Stanford University
EE 102A: Signal Processing and Linear Systems I
Summer 2022
Instructor: Ethan M. Liang

Homework 6, due Friday, August 5

Unless noted otherwise, all references are to *EE 102A Course Reader*, Appendix, Tables 3 and 4.

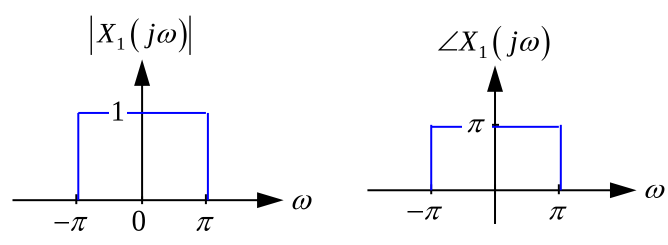
1. *Fourier transforms.* Sketch each signal $x(t)$. Using only tables and properties, obtain an expression for its Fourier transform $X(j\omega)$. Explain how the properties of $x(t)$ (real or imaginary, odd or even, etc.) are reflected in those of $X(j\omega)$.

   a. $x(t) = \Lambda\left(\frac{t}{4}\right)\sin(2\pi t)$. Sketch the real or imaginary part of $X(j\omega)$, whichever is nonzero. *Hint*: use the known FT of the sine function and use the FT multiplication property.

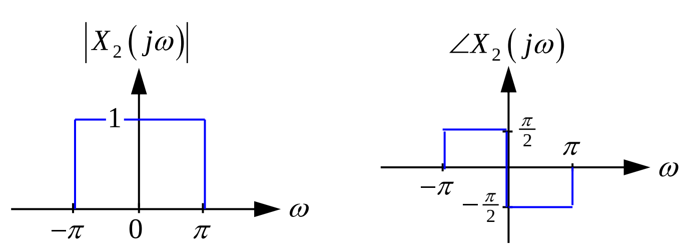
   b. $x(t) = e^{-t}\left[u(t+1) - u(t-1)\right]$. You do not need to sketch $X(j\omega)$. *Hint*: write a term like $e^{-t}u(t+1)$ as $e \cdot e^{-(t+1)}u(t+1)$.

2. *Inverse Fourier transforms.* The Fourier transforms given have identical magnitudes but different phases. Obtain an expression for the inverse Fourier transform of each one (the corresponding time signal) using only tables and properties. Explain how the properties of each time signal (real or imaginary, odd or even, etc.) are reflected in those of its Fourier transform. *Hint*: express each transform in terms of simple functions, such as rectangular pulses, multiplied by constants or frequency-dependent phase factors.

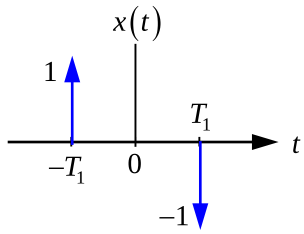
   a.



   b.



3. *Fourier transform integration property*. A signal $x(t)$ is a sum of two scaled, shifted impulses.



a. Obtain an expression for $X(j\omega)$, the Fourier transform of $x(t)$.

b. Now consider $y(t) = \int_{-\infty}^{t} x(\tau) d\tau$. Sketch $y(t)$.

c. Using the Fourier transform integration property, obtain an expression for $Y(j\omega)$, the Fourier transform of $y(t)$. Put $Y(j\omega)$ in the standard form found in the table.

4. *Convolution property and relation between the Fourier series and the Fourier transform of one period.* We are given two rectangular pulses:

$$x_1(t) = \Pi\left(\frac{t}{2T_1}\right) \text{ and } x_2(t) = \Pi\left(\frac{t}{2T_2}\right).$$

Consider the convolution between them:

$$x(t) = x_1(t) * x_2(t).$$

a. Sketch $x(t)$, assuming $T_1 = 1$ and $T_2 = 2$.

b. Obtain an expression for $X(j\omega)$, the Fourier transform of $x(t)$, assuming general values of T_1 and T_2. Sketch $X(j\omega)$, assuming $T_1 = 1$ and $T_2 = 2$ (sketch the real or imaginary part, whichever is nonzero).

