_{-k} \end{cases} ,$$

and the DTFS coefficients have even real parts and odd imaginary parts:

$$x[n] = x^*[n] \overset{FS}{\leftrightarrow} \begin{cases} \mathrm{Re}(a_k) = \mathrm{Re}(a_{-k}) \\ \mathrm{Im}(a_k) = -\mathrm{Im}(a_{-k}) \end{cases} .$$

Real, Even or Real, Odd Signals

- Combining the time reversal and conjugation properties (75) and (76), we find that

$$x[n] \text{ real and even in } n \ \overset{FS}{\leftrightarrow} \ a_k \text{ real and even in } k \tag{78}$$

and

$$x[n] \text{ real and odd in } n \ \overset{FS}{\leftrightarrow} \ a_k \text{ imaginary and odd in } k \ . \tag{79}$$

Fourier Series Synthesis of a Real Signal

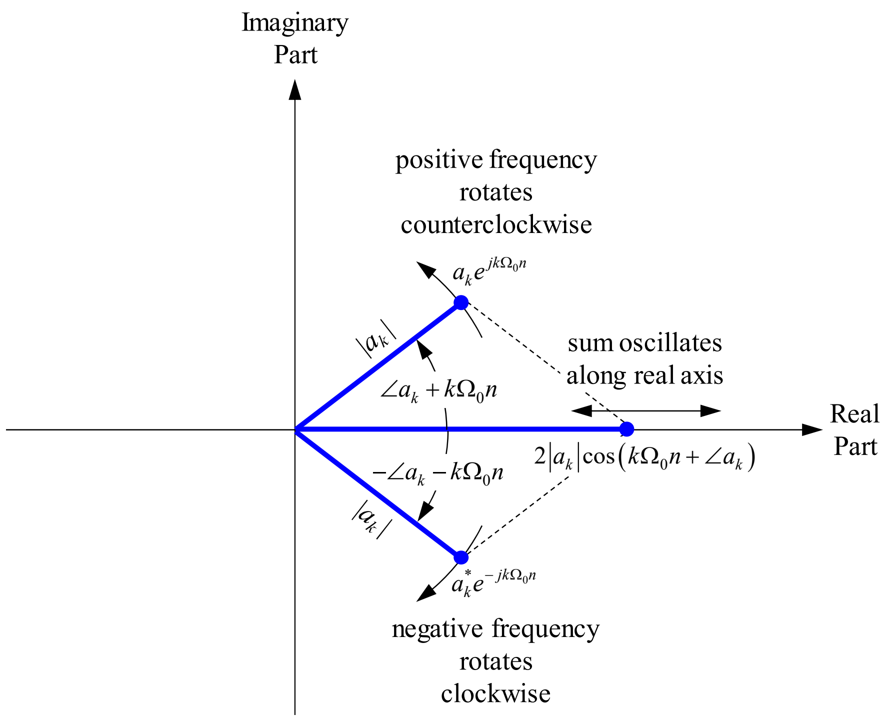
- In order to highlight the roles of positive and negative frequencies and of conjugate-symmetric DTFS coefficients in the synthesis of a real, periodic signal (as on pages 118-119), we consider the synthesis

$$\hat{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n} \ . \tag{59}$$

Assuming a real signal, the DTFS coefficients a_k satisfy (77). Consider a pair of terms in (59) at positive and negative frequencies $\pm k\Omega_0$, $k \neq 0$. Their DTFS coefficients must satisfy $a_k = a_{-k}^*$ (or $a_{-k} = a_k^*$), so they add up to form a real cosine at frequency $k\Omega_0$:

$$\begin{aligned} a_k e^{jk\Omega_0 n} + a_{-k} e^{-jk\Omega_0 n} &= a_k e^{jk\Omega_0 n} + \left(a_k e^{jk\Omega_0 n}\right)^* \\ &= 2\left|a_k\right| \cos\left(\angle a_k + k\Omega_0 n\right) \end{aligned} . \tag{80}$$

- The figure below shows the two terms in (80). The positive-frequency term $a_k e^{jk\Omega_0 n}$ corresponds to a vector rotating *counterclockwise*, with magnitude $\left|a_k\right|$ and phase $\angle a_k + k\Omega_0 n$. The negative-frequency term $a_{-k} e^{-jk\Omega_0 n}$ corresponds to vector rotating *clockwise*, with magnitude $\left|a_k\right|$ and phase $-\angle a_k - k\Omega_0 n$. These two vectors always add up to a real cosine that oscillates along the real axis.



Properties Different from Continuous-Time Fourier Series

Periodicity

- The DTFS coefficients are periodic in the frequency index *k* with period *N*:

$$a_{k+N} = a_k \quad \forall k . \tag{65}$$

Multiplication

- The product of $x[n]$ and $y[n]$ is periodic with the same period N, and has DTFS coefficients that are a *periodic convolution* between the DTFS coefficients a_k and b_k:

$$x[n]\cdot y[n] \overset{FS}{\leftrightarrow} \sum_{l=\langle N\rangle} a_l b_{k-l} \ .$$

The periodic convolution between two discrete sequences is like the ordinary (linear) convolution that we studied in Chapter 2, except that the summation is performed only over N consecutive values of l. The resulting sequence of DTFS coefficients is periodic in k because b_{k-l} is periodic in k.

Parseval's Identity

- The utility of Parseval's identity for the DTFS is similar to that for the CTFS (see pages 93-94). It enables us to compute an inner product between two periodic DT signals, or the power of one periodic DT signal, either in the time domain or in the frequency domain. Depending on the signal(s) we are given, the calculation is often easier in one domain or the other.

Inner Product Between Signals

- The general form of Parseval's identity, for an *inner product between two periodic DT signals*, states that

$$\langle x[n], y[n]\rangle = \sum_{n=\langle N\rangle} x[n]y^*[n] = N\sum_{k=\langle N\rangle} a_k b_k^* \ . \tag{81}$$

The middle expression in (81) is an inner product between the DT signals $x[n]$ and $y[n]$ computed in the time domain. Both are periodic signals, so the summation is performed over only a single period, consisting of any N consecutive values of n. The rightmost expression in (81) is an inner product between the corresponding sequences of DTFS coefficients, a_k and b_k^*. Since these are periodic sequences, the summation is performed over a single period, consisting of any N consecutive values of k.

Signal Power

- Considering the special case of (81) with $x[n]=y[n]$ and $a_k = b_k$, we obtain an expression for the *power of a periodic DT signal*:

$$P = \frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2 \ . \tag{82}$$

The middle expression in (82) represents the power of the periodic signal $x[n]$ computed in the time domain, which is the energy of $x[n]$ in one period divided by the length of the period. As shown by the rightmost expression in (82), we can alternatively compute the power of $x[n]$ by summing the squared magnitudes of its DTFS coefficients a_k over one period.

- In (82), we identify $|a_k|^2$ as the *power density spectrum* of the periodic signal $x[n]$, because $|a_k|^2$ represents the power contained in the signal component at frequency $k\Omega_0$. We may interpret the rightmost expression in (82) as the sum of the powers in N frequency components $k\Omega_0$, $k=\langle N\rangle$.

Response of Discrete-Time Linear Time-Invariant Systems to Periodic Inputs

- Suppose we are given an LTI system whose impulse response is $h[n]$. Assume the sum defining the system frequency response

$$H\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} h[n] e^{-jn\Omega} \ . \tag{46}$$

converges. As stated by (45), the imaginary exponential signals

$$e^{j\Omega n}, \ \Omega \text{ real}, \ -\infty < n < \infty \ .$$

are eigenfunctions of the system, with eigenvalues given by $H\left(e^{j\Omega}\right)$, as illustrated below.

$$x[n] = e^{j\Omega n} \longrightarrow \boxed{h[n] \leftrightarrow H\left(e^{j\Omega}\right)} \longrightarrow y[n] = H\left(e^{j\Omega}\right) \cdot e^{j\Omega n}$$

- Now suppose we input a signal $x[n]$, which is periodic with period $N = 2\pi / \Omega_0$, and which can be represented by a DTFS with coefficients a_k, $k = \langle N \rangle$:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n} \ . \tag{59}$$

Using the system's linearity and the eigenfunction property (45), the output $y[n]$ can be expressed by a DTFS

$$y[n] = \sum_{k=\langle N \rangle} b_k e^{jk\Omega_0 n} = \sum_{k=\langle N \rangle} a_k H\left(e^{jk\Omega_0}\right) e^{jk\Omega_0 n} \ . \tag{83}$$

In (83), the output $y[n]$ has DTFS coefficients b_k given by

$$b_k = a_k H\left(e^{jk\Omega_0}\right), \tag{84}$$

which are the DTFS coefficients of the input $x[n]$, scaled by values of the frequency response $H\left(e^{j\Omega}\right)$ evaluated at frequencies $\Omega = k\Omega_0$. This is shown in the figure below.

$$x[n] \overset{FS}{\leftrightarrow} a_k \longrightarrow \boxed{h[n] \leftrightarrow H\left(e^{j\Omega}\right)} \longrightarrow y[n] \overset{FS}{\leftrightarrow} b_k = a_k H\left(e^{jk\Omega_0}\right)$$

Rewriting (84), we can relate the magnitudes and phases of the input and output DTFS coefficients:

$$\begin{cases} |b_k| = |a_k| \left| H\left(e^{jk\Omega_0}\right) \right| \\ \angle b_k = \angle a_k + \angle H\left(e^{jk\Omega_0}\right) \end{cases} . \tag{84'}$$

Frequency Response of Discrete-Time Linear Time-Invariant Systems

- The frequency response $H\left(e^{j\Omega}\right)$ of a DT LTI system is defined in (46) as the DT Fourier transform of the impulse response $h[n]$, and in (45) as the eigenvalue for an imaginary exponential input $e^{j\Omega n}$. In this section, we examine several important aspects of the frequency response. Most of these are similar to those of CT LTI systems, but there are important differences, as we point out below.

Frequency Response of a Real System

- Consider a DT LTI system whose impulse response $h[n]$ is real, and hence satisfies

$$h[n] = h^*[n]. \tag{85}$$

The frequency response $H\left(e^{j\Omega}\right)$ at frequency Ω is given by (46). To obtain the frequency response at frequency $-\Omega$, we evaluate (46) with the substitution $\Omega \rightarrow -\Omega$, and use (85):

$$\begin{aligned} H\left(e^{-j\Omega}\right) &= \sum_{n=-\infty}^{\infty} h[n] e^{jn\Omega} \\ &= \sum_{n=-\infty}^{\infty} h^*[n] e^{jn\Omega} \\ &= \left(\sum_{n=-\infty}^{\infty} h[n] e^{-jn\Omega}\right)^* \\ &= H^*\left(e^{j\Omega}\right) \end{aligned}. \tag{86}$$

- Our finding can be summarized as

$$h[n] = h^*[n] \leftrightarrow H\left(e^{-j\Omega}\right) = H^*\left(e^{j\Omega}\right). \tag{87}$$

Given a real impulse response, the frequency response has conjugate symmetry, just as for a CT LTI system. Property (87) can be expressed in two alternate ways. First, for a real impulse response, the magnitude and phase of the frequency response are even and odd functions of frequency, respectively:

$$h[n] = h^*[n] \leftrightarrow \begin{cases} \left|H\left(e^{-j\Omega}\right)\right| = \left|H\left(e^{j\Omega}\right)\right| \\ \angle H\left(e^{-j\Omega}\right) = -\angle H\left(e^{j\Omega}\right) \end{cases}. \tag{87a}$$

Second, for a real impulse response, the real and imaginary parts of the frequency response are even and odd functions of frequency, respectively:

$$h[n] = h^*[n] \leftrightarrow \begin{cases} \mathrm{Re}\left[H\left(e^{-j\Omega}\right)\right] = \mathrm{Re}\left[H\left(e^{j\Omega}\right)\right] \\ \mathrm{Im}\left[H\left(e^{-j\Omega}\right)\right] = -\mathrm{Im}\left[H\left(e^{j\Omega}\right)\right] \end{cases}. \tag{87b}$$

These properties of the frequency response are entirely analogous to those for CT LTI systems.

- Now suppose we input a real, periodic signal $x[n]$ to the system. By (77), its DTFS coefficients a_k have conjugate symmetry, $a_k = a_{-k}^*$. As shown by (84), we obtain a periodic output $y[n]$ with DTFS coefficients $b_k = a_k H\left(e^{jk\Omega_0}\right)$. These output DTFS coefficients are conjugate-symmetric:

$$\begin{aligned} b_k &= a_k H\left(e^{jk\Omega_0}\right) \\ &= a_{-k}^* H^*\left(e^{-jk\Omega_0}\right), \\ &= b_{-k}^* \end{aligned}$$

where we used (87) in the second line. The output $y[n]$ is real, as expected. This observation is identical to one we made previously for CT LTI systems.

Periodicity of the Frequency Response

- The frequency response $H\left(e^{j\Omega}\right)$ of a DT LTI system is periodic in frequency Ω with period 2π . This derives from the fact that any two DT imaginary exponential signals are identical if their frequencies differ by a multiple of 2π (Chapter 1, pages 15-16). This periodicity of the frequency response is a major difference between DT LTI systems and their CT counterparts. To prove the periodicity, we evaluate (46) at a frequency $\Omega + 2\pi$:

$$\begin{aligned} H\left(e^{j(\Omega+2\pi)}\right) &= \sum_{n=-\infty}^{\infty} h[n] e^{-jn(\Omega+2\pi)} \\ &= \sum_{n=-\infty}^{\infty} h[n] e^{-jn\Omega} \underbrace{e^{-jn2\pi}}_{=1} . \\ &= H\left(e^{j\Omega}\right) \end{aligned} \tag{88}$$

- In analyzing the response of a DT LTI system to an input that is periodic with period $N = 2\pi / \Omega_0$, we found that the input DTFS coefficients are scaled by values of the frequency response, $H\left(e^{jk\Omega_0}\right)$, to yield the output DTFS coefficients:

$$b_k = a_k H\left(e^{jk\Omega_0}\right) . \tag{84}$$

The $H\left(e^{jk\Omega_0}\right)$ are periodic in *k* with period *N*. We can show this by substituting $k\Omega_0 = k2\pi / N$ for Ω in (88), or by simply writing

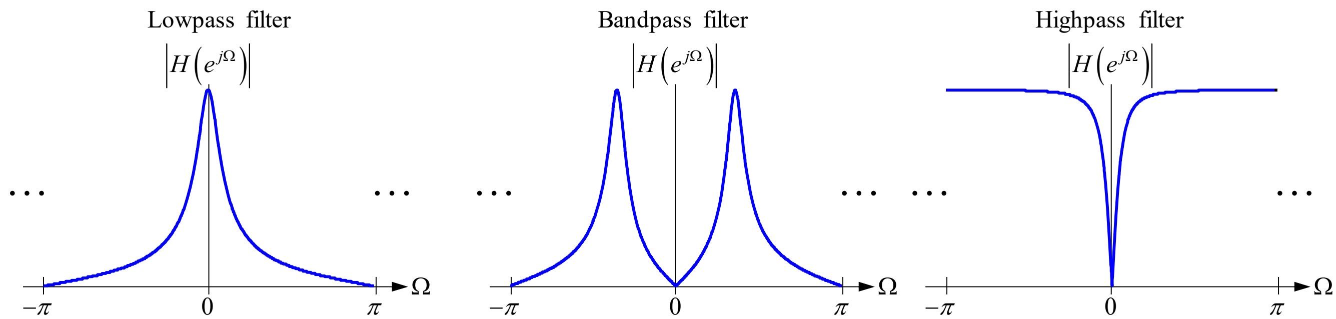
$$H\left(e^{j(k+N)\Omega_0}\right) = H\left(e^{jk\Omega_0} e^{j2\pi}\right) = H\left(e^{jk\Omega_0}\right) .$$

This periodicity of $H\left(e^{jk\Omega_0}\right)$ is necessary so the output DTFS coefficients (84) are periodic in *k* with period *N*.

Types of Linear Distortion and Filters

- Our classification here of DT filters and types of distortion in DT LTI systems is similar to that given for CT (see page 97 above). But there are a few important differences, which we point out below.

- The *magnitude response* $\left|H\left(e^{j\Omega}\right)\right|$ (also known as the *amplitude response*) determines the scaling of different frequency components appearing at the output of a DT LTI system, much as in a CT system.
    - *Magnitude distortion* (also called *amplitude distortion*) occurs if $\left|H\left(e^{j\Omega}\right)\right|$ is frequency-dependent.
    - Filters with a frequency-dependent magnitude response $\left|H\left(e^{j\Omega}\right)\right|$ are often classified as *lowpass*, *bandpass*, or *highpass*, as illustrated here by typical examples. The magnitude $\left|H\left(e^{j\Omega}\right)\right|$ is periodic in Ω with period 2π, so only one period, $-\pi \le \Omega \le \pi$, is shown. The concepts of "low", "medium" and "high" frequencies, which are relevant to classifying DT filters as lowpass, bandpass and highpass, should be interpreted relative to the frequency range $-\pi \le \Omega \le \pi$. By contrast, in classifying CT filters, these concepts are interpreted relative to the frequency range $-\infty < \omega < \infty$.



- The *phase response* $\angle H\left(e^{j\Omega}\right)$ determines the phase shifts of different frequency components appearing at the output of a DT LTI system.
    - If the phase is a linear function of frequency with an integer slope

$$\angle H\left(e^{j\Omega}\right) = -n_0\Omega,$$

    where n_0 is an integer, then all frequency components are subject to an equal time shift n_0.

    - *Phase distortion* occurs when $\angle H\left(e^{j\Omega}\right)$ is not a linear function of frequency with an integer slope.
    - Given a phase response $\angle H\left(e^{j\Omega}\right)$, we define the *group delay* as $-\dfrac{d\angle H\left(e^{j\Omega}\right)}{d\Omega}$. The group delay describes how different frequency components are time-shifted. If the group delay has a constant integer value, the system causes no phase distortion.

- A *distortionless DT system* may scale and time-shift signals, but causes no amplitude or phase distortion. Using (9) from Chapter 2, page 42, its impulse response is of the form

$$h[n] = C\,\delta[n - n_0], \tag{89}$$

where C is a constant and n_0 is an integer. We can compute the corresponding frequency response using (89) in (46), which is the DT Fourier transform of the impulse response:

$$\begin{aligned} H\left(e^{j\Omega}\right) &= \sum_{n=-\infty}^{\infty} h[n]e^{-jn\Omega} \\ &= \sum_{n=-\infty}^{\infty} C\,\delta[n-n_0]e^{-jn\Omega} \, . \\ &= Ce^{-jn_0\Omega} \end{aligned} \tag{90}$$

We have evaluated the sum using the sampling property of the DT impulse function. Applying the frequency response (90) to filtering a periodic signal using (84), we obtain DTFS coefficients that are consistent with the DTFS time-shift property (74). A distortionless system is illustrated below.

$$x[n] \longrightarrow \boxed{h[n] = C\,\delta[n-n_0] \leftrightarrow H\left(e^{j\Omega}\right) = C\,e^{-j\Omega n_0}} \longrightarrow y[n] = C\,x[n-n_0]$$

The output is scaled and time-shifted version of the input: $y[n] = C\,x[n-n_0]$.

- In summary, the concepts of phase response, group delay and distortionless systems for DT are very similar to those for CT, except that in order to avoid distortion, a DT shift n_0 must be integer-valued.

Methods for Evaluating the Frequency Response

- Here we describe two methods for computing the frequency response of a given LTI system. The list is not exhaustive, as other methods exist.

1. *Fourier transform of impulse response*. Suppose a DT LTI system is specified in terms of an *impulse response*. We can find an expression for the frequency response by evaluating the DT Fourier transform of the impulse response:

$$H\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} h[n]e^{-jn\Omega} \, . \tag{46}$$

We used this procedure just above to obtain the frequency response of a distortionless system (90) from its impulse response (89). The sum (46) represents the DT Fourier transform of the impulse response $h[n]$. We will not be able to evaluate (46) for general impulse responses, however, until we study the DT Fourier transform in Chapter 5.

2. *Substitution in difference equation*. Suppose an LTI system is specified by a *linear, constant-coefficient difference equation* in the form (42) in Chapter 2, page 71. If we know that the DT Fourier transform sum (46) converges, then we can compute the frequency response directly from the difference equation, without using the sum (46) by the following simple procedure:

   1. Substitute the following input and output signals in the difference equation:

$$x[n] = e^{j\Omega n} \quad \text{and} \quad y[n] = H\left(e^{j\Omega}\right)e^{j\Omega n} \, .$$

   2. Cancel all factors of $e^{j\Omega n}$ and solve for $H\left(e^{j\Omega}\right)$.

Once we study the DT Fourier transform in Chapter 5, we will understand when the sum (46) converges and this method is valid. Until then, we will apply this method only to judiciously chosen examples.

Examples

- Here we apply Method 2 to three examples.
- For details on computing and plotting the magnitudes and phases of the first-order filter, two-point moving average and edge detector, see Appendix, pages 301-304.

1. *First-Order System*

- We studied a simple first-order DT system in Chapters 1 and 2. It is described by a difference equation

$$y[n]-ay[n-1]=x[n], \tag{91}$$

where a is a real constant. Recall that since difference equation (91) involves feedback of past outputs (recursion), it describes an infinite impulse response (IIR) system, and provides only an implicit description of the system input-output relation. As shown in the table below, the choice of a determines the type of system, whether it is stable, and whether the frequency response exists. (The existence of the frequency response will be explained when we study the DT Fourier transform in Chapter 5.)

System	**Value of *a***	**Stable**	**Frequency Response Exists (in Strict Sense)**
Highpass filter	$-1<a<0$	Yes	Yes
Lowpass filter	$0<a<1$	Yes	Yes
Running summation (accumulation)	$a=1$	No	No
Compound interest	$1<a<\infty$ (typically)	No	No

- Here we assume $|a|<1$ and compute the frequency response from difference equation (91). Substituting for $x[n]$ and $y[n]$:

$$H\left(e^{j\Omega}\right)e^{j\Omega n}-aH\left(e^{j\Omega}\right)e^{j\Omega(n-1)}=e^{j\Omega n}.$$

Cancelling factors of $e^{j\Omega n}$ and solving for the frequency response:

$$H\left(e^{j\Omega}\right)=\frac{1}{1-ae^{-j\Omega}}.$$

- In computing the magnitude and phase, we use the *reciprocal property* (see Appendix, page 289). Given a complex-valued $z=|z|e^{j\angle z}$, its reciprocal is

$$\frac{1}{z}=\frac{1}{|z|e^{j\angle z}}=\frac{1}{|z|}e^{-j\angle z}.$$

The magnitude and phase of $1/z$ are related to those of z as

$$\left|\frac{1}{z}\right| = \frac{1}{|z|} \quad \text{and} \quad \angle\left(\frac{1}{z}\right) = -\angle z \,.$$

- Using the reciprocal property with

$$z = 1 - ae^{-j\Omega} = 1 - a\cos\Omega + ja\sin\Omega \,,$$

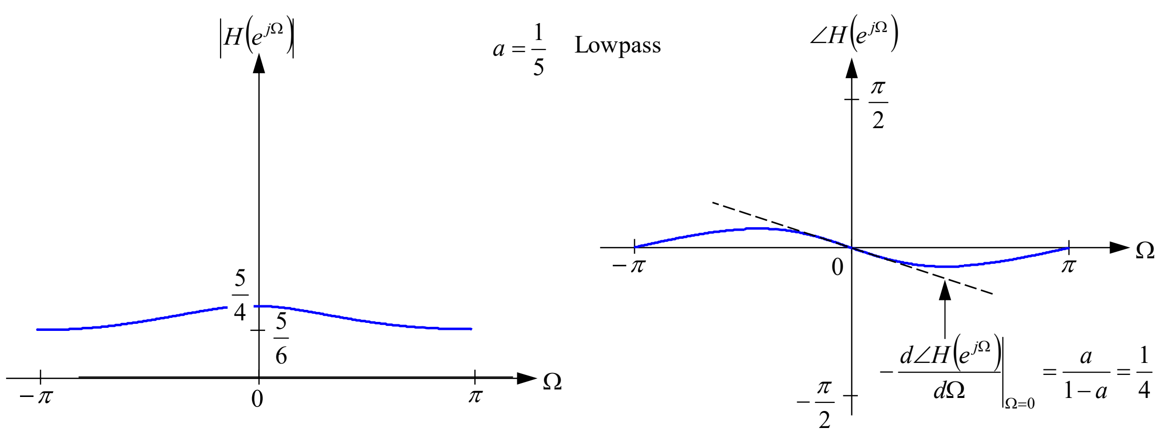
the magnitude and phase responses of the first-order system are

$$\left|H\left(e^{j\Omega}\right)\right| = \frac{1}{\left|1 - ae^{-j\Omega}\right|} = \frac{1}{\left|1 - a\cos\Omega + ja\sin\Omega\right|} = \frac{1}{\sqrt{\left(1 - a\cos\Omega\right)^2 + \left(a\sin\Omega\right)^2}}$$

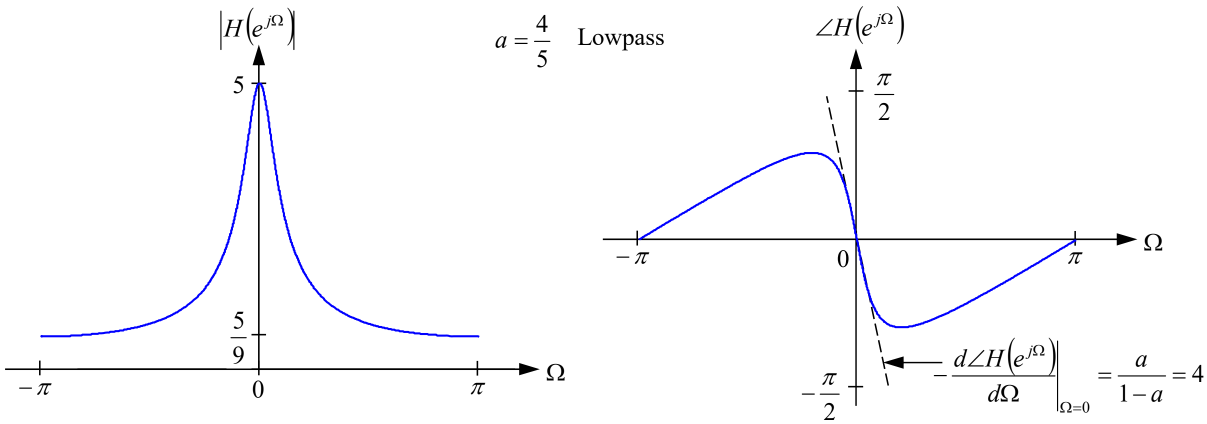
$$\angle H\left(e^{j\Omega}\right) = -\angle\left(1 - ae^{-j\Omega}\right) = -\angle\left(1 - a\cos\Omega + ja\sin\Omega\right) = -\tan^{-1}\left(\frac{a\sin\Omega}{1 - a\cos\Omega}\right).$$

From their mathematical forms, we see that the system causes both magnitude distortion and phase distortion.

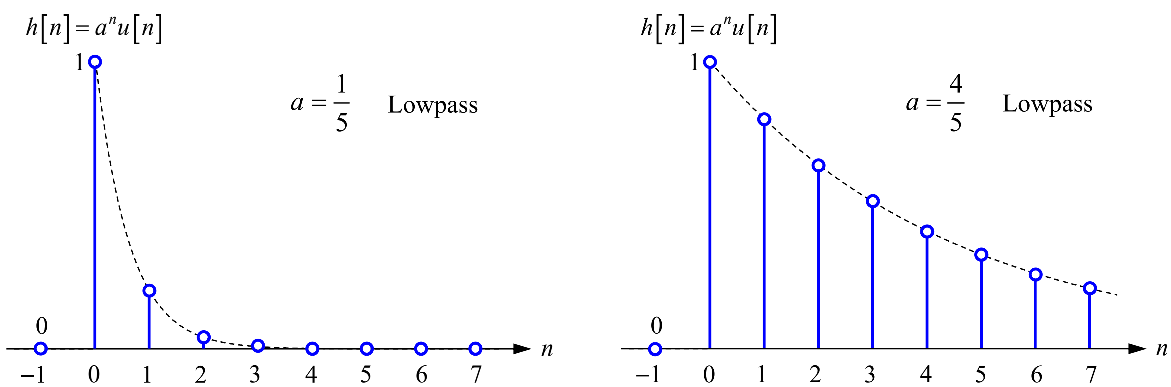
- Here we assume $0 < a < 1$, describing lowpass filters. In the Appendix, pages 303-304, we also consider $-1 < a < 0$, describing highpass filters.
  - $a = 1/5$ is a weak lowpass filter. The ratio between the magnitude responses at $\Omega = 0$ and $\Omega = \pi$ is only $3/2$. Near $\Omega = 0$, where the magnitude response is largest, the group delay is less than one sample: $-d\angle H\left(e^{j\Omega}\right)/d\Omega\Big|_{\Omega=0} = 1/4$.



- a = 4/5 is a stronger lowpass filter. The ratio between the magnitude responses at $\Omega = 0$ and $\Omega = \pi$ is 9. Near $\Omega = 0$, where the magnitude response is largest, the group delay is four samples: $-d\angle H\left(e^{j\Omega}\right)/\,d\Omega\big|_{\Omega=0} = 4$.



- As a increases from 1/5 to 4/5, the group delay near Ω = 0 increases from 1/4 to 4. This can be understood intuitively by comparing the respective impulse responses shown below.



2. *Two-Point Moving Average*

- We studied a two-point moving average in Chapter 2 (pages 73-74). It is described by a difference equation

$$y[n] = \frac{1}{2}\left(x[n] + x[n-1]\right). \tag{92}$$

Since the difference equation (92) is non-recursive, it describes a finite impulse response (FIR) system, and provides an explicit input-output relation for the system. Substituting for $x[n]$ and $y[n]$ in (92):

$$H\left(e^{j\Omega}\right)e^{j\Omega n} = \frac{1}{2}\left[e^{j\Omega n} + e^{j\Omega(n-1)}\right].$$

Cancelling factors of $e^{j\Omega n}$, we immediately obtain the frequency response:

$$H\left(e^{j\Omega}\right) = \frac{1}{2}\left[1 + e^{-j\Omega}\right].$$

- To make it easier to compute the magnitude and phase, we factor out $e^{-j\Omega/2}$:

$$H\left(e^{j\Omega}\right)=e^{-j\frac{\Omega}{2}}\frac{e^{j\frac{\Omega}{2}}+e^{-j\frac{\Omega}{2}}}{2}=e^{-j\frac{\Omega}{2}}\cos\left(\frac{\Omega}{2}\right).$$

- In computing the magnitude and phase, we use the *product property* (see Appendix, page 289). Given complex-valued $z_1=|z_1|e^{j\angle z_1}$ and $z_2=|z_2|e^{j\angle z_2}$, the product is

$$z_1z_2=|z_1|e^{j\angle z_1}\,|z_2|e^{j\angle z_2}=|z_1||z_2|e^{j(\angle z_1+\angle z_2)}.$$

The magnitude and phase of z_1z_2 are related to those of z_1 and z_2 as

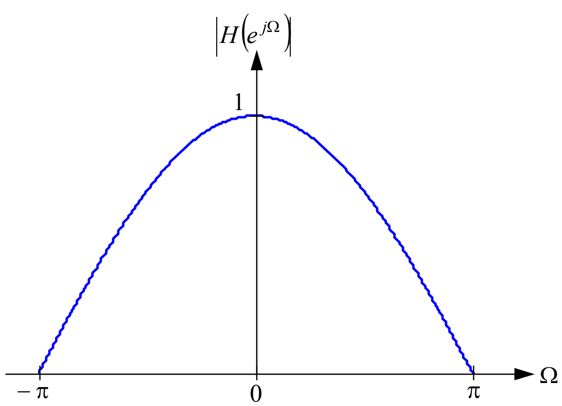
$$|z_1z_2|=|z_1||z_2| \quad \text{and} \quad \angle z_1z_2=\angle z_1+\angle z_2.$$

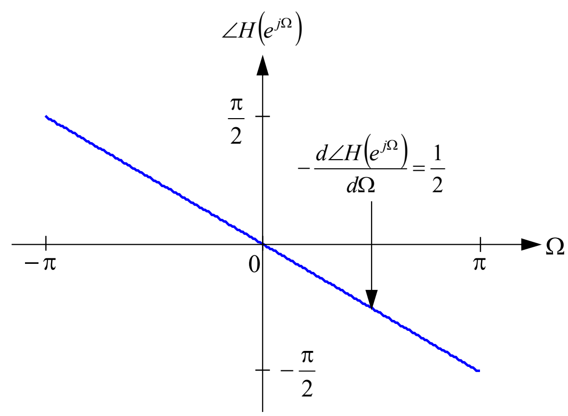
- Using the product property with $z_1=e^{-j\frac{\Omega}{2}}$ and $z_2=\cos\left(\frac{\Omega}{2}\right)$, the magnitude and phase are

$$\left|H\left(e^{j\Omega}\right)\right|=\left|e^{-j\frac{\Omega}{2}}\right|\left|\cos\left(\frac{\Omega}{2}\right)\right|=\left|\cos\left(\frac{\Omega}{2}\right)\right|$$

$$\angle H\left(e^{j\Omega}\right)=\angle e^{-j\frac{\Omega}{2}}+\angle\cos\left(\frac{\Omega}{2}\right)=\begin{cases}-\Omega/2 & \cos(\Omega/2)>0\\ -\Omega/2-\pi & \cos(\Omega/2)<0\end{cases}.$$

These are plotted below. This lowpass filter causes magnitude distortion. The highest frequencies are completely rejected: $\left|H\left(e^{\pm j\pi}\right)\right|=0$. The group delay is less than one sample at all frequencies: $-d\angle H\left(e^{j\Omega}\right)/d\Omega=1/2$. Since the group delay is not integer-valued, the filter causes phase distortion.





3. *Edge Detector*

- We studied an edge detector in Chapter 2 (page 74). It is described by a difference equation

$$y[n]=\frac{1}{2}\left(x[n]-x[n-1]\right). \tag{93}$$

The difference equation (93) is non-recursive, describes an FIR system, and provides an explicit input-output relation for the system, like (92). Substituting for $x[n]$ and $y[n]$ in (93):

$$H\left(e^{j\Omega}\right)\cdot e^{j\Omega n}=\frac{1}{2}\left[e^{j\Omega n}-e^{j\Omega(n-1)}\right].$$

Cancelling factors of $e^{j\Omega n}$, we immediately obtain the frequency response:

$$H\left(e^{j\Omega}\right)=\frac{1}{2}\left[1-e^{-j\Omega}\right].$$

- To help compute the magnitude and phase, we factor out $e^{-j\Omega/2}$ and multiply and divide by *j*:

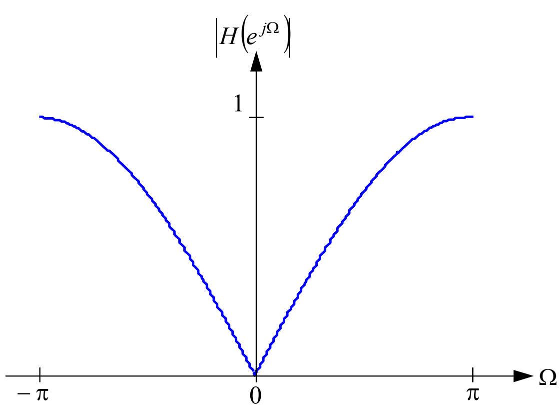
$$H\left(e^{j\Omega}\right)=je^{-j\frac{\Omega}{2}}\frac{e^{j\frac{\Omega}{2}}-e^{-j\frac{\Omega}{2}}}{2j}=je^{-j\frac{\Omega}{2}}\sin\left(\frac{\Omega}{2}\right).$$

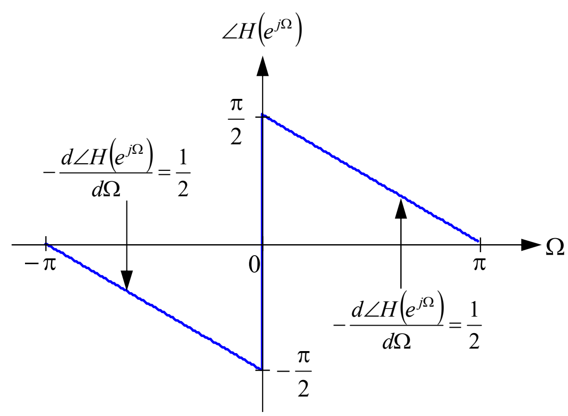
- Using the product property with $z_1=j$, $z_2=e^{-j\frac{\Omega}{2}}$ and $z_3=\sin\left(\frac{\Omega}{2}\right)$, the magnitude and phase are

$$\left|H\left(e^{j\Omega}\right)\right|=|j|\left|e^{-j\frac{\Omega}{2}}\right|\left|\sin\left(\frac{\Omega}{2}\right)\right|=\left|\sin\left(\frac{\Omega}{2}\right)\right|$$

$$\angle H\left(e^{j\Omega}\right)=\angle j+\angle e^{-j\frac{\Omega}{2}}+\angle\sin\left(\frac{\Omega}{2}\right)=\frac{\pi}{2}-\frac{\Omega}{2}+\begin{cases}0 & \sin(\Omega/2)>0\\ -\pi & \sin(\Omega/2)<0\end{cases}=\begin{cases}-\Omega/2+\pi/2 & \sin(\Omega/2)>0\\ -\Omega/2-\pi/2 & \sin(\Omega/2)<0\end{cases}.$$

These are plotted below. This is a highpass filter, and causes magnitude distortion. The filter completely rejects d.c.: $\left|H\left(e^{j0}\right)\right|=0$. The group delay is less than one sample at all frequencies: $-d\angle H\left(e^{j\Omega}\right)/d\Omega=1/2$. The group delay is not integer-valued, so the filter causes phase distortion.

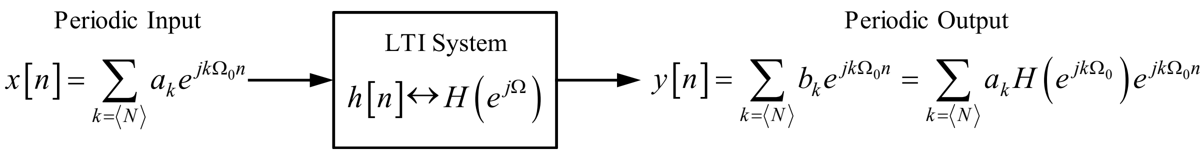




Examples of Filtering Periodic Discrete-Time Signals by Linear Time-Invariant Systems

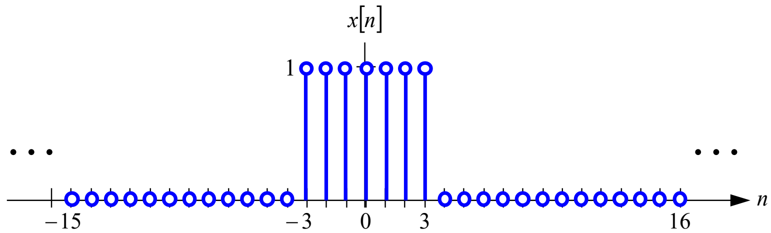
Method of Analysis

- We use the analysis presented on page 124 above, which is summarized in the figure below.



Input Signal

- The input signal $x[n]$ is a rectangular pulse train with period $N=32$ and with $N_1=3$, as shown below. The pulse width is $2N_1+1=7$. The fundamental frequency is $\Omega_0=\frac{2\pi}{N}=\frac{\pi}{16}$.



- Its DTFS coefficients are given by (71). Choosing $N=32$ and $N_1=3$ in (71), the DTFS coefficients become

$$a_k=\frac{1}{32}\frac{\sin(7\pi \mathrm{k}/32)}{\sin(\pi \mathrm{k}/32)}.$$

Throughout this example, we choose $N=32$ consecutive values of *n* and *k* as $n=\langle N\rangle=-15,\ldots,16$ and $k=\langle N\rangle=-15,\ldots,16$. The DTFS representation of the input signal is

$$x[n]=\sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n}=\sum_{k=-15}^{16}\frac{1}{32}\frac{\sin(7\pi k/32)}{\sin(\pi k/32)}e^{jk\left(\frac{\pi}{16}\right)n}.$$

- Because $x[n]$ is real and even in *n*, its DTFS coefficients a_k are real and even in *k* (see (78)). Since the a_k are purely real, their phases can only be an integer multiple of π, and are typically chosen as

$$\angle a_k=\begin{cases}0 & a_k>0\\ \pm\pi & a_k<0\end{cases}.$$

In the plots below, when $a_k<0$, we make the specific choices $\angle a_k=-\pi$ for $k<0$ and $\angle a_k=+\pi$ for $k>0$ so the phase appears with the odd symmetry expected, but this is not necessary. (See Appendix, pages 300-301, for further explanation.)

Linear Time-Invariant Systems

- We consider the three systems whose frequency responses were analyzed on pages 129-133 above.

- First-order system. This is an IIR system. We choose two values of the parameter a:
  - $a = 1/5$: weak lowpass filter
  - $a = 4/5$: stronger lowpass filter
- Two-point moving average: FIR lowpass filter.
- Edge detector: FIR highpass filter.

Output Signal

- Given an LTI system with frequency response $H\left(e^{j\Omega}\right)$, using (84), the output is represented by a DTFS

$$y[n] = \sum_{k=\langle N\rangle} b_k e^{jk\Omega_0 n} = \sum_{k=\langle N\rangle} a_k H\left(e^{jk\Omega_0}\right) e^{jk\Omega_0 n} = \sum_{k=-15}^{16} \frac{1}{32} \frac{\sin(7\pi k/32)}{\sin(\pi k/32)} H\left(e^{jk\left(\frac{\pi}{16}\right)}\right) e^{jk\left(\frac{\pi}{16}\right)n}.$$

The output DTFS coefficients b_k are given by the input DTFS coefficients a_k, scaled by values of the frequency response $H\left(e^{j\Omega}\right)$ evaluated at $\Omega = k\Omega_0$:

$$b_k = a_k H\left(e^{jk\Omega_0}\right). \tag{84}$$

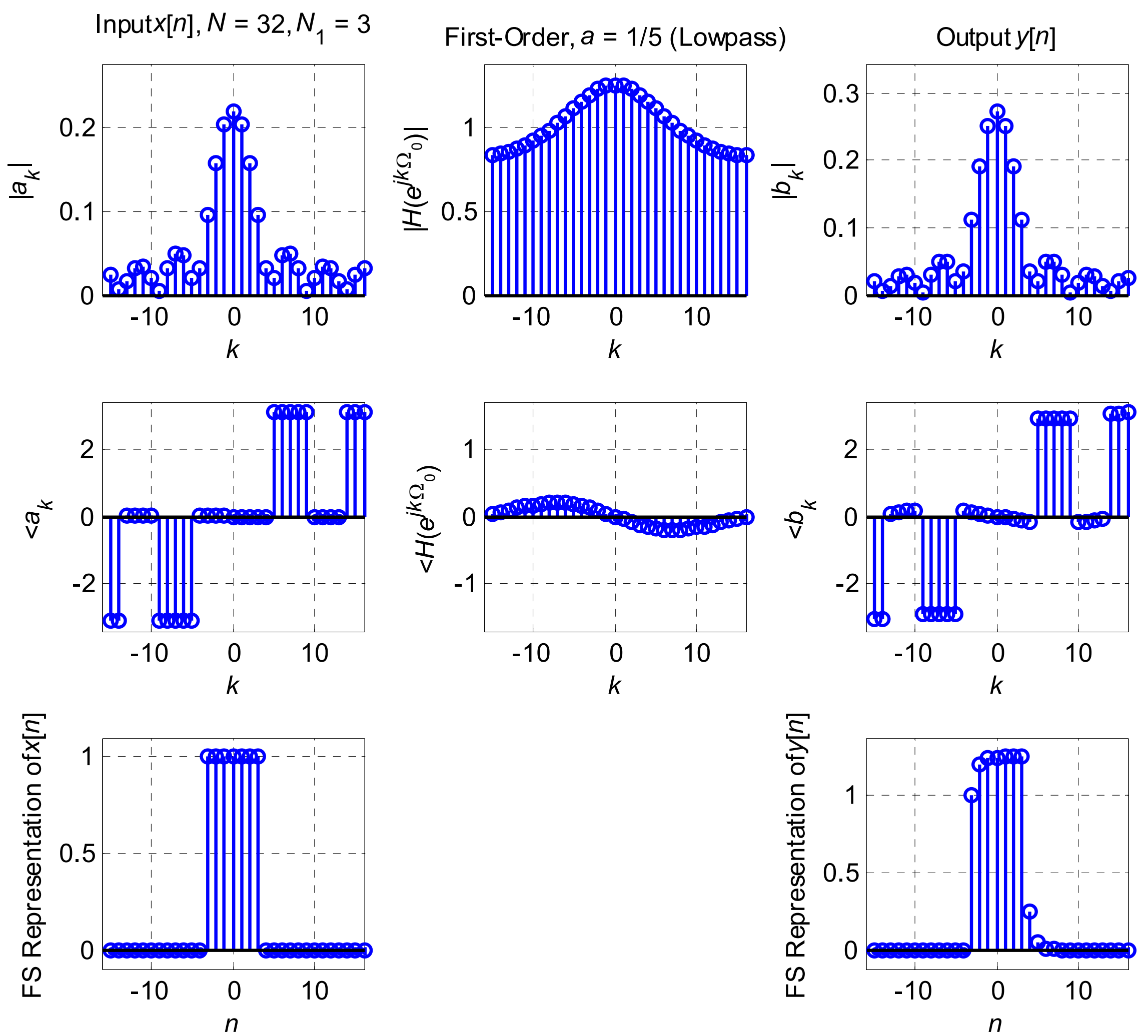
As a result, the magnitudes and phases of the input and output DTFS coefficients are related by

$$\begin{cases} |b_k| = |a_k| \left|H\left(e^{jk\Omega_0}\right)\right| \\ \angle b_k = \angle a_k + \angle H\left(e^{jk\Omega_0}\right) \end{cases}. \tag{84'}$$

In each figure below, the relationship (84') should be evident in the first row (which shows $|a_k|$, $\left|H\left(e^{jk\Omega_0}\right)\right|$ and $|b_k|$) and in the second row (which shows $\angle a_k$, $\angle H\left(e^{jk\Omega_0}\right)$ and $\angle b_k$).

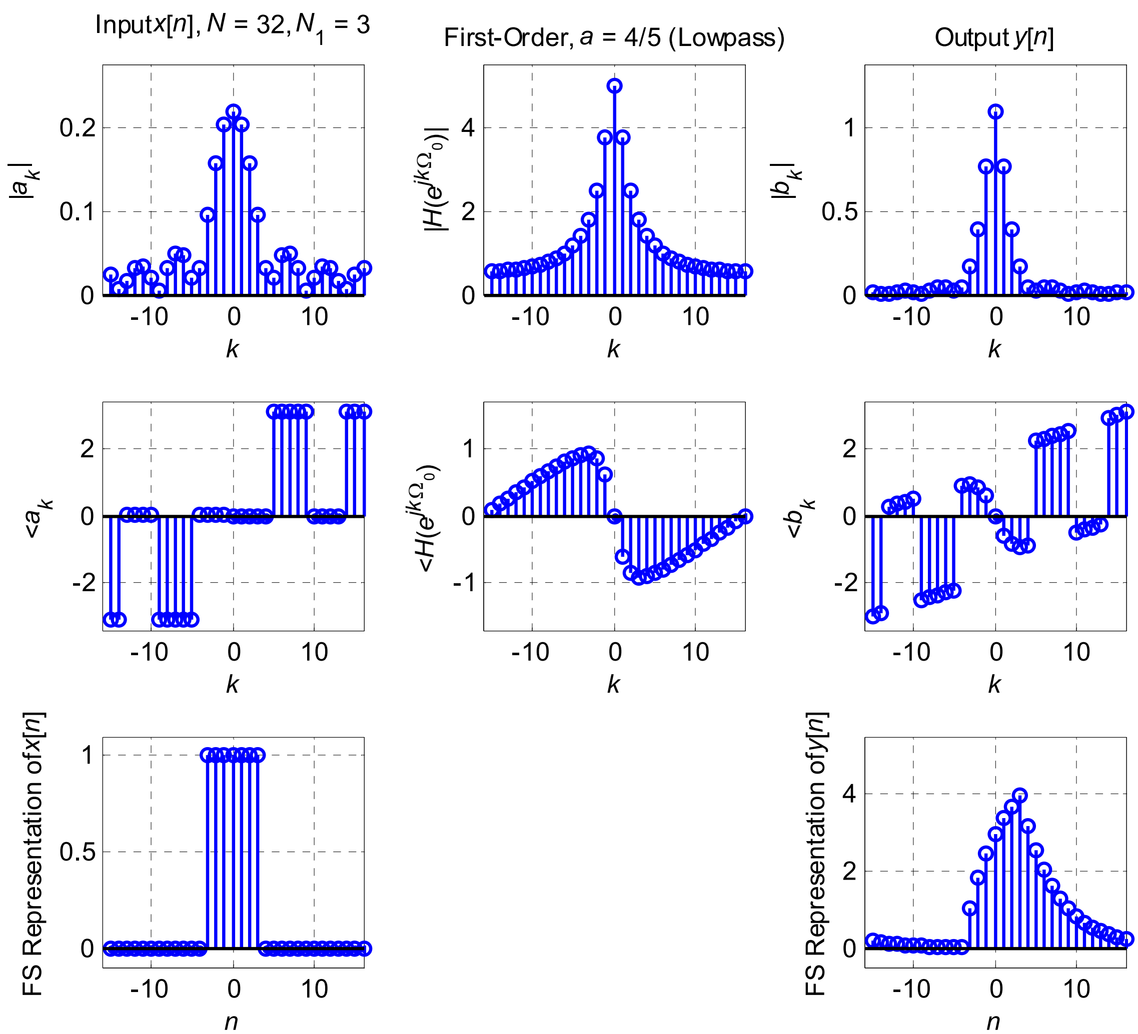
Filtering by First-Order System, a = 1/5 (Infinite Impulse Response, Weak Lowpass Filter)

- The d.c. level (average value) is scaled by $H\left(e^{j0}\right) = 5/4$.
- The rise and fall times are of the order of one sample.
- The pulse centroid, which is determined mainly by low-frequency components, is delayed less than one sample, consistent with the low-frequency group delay $-d\angle H\left(e^{j\Omega}\right)/d\Omega\big|_{\Omega=0} = 1/4$.



Filtering by First-Order System, a = 4/5 (Infinite Impulse Response, Stronger Lowpass Filter)

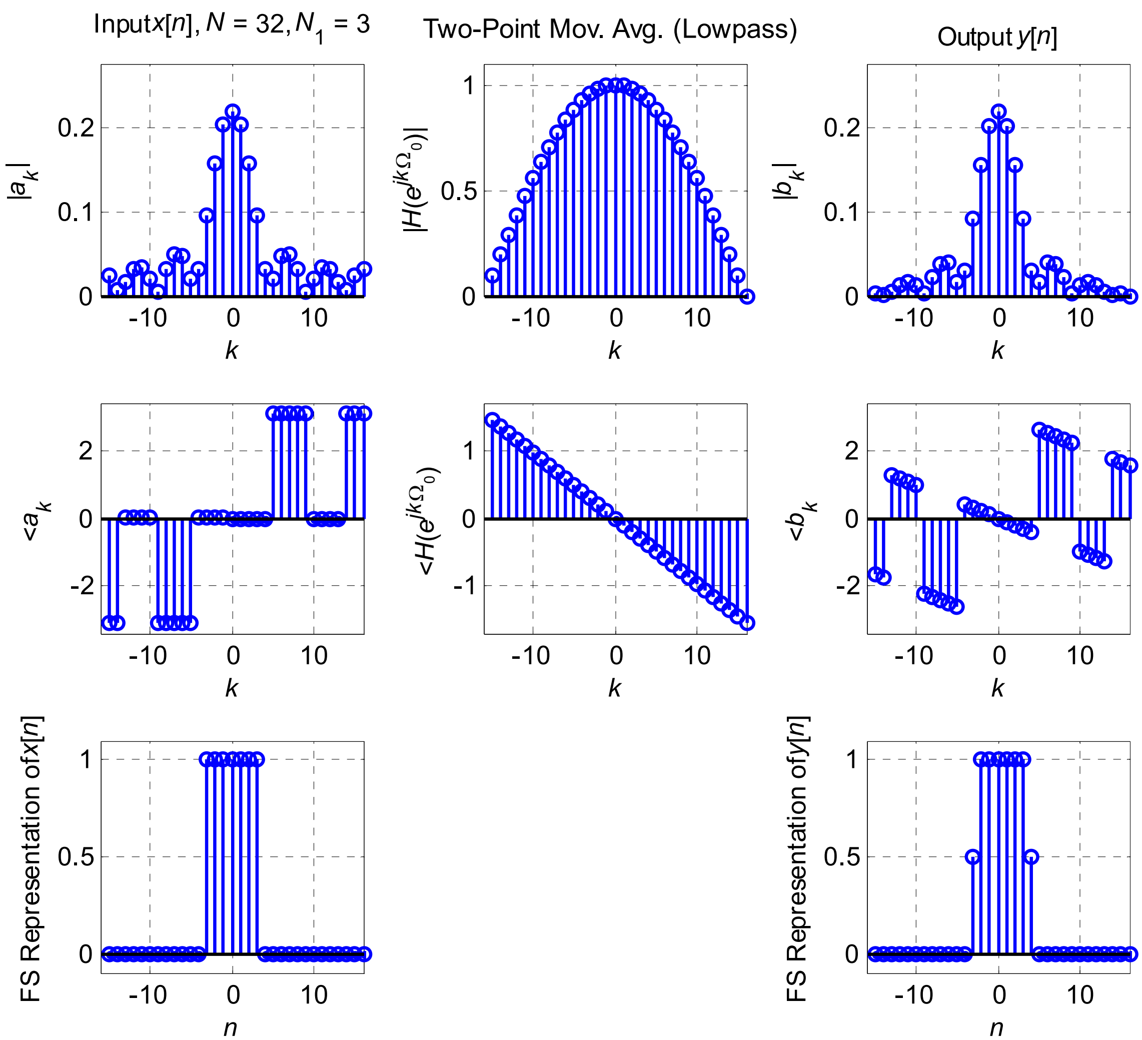
- The d.c. level (average value) is scaled by $H\left(e^{j0}\right)=5$.
- The rise and fall times are about five samples.
- The pulse centroid, which is determined mainly by low-frequency components, is delayed by several samples, consistent with the low-frequency group delay $-d\angle H\left(e^{j\Omega}\right)/d\Omega\Big|_{\Omega=0}=4$.



Filtering by Two-Point Moving Average, (Finite Impulse Response, Lowpass Filter)

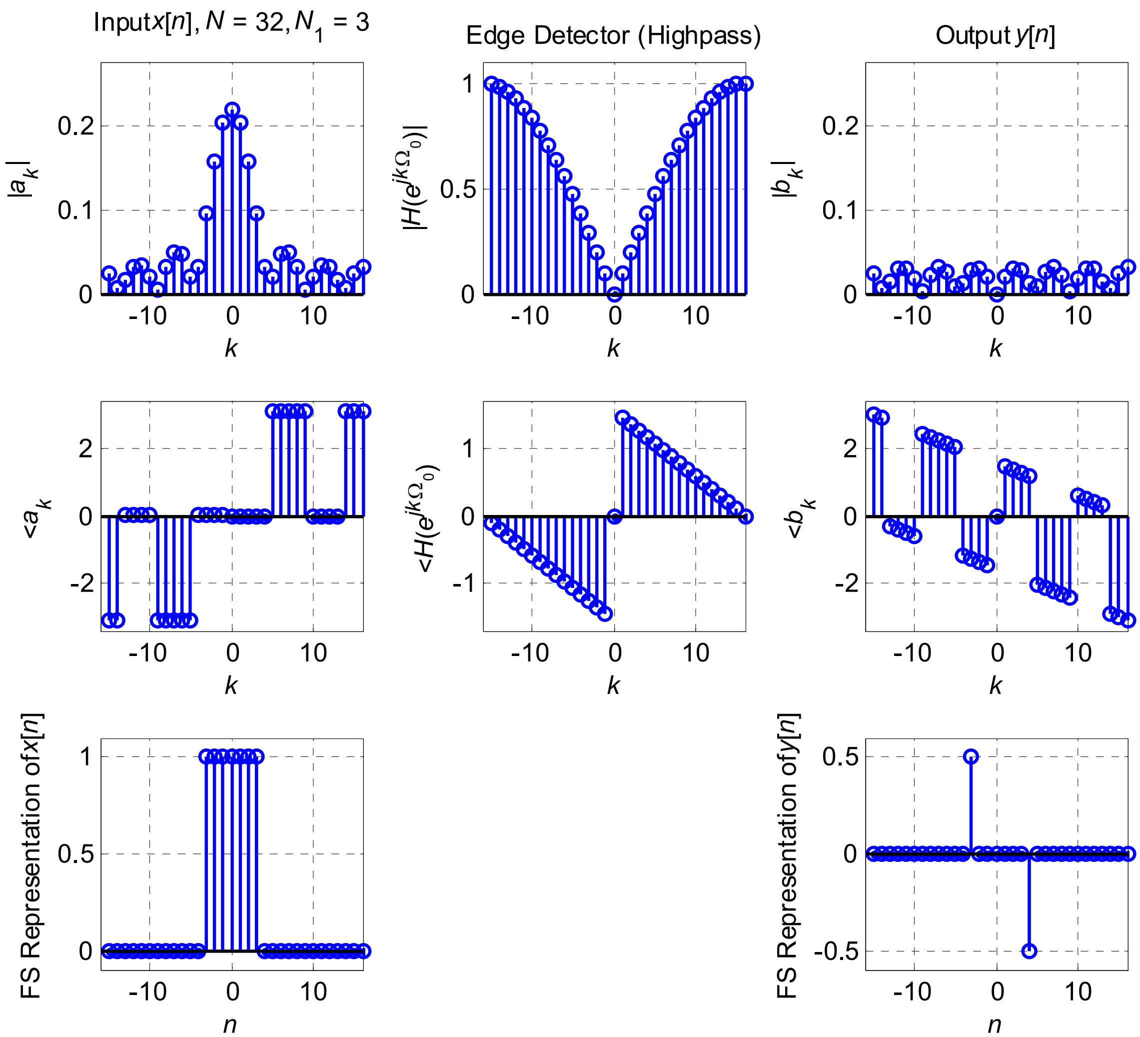
- The d.c. level (average value) is preserved, since $H\left(e^{j0}\right)=1$, while the highest frequencies are attenuated completely, i.e., $H\left(e^{\pm j\pi}\right)=0$.

- The pulse centroid, determined mainly by low-frequency components, is delayed less than one sample, consistent with the group delay $-d\angle H\left(e^{j\Omega}\right)/d\Omega=1/2$.



Filtering by Edge Detector, (Finite Impulse Response, Highpass Filter)

- The d.c. level (average value) is removed, since $H\left(e^{j0}\right)=0$.
- The leading and trailing edges remain, since $H\left(e^{\pm j\pi}\right)=1$.
- The leading and trailing edges are, on average, delayed less than one sample, consistent with $-d\angle H\left(e^{j\Omega}\right)/d\Omega=1/2$.



Comment on Method of Analysis

- We have analyzed these examples using DTFS. We could instead analyze them using convolution methods from Chapter 2 (precisely as we noted for the CTFS examples on pages 103-107). The rectangular pulse train input $x[n]$ could be represented as an infinite sum of scaled and shifted step functions, and the periodic output $y[n]$ could be represented as a corresponding sum of scaled and shifted step responses. The outputs $y[n]$ obtained here can be understood using this approach. The Fourier series method used here offers important advantages, however. (a) It naturally takes account of the overlap between all the scaled, shifted step responses. (b) It is applicable to *any* periodic input $x[n]$ with finite power, even if is not representable in terms of simple functions, such as step functions.

First-Order and Higher-Order Infinite Impulse Response Systems

- First-order systems, in which a single past output $y[n-1]$ is fed back, are easy to analyze and understand. But they are not suitable for many applications. In the first-order system considered here, the single parameter a governs both the time-domain and frequency-domain responses.
- If we feed back older past outputs, such as $y[n-2]$, etc., we obtain systems described by *higher-order difference equations*. Such *higher-order systems* offer more flexibility in their characteristics. For example, they can achieve a sharper passband-stopband transition, and offer more control over tradeoffs between time-domain response and frequency response.
- We will learn about second-order IIR systems when we study the DT Fourier transform in Chapter 5. Higher-order systems are addressed in more depth using the Z transform in EE 102B.

Finite Impulse Response Systems

- FIR filters, in which the output is formed by a linear combination of delayed inputs without feedback of past outputs, offer many options for performing filtering functions:
  - The two-point moving average can be extended to average over any desired number of samples.
  - The two-point edge detector computes a first difference. Discrete-time filters approximating important continuous-time functions, such as the first derivative or a higher derivative, can be realized.
  - Nearly ideal lowpass, bandpass or highpass filters can be realized, which have a nearly flat magnitude response in the passband and a nearly zero response in the stopband, with an abrupt transition between passband and stopband.
  - Filters whose phase response is a linear function of frequency, such that the group delay $-d\angle H\left(e^{j\Omega}\right)/d\Omega$ assumes a constant value in the passband, can be realized.
- In Chapter 6, we will briefly study FIR approximations of a differentiator and of an ideal lowpass filter. FIR approximations of these and other filters are studied in more depth in EE 102B.

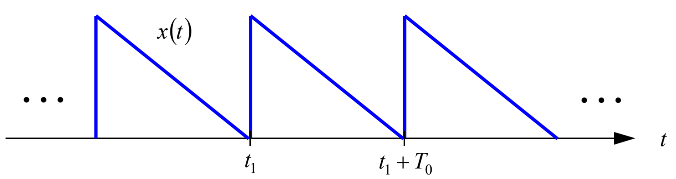
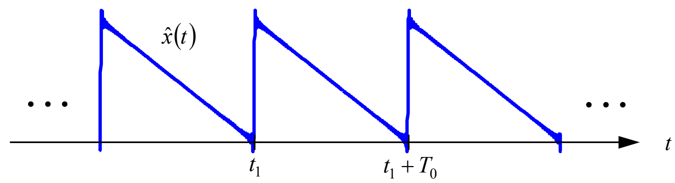
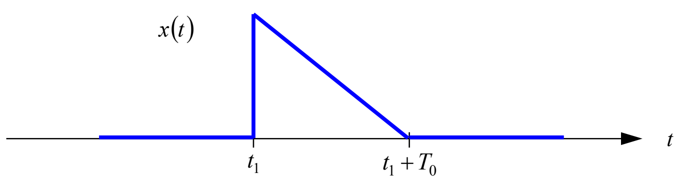
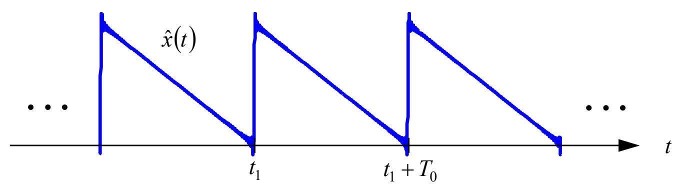
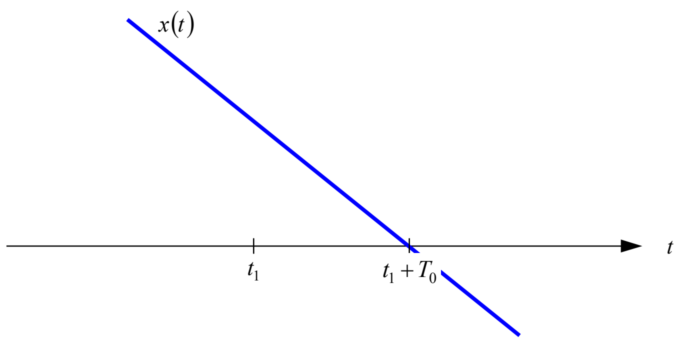
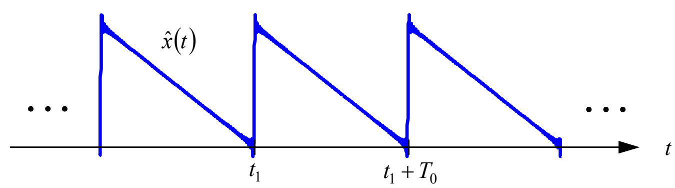
Fourier Series Representation of Different Signal Types

This section may be skipped in a first reading of this chapter.

- In studying the FS for CT or DT signals, we have thus far assumed that the original signal used in analysis, $x(t)$ or $x[n]$, is *periodic*. We found that FS synthesis yields a signal, $\hat{x}(t)$ or $\hat{x}[n]$, that is *periodic* like the original $x(t)$ or $x[n]$. Here we consider what happens if we perform FS analysis on a signal, $x(t)$ or $x[n]$, that is *not periodic*. We will see that FS synthesis still yields a *periodic* $\hat{x}(t)$ or $\hat{x}[n]$. In other words, the FS can be used to obtain a periodic signal from an aperiodic signal or, more generally, to obtain a periodic function from an aperiodic function. A similar method is used in starting from a CT filter (whose frequency response is not generally periodic in frequency) and using it to obtain a DT filter (whose frequency response must be periodic in frequency). The method is described briefly in Chapter 6 below, and in more detail in *EE 102B Course Reader*, Chapter 3.

Continuous-Time Case

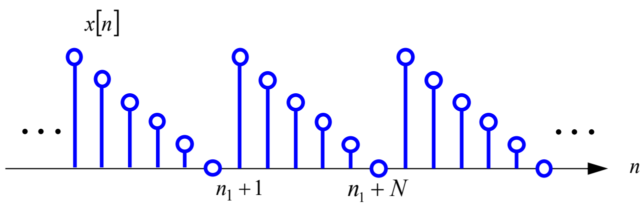
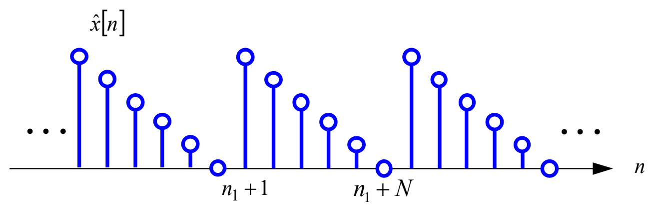
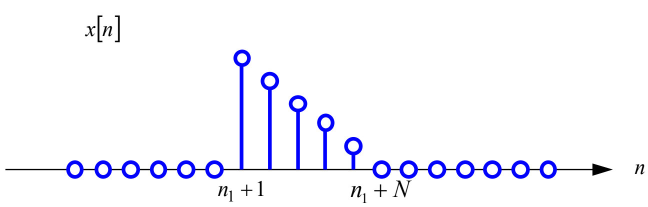
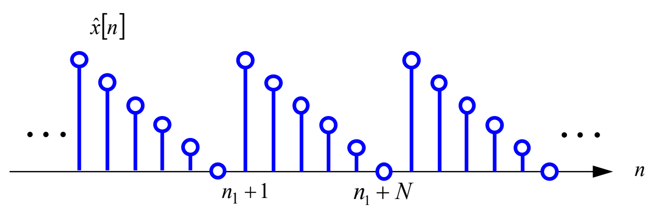
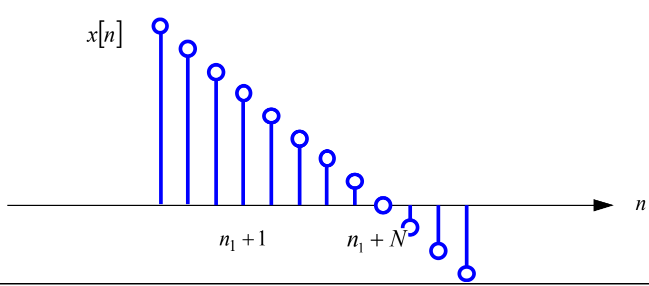
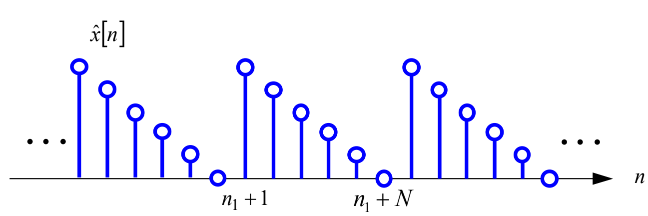
- Given a signal $x(t)$, not necessarily periodic, we choose an interval $t_1 \leq t \leq t_1 + T_0$, and use the analysis equation (19) to compute CTFS coefficients a_k, $-\infty < k < \infty$. We use these CTFS coefficients in the synthesis equation (11) to form a CTFS representation $\hat{x}(t)$, $-\infty < t < \infty$. Assuming the Dirichlet conditions (see pages 82-83) are satisfied over the interval $t_1 \leq t \leq t_1 + T_0$, we know that:
  - $\hat{x}(t)$ reproduces $x(t)$ for $t_1 \leq t \leq t_1 + T_0$, except for differences associated with the Gibbs phenomenon.
  - $\hat{x}(t)$ is periodic with period T_0, i.e., $\hat{x}(t+T_0) = \hat{x}(t)\ \ \forall t$.
- The figure below shows the CTFS representation $\hat{x}(t)$ that results if the starting signal $x(t)$ is of three different types. The three different starting signals $x(t)$ are all *different*. Only one of the three is periodic. However, the three starting signals $x(t)$ are *identical* over the interval $t_1 \leq t \leq t_1 + T_0$ used in the analysis equation (19). As a result, their CTFS representations $\hat{x}(t)$ are all identical. All three CTFS representations $\hat{x}(t)$ are *periodic*.

Signal $x(t)$	Signal approximated by Fourier series representation $\hat{x}(t)$
Periodic: $x(t+T_0)=x(t) \;\; \forall t$ 	Periodic $x(t)$ 
Time-limited: $x(t)=0$ except for $t_1 \le t \le t_1+T_0$ 	Periodic extension of $x(t)$: $\sum_{l=-\infty}^{\infty} x(t-lT_0)$ 
General 	Periodic extension of time-limited version of $x(t)$: $\sum_{l=-\infty}^{\infty} \tilde{x}(t-lT_0)$, where $\tilde{x}(t)=\begin{cases} x(t) & t_1 \le t \le t_1+T_0 \\ 0 & \text{otherwise} \end{cases}$ 

Discrete-Time Case

- Given a signal $x[n]$, not necessarily periodic, we choose an interval $n_1+1 \le n \le n_1+N$, and use the analysis equation (64) to compute DTFS coefficients a_k, $k=\langle N \rangle$. We use these DTFS coefficients in the synthesis equation (59) to form a DTFS representation $\hat{x}[n]$, $-\infty < n < \infty$. We know that:
  - $\hat{x}[n]=x[n]$ exactly for $n_1+1 \le n \le n_1+N$.
  - $\hat{x}[n]$ is periodic with period N, i.e., $x[n]=x[n+N] \;\; \forall n$, .
- The figure below shows the DTFS representation $\hat{x}[n]$ that results if the starting signal $x[n]$ is of three different types. The three different starting signals $x[n]$ are all *different*. Only one of the three is

periodic. The three starting signals $x[n]$ are, however, *identical* over the interval $n_1+1 \le n \le n_1+N$ used in the analysis equation (64). As a result, their DTFS representations $\hat{x}[n]$ are all identical. All three DTFS representations $\hat{x}[n]$ are *periodic*.

<table>
<tr><th>Signal $x[n]$</th><th>Fourier series representation $\hat{x}[n]$</th></tr>
<tr><td>Periodic: $x[n+N]=x[n] \quad \forall n$
</td><td>Identical: $\hat{x}[n]=x[n] \quad \forall n$
</td></tr>
<tr><td>Time-limited: $x[n]=0$ except for $n_1+1 \le n \le n_1+N$
</td><td>Periodic extension: $\hat{x}[n]=\sum_{l=-\infty}^{\infty} x[n-lN]$
</td></tr>
<tr><td>General
</td><td>Periodic extension of time-limited version:
$\tilde{x}[n]=\begin{cases} x[n] & n_1+1 \le n \le n_1+N \\ 0 & \text{otherwise} \end{cases}$
$\hat{x}[n]=\sum_{l=-\infty}^{\infty} \tilde{x}[n-lN]$
</td></tr>
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Stanford University
EE 102A: Signal Processing and Linear Systems I
Professor Joseph M. Kahn

Chapter 4: The Continuous-Time Fourier Transform

Motivations

- In Chapter 3, we studied Fourier series (FS), which allow us to express a *periodic CT or DT signal* as a *discrete sum* of imaginary exponential signals at different frequencies. These frequencies are $e^{jk\omega_0 t}$, ω_0 real, k integer (in CT) or $e^{jk\Omega_0 n}$, Ω_0 real, k integer (in DT). We saw how FS can simplify the analysis of LTI systems. Given a periodic input signal expressed as a FS, we can compute the output signal by multiplying each imaginary exponential by a function of frequency called the *frequency response*, which characterizes an LTI system.
- In this chapter, we extend Fourier analysis to *aperiodic CT signals*. We introduce the CT Fourier transform (CTFT), which expresses an aperiodic CT signal as a *continuous integral* of imaginary exponentials at different frequencies, $e^{j\omega t}$, ω real, $-\infty < \omega < \infty$. The CTFT will allow us to analyze LTI systems with aperiodic inputs. Moreover, the frequency response of an LTI system is the CTFT of its impulse response. Studying the CTFT in detail will enable us to compute the frequency responses for a wide range of systems, including higher-order systems and systems not described by finite-order differential equations. Finally, the CTFT will allow us to study modulation and demodulation, which are essential for communications.
- In Chapter 5, we will extend Fourier analysis to *aperiodic DT signals* by developing the DT Fourier transform (DTFT).

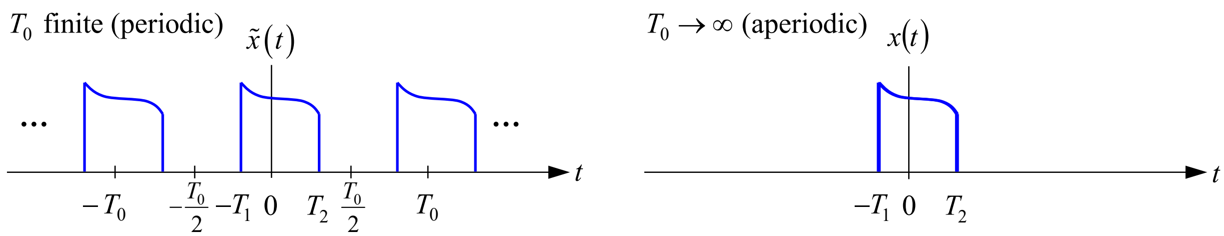
Major Topics in This Chapter

- Continuous-time Fourier transform
  - Derivation for aperiodic signals. Fourier transforms in the limit. Fourier transforms of periodic signals. Properties of Fourier transforms.
- Convolution property and LTI system analysis
  - Frequency response as CTFT of impulse response.
  - LTI systems not described by finite-order differential equations.
    - Time shift, finite-time integrator, ideal lowpass filter.
  - LTI systems described by linear, constant-coefficient differential equations.
    - First-order: integrator, lowpass filter, highpass filter.
    - Second-order: lowpass filter, bandpass filter.
- Multiplication property
  - Amplitude modulation and demodulation.

Continuous-Time Fourier Transform

Derivation of Continuous-Time Fourier Transform

- We are given an aperiodic CT signal $x(t)$, which is nonzero only over an interval $-T_1 \le t \le T_2$. We will consider $x(t)$ to be a periodic CT signal $\tilde{x}(t)$, of period T_0, in the limit that the period becomes infinite, $T_0 \to \infty$. In that limit, $\tilde{x}(t)$ becomes $x(t)$, as illustrated in the figure below.



- To obtain the CT Fourier transform (CTFT) of the aperiodic signal $x(t)$, we start by representing the periodic signal $\tilde{x}(t)$ as a CT Fourier series (CTFS) with fundamental frequency $\omega_0 = \frac{2\pi}{T_0}$ and CTFS coefficients a_k, $-\infty < k < \infty$:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} . \qquad \text{(CTFS synthesis)} \qquad (1)$$

We can obtain the CTFS coefficients by performing analysis over any interval of duration T_0. We assume the analysis interval includes the interval $-T_1 \le t \le T_2$ over which $x(t)$ is nonzero. The CTFS coefficients of $\tilde{x}(t)$ are

$$a_k = \frac{1}{T_0} \int_{T_0} \tilde{x}(t) e^{-jk\omega_o t} dt . \qquad \text{(CTFS analysis)} \qquad (2)$$

Since $\tilde{x}(t) = x(t)$ within the analysis interval and $x(t) = 0$ outside the interval, we can rewrite the CTFS analysis equation (2) as

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_o t} dt = \frac{1}{T_0} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_o t} dt . \qquad (3)$$

- Now we define $X(j\omega)$, a function of a continuous frequency variable ω, which we compute from $x(t)$ using the following integral:

$$X(j\omega) \stackrel{d}{=} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt . \qquad \text{(CTFT or CTFT analysis)} \qquad (4)$$

We refer to $X(j\omega)$ as the *CT Fourier transform* (CTFT) of the aperiodic signal $x(t)$. We refer to the integral (4) as the *CTFT analysis equation*, or simply the *CTFT*.

- Comparing (3) and (4), we observe that we can obtain the CTFS coefficients a_k by sampling the CTFT $X(j\omega)$ at integer multiples of the fundamental frequency and scaling by $1/T_0$:

$$\frac{1}{T_0} X(j\omega)\Big|_{\omega=k\omega_0} = \frac{1}{T_0} X(jk\omega_0) = a_k . \tag{5}$$

Using (5), we can rewrite the CTFS synthesis equation (1) for the periodic signal $\tilde{x}(t)$ as

$$\begin{aligned} \tilde{x}(t) &= \sum_{k=-\infty}^{\infty} \frac{1}{T_0} X(jk\omega_0) e^{jk\omega_0 t} \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0 \end{aligned} . \tag{6}$$

- Now we consider the limit in which the periodic signal $\tilde{x}(t)$ becomes the aperiodic signal $x(t)$:

$$T_0 \to \infty$$

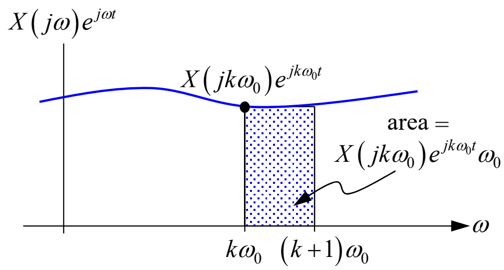
$$\tilde{x}(t) \to x(t)$$

$k\omega_0 \to \omega$, a continuous variable

$\omega_0 \to d\omega$, an infinitesimal increment of ω

$X(j\omega)\Big|_{\omega=k\omega_0} \to X(j\omega)$, a function of a continuous variable

This figure schematically shows $X(j\omega)e^{j\omega t}$ as a function of the continuous frequency variable ω.



With the help of this figure, we see that in the limit we are considering, (6) becomes a Riemann sum approximation of an integral, which allows us to obtain the aperiodic signal $x(t)$ from $X(j\omega)$:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega . \qquad \text{(inverse CTFT or CTFT synthesis)} \tag{7}$$

We refer to the integral (7) as the *inverse CTFT* or the *CTFT synthesis equation*, and refer to $x(t)$ as the *inverse CTFT* of $X(j\omega)$.

- In summary, we have derived the following two expressions.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad \text{(CTFT or CTFT analysis)} \tag{4}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \qquad \text{(inverse CTFT or CTFT synthesis)} \tag{7}$$

The inverse CTFT integral (7) specifies how we can *synthesize* an aperiodic signal $x(t)$ as a weighted sum of imaginary exponentials $e^{j\omega t}$ whose frequency ω is a continuous-valued, real variable, $-\infty < \omega < \infty$. In (7), the imaginary exponential $e^{j\omega t}$ at frequency ω is weighted by a factor $X(j\omega)$. The CTFT integral (4) specifies how, given an aperiodic signal $x(t)$, we can *analyze* $x(t)$ to obtain the weighting factor $X(j\omega)$.

- We may describe (4) and (7) in terms of a *CTFT operator* F and an *inverse CTFT operator* F^{-1}, each of which acts on one function to produce the other function:

$$F\left[x(t)\right] = X(j\omega), \tag{8}$$

and

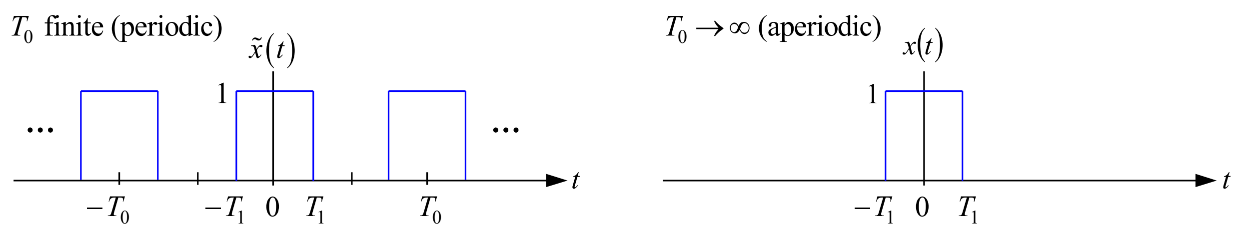
$$F^{-1}\left[X(j\omega)\right] = x(t). \tag{9}$$

We often denote a CT signal $x(t)$ and its CTFT $X(j\omega)$ as a *CTFT pair*:

$$x(t) \overset{F}{\leftrightarrow} X(j\omega). \tag{10}$$

Fourier Series of a Rectangular Pulse Train in the Limit of a Long Period

- In order to illustrate our derivation of the CTFT, we present the example of a periodic rectangular pulse train, as shown in the figure below.



- The periodic pulse train $\tilde{x}(t)$ shown above on the left has period $T_0 = 2\pi / \omega_0$. We compute its CTFS coefficients using the CTFS analysis equation (2) (for mathematical details, see Chapter 3, pages 83-84):

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x}(t)e^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt = \frac{2T_1}{T_0}\text{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\omega_0 T_1}{\pi}\text{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right). \tag{11}$$

Allowing the period T_0 to become infinite, we obtain the single pulse $x(t)$ shown above on the right. Using (4), its CTFT is given by

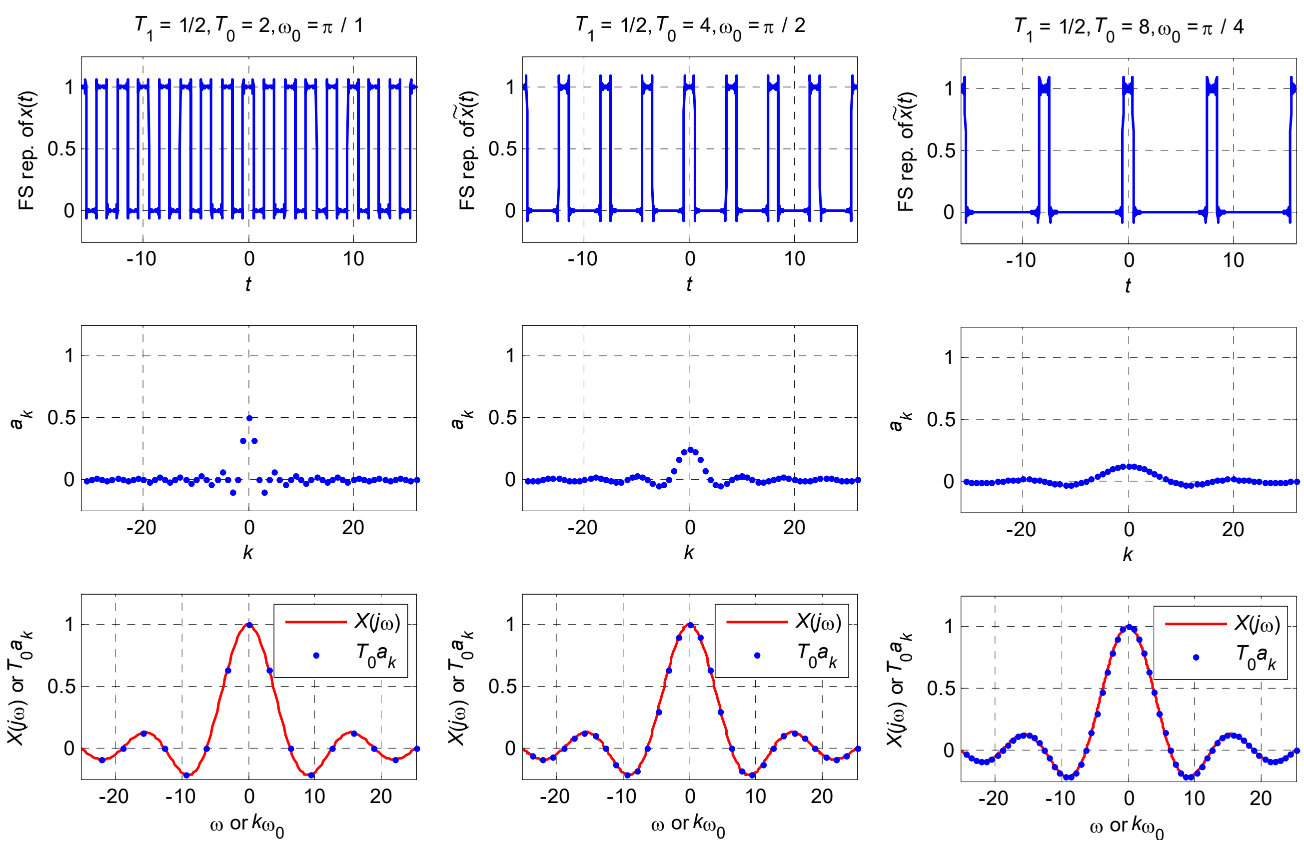
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-T_1}^{T_1} e^{-j\omega t}dt = 2T_1\text{sinc}\left(\frac{\omega T_1}{\pi}\right). \tag{12}$$

By (5), sampling $X(j\omega)$ at $\omega = k\omega_0$ yields the CTFS coefficients of $\tilde{x}(t)$, the a_k, scaled by T_0:

$$X(j\omega)\Big|_{\omega=k\omega_0} = 2T_1 \text{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = T_0 a_k \,. \tag{13}$$

Using $2T_1 / T_0 = \omega_0 T_1 / \pi$, we verify that (13) and (11) provide identical expressions for the a_k.

- The figure below illustrates how the scaled discrete CTFS coefficients $T_0 a_k$ approach the continuous CTFT $X(j\omega)$ as the period T_0 increases. We keep the pulse width $2T_1$ constant and increase the period T_0, thus decreasing the fundamental frequency ω_0. The top row shows the periodic pulse train $\tilde{x}(t)$. The middle row shows its CTFS coefficients a_k vs. k. The bottom row shows the CTFT $X(j\omega)$ vs. ω and the scaled CTFS coefficients $T_0 a_k$ vs. $k\omega_0$, verifying (13). As T_0 increases and ω_0 decreases, the samples $T_0 a_k = X(j\omega)\big|_{\omega=k\omega_0}$ become more closely spaced. In the limit that $T_0 \to \infty$, $\omega_0 \to 0$, the samples reproduce the continuous curve $X(j\omega)$.



Alternate Analysis Method for Continuous-Time Fourier Series

- Our derivation of the CTFT (on pages 146-148) provided an alternate method for computing the CTFS coefficients of periodic signals. We summarize here what we learned in the derivation. Suppose we are given a periodic signal $\tilde{x}(t)$ with period $T_0 = 2\pi / \omega_0$, and wish to find its CTFS coefficients a_k, $-\infty < k < \infty$. We first define an aperiodic signal $x(t)$ that represents one period of $\tilde{x}(t)$:

$$x(t) = \begin{cases} \tilde{x}(t) & t_1 \le t < t_1 + T_0 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

for some t_1. Then we compute the CTFT of the one-period signal $x(t)$ using

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt . \tag{4}$$

Finally, we sample the CTFT $X(j\omega)$ at integer multiples of the fundamental frequency to obtain the CTFS coefficients of the periodic signal $\tilde{x}(t)$:

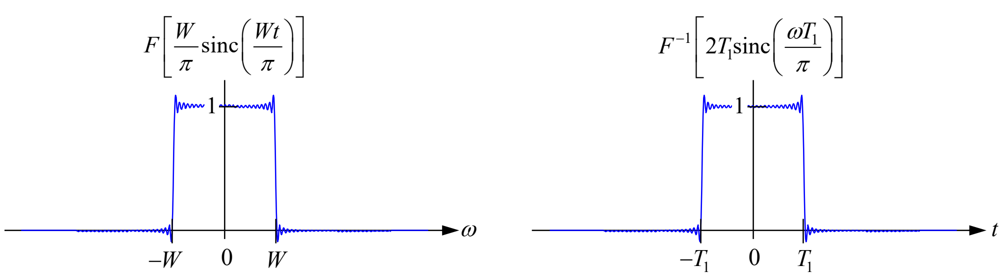
$$a_k = \frac{1}{T_0} X(j\omega)\Big|_{\omega = k\omega_0} . \tag{5}$$

As you become more familiar with the CTFT, you may sometimes find it easier to apply this procedure than to use the CTFS analysis equation (2).

- Perhaps more importantly, our discussion here provides a conceptual linkage between the CTFT and the CTFS. According to (5), every set of CTFS coefficients represents samples of a CTFT. Accordingly, all the properties of the CTFS (Table 1, Appendix) are inherited from properties of the CTFT (Table 3, Appendix). These observations may simplify your learning of Fourier analysis.

Convergence of Continuous-Time Fourier Transform

- The CTFT integral (4) or inverse CTFT integral (7) may not converge for all values of ω or *t*. If $X(j\omega)$ or $x(t)$ have discontinuities, their integral representations will exhibit nonuniform convergence, which will manifest as the Gibbs phenomenon. For example, in the figure below, both the CTFT (on the left) and the inverse CTFT (on the right) are ideally rectangles. Both exhibit the Gibbs phenomenon (made more apparent here by truncating the time signal or CTFT before applying the *F* or F^{-1} operator).



- Here we discuss the convergence of the inverse CTFT. Given a signal $x(t)$ having a CTFT $X(j\omega)$, let us denote the inverse CTFT representation of $x(t)$ by

$$\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega .$$

It can be shown that if $x(t)$ has finite energy (i.e., $x(t)$ is square integrable)

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty ,$$

the energy in the difference between $x(t)$ and $\hat{x}(t)$ vanishes:

$$\int_{-\infty}^{\infty} \left| x(t) - \hat{x}(t) \right|^2 dt = 0 .$$

This does not imply that $\hat{x}(t) = x(t)$ at all t. In fact, $\hat{x}(t)$ differs from $x(t)$ near values of t where $x(t)$ has discontinuities.

- It can be shown that $\hat{x}(t) = x(t)$ except near values of t where $x(t)$ has discontinuities if $x(t)$ satisfies the *Dirichlet conditions*:
  - $x(t)$ is absolutely integrable:

$$\int_{-\infty}^{\infty} \left| x(t) \right| dt < \infty .$$

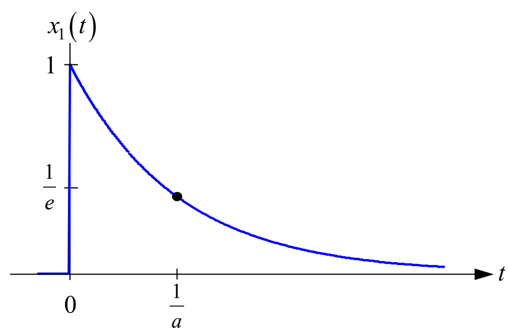
  - $x(t)$ has a finite number of local maxima and minima in any finite interval.
  - $x(t)$ has a finite number of discontinuities in any finite interval.
  - Any discontinuities of $x(t)$ are finite.
- The convergence of the CTFT is entirely analogous to that of the inverse CTFT, owing to the duality between the CTFT integral (4) and the inverse CTFT integral (7) (see page 238 below).
- Many important functions (for example, step functions, constants or sinusoids) are not square integrable or absolutely integrable, but we are still able to compute their CTFTs (or inverse CTFTs) by taking a limit. As we will see shortly, the CTFTs (or inverse CTFTs) we obtain will contain impulse functions.

Examples of Continuous-Time Fourier Transform

1. *Right-sided real exponential*. The signal is given by

$$x_1(t) = e^{-at} u(t), \ a \text{ real}, \ a > 0 ,$$

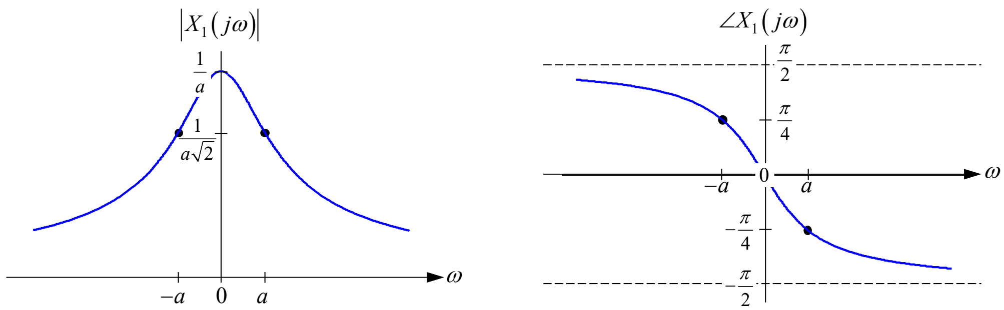
and is shown below.



- We compute its CTFT by evaluating the integral (4):

$$X_1(j\omega) = \int_0^\infty e^{-at} e^{-j\omega t} dt$$
$$= -\frac{1}{a+j\omega} e^{-(a+j\omega)t} \Big|_0^\infty .$$
$$= \frac{1}{a+j\omega}$$

- This complex-valued CTFT is best visualized in terms of magnitude and phase plots, as shown below.



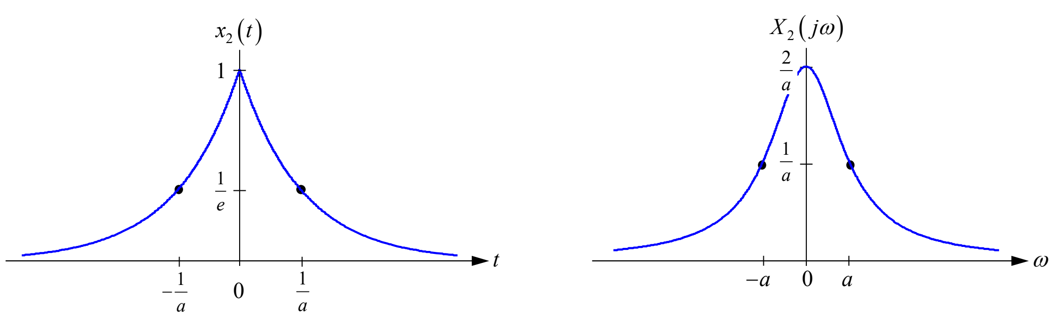
- You may recognize $x_1(t)$ and $X_1(j\omega)$ as the impulse and frequency responses of a first-order lowpass filter that has time constant $\tau = 1/a$ (see Chapter 3, pages 99-100). We will discuss this filter, and several other LTI systems described by differential equations, later in this chapter.
- Observe that:
  - Large values of a describe a signal $x_1(t)$ concentrated in a narrow time span and a CTFT $X_1(j\omega)$ spread out over a wide frequency range.
  - Small values of a describe a signal $x_1(t)$ spread out over a long time span and a CTFT $X_1(j\omega)$ concentrated in a narrow frequency range.

  These observations illustrate the *inverse relationship between time and frequency*, a principle of Fourier analysis we pointed out in Chapter 3. All of the following examples illustrate that principle.

2. *Two-sided real exponential*. The signal is given by

$$x_2(t) = e^{-a|t|}, \ a \text{ real}, \ a > 0 .$$

The signal is shown in the figure below (on the left).



- We compute its CTFT using (4), dividing the integral into two parts, each of a form like the integral we evaluated for $X_1(j\omega)$:

$$\begin{aligned} X_2(j\omega) &= \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt \\ &= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} \\ &= \frac{2a}{a^2+\omega^2} \end{aligned} .$$

The CTFT $X_2(j\omega)$ is purely real, and is shown in the figure above (on the right). Similar to Example 1, large values of *a* correspond to a signal $x_2(t)$ that is concentrated narrowly in time and a CTFT $X_2(j\omega)$ that is spread out widely in frequency (and vice versa).

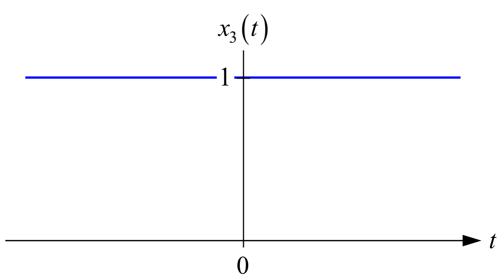
- You may recall that in Chapter 1 (pages 22-23), we encountered a function similar to $X_2(j\omega)$, but as a function of time, not frequency. We used it to represent an *impulse function of time*. In the next example, we will use $X_2(j\omega)$ to represent an *impulse function of frequency*. To help with that example, we compute the value of $x_2(t)$ at $t=0$ by using (7) to compute the inverse CTFT of $X_2(j\omega)$:

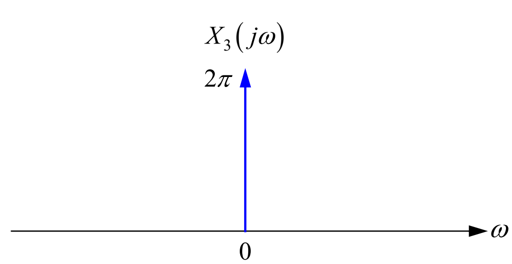
$$\begin{aligned} x_2(0) &= \left. \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(j\omega) e^{j\omega t}\, d\omega \right|_{t=0} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(j\omega)\, d\omega \\ &= 1 \end{aligned} . \qquad (15)$$

3. *Constant*. The signal is given by

$$x_3(t) = 1 \quad \forall t ,$$

and is shown in the figure below (on the left).





- Observe that the two-sided exponential $x_2(t) = e^{-a|t|}$ becomes the constant $x_3(t) = 1$ in the limit that *a* becomes zero:

$$x_3(t) = \lim_{a \to 0} x_2(t).$$

Taking the CTFT of both sides of this expression, we find that

$$\begin{aligned} X_3(j\omega) &= \lim_{a \to 0} X_2(j\omega) \\ &= \lim_{a \to 0} \frac{2a}{a^2 + \omega^2} \end{aligned}.$$

- In this limit, the CTFT $X_2(j\omega)$ has the following properties:
  - A peak of zero width and infinite height at $\omega = 0$.
  - A total area $\int_{-\infty}^{\infty} X_2(j\omega) d\omega = 2\pi x_2(0) = 2\pi$, where we make use of (15).

  Following the same argument as in Chapter 1, pages 22-23, we conclude that $X_3(j\omega)$ is an impulse function of frequency with area 2π:

  $$X_3(j\omega) = 2\pi\delta(\omega).$$

  This is shown in the figure above (on the right).
- In this example, the signal $x_3(t)$ is infinitely wide in time, so its CTFT $X_3(j\omega)$ is infinitely narrow in frequency.

Continuous-Time Fourier Transform in the Limit

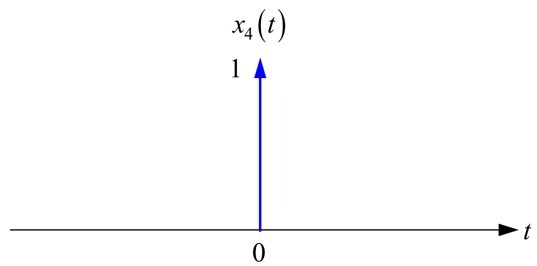
- In Example 3, the constant signal $x_3(t)$ is not absolutely integrable or square integrable, so its CTFT $X_3(j\omega)$ does not exist in the strict sense. Nevertheless, by considering $x_3(t)$ to be the limiting case of a signal whose CTFT does exist, we are able to find an expression for $X_3(j\omega)$, which includes at least one impulse function of frequency. We say that the CTFT $X_3(j\omega)$ *exists in a generalized sense*.
- We can use a similar method to compute generalized CTFTs of other important signals, as in this table.

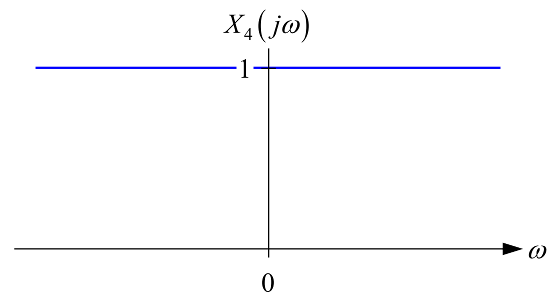
Signal $x(t)$	CTFT $X(j\omega)$
1	$2\pi\delta(\omega)$
$\operatorname{sgn}(t)$	$\frac{2}{j\omega}$
$u(t) = \frac{1}{2}\left[1 + \operatorname{sgn}(t)\right]$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\cos\omega_0 t$	$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
$\sin\omega_0 t$	$\frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$

4. *Unit impulse*. This example is the dual of Example 3. The signal is given by

$$x_4(t) = \delta(t).$$

It is shown in the figure below (on the left).





- We compute its CTFT using (4):

$$\begin{aligned} X_4(j\omega) &= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt \\ &= 1 \quad \forall \omega \end{aligned},$$

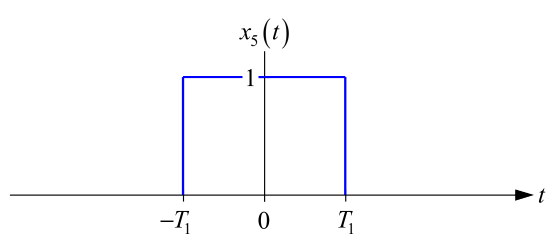
which is shown in the figure above (on the right). We have evaluated the integral using the sampling property of the CT impulse function.

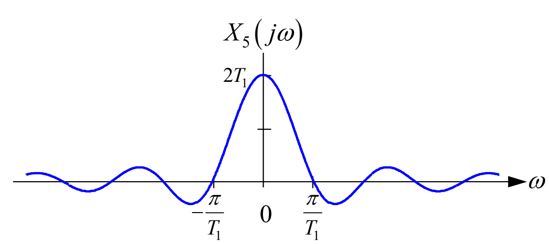
- In this example, the signal $x_4(t)$ is infinitely narrow in time, so its CTFT $X_4(j\omega)$ is infinitely wide in frequency.

5. *Rectangular pulse*. The signal is given by

$$x_5(t) = \Pi\left(\frac{t}{2T_1}\right),$$

and is shown in the figure below (on the left).





- We use (4) to compute its CTFT:

$$\begin{aligned} X_5(j\omega) &= \int_{-\infty}^{\infty} \Pi\left(\frac{t}{2T_1}\right) e^{-j\omega t}\, dt \\ &= \int_{-T_1}^{T_1} e^{-j\omega t}\, dt \\ &= 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right) \end{aligned}. \tag{12}$$

This is shown in the figure above (on the right).

- Note that according to the CTFT integral (4), the value of any CTFT at $\omega = 0$ is given by the area under the corresponding time signal. This observation can be helpful in verifying the correctness of any CTFT we compute. Applying that observation to the present example:

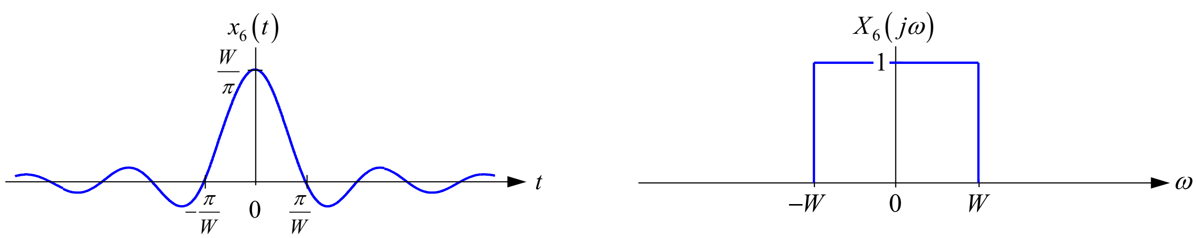
$$\begin{aligned} X_5(j0) &= \int_{-\infty}^{\infty} x_5(t)dt \\ &= 2T_1 \end{aligned}.$$

- Observe that a small value of T_1 corresponds to a signal $x_5(t)$ concentrated in a narrow time span, and corresponds to a CTFT $X_5(j\omega)$ spread out over a wide frequency range (and vice versa).

6. *Sinc function*. This is the dual of Example 5. In this case, it is easiest to start with the CTFT, which is

$$X_6(j\omega) = \Pi\left(\frac{\omega}{2W}\right),$$

and is shown in the figure below (on the right).



- We find the corresponding time signal using the inverse CTFT (7):

$$\begin{aligned} x_6(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Pi\left(\frac{\omega}{2W}\right) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t} d\omega \\ &= \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right) \end{aligned}, \tag{16}$$

which is shown in the figure above (on the left).

- Note that according to the inverse CTFT integral (7), the value of a time signal at $t = 0$ is equal to $1/2\pi$ times the area under the corresponding CTFT. We may use this to check the correctness of any inverse CTFT that we evaluate. In this example, we have

$$\begin{aligned} x_6(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_6(j\omega)d\omega \\ &= \frac{2W}{2\pi} \\ &= \frac{W}{\pi} \end{aligned}.$$

- Finally, observe that a large value of W corresponds to a signal $x_6(t)$ that is concentrated in a narrow time span and a CTFT $X_6(j\omega)$ spread out over a wide frequency range (and vice versa).

Continuous-Time Fourier Transform of Periodic Signals

- In Chapter 3, we saw how to describe a periodic CT signal $x(t)$ by its CTFS coeffcients a_k. As we now show, it is also possible to describe a periodic CT signal $x(t)$ by a generalized CTFT $X(j\omega)$.
- The CTFT of a periodic signal is required to apply Fourier analysis in problems that involve *multiplying* a periodic CT signal by another CT signal (typically aperiodic). Examples of such problems include:
  - *Modulating* a CT signal by a periodic sinusoid. We introduce modulation later in this chapter and discuss it further in Chapter 7.
  - *Sampling* a CT signal to obtain a DT signal. In Chapter 6, we will sometimes model sampling as multiplying the CT signal by a periodic CT impulse train.
- The CTFT of a periodic signal is not needed for problems that involve *convolving* a periodic CT signal with another CT signal (typically aperiodic). We can perform Fourier analysis of such problems using CTFS, as in Chapter 3 (see, e.g., page 95).
- We start by studying an example.

7. *Imaginary exponential*. In this example, as in Example 6, it is easiest to start with the CTFT, which is

$$X_7(j\omega) = 2\pi\delta(\omega - \omega_0),$$

- Using the inverse CTFT (7), we find the corresponding time signal to be

$$\begin{aligned} x_7(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{j\omega t}\,d\omega \\ &= e^{j\omega_0 t} \end{aligned}.$$

  We have evaluated the integral using the sampling property of the impulse function. We have found the CTFT pair:

$$e^{j\omega_0 t} \overset{F}{\leftrightarrow} 2\pi\delta(\omega-\omega_0). \tag{17}$$

- Now we study the *general case* of a periodic signal. Consider a signal $x(t)$ that is periodic with period $T_0 = 2\pi/\omega_0$. We can synthesize $x(t)$ using a CTFS:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \tag{18}$$

  where the a_k, $-\infty < k < \infty$ are the CTFS coefficients for the signal $x(t)$. Now let us compute the CTFT of (18). Since the CTFT is a linear operation, we can compute it term-by-term. Using (17), we obtain

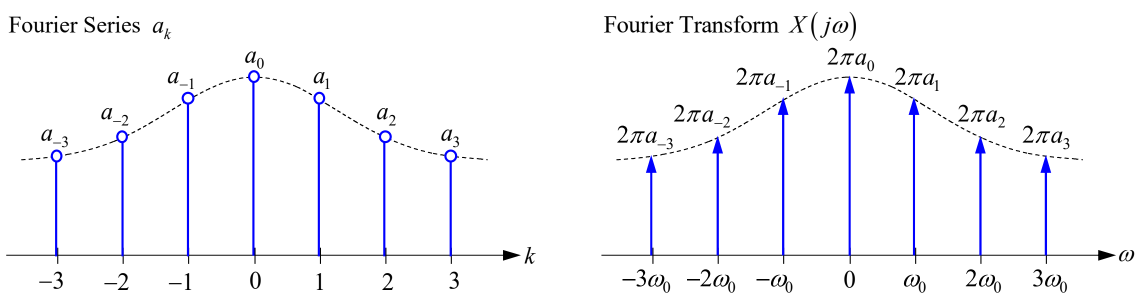
$$X(j\omega) = 2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\omega - k\omega_0). \tag{19}$$

We can summarize our finding as a CTFT pair:

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \stackrel{F}{\leftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - k\omega_0\right). \tag{20}$$

We have found that the CTFT of a periodic signal is a *train of impulses* at frequencies $\omega = k\omega_0$, which are integer multiples of the fundamental frequency ω_0. The impulse at frequency $k\omega_0$ is scaled by 2π times the corresponding CTFS coefficient a_k.

- The relationship between the CTFS coefficients a_k and the CTFT $X\left(j\omega\right)$ for a periodic signal is shown in the figure below.

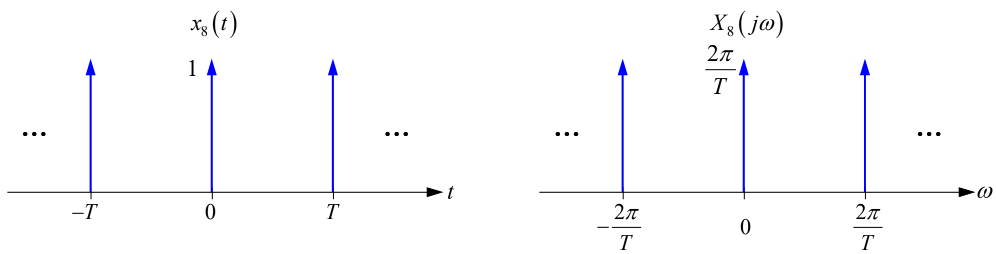


- We close this section by studying an extremely useful example.

8. *Periodic impulse train*. We consider a signal

$$x_8\left(t\right) = \sum_{n=-\infty}^{\infty} \delta\left(t - nT\right), \tag{21}$$

which is shown in the figure below (on the left). The signal $x_8\left(t\right)$ is periodic with period $T_0 = T$ and fundamental frequency $\omega_0 = 2\pi / T$.



- We use the CTFS analysis equation to compute the CTFS coefficients of $x_8\left(t\right)$. Setting the period to be $T_0 = T$ and choosing the integration interval to be $-T/2 \le t < T/2$, we have

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x_8\left(t\right) e^{-jk\omega_o t}\,dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta\left(t\right) e^{-jk\omega_o t}\,dt \\ &= \frac{1}{T} \end{aligned} \tag{22}$$

In the second line of (22), we have used the fact that among all the impulses in $x_8(t)$ given by (21), only the one for $n=0$ lies within the integration interval. In the third line of (22), we have used the sampling property of the CT impulse function.

- Now we use (19) to obtain the CTFT of $x_8(t)$:

$$\begin{aligned} X_8(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0) \\ &= \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right) \end{aligned}. \tag{23}$$

The CTFT $X_8(j\omega)$ is shown in the figure above (on the right). We have found that the *CTFT of a periodic impulse train is a periodic impulse train*.

- Note the inverse relationship between the spacing of the impulses in the time domain, *T*, and the spacing of the impulses in the frequency domain, $2\pi / T$.

- We can summarize the result as a CTFT pair:

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \stackrel{F}{\leftrightarrow} \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right). \tag{24}$$

Expression (24) will be extremely useful in analyzing sampling and reconstruction in Chapter 6.

Properties of Continuous-Time Fourier Transform

- Just like the properties of the CTFS and DTFS that we studied in Chapter 3, these CTFT properties are useful for:
  - Computing the CTFTs for new signals, with minimal effort, by using the CTFTs we already know for other signals.
  - Checking the CTFTs we compute for new signals.
- For a complete list of CTFT properties, see Table 3 in the Appendix. Here, we discuss only a few of the properties.
- We consider one or two signals and their CTFTs. Initially we denote these as

$$x(t) \stackrel{F}{\leftrightarrow} X(j\omega) \quad \text{and} \quad y(t) \stackrel{F}{\leftrightarrow} Y(j\omega).$$

Linearity

- A linear combination of $x(t)$ and $y(t)$ has a CTFT given by the corresponding linear combination of the CTFTs $X(j\omega)$ and $Y(j\omega)$:

$$ax(t) + by(t) \stackrel{F}{\leftrightarrow} aX(j\omega) + bY(j\omega).$$

Time Shift

- A signal time-shifted by t_0 has its CTFT multiplied by a factor $e^{-j\omega t_0}$:

$$x(t-t_0) \overset{F}{\leftrightarrow} e^{-j\omega t_0} X(j\omega). \tag{25}$$

The magnitude and phase of $e^{-j\omega t_0} X(j\omega)$ are related to those of $X(j\omega)$ by

$$\begin{cases} \left|e^{-j\omega t_0} X(j\omega)\right| = \left|X(j\omega)\right| \\ \angle\left(e^{-j\omega t_0} X(j\omega)\right) = \angle X(j\omega) - \omega t_0 \end{cases}. \tag{25$'$}$$

Time-shifting a signal by t_0 affects its CTFT by

- Leaving the magnitude unchanged.
- Adding a phase shift proportional to $-t_0$, which varies linearly with frequency ω.

Proof. Given a time-shifted signal $x(t-t_0)$, we compute its CTFT $F\left[x(t-t_0)\right]$ using (4):

$$F\left[x(t-t_0)\right] = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt .$$

Changing the variable of integration to $\tau = t - t_0$, this becomes

$$\begin{aligned} F\left[x(t-t_0)\right] &= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau+t_0)} d\tau \\ &= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau \\ &= e^{-j\omega t_0} X(j\omega) \end{aligned}$$

QED

Time Scaling

$$x(at) \overset{F}{\leftrightarrow} \frac{1}{|a|} X\left(j\frac{\omega}{a}\right).$$

- Note the inverse relationship between time and frequency:
  - $|a| > 1$ compresses time and expands frequency.
  - $|a| < 1$ expands time and compresses frequency.

Time Reversal

- This is a special case of time scaling with $a = -1$. Reversal in time corresponds to reversal in frequency:

$$x(-t) \overset{F}{\leftrightarrow} X(-j\omega). \tag{26}$$

- If a signal is even in time, its CTFT is even in frequency:

$$x(-t) = x(t) \overset{F}{\leftrightarrow} X(-j\omega) = X(j\omega),$$

while if a signal is odd in time, its CTFT is odd in frequency:

$$x(-t) = -x(t) \overset{F}{\leftrightarrow} X(-j\omega) = -X(j\omega).$$

Conjugation

$$x^*(t) \overset{F}{\leftrightarrow} X^*(-j\omega). \tag{27}$$

Proof. Using (4), the CTFT of $x^*(t)$ is

$$\begin{aligned} F\left[x^*(t)\right] &= \int_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt \\ &= \left(\int_{-\infty}^{\infty} x(t) e^{j\omega t} dt \right)^* . \\ &= X^*(-j\omega) \end{aligned}$$

QED

Conjugate Symmetry for Real Signal

- A real signal $x(t)$ is equal to its complex conjugate $x^*(t)$. In combination with the conjugation property, this implies that

$$x(t) = x^*(t) \overset{F}{\leftrightarrow} X(j\omega) = X^*(-j\omega). \tag{28}$$

If a signal is real, its CTFT is *conjugate symmetric*: the CTFT at positive frequency equals the complex conjugate of the CTFT at negative frequency.

- We can restate the conjugate symmetry property in two ways. First, if a signal is real, the magnitude of its CTFT is even in frequency, while the phase of its CTFT is odd in frequency:

$$x(t) = x^*(t) \overset{F}{\leftrightarrow} \begin{cases} \left|X(j\omega)\right| = \left|X(-j\omega)\right| \\ \angle X(j\omega) = -\angle X(-j\omega) \end{cases}. \tag{28a}$$

Second, if a signal is real, the real part of its CTFT is even in frequency, while the imaginary part of its CTFT is odd in frequency:

$$x(t) = x^*(t) \overset{F}{\leftrightarrow} \begin{cases} \mathrm{Re}\left[X(j\omega)\right] = \mathrm{Re}\left[X(-j\omega)\right] \\ \mathrm{Im}\left[X(j\omega)\right] = -\mathrm{Im}\left[X(-j\omega)\right] \end{cases}. \tag{28b}$$

Real, Even or Real, Odd Signals

- Combining the time reversal and conjugation properties and, we find that

$$x(t) \text{ real and even in } t \overset{F}{\leftrightarrow} X(j\omega) \text{ real and even in } \omega$$

and

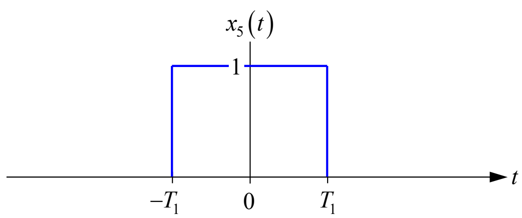
$$x(t) \text{ real and odd in } t \overset{F}{\leftrightarrow} X(j\omega) \text{ imaginary and odd in } \omega \,.$$

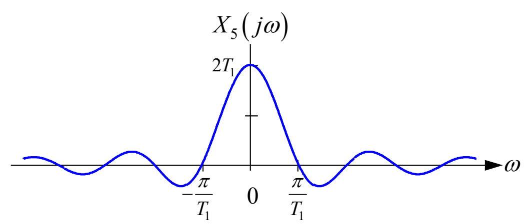
Examples of Symmetry Properties

5. *Real and even signal*. Recall Example 5:

$$x_5(t) = \Pi\left(\frac{t}{2T_1}\right) \overset{F}{\leftrightarrow} X_5(j\omega) = 2T_1 \text{sinc}\left(\frac{\omega T_1}{\pi}\right).$$

The signal and its CTFT are shown again below. The signal is real and even in *t*, and the CTFT is real and even in ω.





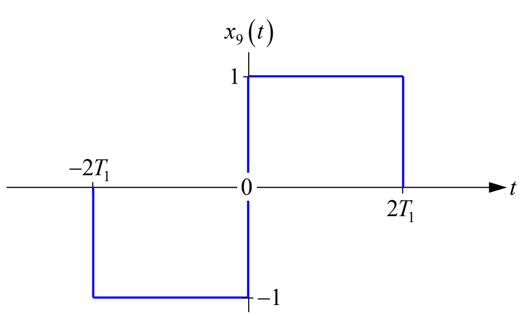
9. *Real and odd signal*. Now consider a signal that is real and odd in *t*:

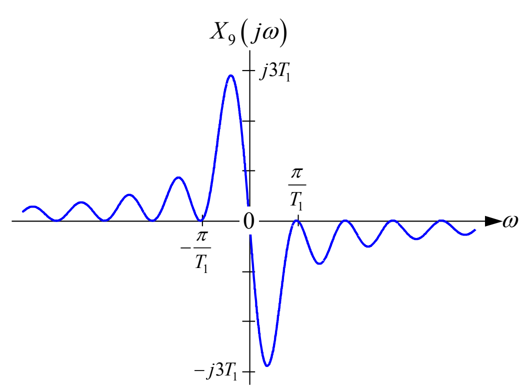
$$x_9(t) = x_5(t - T_1) - x_5(t + T_1)\,.$$

The signal is shown in the figure below (on the left). We compute its CTFT using linearity and the time-shift property:

$$\begin{aligned} X_9(j\omega) &= e^{-j\omega T_1} X_5(j\omega) - e^{j\omega T_1} X_5(j\omega) \\ &= -2j\sin(\omega T_1) X_5(j\omega) \\ &= -4jT_1 \sin(\omega T_1) \text{sinc}\left(\frac{\omega T_1}{\pi}\right) \end{aligned} \,.$$

The CTFT, shown below on the right, is imaginary and odd in ω.



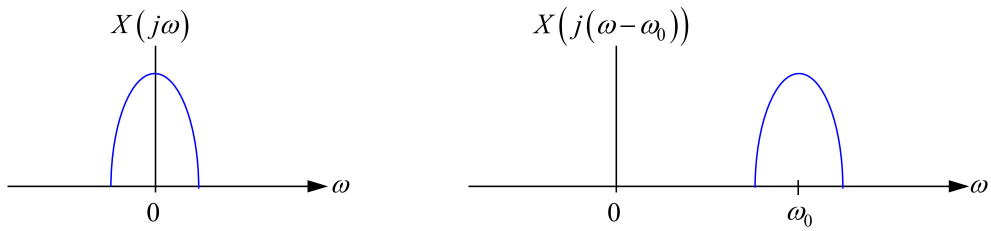


Frequency Shift

- This property is the dual of the time-shift property. It states that a signal multiplied by an imaginary exponential time signal $e^{j\omega_0 t}$ has its CTFT frequency-shifted by ω_0:

$$x(t)e^{j\omega_0 t} \overset{F}{\leftrightarrow} X\left(j\left(\omega-\omega_0\right)\right). \tag{29}$$

  An exemplary CTFT $X(j\omega)$ and the frequency-shifted CTFT $X\left(j\left(\omega-\omega_0\right)\right)$ are shown below.



  Proof. Using (4), the CTFT of $x(t)e^{j\omega_0 t}$ is

$$\begin{aligned} F\left[x(t)e^{j\omega_0 t}\right] &= \int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt \\ &= X\left(j\left(\omega-\omega_0\right)\right) \end{aligned}.$$

  QED

- The frequency-shift property is the basis for amplitude modulation, which we discuss below.

Differentiation in Time

- This property states that *differentiating* a signal in the time domain corresponds to *multiplying* its CTFT by a factor $j\omega$:

$$\frac{dx}{dt} \overset{F}{\leftrightarrow} j\omega X(j\omega). \tag{30}$$

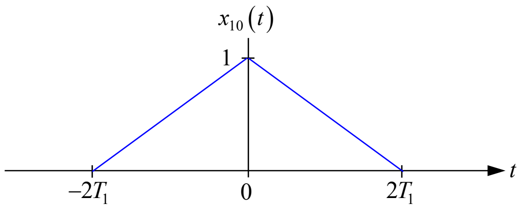
  In order to prove this property, we represent a signal $x(t)$ by an inverse CTFT of $X(j\omega)$, given by (7), and differentiate both sides with respect to *t*.

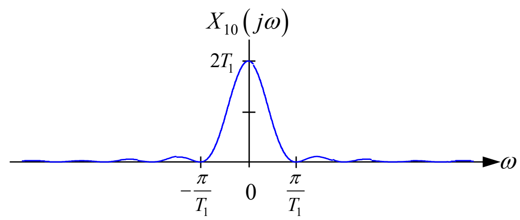
Example of Differentiation Property

10. Triangular pulse. We are given a triangular pulse signal

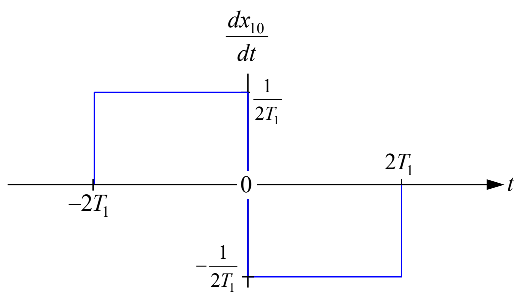
$$x_{10}(t) = \Lambda\left(\frac{t}{2T_1}\right),$$

  which is shown below (on the left).





- In order to compute its CTFT $X_{10}(j\omega)$, we consider the time derivative of the triangular pulse, dx_{10}/dt, which is shown below.



Referring to Example 9 above, we observe that

$$\frac{dx_{10}}{dt} = -\frac{1}{2T_1} x_9(t),$$

from which it follows that

$$\frac{dx_{10}}{dt} \overset{F}{\leftrightarrow} -\frac{1}{2T_1} X_9(j\omega). \tag{31}$$

The differentiation property (30) states that

$$\frac{dx_{10}}{dt} \overset{F}{\leftrightarrow} j\omega X_{10}(j\omega). \tag{32}$$

Combining (31) and (32), we find that

$$j\omega X_{10}(j\omega) = -\frac{1}{2T_1} X_9(j\omega).$$

Solving for $X_{10}(j\omega)$, we find

$$\begin{aligned} X_{10}(j\omega) &= -\frac{1}{2j\omega T_1} X_9(j\omega) \\ &= 2\frac{\sin(\omega T_1)}{\omega}\operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right) \\ &= 2T_1 \frac{\sin\left(\pi \frac{\omega T_1}{\pi}\right)}{\pi \frac{\omega T_1}{\pi}} \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right) \\ &= 2T_1 \operatorname{sinc}^2\left(\frac{\omega T_1}{\pi}\right) \end{aligned}. \tag{33}$$

The CTFT $X_{10}(j\omega)$ is shown in the figure near the top of page 164 (on the right). To summarize, we have found an important CTFT pair:

$$\Lambda\left(\frac{t}{2T_1}\right) \overset{F}{\leftrightarrow} 2T_1 \operatorname{sinc}^2\left(\frac{\omega T_1}{\pi}\right). \tag{34}$$

Integration in Time

- This property states that *integrating* a signal in the time domain corresponds to *dividing* its CTFT by a factor $j\omega$:

$$\begin{aligned} \int_{-\infty}^{t} x(\tau)\,d\tau \overset{F}{\leftrightarrow} & \frac{1}{j\omega} X(j\omega) + \pi X(j\omega)\delta(\omega) \\ & = \frac{1}{j\omega} X(j\omega) + \pi X(j0)\delta(\omega) \end{aligned}. \tag{35}$$

This is intuitively appealing, since differentiating in the time domain corresponds to multiplying the CTFT by $j\omega$. There is, however, an additional term on the right-hand side of (35):

$$\pi X(j\omega)\delta(\omega) = \pi X(j0)\delta(\omega),$$

where the equality follows from the sampling property of the impulse function. This term is nonzero only if the original time-domain signal $x(t)$ has a non-zero d.c. value:

$$X(j0) = \int_{-\infty}^{\infty} x(t)dt \neq 0 .$$

Example of Integration Property

11. *Unit step function*. We are given a unit step signal

$$x_{11}(t) = u(t).$$

Recall Example 4, where we found that the CTFT of an impulse function is a constant:

$$x_4(t) = \delta(t) \overset{F}{\leftrightarrow} X_4(j\omega) = 1 .$$

Recall that the unit step function is the running integral of the impulse function. Applying the integration property, we have

$$\begin{aligned} x_{11}(t) = \int_{-\infty}^{t} x_4(\tau)\,d\tau \overset{F}{\leftrightarrow} X_{11}(j\omega) & = \frac{1}{j\omega} X_4(j\omega) + \pi X_4(j0)\delta(\omega) \\ & = \frac{1}{j\omega} + \pi\delta(\omega) \end{aligned}.$$

In summary, we have found an important CTFT pair:

$$u(t) \overset{F}{\leftrightarrow} \frac{1}{j\omega} + \pi\delta(\omega).$$

Differentiation in Frequency

- This property is the dual of the differentiation-in-time property we discussed earlier. It states that

$$tx(t) \overset{F}{\leftrightarrow} j\frac{dX(j\omega)}{d\omega},$$

i.e., multiplication of a signal by time *t* corresponds to differentiation of its CTFT with respect to frequency ω (and scaling by a factor *j*). In order to prove this property, we differentiate the analysis equation (4) with respect to ω, which shows that $dX(j\omega)/d\omega$ is the CTFT of a signal $-jtx(t)$.

Parseval's Identity

- The importance of inner products between signals is explained on pages 91-93 above. Parseval's identity for the CTFT allows us to compute the inner product between two CT signals, or the energy of one CT signal, either in the time or frequency domain. The computation of an inner product or energy for any particular signal(s) is often much easier in one domain or the other.

Inner Product Between Signals

- The general form of Parseval's identity, for an *inner product between two CT signals*, is

$$\langle x(t), y(t)\rangle = \int_{-\infty}^{\infty} x(t)y^*(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)Y^*(j\omega)\,d\omega \ . \qquad (36)$$

The middle expression in (36) is an inner product between two signals, $x(t)$ and $y(t)$, computed in the time domain. The rightmost expression in (36) is an inner product between the corresponding CTFTs, $X(j\omega)$ and $Y(j\omega)$, computed in the frequency domain.

Proof

We start with the middle expression in (36) and represent $x(t)$ by the inverse CTFT of $X(j\omega)$:

$$\int_{-\infty}^{\infty} x(t)y^*(t)\,dt = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega\right]y^*(t)\,dt \ .$$

We interchange the order of integration and recognize the quantity in square brackets as $Y(j\omega)$:

$$\begin{aligned}\int_{-\infty}^{\infty} x(t)y^*(t)\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\left[\int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt\right]^* d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)Y^*(j\omega)d\omega\end{aligned} \ .$$

QED

Signal Energy

- Now we consider the special case of (36) with $x(t) = y(t)$ and $X(j\omega) = Y(j\omega)$. We obtain an expression for the *energy of a CT signal*:

$$E=\int_{-\infty}^{\infty}\left|x(t)\right|^2 dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|X(j\omega)\right|^2 d\omega \ . \tag{37}$$

- Based on (37), we can identify $\left|X(j\omega)\right|^2$ as the *energy density spectrum* of the signal $x(t)$, since $\left|X(j\omega)\right|^2$ quantifies the energy contained in the component of the signal at a frequency ω. We can interpret the rightmost expression in (37) as the integral of the energies contained in all frequencies ω, $-\infty<\omega<\infty$.

Example: Energy of Sinc Function

- We would like to determine the energy of the signal studied in Example 6:

$$\frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)\overset{F}{\leftrightarrow}\Pi\left(\frac{\omega}{2W}\right).$$

In order to compute the energy in the time domain, we must evaluate the integral

$$E=\left(\frac{W}{\pi}\right)^2\int_{-\infty}^{\infty}\operatorname{sinc}^2\left(\frac{Wt}{\pi}\right)dt,$$

which is difficult. Using Parseval's identity (37), we find it far easier to compute the energy in the frequency domain:

$$\begin{aligned}E&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Pi^2\left(\frac{\omega}{2W}\right)d\omega\\&=\frac{1}{2\pi}\int_{-W}^{W}d\omega\\&=\frac{W}{\pi}\end{aligned}\quad .$$

Convolution Property

- Given two signals and their CTFTs

$$p(t)\overset{F}{\leftrightarrow}P(j\omega) \text{ and } q(t)\overset{F}{\leftrightarrow}Q(j\omega),$$

the convolution property states that

$$p(t)*q(t)\overset{F}{\leftrightarrow}P(j\omega)Q(j\omega), \tag{38}$$

i.e., *convolution in the time domain* corresponds to *multiplication in the frequency domain*.

Proof. We express the convolution as an integral and compute its CTFT using (4):

$$F\left[p(t)*q(t)\right]=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}p(\tau)q(t-\tau)d\tau\right]e^{-j\omega t}dt.$$

Now we interchange the order of integration:

$$F\left[p(t)*q(t)\right]=\int_{-\infty}^{\infty}p(\tau)\left[\int_{-\infty}^{\infty}q(t-\tau)e^{-j\omega t}dt\right]d\tau .$$

We recognize the quantity in square brackets as the CTFT of $q(t-\tau)$. Using the time-shift property (25), this is $Q(j\omega)e^{-j\omega\tau}$. Hence, we have

$$F\left[p(t)*q(t)\right]=Q(j\omega)\int_{-\infty}^{\infty}p(\tau)e^{-j\omega\tau}d\tau .$$

We recognize the integral as $P(j\omega)$, the CTFT of $p(t)$. We have proven (38).

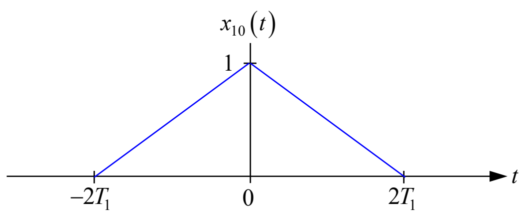
QED

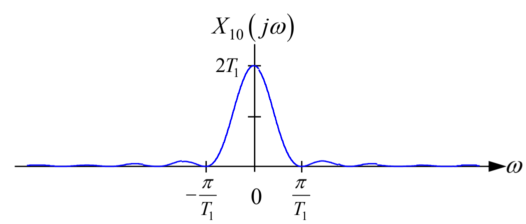
Example of Convolution Property

10. Triangular pulse (alternate approach). Once again we consider a triangular pulse signal

$$x_{10}(t)=\Lambda\left(\frac{t}{2T_1}\right),$$

which is shown again below (on the left).

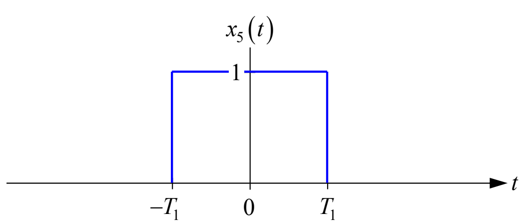


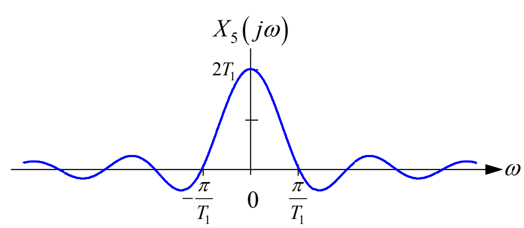


- Recall Example 5:

$$x_5(t)=\Pi\left(\frac{t}{2T_1}\right)\stackrel{F}{\leftrightarrow}X_5(j\omega)=2T_1\text{sinc}\left(\frac{\omega T_1}{\pi}\right).$$

The signal and its CTFT are shown again below.





- Observe that

$$x_{10}(t)=\frac{1}{2T_1}x_5(t)*x_5(t) .$$

Using the convolution property (38), we find that

$$
\begin{aligned}
X_{10}(j\omega) &= \frac{1}{2T_1} X_5(j\omega) X_5(j\omega) \\
&= 2T_1 \text{sinc}^2\left(\frac{\omega T_1}{\pi}\right)
\end{aligned} ,
$$

which is shown in the middle of page 168 (on the right). This agrees with (33) and (34), which we found previously using the differentiation property. The convolution property provides an easier method for computing this particular CTFT.

Frequency Response of Continuous-Time Linear Time-Invariant Systems

- The most important application of the convolution property is to filtering signals by LTI systems. Consider a CT LTI system having an impulse response $h(t)$. As shown in Chapter 3, the system frequency response $H(j\omega)$ is given by the CTFT of $h(t)$ (see (4), page 77). The impulse response and frequency response of a CT LTI system *form a CTFT pair*:

$$
h(t) \overset{F}{\leftrightarrow} H(j\omega). \tag{39}
$$

- Suppose an input signal $x(t) \overset{F}{\leftrightarrow} X(j\omega)$ is passed into the system, yielding an output signal $y(t) \overset{F}{\leftrightarrow} Y(j\omega)$, as shown below.

$$
x(t) \overset{F}{\leftrightarrow} X(j\omega) \longrightarrow \boxed{h(t) \overset{F}{\leftrightarrow} H(j\omega)} \longrightarrow y(t) \overset{F}{\leftrightarrow} Y(j\omega)
$$

  The output signal is obtained in the *time domain* by *convolving* the input signal and the impulse response:

$$
y(t) = h(t) * x(t). \tag{40}
$$

  By the convolution property (38), the CTFT of the output signal (40) is given by the right-hand side of

$$
y(t) = h(t) * x(t) \overset{F}{\leftrightarrow} Y(j\omega) = H(j\omega) X(j\omega). \tag{41}
$$

  According to (41), in the *frequency domain*, the CTFT of the output signal is found by *multiplying* the CTFT of the input signal by the frequency response of the LTI system. This view of LTI filtering as frequency-domain multiplication is intuitively appealing. In many problems, it provides a far easier method of solution than time-domain convolution.