c. Now consider a periodic signal

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} x(t - nT_0),$$

assuming $T_0 \geq 2(T_1 + T_2)$. Sketch two or three periods of $\tilde{x}(t)$, assuming $T_1 = 1$ $T_2 = 2$ and $T_0 = 6$.

d. By sampling $X(j\omega)$, obtain an expression for the Fourier series coefficients of $\tilde{x}(t)$, given by a_k. Sketch the a_k vs. *k*, assuming $T_1 = 1$ $T_2 = 2$ and $T_0 = 6$.

5. *Frequency-shift property and Parseval's Identity.* We are given two signals:

$$x_1(t) = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right) \text{ and } x_2(t) = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)\sin(\omega_0 t), \text{ assuming } \omega_0 \geq 2W.$$

a. Sketch their Fourier transforms, $X_1(j\omega)$ and $X_2(j\omega)$, assuming $W = 2\pi$ and $\omega_0 = 4\pi$.

b. Compute the energies $E_{x_1} = \int_{-\infty}^{\infty} \left|x_1(t)\right|^2 dt$ and $E_{x_2} = \int_{-\infty}^{\infty} \left|x_2(t)\right|^2 dt$.

c. Compute the inner product $\langle x_1(t), x_2(t) \rangle = \int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt$.

CT LTI System Analysis

6. *Ideal filter*. A filter has an impulse response

$$h(t) = \frac{W}{\pi} \operatorname{sinc}\left(\frac{W(t-t_0)}{\pi}\right) \cos\left(\omega_0 (t-t_0)\right), \text{ assuming } \omega_0 \geq W.$$

a. Obtain an expression for the frequency response $H(j\omega)$. Sketch the magnitude and phase, $|H(j\omega)|$ and $\angle H(j\omega)$, assuming $W = \frac{3}{4}\pi$, $\omega_0 = \pi$ and $t_0 = 1$.

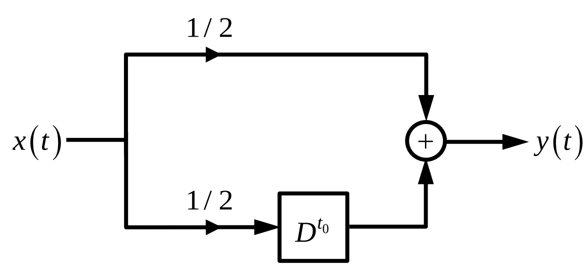
b. What kind of filter is this (lowpass, highpass, etc.)?

c. We input a signal

$$x(t) = 1 + 2\sin(t) + \cos(2\pi t).$$

Obtain an expression for the output signal $y(t)$, assuming the same filter parameters as in part (a).

7. *Delay-and-add system*. A system splits an input signal $x(t)$ into two copies, each scaled by 1/2, delays one copy by t_0, and combines the two copies to obtain an output signal $y(t)$. *Hint*: this is the CT analogue of a DT system we studied in lecture.



a. Obtain an expression for the impulse response $h(t)$.

b. Obtain an expression for the frequency response $H(j\omega)$. Sketch the magnitude and phase $|H(j\omega)|$ and $\angle H(j\omega)$, assuming a general value of the delay t_0.

c. An input signal is $x(t) = \sin(\omega_0 t)$. Specify all values of ω_0 such that the output is $y(t) = 0$, assuming a general value of the delay t_0.