- By (39), the frequency response $H(j\omega)$ is the CTFT of the impulse response $h(t)$. As a consequence, all the frequency response properties studied in Chapter 3 can be understood in terms of the CTFT properties studied in this chapter. For example, if the impulse response $h(t)$ is real, then by (28), the frequency response has conjugate symmetry:

$$h(t) = h^*(t) \overset{F}{\leftrightarrow} H(j\omega) = H^*(-j\omega). \tag{28c}$$

Examples of Frequency Responses

1. Time shift. Consider an LTI system such that the output is a time-shifted version of the input:

$$y(t) = x(t - t_0) = h(t) * x(t). \tag{42}$$

Knowing this is an LTI system, we have expressed the input-output relation as a convolution with an impulse response $h(t)$, although we need not consider an explicit formula for $h(t)$.

- We compute the CTFT of (42) using the time-shift property (25):

$$Y(j\omega) = e^{-j\omega t_0} X(j\omega) = H(j\omega) X(j\omega).$$

We find that the frequency response is

$$H(j\omega) = e^{-j\omega t_0}, \tag{43}$$

which confirms the result found in (42) in Chapter 3, page 98.

- Alternatively, we could use the impulse response given by (23) in Chapter 2, page 53:

$$h(t) = \delta(t - t_0),$$

and evaluate its CTFT using the sampling property of the impulse function, thereby obtaining (43).

2. Integrator. Consider a causal LTI system whose output is a running integral of the input:

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau = h(t) * x(t). \tag{44}$$

We have also expressed the output as a convolution of the input with an impulse response $h(t)$.

- We compute the CTFT of (44) using the integration-in-time property (35):

$$Y(j\omega) = \frac{1}{j\omega} X(j\omega) + \pi X(j\omega) \delta(\omega) = H(j\omega) X(j\omega).$$

We find the frequency response of the integrator to be

$$H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega). \tag{45}$$

Alternatively, we could use the impulse response of the integrator:

$$h(t) = u(t), \tag{46}$$

and evaluate its CTFT to obtain (45).

- Note that because the impulse response (46) is not absolutely integrable

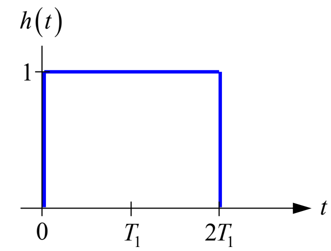
$$\int_{-\infty}^{\infty} \left|h(t)\right| dt = \infty ,$$

the frequency response (45) contains an impulse function, and exists only in the generalized sense.

3. *Finite-time integrator*. Consider a causal LTI system that has an impulse response

$$h(t) = u(t) - u(t - 2T_1) = \Pi\left(\frac{t - T_1}{2T_1}\right) = \Pi\left(\frac{t}{2T_1} - \frac{1}{2}\right), \tag{47}$$

which is shown below.



Given an input signal $x(t)$, the output signal is

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= \int_{-\infty}^{t} x(\tau)\, d\tau - \int_{-\infty}^{t-2T_1} x(\tau)\, d\tau . \\ &= \int_{t-2T_1}^{t} x(\tau)\, d\tau \end{aligned} \tag{48}$$

- We could obtain its frequency response by evaluating the CTFT of the input-output relation (48), using the integration-in-time property (35). Here, we evaluate the CTFT of the impulse response (47) using the CTFT of the rectangular pulse (12) and the time-shift property (25), obtaining

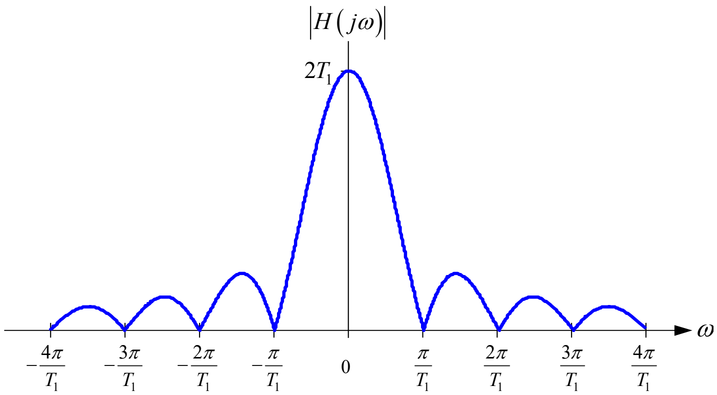
$$H(j\omega) = 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right) e^{-j\omega T_1} . \tag{49}$$

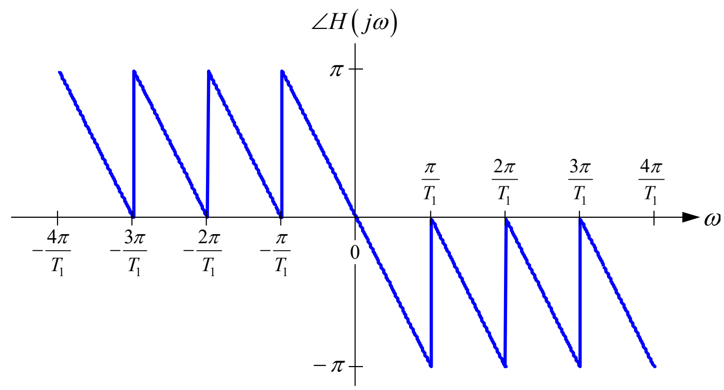
- The frequency response (49) is complex-valued. It is best visualized in plots of its magnitude and phase. Using the product rule (see Appendix, page 289), we find them to be

$$\left|H(j\omega)\right| = \left|2T_1\right| \left|\operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)\right| \left|e^{-j\omega T_1}\right| = 2T_1 \left|\operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)\right| .$$

$$\angle H(j\omega) = \angle 2T_1 + \angle \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right) + \angle e^{-j\omega T_1} = \begin{cases} 0 + k2\pi - \omega T_1 & \operatorname{sinc}(\omega T_1 / \pi) > 0 \\ \pi + k2\pi - \omega T_1 & \operatorname{sinc}(\omega T_1 / \pi) < 0 \end{cases} .$$

These are plotted below. For a detailed explanation of how to plot these, see the Appendix, pages 300-301.





We observe the following:

- The magnitude $|H(j\omega)|$ (on the left) reveals that the finite-time integrator is a lowpass filter, but it is not ideal, as its magnitude falls off slowly as $|\omega|$ increases.
- The phase $\angle H(j\omega)$ (on the right) exhibits jumps of π radians at values of ω where the sinc function in (49) passes through zero and changes sign. Away from these zeros of the sinc function, the phase is linear, with a negative slope corresponding to a group delay $-d\angle H(j\omega)/d\omega = T_1$, which is the average delay of the impulse response (47). The magnitude and phase were plotted by MATLAB, which automatically added multiples of 2π to the phase to keep it in the interval $[-\pi,\pi]$ (this is explained fully in the Appendix, pages 300-301).

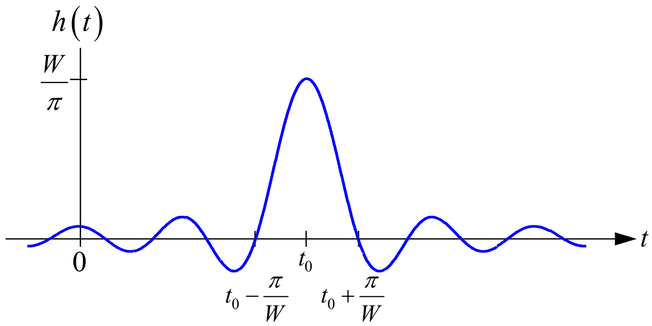
4. *Ideal lowpass filter*. An *ideal lowpass filter* with *cutoff frequency* W and *group delay* t_0 has the following characteristics:

- *Passband*: for $|\omega| < W$, the magnitude $|H(j\omega)|$ is constant, and the phase $\angle H(j\omega)$ is a linear function of ω, $\angle H(j\omega) = -\omega t_0$.
- *Transition*: at $|\omega| = W$, the magnitude $|H(j\omega)|$ goes abruptly to zero.
- *Stopband*: for $|\omega| > W$, the magnitude $|H(j\omega)|$ is zero. (Since the magnitude is zero, the phase can assume any value.)

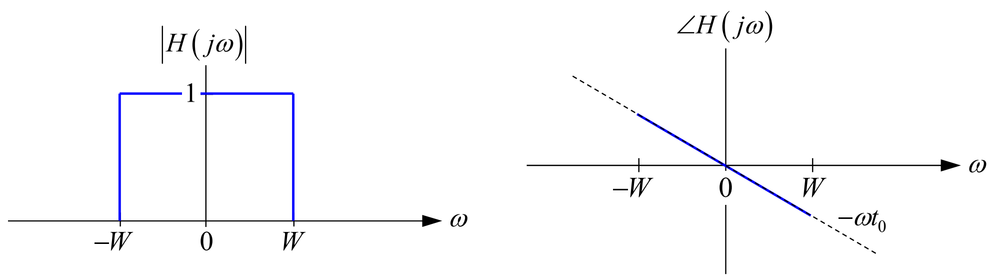
- Using (16) and the time-shift property (25), an ideal lowpass filter has impulse response and frequency response given by

$$h(t) = \frac{W}{\pi}\text{sinc}\left(\frac{W(t-t_0)}{\pi}\right) \overset{F}{\leftrightarrow} H(j\omega) = \Pi\left(\frac{\omega}{2W}\right)e^{-j\omega t_0}. \tag{50}$$

- The impulse response is shown below. Observe that the impulse response is peaked at $t = t_0$, but has tails that extend to $t = \pm\infty$. Hence, an ideal lowpass filter cannot be causal, except in the limit that the group delay becomes infinite, $t_0 \to \infty$.



- The magnitude and phase responses are shown below.



We observe that:

- The magnitude $\left|H(j\omega)\right|$ (on the left) is ideal, as desired.
- The phase $\angle H(j\omega)$ (on the right) is linear in the passband, with a slope corresponding to a constant group delay $-d\angle H(j\omega)/d\omega = t_0$. This plot was made manually, so no multiples of 2π were added to the phase to keep it in the interval $[-\pi,\pi]$.

- We will use ideal lowpass, bandpass and highpass filters throughout EE 102A and 102B. To simplify the analyses, we often set the group delay t_0 to zero. Nevertheless, any causal filter that attempts to approximate the abrupt transition of an ideal filter must have a substantial group delay.

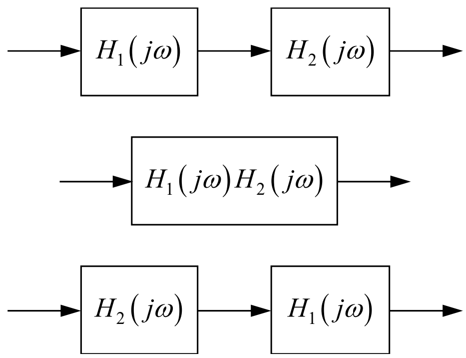
Frequency Response of Cascaded Linear Time-Invariant Systems

- Consider two LTI systems

$$h_1(t) \overset{F}{\leftrightarrow} H_1(j\omega) \quad \text{and} \quad h_2(t) \overset{F}{\leftrightarrow} H_2(j\omega).$$

- Recall from Chapter 2 that when two LTI systems are cascaded, the overall impulse response of the cascade is the convolution of the impulse responses of the two systems, and does not depend on the order in which the two systems are cascaded (see pages 61-62).

- By the convolution property of the CTFT, the overall frequency response of the cascade is the product of the frequency responses of the two systems, and does not depend on the order in which the two systems are connected.
- Hence, the following three LTI systems yield identical input-output relationships.



Multiplication Property

- This property is the dual of the convolution property. Given two signals and their CTFTs

$$p(t)\overset{F}{\leftrightarrow}P(j\omega) \text{ and } q(t)\overset{F}{\leftrightarrow}Q(j\omega),$$

  the multiplication property states that

$$p(t)q(t)\overset{F}{\leftrightarrow}\frac{1}{2\pi}P(j\omega)*Q(j\omega), \tag{51}$$

  i.e., *multiplication in the time domain* corresponds to *convolution in the frequency domain*. Note the factor $1/2\pi$ on the right-hand side of (51), which is not present in the convolution property (38). The proof of (51) is not given here, as it is very similar to the proof of (38), but with the time and frequency variables, t and ω, interchanged.

- We will study several important applications of the multiplication property:
  - Modulation and demodulation (Chapters 4 and 7).
  - Sampling and reconstruction (Chapter 6).
  - Windowing (studied using the DTFT multiplication property in Chapter 6).

Modulation

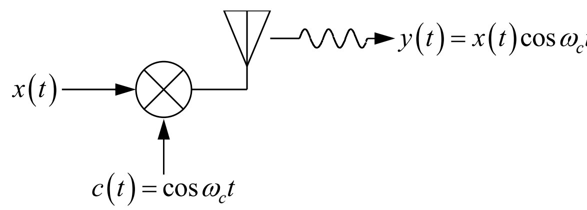
- *Modulation* is a process of embedding an information-bearing *message signal* into another signal called a *carrier signal* in order to create a *modulated signal*. The carrier signal is often a sinusoid at a *carrier frequency* ω_c chosen so the modulated signal can propagate as a wave through a communication medium. Carrier frequencies range from about 1 MHz (broadcast AM radio) to hundreds of THz (optical fiber). We will study various forms of modulation in Chapter 7.

Amplitude Modulation

- In the simplest form of modulation, a message signal $x(t)$ is multiplied by a sinusoidal carrier signal $c(t)=\cos\omega_c t$ to form a modulated signal

$$\begin{aligned} y(t) &= x(t)c(t) \\ &= x(t)\cos\omega_c t \end{aligned}. \tag{52}$$

For now, we will refer to this modulation scheme as *amplitude modulation* (AM). In Chapter 7, we will call it *double-sideband amplitude modulation with suppressed carrier* (DSB-AM-SC) to distinguish it from other related modulation schemes. The system for performing AM, which we may refer to as a *modulator*, is illustrated below. The modulated signal $y(t)$ is shown being radiated by an antenna.



- We would like to compute the spectrum of the modulated signal $y(t)$ using the CTFT. One approach is to write the carrier signal as

$$\begin{aligned} c(t) &= \cos\omega_c t \\ &= \frac{1}{2}e^{j\omega_c t}+\frac{1}{2}e^{-j\omega_c t} \end{aligned} \tag{53}$$

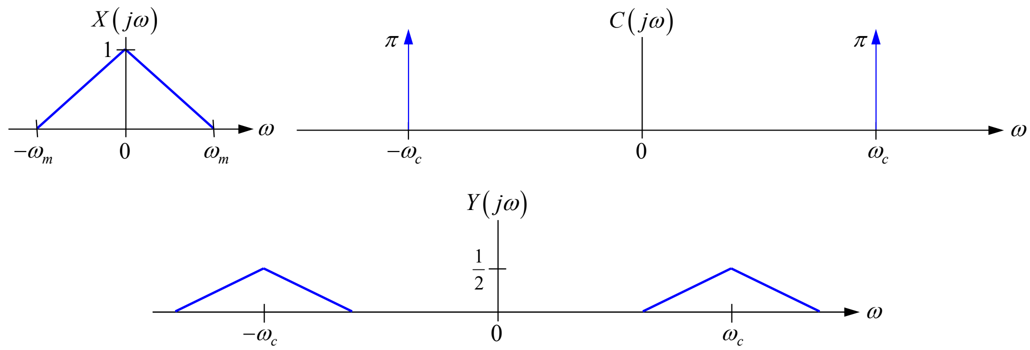
and then apply the frequency-shifting property (29) for each of the two imaginary exponential signals in (53). Instead, we use the CTFT of the carrier signal

$$C(j\omega)=\pi\delta(\omega-\omega_c)+\pi\delta(\omega+\omega_c) \tag{54}$$

and the multiplication property (51) to obtain the spectrum of the modulated signal $y(t)=x(t)c(t)$:

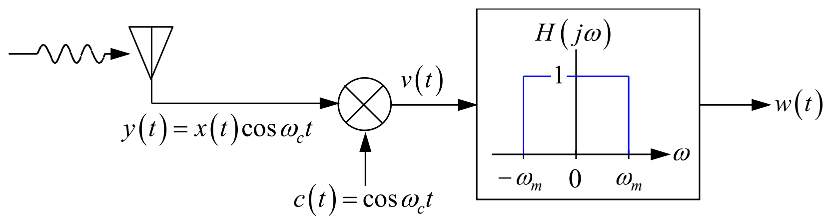
$$\begin{aligned} Y(j\omega) &= \frac{1}{2\pi}X(j\omega)*C(j\omega) \\ &= \frac{1}{2}X\left(j(\omega-\omega_c)\right)+\frac{1}{2}X\left(j(\omega+\omega_c)\right) \end{aligned}. \tag{55}$$

The figure below shows the message signal spectrum $X(j\omega)$, which we assume is nonzero only for $|\omega|<\omega_m$, and the carrier signal spectrum $C(j\omega)$, given by (54). The figure also shows the modulated signal spectrum $Y(j\omega)$, given by (55), which comprises copies of the message signal spectrum $X(j\omega)$, shifted in frequency to $\pm\omega_c$ and scaled by 1/2.



Synchronous Demodulation

- The figure below shows a *demodulator*, a system that receives the AM signal and recovers the message signal. It is called a *synchronous demodulator*, since it must create a replica of the carrier signal $c(t)$ whose frequency and phase are synchronized to the carrier used in the modulator.



- In the demodulator, the first step is to multiply the modulated signal $y(t)$ by the replica carrier signal $c(t)$. The signal obtained at the multiplier output is

$$\begin{aligned} v(t) &= y(t)c(t) \\ &= x(t)\cos^2\omega_c t \\ &= \frac{1}{2}x(t)+\frac{1}{2}x(t)\cos 2\omega_c t \end{aligned} . \tag{56}$$

We have used the trigonometric identity $\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$ to obtain the third line of (56). The multiplier output signal $v(t)$, given by (56), comprises two terms. The first term, $\frac{1}{2}x(t)$, is a scaled replica of the message signal. The second term is that scaled message signal modulated onto a sinusoidal carrier at a frequency $2\omega_c$, twice the original carrier frequency. In the demodulator, the multiplier output signal $v(t)$ is passed to an ideal lowpass filter with unity passband gain and cutoff frequency ω_m, which passes the first term and blocks the second term. The lowpass filter output signal is thus a scaled replica of the message signal, as desired:

$$w(t)=\frac{1}{2}x(t). \tag{57}$$

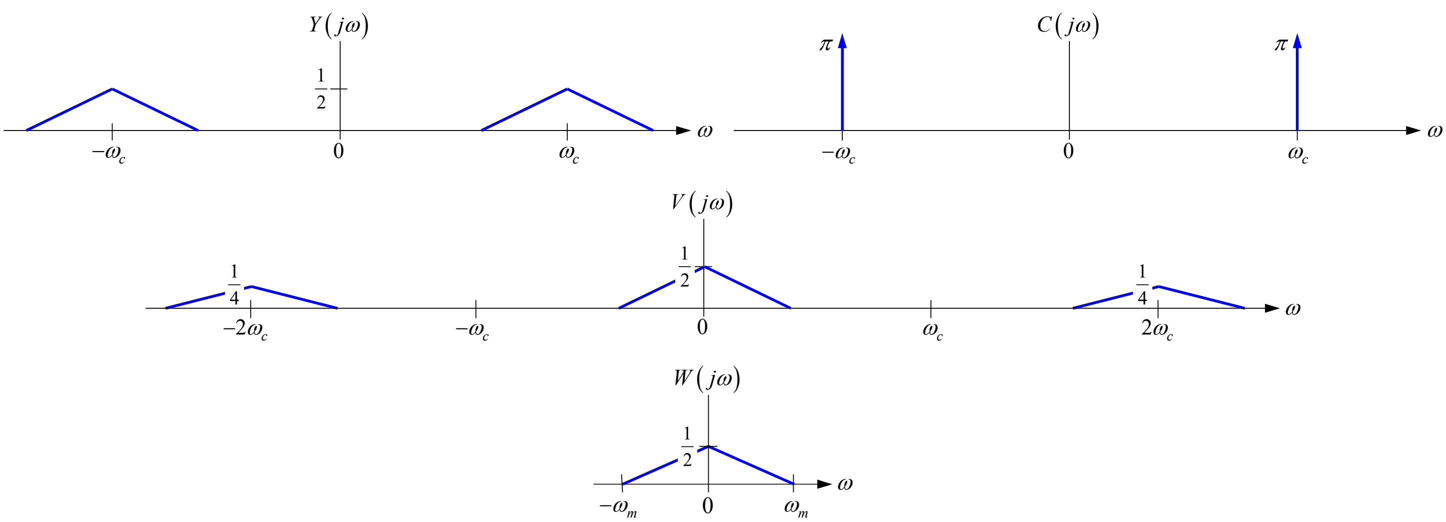
- It is instructive to study the demodulation process in the frequency domain. Using (54) and (55) and the multiplication property (51), the CTFT of the multiplier output $v(t)=y(t)c(t)$ is

$$
\begin{aligned}
V(j\omega) &= \frac{1}{2\pi} Y(j\omega) * C(j\omega) \\
&= \frac{1}{2\pi}\left[\frac{1}{2} X\left(j\left(\omega-\omega_c\right)\right)+\frac{1}{2} X\left(j\left(\omega+\omega_c\right)\right)\right] * \left[\pi\delta\left(\omega-\omega_c\right)+\pi\delta\left(\omega+\omega_c\right)\right] . \\
&= \frac{1}{4} X\left(j\left(\omega-2\omega_c\right)\right)+\frac{1}{2} X(j\omega)+\frac{1}{4} X\left(j\left(\omega+2\omega_c\right)\right)
\end{aligned} \tag{58}
$$

In (58), the convolution in the second line yields four terms, two of which add up to yield the middle term in the third line. In the figure below, the modulated signal spectrum $Y(j\omega)$ and the replica carrier signal spectrum $C(j\omega)$ are shown. The multiplier output spectrum $V(j\omega)$ shown may be visualized as $1/2\pi$ times the convolution between $Y(j\omega)$ and $C(j\omega)$, which yields scaled copies of the message spectrum $X(j\omega)$ centered at frequencies $\omega=0$ and $\omega=\pm 2\omega_c$. Finally, the lowpass filter output spectrum $W(j\omega)$ contains only the scaled message spectrum:

$$
\begin{aligned}
W(j\omega) &= H(j\omega) V(j\omega) \\
&= \frac{1}{2} X(j\omega)
\end{aligned} . \tag{59}
$$

Expression (59) agrees with the result of the time-domain analysis, given by (57).



Linear Time-Invariant Systems Governed by Constant-Coefficient Differential Equations

- In this section, we consider the class of causal LTI CT systems that can be described by a constant-coefficient linear differential equation of the form

$$
\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} . \tag{60}
$$

In (60), $x(t)$ and $y(t)$ denote the input and output signals. The a_k, $k = 0,\ldots,N$ and b_k, $k = 0,\ldots,M$ are constants, which are real-valued in systems that map real inputs to real outputs. We studied this class of LTI systems in Chapter 2 (see pages 67-71).

- Here we describe a method for computing the frequency response of a system described by (60). The method is equivalent to one introduced in Chapter 3 (see Method 2, pages 98-99), and we simply restate it here more formally in terms of the CTFT. In order for the method to be applicable, we require that the impulse response $h(t)$ of the system satisfy the condition of absolute integrability

$$\int_{-\infty}^{\infty} \left|h(t)\right| dt < \infty , \tag{61}$$

such that the system is BIBO stable and the frequency response $H(j\omega)$ exists in the strict sense. For example, consider the integrator discussed on pages 170-171. Although the integrator can be described by (60) with two nonzero coefficients, $a_1 = b_0 = 1$, the impulse response (46) does not satisfy (61), and the frequency response (45) exists only in a generalized sense.

- Assuming (61) is satisfied, the system input-output relation can be described in the time or frequency domain by

$$y(t) = h(t) * x(t) \overset{F}{\leftrightarrow} Y(j\omega) = H(j\omega) X(j\omega) . \tag{41}$$

We can solve the frequency-domain part of (41) to obtain an expression for the frequency response

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} . \tag{62}$$

Given an input signal $x(t) \overset{F}{\leftrightarrow} X(j\omega)$, if we are able to determine through some means the output signal $y(t) \overset{F}{\leftrightarrow} Y(j\omega)$ generated by that input, then we can apply (62) to determine the frequency response $H(j\omega)$ at all frequencies at which $X(j\omega) \neq 0$.

- Now we consider the differential equation (60) and compute its CTFT. We exploit the linearity property to compute the CTFT term-by-term. In order to compute the CTFT of the k^{th} derivative, we apply the differentiation-in-time property (30) k times to obtain

$$\frac{d^k x}{dt^k} \overset{F}{\leftrightarrow} (j\omega)^k X(j\omega) . \tag{63}$$

We find the CTFT of (60) is

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega) . \tag{64}$$

Factoring out the $Y(j\omega)$ and $X(j\omega)$ on the left- and right-hand sides of (64), solving for $Y(j\omega)/X(j\omega)$ and using (62), we obtain an expression for the frequency response:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}. \quad (65)$$

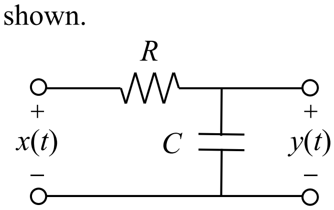
- We have found that for any LTI system described by a differential equation of the form (60), the frequency response (assuming it exists in the strict sense) is given by the ratio of two polynomials in powers of $j\omega$. Such a ratio of two polynomials is called a *rational function* of $j\omega$. The coefficients in the numerator and denominator polynomials in (65) are the same coefficients b_k, $k = 0,\ldots,M$ and a_k, $k = 0,\ldots,N$ that appear in the differential equation (60). In other words, given a differential equation in the form (60), we can find the frequency response (65) *by inspection*. Conversely, given a frequency response in rational form (65), we can find the corresponding differential equation *by inspection*.
- The method derived here is extremely useful in analyzing LTI systems, as we demonstrate through the following examples.

Examples of LTI Systems Governed by Constant-Coefficient Differential Equations

- In each of the following examples, we compute the frequency response $H(j\omega)$ directly from the differential equation. In some examples, we also state the impulse response $h(t)$ and step response $s(t)$ without derivation. While the frequency response could be obtained by evaluating the CTFT of the impulse response, that entails more work than the approach we use here. For more details, especially how to solve the differential equations for the impulse and step responses using Laplace transforms, see the *EE 102B Course Reader*, Chapter 5.
- Our previous plots used linear scales for $|H(j\omega)|$, $\angle H(j\omega)$ and ω. Here we use logarithmic scales for $|H(j\omega)|$ and ω, while using a linear scale for $\angle H(j\omega)$. These choices highlight the asymptotic behavior of $|H(j\omega)|$ and $\angle H(j\omega)$ at low and high frequencies. We show only positive ω, since $|H(j\omega)|$ and $\angle H(j\omega)$ are even and odd functions of ω, respectively for these systems with real impulse responses. We also describe the group delay $-d\angle H(j\omega)/d\omega$. Note that because of the logarithmic frequency scale, the group delay does not correspond to the slope of the plots, which is $-d\angle H(j\omega)/d\log(\omega)$.

First-Order Lowpass Filter

- This can be realized by the circuit shown.



Defining $RC = \tau$, the circuit is described by a first-order differential equation

$$\frac{dy}{dt} + \frac{1}{\tau} y(t) = \frac{1}{\tau} x(t).$$

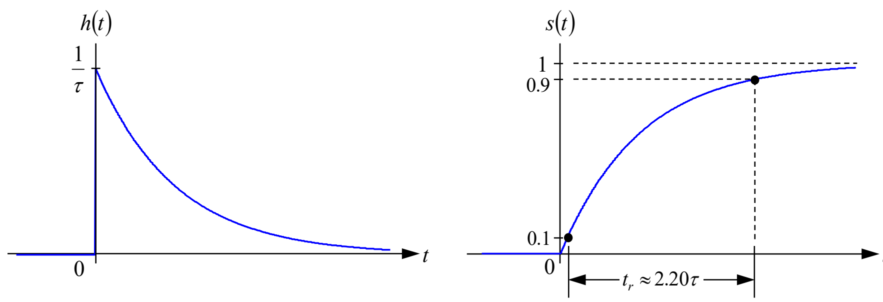
Its impulse and step responses are

$$h(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t) \quad \text{and} \quad s(t) = \left(1 - e^{-\frac{t}{\tau}}\right) u(t),$$

which are plotted below. The step response is often characterized by a *rise time*, which is the time required for the step response to rise from 10% to 90% of its maximum value:

$$t_r = (\ln 0.9 - \ln 0.1)\tau \approx 2.20\tau,$$

which is proportional to τ.



- To find the frequency response, we take the Fourier transform of the differential equation and use the differentiation property, obtaining

$$j\omega Y(j\omega) + \frac{1}{\tau} Y(j\omega) = \frac{1}{\tau} X(j\omega).$$

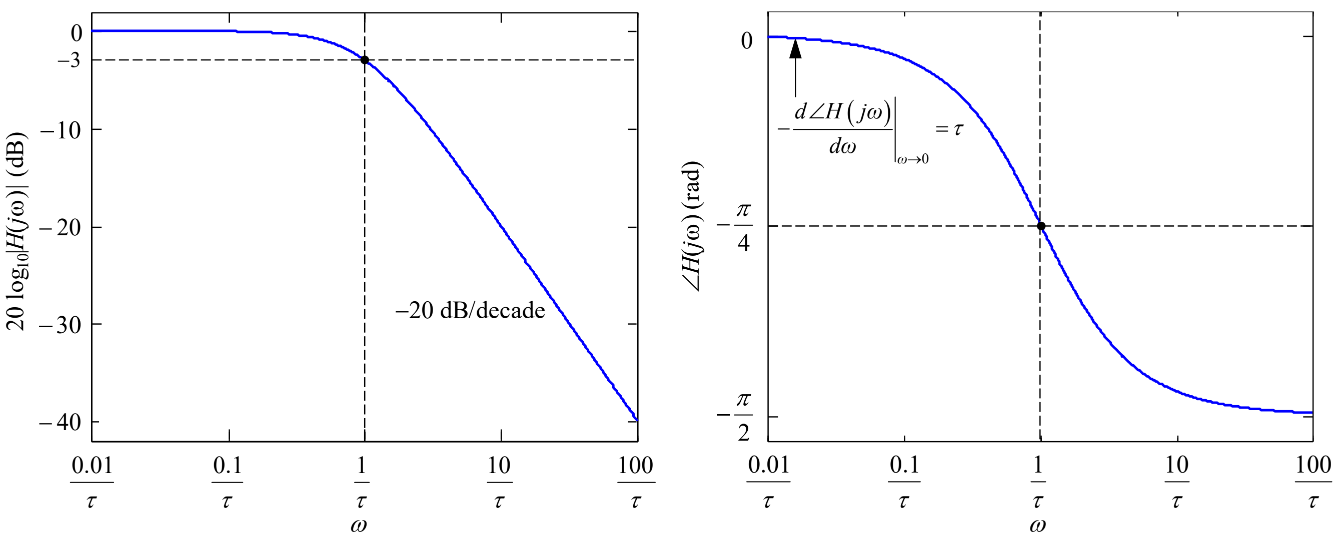
Solving for $H(j\omega) = Y(j\omega)/X(j\omega)$, we obtain

$$H(j\omega) = \frac{1}{1 + j\omega\tau}.$$

The magnitude and phase responses are

$$|H(j\omega)| = \frac{1}{\sqrt{1 + (\tau\omega)^2}} \quad \text{and} \quad \angle H(j\omega) = -\tan^{-1}(\tau\omega),$$

which are plotted below.



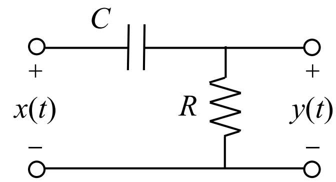
- We can make several observations about the magnitude and phase plots (considering positive ω only).
  - For $\omega << 1/\tau$, $|H(j\omega)| \to 1$, so $20\log_{10}|H(j\omega)| \to 0\,\text{dB}$.
  - For $\omega = 1/\tau$, $|H(j\omega)| = 1/\sqrt{2} \approx 0.707$, so $20\log_{10}|H(j\omega)| \approx -3$ dB.
  - For $\omega >> 1/\tau$, $|H(j\omega)| \propto 1/\omega$, so $20\log_{10}|H(j\omega)|$ decreases at a rate of –20 dB/decade.
  - For $\omega << 1/\tau$, $\angle H(j\omega) \to 0$.
  - For $\omega = 1/\tau$, $\angle H(j\omega) = -\pi/4$.
  - For $\omega >> 1/\tau$, $\angle H(j\omega) \to -\pi/2$.
  - From the phase response, we can compute the group delay

$$-\frac{d\angle H(j\omega)}{d\omega} = \frac{\tau}{1+(\tau\omega)^2}.$$

Near ω = 0, where the magnitude response is largest, the phase $\angle H(j\omega)$ has a slope –τ, corresponding to a group delay τ.

First-Order Highpass Filter

- This can be realized by the circuit shown.



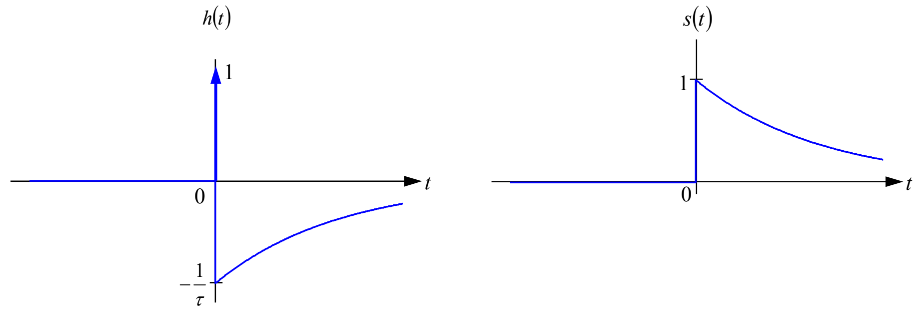
Defining τ = *RC*, the circuit is described by a first-order differential equation

$$\frac{dy}{dt}+\frac{1}{\tau}y(t)=\frac{dx}{dt}.$$

Its impulse and step responses are

$$h(t)=\delta(t)-\frac{1}{\tau}e^{-\frac{t}{\tau}}u(t) \quad \text{and} \quad s(t)=e^{-\frac{t}{\tau}}u(t),$$

which are plotted below.



- To find the frequency response, we take the Fourier transform of the differential equation using the differentiation property, obtaining

$$j\omega Y(j\omega)+\frac{1}{\tau}Y(j\omega)=j\omega X(j\omega).$$

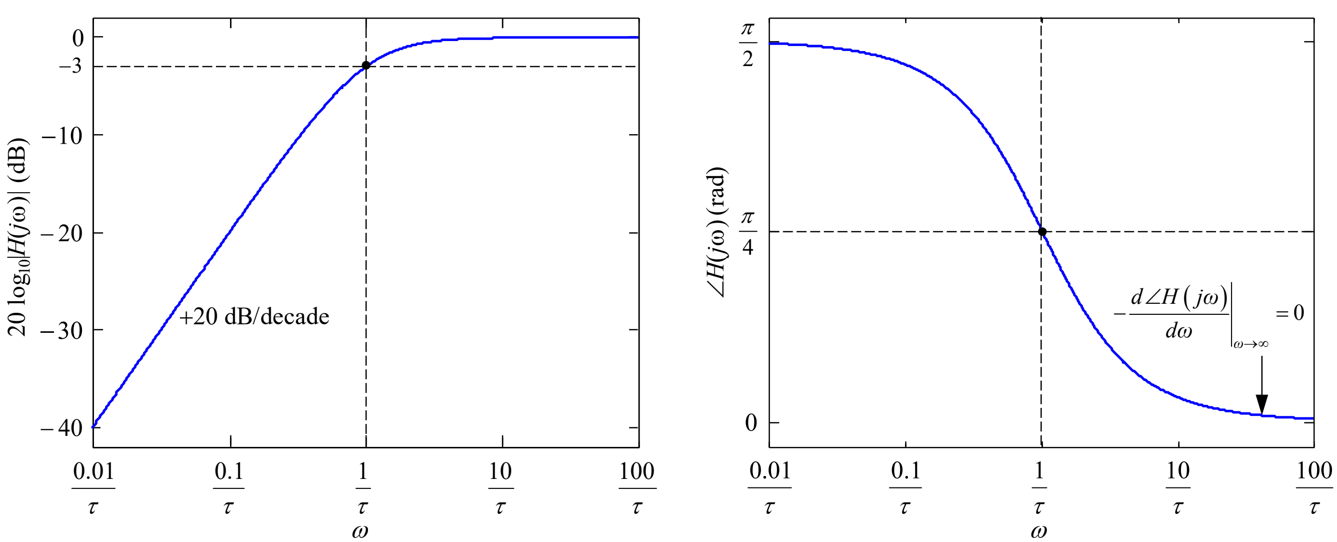
Solving for $H(j\omega)=Y(j\omega)/X(j\omega)$, we obtain

$$H(j\omega)=\frac{j\omega\tau}{j\omega\tau+1}.$$

The magnitude and phase responses are

$$\left|H(j\omega)\right|=\frac{|\omega|\tau}{\sqrt{1+(\omega\tau)^2}} \quad \text{and} \quad \angle H(j\omega)=\frac{\pi}{2}\operatorname{sgn}\omega-\tan^{-1}\omega\tau,$$

which are plotted below.



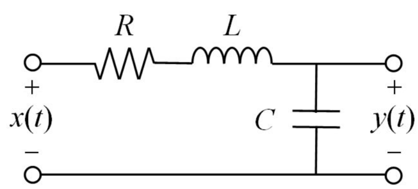
- We can make several observations about the magnitude and phase plots (considering positive ω only).
  - For $\omega << 1/\tau$, $\left|H(j\omega)\right| \propto \omega$, so $20\log_{10}\left|H(j\omega)\right|$ increases at a rate of +20 dB/decade.
  - For $\omega = 1/\tau$, $\left|H(j\omega)\right| = 1/\sqrt{2} \approx 0.707$, so $20\log_{10}\left|H(j\omega)\right| \approx -3$ dB .
  - For $\omega >> 1/\tau$, $\left|H(j\omega)\right| \to 1$, so $20\log_{10}\left|H(j\omega)\right| \to 0\,\text{dB}$.
  - For $\omega << 1/\tau$, $\angle H(j\omega) \to \pi/2$.
  - For $\omega = 1/\tau$, $\angle H(j\omega) = \pi/4$.
  - For $\omega >> 1/\tau$, $\angle H(j\omega) \to 0$.
  - As $\omega \to \infty$, where the magnitude response is largest, the phase $\angle H(j\omega)$ has a zero slope, corresponding to a group delay $-d\angle H(j\omega)/d\omega = 0$.

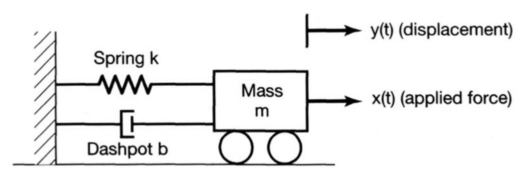
Second-Order Lowpass Filter

- A second-order lowpass system is described by a second-order differential equation

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t).$$

- There are two parameters appearing in the differential equation: the *natural frequency* ω_n and the *damping coefficient* ζ. Two physical systems described by the equation are shown below and the corresponding values of ω_n and ζ are indicated.





$$\omega_n = \frac{1}{\sqrt{LC}} \qquad\qquad \omega_n = \sqrt{\frac{k}{m}}$$

$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}} \qquad\qquad \zeta = \frac{b}{2\sqrt{km}}$$

  - The natural frequency ω_n is the frequency at which the system would exhibit a strong resonant response in the absence of damping, and should be familiar to you from your study of resonant circuits or mechanical motion.
  - The damping coefficient ζ is proportional to the physical quantity that causes energy dissipation in the system, *R* or *b*, respectively. When at least a small amount of damping is present, the system exhibits a lowpass response, and ω_n represents a cutoff frequency, above which the magnitude response of the system decreases.

- The second-order lowpass filter has three regimes, which are distinguished based on the value of the damping constant:
  - Underdamped $0 < \zeta < 1$
  - Critically damped $\zeta = 1$
  - Overdamped $1 < \zeta < \infty$
- The impulse and step responses have different mathematical forms in each regime. Here we provide formulas for the impulse responses.
  - Overdamped, $1 < \zeta < \infty$:

$$h(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left(e^{-\omega_n\left(\zeta - \sqrt{\zeta^2-1}\right)t} - e^{-\omega_n\left(\zeta + \sqrt{\zeta^2-1}\right)t} \right) u(t).$$

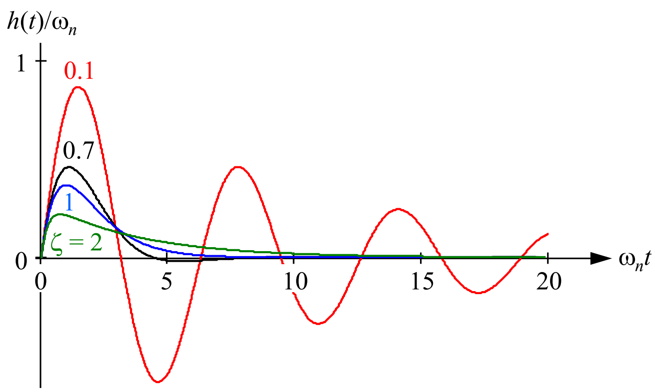
  - Underdamped, $0 < \zeta < 1$:

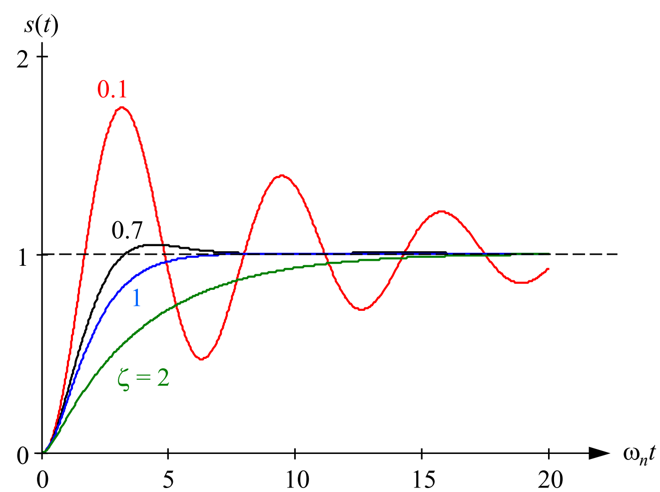
$$h(t) = \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n \sqrt{1-\zeta^2} \cdot t\right) u(t).$$

  - Critically damped, $\zeta = 1$:

$$h(t) = \omega_n^2 t e^{-\omega_n t} u(t).$$

Plots of the impulse response $h(t)$ and step response $s(t)$ are shown below. The time scale is $\omega_n t$, i.e., time is normalized by the natural response time of the system.





- The step response $s(t)$ is especially informative:
  - For all values of the damping coefficient ζ, $0 < \zeta < \infty$, the step response asymptotically approaches unity, $s(t) \rightarrow 1$, as $t \rightarrow \infty$.

- For a small damping coefficient, $\zeta << 1$, the step response exhibits a short rise time, but exhibits significant *overshoot* and *ringing*.
- For a damping coefficient $\zeta > 1$, there is no overshoot or ringing, but as ζ increases, the rise time becomes longer. We note below that the group delay increases with increasing ζ.

- Many filters and feedback control systems are second-order lowpass systems, and are designed to have damping coefficients in the range ζ ~ 0.7-1.0, as this represents a good compromise between overshoot and response time.
- To find the frequency response, we take the Fourier transform of the differential equation using the differentiation property, obtaining

$$(j\omega)^2 Y(j\omega) + 2\zeta\omega_n (j\omega) Y(j\omega) + \omega_n^2 Y(j\omega) = \omega_n^2 X(j\omega) \ .$$

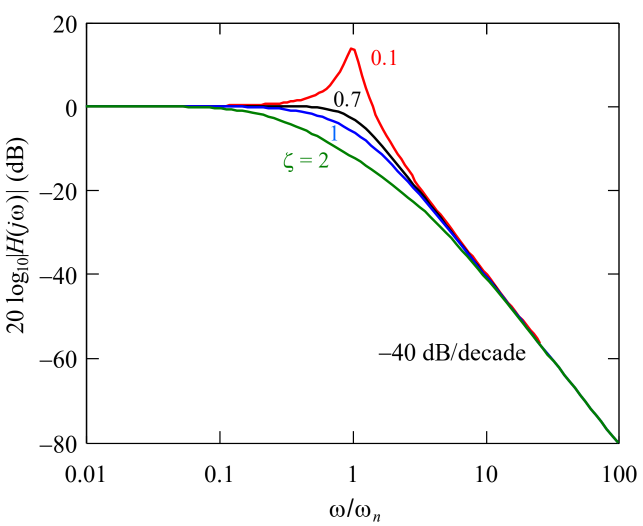
Solving for $H(j\omega) = Y(j\omega)/X(j\omega)$, we obtain

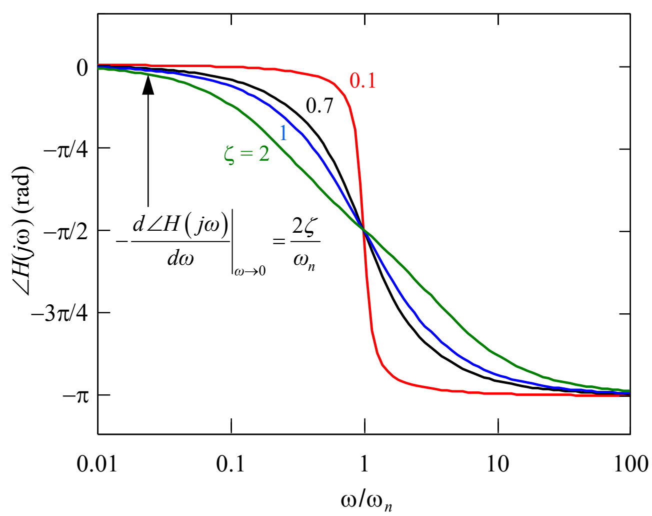
$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

The magnitude and phase are

$$|H(j\omega)| = \frac{\omega_n^2}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2}} \quad \text{and} \quad \angle H(j\omega) = -\tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right),$$

which are plotted below.





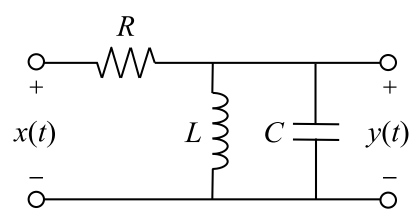
- We can make several observations about the magnitude and phase plots (considering positive ω only).
    - For $\omega << \omega_n$, $|H(j\omega)| \to 1$, so $20\log_{10}|H(j\omega)| \to 0\,\text{dB}$.
    - For $\omega \approx \omega_n$, when $0 < \zeta < 1$ (underdamped regime), $|H(j\omega)|$ increases above 1 ($20\log_{10}|H(j\omega)|$ increases above 0 dB). This effect is called *peaking*, and becomes more pronounced as $\zeta \to 0$.
    - For $\omega >> \omega_n$, $|H(j\omega)| \propto 1/\omega^2$, so $20\log_{10}|H(j\omega)|$ decreases at a rate of –40 dB/decade.
    - For $\omega << \omega_n$, $\angle H(j\omega) \to 0$.
    - For $\omega = \omega_n$, $\angle H(j\omega) = -\pi/2$.
    - For $\omega >> \omega_n$, $\angle H(j\omega) \to -\pi$.
    - In the lowpass filter's passband, $\omega << \omega_n$, the phase response is approximately linear with frequency, and the group delay is given by

$$-\frac{d\angle H(j\omega)}{d\omega} = \frac{2\zeta}{\omega_n}.$$

    This increases with increasing ζ, since damping slows down the system's response.

Second-Order Bandpass Filter

- Bandpass filters are widely used in radio-frequency circuits to control the frequency of an oscillator or to select a channel at a desired frequency while rejecting channels at undesired frequencies. This circuit is one of several different passive *RLC* networks that realize a second-order bandpass filter.



- It is described by a differential equation

$$\frac{d^2y}{dt^2} + \frac{1}{RC}\frac{dy}{dt} + \frac{1}{LC}y(t) = \frac{1}{RC}\frac{dx}{dt}.$$

- We define two parameters to describe the circuit: the *resonance frequency*

$$\omega_n = \frac{1}{\sqrt{LC}}$$

and the *damping coefficient*

$$\zeta = \frac{1}{2R}\sqrt{\frac{L}{C}}.$$

Similar parameters were defined for the second-order lowpass filter above. Here, they have a similar relationship to the frequency response, but have a different dependence on *R*, *L* and *C*. Alternatively, the amount of damping may be described by a *quality factor*

$$Q = \frac{1}{2\zeta} = R\sqrt{\frac{C}{L}}\,.$$

Note that a small value of ζ corresponds to a large value of *Q*. Using the parameters defined above, the differential equation can be rewritten as

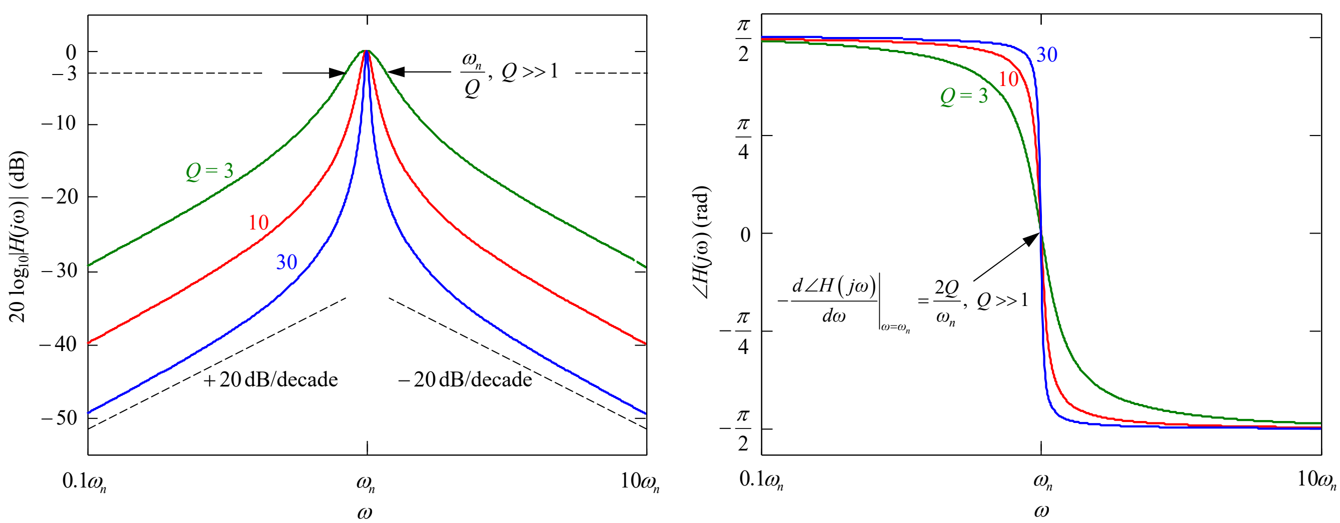
$$\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y(t) = 2\zeta\omega_n\frac{dx}{dt}$$

$$\frac{d^2y}{dt^2} + \frac{\omega_n}{Q}\frac{dy}{dt} + \omega_n^2 y(t) = \frac{\omega_n}{Q}\frac{dx}{dt}$$

- Taking the Fourier transform of the differential equation and solving for $H(j\omega) = Y(j\omega)/X(j\omega)$, we obtain a frequency response

$$H(j\omega) = \frac{2\zeta\omega_n j\omega}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{\frac{\omega_n}{Q}j\omega}{(j\omega)^2 + \frac{\omega_n}{Q}(j\omega) + \omega_n^2}\,.$$

The magnitude and phase responses are plotted below.



- Assuming the filter is highly underdamped, $\zeta << 1$, $Q >> 1$, we can make a few observations (considering positive ω only):
  - $|H(j\omega)| = 1/\sqrt{2}$ at $\omega = \omega_n \pm \frac{\omega_n}{2Q}$, so the bandpass filter has a 3-dB bandwidth $\frac{\omega_n}{Q}$, which decreases as *Q* increases.

- At the resonance frequency $\omega = \omega_n$, the group delay is

$$-\left.\frac{d\angle H(j\omega)}{d\omega}\right|_{\omega=\omega_n} = \frac{2Q}{\omega_n},$$

which increases as Q increases. The input signal is stored in the *L*-*C* resonator, delaying the output relative to the input.

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EE 102A: Signal Processing and Linear Systems I
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Chapter 5: The Discrete-Time Fourier Transform

Motivations

- In this chapter, we extend Fourier analysis to *aperiodic DT signals*. We introduce the DT Fourier transform (DTFT), which expresses an aperiodic DT signal as a *continuous integral* of imaginary exponential signals at different frequencies, $e^{j\Omega n}$, where Ω is real. The integral is performed over any frequency interval of length 2π, such as $-\pi \le \Omega \le \pi$, since DT imaginary exponentials with frequencies differing by any multiple of 2π, $e^{j\Omega n}$ and $e^{j(\Omega+k2\pi)n}$, are indistinguishable. The DTFT will allow us to analyze DT LTI systems with aperiodic inputs. Furthermore, the frequency response of an DT LTI system is the DTFT of its impulse response. Detailed study of the DTFT will equip us to compute the frequency responses for a wide range of systems, including higher-order systems and systems not described by finite-order difference equations.
- We conclude this chapter by studying the overall schema of Fourier representations in CT and DT, including the organizing principles and the dualities that exist within and between the various Fourier series and transforms.

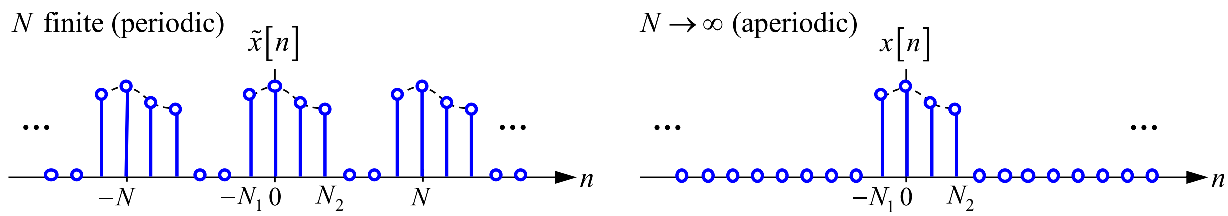
Major Topics in This Chapter

- Discrete-time Fourier transform
  - Derivation for aperiodic signals. Fourier transforms in the limit. Fourier transforms of periodic signals. Properties of Fourier transforms.
- Convolution property and LTI system analysis
  - Frequency response as DTFT of impulse response.
  - LTI systems not described by finite-order difference equations.
    - Ideal lowpass filter.
  - LTI systems described by linear, constant-coefficient difference equations.
    - Infinite impulse response: first-order system, second-order system.
    - Finite impulse response: moving average, approximation of ideal lowpass filter.
- Overview of CT and DT Fourier representations
  - Organizing principles: periodic vs. aperiodic, continuous vs. discrete.
  - Dualities: in CTFT, in DTFS, between CTFS and DTFT.

Discrete-Time Fourier Transform

Derivation of Discrete-Time Fourier Transform

- Our derivation of the DT Fourier transform (DTFT) is similar to the derivation of the CT Fourier transform (CTFT) presented in Chapter 4 (see pages 146-148). We are given an aperiodic DT signal $x[n]$. We assume $x[n]$ is nonzero only within an interval $-N_1 \le n \le N_2$. We consider the aperiodic $x[n]$ to be a periodic DT signal $\tilde{x}[n]$, which has a period N, in the limit that the period becomes infinite, $N \to \infty$. In that limit, the periodic signal $\tilde{x}[n]$ becomes the aperiodic signal $x[n]$, as shown in the figure below.



- In order to derive the DTFT of the aperiodic signal $x[n]$, we start by representing the periodic signal $\tilde{x}[n]$ as a DT Fourier series (DTFS) with a fundamental frequency $\Omega_0 = \dfrac{2\pi}{N}$ and DTFS coefficients a_k, $k = \langle N \rangle$:

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n} . \qquad \text{(DTFS synthesis)} \qquad (1)$$

  Expression (1) synthesizes the periodic $\tilde{x}[n]$ as a linear combination of $e^{jk\Omega_0 n}$ over any *N* consecutive values of *k*. Recall from Chapter 3 that both the imaginary exponentials and the DTFS coefficients are periodic in *k* with period *N*, i.e., $e^{j(k+N)\Omega_0 n} = e^{jk\Omega_0 n}$ and $a_{k+N} = a_k$. We may obtain the DTFS coefficients by performing analysis over any *N* consecutive values of *n*:

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk\Omega_0 n} . \qquad \text{(DTFS analysis)} \qquad (2)$$

  We assume the interval $n = \langle N \rangle$ includes the interval $-N_1 \le n \le N_2$ over which the aperiodic signal $x[n]$ is nonzero. Because $\tilde{x}[n] = x[n]$ over the interval $-N_1 \le n \le N_2$ and $x[n] = 0$ outside this interval, we may rewrite the DTFS analysis equation (2) as

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk\Omega_0 n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\Omega_0 n} . \qquad (3)$$

- Now we define $X\left(e^{j\Omega}\right)$, a function of a continuous frequency variable Ω, which is computed from $x[n]$ using the following summation:

$$X\left(e^{j\Omega}\right) \stackrel{d}{=} \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} . \qquad \text{(DTFT or DTFT analysis)} \qquad (4)$$

We refer to $X\left(e^{j\Omega}\right)$ as the *DT Fourier transform* (DTFT) of the aperiodic signal $x[n]$. We refer to the summation (4) as the *DTFT analysis equation*, or simply the *DTFT*. Note that $X\left(e^{j\Omega}\right)$ is a periodic function of Ω with period 2π:

$$X\left(e^{j(\Omega+2\pi)}\right)=\sum_{n=-\infty}^{\infty}x[n]e^{-j(\Omega+2\pi)n}=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}\underbrace{e^{-j2\pi n}}_{=1}=X\left(e^{j\Omega}\right). \tag{5}$$

- If we compare (3) and (4), we observe that we can obtain the DTFS coefficients a_k by sampling the DTFT $X\left(e^{j\Omega}\right)$ at integer multiples of the fundamental frequency and scaling by $1/N$:

$$\frac{1}{N}X\left(e^{j\Omega}\right)\Big|_{\Omega=k\Omega_0}=\frac{1}{N}X\left(e^{jk\Omega_0}\right)=a_k\ . \tag{6}$$

Using (6), we can rewrite the DTFS synthesis equation (1) for the periodic signal $\tilde{x}[n]$ as

$$\begin{aligned}\tilde{x}[n]&=\sum_{k=\langle N\rangle}\frac{1}{N}X\left(e^{jk\Omega_0}\right)e^{jk\Omega_0 n}\\&=\frac{1}{2\pi}\sum_{k=\langle N\rangle}X\left(e^{jk\Omega_0}\right)e^{jk\Omega_0 n}\Omega_0\end{aligned}. \tag{7}$$

- Now we consider the limit in which the periodic signal $\tilde{x}[n]$ becomes the aperiodic signal $x[n]$:

$$N\to\infty$$

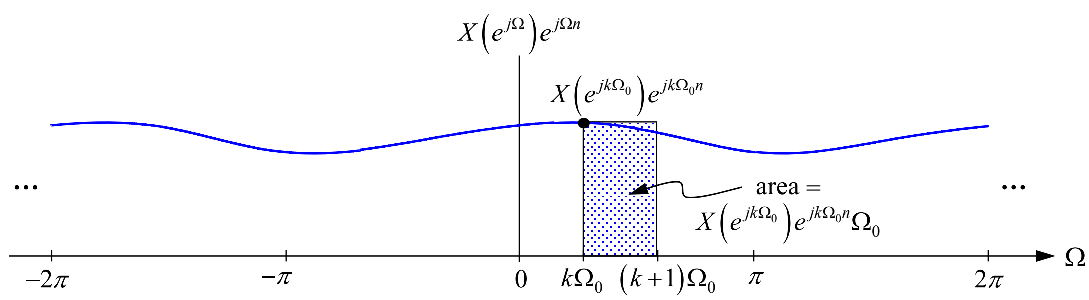
$$\tilde{x}[n]\to x[n]$$

$$k\Omega_0\to\Omega\text{, a continuous variable}$$

$$\Omega_0\to d\Omega\text{, an infinitesimal increment of }\Omega$$

$$X\left(e^{j\Omega}\right)\Big|_{\Omega=k\Omega_0}\to X\left(e^{j\Omega}\right)\text{, a function of a continuous variable}$$

The figure below shows $X\left(e^{j\Omega}\right)e^{j\Omega n}$ as a function of the continuous frequency variable Ω.



With the aid of the figure, we see that in the limit we are considering, (7) becomes a Riemann sum approximation of an integral, which allows us to obtain the aperiodic signal $x[n]$ from $X\left(e^{j\Omega}\right)$:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\Omega}\right) e^{j\Omega n} d\Omega . \qquad \text{(inverse DTFT or DTFT synthesis)} \qquad (8)$$

Since (7) sums over any N consecutive frequency intervals of length $\Omega_0 = 2\pi / N$, the integral (8) may be performed over any interval of length 2π. We refer to the integral (8) as the *inverse DTFT* or the *DTFT synthesis equation*, and refer to $x[n]$ as the *inverse DTFT* of $X\left(e^{j\Omega}\right)$.

- To summarize, we have derived the following two expressions.

$$X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \qquad \text{(DTFT or DTFT analysis)} \qquad (4)$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\Omega}\right) e^{j\Omega n} d\Omega \qquad \text{(inverse DTFT or DTFT synthesis)} \qquad (8)$$

The inverse DTFT integral (8) specifies how to *synthesize* an aperiodic signal $x[n]$ as a weighted sum of imaginary exponentials $e^{j\Omega n}$ whose frequency Ω is a continuous-valued, real variable spanning any interval of length 2π. In (8), the imaginary exponential $e^{j\Omega n}$ at frequency Ω is weighted by a factor $X\left(e^{j\Omega}\right)$. The DTFT sum (4) specifies how, given an aperiodic signal $x[n]$, we may *analyze* $x[n]$ to obtain the weighting factor $X\left(e^{j\Omega}\right)$.

- We often describe (4) and (8) in terms of a *DTFT operator F* and an *inverse DTFT operator* F^{-1}, each of which acts on one function to produce the other function:

$$F\left[x[n]\right] = X\left(e^{j\Omega}\right), \qquad (9)$$

and

$$F^{-1}\left[X\left(e^{j\Omega}\right)\right] = x[n]. \qquad (10)$$

We often denote a DT signal $x[n]$ and its DTFT $X\left(e^{j\Omega}\right)$ as a *DTFT pair*:

$$x[n] \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right). \qquad (11)$$

Alternate Analysis Method for Discrete-Time Fourier Series

- The derivation of the DTFT (see pages 190-192) provided an alternate method for computing the DTFS coefficients of periodic signals. Here we summarize that method. Suppose we are given a periodic signal $\tilde{x}[n]$ with period $N = 2\pi / \Omega_0$, and want to determine its DTFS coefficients a_k. First, we define an aperiodic signal $x[n]$ that represents one period of $\tilde{x}[n]$:

$$x[n] = \begin{cases} \tilde{x}[n] & n_1 + 1 \le n \le n_1 + N \\ 0 & \text{otherwise} \end{cases}. \qquad (12)$$

for some n_1. Next, we compute the DTFT of the one-period signal $x[n]$ using

$$X\left(e^{j\Omega}\right)=\sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}. \tag{4}$$

Finally, we sample the DTFT $X\left(e^{j\Omega}\right)$ at integer multiples of the fundamental frequency to obtain the DTFS coefficients of the periodic signal $\tilde{x}[n]$:

$$a_k=\frac{1}{N} X\left(e^{j\Omega}\right)\Big|_{\Omega=k\Omega_0}. \tag{6}$$

In some cases, you may find it easier to apply this method than to use the DTFS analysis equation (2).

- Even more importantly, according to (6), every set of DTFS coefficients corresponds to the samples of a DTFT. Hence, all the properties of the DTFS (Table 2, Appendix) are derived from properties of the DTFT (Table 5, Appendix). Understanding this relationship can streamline your learning of DT Fourier analysis.

Convergence of Discrete-Time Fourier Transform

- Consider a signal and its DTFT

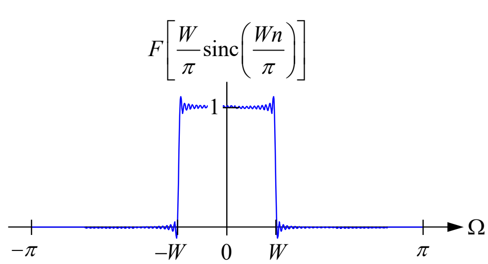
$$x[n] \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right).$$

- If the DTFT $X\left(e^{j\Omega}\right)$ exists, then the inverse DTFT $F^{-1}\left[X\left(e^{j\Omega}\right)\right]$ always converges to the original signal $x[n]$.

- The DTFT

$$X\left(e^{j\Omega}\right)=\sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}. \tag{4}$$

can be considered a Fourier series representation of a periodic function of a continuous-valued variable Ω with Fourier series coefficients given by the $x[n]$. (This is a consequence of a duality between the DTFT and the CTFS, as described on pages 238-239 below.) Hence, the convergence of the DTFT $X\left(e^{j\Omega}\right)$ is analogous to the convergence of a CTFS synthesis $\hat{x}(t)$ of a periodic CT signal $x(t)$. In particular, if the DTFT $X\left(e^{j\Omega}\right)$ has any discontinuities, it will exhibit the Gibbs phenomenon (made more visible here by truncating the time signal before applying the F operator).



- As we will see later in this chapter, the Gibbs phenomenon can negatively impact the performance of finite impulse response (FIR) DT filters. For example, if the DTFT shown above represented the frequency response of an FIR lowpass filter, the Gibbs phenomenon would cause *distortion* of desired signals in the passband, as well as *leakage* of undesired signals in the stopband. As we will learn in Chapter 6, these effects can be mitigated by *windowing* the FIR filter's impulse response.

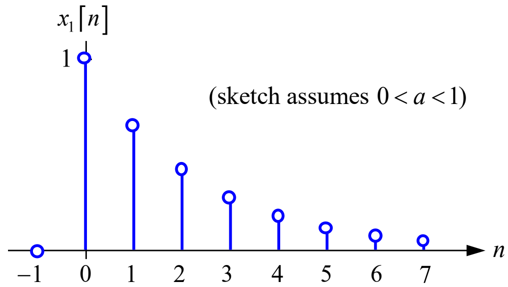
Examples of Discrete-Time Fourier Transform

- Since all DTFTs are periodic functions of Ω with period 2π, we often plot them over a single period, such as $-\pi \le \Omega \le \pi$.

1. *Right-sided real exponential*. The signal is given by

$$x_1[n] = a^n u[n],\ a \text{ real},\ |a| < 1 .$$

This signal is shown in the figure below, which assumes $0 < a < 1$.



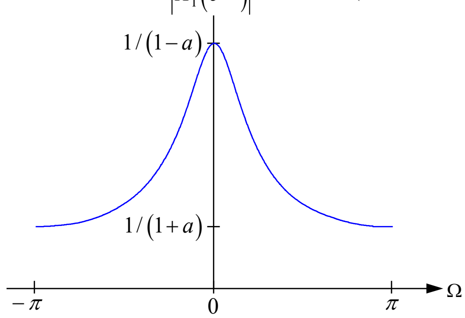
- To compute its DTFT, we evaluate the sum (4):

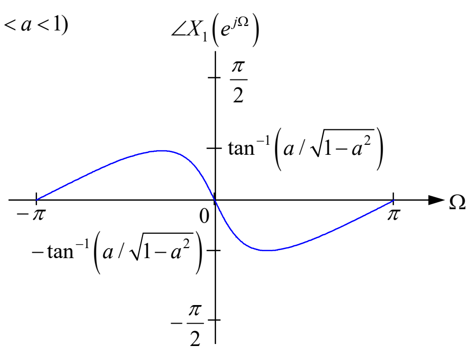
$$\begin{aligned} X_1\left(e^{j\Omega}\right) &= \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} \left(ae^{-j\Omega}\right)^n \\ &= \frac{1}{1-ae^{-j\Omega}} \end{aligned} .$$

The second line above expresses $X_1\left(e^{j\Omega}\right)$ as a geometric series. In the third line above, we have used the fact that $\left|ae^{-j\Omega}\right| < 1$ to sum the geometric series. We obtained a DTFT of the same form when we analyzed the frequency response of a first-order system in Chapter 3 (see pages 129-131).

- This DTFT is complex-valued, so it is best visualized in terms of magnitude and phase plots, as shown below. These plots assume $0 < a < 1$.

(sketches assume $0 < a < 1$)



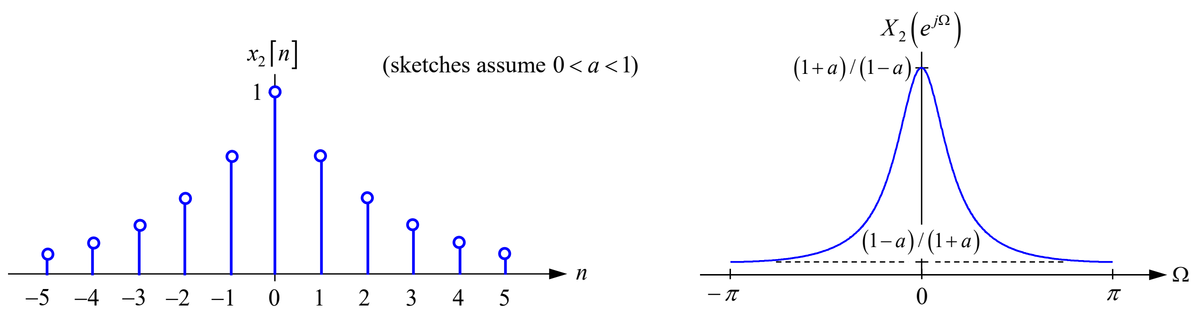


- Observe that a value of $|a|$ close to unity describes a signal $x_1[n]$ that is spread out in time and a DTFT $X_1(e^{j\Omega})$ that is concentrated in frequency near $\Omega = 0$ (and $\Omega = \pm 2\pi, \pm 4\pi, \ldots$). Conversely, a value of $|a|$ close to zero describes a signal concentrated in time near $n = 0$ and a DTFT that is spread out in frequency. These observations exemplify an *inverse relationship between time and frequency* in the DTFT, similar to what we noted for the CTFT in Chapter 4.

2. *Two-sided real exponential*. This signal is given by

$$x_2[n] = a^{|n|}, \; a \text{ real}, \; |a| < 1 .$$

This signal is shown in the figure below (on the left).



- In order to compute its DTFT, we use (4), dividing the sum into two parts, each in a form like the sum we evaluated for $X_1(e^{j\Omega})$:

$$\begin{aligned} X_2(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} a^n e^{-j\Omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\Omega n} \end{aligned} .$$

To evaluate the second sum, we change the summation variable to $m = -1-n$, i.e., $n = -1-m$:

$$\begin{aligned} X_2(e^{j\Omega}) &= \sum_{n=0}^{\infty} \left(ae^{-j\Omega}\right)^n + ae^{j\Omega} \sum_{m=0}^{\infty} \left(ae^{j\Omega}\right)^m \\ &= \frac{1}{1-ae^{-j\Omega}} + \frac{ae^{j\Omega}}{1-ae^{j\Omega}} \\ &= \frac{1-a^2}{1-2a\cos\Omega + a^2} \end{aligned} .$$

In the second line, we have summed the two geometric series, which converge because $\left|ae^{-j\Omega}\right| = \left|ae^{j\Omega}\right| < 1$. In the third line, we have added the two terms to obtain a purely real expression for $X_2(e^{j\Omega})$. The DTFT $X_2(e^{j\Omega})$ is shown in the figure above (on the right). As in Example 1, a value of $|a|$ close to unity describes a signal that is spread out over time and a DTFT that is concentrated in frequency, while values of $|a|$ close to zero describe a signal concentrated in time and a DTFT spread out over frequency.

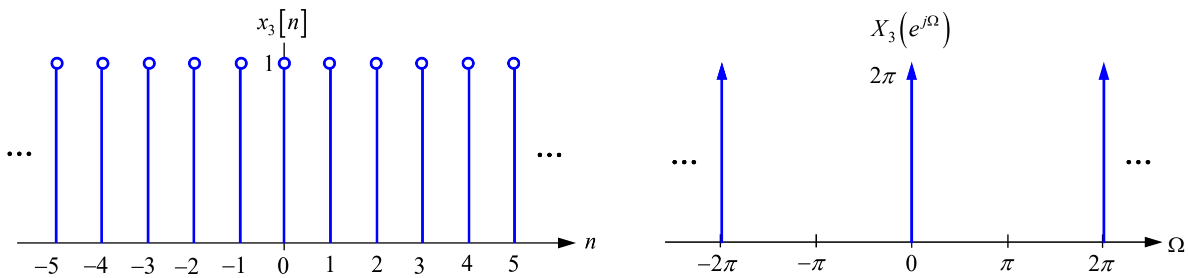
- In the next example, we will use $X_2\left(e^{j\Omega}\right)$ to represent an impulse train in frequency. To help with that example, we express the value of the signal $x_2[n]$ at $n=0$ by using $X_2\left(e^{j\Omega}\right)$ in the inverse DTFT integral (8):

$$x_2[0]=\frac{1}{2\pi}\int_{2\pi} X_2\left(e^{j\Omega}\right)e^{j\Omega n}d\Omega\bigg|_{n=0}=\frac{1}{2\pi}\int_{2\pi} X_2\left(e^{j\Omega}\right)d\Omega=1\,. \tag{13}$$

3. *Constant*. The signal is given by

$$x_3[n]=1\ \forall n\,,$$

and is shown in the figure below (on the left).



- Observe that the two-sided exponential signal $x_2[n]=a^{|n|}$ becomes the constant signal $x_3[n]=1$ in the limit that a becomes unity:

$$x_3[n]=\lim_{a\to 1}x_2[n]\,.$$

Taking the DTFT of both sides of this expression, we find that

$$\begin{aligned}X_3\left(e^{j\Omega}\right)&=\lim_{a\to 1}X_2\left(e^{j\Omega}\right)\\&=\lim_{a\to 1}\frac{1-a^2}{1-2a\cos\Omega+a^2}\end{aligned}.$$

- In the limit $a\to 1$, $X_2\left(e^{j\Omega}\right)$ has the following characteristics:
  - It has peaks of zero width and infinite height at $\Omega=0,\pm 2\pi,\pm 4\pi,\ldots$.
  - Over any interval of length 2π, using (13), the area is $\int_{2\pi} X_2\left(e^{j\Omega}\right)d\Omega=2\pi$.

Using an argument similar to Chapter 1, pages 22-23, we conclude that as $a\to 1$, each peak of $X_3\left(e^{j\Omega}\right)$ becomes an impulse function of frequency with area 2π. Including the peaks at $\Omega=0,\pm 2\pi,\pm 4\pi,\ldots$, the DTFT $X_3\left(e^{j\Omega}\right)$ becomes a periodic train of impulses at $\Omega=0,\pm 2\pi,\pm 4\pi,\ldots$:

$$X_3\left(e^{j\Omega}\right)=2\pi\sum_{l=-\infty}^{\infty}\delta\left(\Omega-l2\pi\right).$$

This is shown in the figure above (on the right).

- Observe that the signal $x_3[n]$ is maximally spread out in time, while the DTFT $X_3(e^{j\Omega})$ is maximally concentrated in frequency.

Discrete-Time Fourier Transform in the Limit

- The constant signal $x_3[n]$ is not absolutely summable or absolute-square summable, so its DTFT $X_3(e^{j\Omega})$ does not exist in the strict sense. By considering $x_3[n]$ as the limiting case of a signal $x_2[n]$ whose DTFT does exist, we are able to obtain an expression for $X_3(e^{j\Omega})$, which includes impulse functions of frequency. We may say that the DTFT $X_3(e^{j\Omega})$ *exists in a generalized sense.*
- We can employ a similar technique to obtain generalized DTFTs of other signals, as shown this table.

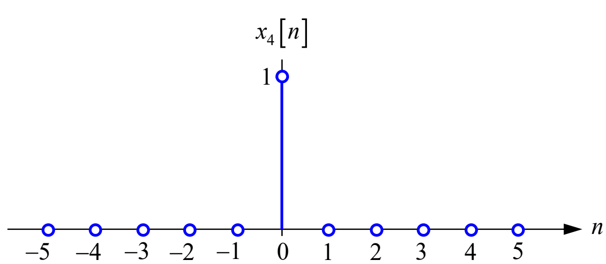
Signal $x[n]$	DTFT $X(e^{j\Omega})$
1	$2\pi\sum_{l=-\infty}^{\infty}\delta(\Omega-l2\pi)$
$\text{sgn}[n]$	$\dfrac{1+e^{-j\Omega}}{1-e^{-j\Omega}}$
$u[n]=\frac{1}{2}(1+\text{sgn}[n]+\delta[n])$	$\dfrac{1}{1-e^{-j\Omega}}+\pi\sum_{l=-\infty}^{\infty}\delta(\Omega-l2\pi)$
$e^{j\Omega_0 n}$	$2\pi\sum_{l=-\infty}^{\infty}\delta(\Omega-\Omega_0-l2\pi)$
$\cos\Omega_0 n$	$\pi\sum_{l=-\infty}^{\infty}\left[\delta(\Omega-\Omega_0-l2\pi)+\delta(\Omega+\Omega_0-l2\pi)\right]$
$\sin\Omega_0 n$	$\dfrac{\pi}{j}\sum_{l=-\infty}^{\infty}\left[\delta(\Omega-\Omega_0-l2\pi)-\delta(\Omega+\Omega_0-l2\pi)\right]$

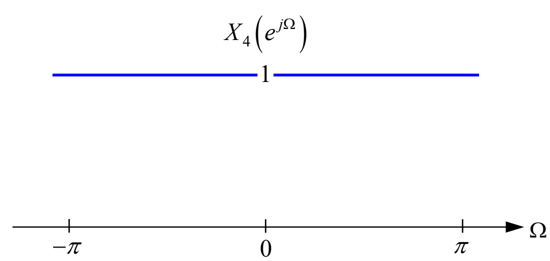
Examples of Discrete-Time Fourier Transform (Continued)

4. *Unit impulse*. This can be considered the dual of Example 3. The signal is given by

$$x_4[n]=\delta[n],$$

as shown in the figure below (on the left).





- To compute its DTFT, we use (4):

$$\begin{aligned} X_4\left(e^{j\Omega}\right) &= \sum_{n=-\infty}^{\infty} \delta[n] e^{-jn\Omega} \\ &= 1 \ \forall \Omega \end{aligned}.$$

We have used the sampling property of the DT impulse to evaluate the sum. The DTFT $X_4\left(e^{j\Omega}\right)$ is shown in the figure above (on the right). As $x_4[n]$ is maximally concentrated in time, $X_4\left(e^{j\Omega}\right)$ is maximally spread out in frequency.

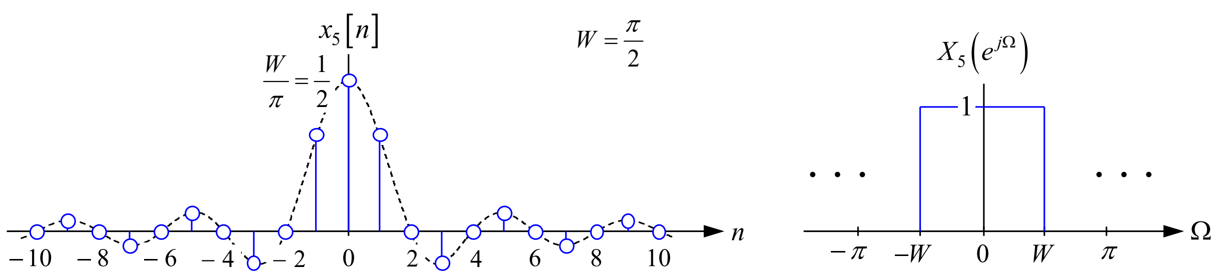
5. *Sinc function*. This signal is the impulse response of an ideal lowpass filter. We start with its DTFT, which is

$$X_5\left(e^{j\Omega}\right) = \begin{cases} 1 & |\Omega| \le W < \pi \\ 0 & W < |\Omega| < \pi \end{cases}, \ X_5\left(e^{j(\Omega+2\pi)}\right) = X_5\left(e^{j\Omega}\right).$$

This DTFT is a periodic rectangular pulse train in frequency, which we can express as

$$X_5\left(e^{j\Omega}\right) = \sum_{l=-\infty}^{\infty} \Pi\left(\frac{\Omega - l2\pi}{2W}\right).$$

The DTFT $X_5\left(e^{j\Omega}\right)$ is shown in the figure below (on the right).



- To find the corresponding time signal, we use the inverse DTFT (8), choosing the integration interval to be $-\pi \le \Omega < \pi$:

$$\begin{aligned} x_5[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X_5\left(e^{j\Omega}\right) e^{j\Omega n} d\Omega \\ &= \frac{1}{2\pi} \int_{-W}^{W} e^{j\Omega n} d\Omega \\ &= \frac{W}{\pi} \operatorname{sinc}\left(\frac{W}{\pi} n\right) \end{aligned}.$$

The signal $x_5[n]$ is shown in the figure above (on the left). In summary, we have found the DTFT pair:

$$\frac{W}{\pi} \operatorname{sinc}\left(\frac{W}{\pi} n\right) \stackrel{F}{\leftrightarrow} \sum_{l=-\infty}^{\infty} \Pi\left(\frac{\Omega - l2\pi}{2W}\right). \tag{14}$$

- Values of W close to zero describe a signal spread out in time and a DTFT concentrated in frequency, while values of W close to π describe a signal concentrated in time and a DTFT spread out in frequency. In the special case $W = \pi$, $x_5[n] = \text{sinc}(n) = \delta[n]$ and $X_5(e^{j\Omega}) = 1\ \forall\Omega$, corresponding to Example 4.

- Using the inverse DTFT integral (8), the value of a time signal at $n = 0$ equals 1/2π times the area under one period of the corresponding DTFT. We can use this to check the correctness of any inverse DTFT we compute. In this example, we have

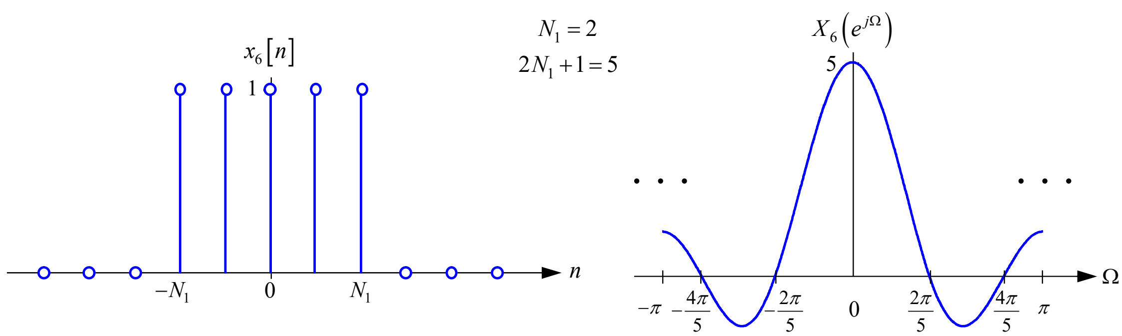
$$\begin{aligned} x_5[0] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_5(e^{j\Omega})e^{j\Omega n}d\Omega\Big|_{n=0} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_5(e^{j\Omega})d\Omega \\ &= \frac{W}{\pi} \end{aligned},$$

which agrees with (14).

6. *Rectangular pulse*. This can be considered the dual of Example 5. The signal is given by

$$x_6[n] = \Pi\left(\frac{n}{2N_1}\right) = \begin{cases} 1 & |n| \le N_1 \\ 0 & |n| > N_1 \end{cases},$$

and is shown in the figure below (on the left).



- We compute its DTFT using (4):

$$\begin{aligned} X_6(e^{j\Omega}) &= \sum_{n=-N_1}^{N_1} e^{-jn\Omega} \\ &= e^{j\Omega N_1}\sum_{l=0}^{2N_1} e^{-jl\Omega} \end{aligned}, \tag{15}$$

where we have changed the summation variable from n to $l = n + N_1$. In evaluating the sum (15), we consider two different cases:

  - When $\Omega = 0, \pm 2\pi, \pm 4\pi, \ldots$, we have $e^{j\Omega N_1} = 1$ and $e^{-jl\Omega} = 1$, so

$$X_6(e^{j\Omega}) = 2N_1 + 1. \tag{16}$$

  - When $\Omega \ne 0, \pm 2\pi, \pm 4\pi, \ldots$, we sum the geometric series (15) to obtain

$$X_6\left(e^{j\Omega}\right) = e^{j\Omega N_1}\frac{1-e^{-j\Omega(2N_1+1)}}{1-e^{-j\Omega}}.$$

Multiplying the numerator and denominator by $e^{j\Omega/2}$, we obtain

$$\begin{aligned} X_6\left(e^{j\Omega}\right) &= \frac{e^{j\Omega\left(N_1+\frac{1}{2}\right)} - e^{-j\Omega\left(N_1+\frac{1}{2}\right)}}{e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}} \\ &= \frac{\sin\left(\Omega\left(N_1+\frac{1}{2}\right)\right)}{\sin\left(\frac{\Omega}{2}\right)} \end{aligned}. \tag{17}$$

- Note that in the limit that $\Omega \to 0, \pm 2\pi, \pm 4\pi, \ldots$, expression (17) approaches (16). So we can express $X_6\left(e^{j\Omega}\right)$ using a single expression for all values of Ω:

$$X_6\left(e^{j\Omega}\right) = \frac{\sin\left(\Omega\left(N_1+\frac{1}{2}\right)\right)}{\sin\left(\frac{\Omega}{2}\right)} \ \forall\Omega. \tag{17}$$

- The DTFT $X_6\left(e^{j\Omega}\right)$ is shown in the figure above (on the right). Over the frequency range $-\pi \le \Omega \le \pi$ shown, it appears similar to a sinc function, peaking at $\Omega = 0$ and decaying in amplitude and oscillating as $|\Omega|$ increases. It has its first zeros at $|\Omega| = \dfrac{2\pi}{2N_1+1}$, which lie closer to $\Omega = 0$ when N_1 is larger. In other words, large values of N_1 correspond to a long time-domain pulse $x_6[n]$ and a DTFT $X_6\left(e^{j\Omega}\right)$ that is concentrated in a narrow frequency range (and vice versa).
- Unlike a sinc function, the DTFT $X_6\left(e^{j\Omega}\right)$, given by (17), is a *periodic* function of Ω with period 2π.
- Using the DTFT sum (4), for any signal and its DTFT, the sum of the samples of the time signal equals the value of the DTFT at $\Omega = 0, \pm 2\pi, \pm 4\pi, \ldots$. We can use this to verify the correctness of any DTFT we compute. In this example, we have

$$\begin{aligned} X_6\left(e^{j\Omega}\right)\Big|_{\Omega=0,\pm 2\pi,\pm 4\pi,\ldots} &= \sum_{n=-\infty}^{\infty} x_6[n]e^{-j\Omega n}\Big|_{\Omega=0,\pm 2\pi,\pm 4\pi,\ldots} \\ &= \sum_{n=-\infty}^{\infty} x_6[n] \\ &= 2N_1+1 \end{aligned},$$

which agrees with the expression (16) that we found earlier.

Discrete-Time Fourier Transform of Periodic Signals

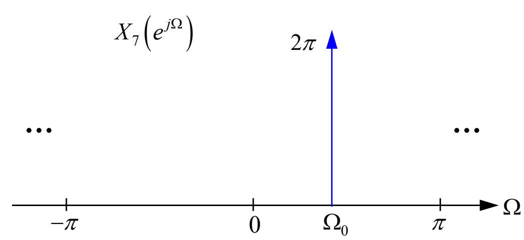
- In Chapter 3, we studied how to describe a periodic DT signal $x[n]$ by its DTFS coefficients a_k. As we show in this section, we can also describe a periodic DT signal $x[n]$ by a generalized DTFT $X\left(e^{j\Omega}\right)$.

- Just like the CTFT of a periodic CT signal described in Chapter 4, the DTFT of a periodic DT signal is needed to use Fourier analysis when *multiplying* a periodic DT signal by another DT signal (typically aperiodic). Examples include:
  - *Modulating* a DT signal by a periodic DT sinusoid.
  - *Sampling* a DT signal by multiplying it by a DT impulse train.
- Just as in CT, the DTFT of a periodic DT signal is not needed when *convolving* a periodic DT signal with another DT signal (typically aperiodic). We can perform Fourier analysis of such problems using the DTFS, as in Chapter 3 (see, e.g., page 124).
- We begin the discussion by studying an example.

7. *Imaginary exponential*. As in Example 5, we start with the DTFT, which is

$$X_7\left(e^{j\Omega}\right) = 2\pi \sum_{l=-\infty}^{\infty} \delta\left(\Omega - \Omega_0 - l2\pi\right),$$

as shown in the figure below.



- Using the inverse DTFT (8), we find the corresponding time signal to be

$$\begin{aligned} x_7[n] &= \frac{1}{2\pi} \int_{2\pi} X_7\left(e^{j\Omega}\right) e^{j\Omega n} d\Omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi\delta\left(\Omega - \Omega_0\right) e^{j\Omega n} d\Omega, \\ &= e^{j\Omega_0 n} \end{aligned}$$

where we have used the sampling property of the impulse function in evaluating the integral. We have found the DTFT pair:

$$e^{j\Omega_0 n} \stackrel{F}{\leftrightarrow} 2\pi \sum_{l=-\infty}^{\infty} \delta\left(\Omega - \Omega_0 - l2\pi\right). \tag{18}$$

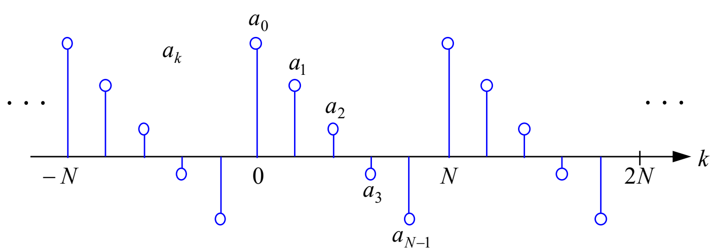
- We now study the *general case* of a periodic DT signal. Consider a signal $x[n]$ that is periodic with period $N = 2\pi / \Omega_0$. We can synthesize $x[n]$ using a DTFS:

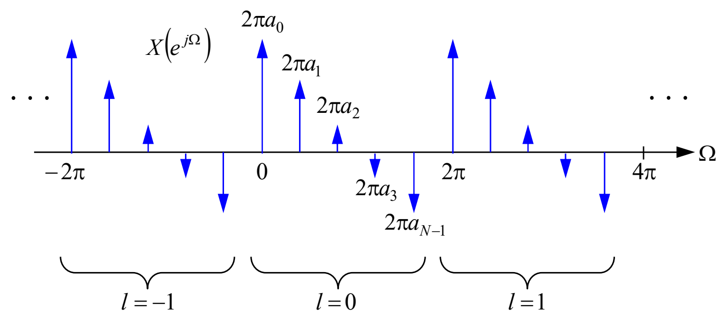
$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n} . \tag{19}$$

The a_k are the DTFS coefficients for the signal $x[n]$, and are periodic in k with period N, i.e., $a_{k+N} = a_k$. Now we compute the DTFT of (19). Since the DTFT is a linear operation, we compute it term-by-term using (18), and obtain

$$X\left(e^{j\Omega}\right) = \sum_{k=\langle N\rangle} 2\pi a_k \sum_{l=-\infty}^{\infty} \delta\left(\Omega - k\Omega_0 - l2\pi\right). \tag{20}$$

- The figure below helps us interpret the expression (20). Some exemplary DTFS coefficients a_k (not corresponding to a real signal $x[n]$) are shown on the left, while the corresponding DTFT (20) is shown on the right. To simplify the interpretation, we have chosen the N consecutive values of k, $k = \langle N\rangle$, to be $0 \le k \le N-1$. We observe that each value of l in (20) contributes a set of N impulses scaled by $2\pi a_k$, $0 \le k \le N-1$. We can replace the double summation over k and l in (20) by a single summation over k, $-\infty < k < \infty$.





We can express our result as a DTFT pair:

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n} \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - k\Omega_0\right). \tag{21}$$

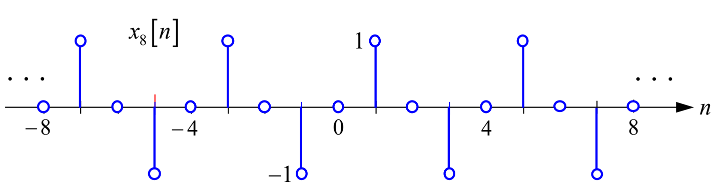
The DTFT of a periodic signal is a *train of impulses* at frequencies $\Omega = k\Omega_0$, which are integer multiples of the fundamental frequency $\Omega_0 = 2\pi / N$. Each impulse is scaled by 2π times the corresponding DTFS coefficient a_k.

- Now we present two more examples of DTFTs of periodic DT signals.

8. *Sine function*. We consider a signal

$$x_8[n] = \sin\left(\frac{\pi}{2} n\right),$$

which is shown below.



We can find the DTFS coefficients by inspection, as shown in Chapter 3 (see pages 114-115). The signal is periodic with period $N=4$, so we can express it as a linear combination of imaginary exponentials with frequencies $k\Omega_0 = k\dfrac{2\pi}{N} = k\dfrac{\pi}{2}$:

$$x_8[n] = \frac{1}{2j}\left(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}\right).$$

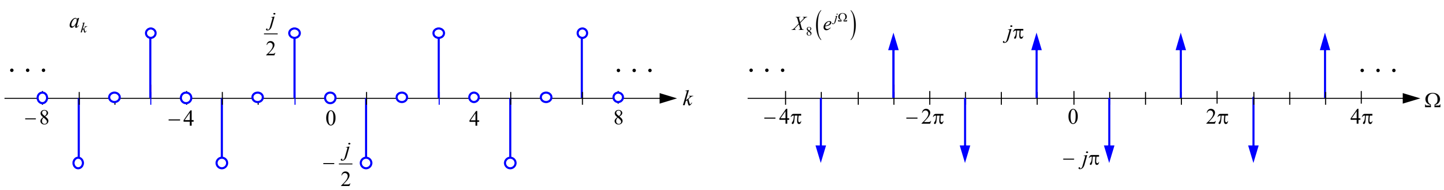
We find the DTFS coefficients to be

$$a_k = \begin{cases} \frac{j}{2} & k=-1 \\ -\frac{j}{2} & k=1 \\ 0 & k=0,2 \end{cases}.$$

Finally, we obtain the DTFT $X_8\left(e^{j\Omega}\right)$ using (21):

$$X_8\left(e^{j\Omega}\right) = \frac{\pi}{j}\sum_{l=-\infty}^{\infty}\left[\delta\left(\Omega - \frac{\pi}{2} - l2\pi\right) - \delta\left(\Omega + \frac{\pi}{2} - l2\pi\right)\right].$$

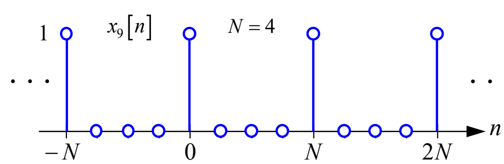
The figure below shows the DTFS coefficients a_k (on the left) and the DTFT $X_8\left(e^{j\Omega}\right)$ (on the right).



9. *Periodic impulse train*. We consider a signal

$$x_9[n] = \sum_{k=-\infty}^{\infty}\delta[n-kN], \tag{22}$$

as shown in the figure below for $N=4$. The signal $x_9[n]$ is periodic with period N and fundamental frequency $\Omega_0 = 2\pi/N$.



- We can compute the DTFS coefficients using the DTFS analysis equation:

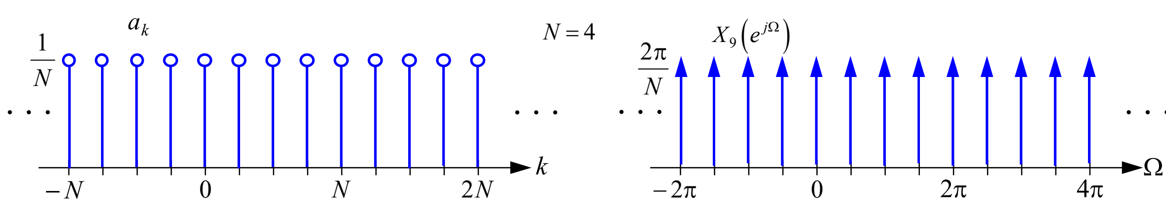
$$\begin{aligned} a_k &= \frac{1}{N}\sum_{n=\langle N\rangle} x_9[n]e^{-jk\Omega_0 n} \\ &= \frac{1}{N}\sum_{n=\langle N\rangle}\delta[n]e^{-jk\Omega_0 n} \\ &= \frac{1}{N}\quad \forall k \end{aligned}. \tag{23}$$

We have chosen the summation interval $n = \langle N \rangle$ to include the origin $n = 0$. In the second line of (23), we used the fact that among all the impulses in the infinite sum over k given by (22), only the one for $k = 0$ lies within the summation interval. In the third line of (23), we used the sampling property of the DT impulse function.

- Finally, we obtain the DTFT of $x_9[n]$ using (21):

$$\begin{aligned} X_9\left(e^{j\Omega}\right) &= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - k\Omega_0\right) \\ &= \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{N}\right) \end{aligned}. \tag{24}$$

We have found that the *DTFT of a periodic impulse train is a periodic impulse train*. The DTFS coefficients a_k and the DTFT $X_9\left(e^{j\Omega}\right)$ are shown in the figure below, assuming $N = 4$.



- We observe an inverse relationship between the spacing of impulses in the time domain, N, and the spacing between the impulses in the frequency domain, $2\pi / N$, much as we observed for the CT impulse train. We can summarize the result of this example as a DTFT pair:

$$\sum_{k=-\infty}^{\infty} \delta[n - kN] \overset{F}{\leftrightarrow} \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{N}\right). \tag{25}$$

Properties of Discrete-Time Fourier Transform

- Much like the CTFS, DTFS and CTFT properties we have already studied, these DTFT properties are helpful for:
  - Computing DTFTs for new signals, with minimal effort, by using the DTFTs we already know for other signals.
  - Verifying DTFTs that we compute for new signals.
- A complete list of DTFT properties is given in Table 5 in the Appendix. We discuss some of the most important properties here.
- We consider one or two signals and their DTFTs. Initially we denote these as

$$x[n] \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right) \quad \text{and} \quad y[n] \overset{F}{\leftrightarrow} Y\left(e^{j\Omega}\right).$$

- Many properties of the DTFT are similar to CTFT properties, and we present those first. Then we discuss several DTFT properties that are significantly different from CTFT properties.

Properties Similar to Continuous-Time Fourier Transform

Linearity

- A linear combination of $x[n]$ and $y[n]$ has a DTFT given by the corresponding linear combination of the DTFTs $X\left(e^{j\Omega}\right)$ and $Y\left(e^{j\Omega}\right)$:

$$ax[n]+by[n]\overset{F}{\leftrightarrow}aX\left(e^{j\Omega}\right)+bY\left(e^{j\Omega}\right).$$

Time Shift

- Time-shifting a signal by an integer n_0 corresponds to multiplication of its DTFT by a factor $e^{-j\Omega n_0}$:

$$x[n-n_0]\overset{F}{\leftrightarrow}e^{-j\Omega n_0}X\left(e^{j\Omega}\right). \tag{26}$$

The magnitude and phase of $e^{-j\Omega n_0}X\left(e^{j\Omega}\right)$ are related to those of $X\left(e^{j\Omega}\right)$ by

$$\begin{cases} \left|e^{-j\Omega n_0}X\left(e^{j\Omega}\right)\right|=\left|X\left(e^{j\Omega}\right)\right| \\ \angle\left(e^{-j\Omega n_0}X\left(e^{j\Omega}\right)\right)=\angle X\left(e^{j\Omega}\right)-\Omega n_0 \end{cases}. \tag{26$'$}$$

Time-shifting a signal by n_0 modifies its DTFT by

- Leaving the magnitude unchanged.
- Adding a phase shift proportional to $-n_0$, which varies linearly with frequency Ω.

Proof. Given a time-shifted signal $x[n-n_0]$, we compute its DTFT $F\left[x[n-n_0]\right]$ using (4):

$$F\left[x[n-n_0]\right]=\sum_{n=-\infty}^{\infty}x[n-n_0]e^{-j\Omega n}.$$

We change the variable of summation to $m=n-n_0$, and the DTFT becomes

$$\begin{aligned} F\left[x[n-n_0]\right]&=\sum_{m=-\infty}^{\infty}x[m]e^{-j\Omega(m+n_0)} \\ &=e^{-j\Omega n_0}\sum_{m=-\infty}^{\infty}x[m]e^{-j\Omega m} \\ &=e^{-j\Omega n_0}X\left(e^{j\Omega}\right) \end{aligned}$$

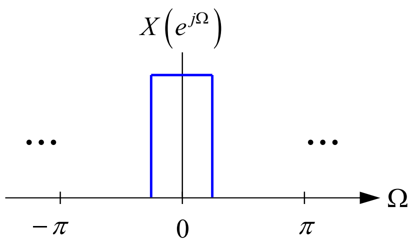
QED

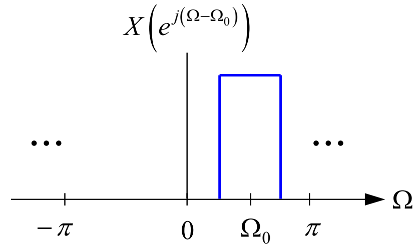
Frequency Shift

- The frequency-shift property is the dual of the time-shift property. It states that multiplying a signal by an imaginary exponential time signal $e^{j\Omega_0 n}$ causes its DTFT to be frequency-shifted by Ω_0:

$$x[n]e^{j\Omega_0 n}\overset{F}{\leftrightarrow}X\left(e^{j(\Omega-\Omega_0)}\right). \tag{27}$$

A DTFT $X\left(e^{j\Omega}\right)$ and the frequency-shifted DTFT $X\left(e^{j(\Omega-\Omega_0)}\right)$ are shown in the figure below.





Proof. Using (4), the DTFT of $x[n]e^{j\Omega_0 n}$ is

$$\begin{aligned} F\left[x[n]e^{j\Omega_0 n}\right] &= \sum_{n=-\infty}^{\infty} x[n]e^{j\Omega_0 n}e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\Omega-\Omega_0)n} \\ &= X\left(e^{j(\Omega-\Omega_0)}\right) \end{aligned}.$$

QED

- Frequency shifting is the basis for amplitude modulation of DT signals. It is also used to transform one type of DT filter into another; e.g., to change a lowpass filter into a bandpass filter or a highpass filter.

Time Reversal

- Reversal in time corresponds to reversal in frequency:

$$x[-n] \overset{F}{\leftrightarrow} X\left(e^{-j\Omega}\right). \tag{28}$$

- If a signal is even in time, its DTFT is even in frequency:

$$x[-n] = x[n] \overset{F}{\leftrightarrow} X\left(e^{-j\Omega}\right) = X\left(e^{j\Omega}\right),$$

while if a signal is odd in time, its DTFT is odd in frequency:

$$x[-n] = -x[n] \overset{F}{\leftrightarrow} X\left(e^{-j\Omega}\right) = -X\left(e^{j\Omega}\right).$$

Conjugation

$$x^*[n] \overset{F}{\leftrightarrow} X^*\left(e^{-j\Omega}\right). \tag{29}$$

Proof. Using (4), the DTFT of $x^*[n]$ is

$$\begin{aligned} F\left[x^*[n]\right] &= \sum_{n=-\infty}^{\infty} x^*[n] e^{-j\Omega n} \\ &= \left(\sum_{n=-\infty}^{\infty} x[n] e^{j\Omega n}\right)^* . \\ &= X^*\left(e^{-j\Omega}\right) \end{aligned}$$

QED

Conjugate Symmetry for Real Signal

- A real signal $x[n]$ is equal to its complex conjugate $x^*[n]$. This implies, in combination with the conjugation property, that

$$x[n] = x^*[n] \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right) = X^*\left(e^{-j\Omega}\right). \tag{30}$$

If a signal is real, its DTFT is *conjugate symmetric*: the DTFT at positive frequency is equal to the complex conjugate of the DTFT at negative frequency.

- The conjugate symmetry property can be restated in two ways. If a signal is real, the magnitude of its DTFT is even in frequency, while the phase of its DTFT is odd in frequency:

$$x[n] = x^*[n] \overset{F}{\leftrightarrow} \begin{cases} \left|X\left(e^{j\Omega}\right)\right| = \left|X\left(e^{-j\Omega}\right)\right| \\ \angle X\left(e^{j\Omega}\right) = -\angle X\left(e^{-j\Omega}\right) \end{cases}. \tag{30a}$$

Also, if a signal is real, the real part of its DTFT is even in frequency, while the imaginary part of its DTFT is odd in frequency:

$$x[n] = x^*[n] \overset{F}{\leftrightarrow} \begin{cases} \mathrm{Re}\left[X\left(e^{j\Omega}\right)\right] = \mathrm{Re}\left[X\left(e^{-j\Omega}\right)\right] \\ \mathrm{Im}\left[X\left(e^{j\Omega}\right)\right] = -\mathrm{Im}\left[X\left(e^{-j\Omega}\right)\right] \end{cases}. \tag{30b}$$

Real, Even or Real, Odd Signals

- By combining the time reversal and conjugation properties, we find that

$$x[n] \text{ real and even in } n \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right) \text{ real and even in } \Omega$$

and

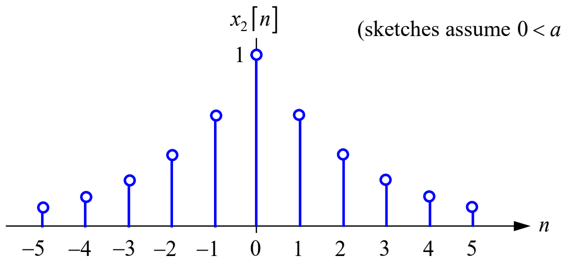
$$x[n] \text{ real and odd in } n \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right) \text{ imaginary and odd in } \Omega .$$

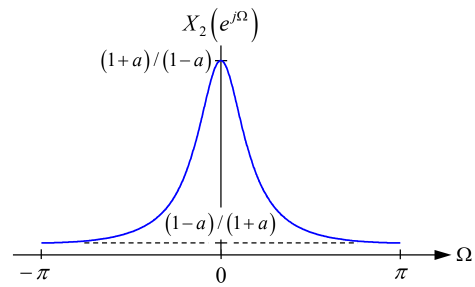
Examples of Symmetry Properties

2. *Real and even signal*. Recall Example 2:

$$x_2[n] = a^{|n|} \overset{F}{\leftrightarrow} X_2\left(e^{j\Omega}\right) = \frac{1-a^2}{1-2a\cos\Omega + a^2} ,$$

where we require $|a|<1$ .The signal and its DTFT are shown again below. The signal is real and even in *n*, so the DTFT is real and even in Ω.

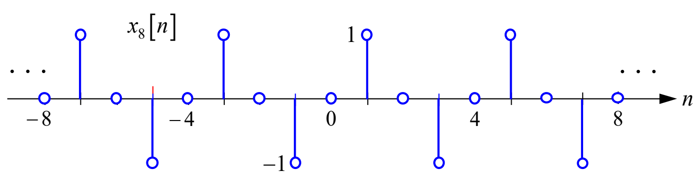


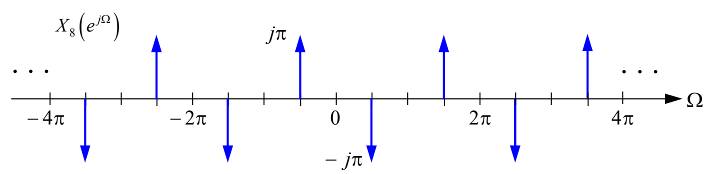


8. *Real and odd signal*. Recall Example 8:

$$x_8[n]=\sin\left(\frac{\pi}{2}n\right)\overset{F}{\leftrightarrow}X_8\left(e^{j\Omega}\right)=\frac{\pi}{j}\sum_{l=-\infty}^{\infty}\left[\delta\left(\Omega-\frac{\pi}{2}-l2\pi\right)-\delta\left(\Omega+\frac{\pi}{2}-l2\pi\right)\right].$$

The signal and its DTFT are shown once again below. The signal is real and odd in *n*, so the DTFT is imaginary and odd in Ω.





Differentiation in Frequency

- This property states that

$$nx[n]\overset{F}{\leftrightarrow}j\frac{dX\left(e^{j\Omega}\right)}{d\Omega}. \tag{31}$$

Multiplication of a signal by time *n* corresponds to differentiation of its DTFT with respect to frequency Ω (and scaling by a factor *j*). In order to prove this property, we differentiate the analysis equation (4) with respect to Ω, and find that $dX\left(e^{j\Omega}\right)/d\Omega$ is the DTFT of a signal $-jnx[n]$.

Example of Differentiation-in-Frequency Property

10. *Impulse response of second-order system*. Later in this chapter, we will see that the following DTFT pair describes the impulse and frequency responses for a second-order system with $0\le r<1$ and $\theta=0$ or π (see pages 227-229 below):

$$x_{10}[n]=(n+1)a^n u[n]\overset{F}{\leftrightarrow}X_{10}\left(e^{j\Omega}\right)=\frac{1}{\left(1-ae^{-j\Omega}\right)^2}, \tag{32}$$

where we assume *a* is real and $|a|<1$. In this example, we derive (32). We start with the DTFT pair derived in Example 1:

$$x_1[n] = a^n u[n] \overset{F}{\leftrightarrow} X_1\left(e^{j\Omega}\right) = \frac{1}{1 - ae^{-j\Omega}}.$$

Applying the differentiation-in-frequency property:

$$nx_1[n] = na^n u[n] \overset{F}{\leftrightarrow} j\frac{dX_1\left(e^{j\Omega}\right)}{d\Omega} = \frac{ae^{-j\Omega}}{\left(1 - ae^{-j\Omega}\right)^2}.$$

Using the time-shifting property with $n_0 = -1$:

$$(n+1)a^{n+1}u[n+1] \overset{F}{\leftrightarrow} \frac{a}{\left(1 - ae^{-j\Omega}\right)^2}.$$

We note that at time $n = -1$, the signal $(n+1)a^{n+1}u[n+1]$ vanishes, so it can be rewritten as $(n+1)a^{n+1}u[n]$. Dividing both sides by a, we obtain (32).

Convolution Property

- This property is identical in form to the convolution property for the CTFT. Given two DT signals and their DTFTs

  $$p[n] \overset{F}{\leftrightarrow} P\left(e^{j\Omega}\right) \text{ and } q[n] \overset{F}{\leftrightarrow} Q\left(e^{j\Omega}\right),$$

  the convolution property states that

  $$p[n] * q[n] \overset{F}{\leftrightarrow} P\left(e^{j\Omega}\right)Q\left(e^{j\Omega}\right), \tag{33}$$

  in other words, *convolution in the time domain* corresponds to *multiplication in the frequency domain*.

  Proof. We express the convolution as a sum and compute its DTFT using (4):

  $$F\left[p[n] * q[n]\right] = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} p[k]q[n-k]\right]e^{-jn\Omega}.$$

  We now interchange the order of summation:

  $$F\left[p[n] * q[n]\right] = \sum_{k=-\infty}^{\infty} p[k]\left[\sum_{n=-\infty}^{\infty} q[n-k]e^{-jn\Omega}\right].$$

  We recognize the quantity in square brackets as the DTFT of $q[n-k]$. By the time-shift property (26), this is $Q\left(e^{j\Omega}\right)e^{-jk\Omega}$. Thus, we have

  $$F\left[p[n] * q[n]\right] = Q\left(e^{j\Omega}\right)\sum_{k=-\infty}^{\infty} p[k]e^{-jk\Omega}.$$

  We recognize the sum as $P\left(e^{j\Omega}\right)$, the DTFT of $p[n]$. We have proven (33).

  QED

Example of Convolution Property

- In a homework problem studying the cascade of two first-order DT systems, you derived the convolution

$$a^n u[n] * b^n u[n] = \frac{b^{n+1} - a^{n+1}}{b-a} u[n], \quad a \neq b .$$

Here we study the case in which $a = b$. We start with the result of Example 1:

$$x_1[n] = a^n u[n] \overset{F}{\leftrightarrow} X_1\left(e^{j\Omega}\right) = \frac{1}{1 - ae^{-j\Omega}} .$$

Applying the convolution property:

$$x_1[n] * x_1[n] \overset{F}{\leftrightarrow} X_1^2\left(e^{j\Omega}\right)$$

or

$$a^n u[n] * a^n u[n] \overset{F}{\leftrightarrow} \frac{1}{\left(1 - ae^{-j\Omega}\right)^2} . \tag{34}$$

But we know from Example 10 that

$$(n+1) a^n u[n] \overset{F}{\leftrightarrow} \frac{1}{\left(1 - ae^{-j\Omega}\right)^2} . \tag{32}$$

Since the right-hand sides of (34) and (32) are equal, the left-hand sides must be equal:

$$a^n u[n] * a^n u[n] = (n+1) a^n u[n] .$$

We have shown that a second-order system described by (32) is equivalent to a cascade of two identical first-order systems.

Frequency Response of Discrete-Time Linear Time-Invariant Systems

- As in CT, the most important application of the DTFT convolution property is to filtering signals by LTI systems. Consider a DT LTI system that has an impulse response $h[n]$. As we saw in Chapter 3, the system's frequency response $H\left(e^{j\Omega}\right)$ is the DTFT of $h[n]$ (see (46), page 109). The impulse and frequency responses of a DT LTI system *form a DTFT pair*:

$$h[n] \overset{F}{\leftrightarrow} H\left(e^{j\Omega}\right) . \tag{35}$$

- Now suppose an input signal $x[n] \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right)$ is fed into the system, yielding an output signal $y[n] \overset{F}{\leftrightarrow} Y\left(e^{j\Omega}\right)$, as shown below.

$$x[n] \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right) \longrightarrow \boxed{h[n] \overset{F}{\leftrightarrow} H\left(e^{j\Omega}\right)} \longrightarrow y[n] \overset{F}{\leftrightarrow} Y\left(e^{j\Omega}\right)$$

In the *time domain*, we obtain the output signal by *convolving* the input signal and the impulse response:

$$y[n] = h[n] * x[n]. \tag{36}$$

According to the convolution property (33), the DTFT of the output signal (36) is the right-hand side of

$$y[n] = h[n] * x[n] \overset{F}{\leftrightarrow} Y\left(e^{j\Omega}\right) = H\left(e^{j\Omega}\right) X\left(e^{j\Omega}\right). \tag{37}$$

Expression (37) states that in the *frequency domain*, the DTFT of the output signal is found by *multiplying* the DTFT of the input signal by the frequency response of the LTI system. Just as in CT, this view of LTI filtering as frequency-domain multiplication is intuitively appealing and, in many problems, it represents an easier method of solution than time-domain convolution.

- According to (35), the frequency response $H\left(e^{j\Omega}\right)$ is the DTFT of the impulse response $h[n]$. Accordingly, all the properties of the frequency response discussed in Chapter 3 may be understood in terms of DTFT properties studied in this chapter. For example, the frequency response is always periodic in frequency Ω with period 2π:

$$H\left(e^{j(\Omega+2\pi)}\right) = H\left(e^{j\Omega}\right). \tag{5′}$$

Also, if the impulse response $h[n]$ is real, then by (30), the frequency response has conjugate symmetry:

$$h[n] = h^*[n] \overset{F}{\leftrightarrow} H\left(e^{j\Omega}\right) = H^*\left(e^{-j\Omega}\right). \tag{30c}$$

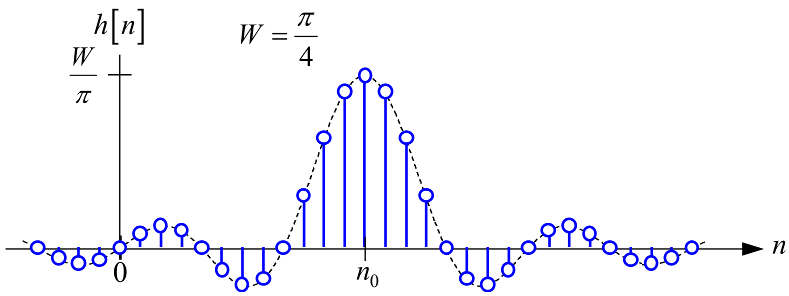
Example: Ideal Lowpass Filter

- An ideal DT lowpass filter is similar to an ideal CT lowpass filter (see pages 172-173) except that the DT frequency response must be periodic in frequency Ω. Over the frequency range $-\pi \le \Omega \le \pi$, an ideal DT lowpass filter with *cutoff frequency* W ($W < \pi$) and *group delay* n_0 (n_0 integer) has the following characteristics:
    - *Passband*: for $|\Omega| < W$, the magnitude $\left|H\left(e^{j\Omega}\right)\right|$ is constant, and the phase $\angle H\left(e^{j\Omega}\right)$ is a linear function of Ω with integer slope, $\angle H\left(e^{j\Omega}\right) = -\Omega n_0$.
    - *Transition*: at $|\Omega| = W$, the magnitude $\left|H\left(e^{j\Omega}\right)\right|$ goes abruptly to zero.
    - *Stopband*: for $|\Omega| > W$, the magnitude $\left|H\left(e^{j\Omega}\right)\right|$ is zero. (The phase can assume any value.)
- Using (14) and the time-shift property (26), the impulse and frequency responses of an ideal lowpass filter are

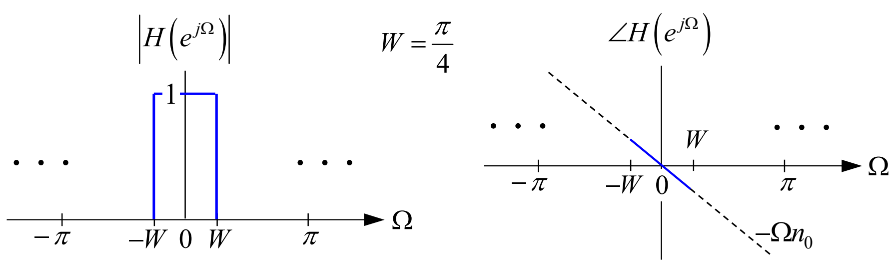
$$h[n]=\frac{W}{\pi}\operatorname{sinc}\left(\frac{W(n-n_0)}{\pi}\right)\overset{F}{\leftrightarrow}H\left(e^{j\Omega}\right)=e^{-j\Omega n_0}\sum_{l=-\infty}^{\infty}\Pi\left(\frac{\Omega-l2\pi}{2W}\right). \tag{38}$$

As in Example 5, the summation ensures that $H\left(e^{j\Omega}\right)$ is periodic in Ω with period 2π. The linear phase factor need not be inside the summation, since $e^{-j(\Omega-l2\pi)n_0}=e^{-j\Omega n_0}e^{jl2\pi n_0}=e^{-j\Omega n_0}$.

- The impulse response is shown below, assuming $W=\pi/4$. The impulse response peaks at $n=n_0$, but extends to $n=\pm\infty$. Like an ideal CT lowpass filter, an ideal DT lowpass filter cannot be causal, except in the limit of infinite group delay, $n_0\to\infty$.



- The magnitude and phase responses are shown below.



The magnitude $\left|H\left(e^{j\Omega}\right)\right|$ is ideal, as desired. The phase $\angle H\left(e^{j\Omega}\right)$ is linear in the passband, with an integer group delay $-d\angle H\left(e^{j\Omega}\right)/d\Omega=n_0$.

- We will use ideal DT lowpass, bandpass and highpass filters frequently in EE 102A and 102B, and will often set the group delay n_0 to zero in order to simplify our analyses. Nevertheless, a causal filter that is designed to approximate the abrupt transition of an ideal filter must have a long group delay.

Example: Ideal Lowpass Filter with Sinc Function Input

- To illustrate how frequency-domain multiplication can provide an easier method of solution than time-domain convolution, we consider inputting a sinc function

$$x[n]=\frac{1}{2}\operatorname{sinc}\left(\frac{n}{2}\right)$$

to a lowpass filter with impulse response

$$h[n] = \frac{1}{4}\operatorname{sinc}\left(\frac{n}{4}\right),$$

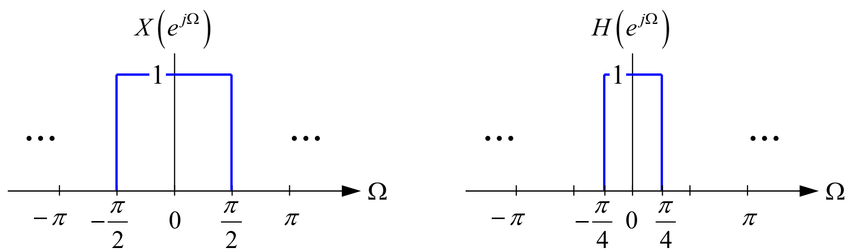
corresponding to (38) with $W = \pi/4$ and $n_0 = 0$.

- To compute the output signal $y[n]$ in the time domain, we should perform the convolution

$$y[n] = x[n] * h[n] = \frac{1}{2}\operatorname{sinc}\left(\frac{n}{2}\right) * \frac{1}{4}\operatorname{sinc}\left(\frac{n}{4}\right),$$

but this is difficult to evaluate.

- To compute the output in the frequency domain, we use the DTFT of the input, $X\left(e^{j\Omega}\right)$, and the system frequency response $H\left(e^{j\Omega}\right)$, which are shown below.



The DTFT of the output is

$$Y\left(e^{j\Omega}\right) = X\left(e^{j\Omega}\right)H\left(e^{j\Omega}\right) = H\left(e^{j\Omega}\right),$$

so the output signal is

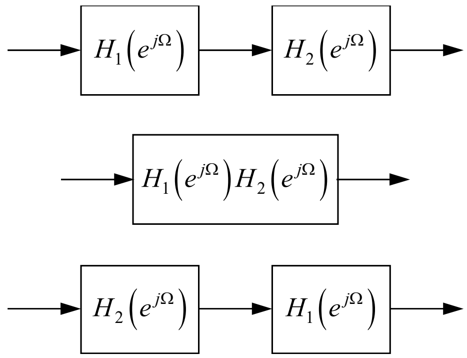
$$y[n] = h[n] = \frac{1}{4}\operatorname{sinc}\left(\frac{n}{4}\right).$$

Frequency Response of Cascaded Linear Time-Invariant Systems

- Consider two LTI systems

$$h_1[n] \overset{F}{\leftrightarrow} H_1\left(e^{j\Omega}\right) \quad \text{and} \quad h_2[n] \overset{F}{\leftrightarrow} H_2\left(e^{j\Omega}\right).$$

- As shown in Chapter 2 (see page 62), when two LTI systems are cascaded, the overall impulse response of the cascade is the convolution of the impulse responses of the two systems, and is independent of the order in which the two systems are cascaded.
- By the convolution property of the DTFT, the overall frequency response of the cascade is the product of the frequency responses of the two systems, and does not depend on the order in which the two systems are cascaded.
- Thus, the following three LTI systems have identical input-output relationships.



Properties Different from Continuous-Time Fourier Transform

- Now we study properties of the DTFT that differ significantly from the corresponding CTFT properties. The differences arise because DT signals are functions of a *discrete* time variable *n*, and their DTFTs are *periodic* functions of frequency Ω.
- Once again, we consider one or two signals and their DTFTs. For now, we denote these as

$$x[n] \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right) \quad \text{and} \quad y[n] \overset{F}{\leftrightarrow} Y\left(e^{j\Omega}\right).$$

Periodicity

- Any DTFT must be a periodic function of frequency Ω with period 2π:

$$X\left(e^{j(\Omega+2\pi)}\right) = X\left(e^{j\Omega}\right).$$

First Difference

- Taking the first difference of a DT signal corresponds to *multiplying* its DTFT by a factor $1-e^{-j\Omega}$:

$$x[n]-x[n-1] \overset{F}{\leftrightarrow} \left(1-e^{-j\Omega}\right) X\left(e^{j\Omega}\right).$$

  This is not a separate property, but is simply a consequence of linearity and the time-shift property (26). We mention it because the first difference of a DT signal may be considered somewhat analogous to the time derivative of a CT signal. In Chapter 6, we will learn how to design an FIR DT filter that approximates a CT differentiator far better than a DT first-difference system does.

Running Summation (Accumulation)

- Taking the running summation of a time-domain signal corresponds to *dividing* its DTFT by a factor $1-e^{-j\Omega}$:

$$\begin{aligned} \sum_{m=-\infty}^{n} x[m] \overset{F}{\leftrightarrow} & \frac{1}{1-e^{-j\Omega}} X\left(e^{j\Omega}\right) + \pi X\left(e^{j\Omega}\right) \sum_{k=-\infty}^{\infty} \delta(\Omega - k2\pi) \\ = & \frac{1}{1-e^{-j\Omega}} X\left(e^{j\Omega}\right) + \pi X\left(e^{j0}\right) \sum_{k=-\infty}^{\infty} \delta(\Omega - k2\pi) \end{aligned}. \tag{39}$$

This is to be expected, since time-domain first differencing corresponds to multiplying the DTFT by $1-e^{-j\Omega}$. As in the CTFT integration property, there is an additional term on the right-hand side of (39):

$$\pi X\left(e^{j\Omega}\right)\sum_{k=-\infty}^{\infty}\delta(\Omega-k2\pi)=\pi X\left(e^{j0}\right)\sum_{k=-\infty}^{\infty}\delta(\Omega-k2\pi),$$

which is nonzero if the original time-domain signal $x[n]$ has a non-zero d.c. value:

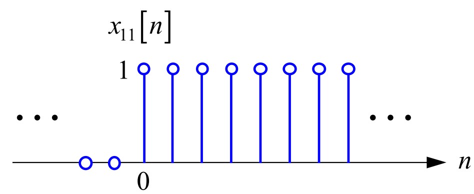
$$X\left(e^{j0}\right)=\sum_{n=-\infty}^{\infty}x[n]\neq 0.$$

Example of Accumulation Property

11. Unit step function. We are given a unit step signal

$$x_{11}[n]=u[n],$$

as shown below.



We start with Example 4, which studied the unit impulse:

$$x_4[n]=\delta[n]\overset{F}{\leftrightarrow}X_4\left(e^{j\Omega}\right)=1\quad\forall\Omega$$

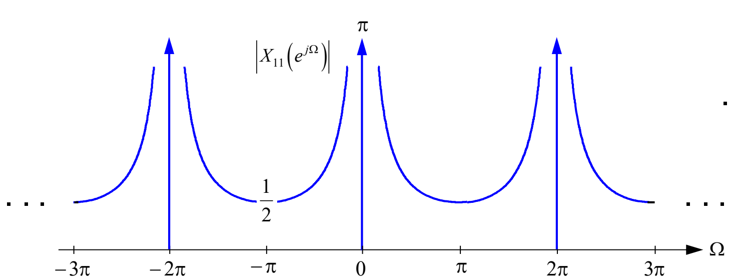
and recall from Chapter 1 that the unit step is the running summation of the unit impulse:

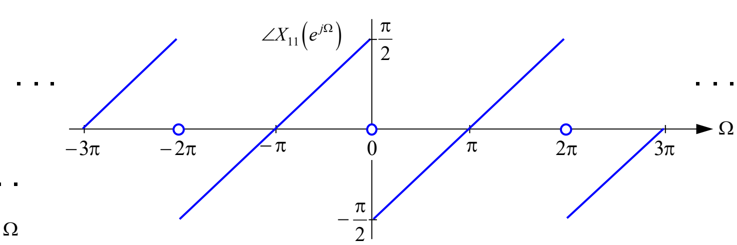
$$u[n]=\sum_{m=-\infty}^{n}\delta[m].$$

Using the accumulation property (39), we find the DTFT of the unit step to be

$$X_{11}\left(e^{j\Omega}\right)=\frac{1}{1-e^{-j\Omega}}+\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-k2\pi).$$

The DTFT $X_{11}\left(e^{j\Omega}\right)$ is complex-valued, and its magnitude and phase are shown below.





Time Scaling

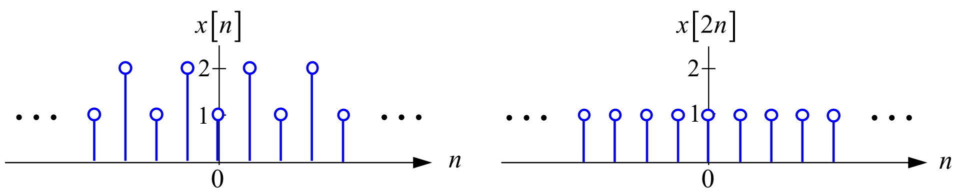
- We studied the operations of time compression and time expansion for DT signals in Chapter 1 (see page 9), and review them here.

Time Compression

- Consider a positive integer $k \geq 1$. Given a DT signal $x[n]$, the *compressed signal* is

$$x[kn].$$

- If $k > 1$, samples of the signal are lost. An example is shown for $k = 2$.



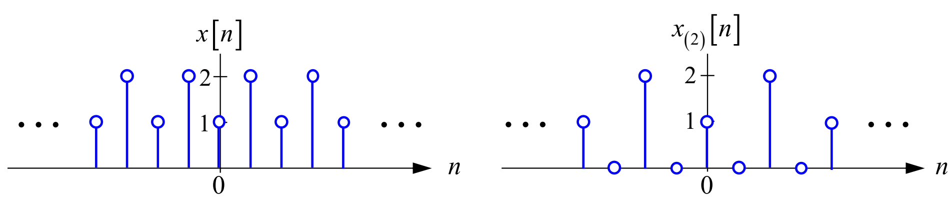
- We will not consider time compression further in EE 102A.

Time Expansion

- Consider a positive integer $m \geq 1$. Given a DT signal $x[n]$, the *expanded signal* is

$$x_{(m)}[n] = \begin{cases} x\left[\frac{n}{m}\right] & \frac{n}{m} \text{ integer} \\ 0 & \text{otherwise} \end{cases}. \tag{40}$$

- For any positive integer *m*, no samples of the signal are lost. An example is shown for $m = 2$.



- Now we compute the DTFT of the time-expanded signal (40), which we denote by $X_{(m)}\left(e^{j\Omega}\right)$. Using the DTFT analysis equation (4), we have

$$X_{(m)}\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x_{(m)}[n] e^{-j\Omega n}.$$

Noting that $x_{(m)}[n] = 0$ unless n/m is an integer, we change the summation index to $l = n/m$ (so that $n = lm$):

$$X_{(m)}\left(e^{j\Omega}\right) = \sum_{l=-\infty}^{\infty} x_{(m)}[lm] e^{-j\Omega lm}.$$

Now we use the fact that by (40), $x_{(m)}[lm] = x[l]$ to write

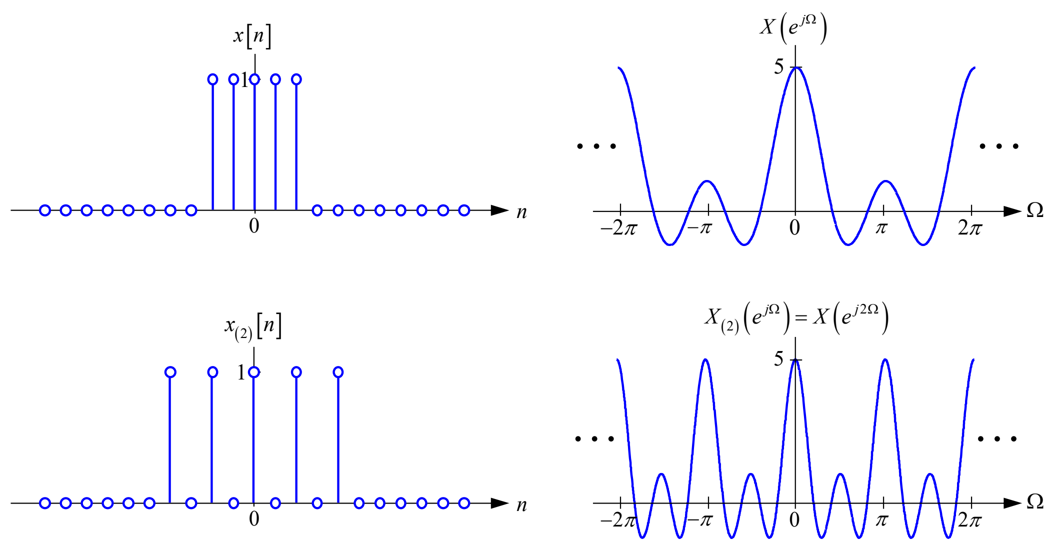
$$
\begin{aligned}
X_{(m)}\left(e^{j\Omega}\right) &= \sum_{l=-\infty}^{\infty} x[l]e^{-j(m\Omega)l} \\
&= X\left(e^{jm\Omega}\right)
\end{aligned}
.
$$

In summary, we have found that

$$
x_{(m)}[n] \overset{F}{\leftrightarrow} X\left(e^{jm\Omega}\right), \tag{41}
$$

in other words, *expanding time* by a factor $m \geq 1$ *compresses frequency* by a factor *m* in the DTFT.

- In the figures below, we use the rectangular pulse signal from Example 6 to illustrate time expansion and frequency compression by a factor $m = 2$.



Parseval's Identity

- The significance of inner products between signals is explained on pages 91-93 above. Parseval's identity for the DTFT enables us to compute the inner product between two DT signals, or the energy of one DT signal, either in the time or frequency domain. Computing an inner product or energy for any particular signal(s) can be far easier in one domain or the other.

Inner Product Between Signals

- The general form of Parseval's identity, for an *inner product between two DT signals*, states that

$$
\langle x[n], y[n] \rangle = \sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\Omega}\right)Y^*\left(e^{j\Omega}\right)d\Omega . \tag{42}
$$

The middle expression in (42) is an inner product between two signals, $x[n]$ and $y[n]$, calculated in the time domain. The rightmost expression in (42) is an inner product between the corresponding DTFTs, $X\left(e^{j\Omega}\right)$ and $Y\left(e^{j\Omega}\right)$, calculated in the frequency domain.

Proof

We start with the middle expression in (42) and represent $x[n]$ by the inverse DTFT of $X\left(e^{j\Omega}\right)$:

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{2\pi} X\left(e^{j\Omega}\right)e^{j\Omega n}d\Omega\right]y^*[n].$$

Now we interchange the order of summation and integration, and recognize the quantity in square brackets as $Y\left(e^{j\Omega}\right)$:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x[n]y^*[n] &= \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\Omega}\right)\left[\sum_{n=-\infty}^{\infty} y[n]e^{-j\Omega n}\right]^* d\Omega \\ &= \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\Omega}\right)Y^*\left(e^{j\Omega}\right)d\Omega\end{aligned}.$$

QED

Signal Energy

- Now we consider the special case of (42) with $x[n]=y[n]$ and $X\left(e^{j\Omega}\right)=Y\left(e^{j\Omega}\right)$, and obtain an expression for the *energy of a DT signal*:

$$E = \sum_{n=-\infty}^{\infty}\left|x[n]\right|^2 = \frac{1}{2\pi}\int_{2\pi}\left|X\left(e^{j\Omega}\right)\right|^2 d\Omega \ . \tag{43}$$

- In expression (43), we can identify $\left|X\left(e^{j\Omega}\right)\right|^2$ as the *energy density spectrum* of the signal $x[n]$, because $\left|X\left(e^{j\Omega}\right)\right|^2$ measures the energy in the Fourier component at a frequency Ω. We can interpret the rightmost expression in (43) as an integral of the energies in all the Fourier components over a frequency interval of length 2π.

Example: Energy of Sinc Function

- We would like to compute the energy of a signal

$$x[n] = \frac{1}{3}\text{sinc}\left(\frac{n}{3}\right).$$

- To compute the energy in the time domain, we must evaluate the sum

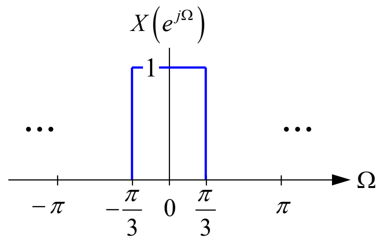
$$E = \sum_{n=-\infty}^{\infty}\left|x[n]\right|^2 = \sum_{n=-\infty}^{\infty}\left(\frac{1}{3}\right)^2 \text{sinc}^2\left(\frac{n}{3}\right),$$

which is difficult.

- It is much easier to compute the energy in the frequency domain using Parseval's identity (43). Using Example 5 with $W=\pi/3$, the DTFT of $x[n]$ is

$$X\left(e^{j\Omega}\right)=\sum_{l=-\infty}^{\infty}\Pi\left(\frac{\Omega-l2\pi}{2\pi/3}\right)$$

which is shown in the figure below.



Using the rightmost expression in (43) and choosing an integration interval $-\pi \le \Omega \le \pi$, the energy is

$$\begin{aligned}E &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X\left(e^{j\Omega}\right)\right|^2 d\Omega \\ &= \frac{1}{2\pi}\int_{-\pi/3}^{\pi/3}(1)^2 d\Omega \\ &= \frac{1}{3}\end{aligned}.$$

Example: Inner Product Between Two Signals

- We would like to compute the inner product between two signals

$$x[n]=\frac{1}{4}\text{sinc}\left(\frac{n}{4}\right)\cos\left(\frac{\pi}{2}n\right) \quad \text{and} \quad y[n]=\frac{1}{4}\text{sinc}\left(\frac{n}{4}\right)\cos(\pi n).$$

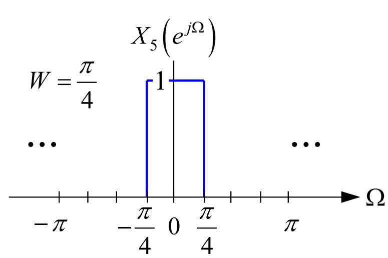
- To compute the inner product in the time domain, we need to evaluate the summation

$$\sum_{n=-\infty}^{\infty}x[n]y^*[n]=\sum_{n=-\infty}^{\infty}\left(\frac{1}{4}\right)^2\text{sinc}^2\left(\frac{n}{4}\right)\cos\left(\frac{\pi}{2}n\right)\cos(\pi n).$$

- We find it easier to evaluate the inner product in the frequency domain.
  - First, we use Example 5 with $W=\pi/4$ to obtain the DTFT pair

$$x_5[n]=\frac{1}{4}\text{sinc}\left(\frac{n}{4}\right)\overset{F}{\leftrightarrow}X_5\left(e^{j\Omega}\right)=\sum_{l=-\infty}^{\infty}\Pi\left(\frac{\Omega-l2\pi}{\pi/2}\right).$$

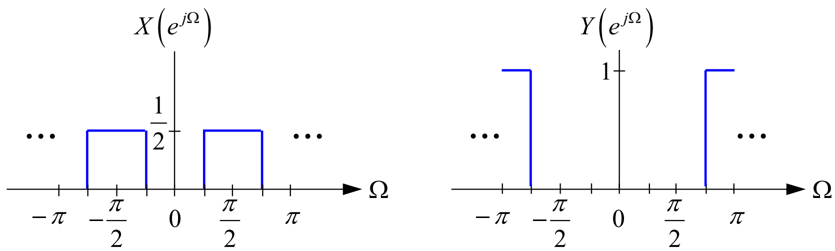
  The DTFT $X_5\left(e^{j\Omega}\right)$ is shown below.



- Both $x[n]$ and $y[n]$ are of the form of signal $x_5[n]$ multiplied by a cosine signal at some frequency Ω_0. We can compute their DTFTs with the help of the frequency shift property (27):

$$x_5[n]\cos(\Omega_0 n) = \frac{1}{2}x_5[n]\left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right) \overset{F}{\leftrightarrow} \frac{1}{2}\left[X_5\left(e^{j(\Omega-\Omega_0)}\right) + X_5\left(e^{j(\Omega+\Omega_0)}\right)\right].$$

  The DTFTs $X\left(e^{j\Omega}\right)$ and $Y\left(e^{j\Omega}\right)$ are shown below.



  The DTFT $X\left(e^{j\Omega}\right)$ (on the left) comprises copies of $X_5\left(e^{j\Omega}\right)$ scaled by 1/2 and shifted to $\pm\Omega_0 = \pm\pi/2$. The DTFT $Y\left(e^{j\Omega}\right)$ (on the right) comprises copies of $X_5\left(e^{j\Omega}\right)$ scaled by 1/2 and shifted to $\pm\Omega_0 = \pm\pi$; these copies overlap and add together to yield a height of 1.

- Finally, we compute the inner product between the two signals using the rightmost expression in (42), choosing an integration interval $-\pi \le \Omega \le \pi$. Observe that the two DTFTs $X\left(e^{j\Omega}\right)$ and $Y\left(e^{j\Omega}\right)$ are nonzero over disjoint intervals of Ω, so the product $X\left(e^{j\Omega}\right)Y^*\left(e^{j\Omega}\right)$ is zero at all Ω. Hence, the integral vanishes:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\Omega}\right)Y^*\left(e^{j\Omega}\right)d\Omega = 0.$$

- By (42), the inner product between $x[n]$ and $y[n]$ is zero. The two signals are mutually orthogonal.

Multiplication Property

- The DTFT multiplication property, much like the corresponding CTFT property, has several important applications:
  - Modulation and demodulation of DT signals.
  - Sampling of DT signals (discussed in EE 102B).
  - Windowing (discussed in Chapter 6).
- Given two DT signals and their DTFTs

$$p[n] \overset{F}{\leftrightarrow} P\left(e^{j\Omega}\right) \text{ and } q[n] \overset{F}{\leftrightarrow} Q\left(e^{j\Omega}\right),$$

  the multiplication property states that

$$p[n]q[n] \overset{F}{\leftrightarrow} \frac{1}{2\pi}\int_{2\pi} P\left(e^{j\theta}\right)Q\left(e^{j(\Omega-\theta)}\right)d\theta . \tag{44}$$

As one might expect, *multiplication in the time domain* corresponds to *convolution in the frequency domain*. However, because the DTFTs $P\left(e^{j\Omega}\right)$ and $Q\left(e^{j\Omega}\right)$ are periodic functions, the convolution on the right-hand side of (44) is a *periodic convolution*. It is the same as an ordinary convolution, except the integration is performed only over *one period* – any interval of length 2π – instead of the interval from $-\infty$ to ∞. The result of the periodic convolution is nevertheless a periodic function of Ω, because the factor $Q\left(e^{j(\Omega-\theta)}\right)$ in the integrand is periodic in Ω.

Proof. Using the DTFT analysis equation, the DTFT of $p[n]q[n]$ is

$$F\left[p[n]q[n]\right] = \sum_{n=-\infty}^{\infty} p[n]q[n]e^{-jn\Omega} .$$

Using (8), we express $p[n]$ as the inverse DTFT of $P\left(e^{j\Omega}\right)$, employing an integration variable θ:

$$F\left[p[n]q[n]\right] = \sum_{n=-\infty}^{\infty} q[n]\left[\frac{1}{2\pi}\int_{2\pi} P\left(e^{j\theta}\right)e^{j\theta n}d\theta\right]e^{-jn\Omega} .$$

Now we interchange the order of summation and integration:

$$F\left[p[n]q[n]\right] = \frac{1}{2\pi}\int_{2\pi} P\left(e^{j\theta}\right)\left[\sum_{n=-\infty}^{\infty} q[n]e^{-jn(\Omega-\theta)}\right]d\theta .$$

We recognize the sum as $Q\left(e^{j(\Omega-\theta)}\right)$, the DTFT of $q[n]$ at frequency $\Omega-\theta$. We have proven (44).

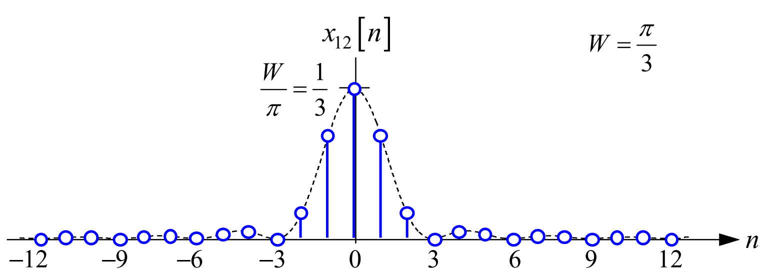
QED

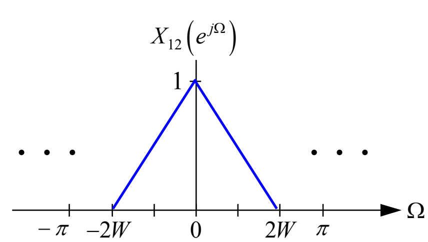
Example of Multiplication Property

12. *Sinc-squared function*. We are given a signal

$$x_{12}[n] = \frac{W}{\pi}\mathrm{sinc}^2\left(\frac{W}{\pi}n\right),$$

which is shown in the figure below (on the left). We wish to compute its DTFT $X_{12}\left(e^{j\Omega}\right)$.

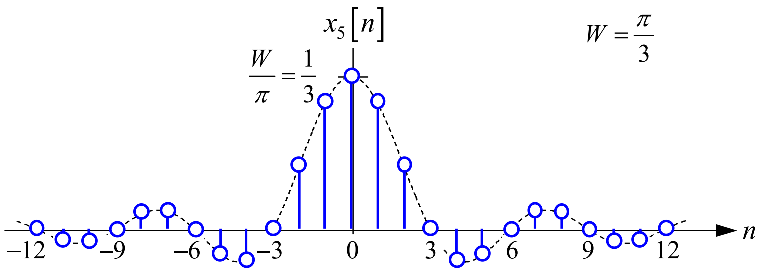


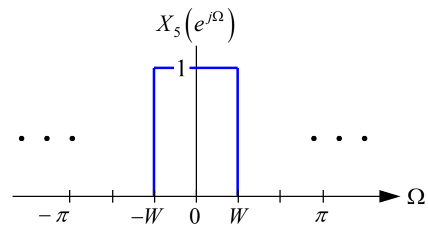


- We start by recalling Example 5:

$$x_5[n] = \frac{W}{\pi}\operatorname{sinc}\left(\frac{W}{\pi}n\right) \overset{F}{\leftrightarrow} X_5\left(e^{j\Omega}\right) = \sum_{l=-\infty}^{\infty} \Pi\left(\frac{\Omega - l2\pi}{2W}\right). \tag{14}$$

The signal $x_5[n]$ and its DTFT $X_5\left(e^{j\Omega}\right)$ are shown in the figure below.





- We observe that signal $x_{12}[n]$ is π / W times the square of signal $x_5[n]$:

$$x_{12}[n] = \frac{\pi}{W} x_5^2[n].$$

Using the multiplication property (44), the DTFT $X_{12}\left(e^{j\Omega}\right)$ is proportional to a periodic convolution between $X_5\left(e^{j\Omega}\right)$ and itself:

$$X_{12}\left(e^{j\Omega}\right) = \frac{\pi}{W} \times \frac{1}{2\pi} \int_{2\pi} X_5\left(e^{j\theta}\right) X_5\left(e^{j(\Omega-\theta)}\right) d\theta.$$

Since $X_5\left(e^{j\Omega}\right)$ is a rectangular pulse train, the periodic convolution yields a triangular pulse train. The DTFT $X_{12}\left(e^{j\Omega}\right)$ is shown in the figure near the bottom of page 221 (on the right). We have found the DTFT pair:

$$\frac{W}{\pi}\operatorname{sinc}^2\left(\frac{W}{\pi}n\right) \overset{F}{\leftrightarrow} \sum_{l=-\infty}^{\infty} \Lambda\left(\frac{\Omega - l2\pi}{2W}\right). \tag{45}$$

Linear Time-Invariant Systems Governed by Constant-Coefficient Difference Equations

- In this section, we discuss causal LTI DT systems that can be described by constant-coefficient linear difference equations in the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \tag{46}$$

In (46), $x[n]$ and $y[n]$ denote the input and output signals. The a_k, $k = 0,\ldots,N$ and b_k, $k = 0,\ldots,M$ are constants, and are real-valued in systems that map real inputs to real outputs. We studied this class of LTI systems in Chapter 2 (see pages 71-74).

- Here, we describe a technique for determining the frequency response of a system described by (46). The technique is equivalent to one introduced in Chapter 3 (see Method 2, pages 128-129), and we restate it here more formally in terms of the DTFT. In order for the technique to be applicable, we require that the system impulse response $h[n]$ be absolutely summable

$$\sum_{n=-\infty}^{\infty} \left| h[n] \right| < \infty , \tag{47}$$

so the system is BIBO stable and the frequency response $H\left(e^{j\Omega}\right)$ exists in a strict sense. As an example in which condition (47) is not satisfied, consider an accumulator system, whose output is the running summation of the input:

$$y[n] = \sum_{m=-\infty}^{n} x[m] . \tag{48}$$

Recall that the accumulator impulse response is

$$h[n] = u[n] , \tag{49}$$

which you can verify by showing that convolution of $x[n]$ with (49) yields an output $y[n]$ given by (48). The accumulator can be described by the general difference equation (46) with three nonzero coefficients: $a_0 = 1$, $a_1 = -1$ and $b_0 = 1$ (see the difference equation (10) on page 42). However, the impulse response (49) does not satisfy condition (47). The frequency response

$$H\left(e^{j\Omega}\right) = \frac{1}{1-e^{-j\Omega}} + \pi \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k2\pi\right) , \tag{50}$$

which can be obtained using (49) and Example 11, exists only in a generalized sense.

- Hereafter we assume condition (47) is satisfied. Then the system input-output relation can be described in the time or frequency domain by

$$y[n] = h[n] * x[n] \overset{F}{\leftrightarrow} Y\left(e^{j\Omega}\right) = H\left(e^{j\Omega}\right) X\left(e^{j\Omega}\right) . \tag{37}$$

  Solving the frequency-domain part of (37), we can obtain an expression for the frequency response

$$H\left(e^{j\Omega}\right) = \frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)} . \tag{51}$$

  If we are given an input signal $x[n] \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right)$ and can determine, by some method, the output signal $y[n] \overset{F}{\leftrightarrow} Y\left(e^{j\Omega}\right)$ induced by that input, then we can use (51) to compute the frequency response $H\left(e^{j\Omega}\right)$ at all frequencies at which $X\left(e^{j\Omega}\right) \neq 0$.

- Now we compute the DTFT of the difference equation (46). Using the linearity property and the time-shift property

$$x[n-k] \overset{F}{\leftrightarrow} e^{-jk\Omega} X\left(e^{j\Omega}\right) , \tag{26'}$$

  the DTFT of (46) is

$$\sum_{k=0}^{N} a_k e^{-jk\Omega} Y\left(e^{j\Omega}\right) = \sum_{k=0}^{M} b_k e^{-jk\Omega} X\left(e^{j\Omega}\right) . \tag{52}$$

Factoring out the $Y\left(e^{j\Omega}\right)$ and $X\left(e^{j\Omega}\right)$ on the left- and right-hand sides of (52), solving for $Y\left(e^{j\Omega}\right)/X\left(e^{j\Omega}\right)$ and using (51), we find the frequency response is

$$H\left(e^{j\Omega}\right)=\frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)}=\frac{\sum_{k=0}^{M} b_k e^{-jk\Omega}}{\sum_{k=0}^{N} a_k e^{-jk\Omega}}. \tag{53}$$

- We have found that for any LTI system described by a difference equation in the form (46), the frequency response (if it exists in the strict sense) can be expressed as a ratio of two polynomials in powers of $e^{-j\Omega}$. Such a ratio of two polynomials is called a *rational function* of $e^{-j\Omega}$. The coefficients appearing in the numerator and denominator polynomials in (53) are the same coefficients b_k, $k=0,\ldots,M$ and a_k, $k=0,\ldots,N$ that appear in the difference equation (46). Hence, if we are given a difference equation in the form (46), we can find the frequency response (53) *by inspection*. Conversely, if we are given a frequency response in the rational form (53), we can find the corresponding difference equation *by inspection*.
- The technique derived here is useful in analyzing LTI systems, as the following examples demonstrate.

Examples of LTI Systems Governed by Constant-Coefficient Difference Equations

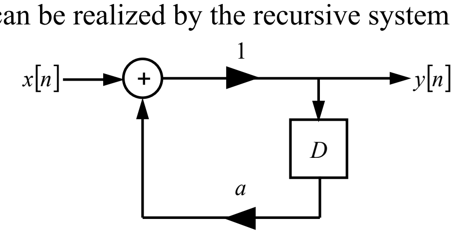
- In plotting the frequency response $H\left(e^{j\Omega}\right)$ in each of the following examples, we use a linear scale for frequency Ω, since $H\left(e^{j\Omega}\right)$ is a periodic function of Ω. We use a logarithmic scale for the magnitude $\left|H\left(e^{j\Omega}\right)\right|$ and a linear scale for the phase $\angle H\left(e^{j\Omega}\right)$.

Infinite Impulse Response Systems

- These systems are recursive, being described by constant-coefficient linear difference equations of order N = 1 or 2. We state the impulse and step responses without deriving them. For derivations of the impulse and step responses using Z transforms, refer to the *EE 102B Course Reader*, Chapter 7. In each case, we obtain the frequency response by analyzing the difference equation using the discrete-time Fourier transform (DTFT) and time-shift property. Alternatively, we could obtain the frequency response by computing the DTFT of the impulse response.

First-Order System

- A simple first-order system can be realized by the recursive system shown.



The system is described by a first-order difference equation

$$y\left[n\right]-ay\left[n-1\right]=x\left[n\right].$$

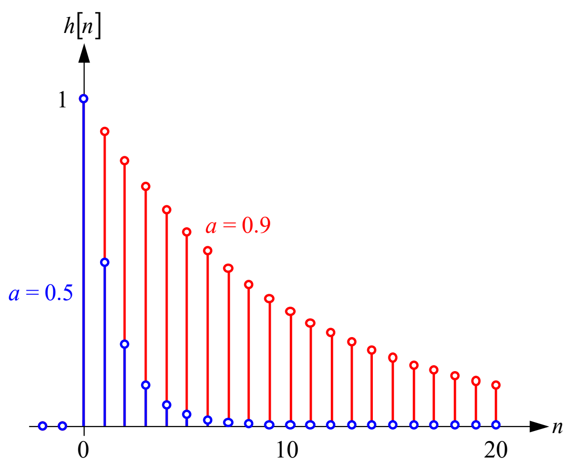
Its impulse response is

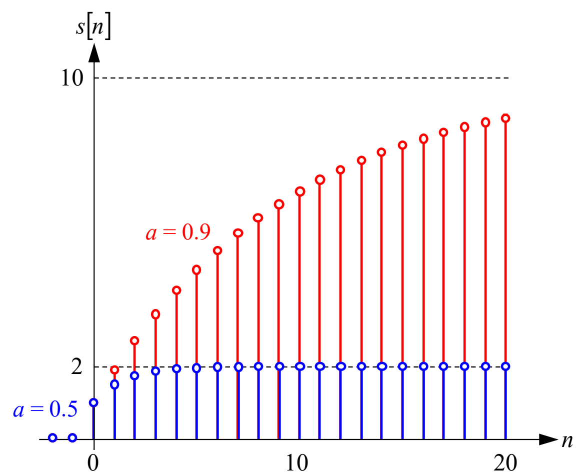
$$h[n] = a^n u[n],$$

and is step response is

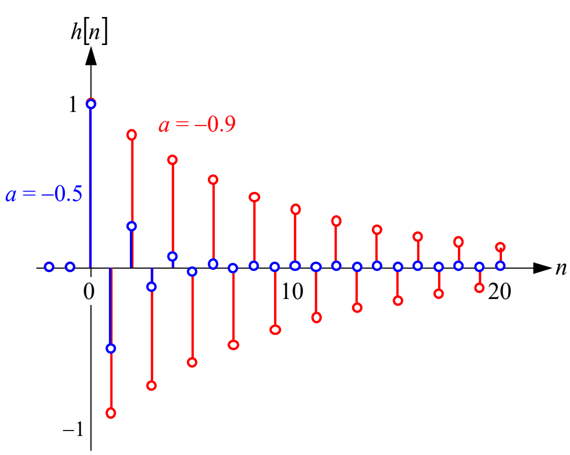
$$s[n] = \begin{cases} \dfrac{1-a^{n+1}}{1-a} u[n] & a \neq 1 \\ (n+1) u[n] & a = 1 \end{cases}.$$

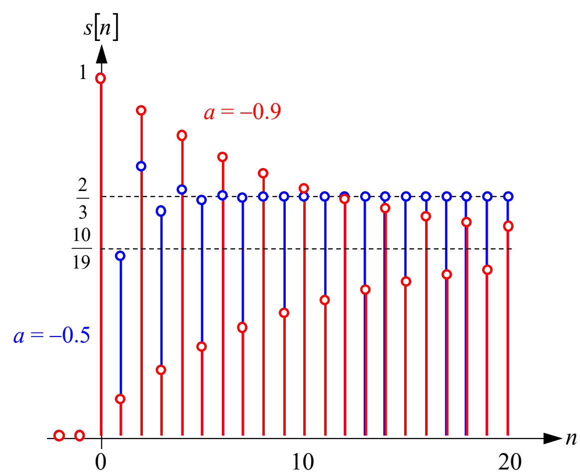
- The system is stable, and the frequency response exists, only for $|a| < 1$. Hereafter, we consider only $|a| < 1$. When $0 < a < 1$, as shown below, the system is a lowpass filter. The impulse response $h[n]$ decays monotonically to zero, while the step response $s[n]$ asymptotically approaches a limiting value $(1-a)^{-1} > 1$.





- When $-1 < a < 0$, as shown below, the system is a highpass filter. The impulse response $h[n]$ alternates sign as it decays to zero, while the step response $s[n]$ oscillates as it approaches a limiting value $(1-a)^{-1} < 1$.





- To find the frequency response, we take the DTFT of the difference equation using the time-shift property, obtaining

$$Y\left(e^{j\Omega}\right) - aY\left(e^{j\Omega}\right)e^{-j\Omega} = X\left(e^{j\Omega}\right).$$

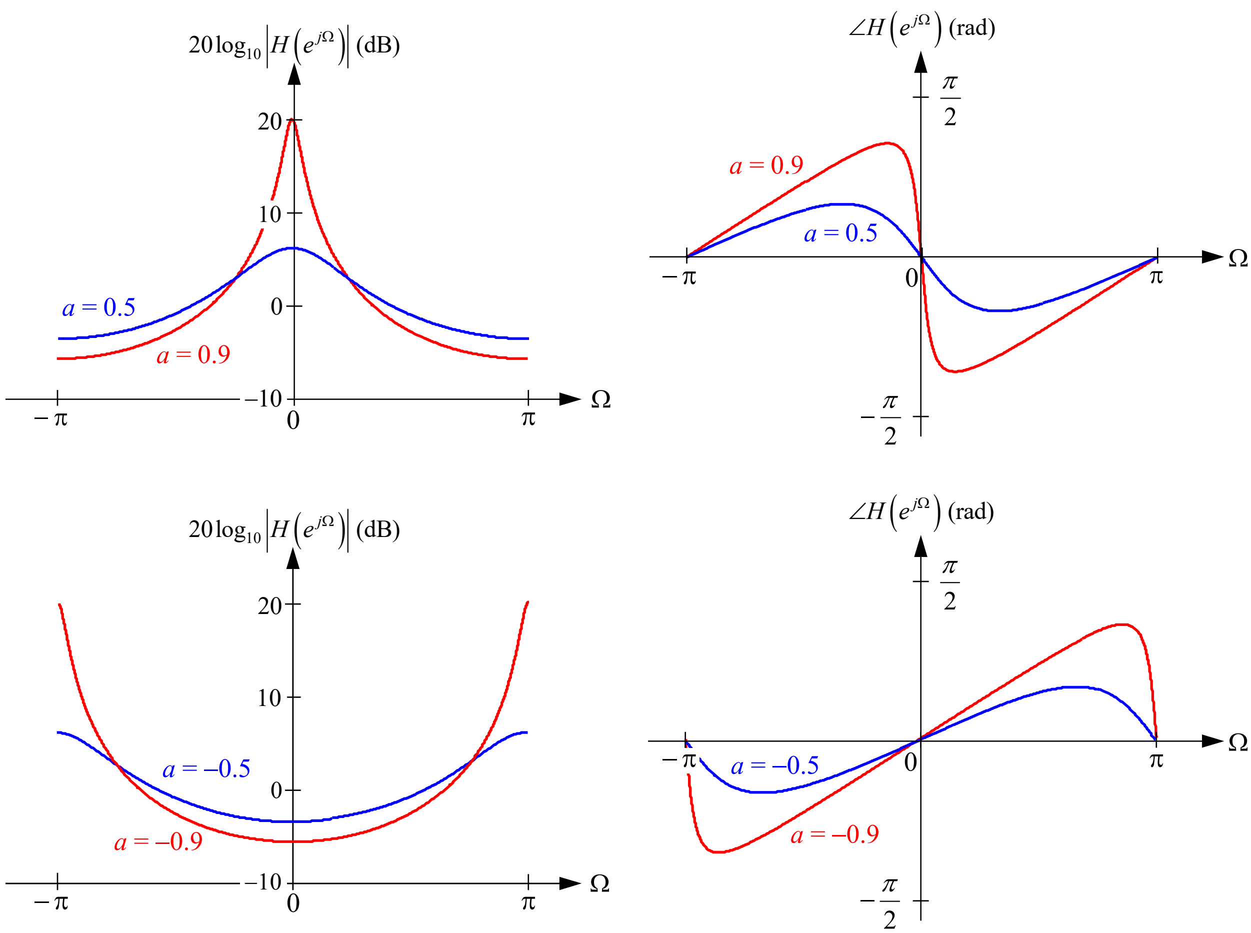
Solving for $H\left(e^{j\Omega}\right) = Y\left(e^{j\Omega}\right) / X\left(e^{j\Omega}\right)$, we obtain

$$H\left(e^{j\Omega}\right) = \frac{1}{1 - ae^{-j\Omega}}.$$

The magnitude and phase responses are

$$\left|H\left(e^{j\Omega}\right)\right| = \frac{1}{\sqrt{\left(1 - a\cos\Omega\right)^2 + \left(a\sin\Omega\right)^2}} \quad \text{and} \quad \angle H\left(e^{j\Omega}\right) = -\tan^{-1}\left(\frac{a\sin\Omega}{1 - a\cos\Omega}\right),$$

which are plotted below for $0 < a < 1$ (lowpass filter) and $-1 < a < 0$ (highpass filter).



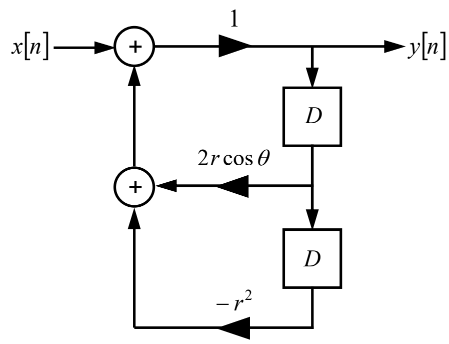
- We can make several observations about the magnitude and phase plots.
  - Given a value of $|a|$, the frequency responses for $a = |a|$ (lowpass) and $a = -|a|$ (highpass) are identical, except for a frequency shift of π. The impulse responses of the two filters are identical except for a factor $(-1)^n = e^{j\pi n}$, so by the DTFT frequency-shift property with $\Omega_0 = \pi$, their frequency responses are related by a frequency shift of $\Omega_0 = \pi$.

- For any value of *a*, the magnitude $\left|H\left(e^{j\Omega}\right)\right|$ is not constant, so the system causes amplitude distortion. The peaking of the magnitude $\left|H\left(e^{j\Omega}\right)\right|$ becomes more pronounced as $|a| \to 1$, consistent with the increased duration of the impulse response.

- For any value of *a*, the phase $\angle H\left(e^{j\Omega}\right)$ is not a linear function of Ω, so the system causes phase distortion. Nevertheless, for values of Ω close to the peak of the magnitude $\left|H\left(e^{j\Omega}\right)\right|$, the phase is approximately linear in Ω. At these values of Ω, the group delay $-d\angle H\left(e^{j\Omega}\right)/d\Omega$ increases as $|a| \to 1$, consistent with the increased duration of the impulse response.

Second-Order System

- A simple second-order system can be realized by the recursive system shown.



The system can be described by a difference equation

$$y[n] - 2r\cos\theta\, y[n-1] + r^2 y[n-2] = x[n],$$

where the parameters *r* and θ are real. All cases of interest can be described by considering $0 \le r < \infty$ and $0 \le \theta \le \pi$. Similar to the CT second-order lowpass filter, this DT system has three regimes, depending on the value of θ. The impulse and step responses have different mathematical forms in each regime. Here we provide formulas for the impulse responses.

- $\theta = 0$ (critically damped):

$$h[n] = \left[(n+1)r^n\right]u[n].$$

- $0 < \theta < \pi$ (underdamped):

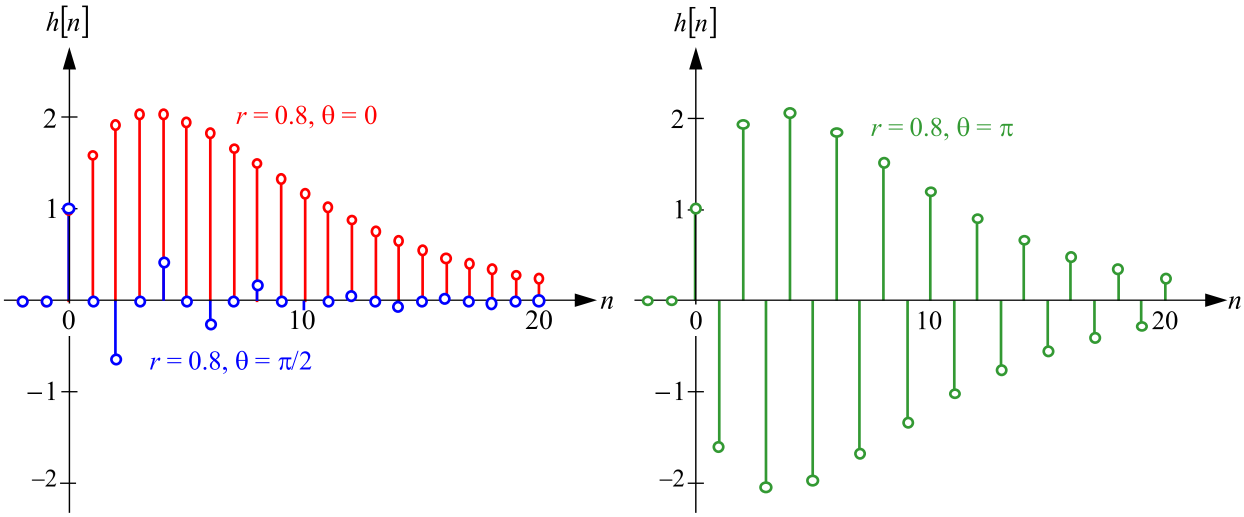
$$h[n] = \frac{1}{\sin\theta}\left[r^n \sin\left((n+1)\theta\right)\right]u[n].$$

- $\theta = \pi$ (underdamped):

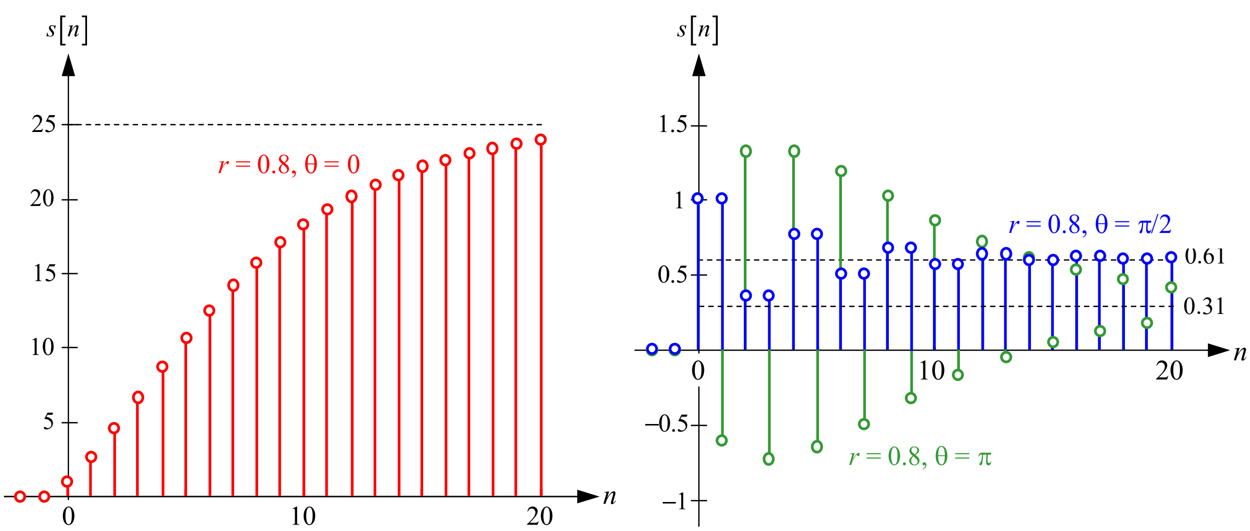
$$h[n] = \left[(n+1)(-r)^n\right]u[n].$$

- The system is stable, and the frequency response exists, only for $|r| < 1$. Hereafter, we consider only $0 \le r < 1$. The impulse response $h[n]$ is shown below for $r = 0.8$ for $\theta = 0,\ \pi/2$ and π. We note the following:

- $\theta = 0$: the system is a lowpass filter, and $h[n]$ is similar to a decaying exponential multiplied by a factor $n + 1$.
- $\theta = \pi / 2$: the system is a bandpass filter, and $h[n]$ is similar to a decaying exponential multiplied by a sinusoidal function of time n.
- $\theta = \pi$: the system is a highpass filter, and $h[n]$ is similar to a decaying exponential multiplied by a factor $n + 1$ and an alternating sequence $(-1)^n = e^{j\pi n}$.
- Although other values of r are not shown, the impulse response duration decreases as $r \to 0$ and increases as $r \to 1$.



- The step response $s[n]$ is shown here for $r = 0.8$ for $\theta = 0, \pi / 2$ and π.



- To find the frequency response, we take the DTFT of the difference equation using the time-shift property, obtaining

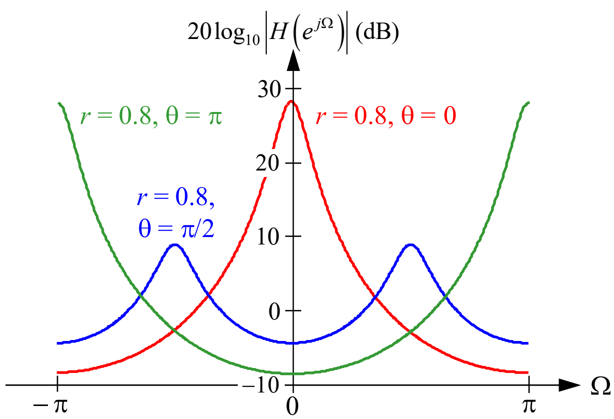
$$Y\left(e^{j\Omega}\right) - 2r\cos\theta \cdot Y\left(e^{j\Omega}\right)e^{-j\Omega} + r^2 \cdot Y\left(e^{j\Omega}\right)e^{-j2\Omega} = X\left(e^{j\Omega}\right).$$

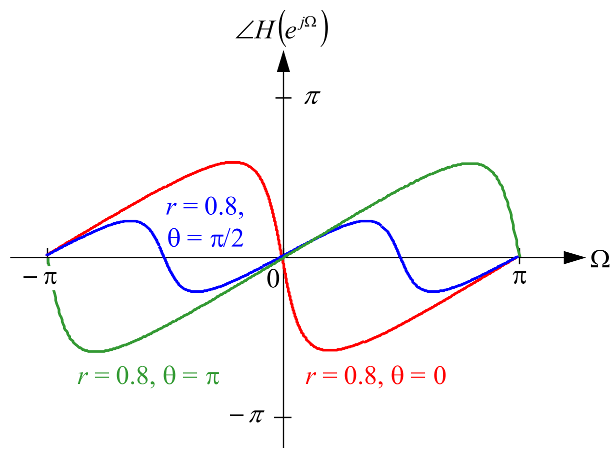
Solving for $H(e^{j\Omega}) = Y(e^{j\Omega}) / X(e^{j\Omega})$, we obtain

$$H(e^{j\Omega}) = \frac{1}{1 - 2r\cos\theta e^{-j\Omega} + r^2 e^{-j2\Omega}}.$$

Note that in the critically damped case $\theta = 0$, $H(e^{j\Omega}) = 1/\left(1 - re^{-j\Omega}\right)^2$, and $h[n] \overset{F}{\leftrightarrow} H(e^{j\Omega})$ corresponds to (32) with the substitution $a \to r$.

- The magnitude and phase responses are plotted below for $r = 0.8$ for $\theta = 0$ (lowpass filter), $\theta = \pi/2$ (bandpass filter) and $\theta = \pi$ (highpass filter).





- We can make several observations about the magnitude and phase plots.
  - Given a value of *r*, the frequency responses for $\theta = 0$ (lowpass) and $\theta = \pi$ (highpass) are identical, except for a frequency shift of π. The impulse responses of the two filters are identical except for a factor $(-1)^n = e^{j\pi n}$, so by the DTFT frequency-shift property with $\Omega_0 = \pi$, their frequency responses are related by a frequency shift of $\Omega_0 = \pi$.
  - For any values of *r* and θ, the magnitude $\left|H(e^{j\Omega})\right|$ is not constant, so the system causes amplitude distortion. The peaking of the magnitude $\left|H(e^{j\Omega})\right|$ becomes more pronounced as $r \to 1$, consistent with the increased duration of the impulse response.
  - For any values of *r* and θ, the phase $\angle H(e^{j\Omega})$ is not a linear function of Ω, so the system causes phase distortion. Nevertheless, for values of Ω close to the peak of the magnitude $\left|H(e^{j\Omega})\right|$, the phase is approximately linear in Ω. At these values of Ω, the group delay $-d\angle H(e^{j\Omega}) / d\Omega$ increases as $r \to 1$, consistent with the increased duration of the impulse response.

Finite Impulse Response Systems

- These systems are non-recursive, and are described by constant-coefficient linear difference equations of order $N = 0$.

Symmetric Moving Average

- A system averaging the input $x[n]$ over $2N_1+1$ times centered at the present time n is described by a difference equation

$$y[n] = \frac{1}{2N_1+1}\sum_{k=-N_1}^{N_1} x[n-k],$$

and has an impulse response

$$\begin{aligned} h_{\text{ma, sym}}[n] &= \frac{1}{2N_1+1}\sum_{k=-N_1}^{N_1} \delta[n-k] \\ &= \frac{1}{2N_1+1}\Pi\left(\frac{n}{2N_1}\right) \end{aligned}. \tag{54}$$

This system is not causal, so it cannot be employed to average real-time signals, but it can be used to average signals that have been recorded previously.

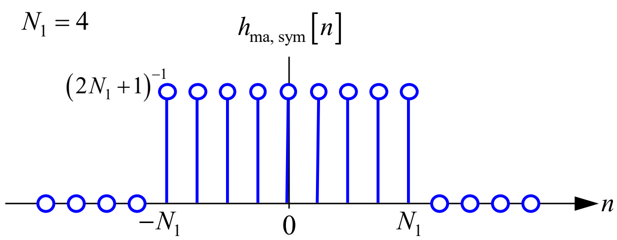
- In order to compute the frequency response, it is easiest to use the known DTFT of the rectangular pulse function $\Pi(n/2N_1)$, given by (17), scaling it by $1/(2N_1+1)$:

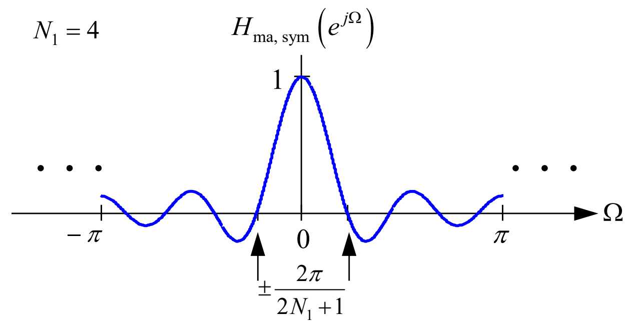
$$H_{\text{ma, sym}}\left(e^{j\Omega}\right) = \frac{1}{2N_1+1}\frac{\sin\left(\Omega\left(N_1+\frac{1}{2}\right)\right)}{\sin(\Omega/2)} . \tag{55}$$

Alternatively, we can take the DTFT of the difference equation using the time-shift property and solve for $H\left(e^{j\Omega}\right) = Y\left(e^{j\Omega}\right)/X\left(e^{j\Omega}\right)$ to obtain

$$H_{\text{ma, sym}}\left(e^{j\Omega}\right) = \frac{1}{2N_1+1}\sum_{k=-N_1}^{N_1} e^{-jk\Omega} ,$$

and then sum this geometric series to obtain (55). The impulse response $h_{\text{ma, sym}}[n]$ and frequency response $H_{\text{ma, sym}}\left(e^{j\Omega}\right)$ are shown here. The impulse response $h_{\text{ma, sym}}[n]$ is real and even in n, so the frequency response $H_{\text{ma, sym}}\left(e^{j\Omega}\right)$ is real and even in Ω. The frequency response $H_{\text{ma, sym}}\left(e^{j\Omega}\right)$ has a peak value of 1 at $\Omega = 0$, and decreases to zero at $\Omega = \pm 2\pi/(2N_1+1)$. As we increase the averaging window length $2N_1+1$, the width of the passband decreases. The frequency response $H_{\text{ma, sym}}\left(e^{j\Omega}\right)$ is periodic in Ω with period 2π, owing to the sinusoidal functions appearing in (55).





Causal Moving Average

- In order to compute a moving average in real time, we want a causal system that computes an average over the present and $2N_1$ past samples of the input. This system is described by a difference equation

$$y[n] = \frac{1}{2N_1+1}\sum_{k=0}^{2N_1} x[n-k],$$

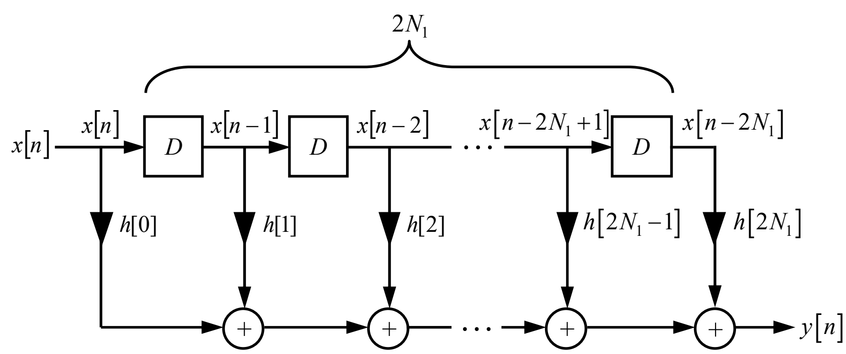
and has an impulse response

$$\begin{aligned} h_{\text{ma, causal}}[n] &= \frac{1}{2N_1+1}\sum_{k=0}^{2N_1}\delta[n-k] \\ &= \frac{1}{2N_1+1}\Pi\left(\frac{n-N_1}{2N_1}\right) \end{aligned}. \tag{56}$$

The causal impulse response (56) is simply the symmetric impulse response (54) delayed by N_1 samples:

$$h_{\text{ma, causal}}[n] = h_{\text{ma, sym}}[n-N_1]. \tag{57}$$

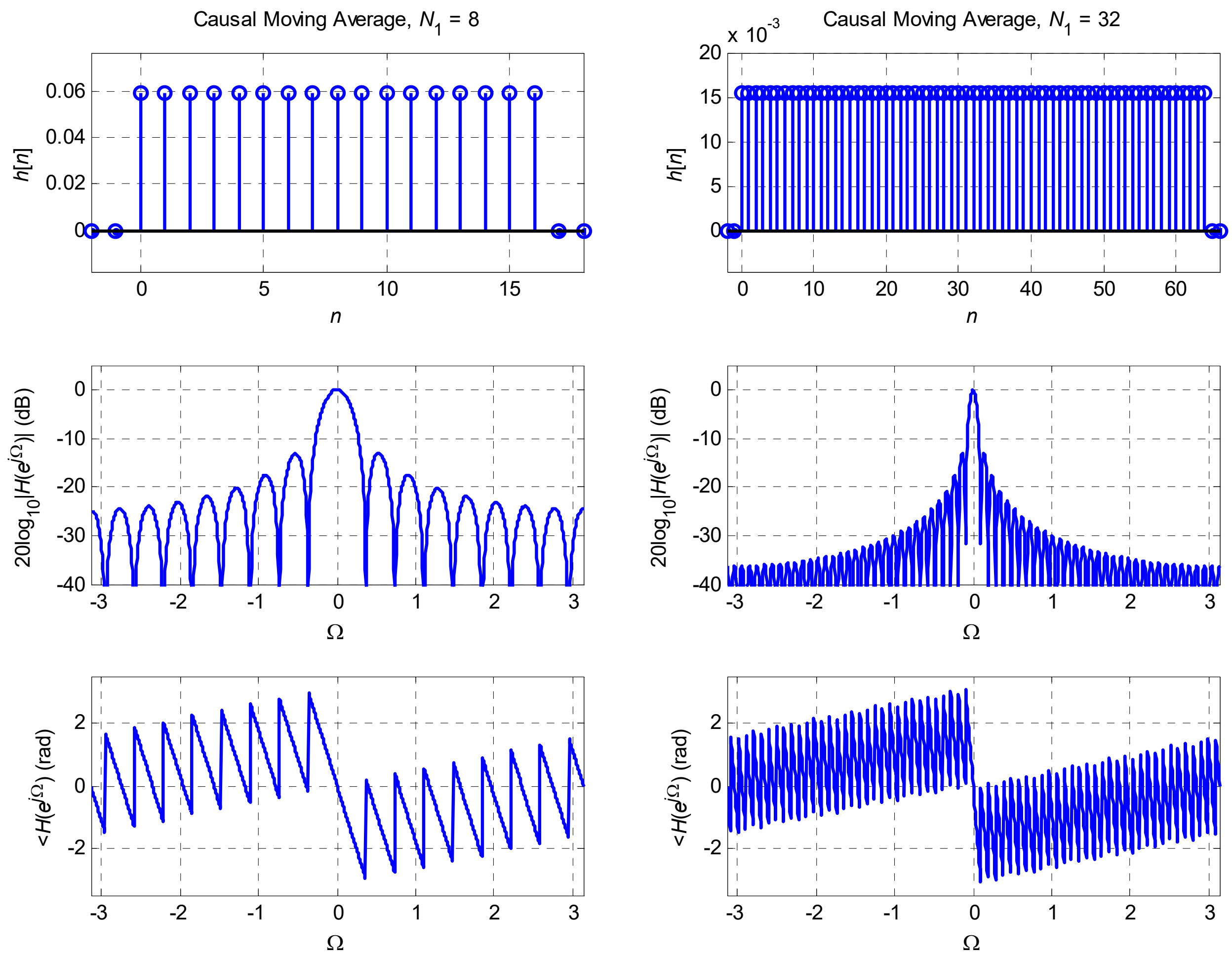
The causal moving average system can be realized by the system shown. The coefficients $h[0], h[1], h[2], \ldots$ correspond to the samples of $h_{\text{ma, causal}}[n]$.



- From (57), we know that the causal moving average frequency response is the same as (55), but with a linear phase factor arising from the N_1-sample delay:

$$
\begin{aligned}
H_{\text{ma, causal}}\left(e^{j\Omega}\right) &= H_{\text{ma, sym}}\left(e^{j\Omega}\right)e^{-jN_1\Omega} \\
&= \frac{1}{2N_1+1}\frac{\sin\left(\Omega\left(N_1+\frac{1}{2}\right)\right)}{\sin\left(\Omega/2\right)}e^{-jN_1\Omega}
\end{aligned}. \tag{58}
$$

The impulse response $h_{\text{ma, causal}}\left[n\right]$ and frequency response $H_{\text{ma, causal}}\left(e^{j\Omega}\right)$ are shown below for $N_1 = 8$ and $N_1 = 32$.



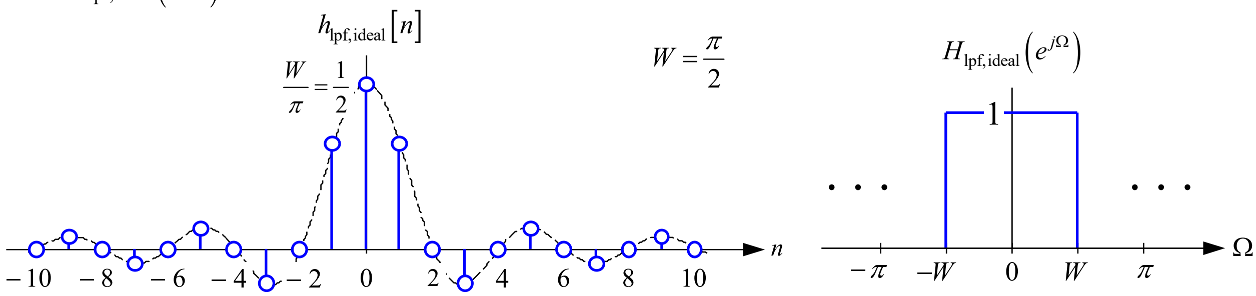
- As we increase N_1, the passband width, proportional to $1/\left(2N_1+1\right)$, decreases as expected, while the group delay in the passband, given by N_1, increases as expected. These moving average filters obviously induce significant amplitude distortion. But their phases are linear with integer slope, so they induce no phase distortion.

Symmetric Finite Approximation of Ideal Lowpass Filter

- We would like to realize an ideal lowpass filter. An ideal filter with cutoff frequency $W < \pi$ has an impulse response and frequency response given by (38) with $n_0 = 0$:

$$h_{\text{lpf,ideal}}[n]=\frac{W}{\pi}\operatorname{sinc}\left(\frac{W}{\pi}n\right)\leftrightarrow H_{\text{lpf,ideal}}\left(e^{j\Omega}\right)=\sum_{l=-\infty}^{\infty}\Pi\left(\frac{\Omega-l2\pi}{2W}\right). \tag{59}$$

Recall that the summation on the right-hand side of (59) makes the frequency response $H_{\text{lpf,ideal}}\left(e^{j\Omega}\right)$ periodic in Ω with period 2π. Here we show the impulse and frequency responses given by (59) for an ideal filter with cutoff frequency $W=\pi/2$. We observe that $h_{\text{lpf,ideal}}[n]$ is real and even in n, so $H_{\text{lpf,ideal}}\left(e^{j\Omega}\right)$ is real and even in Ω.



- This ideal lowpass filter cannot be implemented, however. It is non-causal, and its impulse response has infinite extent over both positive and negative time n. As a first step toward a realizable filter, we truncate the impulse response to have a finite duration. To achieve this, we multiply $h_{\text{lpf,ideal}}[n]$ by a rectangular function $\Pi(n/2N_1)$, making it nonzero only over a symmetric interval $-N_1\leq n\leq N_1$, a total length of $2N_1+1$ samples. The truncated impulse response is

$$\begin{aligned}h_{\text{lpf, trunc}}[n]&=\Pi\left(\frac{n}{2N_1}\right)\cdot h_{\text{lpf, ideal}}[n]\\&=\begin{cases}\frac{W}{\pi}\operatorname{sinc}\left(\frac{W}{\pi}n\right) & -N_1\leq n\leq N_1\\ 0 & \text{otherwise}\end{cases}.\end{aligned} \tag{60}$$

The truncated lowpass filter is described by a difference equation

$$y[n]=\frac{W}{\pi}\sum_{k=-N_1}^{N_1}\operatorname{sinc}\left(\frac{W}{\pi}k\right)x[n-k]. \tag{61}$$

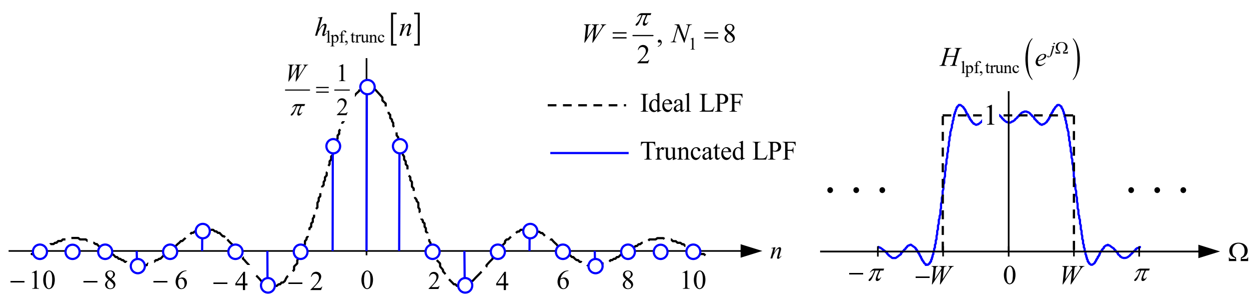
We can see that this difference equation generates the same output as a convolution of the input $x[n]$ with the impulse response (60):

$$\begin{aligned}y[n]&=h_{\text{lpf, trunc}}[n]*x[n]\\&=\sum_{k=-\infty}^{\infty}h_{\text{lpf, trunc}}[k]x[n-k]\end{aligned}.$$

- In order to obtain the frequency response $H_{\text{lpf,trunc}}\left(e^{j\Omega}\right)$, we can compute the DTFT of the impulse response (60):

$$
\begin{aligned}
H_{\text{lpf, trunc}}\left(e^{j\Omega}\right) &= \sum_{n=-\infty}^{\infty} h_{\text{lpf, trunc}}[n] e^{-jn\Omega} \\
&= \frac{W}{\pi} \sum_{n=-N_1}^{N_1} \operatorname{sinc}\left(\frac{W}{\pi} n\right) e^{-jn\Omega}
\end{aligned} \tag{62}
$$

Alternatively, we can analyze the difference equation (61) using the DTFT and the time-shift property. The expression (62) can be evaluated numerically, but cannot be simplified into a closed-form analytical expression. The impulse response $h_{\text{lpf, trunc}}[n]$ and frequency response $H_{\text{lpf, trunc}}\left(e^{j\Omega}\right)$ are shown here for a filter with cutoff frequency $W = \pi/2$ and $N_1 = 8$, a total length $2N_1 + 1 = 17$. We note that $h_{\text{lpf, trunc}}[n]$ is real and even in n, so $H_{\text{lpf, trunc}}\left(e^{j\Omega}\right)$ is real and even in Ω, and that $H_{\text{lpf, trunc}}\left(e^{j\Omega}\right)$ is periodic in Ω, owing to the $e^{-jn\Omega}$ appearing in the DTFT expression (62).



- The ideal frequency response $H_{\text{lpf, ideal}}\left(e^{j\Omega}\right)$ and truncated filter frequency response $H_{\text{lpf, trunc}}\left(e^{j\Omega}\right)$ are indicated on the right above by dashed and solid lines, respectively. Unlike the ideal response $H_{\text{lpf, ideal}}\left(e^{j\Omega}\right)$, the truncated response $H_{\text{lpf, trunc}}\left(e^{j\Omega}\right)$ exhibits a gradual transition between passband and stopband at $\Omega = \pm W$. Moreover, $H_{\text{lpf, trunc}}\left(e^{j\Omega}\right)$ exhibits passband ripple and stopband leakage.

These non-ideal characteristics of $H_{\text{lpf, trunc}}\left(e^{j\Omega}\right)$ can be understood in two different ways.

- First, the DTFT expression (62) is a Fourier series synthesis of $H_{\text{lpf, trunc}}\left(e^{j\Omega}\right)$, a periodic function of Ω, with Fourier series coefficients given by the samples of the time signal, in this case, the impulse response $h_{\text{lpf, trunc}}[n]$. Here, the Fourier series synthesis is intended to represent an ideal rectangular pulse train in frequency Ω. Like any Fourier series synthesis, it exhibits the Gibbs phenomenon at discontinuities, in this case, at $\Omega = \pm W$.

- Second, recall that the finite impulse response is obtained by multiplying the ideal impulse response by a rectangular pulse function in (60): $h_{\text{lpf, trunc}}[n] = h_{\text{lpf, ideal}}[n] \cdot \Pi\left(n/2N_1\right)$. By the multiplication property of the DTFT, the frequency response $H_{\text{lpf, trunc}}\left(e^{j\Omega}\right)$ is a periodic convolution between $H_{\text{lpf, ideal}}\left(e^{j\Omega}\right)$ and the DTFT of $\Pi\left(n/2N_1\right)$, which is given by $\sin\left(\Omega\left(N_1 + \tfrac{1}{2}\right)\right)/\sin\left(\Omega/2\right)$:

$$H_{\text{lpf,trunc}}\left(e^{j\Omega}\right)=\frac{1}{2\pi}\int_{2\pi} H_{\text{lpf,ideal}}\left(e^{j(\Omega-\theta)}\right)\frac{\sin\left(\theta\left(N_1+\frac{1}{2}\right)\right)}{\sin\left(\theta/2\right)}d\theta . \tag{63}$$

Recall that a periodic convolution is like an ordinary convolution, except the integration is performed only over one period, in this case, any interval of length 2π. In (63), the first function is a periodic rectangular pulse train, while the second function is analogous to a periodic sinc function. The convolution between them gives rise to the gradual transition and the ripple and leakage observed in $H_{\text{lpf,trunc}}\left(e^{j\Omega}\right)$.

Causal Finite Approximation of Ideal Lowpass Filter

- While the symmetric finite approximation of the lowpass filter is usable on signals that have been recorded and stored, many applications require a causal system to filter signals in real time. We obtain a causal filter by delaying the symmetric filter by N_1 samples. The causal filter's impulse response is $h_{\text{lpf, trunc}}\left[n\right]$, given by (60), delayed by N_1 samples:

$$\begin{aligned} h_{\text{lpf, causal}}\left[n\right] &= h_{\text{lpf, trunc}}\left[n-N_1\right] \\ &= \begin{cases} \dfrac{W}{\pi}\operatorname{sinc}\left(\dfrac{W}{\pi}\left(n-N_1\right)\right) & 0\le n\le 2N_1 \\ 0 & \text{otherwise} \end{cases} . \end{aligned} \tag{64}$$

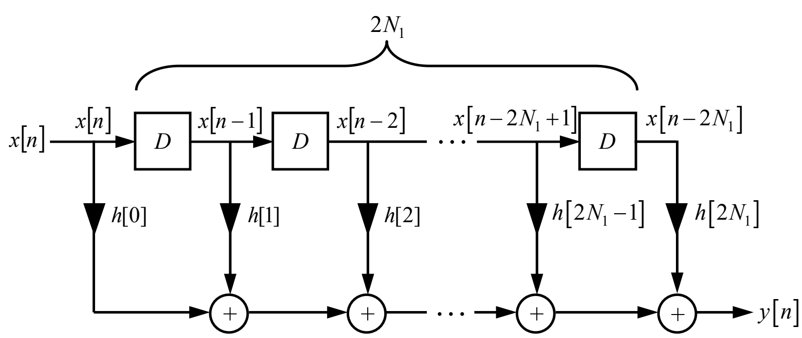
  The causal lowpass filter is described by a difference equation

$$y\left[n\right]=\frac{W}{\pi}\sum_{k=0}^{2N_1}\operatorname{sinc}\left(\frac{W}{\pi}\left(k-N_1\right)\right)x\left[n-k\right]. \tag{65}$$

  In (65), the output $y\left[n\right]$ is a linear combination of the present and $2N_1$ past inputs. It is the same as a convolution of the input $x\left[n\right]$ with the impulse response in (64):

$$\begin{aligned} y\left[n\right] &= h_{\text{lpf, causal}}\left[n\right]*x\left[n\right] \\ &= \sum_{k=-\infty}^{\infty} h_{\text{lpf, causal}}\left[k\right]x\left[n-k\right] \end{aligned} .$$

- The causal lowpass filter can be realized by the system shown. The coefficients $h\left[0\right],h\left[1\right],h\left[2\right],\ldots$ correspond to the samples of $h_{\text{lpf, causal}}\left[n\right]$.



- Using the first line of (64) and the DTFT time-shift property, we know that the causal lowpass filter's frequency response is the same as that of the symmetric truncated lowpass filter, given by (62) or (63), but with a linear phase factor arising from the N_1-sample delay:

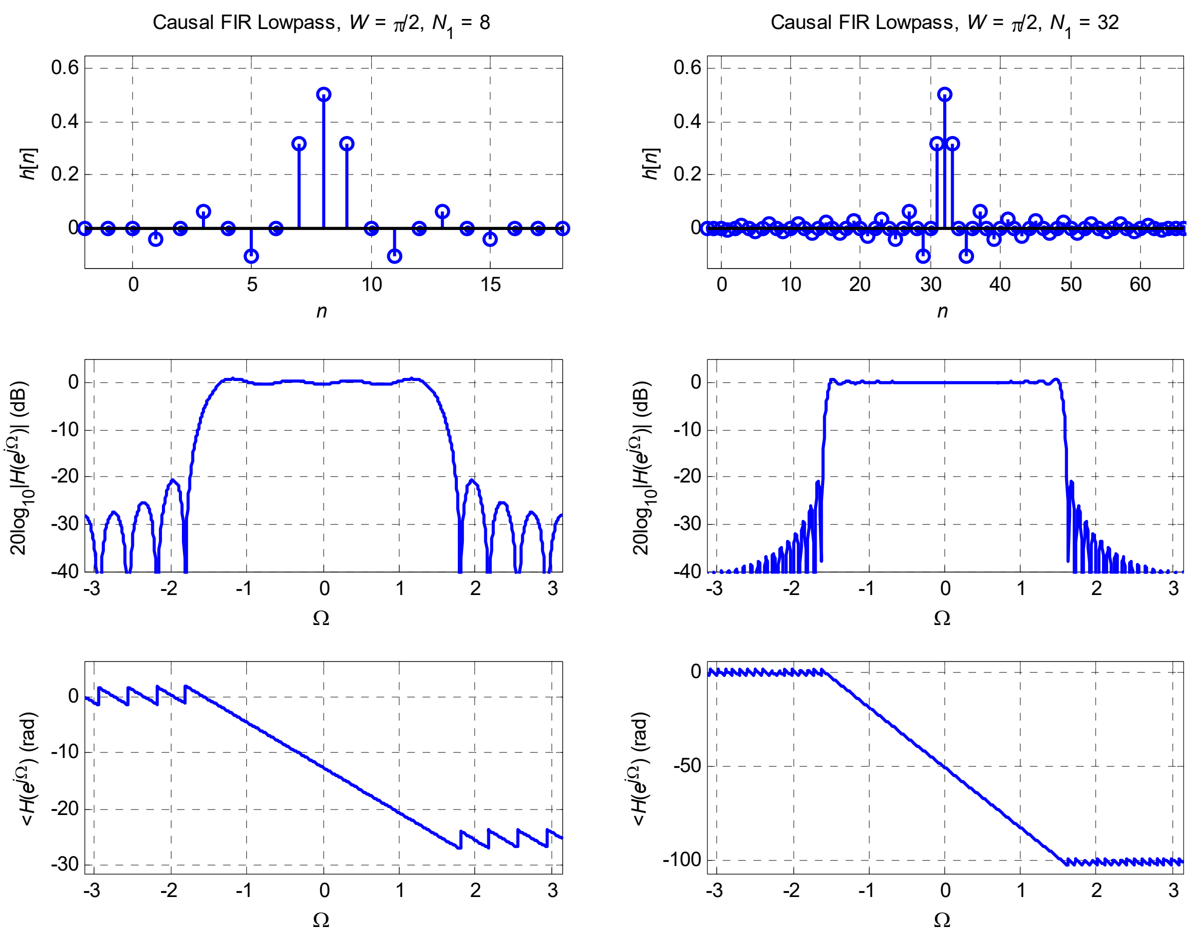
$$H_{\text{lpf, causal}}\left(e^{j\Omega}\right) = H_{\text{lpf, trunc}}\left(e^{j\Omega}\right) e^{-jN_1\Omega} . \tag{66}$$

Alternatively, the frequency response can be obtained by taking the DTFT of impulse response (64):

$$\begin{aligned} H_{\text{lpf,causal}}\left(e^{j\Omega}\right) &= \sum_{n=-\infty}^{\infty} h_{\text{lpf, causal}}[n] e^{-jn\Omega} \\ &= \frac{W}{\pi} \sum_{n=0}^{2N_1} \operatorname{sinc}\left(\frac{W}{\pi}(n - N_1)\right) e^{-jn\Omega} \end{aligned} . \tag{67}$$

Finally, the frequency response $H_{\text{lpf,causal}}\left(e^{j\Omega}\right)$ could instead be obtained by analyzing the difference equation (65) using the DTFT time-shift property.

- The impulse response $h_{\text{lpf, causal}}[n]$ and frequency response $H_{\text{lpf, causal}}\left(e^{j\Omega}\right)$ are shown here for a cutoff frequency $W = \pi/2$ for $N_1 = 8$ and $N_1 = 32$. We used the MATLAB **`unwrap`** command on the phase $\angle H_{\text{lpf, causal}}\left(e^{j\Omega}\right)$ to avoid 2π phase jumps, helping highlight the linearity of the phase in the passband.



- Note that as we increase N_1:
  - The nominal cutoff frequencies of the passband at $\Omega = \pm W$ do not change.
  - The transition between the passband and the stopband at $\Omega = \pm W$ becomes more abrupt.
  - The passband ripple and stopband leakage do not diminish in peak magnitude, but become confined to a narrower frequency range near $\Omega = \pm W$.
  - The passband group delay $-d\angle H_{\text{lpf, causal}}\left(e^{j\Omega}\right)/d\Omega = N_1$ increases.
- Note that the causal approximation of the ideal lowpass filter:
  - Induces only minimal amplitude distortion for signals within the passband.
  - Induces no phase distortion.

Overview of Fourier Representations

Organizing Principles

- The four Fourier representations we have studied are summarized in the figure below.

	Continuous Time		**Discrete Time**	
	Time Domain	**Frequency Domain**	**Time Domain**	**Frequency Domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ Continuous Periodic	$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt$ Discrete Aperiodic	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n}$ Discrete Periodic	$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\Omega_0 n}$ Discrete Periodic
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$ Continuous Aperiodic	$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ Continuous Aperiodic	$x[n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\Omega}\right)e^{j\Omega n}d\Omega$ Discrete Aperiodic	$X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$ Continuous Periodic

duality

duality

duality

- The most important attributes of these Fourier representations – discrete vs. continuous and periodic vs. aperiodic – conform to a simple pattern:

discrete in one domain ↔ periodic in the other domain

continuous in one domain ↔ aperiodic in the other domain.

In the above, "one domain" can denote time or frequency, and likewise, "the other domain" can denote frequency or time. For example, the CTFS describes a signal $x(t)$ that is a periodic function of a continuous-valued time variable *t*. Since the time-domain description is *periodic*, the frequency-domain description must be *discrete*. And since the time-domain description is a function of a *continuous* variable, the frequency-domain description must be *aperiodic*. These observations are consistent with the fact that the CTFS coefficients a_k are an aperiodic function of a discrete variable *k*.

Dualities

- There are three *dualities* evident in the figure above. First, we observe a *duality in the CTFT*.
  - The time-domain description $x(t)$ is continuous and aperiodic. Likewise, the frequency-domain description $X(j\omega)$ is continuous and aperiodic.
  - The CTFT (analysis) and inverse CTFT (synthesis) are of very similar mathematical forms. As a consequence, every CTFT pair $x(t) \overset{F}{\leftrightarrow} X(j\omega)$ corresponds to another pair $y(t) \overset{F}{\leftrightarrow} Y(j\omega)$ in which the functional forms of $x(t)$ and $X(j\omega)$ are similar to the functional forms of $Y(j\omega)$ and $y(t)$, respectively. We saw examples of this in Chapter 4. One example is $1 \overset{F}{\leftrightarrow} 2\pi\delta(\omega)$ and $\delta(t) \overset{F}{\leftrightarrow} 1$. Another example is $\Pi(t/2T_1) \overset{F}{\leftrightarrow} 2T_1 \text{sinc}(\omega T_1/\pi)$ and $(W/\pi)\text{sinc}(Wt/\pi) \overset{F}{\leftrightarrow} \Pi(\omega/2W)$.
  - As another consequence of the CTFT and inverse CTFT having similar forms, every property of the CTFT corresponds to a property for the inverse CTFT. For example, the convolution property $x(t)*y(t) \overset{F}{\leftrightarrow} X(j\omega)Y(j\omega)$ is the dual of the multiplication property $x(t)y(t) \overset{F}{\leftrightarrow} (1/2\pi)X(j\omega)*Y(j\omega)$. Other examples are evident in Table 3, Appendix.
- Second, we observe a *duality in the DTFS*.
  - The time-domain description $x[n]$ is discrete and periodic and likewise, the frequency-domain description a_k is discrete and periodic.
  - The DTFS synthesis and analysis equations are of very similar mathematical forms. Every DTFS pair $x[n] \overset{FS}{\leftrightarrow} a_k$ corresponds to another pair $y[n] \overset{FS}{\leftrightarrow} b_k$ in which the functional forms of $x[n]$ and a_k are similar to the functional forms of b_k and $y[n]$, respectively.
  - Every property of the DTFS synthesis corresponds to a property of the DTFS analysis. For examples, refer to Table 2, Appendix.
- Third, we observe a *duality between the CTFS and the DTFT*.
  - In the CTFS, the time-domain description is continuous and periodic, while the frequency-domain description is discrete and aperiodic. The DTFT is similar, but with the time- and frequency-domain descriptions interchanged. In other words, in the DTFT, the time-domain description is discrete and aperiodic, while the frequency-domain description is continuous and periodic.
  - The CTFS synthesis equation is of a mathematical form similar to the DTFT analysis equation, namely, an infinite sum over a discrete variable. Likewise, the CTFS analysis equation has a mathematical form similar to the DTFT synthesis equation. Both are integrals over one period of a periodic function of a continuous variable.

- As a consequence, every CTFS pair $x(t) \overset{FS}{\leftrightarrow} a_k$ corresponds to a DTFT pair $y[n] \overset{F}{\leftrightarrow} Y\left(e^{j\Omega}\right)$ in which the functional forms of $x(t)$ and $Y\left(e^{j\Omega}\right)$ are similar, while the functional forms of a_k and $y[n]$ are similar. An example is the CTFS pair describing a rectangular pulse train:

$$\sum_{l=-\infty}^{\infty} \Pi\left(\frac{t-lT_0}{2T_1}\right) \overset{FS}{\leftrightarrow} \frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{\omega_0 T_1}{\pi} k\right)$$

  and the DTFT pair describing an ideal lowpass filter:

$$\frac{W}{\pi} \operatorname{sinc}\left(\frac{W}{\pi} n\right) \overset{F}{\leftrightarrow} \sum_{l=-\infty}^{\infty} \Pi\left(\frac{\Omega - l2\pi}{2W}\right).$$

- Finally, every property of the CTFS synthesis equation corresponds to a property of the DTFT analysis equation and likewise, every property of the CTFS analysis equation corresponds to a property of the DTFT synthesis equation. This duality is evident in comparing the CTFS properties (Table 1, Appendix) to the DTFT properties (Table 5, Appendix).

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EE 102A: Signal Processing and Linear Systems I
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Chapter 6: Sampling and Reconstruction

Motivations

- *Sampling* is the conversion of a CT signal (usually not quantized) to a DT signal (usually quantized). If the DT signal is quantized, this process may be called *digitization* or *analog-to-digital conversion*.
- *Reconstruction* is the conversion of a DT signal (usually quantized) to a CT signal. If the DT signal is quantized, this process may be called *digital-to-analog conversion*.
- Sampling and reconstruction are essential to many digital technologies:
  - *Capture*: audio, images, video
  - *Signal processing*: enhancement, synthesis and compression of audio, images, video
  - *Storage*: CD, DVD, Blu-Ray, MP3, JPEG, MPEG
  - *Communication*: optical fiber links and networks, wireless links and networks
  - *Network applications*: VoIP telephony, video telephony, streaming music and video
- We will address several questions in this chapter.
  1. Given a CT signal, under what conditions do the samples the signal represent sufficient information to reconstruct the signal perfectly?
  2. Assuming a CT signal is sampled satisfying those conditions, how can it be reconstructed perfectly from the samples?
  3. What are the practical challenges in reconstruction, and how can we overcome them?
  4. How can we perform DT filtering on the samples of a CT signal to approximate CT filtering of the original CT signal?
- We will not study quantization in EE 102A. Quantization causes noise, limiting the signal-to-noise ratio to about 6 dB per bit. For example, some common digital audio standards use 16-bit quantization, limiting the signal-to-noise ratio to about 96 dB.

Major Topics in This Chapter

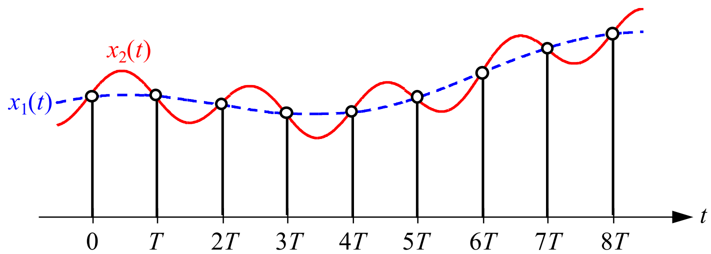
- Sampling
  - Sampling as conversion from CT to DT. Equivalent all-CT model for sampling.
  - Spectrum of sampled signal. Potential for aliasing.
  - Nyquist requirement: sampling rate required to avoid aliasing.
- Reconstruction
  - Reconstruction as conversion from DT to CT. Equivalent all-CT model for reconstruction.
  - Ideal bandlimited reconstruction. Practical reconstruction with oversampling or digital upsampling.

- DT processing of sampled CT signals.
  - Design of an FIR DT filter to approximate a given CT filter. Windowing.
  - Examples: ideal lowpass filter, differentiator.

Sampling

- *Sampling* is the conversion of a CT signal to a DT signal. We record samples at integer multiples of a *sampling interval* T (in units of s). Equivalently, we use a *sampling rate* $1/T$ (in units of s^{-1}).

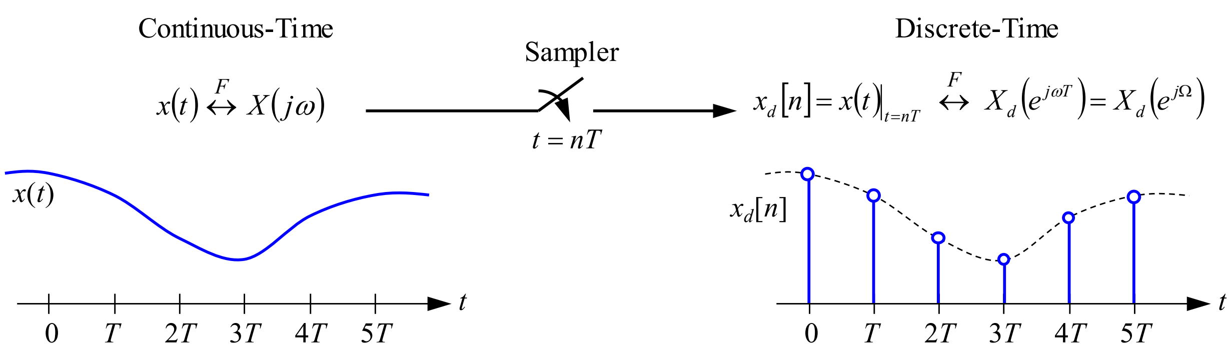
- After sampling a CT signal, we would like to be able to reconstruct the CT signal from its samples. There exist an infinite number of different signals which, if sampled at rate $1/T$, generate identical samples. This figure shows two CT signals whose samples are identical, i.e., $x_1(t)\big|_{t=nT} = x_2(t)\big|_{t=nT}$.



- Our initial goal is to determine conditions under which the samples of a CT signal represent sufficient information to reconstruct the signal unambiguously. To this end, we will use Fourier analysis to study the spectrum of a sampled signal.

Actual Sampling System

- An actual sampling system, as shown below, has a CT input $x(t)$ and a DT output $x_d[n]$.



- The CT signal $x(t)$ is defined for all t.

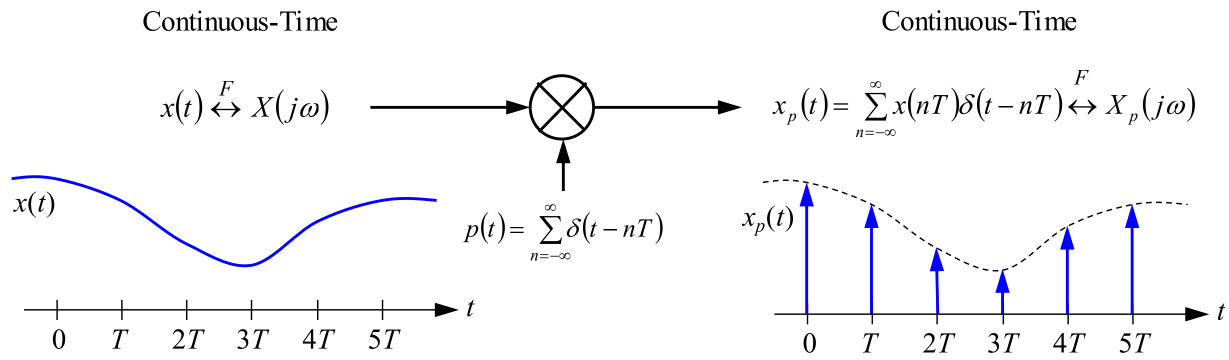
- Its spectrum is the CTFT $X(j\omega)$.

- The DT signal $x_d[n]$ is defined only at $t = nT$, $n = 0, \pm 1, \pm 2, \ldots$.

- Its spectrum is the DTFT $X_d(e^{j\Omega}) = X_d(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x_d[n] e^{-jn\omega T}$, where $\Omega = \omega T$.

Equivalent Continuous-Time System

- In order to relate the spectrum of the DT sampled signal to the spectrum of the original CT signal, it is helpful to analyze an equivalent system involving only CT, as shown below.



- The CT signal $x(t)$ is defined for all *t*.
- Its spectrum is the CTFT $X(j\omega)$.
- The CT signal $x_p(t)$ is defined for all *t*, but is nonzero only at $t = nT$, $n = 0, \pm 1, \pm 2, \ldots$ ..
- Its spectrum is the CTFT $X_p(j\omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-jn\omega T}$.
  Notice that $X_p(j\omega) = X_d(e^{j\omega T})$.

Spectrum of Sampled Signal

- In the actual sampling system, $x(t)$ is sampled at $t = nT$, $n = 0, \pm 1, \pm 2, \ldots$, which yields a DT *sampled signal* $x_d[n]$:

$$x_d[n] = x(t)\big|_{t=nT} = x(nT). \tag{1}$$

  Since $x_d[n]$ is a DT signal, by definition its spectrum is given by the DTFT

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-jn\Omega}. \tag{2}$$

  Like any DTFT, $X_d(e^{j\Omega})$ is periodic in Ω with period 2π, since $e^{-jn(\Omega+2\pi)} = e^{-jn\Omega}$.

- In the equivalent CT model, $x(t)$ is multiplied by a CT impulse train $p(t)$ to obtain a CT *impulse-sampled signal* $x_p(t)$:

$$x_p(t) = x(t)\cdot p(t) = x(t)\cdot \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT), \tag{3}$$

  where we have used the sampling property of the CT impulse.

- Now we will derive two different expressions for the spectrum of the sampled signal. These two expressions will serve two distinct functions.

Spectrum to Validate Equivalent Continuous-Time System

- First, we compute an expression for the spectrum of the CT impulse-sampled signal $x_p(t)$ in order to validate the equivalent CT system shown on the top of page 243. We do this by evaluating the CTFT of (3):

$$X_p(j\omega) = F\left(\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)\right). \tag{4}$$

Using the linearity property of the CTFT and noting that the $x(nT)$ are constants and not functions of time *t*, (4) becomes

$$X_p(j\omega) = \sum_{n=-\infty}^{\infty} x(nT) F\left(\delta(t-nT)\right).$$

Using the CTFT pair $\delta(t) \overset{F}{\leftrightarrow} 1$ and the CTFT time-shift property $z(t-nT) \overset{F}{\leftrightarrow} Z(j\omega)e^{-j\omega nT}$, we find that $\delta(t-nT) \overset{F}{\leftrightarrow} e^{-j\omega nT}$, so that

$$X_p(j\omega) = \sum_{n=-\infty}^{\infty} x(nT) e^{-j\omega nT}. \tag{5}$$

If we draw an association between the DT and CT frequencies

$$\Omega = \omega T, \tag{6}$$

then we notice that

$$X_p(j\omega) = X_d\left(e^{j\omega T}\right), \tag{7}$$

where $X_d\left(e^{j\omega T}\right) = X_d\left(e^{j\Omega}\right)$ is the DTFT (2). In other words, the equivalent CT system (top of page 243), which is analyzed using the CTFT, yields the *same spectrum* as the actual sampling system (bottom of page 242), which has a CT input and a DT output, and is analyzed using the DTFT. Hence we know that $X_p(j\omega)$ is periodic in ωT with period 2π, i.e., it is periodic in ω with period $2\pi/T$. This period corresponds to a *sampling frequency*:

$$\omega_s = \frac{2\pi}{T}. \tag{8}$$

Spectrum to Study Sampling Rate Requirement

- Now that we have validated the equivalent CT system, we derive another expression for the spectrum of the CT impulse-sampled signal $x_p(t)$, relating it to the spectrum of the original CT signal $x(t)$. Using the CTFT multiplication property, the CTFT of the impulse-sampled signal (3) is the convolution

$$X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega). \tag{9}$$

Recall that the CTFT of the impulse train is an impulse train (see (24), Chapter 4, page 159):

$$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT) \overset{F}{\leftrightarrow} P(j\omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-k\frac{2\pi}{T}\right). \tag{10}$$

Performing the convolution on the right-hand side of (9), we obtain the spectrum of the impulse-sampled signal:

$$X_p(j\omega)=\frac{1}{T}\sum_{k=-\infty}^{\infty}X\left(j\left(\omega-k\frac{2\pi}{T}\right)\right)=\frac{1}{T}\sum_{k=-\infty}^{\infty}X\left(j(\omega-k\omega_s)\right). \tag{11}$$

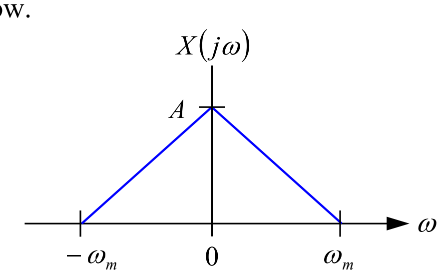
The spectrum $X_p(j\omega)$ consists of copies of $X(j\omega)$ scaled by $1/T$ and shifted in frequency to all multiples of the sampling frequency $\omega_s = 2\pi/T$. As expected from (7), $X_p(j\omega)$ is periodic in ω with period $\omega_s = 2\pi/T$.

Sampling Rate Required to Avoid Aliasing: the Nyquist Sampling Theorem

- Throughout our analysis of sampling and reconstruction, we assume the CT signal $x(t)$ being sampled is bandlimited to a frequency range $|\omega| \le \omega_m$, so its CTFT is zero outside that range:

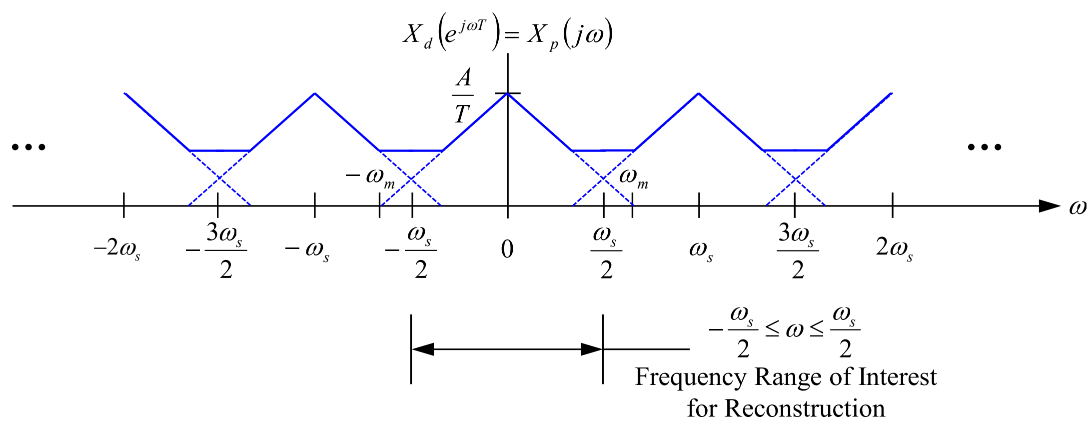
$$X(j\omega)=0,\ |\omega|>\omega_m, \tag{12}$$

as in the example shown below.



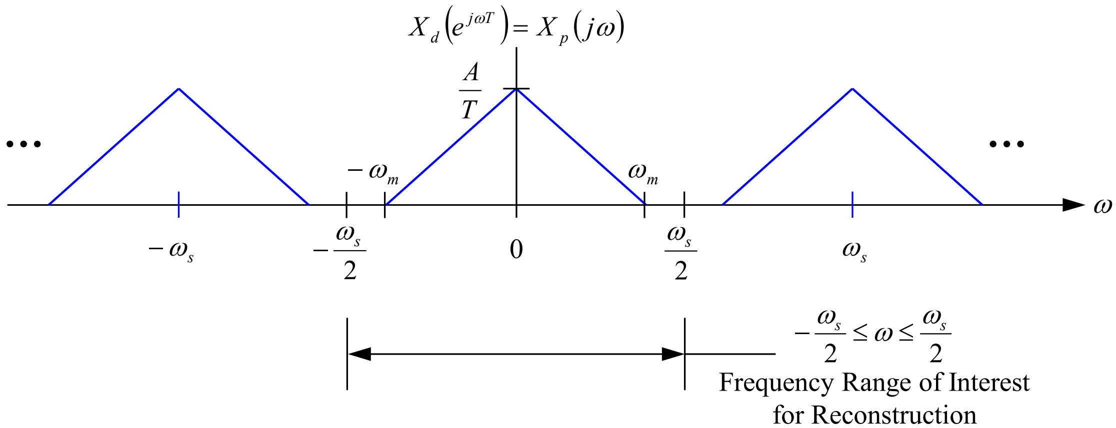
Most practical sampling systems incorporate an *anti-aliasing filter*, which is a lowpass filter designed to bandlimit the signal being sampled. Many analog-to-digital converters include an anti-aliasing lowpass filter. In many digital cameras, an optical lowpass filter is placed in front of the image sensor in order to bandlimit the spatial frequency spectrum of the image prior to sampling by the image sensor.

- Given a CT signal $x(t)$ bandlimited to $|\omega| \le \omega_m$, let us try sampling at a sampling frequency ω_s less than twice ω_m ($\omega_s < 2\omega_m$). The spectrum of the sampled signal, obtained using (11), is as shown.



Reconstruction methods use information in the frequency range $|\omega| \le \omega_s/2$. Over this frequency range, the spectrum $X_d\left(e^{j\omega T}\right) = X_p\left(j\omega\right)$ is *not* a faithful representation of the original CT signal spectrum $X\left(j\omega\right)$. Some components of $X\left(j\omega\right)$ from the frequency range $|\omega| > \omega_s/2$ overlap with other components in the frequency range $|\omega| \le \omega_s/2$. This effect is called *aliasing*. Once aliasing has occurred, it becomes impossible to reconstruct the original signal $x\left(t\right)$ perfectly (except in special cases).

- Now let us try sampling at a sampling frequency ω_s at least twice ω_m ($\omega_s > 2\omega_m$). The spectrum of the sampled signal, computed using (11), is as shown.



Now no aliasing occurs, and it is possible to reconstruct original CT signal $x\left(t\right)$ perfectly, as we show below.

- The fact that a signal $x\left(t\right)$ bandlimited to $|\omega| \le \omega_m$ can be reconstructed perfectly from its samples taken at a rate $\omega_s > 2\omega_m$ is often called the *Nyquist Sampling Theorem*.

- Hereafter, we always assume the sampling rate is sufficient to prevent aliasing and permit perfect reconstruction.

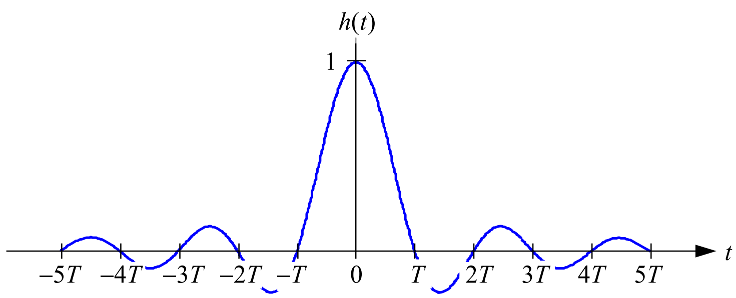
Ideal Bandlimited Reconstruction

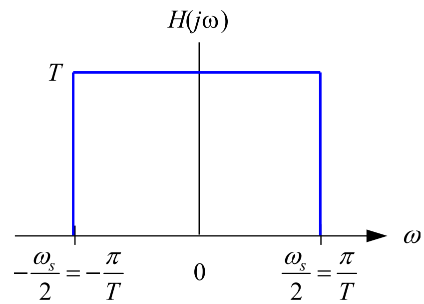
- This method is easy to analyze. The analysis provides a simple, elegant proof that perfect reconstruction of a signal from its samples is possible, in principle. This method is difficult to implement, however, because the time-domain reconstruction pulses have infinite duration.

- The reconstruction pulse and its CTFT are given by

$$h\left(t\right) = \text{sinc}\left(\frac{t}{T}\right) \stackrel{F}{\leftrightarrow} H\left(j\omega\right) = T\Pi\left(\frac{\omega T}{2\pi}\right) = T\Pi\left(\frac{\omega}{\omega_s}\right). \tag{13}$$

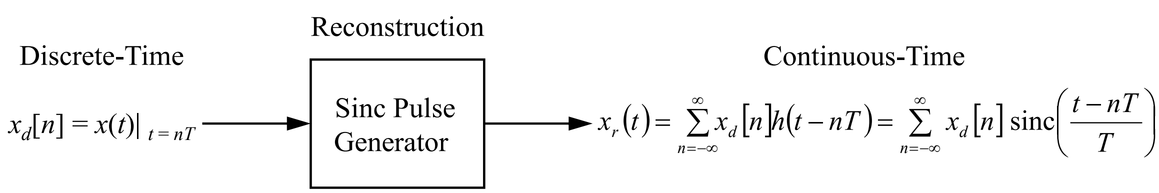
The reconstruction pulse $h(t)$ is a sinc function with zero crossings at all nonzero multiples of the sampling interval T. Its CTFT $H(j\omega)$ is an ideal lowpass filter with cutoff frequency equal to half the sampling frequency ω_s. These are shown in the figure below.





Actual Reconstruction System

- The actual reconstruction system, shown below, has a DT input $x_d[n]$ and a CT output $x_r(t)$.



The reconstructed signal $x_r(t)$ is a train of sinc pulses delayed to multiples of the sampling interval T, with the sinc pulse centered at time nT scaled by the signal sample $x_d[n]$:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x_d[n]h(t-nT) = \sum_{n=-\infty}^{\infty} x_d[n]\,\mathrm{sinc}\left(\frac{t-nT}{T}\right). \tag{14}$$

Note that each sinc pulse extends infinitely forward and backward in time.

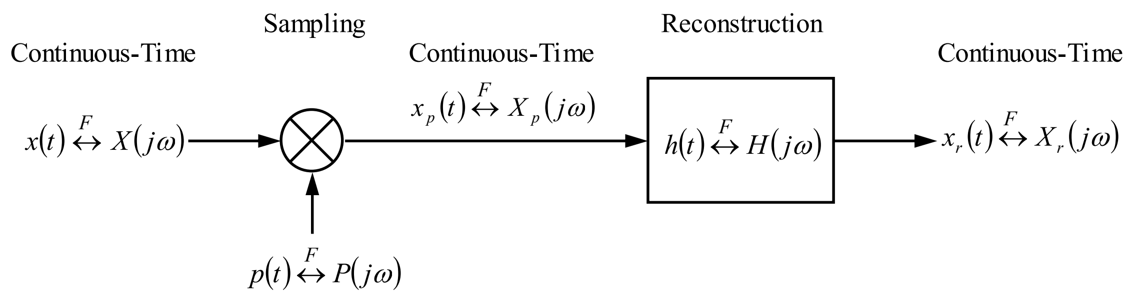
Equivalent Continuous-Time System

- In order to show that this method can reconstruct the original CT signal $x(t)$ perfectly, we analyze an equivalent system involving only CT. We can write the reconstructed signal (14) as

$$\begin{aligned} x_r(t) &= \sum_{n=-\infty}^{\infty} x_d[n]h(t-nT) \\ &= h(t) * \sum_{n=-\infty}^{\infty} x_d[n]\delta(t-nT) \quad . \\ &= h(t) * \left[x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-nT)\right] \end{aligned} \tag{15}$$

It may be easiest to understand (15) in reverse order. Using the sampling property of the CT impulse and definition (1), the second line follows from the third line. After performing the convolution in the second line, we obtain the first line.

- The equivalent CT system, as shown below, first samples the CT signal $x(t)$ using an impulse train to obtain an impulse-sampled signal $x_p(t)$, given by (3), then filters $x_p(t)$ by convolving it with a reconstruction filter $h(t)$ to obtain a reconstructed signal $x_r(t)$.

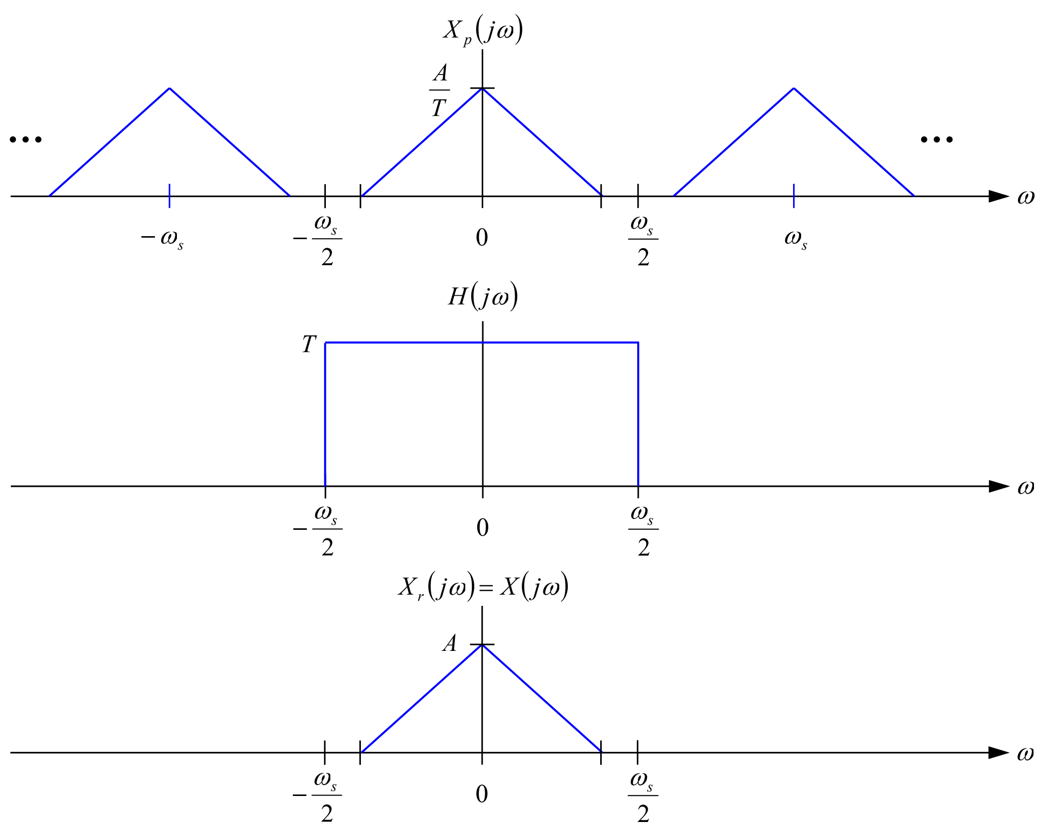


Proof of Perfect Reconstruction

- In order to show that perfect reconstruction is achieved, we compute the CTFT of (15) using the CTFT convolution and multiplication properties:

$$\begin{aligned} X_r(j\omega) &= H(j\omega)\cdot\left[\frac{1}{2\pi}X(j\omega) * \omega_s \sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)\right] \\ &= H(j\omega)\cdot\left[\frac{1}{T}\sum_{k=-\infty}^{\infty} X\big(j(\omega - k\omega_s)\big)\right] \\ &= H(j\omega)\cdot X_p(j\omega) \end{aligned} \quad . \tag{16}$$

The figure below illustrates how the reconstruction filter $H(j\omega)$ filters the CTFT of the impulse-sampled signal, $X_p(j\omega)$. This filtering removes all the shifted replicas of $X(j\omega)$, leaving only $k = 0$, such that $X_r(j\omega) = X(j\omega)$. *The original CT signal is reconstructed perfectly.*



- We consider two examples of ideal bandlimited reconstruction.
  - The first signal (on the left below) is

$$x(t) = \mathrm{sinc}^2\left(\frac{t-5}{4}\right) \overset{F}{\leftrightarrow} X(j\omega) = 4\Lambda\left(\frac{2\omega}{\pi}\right)e^{-j5\omega}.$$

In computing $X(j\omega)$, we have used the CTFT pair:

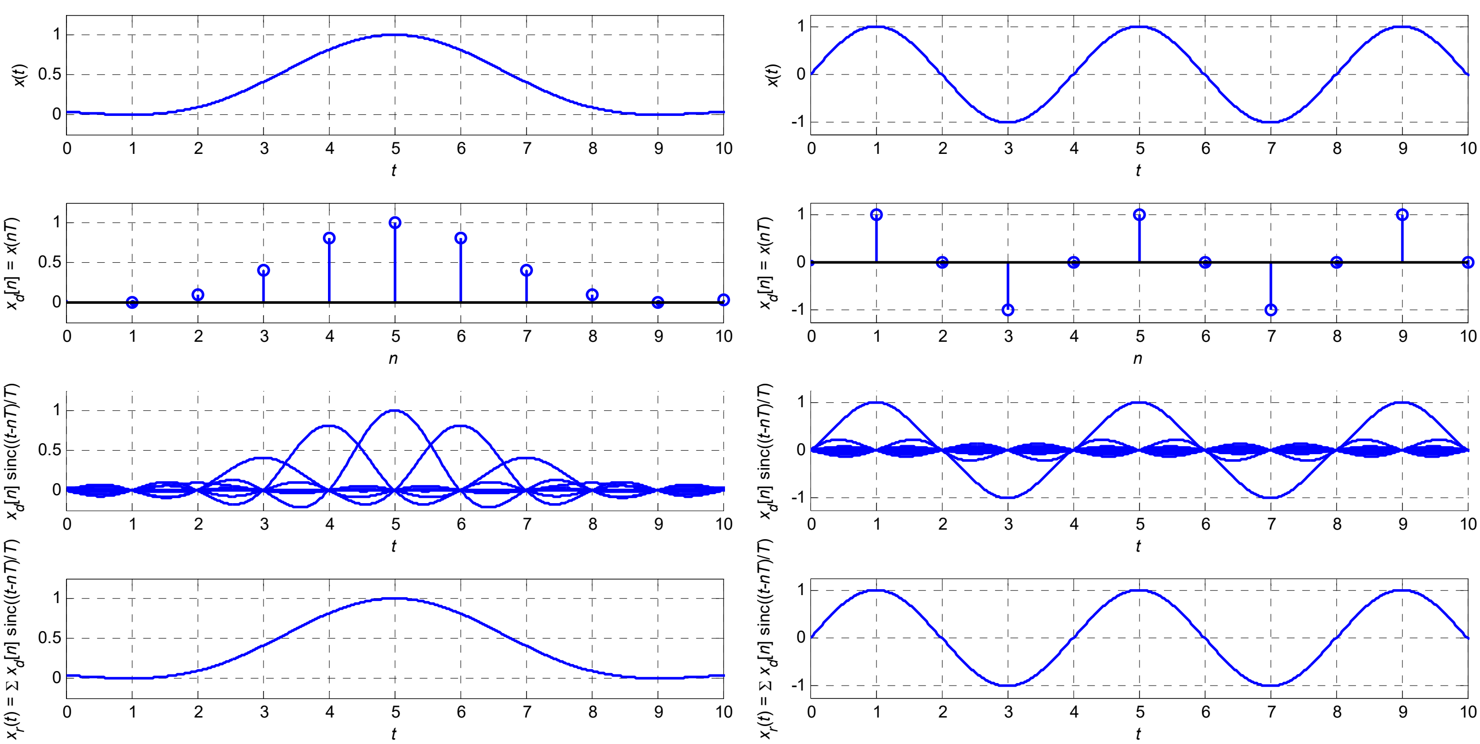
$$\frac{W}{\pi}\mathrm{sinc}^2\left(\frac{Wt}{\pi}\right) \overset{F}{\leftrightarrow} \Lambda\left(\frac{\omega}{2W}\right)$$

and the time-shift property of the CTFT. The CTFT $X(j\omega)$ is bandlimited to $|\omega| \leq \omega_m = \pi/2$ rad/s ($\omega_m / 2\pi = 1/4$ Hz). We sample with a sampling interval T = 1 s, corresponding to a sampling frequency $\omega_s = 2\pi$ rad/s ($\omega_s / 2\pi = 1$ Hz), which is twice the Nyquist rate.

  - The second signal (on the right below) is

$$x(t) = \sin\left(\frac{\pi t}{2}\right).$$

This is a sinusoid at a frequency $\pi/2$ rad/s (1/4 Hz). As in the previous example, we use a sampling interval $T = 1$ s and sampling frequency $\omega_s = 2\pi$ rad/s ($\omega_s / 2\pi = 1$ Hz), which is twice the Nyquist rate.

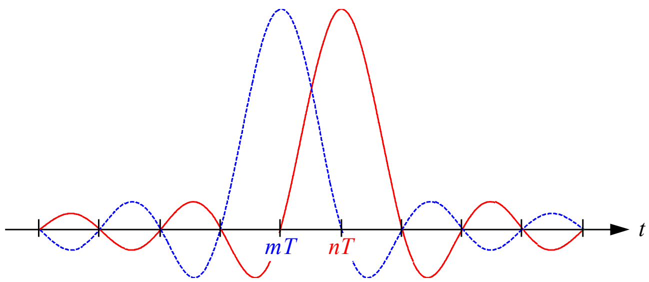


Sinc Pulses as an Orthonormal Basis for Bandlimited Signals

- Consider the set of time-translated sinc pulses used in the reconstruction (14):

$$\left\{ \operatorname{sinc}\left(\frac{t-nT}{T}\right),\ n=0,\pm 1,\pm 2,\ldots \right\}. \tag{17}$$

Two of these sinc pulses are shown in the figure below.



- One can show that these sinc pulses are mutually orthogonal for $m \neq n$:

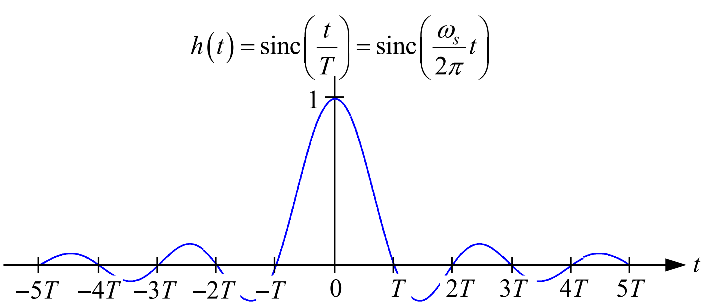
$$\int_{-\infty}^{\infty} \operatorname{sinc}\left(\frac{t-mT}{T}\right)\operatorname{sinc}\left(\frac{t-nT}{T}\right) dt = \begin{cases} T & m=n \\ 0 & m\neq n \end{cases}. \tag{18}$$

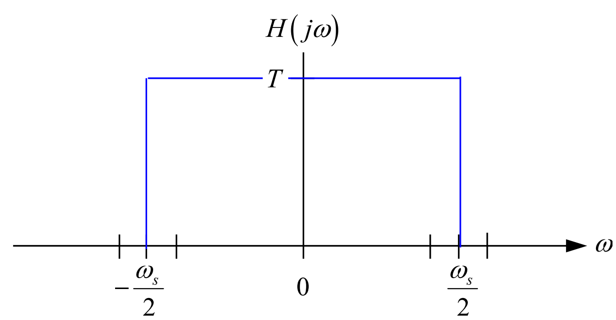
In a homework problem, you will be asked to prove (18) using Parseval's identity.

- One can further show that the set of pulses (17) forms a complete basis for representing the set of bandlimited CT signals $x(t) \overset{F}{\leftrightarrow} X(j\omega)$ such that $X(j\omega)=0$, $|\omega| \geq \pi/T$.

Practical Approximation of Ideal Reconstruction

- Ideal reconstruction using sinc pulses cannot be implemented in practice without modification, for the reasons summarized below.





- Time-domain pulse magnitude $|h(t)|$ falls off slowly, proportional to $1/|t|$.
- Truncating the pulses at a finite length causes significant error.

- Frequency-domain magnitude $|H(j\omega)|$ has abrupt transitions at $|\omega| = \omega_s/2$.
- Approximating the abrupt transition well requires very high delay and complexity.

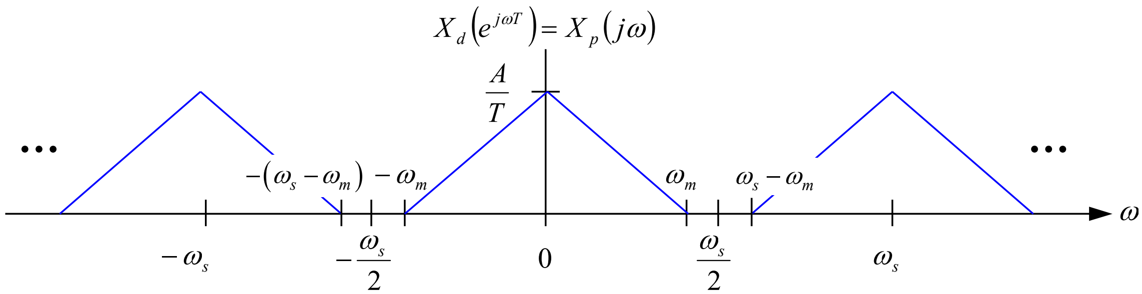
- In this section, we briefly discuss practical reconstruction methods that can achieve near-ideal results. Some of these methods are studied in greater depth in *EE 102B Course Reader*, Chapter 2.

Exploiting Oversampling

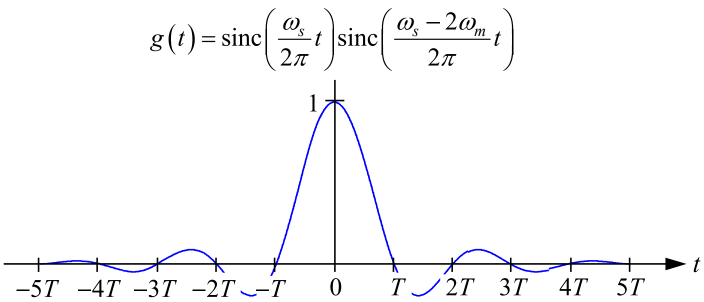
- Reconstruction is made significantly easier by *oversampling*. Given a CT signal $x(t) \overset{F}{\leftrightarrow} X(j\omega)$ that is bandlimited to $|\omega| \le \omega_m$, as in (12), oversampling corresponds to choosing a sampling frequency ω_s that is higher than the minimum value $2\omega_m$ that is required to avoid aliasing.
- For example, in the compact disc (CD) audio standard, developed in the early 1980s, the parameters are approximately

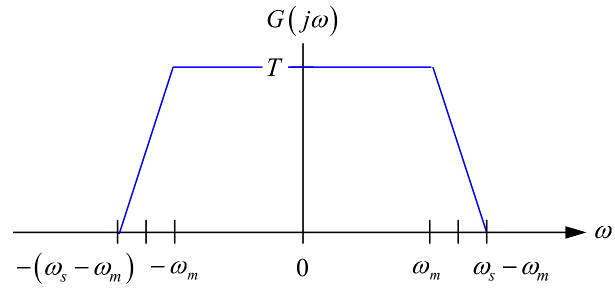
$$\frac{\omega_m}{2\pi} \approx 18 \text{ kHz} \quad \text{and} \quad \frac{\omega_s}{2\pi} \approx 44.1 \text{ kHz}.$$

The value of $\omega_m / 2\pi$ is related to the upper limit of human hearing, while the value of $\omega_s / 2\pi$ is chosen to facilitate reconstruction. The figure below, showing the spectrum of the sampled signal, is drawn to scale with these values.



- Because of oversampling, there are gaps in the spectrum between the frequency-shifted, scaled replicas of the original CT signal spectrum $X(j\omega)$. It is possible to achieve perfect reconstruction (as on page 248 above) while allowing the reconstruction filter frequency response to assume *any value* in the frequency range $\omega_m < |\omega| < \omega_s - \omega_m$.
- For example, we can achieve perfect reconstruction using the pulse shape and CTFT $g(t) \overset{F}{\leftrightarrow} G(j\omega)$ shown below. The time-domain pulse shape $g(t)$ is the product of two sinc functions. As a result, the pulse magnitude $|g(t)|$ falls off in proportion to $1/|t|^2$ for large $|t|$. This significantly reduces errors that arise when the reconstruction pulse is truncated to a finite length. The frequency response $G(j\omega)$ exhibits a gradual transition over the frequency range $\omega_m < |\omega| < \omega_s - \omega_m$.
- In a homework problem, you will be asked to derive the CTFT pair $g(t) \overset{F}{\leftrightarrow} G(j\omega)$ using the CTFT multiplication property. In a MATLAB exercise, you will be asked to study reconstruction errors arising from truncating the reconstruction pulses, comparing the pulse shapes $h(t)$ and $g(t)$.

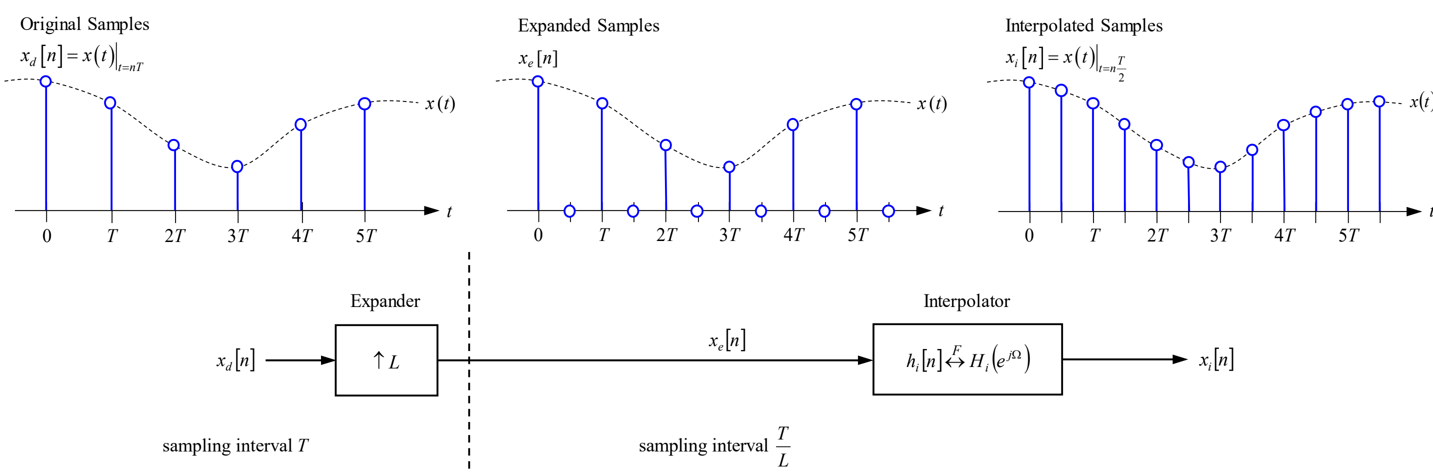




- Time-domain pulse magnitude $|g(t)|$ falls off more rapidly, proportional to $1/|t|^2$.
- Truncating the pulses at a finite length causes less error.
- Frequency-domain magnitude $|G(j\omega)|$ has gradual transition over the range $\omega_m < |\omega| < \omega_s - \omega_m$.
- The gradual transition can be approximated well with less delay and lower complexity.

Digital Upsampling

- Given a CT signal $x(t) \overset{F}{\leftrightarrow} X(j\omega)$ bandlimited to $|\omega| < \omega_m$, reconstruction becomes progressively easier as we increase the sampling frequency ω_s. It is often impractical, however, to increase ω_s as much as we would like in order to simplify reconstruction. Increasing ω_s requires using faster analog-to-digital converters, storing more data, and transmitting more data through communication networks. In practice, we usually employ only moderate oversampling ($\omega_s / 2\omega_m \approx 1.22$, as in the CD audio standard, is typical) and store only the samples obtained at sampling frequency ω_s. Then, just prior to reconstruction, we use *digital upsampling* to increase the sampling frequency to $L\omega_s$, where L is called an *upsampling ratio*. Upsampling ratios of $L = 8$ or 16 are common in audio reconstruction. Here we discuss digital upsampling very briefly.
- Digital upsampling is illustrated in the figure below (an upsampling ratio $L = 2$ is assumed).



- We start with an *original signal*

$$x_d[n] = x(t)\big|_{t=nT}, \tag{1}$$

representing the samples of a CT signal $x(t)$ recorded with sampling interval T and sampling frequency $\omega_s = 2\pi / T$.

- The *expander* inserts $L-1$ zeros between successive samples of $x_d[n]$, yielding an *expanded signal*

$$x_e[n] = \begin{cases} x_d\left[\dfrac{n}{L}\right] & \dfrac{n}{L} \text{ integer} \\ 0 & \text{otherwise} \end{cases}. \tag{19}$$

Assuming the signals are streaming in real time, the interval between samples (including zero samples) has been reduced to T/L, and the sampling frequency has been effectively increased to $2\pi/(T/L) = L\omega_s$. Recall that we defined DT signal expansion in Chapter 1 (page 9) and analyzed the DTFT of expanded signals in Chapter 5 (pages 216-217).

- The *interpolator* is, in principle, an ideal DT lowpass filter with cutoff frequency $W = \pi / L$ and passband gain L. It has an impulse response

$$h_i[n] = \text{sinc}\left(\frac{n}{L}\right). \tag{20}$$

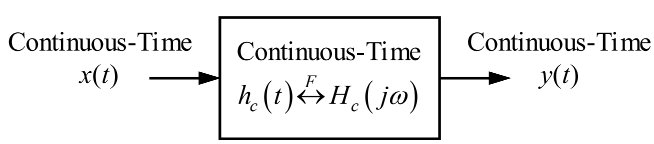
Assuming the original CT signal (1) was sampled without aliasing, the interpolator replaces the zero samples in $x_e[n]$ by the values that would be obtained by sampling the original CT signal with a shorter sampling interval T/L and a higher sampling frequency $L\omega_s$. At the interpolator output, the *interpolated signal* is

$$x_i[n] = x(t)\big|_{t=n\frac{T}{L}}. \tag{21}$$

- We have outlined the concept of digital upsampling only very briefly. See *EE 102B Course Reader*, Chapter 2 for a mathematical derivation and more detailed explanation.
- Later in this chapter, we show how to realize a practical filter approximating the ideal interpolator (20).

Discrete-Time Processing of Sampled Continuous-Time Signals

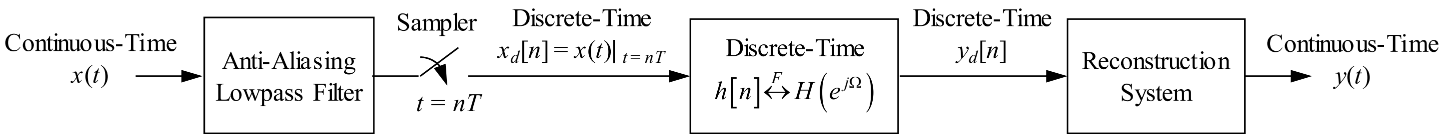
- Traditionally, signal processing, control or communication systems operated on continuous-valued, CT (i.e., analog) signals, performing only analog signal processing, like the LTI filtering shown below.



- Over time, most analog signal processing has been replaced by DT signal processing. It is possible to implement continuous-valued, DT signal processing, for example, using delay lines and multipliers.

Nevertheless, almost all DT signal processing is now performed using numerical calculations on quantized signals, in which case, it is called *digital signal processing*.

- We will focus here on implementing LTI filtering. The figure below provides a framework for our discussion. A CT signal $x(t)$ is sampled, yielding DT samples $x_d[n]$. DT LTI filtering is performed, resulting in filtered DT samples $y_d[n]$. Finally, a CT signal $y(t)$ is reconstructed from the filtered DT samples. We will design the DT LTI filtering so the overall system, with CT input $x(t)$ and CT output $y(t)$, functions similarly to a reference analog system, like the one shown on the previous page.



- Some important questions arise in this context:
  - How can we design a DT LTI filter to approximate well a given CT LTI filter?
  - How can we minimize the required sampling rate, bit precision and computational complexity?

  We address some of these questions briefly in this section.

Types of Discrete-Time Filters

- Recall that DT LTI filters may be classified as having an infinite impulse response (IIR) or a finite impulse response (FIR).
- The principal advantages of each type of filter are indicated by thick lines in the table below. The advantages of FIR filters – less error from finite bit precision, ease of ensuring stability, and ease of ensuring linear phase – make them the most popular type of DT filter.

	Infinite Impulse Response	**Finite Impulse Response**
Impulse response duration	Infinite	Finite
Implemented using recursion	Yes	No
Delay-and-multiply operations required for given performance	Fewer	More
Error from finite bit precision	More	Less
Stability	Stable iff $\sum_{n=-\infty}^{\infty} \lvert h[n] \rvert < \infty$. Finite bit precision can cause some stable filters to become unstable.	An FIR filter with finite coefficients is always stable. $\sum_{n=-\infty}^{\infty} \lvert h[n] \rvert < \infty$ always.
Linear phase response	Generally no.	Can always be achieved.

- Here we provide a brief discussion on the design of FIR filters using the Fourier series method.

Finite Impulse Response Filter Design by Fourier Series Method

- As a starting point, we are given a CT filter, which has impulse response and frequency response

$$h_c(t) \overset{F}{\leftrightarrow} H_c(j\omega). \tag{22}$$

  Except in special cases, $H_c(j\omega)$ is not a periodic function of frequency ω.

- We would like to approximate the CT system by a DT FIR filter, which has impulse response and frequency response

$$h[n] \overset{F}{\leftrightarrow} H\left(e^{j\Omega}\right). \tag{23}$$

  $H\left(e^{j\Omega}\right)$ must be a periodic function of frequency Ω. We would like $h[n]$ to have a finite duration.

- We choose a sampling rate $1/T$ sufficiently high to avoid aliasing of the CT signals that are to be filtered. The CT and DT frequencies, ω and Ω, are related by

$$\Omega = \omega T. \tag{24}$$

  The DT FIR filter frequency response $H\left(e^{j\Omega}\right)$ will be periodic in Ω with period 2π, so $H\left(e^{j\omega T}\right)$ will be periodic in ω with period $2\pi/T$.

- We choose a filter length $2N_1 + 1$. The DT FIR impulse response $h[n]$ will be nonzero for $-N_1 \le n \le N_1$, i.e., it will be noncausal. After designing $h[n]$ (and applying a window function, as described later), we can delay it by N_1 samples to obtain a causal filter.

- Now we express the CT frequency response $H_c(j\omega)$ in terms of the DT frequency variable Ω, as $H_c(j\Omega/T)$. In order to obtain the DT FIR filter impulse response $h[n]$, $-N_1 \le n \le N_1$, we compute the inverse DTFT of the CT frequency response $H_c(j\omega) = H_c(j\Omega/T)$ over a CT frequency interval $-\pi/T \le \omega \le \pi/T$, which corresponds to a DT frequency interval $-\pi \le \Omega \le \pi$:

$$h[n] = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} H_c\left(j\dfrac{\Omega}{T}\right) e^{jn\Omega} d\Omega & |n| \le N_1 \\ 0 & |n| > N_1 \end{cases}. \tag{25}$$

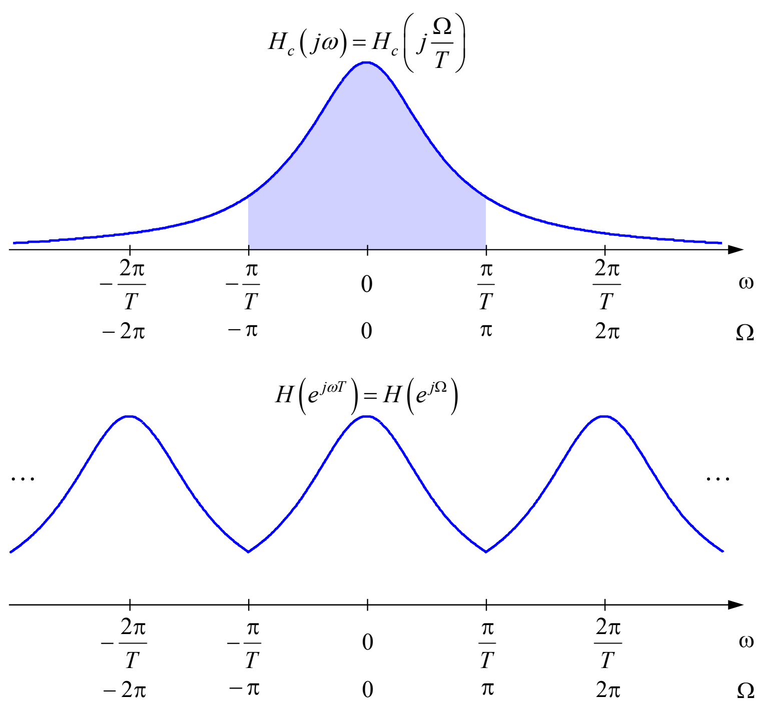
  Equation (25) is the key equation of the *Fourier series method* for designing FIR filters. The method is so named because (25) is mathematically equivalent to the CTFS analysis equation. Recall the duality between the DTFT and the CTFS that we studied at the end of Chapter 5.

- After obtaining the DT FIR impulse response $h[n]$ using (25), we can obtain its frequency response $H\left(e^{j\Omega}\right)=H\left(e^{j\omega T}\right)$ by computing the DTFT of $h[n]$:

$$H\left(e^{j\Omega}\right)=\sum_{n=-N_1}^{N_1} h[n]e^{-jn\Omega}=\sum_{n=-N_1}^{N_1} h[n]e^{-jn\omega T}=H\left(e^{j\omega T}\right). \tag{26}$$

  Expression (26) is mathematically equivalent to the CTFS synthesis equation (invoking the DTFT/CTFS duality again). Thus, it generates a periodic function of frequency Ω or ω.

- The figure below illustrates an example of the FS design method. The frequency scales are labeled in terms of both CT frequency ω and DT frequency $\Omega=\omega T$. We begin with a CT filter, whose frequency response $H_c(j\omega)=H_c(j\Omega/T)$ (shown in the upper row) is *not periodic in frequency*. We compute the DT FIR impulse response $h[n]$ using (25), integrating over the shaded region shown. Finally, we compute the DT FIR frequency response using (26). The DT filter frequency response $H\left(e^{j\omega T}\right)=H\left(e^{j\Omega}\right)$ (shown in the lower row) is *periodic in frequency* ω with period 2π/*T*, and in Ω with period 2π.



- The FS design method has converted an aperiodic function of frequency, $H_c(j\omega)=H_c(j\Omega/T)$, to a periodic function of frequency, $H\left(e^{j\omega T}\right)=H\left(e^{j\Omega}\right)$. This is entirely analogous to what happens if we start with an aperiodic CT signal $x(t)$, compute its CTFS coefficients over a finite time interval, and then perform a CTFS synthesis to obtain a signal $\hat{x}(t)$, which must always be a periodic function of time *t* (recall the discussion in Chapter 3, pages 141-142).

- Note that although we started with a CT frequency response $H_c(j\omega)=H_c(j\Omega/T)$ that is not bandlimited, *no aliasing has occurred*. This is because the filter design equation (25) integrates only over the interval $-\pi/T \le \omega \le \pi/T$ (corresponding to $-\pi \le \Omega \le \pi$), which is shaded in the figure above. As a result, the DT FIR frequency response $H\left(e^{j\omega T}\right)=H\left(e^{j\Omega}\right)$ reflects only the part of the CT response $H_c(j\omega)=H_c(j\Omega/T)$ over the shaded frequency interval.
- If the DT FIR frequency response $H\left(e^{j\omega T}\right)=H\left(e^{j\Omega}\right)$ has any discontinuities, it will exhibit the *Gibbs phenomenon*, which will manifest as ripple in $H\left(e^{j\omega T}\right)=H\left(e^{j\Omega}\right)$ near the discontinuities. Even if the initial CT frequency response $H_c(j\omega)=H_c(j\Omega/T)$ is continuous at all ω or Ω, the Gibbs phenomenon will occur if $H_c(j\omega)=H_c(j\Omega/T)$ assumes different values at the endpoints of the integration interval in (25), corresponding to $\omega=\pm\pi/T$ (or $\Omega=\pm\pi$). This does not occur in the example shown in the figure above, but it does occur in the differentiator example presented below.
- Assuming we start with a CT system whose impulse response and frequency response are symmetric, namely, either

$$h_c(t) \text{ real, even in } t \overset{F}{\leftrightarrow} H_c(j\omega) \text{ real, even in } \omega \tag{27a}$$

or

$$h_c(t) \text{ real, odd in } t \overset{F}{\leftrightarrow} H_c(j\omega) \text{ imaginary, odd in } \omega, \tag{27b}$$

then the DT FIR filter $h[n] \overset{F}{\leftrightarrow} H\left(e^{j\Omega}\right)$ will retain the corresponding symmetry, and will be *free of phase distortion*. For example, if (27a) is satisfied, then $h[n]$ will be real and even in *n*, and $H\left(e^{j\Omega}\right)$ will be real and even in Ω. Since $H\left(e^{j\Omega}\right)$ is real, its phase $\angle H\left(e^{j\Omega}\right)$ can only assume values from the set $\{0,\pi\}$ (plus any multiple of 2π). Since $\angle H\left(e^{j\Omega}\right)$ is a piecewise-constant function of Ω, the group delay $-d\angle H\left(e^{j\Omega}\right)/d\Omega$ is zero at all Ω, except where $H\left(e^{j\Omega}\right)$ changes sign, such that $\left|H\left(e^{j\Omega}\right)\right|=0$. Likewise, if (27b) is satisfied, then $h[n]$ will be real and odd in *n*, and $H\left(e^{j\Omega}\right)$ will be imaginary and odd in Ω. As a consequence, $\angle H\left(e^{j\Omega}\right)$ is a piecewise-constant function of Ω, assuming only values from $\{-\pi/2,\pi/2\}$ (plus any multiple of 2π), so $-d\angle H\left(e^{j\Omega}\right)/d\Omega$ is zero at all Ω, except where $\left|H\left(e^{j\Omega}\right)\right|=0$.

Windowing

- The Gibbs phenomenon, if present in a DT FIR frequency response $H\left(e^{j\omega T}\right)=H\left(e^{j\Omega}\right)$, can cause:
    - Distortion of desired signals in the passband.
    - Leakage of undesired signals in the stopband.

Windowing is a method for mitigating the Gibbs phenomenon.

- As a first step, we choose a *window function*:

$$w[n] \overset{F}{\leftrightarrow} W\left(e^{j\Omega}\right). \tag{28}$$

The time-domain window function $w[n]$ is chosen to be real and even in *n*, so its DTFT $W\left(e^{j\Omega}\right)$ is real and even in Ω. The window function $w[n]$ is nonzero for $-N_1 \le n \le N_1$, the same values of *n* as the FIR impulse response $h[n]$.

- Once we have chosen the window function, we multiply $w[n]$ by the FIR impulse response $h[n]$ to obtain a *windowed FIR* impulse response

$$h_w[n] = h[n]w[n]. \tag{29}$$

In order to understand the corresponding frequency response $H_w\left(e^{j\Omega}\right)$, we use the multiplication property of the DTFT (see (44) in Chapter 5, page 221) to obtain

$$h_w[n] = h[n]w[n] \overset{F}{\leftrightarrow} H_w\left(e^{j\Omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j(\Omega-\theta)}\right)W\left(e^{j\theta}\right)d\theta. \tag{30}$$

Because $h_w[n]$ involves *multiplication in time*, $H_w\left(e^{j\Omega}\right)$ is a *periodic convolution in frequency* between the FIR frequency response $H\left(e^{j\Omega}\right)$ and the DTFT of the window function, $W\left(e^{j\Omega}\right)$.

- Before proceeding further, we consider an example. Suppose we start with a very long-duration DT impulse response $h[n]$ and truncate it to be nonzero for $-N_1 \le n \le N_1$. This truncation corresponds to multiplying $h[n]$ by a *rectangular window function*, which is described in time and frequency by

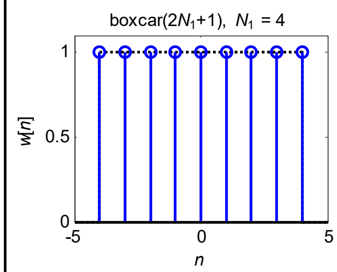
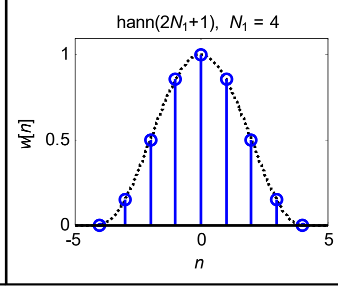
$$w_{\text{rect}}[n] = \Pi\left(\frac{n}{2N_1}\right) \overset{F}{\leftrightarrow} W_{\text{rect}}\left(e^{j\Omega}\right) = \frac{\sin\left(\Omega\left(N_1+\frac{1}{2}\right)\right)}{\sin(\Omega/2)}. \tag{31}$$

Recall our discussion of an FIR approximation of an ideal lowpass filter in Chapter 5, where we explained the Gibbs phenomenon as a periodic convolution of an ideal lowpass response $H_{\text{lpf,ideal}}\left(e^{j\Omega}\right)$ with the function $W_{\text{rect}}\left(e^{j\Omega}\right)$ appearing in (31) (see pages 234-235).

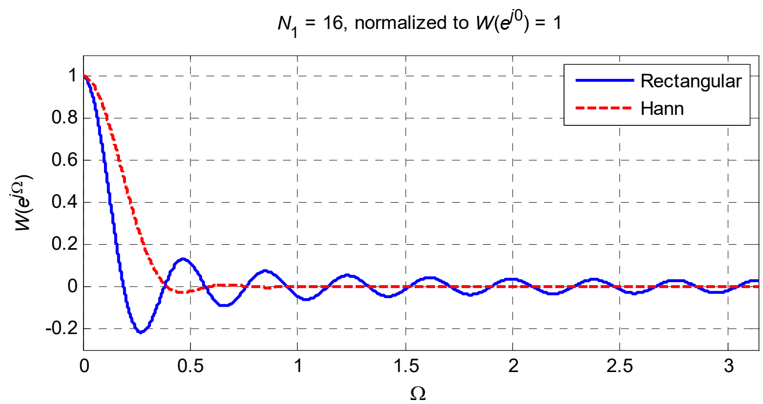
- MATLAB defines many different window functions (my version has at least 17). To see a full listing, type the following at the command prompt:

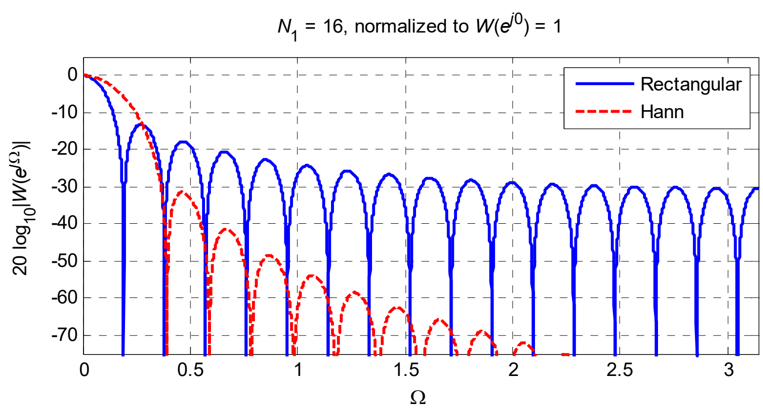
```
>> help window
```

Here we compare just two: the *rectangular window* and the *Hann window*. The figure below shows the MATLAB command and time-domain $w[n]$ for each window, for $N_1 = 4$. The rectangular window goes abruptly to zero at the endpoints, whereas the Hann window tapers to zero at the endpoints.

Name(s)	Definition	MATLAB Command	Graph ($N_1 = 4$)
No Window Rectangular Boxcar Fourier	$w[n] = \begin{cases} 1 & \lvert n \rvert \le N_1 \\ 0 & \lvert n \rvert > N_1 \end{cases}$	`boxcar(2*N1+1)`	
Hann	$w[n] = \begin{cases} \frac{1}{2}\left(1+\cos\left(\frac{\pi n}{N_1}\right)\right) & \lvert n \rvert \le N_1 \\ 0 & \lvert n \rvert > N_1 \end{cases}$	`hann(2*N1+1)`	

- The figure below shows the DTFT $W\left(e^{j\Omega}\right)$ for each window, for $N_1 = 16$. The upper plot shows $W\left(e^{j\Omega}\right)$ on a linear scale, and the lower plot shows $20\log_{10}\left|W\left(e^{j\Omega}\right)\right|$ on a decibel scale. The responses have been normalized to unit d.c.gain, i.e., $W\left(e^{j0}\right) = 1$





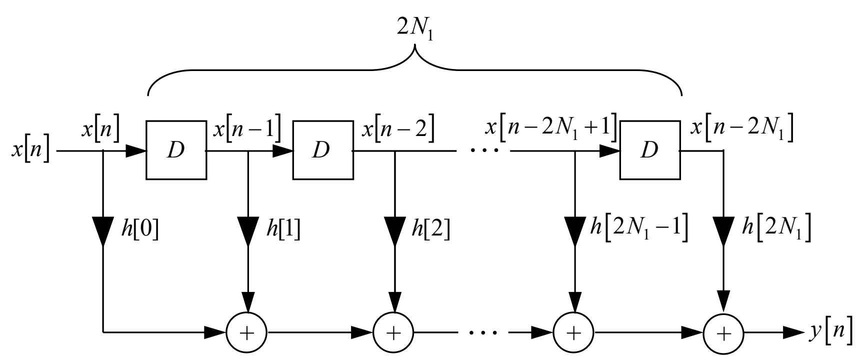
- Two features of the plots are especially important.
  - The *main lobe* of $W\left(e^{j\Omega}\right)$ is the peak close to $\Omega = 0$. Its width determines the abruptness of transitions between the passband and the stopband in a windowed frequency response $H_w\left(e^{j\Omega}\right)$.
  - The *side lobes* of $W\left(e^{j\Omega}\right)$ are the subsidiary peaks (positive and negative) away from the main lobe. Their heights affect the amount of passband distortion and stopband leakage in a windowed frequency response $H_w\left(e^{j\Omega}\right)$.
- The rectangular window function offers a narrow main lobe, but has high side lobes.
- The Hann window function has a wider main lobe, but offers far lower side lobes than the rectangular window. The latter is especially evident on the decibel scale. The Hann window sacrifices abrupt transitions to achieve very low passband distortion and stopband leakage.

Realization of Causal Finite Impulse Response Filters

- The windowed FIR impulse response $h_w[n]$ is nonzero for $-N_1 \le n \le N_1$, so it is not causal. In order to make it usable for real-time filtering, we delay it by N_1 samples, so it becomes nonzero for $0 \le n \le 2N_1$. This adds a linear phase to the frequency response. The causal FIR filter we implement is

$$h_{w,\,\text{causal}}[n] = h_w\left[n - N_1\right] \stackrel{F}{\leftrightarrow} H_{w,\,\text{causal}}\left(e^{j\Omega}\right) = H_w\left(e^{j\Omega}\right)e^{-jN_1\Omega}. \tag{32}$$

- We can realize the causal FIR response (32) using the following structure. The coefficients $h[0], h[1], h[2], \ldots$ correspond to the samples of $h_{w,\,\text{causal}}[n]$.



Finite Impulse Response Filter Examples

Ideal Lowpass Filter

- The CT system is an ideal lowpass filter with cutoff frequency W (in units of rad/s). It has a frequency response

$$H_c(j\omega) = \Pi\left(\frac{\omega}{2W}\right),$$

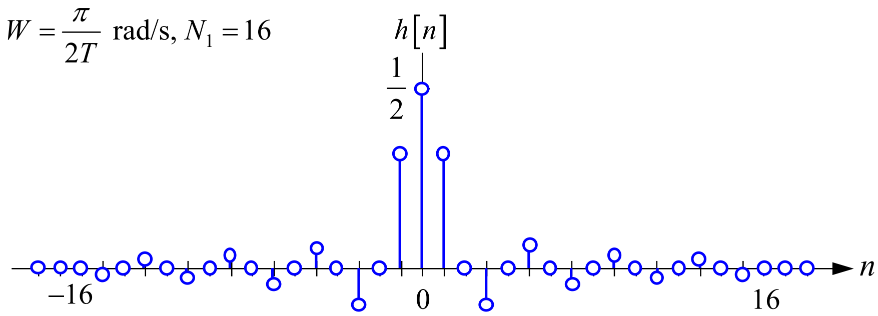
which is real and even in ω. It is not periodic in ω.

- Choosing a sampling rate $1/T$ and a filter length $2N_1+1$, using (25), the symmetric FIR approximation has an impulse response for $-N_1 \le n \le N_1$ given by

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_c\left(j\frac{\Omega}{T}\right)e^{jn\Omega}d\Omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Pi\left(\frac{\Omega}{2WT}\right)e^{jn\Omega}d\Omega \\ &= \frac{1}{2\pi}\int_{-WT}^{WT} e^{jn\Omega}d\Omega \\ &= \frac{WT}{\pi}\operatorname{sinc}\left(\frac{WT}{\pi}n\right) \end{aligned} . \tag{33}$$

The impulse response (33) is real and even in n, so it is not causal. We recognize (33) as the impulse response of an ideal DT lowpass filter with cutoff frequency WT (in units of rad), truncated to $-N_1 \le n \le N_1$. We obtained it previously by simply truncating the impulse response of an ideal DT lowpass filter (see (60) in Chapter 5, page 233).

- We consider an example in which the CT cutoff frequency is $W = \pi/2T$ rad/s, so the DT cutoff frequency is $WT = \pi/2$ rad. We consider a filter length $2N_1+1=33$. The figure below shows the symmetric FIR filter impulse response, which is $h[n]=\frac{1}{2}\operatorname{sinc}\left(\frac{n}{2}\right)$, $-16 \le n \le 16$.



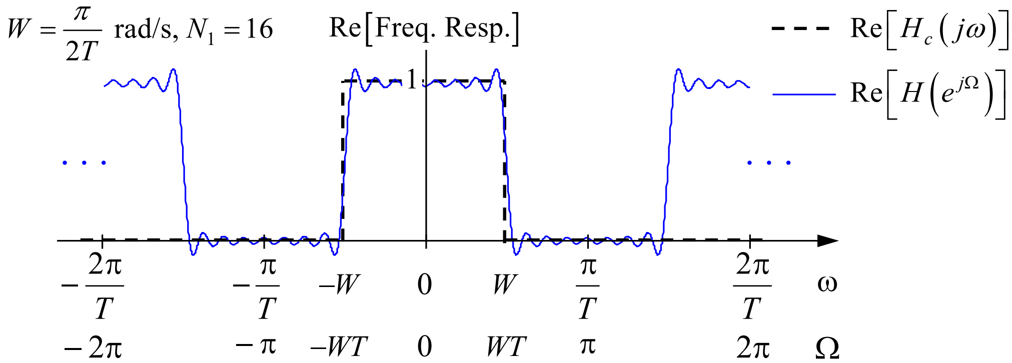
- Using (26), the symmetric FIR approximation has a frequency response

$$\begin{aligned} H\left(e^{j\Omega}\right) &= \sum_{n=-N_1}^{N_1} h[n]e^{-jn\Omega} \\ &= \frac{WT}{\pi}\sum_{n=-N_1}^{N_1}\operatorname{sinc}\left(\frac{WT}{\pi}n\right)e^{-jn\Omega} \end{aligned} , \tag{34}$$

which is real and even in Ω. It is periodic in Ω.

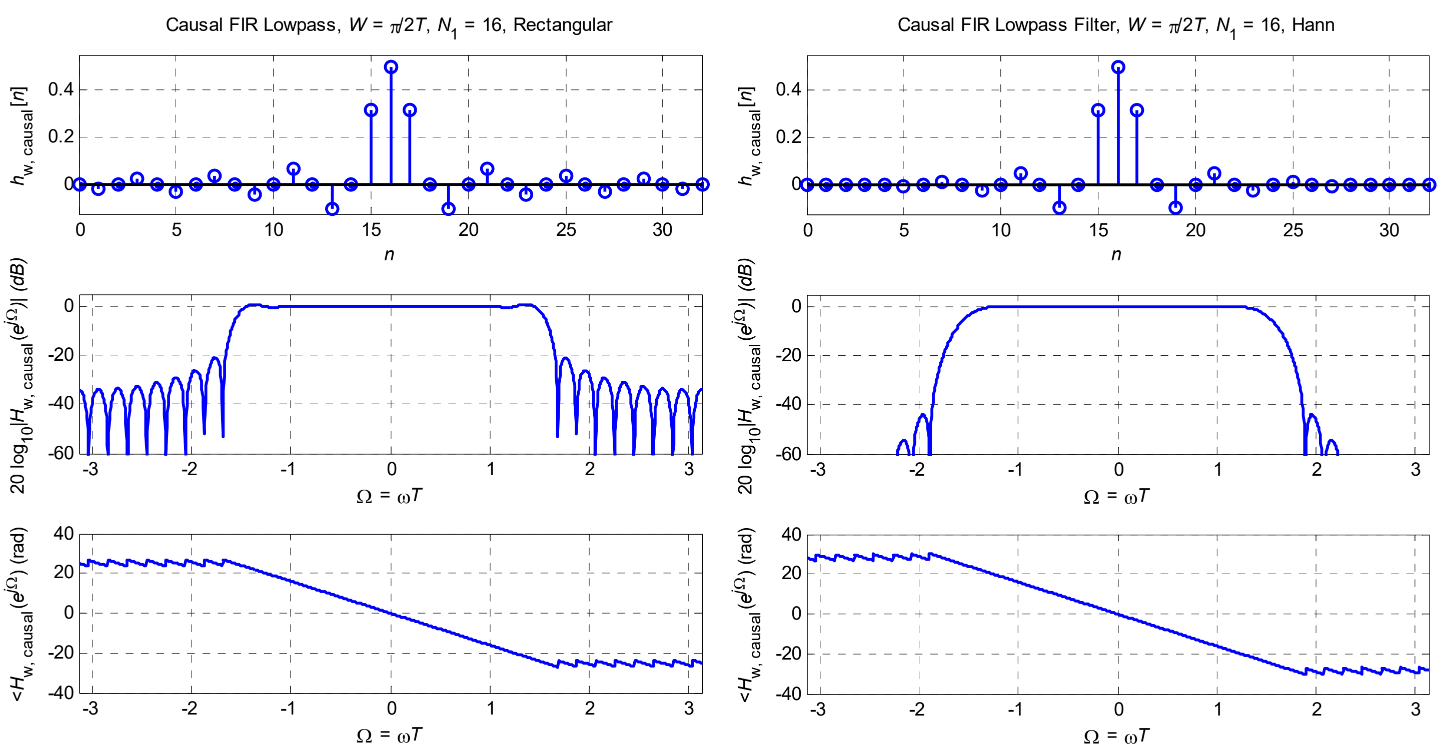
- The figure below shows the CT frequency response $H_c(j\omega)$ and the symmetric FIR frequency response $H\left(e^{j\Omega}\right)$. Only real parts are shown, since both are purely real. $H_c(j\omega)$ is an ideal CT lowpass response, which is not periodic. $H\left(e^{j\Omega}\right)$ is a Fourier series approximation of it, so it is periodic. At frequencies where $H\left(e^{j\Omega}\right)$ has discontinuities, it exhibits the Gibbs phenomenon.

- Since the symmetric FIR frequency response $H\left(e^{j\Omega}\right)$ is real and positive in the passband, $\angle H\left(e^{j\Omega}\right)=0$ in the passband.



- Now we window the impulse response and delay it to make it causal. Using (30), the windowed symmetric FIR impulse response is $h_w[n]=h[n]w[n]$, where $h[n]$ is given by (33) and $w[n]$ is the window function. The corresponding frequency response is $H_w\left(e^{j\Omega}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}W\left(e^{j\theta}\right)H\left(e^{j(\Omega-\theta)}\right)d\theta$. Delaying $h_w[n]$ by N_1, we obtain the causal windowed FIR impulse response $h_{w,\text{ causal}}[n]=h_w[n-N_1]$, corresponding to a frequency response $H_{w,\text{ causal}}\left(e^{j\Omega}\right)=H_w\left(e^{j\Omega}\right)e^{-jN_1\Omega}$, as given by (32).

- The figures below show filters designed with the same cutoff frequency and length as above, comparing rectangular and Hann window functions.



- The magnitude responses are shown in terms of $20\log_{10}\left|H_{w,\text{ causal}}\left(e^{j\Omega}\right)\right|$ (dB) to best illustrate stopband leakage. The rectangular window (on the left) yields a more abrupt cutoff but large passband ripple and stopband leakage, while the Hann window yields small passband ripple and stopband leakage, but at the cost of a more gradual cutoff.

- In the phase response $\angle H_{w,\text{ causal}}\left(e^{j\Omega}\right)$, for both window functions, we observe a linear phase in the passband, whose slope corresponds to a group delay of $N_1 = 16$ samples. In plotting the phase responses, we have used the MATLAB **`unwrap`** command to eliminate 2π phase jumps, in order to highlight the linearity of the phase in the passband.

Differentiator

- The CT system is a differentiator, which has a frequency response

$$H_c(j\omega) = j\omega,$$

which is imaginary and odd in ω. It is not periodic in ω.

- Choosing a sampling rate $1/T$ and a filter length $2N_1+1$, using (25), the symmetric FIR approximation has an impulse response for $-N_1 \le n \le N_1$ given by

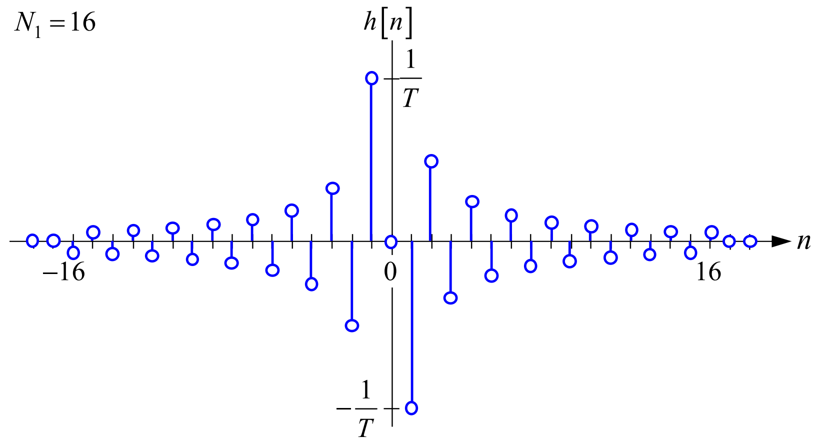
$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_c\left(j\frac{\Omega}{T}\right)e^{jn\Omega}d\Omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} j\frac{\Omega}{T}e^{jn\Omega}d\Omega \end{aligned} . \tag{35}$$

We evaluate the integral using integration by parts:

$$\begin{aligned} h[n] &= \frac{j}{2\pi T}\left(\frac{\Omega}{jn}e^{jn\Omega}\Big|_{-\pi}^{\pi} - \frac{1}{jn}\int_{-\pi}^{\pi} e^{jn\Omega}d\Omega\right) \\ &= \frac{1}{Tn}\left[\frac{1}{2}\left(e^{j\pi n}+e^{-j\pi n}\right) - \frac{1}{j2\pi n}\left(e^{j\pi n}-e^{-j\pi n}\right)\right] \\ &= \frac{1}{Tn}\left(\cos\pi n - \frac{\sin\pi n}{\pi n}\right) \\ &= \frac{1}{Tn}\left[(-1)^n - \delta[n]\right] \\ &= \begin{cases} 0 & n=0 \\ \frac{1}{Tn}(-1)^n & n \ne 0 \end{cases} \end{aligned} , \tag{36}$$

The impulse response $h[n]$ is real and odd in n. It is not causal.

- We consider a filter length $2N_1+1=33$. The figure below shows the DT FIR filter impulse response $h[n]$, $-16 \le n \le 16$.



- Using (26), the symmetric FIR approximation has a frequency response

$$\begin{aligned} H\left(e^{j\Omega}\right) &= \sum_{n=-N_1}^{N_1} h[n] e^{-jn\Omega} \\ &= \frac{1}{T} \sum_{n=-N_1}^{N_1} \frac{1}{n}\left[(-1)^n - \delta[n]\right] e^{-jn\Omega} \end{aligned}, \tag{37}$$

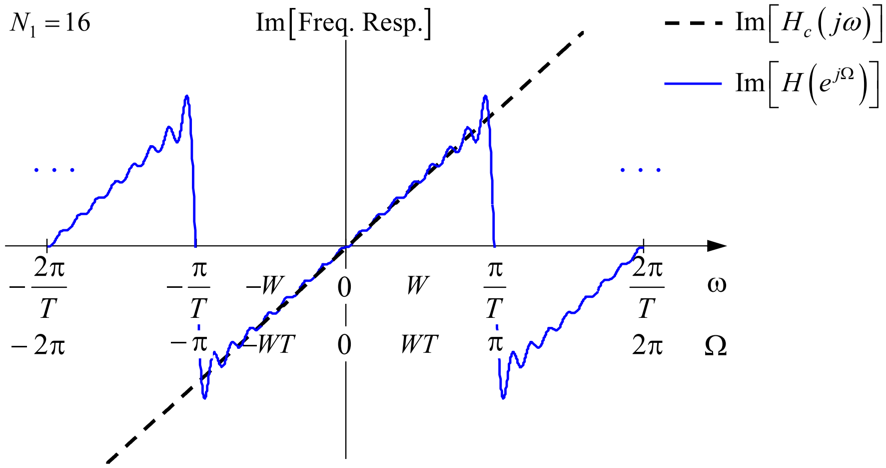
which is imaginary and odd in Ω. It is periodic in Ω.

- The figure below shows the CT frequency response $H_c(j\omega)$ and the symmetric FIR frequency response $H\left(e^{j\Omega}\right)$. Only imaginary parts are shown, since both are purely imaginary. $H_c(j\omega)$ is a linear function of ω, so it is nonperiodic. $H\left(e^{j\Omega}\right)$ is a Fourier series approximation to the linear function over the interval $-\pi \le \Omega < \pi$, and is thus a periodic sawtooth function. Although $H_c(j\omega)$ has no discontinuities, $H\left(e^{j\Omega}\right)$ has discontinuities at $\Omega = \pm\pi, \pm 3\pi, \ldots$, where it exhibits the Gibbs phenomenon.

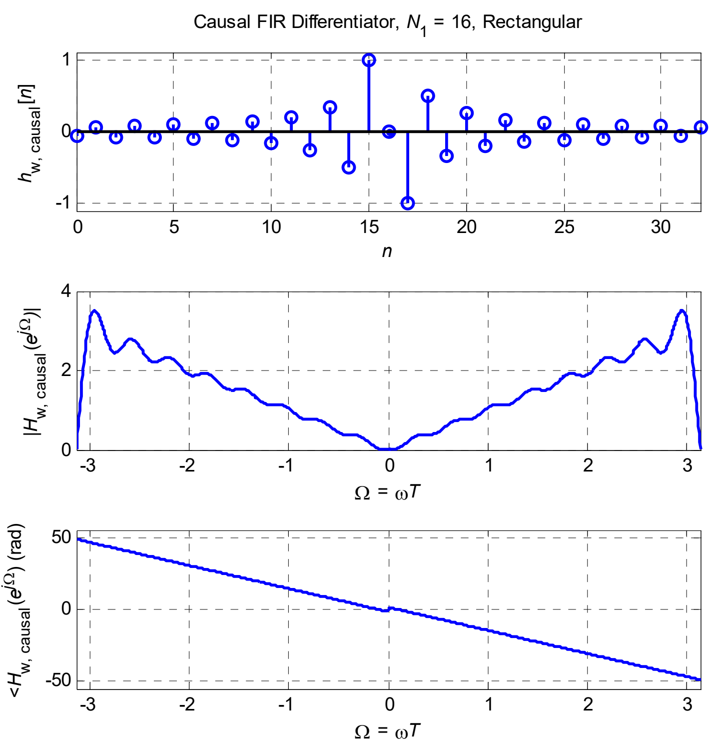
- The symmetric FIR frequency response $H\left(e^{j\Omega}\right)$ is purely imaginary, and has a phase given by

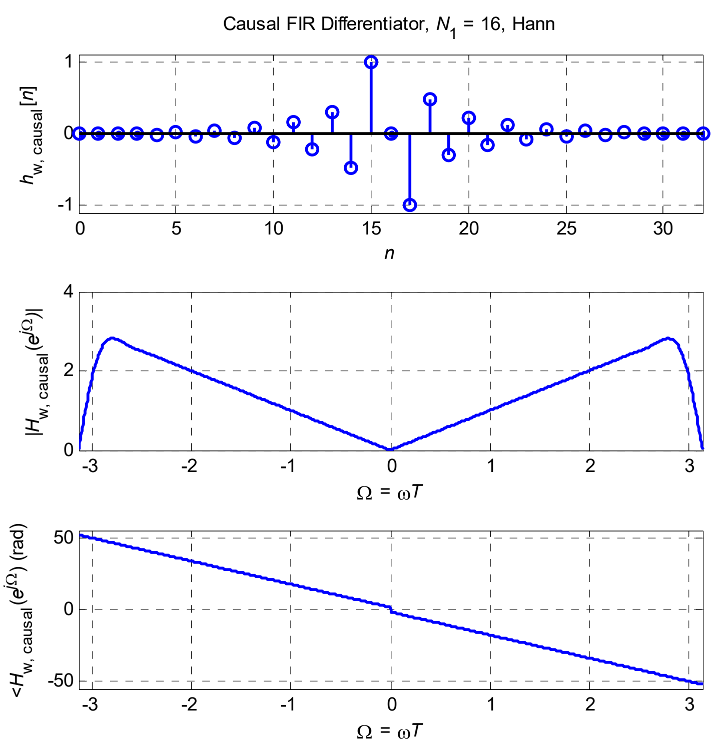
$$\angle H\left(e^{j\Omega}\right) = \begin{cases} +\frac{\pi}{2} & \operatorname{Im}\left[H\left(e^{j\Omega}\right)\right] > 0 \\ -\frac{\pi}{2} & \operatorname{Im}\left[H\left(e^{j\Omega}\right)\right] < 0 \end{cases}. \tag{38}$$

Since the phase (38) is piecewise constant, the corresponding group delay $-d\angle H\left(e^{j\Omega}\right)/d\Omega$ is zero, except at discontinuities of the phase, where $H\left(e^{j\Omega}\right)$ changes sign, and thus passes through zero.



- Now we window the impulse response and delay it to make it causal. The figures below show filters designed with the same length as above, comparing rectangular and Hann window functions.





- In this example, the magnitude responses $\left|H_{w,\text{ causal}}\left(e^{j\Omega}\right)\right|$ are shown on a linear scale to best illustrate the linear dependence on frequency. The rectangular window yields large passband ripple and a passband extending close to $\Omega = \pm\pi$, while the Hann window yields small passband ripple, but the passband does not extend as close to $\Omega = \pm\pi$.

- In the phase response $\angle H_{w,\text{ causal}}\left(e^{j\Omega}\right)$, for both window functions, we observe a linear phase in the passband, whose slope corresponds to a group delay of $N_1 = 16$ samples. In plotting the phase responses, we have used the MATLAB **`unwrap`** command to eliminate 2π phase jumps, helping to highlight the linearity of the phase in the passband.

Stanford University
EE 102A: Signal Processing and Linear Systems I
Professor Joseph M. Kahn

Chapter 7: Communication Systems

Motivations

- Electrical and electromagnetic communication systems have been used widely for well over a century. The earliest systems, including wired telephones, and broadcast radio and television, were analog. Over the past 50 years, these have been largely supplanted by digital communication systems. Digital optical fiber cables interconnect servers in data centers and form the backbone of the global Internet. Digital wireless networks provide telephony and Internet access in buildings and across vast outdoor areas.
- *Modulation* and *demodulation* serve key functions in these communication systems, and are the main aspects of communications we study in EE 102A. *Modulation* is the process of embedding an information-bearing *message signal* into another signal called a *carrier signal* in order to create a *modulated signal*. The carrier signal is often a sinusoid at a *carrier frequency* ω_c chosen so the modulated signal can propagate as a wave through a communication medium. Commonly used carrier frequencies range from about 1 MHz (broadcast AM radio) to hundreds of THz (optical fiber). *Demodulation* is the process of receiving a modulated signal and recovering the message signal from it.
- Modulation may be performed on the *amplitude*, *phase* or *frequency* of the carrier signal. In EE 102A, we mainly study amplitude modulation. We can analyze amplitude modulation easily in the time domain, or in the frequency domain using the CTFT. By contrast, phase or frequency modulation are fundamentally nonlinear operations, making them harder to analyze.

Major Topics in This Chapter

- Communication systems and carrier frequencies employed
- Amplitude modulation techniques
  - Double-sideband amplitude modulation with suppressed carrier. Synchronous demodulation.
  - Quadrature amplitude modulation. Synchronous demodulation.
  - Double-sideband amplitude modulation with large carrier. Asynchronous demodulation.
- Angle modulation techniques
  - Phase modulation
  - Frequency modulation
- Communication system concepts
  - Analog vs. digital communication
  - Multiplexing and reuse

Electrical and Electromagnetic Communication Systems

- The table below lists several important communication systems and the carrier frequencies they typically use.
- As the carrier frequency increases, the bandwidth available for transmission tends to increase. For example, while cellular data networks can access up to hundreds of MHz of bandwidth, optical fibers provide tens of THz of bandwidth.
- Some of these systems, including digital subscriber lines, television cables and optical fibers, propagate modulated signals through wires, coaxial cables or dielectric waveguides. By contrast, wireless systems allow modulated signals to propagate through free space. One type of system – extremely low frequency submarine links – propagates signals through the Earth itself.

System	Approximate Carrier Frequency $\frac{\omega_c}{2\pi}$
Extremely low frequency links to submarines	3–300 Hz
Digital subscriber lines	4–4000 kHz
Broadcast AM radio	540–1610 kHz
Broadcast FM radio	88–108 MHz
Broadcast television	54–88, 174–216, 470–890 MHz
Cable television and Internet access	7–1000 MHz
Cellular telephone and data networks	700, 800, 850, 1700, 1900, 2100, 2500 MHz
Wireless local area networks (WiFi)	2.4, 5 GHz
Satellite communications	1–2, 2–4, 4–8, 8–12, 12–18, 26–40 GHz
Wireless HDMI interface	60 GHz
Optical fibers for telecommunications	185–196 THz
Optical fibers for data communications	300–350 THz

Amplitude Modulation Techniques

- We can easily analyze amplitude modulation and its demodulation in the time domain, or in the frequency domain using the CTFT, owing to the simple operations employed, such as multiplication and addition.

Double-Sideband Amplitude Modulation with Suppressed Carrier

- Double-sideband amplitude modulation is used for broadcast AM radio, but with a large carrier added. While transmitting a large carrier enables the message signal to be recovered using asynchronous demodulation, it leads to poor power efficiency. In a later section, we will study *double-sideband amplitude modulation with large carrier* (DSB-AM-LC) and its asynchronous demodulation.

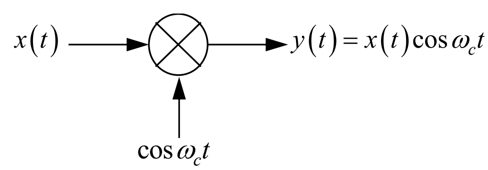
- The modulation scheme we consider here contains no carrier component, and is called *double-sideband amplitude modulation with suppressed carrier* (DSB-AM-SC). While it has better power efficiency than DSB-AM-LC, it requires synchronous demodulation. We already analyzed the modulation and demodulation of DSB-AM-SC in Chapter 4 (pages 175-177). We review these here, and then proceed to study some aspects we did not discuss previously.

Modulation

- In DSB-AM-SC, we multiply a real message signal $x(t)$ is by a sinusoidal carrier signal $\cos\omega_c t$ to form a modulated signal

$$y(t) = x(t)\cos\omega_c t\ , \tag{1}$$

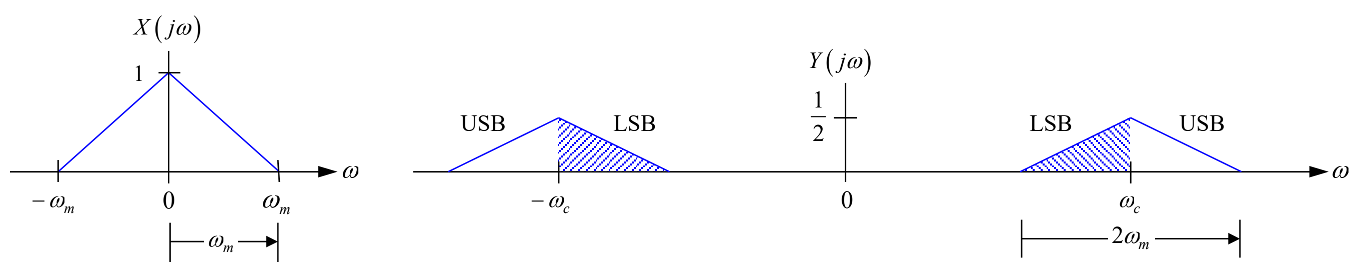
as shown in the figure below.



- To compute the spectrum of the modulated signal $y(t)$, we use the CTFT multiplication property $p(t)q(t) \overset{F}{\leftrightarrow} (1/2\pi)P(j\omega) * Q(j\omega)$ and the CTFT pair $\cos\omega_c t \overset{F}{\leftrightarrow} \pi\delta(\omega-\omega_c) + \pi\delta(\omega+\omega_c)$ to obtain

$$Y(j\omega) = \frac{1}{2}X\big(j(\omega-\omega_c)\big) + \frac{1}{2}X\big(j(\omega+\omega_c)\big). \tag{2}$$

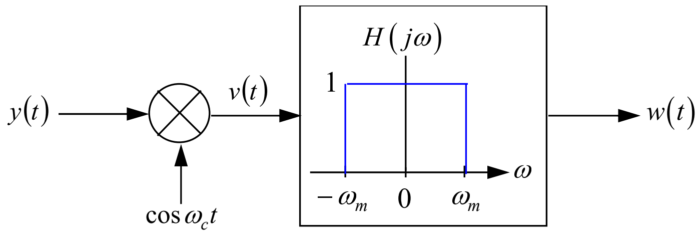
- The figure below shows the message signal spectrum $X(j\omega)$ (on the left), and the modulated signal spectrum $Y(j\omega)$ (on the right). Observe that the modulated spectrum $Y(j\omega)$ comprises copies of the message spectrum $X(j\omega)$, shifted in frequency to $\pm\omega_c$ and scaled by 1/2.



We will discuss other aspects of the spectrum, including the bandwidths ω_m and $2\omega_m$ indicated, and the features labeled "USB" and "LSB", in a subsection below on *Spectral Efficiency*.

Synchronous Demodulation

- The figure below shows a *synchronous demodulator* for DSB-AM-SC. The demodulator must create a replica of the carrier signal $\cos\omega_c t$ that is synchronized in frequency and phase to the received modulated signal $y(t)$.



- The demodulator multiplies the received modulated signal $y(t)$ by the carrier replica $\cos\omega_c t$, obtaining a signal

$$\begin{aligned} v(t) &= y(t)\cos\omega_c t \\ &= x(t)\cos^2\omega_c t \\ &= \frac{1}{2}x(t)+\frac{1}{2}x(t)\cos 2\omega_c t \end{aligned} \quad . \tag{3}$$

We used the identity $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ to obtain the third line of (3). The multiplier output $v(t)$, given by (3), contains two terms. The first, $\frac{1}{2}x(t)$, is a scaled replica of the message signal. The second, $\frac{1}{2}x(t)\cos 2\omega_c t$, is the scaled message signal modulated onto a carrier at frequency $2\omega_c$.

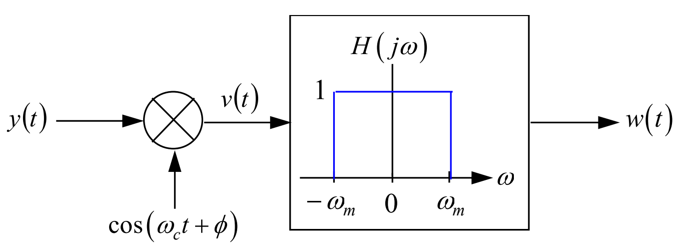
- The demodulator sends the multiplier output $v(t)$ to an ideal lowpass filter, which passes the first term and blocks the second term. The lowpass filter output signal $w(t)$ is a scaled replica of the original message signal:

$$w(t)=\frac{1}{2}x(t). \tag{4}$$

- We have presented only the time-domain analysis of demodulation here. To review the frequency-domain analysis, please see Chapter 4, pages 176-177.

Effect of Phase Offset

- Synchronous demodulation requires a carrier replica synchronized in both frequency and phase to the received modulated signal $y(t)$. Here we study the effect of a phase error.



As shown in the figure above, the demodulator uses a carrier replica $\cos(\omega_c t+\phi)$. We assume that the phase error ϕ is constant. The demodulator multiplies $y(t)$ by the carrier replica, obtaining a signal

$$\begin{aligned} v(t) &= y(t)\cos(\omega_c t+\phi) \\ &= x(t)\cos\omega_c t\cdot\cos(\omega_c t+\phi) \\ &= \frac{1}{2}x(t)\cos\phi+\frac{1}{2}x(t)\cos(2\omega_c t+\phi) \end{aligned} \quad . \tag{5}$$

We used the identity $\cos A\cos B=\frac{1}{2}\cos(A+B)+\frac{1}{2}\cos(A-B)$ in the third line of (5). The multiplier output (5) again comprises two terms. The first term, $\frac{1}{2}x(t)\cos\phi$, is a scaled replica of the message, but multiplied by a phase-dependent factor $\cos\phi$. The second term, $\frac{1}{2}x(t)\cos(2\omega_c t+\phi)$, is the scaled message signal modulated onto a carrier at frequency $2\omega_c$, which is phase-shifted by ϕ.

- After lowpass filtering the multiplier output $v(t)$, we obtain

$$w(t)=\frac{1}{2}x(t)\cos\phi . \tag{6}$$

The lowpass filter output (6) contains the scaled message signal, as desired, but the phase-dependent factor $\cos\phi$ is problematic. As the phase shift ϕ varies, we may obtain the following:

$$w(t)=\begin{cases} \frac{1}{2}x(t) & \phi=0 \\ 0 & \phi=\pi/2 \\ -\frac{1}{2}x(t) & \phi=\pi \end{cases} . \tag{7}$$

Depending on the phase shift, the message signal may be inverted ($\phi=\pi$) or lost entirely ($\phi=\pi/2$).

- Accurate frequency and phase synchronization of the demodulator to the received signal can be achieved by various means, such as phase-locked loops. In the early 1900s, however, such synchronization methods were prohibitively complex. Broadcast AM radio adopted a DSB-AM-LC method compatible with asynchronous demodulation, which is easier to implement. We will discuss DSB-AM-LC and its asynchronous demodulation in a section below.

Spectral Efficiency

- We discuss the spectrum of a DSB-AM-SC signal once again, referring to the figure near the bottom of page 269. We assume the message signal spectrum $X(j\omega)$ (shown on the left) is nonzero only for $|\omega|\le\omega_m$. Considering positive frequencies only, we say that the message signal $x(t)$ *occupies a bandwidth* ω_m. The modulated signal spectrum $Y(j\omega)$ (shown on the right) contains scaled copies of $X(j\omega)$ shifted in frequency to $\pm\omega_c$. Considering positive frequencies, the modulated signal $y(t)$ occupies a bandwidth $2\omega_m$. Since the modulated signal occupies twice as much bandwidth as the message signal, the DSB-AM-SC method has poor *spectral efficiency*. This limits the amount of information that can be transmitted in a given bandwidth. This is particularly problematic in wireless systems, where the bandwidth available for transmission is a scarce resource.
- There are two common methods for doubling the spectral efficiency of AM.

- The first method is *single-sideband amplitude modulation* (SSB-AM). In order to understand this method, we refer again to the spectra shown near the bottom of page 269. In the DSB-AM-SC signal spectrum $Y(j\omega)$, the lower sideband (LSB) and upper sideband (USB) parts of the spectrum, corresponding to $|\omega| < \omega_c$ and $|\omega| > \omega_c$, respectively, contain redundant information. In SSB-AM, we transmit only one of the two sidebands, thus doubling the spectral efficiency. This can be achieved, for example, by filtering the modulated signal $y(t)$ to select one of sidebands, using a filter having abrupt transitions near $|\omega| = \omega_c$.
- SSB-AM is used in amateur radio transmission. In modified form, SSB-AM has been used in various applications.
  - The video portion of analog television, broadcast over the air and in coaxial cable, used a method called vestigial sideband amplitude modulation (VSB-AM). This is now obsolete.
  - Digital high-definition television broadcast over the air in the US uses a method called vestigial sideband 8-level pulse-amplitude modulation (8-VSB).
- The second method for doubling the spectral efficiency of AM is *quadrature amplitude modulation* (QAM), and is the subject of the next section.

Quadrature Amplitude Modulation

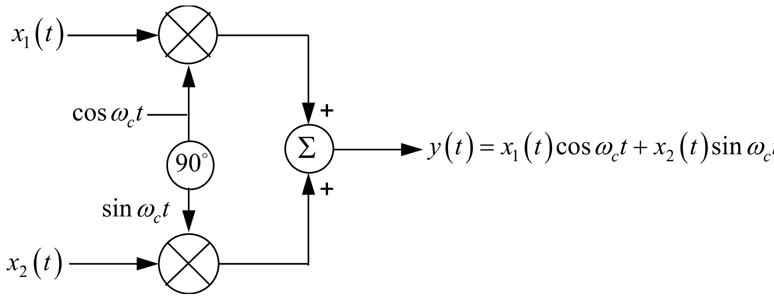
- In *quadrature amplitude modulation* (QAM), we modulate two independent message signals, $x_1(t)$ and $x_2(t)$ onto mutually orthogonal cosine and sine carriers at the same carrier frequency. QAM is able to transmit message signals occupying a total bandwidth $2\omega_m$ in a modulated signal $y(t)$ occupying a bandwidth $2\omega_m$, achieving twice the spectral efficiency of DSB-AM (whether -SC or -LC).
- QAM is used in many systems transmitting digital data, including:
  - Optical fiber trunk lines for telecommunications
  - Digital high-definition television signals and Internet Protocol data transmitted in coaxial cables
  - Point-to-point radio or microwave links

Modulation

- Given two independent message signals, $x_1(t)$ and $x_2(t)$, a QAM signal is expressed as

$$y(t) = x_1(t)\cos\omega_c t + x_2(t)\sin\omega_c t \,. \tag{8}$$

  We can realize a QAM modulator as shown below. Typically, we use a single oscillator at carrier frequency ω_c, splitting it into copies with a relative phase shift of $90°$ to obtain the cosine and sine carriers.

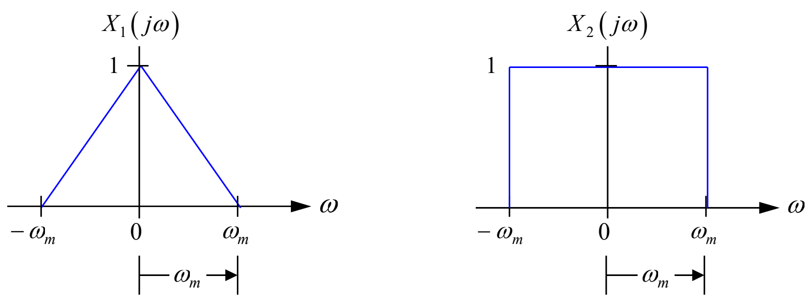


- In order to compute the modulated signal spectrum $Y(j\omega)$, we use the CTFT multiplication property $p(t)q(t) \overset{F}{\leftrightarrow} (1/2\pi)P(j\omega)*Q(j\omega)$ and the CTFT pairs $\cos\omega_c t \overset{F}{\leftrightarrow} \pi\delta(\omega-\omega_c)+\pi\delta(\omega+\omega_c)$ and $\sin\omega_c t \overset{F}{\leftrightarrow} \frac{\pi}{j}\delta(\omega-\omega_c)-\frac{\pi}{j}\delta(\omega+\omega_c)$, obtaining

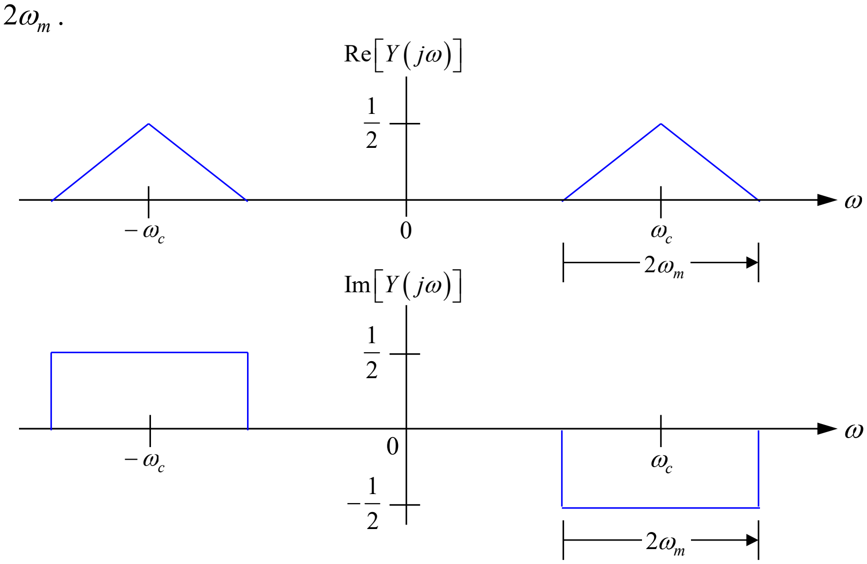
$$Y(j\omega)=\frac{1}{2}\left[X_1\left(j(\omega-\omega_c)\right)+X_1\left(j(\omega+\omega_c)\right)\right]+\frac{1}{2j}\left[X_2\left(j(\omega-\omega_c)\right)-X_2\left(j(\omega+\omega_c)\right)\right], \tag{9}$$

  where $X_1(j\omega)$ and $X_2(j\omega)$ are the CTFTs of the message signals.

- To illustrate the spectrum (9), we assume the message signal spectra $X_1(j\omega)$ and $X_2(j\omega)$ shown below.



- The figure below shows the modulated signal spectrum $Y(j\omega)$, given by (9). As desired, we transmit two message signals occupying a total bandwidth $2\omega_m$ in one modulated signal $y(t)$ occupying a bandwidth $2\omega_m$.



- Using Parseval's identity for the CTFT

$$\int_{-\infty}^{\infty} p(t)q^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(j\omega)Q^*(j\omega)d\omega \tag{10}$$

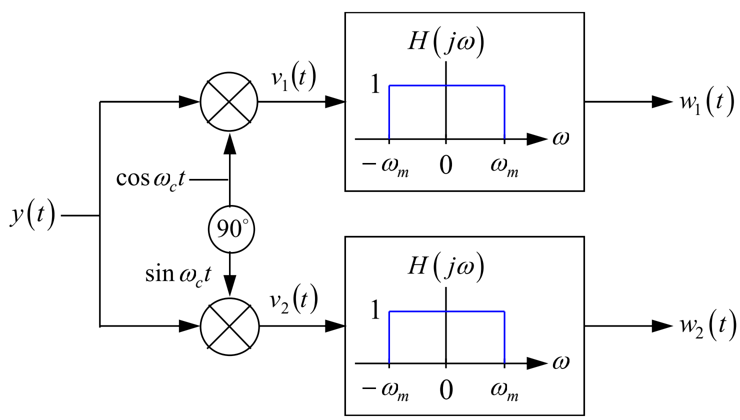
and the plots of $\mathrm{Re}\left[Y(j\omega)\right]$ and $\mathrm{Im}\left[Y(j\omega)\right]$, we can easily see that the two components of the QAM signal (8) are mutually orthogonal

$$\int_{-\infty}^{\infty} x_1(t)\cos\omega_c t \cdot x_2(t)\sin\omega_c t dt = 0, \tag{11}$$

and therefore do not interfere with each other. In order to simplify the figures, we have assumed the message signals $x_1(t)$ and $x_2(t)$ are real and even, so their CTFTs $X_1(j\omega)$ and $X_2(j\omega)$ are real and even. Nevertheless, the orthogonality (11) is satisfied for any real $x_1(t)$ and $x_2(t)$ whose CTFTs $X_1(j\omega)$ and $X_2(j\omega)$ are bandlimited to $|\omega| \le \omega_m$, where $\omega_m < \omega_c$.

Synchronous Demodulation

- This figure shows a synchronous demodulator for QAM. The demodulator must create a carrier replica $\cos\omega_c t$ and a 90°-shifted carrier replica $\sin\omega_c t$.



- Multiplying the received modulated signal $y(t)$ by the carrier replicas, the demodulator obtains signals

$$v_1(t) = y(t)\cos\omega_c t = \frac{1}{2}x_1(t)(1+\cos 2\omega_c t) + \frac{1}{2}x_2(t)\sin 2\omega_c t \tag{12}$$

and

$$v_2(t) = y(t)\sin\omega_c t = \frac{1}{2}x_1(t)\sin 2\omega_c t + \frac{1}{2}x_2(t)(1-\cos 2\omega_c t). \tag{13}$$

We used the identities $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ and $\cos\theta\sin\theta = \frac{1}{2}\sin 2\theta$ to obtain (12) and (13). The multiplier output signals $v_1(t)$ and $v_2(t)$ contain scaled copies of the desired message signals, $\frac{1}{2}x_1(t)$ and $\frac{1}{2}x_2(t)$, respectively. They also contain scaled copies of the message signals modulated onto carriers $\cos 2\omega_c t$ and $\sin 2\omega_c t$, at a frequency $2\omega_c$.

- The multiplier output signals $v_1(t)$ and $v_2(t)$ are passed to ideal lowpass filters, which block the signals at a frequency $2\omega_c$, yielding outputs $w_1(t)$ and $w_2(t)$ that are scaled copies of the original message signals:

$$w_1(t) = \frac{1}{2} x_1(t) \tag{14}$$

and

$$w_2(t) = \frac{1}{2} x_2(t) . \tag{15}$$

Effect of Phase Offset

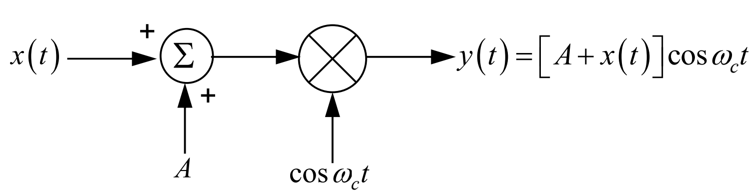
- Suppose the demodulator has a constant phase error ϕ, so it demodulates using carrier replicas $\cos(\omega_c t + \phi)$ and $\sin(\omega_c t + \phi)$. Depending on the value of ϕ, the lowpass filter output signals outputs $w_1(t)$ and $w_2(t)$ may contain different linear combinations of the desired message signals $x_1(t)$ and $x_2(t)$. For example, when $\phi = \pi/4$, $w_1(t)$ and $w_2(t)$ each contain contributions of equal magnitude from $x_1(t)$ and $x_2(t)$, and the two message signals strongly interfere with each other. In order to avoid mutual interference between the two message signals, QAM requires more precise carrier synchronization than DSB-AM. Nevertheless, sufficiently precise synchronization can be achieved in many scenarios, and QAM is used in numerous applications.

Double-Sideband Amplitude Modulation with Large Carrier

- As explained above, DSB-AM-LC is used in AM radio broadcasting. DSB-AM-LC transmits a strong carrier along with the modulated message signal. This leads to poor transmitter power efficiency, but allows a receiver to avoid carrier phase synchronization, simplifying the receiver design. Like DSB-AM-SC, DSB-AM-LC encodes one message signal in a modulated signal with redundant upper and lower sidebands, so the two techniques offer similar spectral efficiencies.

Modulation

- A modulator for DSB-AM-LC is shown in the figure below.



The modulated signal is given by

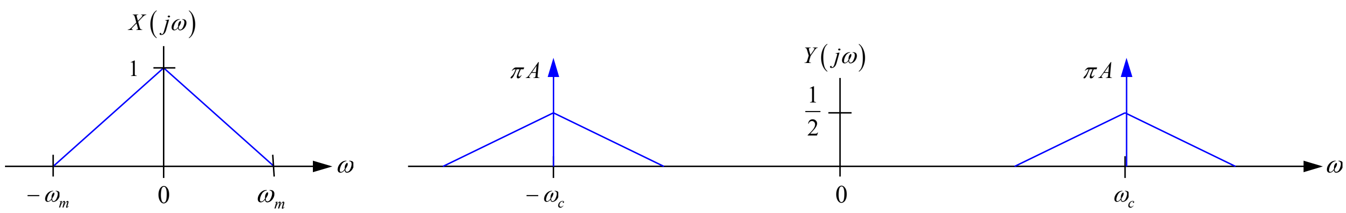
$$y(t) = [A + x(t)]\cos\omega_c t , \tag{16}$$

where $x(t)$ is a real message signal and A is a real constant, $A > 0$. The signal in (16) contains a message-bearing component, $x(t)\cos\omega_c t$, and a large carrier component, $A\cos\omega_c t$.

- The spectrum of the DSB-AM-LC signal (16) is given by

$$Y(j\omega)=\frac{1}{2}\left[X\left(j(\omega-\omega_c)\right)+X\left(j(\omega+\omega_c)\right)\right]+\pi A\left[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)\right]. \tag{17}$$

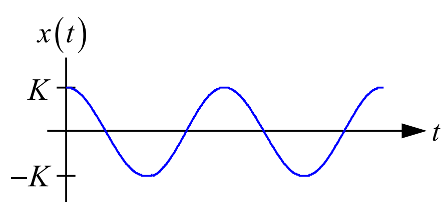
The figure below shows the message signal spectrum $X(j\omega)$ (on the left), and the modulated signal spectrum $Y(j\omega)$ (on the right). The modulated signal spectrum $Y(j\omega)$ is similar to that of a DSB-AM-SC signal (near the bottom of page 269), but also includes impulses of area πA at frequencies $\pm\omega_c$, corresponding to the large carrier.



- It is instructive to examine the modulated signal (16), in the time domain. As we will see in the next section, given a modulated signal $y(t)=\left[A+x(t)\right]\cos\omega_c t$, the asynchronous demodulator will yield a signal proportional to the *envelope* of the signal, which is the absolute value of the factor multiplying the cosine, namely $\left|A+x(t)\right|$. In order for the envelope to contain a faithful representation of the message signal, we require $\left|A+x(t)\right|=A+x(t)$, which requires satisfying a condition

$$A+x(t)>0. \tag{18}$$

Let us assume that $\left|x(t)\right|\leq K$, for some positive real constant *K*, as in the figure below.

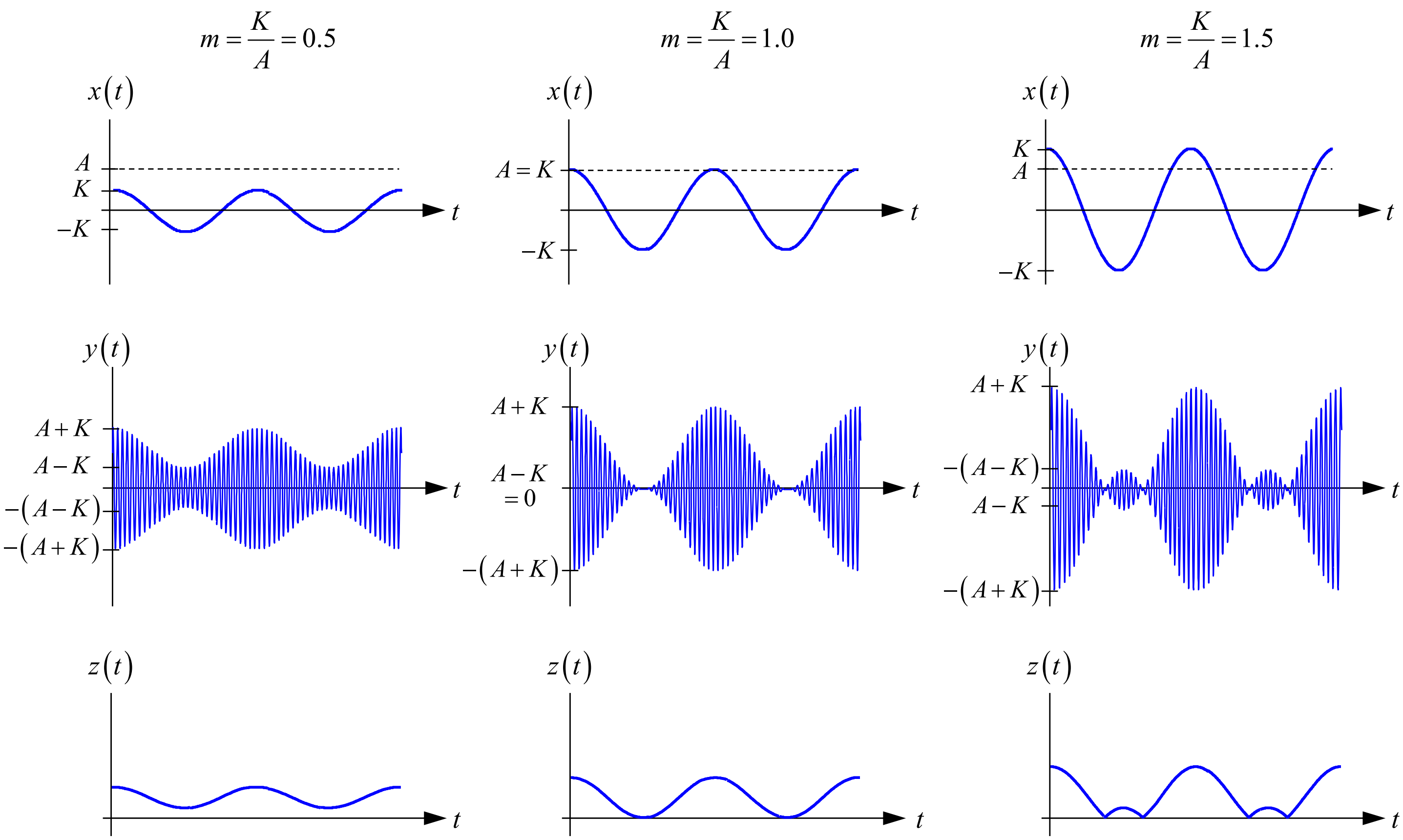


Then satisfying (18) requires $A\geq K$, i.e., the carrier amplitude must be at least as large as the message signal amplitude. Defining a *modulation index*

$$m\overset{d}{=}\frac{K}{A}, \tag{19}$$

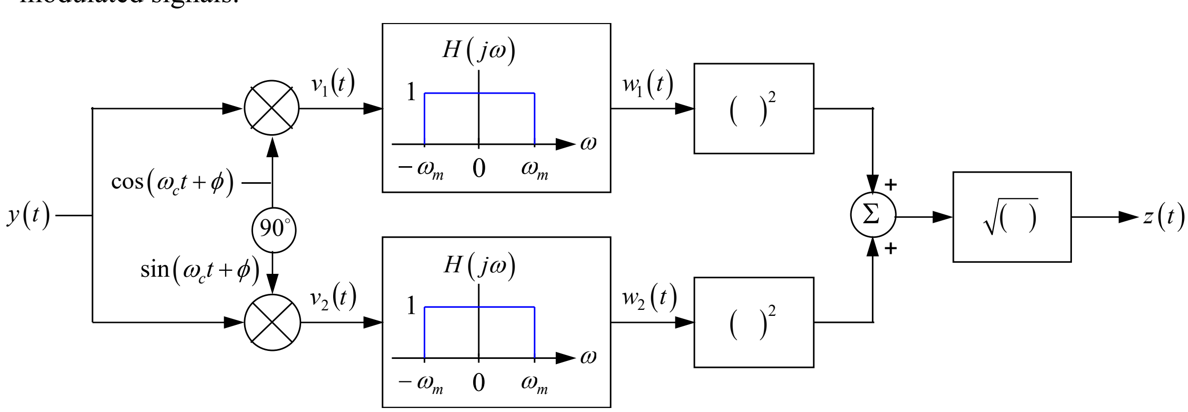
we see that satisfying (18) requires $m\leq 1$. Choosing a modulation index *m* close to unity optimizes power efficiency, while choosing a smaller *m* makes it easier to implement an envelope detector, particularly using simple analog circuitry.

- In the figure below, we fix the carrier amplitude *A* and vary *K*, so *m* assumes values of 0.5, 1.0 and 1.5. The top row shows the message signals $x(t)$, and the middle row shows the corresponding modulated signals $y(t)$. Finally, the bottom row shows the signal $z(t)$ extracted by the envelope detector (which we study in the next section). Observe that the modulation index value $m=1$, shown in the middle column, is the largest value for which $z(t)$ faithfully represents $x(t)$. When the modulation index m exceeds unity, as in the right column, $z(t)$ contains obvious distortion.



Asynchronous Demodulation

- The figure below shows an asynchronous demodulator based on an *envelope detector*. While some of the earliest asynchronous demodulators used a half- or full-wave rectifier followed by a lowpass filter, the envelope detector is more modern and efficient, and is widely used in receivers for digitally modulated signals.



- Notice that the first part of the asynchronous receiver – up to and including the lowpass filters – is very similar to the synchronous receiver for QAM, shown on page 274. Here, however, instead of requiring carrier replicas $\cos\omega_c t$ and $\sin\omega_c t$ that are synchronized to $y(t)$, we allow the carrier replicas to have a phase error ϕ, so they are $\cos(\omega_c t+\phi)$ and $\sin(\omega_c t+\phi)$. We assume ϕ is fixed.

- Multiplying the received modulated signal $y(t)$ by the carrier replicas, the demodulator obtains

$$\begin{aligned} v_1(t) &= y(t)\cos(\omega_c t+\phi) \\ &= \left[A+x(t)\right]\cos\omega_c t\cdot\cos(\omega_c t+\phi) \\ &= \frac{A+x(t)}{2}\left[\cos\phi+\cos(2\omega_c t+\phi)\right] \end{aligned} \quad (20)$$

and

$$\begin{aligned} v_2(t) &= y(t)\sin(\omega_c t+\phi) \\ &= \left[A+x(t)\right]\cos\omega_c t\cdot\sin(\omega_c t+\phi) \\ &= \frac{A+x(t)}{2}\left[\sin(2\omega_c t+\phi)+\sin\phi\right] \end{aligned} \quad (21)$$

We have used the trigonometric identities $\cos A\cos B=\frac{1}{2}\cos(A+B)+\frac{1}{2}\cos(A-B)$ and $\cos A\sin B=\frac{1}{2}\sin(A+B)-\frac{1}{2}\sin(A-B)$ in the third lines of (20) and (21), respectively. The multiplier output signals $v_1(t)$ and $v_2(t)$ contain copies of $\frac{1}{2}\left[A+x(t)\right]$ scaled by $\cos\phi$ and $\sin\phi$, respectively, and also contain copies modulated onto cosine and sine carriers at a frequency $2\omega_c$. The lowpass filters block the latter terms, so the lowpass filter output signals are

$$w_1(t)=\frac{1}{2}\left[A+x(t)\right]\cos\phi \quad (22)$$

and

$$w_2(t)=\frac{1}{2}\left[A+x(t)\right]\sin\phi . \quad (23)$$

- The asynchronous demodulator squares the lowpass filter outputs $w_1(t)$ and $w_2(t)$, adds the squared signals, and takes the square root to obtain its output, which is

$$\begin{aligned} z(t) &= \left[w_1^2(t)+w_2^2(t)\right]^{1/2} \\ &= \left[\left(\frac{A+x(t)}{2}\right)^2\left(\cos^2\phi+\sin^2\phi\right)\right]^{1/2} \\ &= \frac{1}{2}\left|A+x(t)\right| \end{aligned} \quad (24)$$

We have used $\cos^2\phi + \sin^2\phi = 1$ to obtain the third line of (24). The asynchronous demodulator output $z(t)$, given by (24), is *independent of the phase offset* ϕ. The demodulator output $z(t)$ was shown in the figure on page 277 for three different values of the modulation index $m = K / A$. Recall that if we choose $m \le 1$, then the condition $A + x(t) > 0$ is satisfied, and the demodulator output $z(t)$ provides a faithful representation of the message signal $x(t)$.

Angle Modulation Techniques

- *Angle modulation* includes *phase modulation* (PM) and *frequency modulation* (FM). In general, an angle-modulated signal $y(t)$ is given by

$$y(t) = A\cos\left(\omega_c t + \theta_c(t)\right), \tag{25}$$

where $\theta_c(t)$ is a *phase signal* derived from the message signal $x(t)$ in some way. Since $\theta_c(t)$ appears in the argument of a cosine function, the relationship between $y(t)$ and $x(t)$ is *nonlinear*, and we can say that an angle modulator is a *nonlinear system*. This can make it difficult to analytically relate the modulated signal Fourier transform $Y(j\omega)$ to the message signal Fourier transform $X(j\omega)$. In the examples shown below, the modulated signal spectra have been computed numerically.

Phase Modulation

- Phase modulation is often used to transmit digital signals, for example, in optical fibers or in microwave links to satellites.
- We define a *phase sensitivity factor* of the modulator, k_p. The phase signal is given by

$$\theta_c(t) = k_p x(t), \tag{26}$$

and the phase-modulated signal is given by

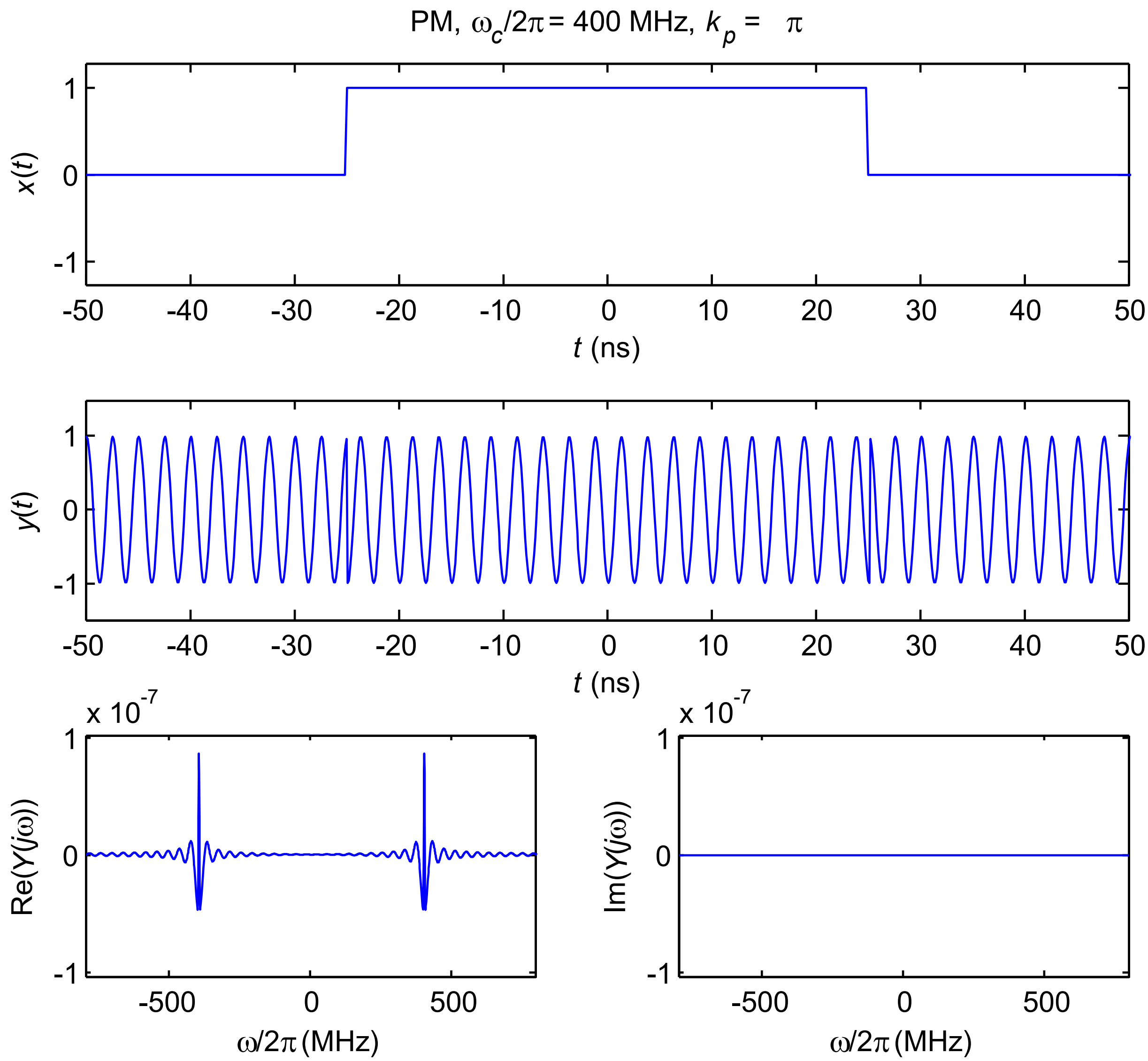
$$y(t) = A\cos\left(\omega_c t + k_p x(t)\right). \tag{27}$$

Observe that *the message signal* $x(t)$ *is modulated directly onto the phase signal* $\theta_c(t)$. If the messsage $x(t)$ has discontinuities, then the phase of the modulated signal $y(t)$ has discontinuities.

- Demodulation
  - Synchronous demodulation can directly yield the message signal $x(t)$.

Numerical Example

- In this example, the message signal $x(t)$ has two discontinuities, as does the phase of the modulated signal $y(t)$. The modulated signal $y(t)$ is real and even, so its Fourier transform $Y(j\omega)$ is real and even.



Frequency Modulation

- Frequency modulation is often used to transmit analog signals, as in FM radio broadcasting. It was used for the audio portion of analog broadcast television. It is also used to transmit digital signals, for example, in some cellular telephone systems and in the basic mode of Bluetooth links.

- We define a *frequency sensitivity factor* of the modulator, k_f. The phase signal is given by k_f times a running integral of the message signal $x(t)$:

$$\theta_c(t) = k_f \int_{-\infty}^{t} x(\tau) d\tau \,, \tag{28}$$

  and the modulated signal is given by

$$y(t) = A\cos\left(\omega_c t + k_f \int_{-\infty}^{t} x(\tau) d\tau\right). \tag{29}$$

  Even if the message signal $x(t)$ has finite discontinuities, the phase of the modulated signal $y(t)$ is continuous.

- In order to visualize frequency modulation, it is helpful to define an *instantaneous frequency* $\omega_i(t)$, which is the time derivative of the argument of the cosine function in $y(t)$:

$$\begin{aligned} \omega_i(t) &= \frac{d}{dt}\left(\omega_c t + k_f \int_{-\infty}^{t} x(\tau) d\tau\right) \\ &= \omega_c + k_f x(t) \end{aligned}. \tag{30}$$

  Observe that *the message signal* $x(t)$ *is modulated directly onto the instantaneous frequency* $\omega_i(t)$.

- If the frequency sensitivity factor k_f is large, the bandwidth occupied by the modulated signal $y(t)$ is much larger than the bandwidth of the message signal $x(t)$, and the spectral efficiency is low. For example, in broadcast FM radio:

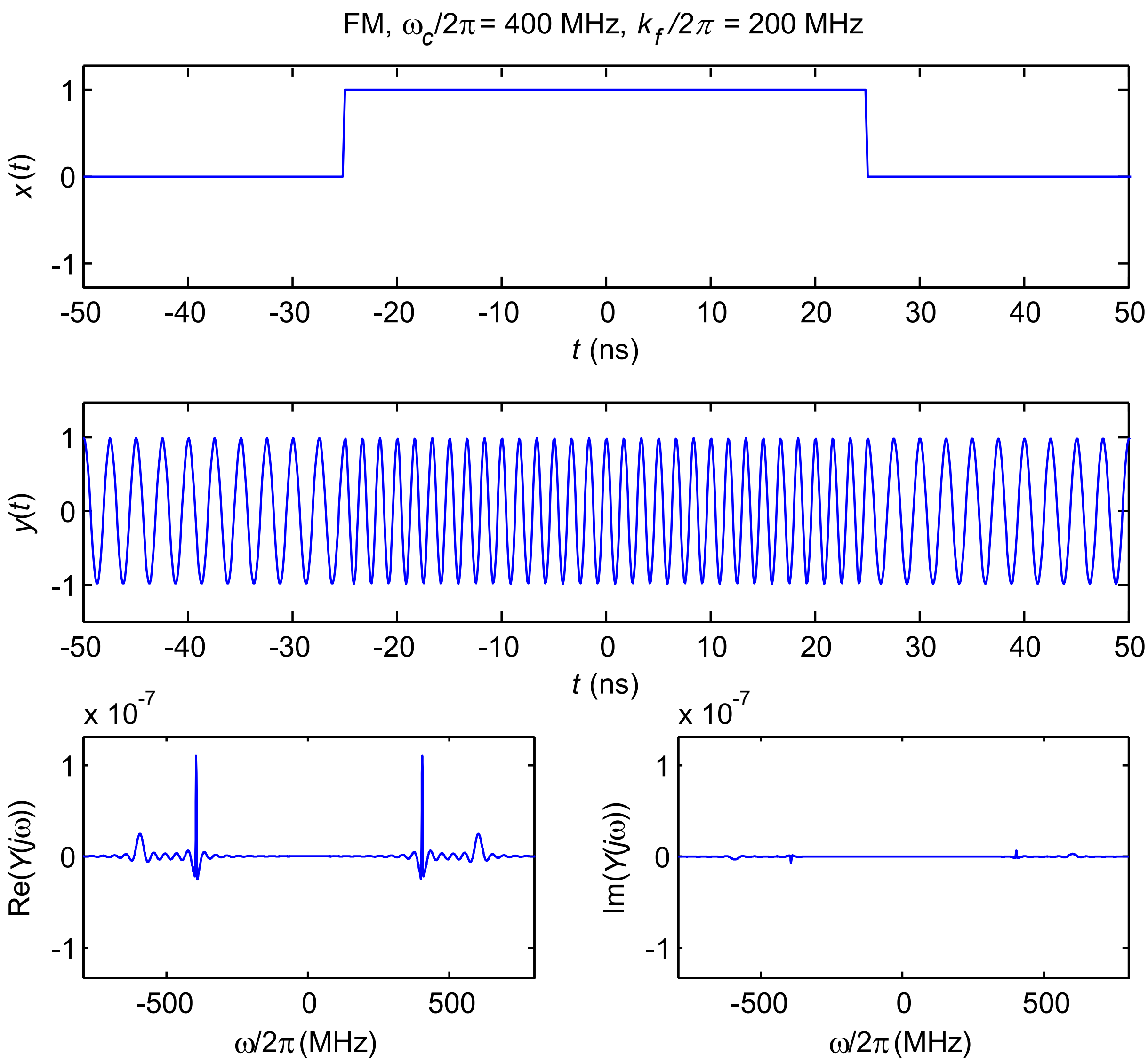
$$B_x \approx 15 \text{ kHz} \quad \text{and} \quad B_y \approx 180 \text{ kHz}.$$

  Despite the low spectral efficiency, frequency modulation is used because it provides excellent immunity against noise.

- Demodulation
  - *Asynchronous*: a *differentiator* or a *frequency discriminator* can be used to convert FM to AM, which can be demodulated using an envelope detector, much like DSB-AM-LC is demodulated.
  - *Synchronous*: a *phase-locked loop* can be used to track the instantaneous frequency $\omega_i(t)$. When the loop is locked, the voltage inside the loop, which drives a *voltage-controlled oscillator*, directly yields the message signal $x(t)$.

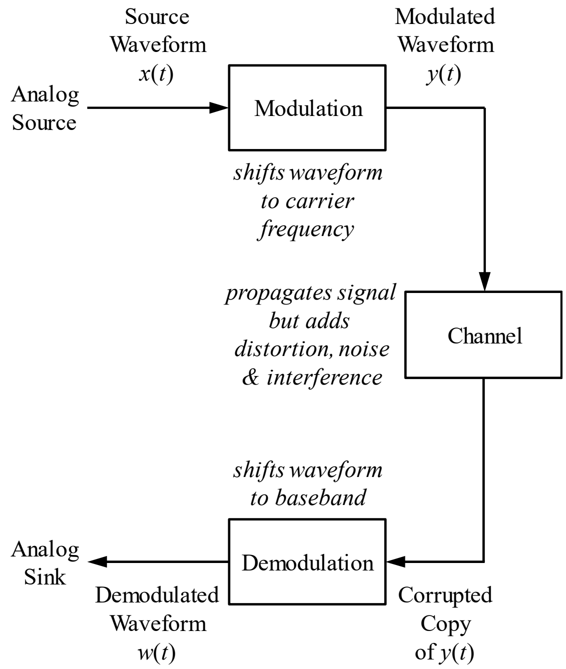
Numerical Example

- In this example, the message $x(t)$ has two discontinuities, but the phase of the modulated signal $y(t)$ is continuous. The modulated signal $y(t)$ is very nearly real and even, so its Fourier transform $Y(j\omega)$ is very nearly real and even.



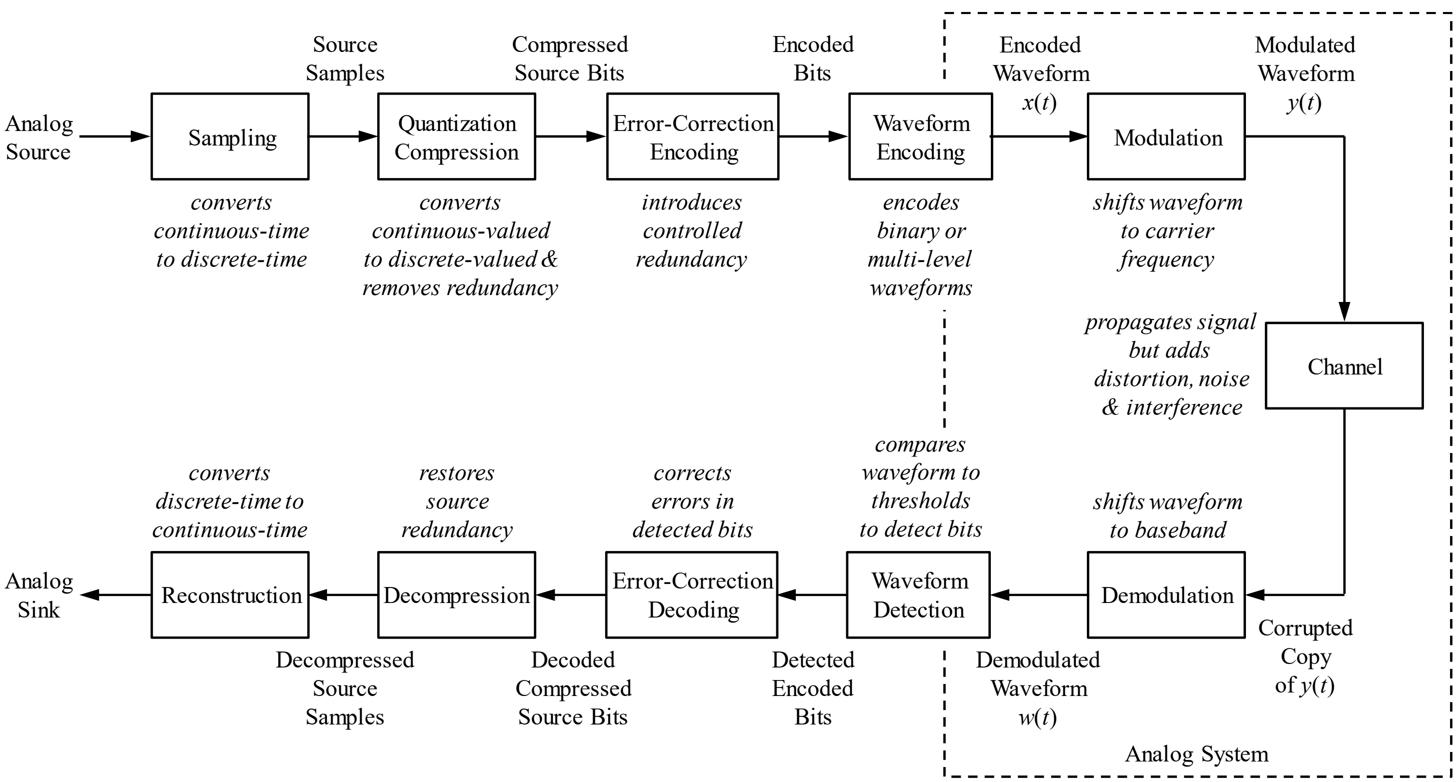
Analog Communications

- An *analog communication system* starts with a continuous-valued, continuous-time *source waveform* $x(t)$, such as an audio or video signal. Earlier, we referred to $x(t)$ as a *message waveform*. The source waveform $x(t)$ is modulated onto a carrier, yielding a *modulated waveform* $y(t)$. The modulated waveform $y(t)$ propagates through the communication medium, which is often referred to as a *communication channel*. During propagation, $y(t)$ may become attenuated and distorted, and noise and interference may be added to it. At the receiving end, a corrupted replica of $y(t)$ is demodulated to yield an analog waveform $w(t)$.

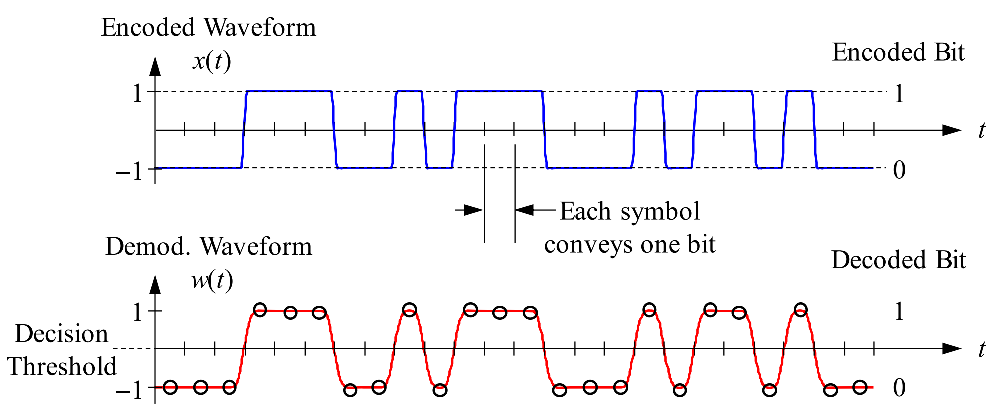


Digital Communications

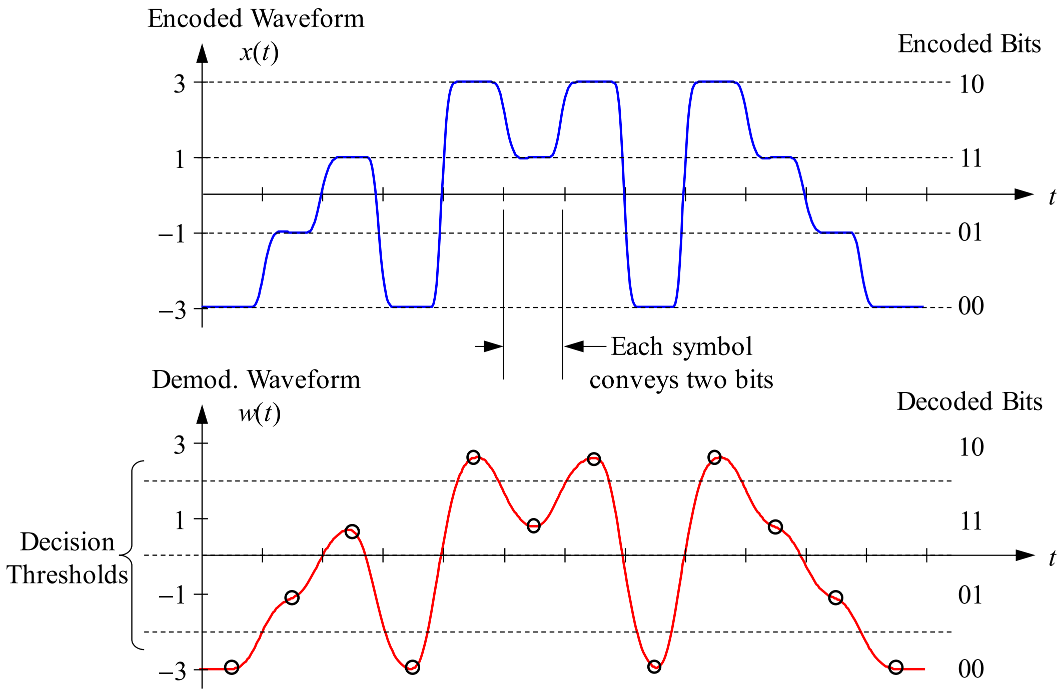
- A typical *digital communication system*, as shown in the figure below, is far more complicated.



- Assuming the original source signal is analog, the source is *sampled* and *quantized*, to yield a discrete-time, discrete-valued signal. The process of sampling and quantization is often called *digitization*. A digitized signal may be represented in hardware as a serial or parallel bit stream.
- In order to reduce the number of bits required to represent the source, the source bits are typically *compressed*. Compression methods reduce the redundancy present in the source, and may be *lossless*, where all the information in a quantized source is preserved and compression is reversible, or *lossy*, where some source information is irretrievably lost. Different compression methods are used for different sources, including data files, speech, music, still images and video. For typical analog sources, the number of bits representing the source may be reduced by a *large factor*, as high as 10 or more in some cases, depending on the source, the compression method, and the quality required.
- *Error-correction encoding* adds redundancy to the compressed source bits in a controlled way that makes it possible to correct for errors caused by transmission through a channel. Typically, the added redundancy increases the number of bits by a *small factor* of 1.1 to 2, and enables correction of a moderate fraction of bit errors, 10^{-4} to 10^{-1}, depending on the encoding and decoding method. Error-correction encoding is essential for achieving efficiency in most digital communication systems.
- After error-correction encoding, bits are encoded into an *encoded waveform* $x(t)$. The encoded waveform is modulated onto a carrier to yield a *modulated waveform* $y(t)$. (Methods such as QAM can modulate two independent encoded waveforms, $x_1(t)$ and $x_2(t)$, onto $y(t)$.) After transmission through the channel, a corrupted replica of $y(t)$ is obtained, and demodulation yields a *demodulated waveform* $w(t)$ (or two demodulated waveforms, $w_1(t)$ and $w_2(t)$). The entire process of modulation, transmission and demodulation, shown enclosed in the dashed box, is an *analog communication system*.
- Encoded waveforms are composed of sequences of discrete-valued pulses called *symbols*. If the symbols are *binary*, assuming one of two values, then each symbol encodes one bit. In a process called *detection*, samples of the demodulated waveform $w(t)$ are compared to a single *decision threshold* to yield detected bits.



- If the encoded symbols are *quaternary*, assuming one of four levels, then each symbol encodes two bits. Samples of the demodulated waveform $w(t)$ are compared to three decision thresholds to yield the detected bits. A system using quaternary symbols can send symbols at half the rate of a system using binary signals to achieve the same bit rate, so a quaternary system has twice the spectral efficiency of a binary system. On the other hand, a quaternary system must transmit more power than a binary system to achieve the same spacing between levels, and thus achieve the same noise immunity, so a quaternary system has lower power efficiency than a binary system.

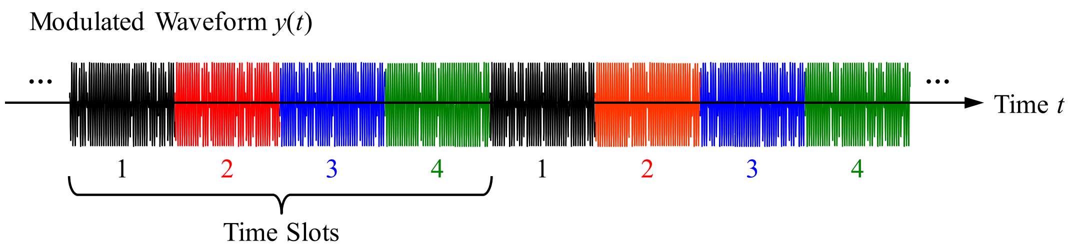


Challenges and Solutions

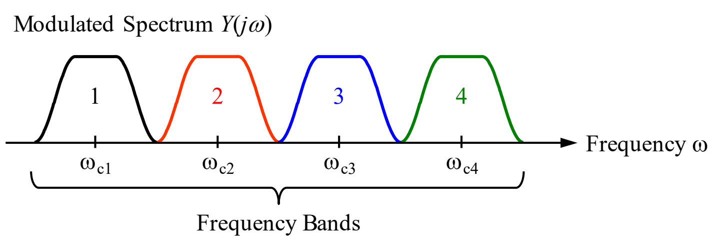
Challenge	Analog Solutions	Drawbacks	Digital Solutions	Drawbacks
Noise and interference	Wideband FM	More bandwidth required.	Wideband FM. Few levels in $x(t)$. High redundancy in error-correction code.	More bandwidth required. High decoding complexity.
Limited bandwidth available	SSB or QAM	Less noise or interference tolerated. Synchronous detection required.	SSB or QAM. Many levels in $x(t)$. Low redundancy in error-correction code.	Less noise or interference tolerated. Synchronous detection required.
			Equalization or OFDM.	Limited effectiveness. High signal processing complexity.
			Multi-input, multi-output transmission.	High hardware and signal processing complexity.
			Reuse.	Cost of base stations.
			Source compression.	Quality reduction.

Multiplexing

- *Multiplexing* allows several messages to be transmitted through a shared communication medium. It is desirable for the signals conveying different messages to be *mutually orthogonal*, so they do not interfere with each other.
- In *time-division multiplexing*, all the signals are transmitted at the same carrier frequency, but in different time slots. Since they do not overlap in time, they are mutually orthogonal.



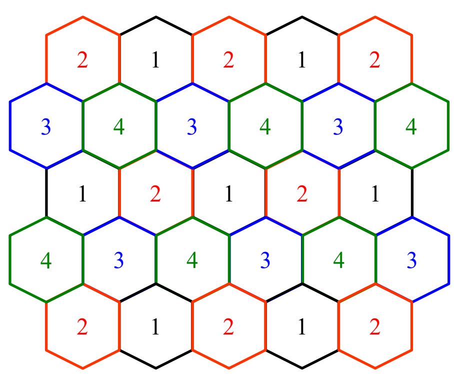
- In *frequency-division multiplexing*, all the signals are transmitted at the same time, but on different carrier frequencies. Since they do not overlap in frequency, they are mutually orthogonal.



- It is also possible for transmissions to overlap in both time and frequency and be mutually orthogonal. Examples include orthogonal frequency-division multiplexing (OFDM) and code-division multiplexing (CDM).

Reuse

- *Reuse* allows the same resource (such as time slots or frequency bands) to be used simultaneously in different regions that are spatially separated. The different regions are often called *cells*. Provided that cells using the same resource are sufficiently far apart, mutual interference between cells can be acceptably small. Reuse is employed extensively in cellular wireless telephone/data networks and in wireless local area (WiFi) networks. The figure below shows a network with idealized hexagonal cells. Since four different time slots or frequency bands are re-used, this system has a *reuse factor* of four.



Stanford University
EE 102A: Signal Processing and Linear Systems I
Professor Joseph M. Kahn

Appendix

Geometric Series

Let a be a complex number.

Finite Sum

$$\sum_{k=0}^{n} a^k = \begin{cases} 0 & \text{any } a, n<0 \\ \dfrac{1-a^{n+1}}{1-a} & a \neq 1,\ n \geq 0 \\ n+1 & a=1,\ n\geq 0 \end{cases}$$

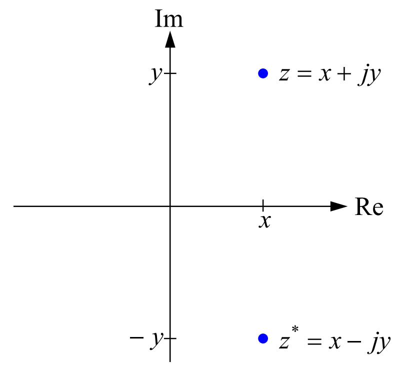
Infinite Sum

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a},\ |a|<1$$

Complex Numbers

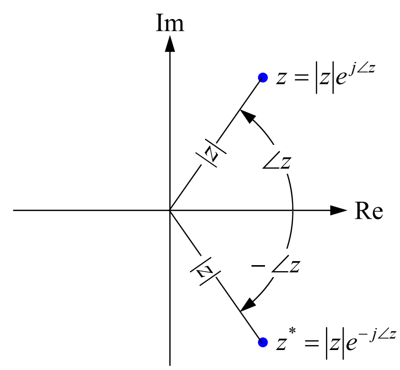
Cartesian Form (top right)

- Complex number $\quad z = x + jy$
- Real part $\quad x = \operatorname{Re} z = \frac{1}{2}\left(z + z^*\right)$
- Imaginary part $\quad y = \operatorname{Im} z = \frac{1}{2j}\left(z - z^*\right)$
- Imaginary unit $\quad j = \sqrt{-1}$

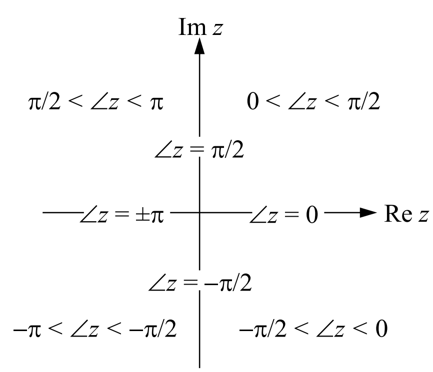


Polar Form (middle right)

- Complex number $\quad z = |z| e^{j\angle z}$
- Magnitude or modulus $\quad |z| = \sqrt{x^2 + y^2}$
- Phase or argument $\quad \angle z = \arg z = \tan^{-1}\left(\frac{y}{x}\right)$



(*Note:* $\tan^{-1}$ is multi-valued. See bottom right for choice of value. Can add any integer multiple of 2π to values shown.)



Complex Conjugate

- Cartesian form (top right) $\quad z^* = x - jy$
- Polar form (middle right) $\quad z^* = |z| e^{-j\angle z}$

Euler's Relation

- Imaginary exponential:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

- Real and imaginary parts:

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \text{ and } \sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

- Complex exponential:

$$e^{z} = e^{x+jy} = e^{x}e^{jy} = e^{x}\left(\cos y + j\sin y\right)$$

Nth Roots

- Given a complex number z, to compute its Nth root, write z in polar form:

$$z = |z|e^{j(\angle z + k2\pi)},$$

where k may be any integer. We can add any integer multiple of 2π to the phase of z and still recover z:

$$|z|e^{j(\angle z + k2\pi)} = |z|e^{j\angle z}e^{jk2\pi} = |z|e^{j\angle z}.$$

- The Nth root of z is

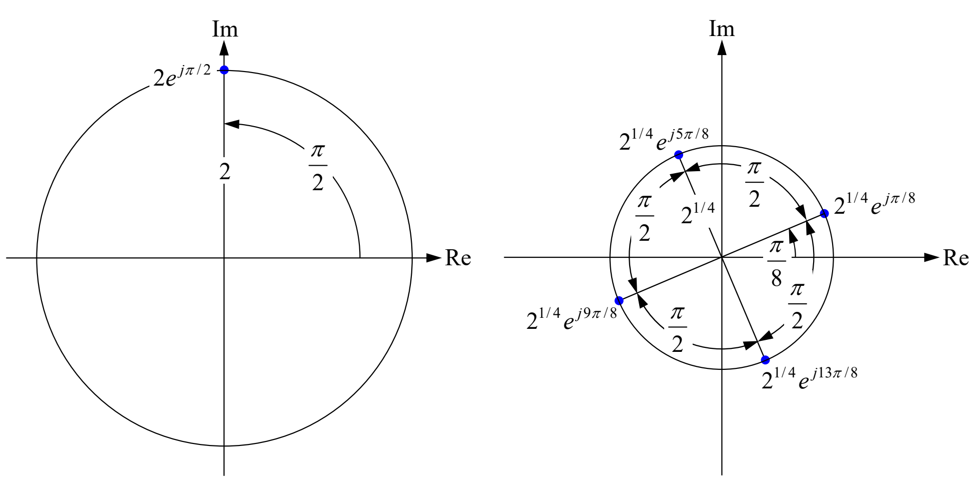
$$\sqrt[N]{z} = z^{\frac{1}{N}} = |z|^{\frac{1}{N}} e^{j\frac{1}{N}(\angle z + k2\pi)},\ k = 0,\ldots,N-1.$$

The magnitude of $z^{1/N}$ is $|z|^{1/N}$, the Nth root of the magnitude of z. The phase of $z^{1/N}$ is $\angle z / N + k2\pi / N$, which is $1/N$ times the phase of z plus an integer k times $2\pi / N$. We obtain N distinct roots by choosing the integer values to be $k = 0,\ldots,N-1$.

- As an example, we compute the fourth roots of $2j$:

$$\left(2j\right)^{\frac{1}{4}} = \left(2e^{j\frac{\pi}{2}}\right)^{\frac{1}{4}} = \left(2\right)^{\frac{1}{4}} e^{j\frac{1}{4}\left(\frac{\pi}{2}+k2\pi\right)} = \left(2\right)^{\frac{1}{4}} e^{j\left(\frac{\pi}{8}+k\frac{\pi}{2}\right)},\ k = 0,\ldots,3.$$

The number $2j$ and its four fourth roots are shown below.



Magnitude and Phase of Product, Quotient or Reciprocal

- These relations are especially useful for frequency responses expressed as products or quotients, which occur frequently in studying LTI systems. Consider two complex numbers:

$$z_1 = |z_1| e^{j\angle z_1} \text{ and } z_2 = |z_2| e^{j\angle z_2}.$$

  Using the properties of exponentials, it is easy to show the following.

Product

- The magnitude of a product is the product of the magnitudes

$$|z_1 z_2| = |z_1||z_2|,$$

  while the phase of a product is the sum of the phases

$$\angle(z_1 z_2) = \angle z_1 + \angle z_2.$$

Quotient

- The magnitude of a quotient is the quotient of the magnitudes

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|},$$

  while the phase of a quotient is the difference between the phases

$$\angle\left(\frac{z_1}{z_2}\right) = \angle z_1 - \angle z_2.$$

Reciprocal

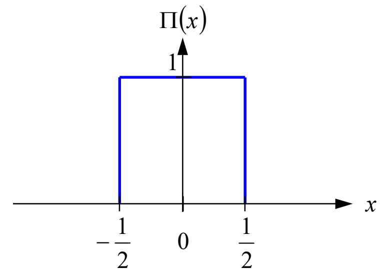
- The magnitude and phase of a reciprocal are a special case of the quotient:

$$\left|\frac{1}{z}\right| = \frac{1}{|z|} \text{ and } \angle\left(\frac{1}{z}\right) = -\angle z.$$

Special Functions

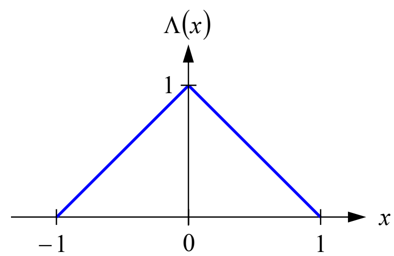
Rectangular Pulse

$$\Pi(x) \stackrel{d}{=} \begin{cases} 1 & |x| \le \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases}$$



Triangular Pulse

$$\Lambda(x) \stackrel{d}{=} \begin{cases} 1-|x| & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

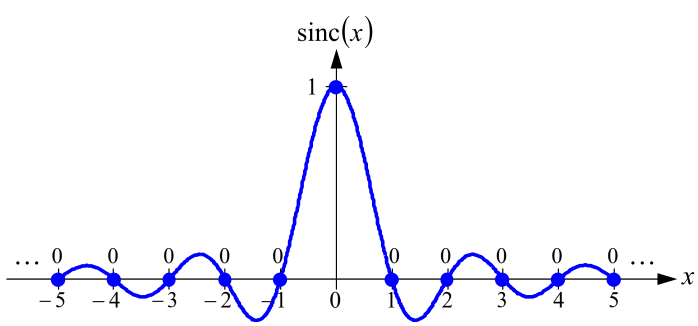


- Note that the convolution of a rectangular pulse (width 1) with itself yields a triangular pulse (width 2):

$$\Pi(x) * \Pi(x) = \int_{-\infty}^{\infty} \Pi(x')\Pi(x-x')\,dx' = \Lambda(x).$$

Sinc Function

$$\operatorname{sinc}(x) \stackrel{d}{=} \frac{\sin(\pi x)}{\pi x}$$



- Note that for integer-valued arguments, the sinc function assumes values of zero or one. When x assumes nonzero integer values, $\operatorname{sinc}(x)$ is zero:

$$\operatorname{sinc}(x)\Big|_{x=\pm1,\pm2,\ldots} = 0 .$$

As x approaches zero, $\operatorname{sinc}(x)$ approaches zero over zero, so its value must be evaluated using L'Hôpital's rule:

$$\lim_{x\to0} \operatorname{sinc}(x) = \lim_{x\to0} \frac{\sin(\pi x)}{\pi x} = \lim_{x\to0} \frac{\frac{d}{dx}\left[\sin(\pi x)\right]}{\frac{d}{dx}\left[\pi x\right]} = 1 .$$

Frequency Variables

Case	EE 102A Lectures	Oppenheim, Willsky, Nawab
Continuous time (CT)	ω (rad/s)	ω (rad/s)
Discrete time (DT)	Ω (rad)	ω (rad)
Mixed CT + DT	CT: ω (rad/s) DT: Ω (rad) Sampling interval: T (s) Relation: $\Omega = \omega T$	CT: ω (rad/s) DT: Ω (rad) Sampling interval: T (s) Relation: $\Omega = \omega T$

Overview of Fourier Representations

	Continuous Time		Discrete Time	
	Time Domain	**Frequency Domain**	**Time Domain**	**Frequency Domain**
Fourier Series	$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ Continuous Periodic	$a_k=\frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt$ Discrete Aperiodic	$x[n]=\sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n}$ Discrete Periodic	$a_k=\frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\Omega_0 n}$ Discrete Periodic
Fourier Transform	$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$ Continuous Aperiodic	$X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ Continuous Aperiodic	$x[n]=\frac{1}{2\pi}\int_{2\pi} X\left(e^{j\Omega}\right)e^{j\Omega n}d\Omega$ Discrete Aperiodic	$X\left(e^{j\Omega}\right)=\sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$ Continuous Periodic

duality

duality

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Table 1. Continuous-Time Fourier Series Properties

Property	Periodic Signal	Fourier Series Coefficients
	$x(t)$ $y(t)$ Both periodic with period T_0 and fundamental frequency $\omega_0 = 2\pi / T_0$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	$x(t - t_0)$	$e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T_0)t_0} a_k$
Frequency Shifting	$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T_0)t} x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a_{-k}^*
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t)$, $\alpha > 0$ (periodic with period T / α)	a_k
Periodic Convolution	$\int_{T_0} x(\tau) y(t - \tau) d\tau$	$T_0 a_k b_k$
Multiplication	$x(t) y(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk(2\pi / T_0) a_k$
Running Integration	$\int_{-\infty}^{t} x(\tau) d\tau$ (finite and periodic only if $a_0 = 0$)	$\frac{1}{jk\omega_0} a_k = \frac{1}{jk(2\pi / T_0)} a_k$
Conjugate Symmetry for Real Signals	$x(t)$ real	$a_k = a_{-k}^*$ $\begin{cases} \mathrm{Re}\{a_k\} = \mathrm{Re}\{a_{-k}\} \\ \mathrm{Im}\{a_k\} = -\mathrm{Im}\{a_{-k}\} \end{cases}$ $\begin{cases} \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \angle a_k = -\angle a_{-k} \end{cases}$
Real and Even Signals	$x(t)$ real and even	a_k real and even
Real and Odd Signals	$x(t)$ real and odd	a_k imaginary and odd
Even-Odd Decomposition for Real Signals	$x(t)$ real $x_e(t) = \frac{1}{2}[x(t) + x(-t)]$ $x_0(t) = \frac{1}{2}[x(t) - x(-t)]$	$\mathrm{Re}\{a_k\}$ $j\,\mathrm{Im}\{a_k\}$
Parseval's Identity for Inner Product	$\langle x(t), y(t) \rangle = \int_{T_0} x(t) y^*(t) dt = T_0 \sum_{k=-\infty}^{\infty} a_k b_k^*$	
Parseval's Identity for Power	$P = \frac{1}{T_0} \int_{T_0} \lvert x(t) \rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert a_k \rvert^2$	

Table 2. Discrete-Time Fourier Series Properties

Property	Periodic Signal	Fourier Series Coefficients
	$x[n]$ $y[n]$ Both periodic with period N and fundamental frequency $\Omega_0 = 2\pi / N$	a_k b_k Both periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$e^{-jk\Omega_0 n_0} a_k = e^{-jk(2\pi / N)n_0} a_k$
Frequency Shifting	$e^{jM\Omega_0 n} x[n] = e^{jM(2\pi / N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m] & \text{if } n \text{ is a multiple of } m \\ 0 & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k$ (viewed as periodic with period mN)
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r] y[n-r]$	$N a_k b_k$
Multiplication	$x[n] y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$\left(1 - e^{-jk\Omega_0}\right) a_k = \left(1 - e^{-jk(2\pi / N)}\right) a_k$
Running Summation	$\sum_{k=-\infty}^{n} x[k]$ (finite and periodic only if $a_0 = 0$)	$\frac{1}{1 - e^{-jk\Omega_0}} a_k = \frac{1}{1 - e^{-jk(2\pi / N)}} a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$a_k = a^*_{-k}$ $\begin{cases} \mathrm{Re}\{a_k\} = \mathrm{Re}\{a_{-k}\} \\ \mathrm{Im}\{a_k\} = -\mathrm{Im}\{a_{-k}\} \end{cases}$ $\begin{cases} \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \angle a_k = -\angle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k imaginary and odd
Even-Odd Decomposition for Real Signals	$x[n]$ real $x_e[n] = \frac{1}{2}(x[n] + x[-n])$ $x_o[n] = \frac{1}{2}(x[n] - x[-n])$	 $\mathrm{Re}\{a_k\}$ $j\,\mathrm{Im}\{a_k\}$
Parseval's Identity for Inner Product	$\langle x[n], y[n] \rangle = \sum_{n=\langle N \rangle} x[n] y^*[n] = N \sum_{k=\langle N \rangle} a_k b_k^*$	
Parseval's Identity for Power	$P = \frac{1}{N} \sum_{n=\langle N \rangle} \lvert x[n] \rvert^2 = \sum_{k=\langle N \rangle} \lvert a_k \rvert^2$	

Table 3. Continuous-Time Fourier Transform Properties

Property	Aperiodic Signal	Fourier Transform
	$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
Linearity	$Ax(t)+By(t)$	$AX(j\omega)+BY(j\omega)$
Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
Differentiation in Time	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Running Integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{1}{j\omega}X(j\omega)+\pi X(j0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{dX(j\omega)}{d\omega}$
Conjugate Symmetry for Real Signals	$x(t)$ real	$X(j\omega)=X^*(-j\omega)$ $\begin{cases}\mathrm{Re}\{X(j\omega)=\}=\mathrm{Re}\{X(-j\omega)\}\\ \mathrm{Im}\{X(j\omega)=\}=-\mathrm{Im}\{X(-j\omega)\}\end{cases}$ $\begin{cases}\lvert X(j\omega)\rvert=\lvert X(-j\omega)\rvert\\ \angle X(j\omega)=-\angle X(-j\omega)\end{cases}$
Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ imaginary and odd
Even-Odd Decomposition for Real Signals	$x(t)$ real $x_e(t)=\frac{1}{2}[x(t)+x(-t)]$ $x_0(t)=\frac{1}{2}[x(t)-x(-t)]$	$\mathrm{Re}\{X(j\omega)\}$ $j\,\mathrm{Im}\{X(j\omega)\}$
Parseval's Identity for Inner Product	$\langle x(t),y(t)\rangle=\int_{-\infty}^{\infty}x(t)y^*(t)dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)Y^*(j\omega)d\omega$	
Parseval's Identity for Energy	$E=\int_{-\infty}^{\infty}\lvert x(t)\rvert^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(j\omega)\rvert^2d\omega$	

Table 4. Continuous-Time Fourier Transforms and Fourier Series Coefficients

Signal	Fourier Transform	Fourier Series Coefficients (if Periodic)
$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
1	$2\pi\delta(\omega)$	$a_0 = 1$ $a_k = 0$, otherwise (valid for any T_0)
Rectangular pulse train $x(t)=\begin{cases}1 & \lvert t\rvert \le T_1 \\ 0 & T_1 < \lvert t\rvert \le \frac{T_0}{2}\end{cases}$ $x(t+T_0) = x(t)$	$2\omega_0 T_1 \sum_{k=-\infty}^{\infty} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right)\delta(\omega - k\omega_0)$ $\omega_0 = \frac{2\pi}{T_0}$	$\frac{\omega_0 T_1}{\pi}\operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right)$
$\sum_{n=-\infty}^{\infty} \delta(t - nT_0)$	$\frac{2\pi}{T_0}\sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T_0}\right)$	$a_k = \frac{1}{T_0}$ for all k
$\Pi\left(\frac{t}{2T_1}\right) = \begin{cases}1 & \lvert t\rvert \le T_1 \\ 0 & \lvert t\rvert > T_1\end{cases}$	$2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)$	–
$\frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)$	$\Pi\left(\frac{\omega}{2W}\right) = \begin{cases}1 & \lvert\omega\rvert \le W \\ 0 & \lvert\omega\rvert > W\end{cases}$	–
$\delta(t)$	1	–
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	–
$\delta(t - t_0)$	$e^{-j\omega t_0}$	–
$e^{-at}u(t)$, $\operatorname{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$	–
$te^{-at}u(t)$, $\operatorname{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	–
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\operatorname{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	–

Table 5. Discrete-Time Fourier Transform Properties

Property	Aperiodic Signal	Fourier Transform
	$x[n]$ $y[n]$	$X(e^{j\Omega})$ $Y(e^{j\Omega})$ Both periodic with period 2π
Linearity	$Ax[n]+By[n]$	$AX(e^{j\Omega})+BY(e^{j\Omega})$
Time Shifting	$x[n-n_0]$	$e^{-j\Omega n_0}X(e^{j\Omega})$
Frequency Shifting	$e^{j\Omega_0 n}x[n]$	$X(e^{j(\Omega-\Omega_0)})$
Conjugation	$x^*[n]$	$X^*(e^{-j\Omega})$
Time Reversal	$x[-n]$	$X(e^{-j\Omega})$
Time Expansion	$x_{(m)}[n]=\begin{cases} x[n/m] & \text{if } n \text{ is a multiple of } m \\ 0 & \text{if } n \text{ is not a multiple of } m \end{cases}$	$X(e^{jm\Omega})$
Convolution	$x[n]*y[n]$	$X(e^{j\Omega})Y(e^{j\Omega})$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi}X(e^{j\theta})Y(e^{j(\Omega-\theta)})d\theta$
First Difference	$x[n]-x[n-1]$	$(1-e^{-j\Omega})X(e^{j\Omega})$
Running Summation	$\sum_{k=-\infty}^{n}x[k]$	$\frac{1}{1-e^{-j\Omega}}X(e^{j\Omega})+\pi X(e^{j0})\sum_{k=-\infty}^{\infty}\delta(\Omega-k2\pi)$
Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\Omega})}{d\Omega}$
Conjugate Symmetry for Real Signals	$x[n]$ real	$X(e^{j\Omega})=X^*(e^{-j\Omega})$ $\begin{cases} \mathrm{Re}\{X(e^{j\Omega})=\}=\mathrm{Re}\{X(e^{-j\Omega})\} \\ \mathrm{Im}\{X(e^{j\Omega})=\}=-\mathrm{Im}\{X(e^{-j\Omega})\} \end{cases}$ $\begin{cases} \lvert X(e^{j\Omega})\rvert=\lvert X(e^{-j\Omega})\rvert \\ \angle X(e^{j\Omega})=-\angle X(e^{-j\Omega}) \end{cases}$
Real and Even Signals	$x[n]$ real and even	$X(e^{j\Omega})$ real and even
Real and Odd Signals	$x[n]$ real and odd	$X(e^{j\Omega})$ imaginary and odd
Even-Odd Decomposition for Real Signals	$x[n]$ real $x_e[n]=\frac{1}{2}(x[n]+x[-n])$ $x_o[n]=\frac{1}{2}(x[n]-x[-n])$	$\mathrm{Re}\{X(e^{j\Omega})\}$ $j\,\mathrm{Im}\{X(e^{j\Omega})\}$
Parseval's Identity for Inner Product	$\langle x[n],y[n]\rangle=\sum_{n=-\infty}^{\infty}x[n]y^*[n]=\frac{1}{2\pi}\int_{2\pi}X(e^{j\Omega})Y^*(e^{j\Omega})d\Omega$	
Parseval's Identity for Energy	$E=\sum_{n=-\infty}^{\infty}\lvert x[n]\rvert^2=\frac{1}{2\pi}\int_{2\pi}\lvert X(e^{j\Omega})\rvert^2 d\Omega$	

Table 6. Discrete-Time Fourier Transforms and Fourier Series Coefficients

Signal	Fourier Transform	Fourier Series Coefficients (if Periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\Omega_0 n}$	$2\pi \sum_{l=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi l)$	(a) $\frac{\Omega_0}{2\pi} = \frac{m}{N}$: $a_k = \begin{cases} 1 & k = m, m \pm N, m \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}$ (b) $\frac{\Omega_0}{2\pi}$ irrational: signal not periodic.
$\cos \Omega_0 n$	$\pi \sum_{l=-\infty}^{\infty} [\delta(\Omega - \Omega_0 - 2\pi l) + \delta(\Omega + \Omega_0 - 2\pi l)]$	(a) $\frac{\Omega_0}{2\pi} = \frac{m}{N}$: $a_k = \begin{cases} \frac{1}{2} & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}$ (See footnote 1.) (b) $\frac{\Omega_0}{2\pi}$ irrational: signal not periodic.
$\sin \Omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{\infty} [\delta(\Omega - \Omega_0 - 2\pi l) - \delta(\Omega + \Omega_0 - 2\pi l)]$	(a) $\frac{\Omega_0}{2\pi} = \frac{m}{N}$: $a_k = \begin{cases} \frac{1}{2j} & k = m, m \pm N, m \pm 2N, \ldots \\ -\frac{1}{2j} & k = -m, -m \pm N, -m \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}$ (See footnote 1.) (b) $\frac{\Omega_0}{2\pi}$ irrational: signal not periodic.
1	$2\pi \sum_{l=-\infty}^{\infty} \delta(\Omega - 2\pi l)$	$a_k = \begin{cases} 1 & k = 0, \pm N, \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}$
Rectangular pulse train $x[n] = \begin{cases} 1 & \|n\| \le N_1 \\ 0 & N_1 < \|n\| \le \frac{N}{2} \end{cases}$ $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{\infty} \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin(2\pi k/2N)} \delta\left(\Omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin(2\pi k/2N)}$ Note that $a_k = \frac{2N_1 + 1}{N}$, $k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$, for all k
$\Pi\left(\frac{n}{2N_1}\right) = \begin{cases} 1 & \|n\| \le N_1 \\ 0 & \|n\| > N_1 \end{cases}$	$\frac{\sin[\Omega(N_1 + \frac{1}{2})]}{\sin(\Omega/2)}$	–
$\frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$, $0 < W < \pi$	$X(e^{j\Omega}) = \begin{cases} 1 & \|\Omega\| \le W \\ 0 & W < \|\Omega\| \le \pi \end{cases}$ $X(e^{j(\Omega + 2\pi)}) = X(e^{j\Omega})$	–
$\delta[n]$	1	–
$u[n]$	$\frac{1}{1 - e^{-j\Omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$	–
$\delta[n - n_0]$	$e^{-j\Omega n_0}$	–
$a^n u[n]$, $\|a\| < 1$	$\frac{1}{1 - ae^{-j\Omega}}$	–
$(n+1)a^n u[n]$, $\|a\| < 1$	$\frac{1}{(1 - ae^{-j\Omega})^2}$	–
$\frac{(n+m-1)!}{n!(m-1)!} a^n u[n]$, $\|a\| < 1$, $m \ge 1$	$\frac{1}{(1 - ae^{-j\Omega})^m}$	–

1. Careful consideration is required in the special cases when Ω_0 is a integer multiple of π, such that the signal becomes a constant or an alternating sequence.

Frequency Response Magnitude and Phase Plots

- In order to understand and analyze LTI systems with memory, it is important to be proficient in sketching magnitudes and phases of frequency responses. Here we provide several examples of CT and DT systems. In all cases, *x* and *y* denote the input and output, respectively.
- The product, quotient and reciprocal properties used to evaluate magnitudes and phases are stated on page 289 above.

Continuous-Time Systems

- The frequency responses are not periodic functions of the frequency ω.

CT First-Order Lowpass Filter (Chapter 3, pages 99-100)

- It is described by a differential equation

$$\tau \frac{dy}{dt} + y(t) = x(t),$$

and its frequency response is

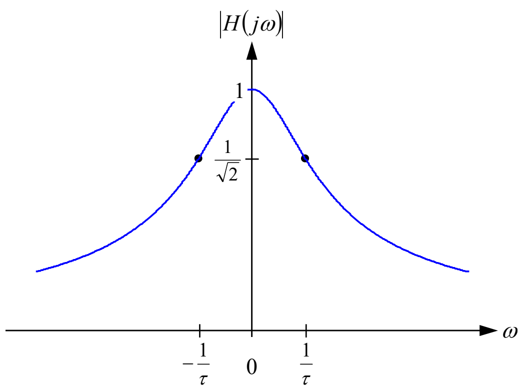
$$H(j\omega) = \frac{1}{1 + j\tau\omega}.$$

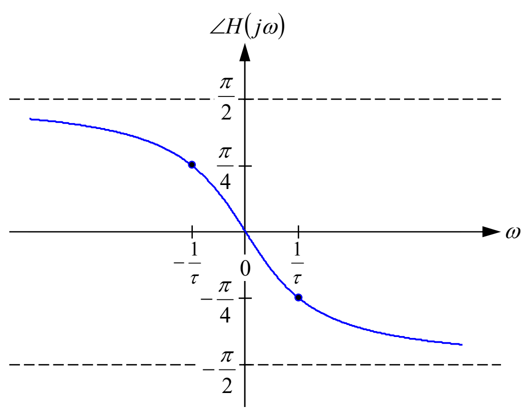
- Using the reciprocal property, we write its magnitude and phase as

$$|H(j\omega)| = \frac{1}{|1 + j\tau\omega|} = \frac{1}{\sqrt{1 + (\tau\omega)^2}}$$

$$\angle H(j\omega) = -\angle(1 + j\tau\omega) = -\tan^{-1}(\tau\omega).$$

These are plotted below.





CT First-Order Highpass Filter (Chapter 3, pages 100-102)

- It is described by a differential equation

$$\frac{dy}{dt} + \frac{1}{\tau} y(t) = \frac{dy}{dt},$$

and its frequency response is

$$H(j\omega)=\frac{j\tau\omega}{1+j\tau\omega}.$$

- Using the quotient property, we write its magnitude and phase as

$$\left|H(j\omega)\right|=\frac{\left|j\tau\omega\right|}{\left|1+j\tau\omega\right|}=\frac{\tau\left|\omega\right|}{\sqrt{1+(\tau\omega)^2}}$$

$$\angle H(j\omega)=\angle(j\tau\omega)-\angle(1+j\tau\omega).$$

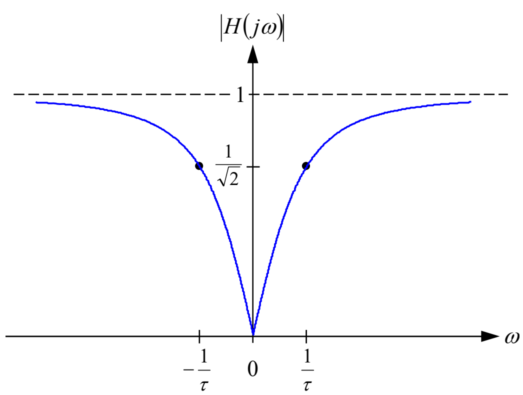
The numerator $j\tau\omega$ is purely imaginary, so its phase assumes one of two values, depending on the sign of ω:

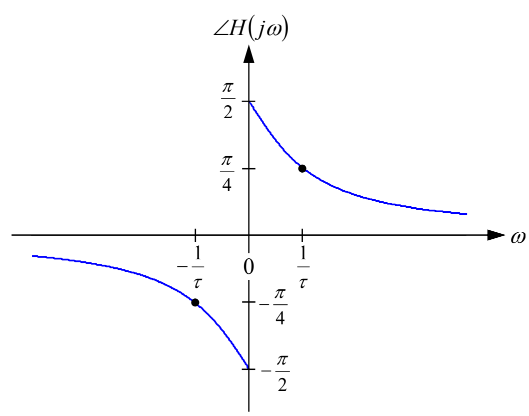
$$\angle(j\tau\omega)=\begin{cases}+\dfrac{\pi}{2} & \omega>0\\ -\dfrac{\pi}{2} & \omega<0\end{cases}=\frac{\pi}{2}\operatorname{sgn}\omega.$$

The phase of the denominator, $\angle(1+j\tau\omega)=\tan^{-1}(\tau\omega)$, was computed in the previous example. Thus

$$\angle H(j\omega)=\frac{\pi}{2}\operatorname{sgn}\omega-\tan^{-1}(\tau\omega).$$

The magnitude and phase are plotted below.





CT Finite-Time Integrator (Chapter 4, pages 171-172)

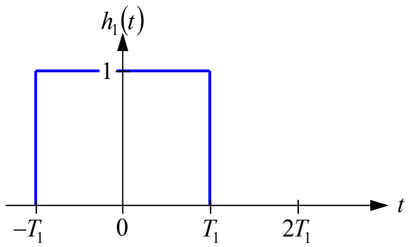
- We consider systems that integrate an input signal for $2T_1$ seconds. The first system is non-causal:

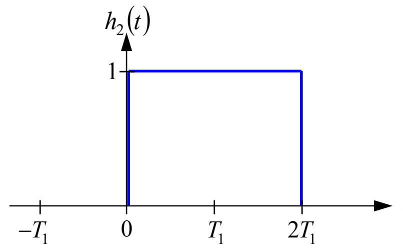
$$h_1(t)=\Pi\left(\frac{t}{2T_1}\right).$$

To form the second system, this impulse response is delayed by T_1 seconds to make it causal:

$$h_2(t)=h_1(t-T_1)=\Pi\left(\frac{t-T_1}{2T_1}\right)=\Pi\left(\frac{t}{2T_1}-\frac{1}{2}\right).$$

These impulse responses are plotted below.





- Their frequency responses are given by the CT Fourier transforms of their impulse responses:

$$H_1(j\omega) = 2T_1 \text{sinc}\left(\frac{\omega T_1}{\pi}\right)$$

$$H_2(j\omega) = 2T_1 \text{sinc}\left(\frac{\omega T_1}{\pi}\right) e^{-j\omega T_1}.$$

By the CT Fourier transform time-shift property, because $h_2(t) = h_1(t - T_1)$, it follows that $H_2(j\omega) = H_1(j\omega)e^{-j\omega T_1}$. The two frequency responses are identical except for a phase factor that varies linearly with frequency. The two systems have the same magnitude response:

$$|H_1(j\omega)| = |H_2(j\omega)| = 2T_1 \left|\text{sinc}\left(\frac{\omega T_1}{\pi}\right)\right|.$$

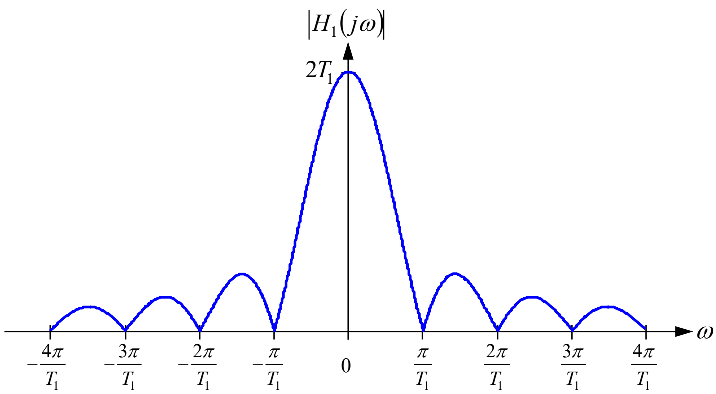
The first system's phase response is

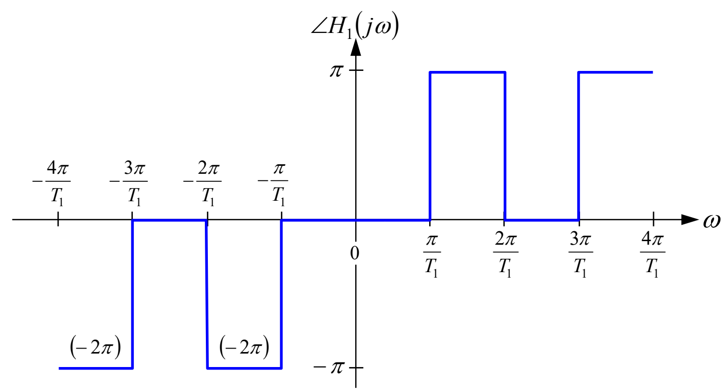
$$\angle H_1(j\omega) = \angle \text{sinc}\left(\frac{\omega T_1}{\pi}\right) = \begin{cases} 0 + k2\pi & \text{sinc}(\omega T_1 / \pi) > 0 \\ \pi + k2\pi & \text{sinc}(\omega T_1 / \pi) < 0 \end{cases},$$

while the second system's phase response is

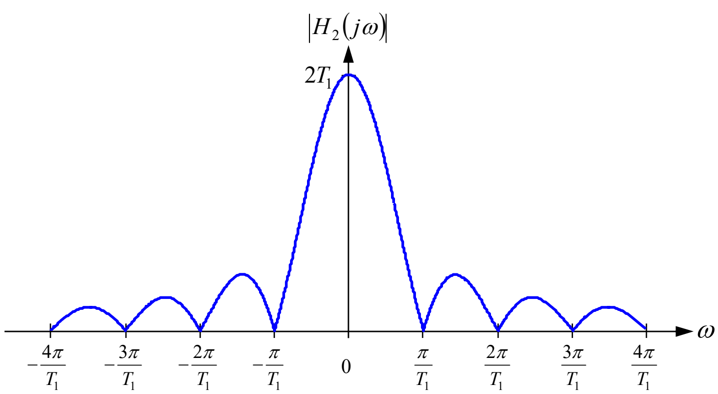
$$\angle H_2(j\omega) = \angle H_1(j\omega) - \omega T_1 = \angle \text{sinc}\left(\frac{\omega T_1}{\pi}\right) - \omega T_1 = \begin{cases} 0 + k2\pi - \omega T_1 & \text{sinc}(\omega T_1 / \pi) > 0 \\ \pi + k2\pi - \omega T_1 & \text{sinc}(\omega T_1 / \pi) < 0 \end{cases}.$$

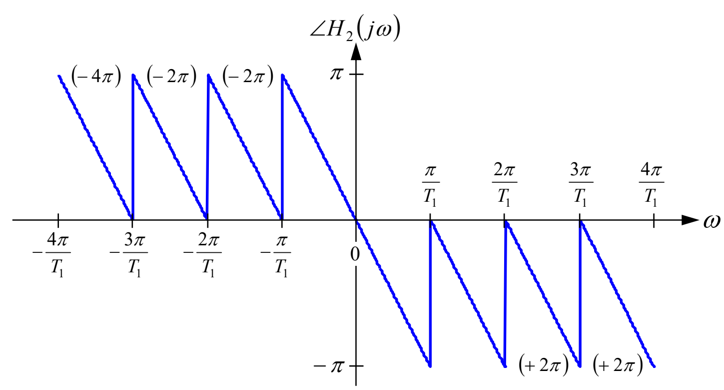
- We can add any integer k multiple of 2π to a phase plot (in CT or DT). It is common practice to add different multiples of 2π to the phase over different frequency ranges so that the phase lies between $-\pi$ and $+\pi$ (MATLAB does this by default). We may also add different multiples of 2π so the phase appears with the symmetry expected (e.g., an odd function of frequency for a real-valued system), although this is not strictly necessary.
- The first system's magnitude and phase are plotted below. At frequencies that are nonzero multiples of π/T_1, where the sinc function passes through zero and changes sign, the phase exhibits jumps of π radians. In order to make the phase plot appear odd, -2π has been added over the intervals indicated.





- The second system's magnitude and phase are plotted below. Again, the phase exhibits jumps of π radians at frequencies where the sinc function changes sign. The phase exhibits a negative slope of $d\angle H_2(j\omega)/d\omega = -T_1$, which corresponds to the delay by T_1 seconds. In order to keep the phase values between −π and +π, various negative and positive multiples of 2π have been added in the intervals indicated.





Discrete-Time Systems

- The frequency responses are periodic functions of the frequency Ω with period 2π, so it is sufficient to plot them for $-\pi \le \Omega < \pi$.

DT Two-Point Moving Average Lowpass Filter (Chapter 3, pages 131-132)

- This finite impulse response filter is described by a difference equation

$$y[n] = \frac{1}{2}\left(x[n] + x[n-1]\right),$$

and its frequency response is

$$H\left(e^{j\Omega}\right) = \frac{1}{2}\left(1 + e^{-j\Omega}\right).$$

To make it easier to compute the magnitude and phase, we factor out $e^{-j\Omega/2}$:

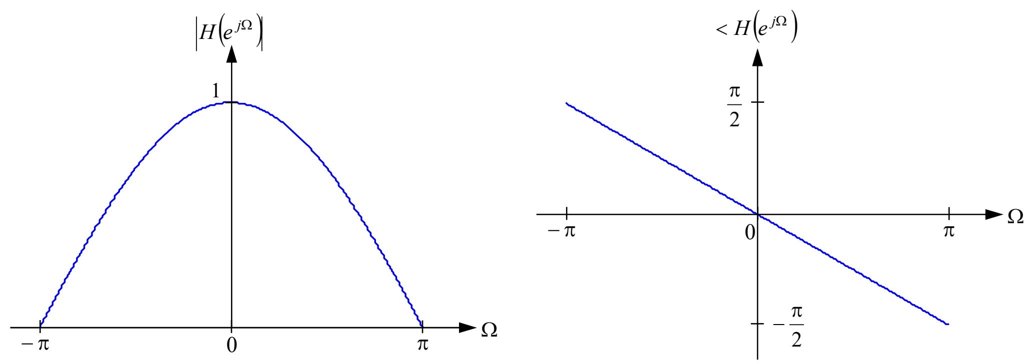
$$H\left(e^{j\Omega}\right) = e^{-j\frac{\Omega}{2}} \frac{e^{j\frac{\Omega}{2}} + e^{-j\frac{\Omega}{2}}}{2} = e^{-j\frac{\Omega}{2}} \cos\left(\frac{\Omega}{2}\right).$$

- Using the product property, we write the magnitude and phase as

$$\left|H\left(e^{j\Omega}\right)\right| = \left|e^{-j\frac{\Omega}{2}}\right| \left|\cos\left(\frac{\Omega}{2}\right)\right| = \left|\cos\left(\frac{\Omega}{2}\right)\right|$$

$$\angle H\left(e^{j\Omega}\right) = \angle e^{-j\frac{\Omega}{2}} + \angle\cos\left(\frac{\Omega}{2}\right) = -\frac{\Omega}{2} + \begin{cases} 0 & \cos(\Omega/2) > 0 \\ -\pi & \cos(\Omega/2) < 0 \end{cases} = \begin{cases} -\Omega/2 & \cos(\Omega/2) > 0 \\ -\Omega/2 - \pi & \cos(\Omega/2) < 0 \end{cases}.$$

The magnitude and phase of its frequency response over one period, $-\pi \le \Omega < \pi$, are plotted below.



DT Edge Detector Highpass Filter (Chapter 3, page 133)

- This finite impulse response filter is described by a difference equation

$$y[n] = \frac{1}{2}\left(x[n] - x[n-1]\right),$$

and its frequency response is

$$H\left(e^{j\Omega}\right) = \frac{1}{2}\left(1 - e^{-j\Omega}\right).$$

- To make it easier to compute the magnitude and phase, we factor out $e^{-j\Omega/2}$ and multiply and divide by j:

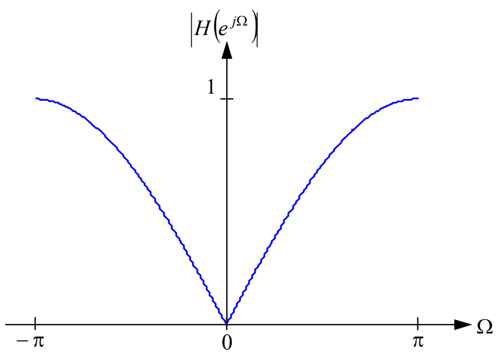
$$H\left(e^{j\Omega}\right) = je^{-j\frac{\Omega}{2}} \frac{e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}}{2j} = je^{-j\frac{\Omega}{2}} \sin\left(\frac{\Omega}{2}\right).$$

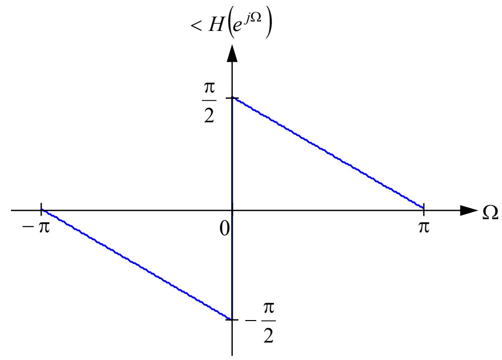
Using the product property, we write its magnitude and phase as

$$\left|H\left(e^{j\Omega}\right)\right| = |j| \left|e^{-j\frac{\Omega}{2}}\right| \left|\sin\left(\frac{\Omega}{2}\right)\right| = \left|\sin\left(\frac{\Omega}{2}\right)\right|$$

$$\angle H\left(e^{j\Omega}\right) = \angle j + \angle e^{-j\frac{\Omega}{2}} + \angle\sin\left(\frac{\Omega}{2}\right) = \frac{\pi}{2} - \frac{\Omega}{2} + \begin{cases} 0 & \sin(\Omega/2) > 0 \\ -\pi & \sin(\Omega/2) < 0 \end{cases} = \begin{cases} -\Omega/2 + \pi/2 & \sin(\Omega/2) > 0 \\ -\Omega/2 - \pi/2 & \sin(\Omega/2) < 0 \end{cases}.$$

The magnitude and phase of its frequency response over one period, $-\pi \le \Omega < \pi$, are plotted below.





- Observe that the frequency responses of the two-point moving average (lowpass filter) and edge detector (highpass filter) are identical, except for a frequency shift of π, which transforms a lowpass filter to a highpass filter, or vice versa. We can understand this by comparing their impulse responses:

$$h_{\text{mov. avg.}}[n] = \frac{1}{2}\left(\delta[n] + \delta[n-1]\right) \text{ and } h_{\text{edge det.}}[n] = \frac{1}{2}\left(\delta[n] - \delta[n-1]\right),$$

which are related by

$$h_{\text{edge det.}}[n] = (-1)^n h_{\text{mov. avg.}}[n] = e^{j\pi n} h_{\text{mov. avg.}}[n].$$

Using the frequency-shift property of the DT Fourier transform with $\Omega_0 = \pi$, the frequency responses are related by

$$H_{\text{edge det.}}\left(e^{j\Omega}\right) = H_{\text{mov. avg.}}\left(e^{j(\Omega-\pi)}\right).$$

DT First-Order Lowpass or Highpass Filter (Chapter 3, pages 129-131)

- This infinite impulse response filter is described by a difference equation

$$y[n] - ay[n-1] = x[n],$$

where a is real. The system can be useful for any a, but the frequency response only exists for $|a| < 1$, in which case, its frequency response is

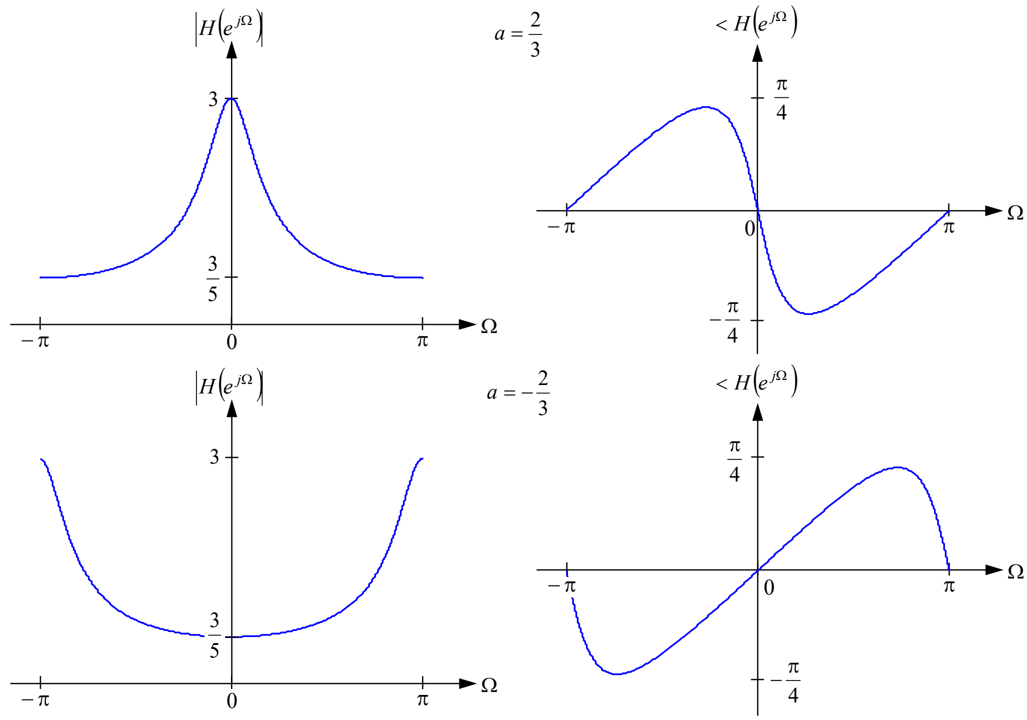
$$H\left(e^{j\Omega}\right) = \frac{1}{1 - ae^{-j\Omega}}.$$

- Using the reciprocal property, we can write its magnitude and phase as

$$\left|H\left(e^{j\Omega}\right)\right| = \frac{1}{\left|1 - ae^{-j\Omega}\right|} = \frac{1}{\sqrt{(1 - a\cos\Omega)^2 + (a\sin\Omega)^2}}$$

$$\angle H\left(e^{j\Omega}\right) = -\angle\left(1 - ae^{-j\Omega}\right) = -\tan^{-1}\left(\frac{a\sin\Omega}{1 - a\cos\Omega}\right).$$

Here we plot the magnitude and phase of its frequency response over one period, $-\pi \le \Omega < \pi$. We consider two different values of a: $a = 2/3$ (lowpass filter) and $a = -2/3$ (highpass filter).



- We notice that the expressions for the magnitude and phase are more complicated than for the other examples above, and it is difficult to make accurate plots without a computer or calculator. Nevertheless, we can sketch the general shape of the magnitude response by evaluating $\left|H\left(e^{j\Omega}\right)\right|$ for a few values of Ω, such as 0, π/2 and π. For example, for $a = 2/3$:

Ω	0	$\pi/2$	π
$\left\|H\left(e^{j\Omega}\right)\right\|$	$\left[\left(1-\frac{2}{3}\right)^2+\left(0\right)^2\right]^{-1/2}=3$	$\left[\left(1\right)^2+\left(\frac{2}{3}\right)^2\right]^{-\frac{1}{2}}=\frac{3}{\sqrt{13}}$	$\left(1+\frac{2}{3}\right)^2+\left(0\right)^{2-\frac{1}{2}}=\frac{3}{5}$

- The frequency responses for $a = 2/3$ (lowpass filter) and $a = -2/3$ (highpass filter) are identical, except for a frequency shift of π. The impulse responses of the two filters

$$h_{a=2/3}[n]=\left(\frac{2}{3}\right)^n u[n] \quad \text{and} \quad h_{a=-2/3}[n]=\left(-\frac{2}{3}\right)^n u[n]$$

are related by

$$h_{a=-2/3}[n]=(-1)^n h_{a=2/3}[n]=e^{j\pi n}h_{a=2/3}[n],$$

so by the frequency-shift property of the DT Fourier transform with $\Omega_0=\pi$, their frequency responses are related by

$$H_{a=-2/3}\left(e^{j\Omega}\right)=H_{a=2/3}\left(e^{j(\Omega-\pi)}\right).$$