8. *Critically damped second-order lowpass filter*. We discuss a second-order lowpass filter in the *EE 102A Course Reader*, Chapter 4, pages 183-186. Assuming a natural frequency ω_n and a damping constant $\zeta = 1$ (critical damping), the impulse and frequency responses are given by

$$h(t)=\omega_n^2 t e^{-\omega_n t}u(t) \overset{F}{\leftrightarrow} H(j\omega)=\frac{1}{\left(1+j\frac{\omega}{\omega_n}\right)^2}=\frac{\omega_n^2}{(j\omega)^2+2\omega_n(j\omega)+\omega_n^2}\,. \tag{1}$$

We derive the CTFT pair (1) in this problem. To facilitate using earlier results, we make a substitution $\omega_n \to 1/\tau$, so (1) becomes

$$h(t)=\frac{t}{\tau^2}e^{-\frac{t}{\tau}}u(t) \overset{F}{\leftrightarrow} H(j\omega)=\frac{1}{(1+j\omega\tau)^2}\,. \tag{1'}$$

a. Use the CTFT pair

$$x(t)=\frac{1}{\tau}e^{-\frac{t}{\tau}}u(t) \overset{F}{\leftrightarrow} X(j\omega)=\frac{1}{1+j\omega\tau} \tag{2}$$

and the CTFT differentiation-in-frequency property to prove (1').

b. In Homework 3 Problem 6, we studied the cascade of two CT first-order lowpass filters with time constants τ_1 and τ_2, $\tau_1 \neq \tau_2$, and derived the resulting impulse response

$$\frac{1}{\tau_1}e^{-\frac{t}{\tau_1}}u(t) * \frac{1}{\tau_2}e^{-\frac{t}{\tau_2}}u(t)=\frac{1}{\tau_1-\tau_2}\left(e^{-\frac{t}{\tau_1}}-e^{-\frac{t}{\tau_2}}\right)u(t)\,.$$

Here we study the case $\tau_1=\tau_2=\tau$. Use (1'), (2) and the CTFT convolution property to show that the cascade has an impulse response

$$\frac{1}{\tau}e^{-\frac{t}{\tau}}u(t) * \frac{1}{\tau}e^{-\frac{t}{\tau}}u(t)=\frac{t}{\tau^2}e^{-\frac{t}{\tau}}u(t)\,. \tag{3}$$

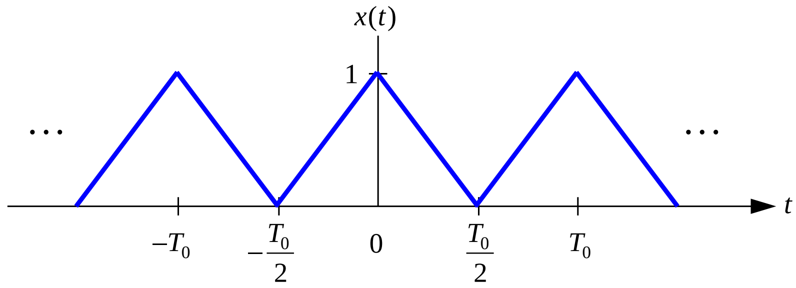
This shows that a critically damped second-order lowpass filter is equivalent to a cascade of two identical first-order lowpass filters.

Continuous-Time Fourier Transforms

9. *CT FT symmetry properties*. Consider a real, right-sided signal $x(t)$, which satisfies $x(t)=0$, $t \le 0$ (note that $x(t)=0$ at $t=0$). It has a Fourier transform $X(j\omega)$. We are given the real part of its Fourier transform, $\mathrm{Re}\left(X(j\omega)\right)$. Explain how we can compute $x(t)$. *Hint*: consider decomposing $x(t)$ into even and odd parts.

10. *Convolution or multiplication of periodic and aperiodic signals*.

a. $x(t)$ is the periodic triangular pulse train shown.



Sketch $X(j\omega)$, the CT FT of $x(t)$. *Hint*: you computed the CT FS coefficients of a scaled version of $x(t)$ in Homework 4.

b. $h(t)$ is an aperiodic signal

$$h(t) = \frac{1}{T_0}\mathrm{sinc}\left(\frac{t}{T_0}\right) .$$

Sketch $H(j\omega)$, the CT FT of $h(t)$.

c. Let $y(t) = x(t) * h(t)$. Is $y(t)$ periodic or aperiodic? Sketch $Y(j\omega)$, the CT FT of $y(t)$.

d. Let $z(t) = x(t) \times h(t)$. Is $z(t)$ periodic or aperiodic? Sketch $Z(j\omega)$, the CT FT of $z(t)$.

Laboratory 6

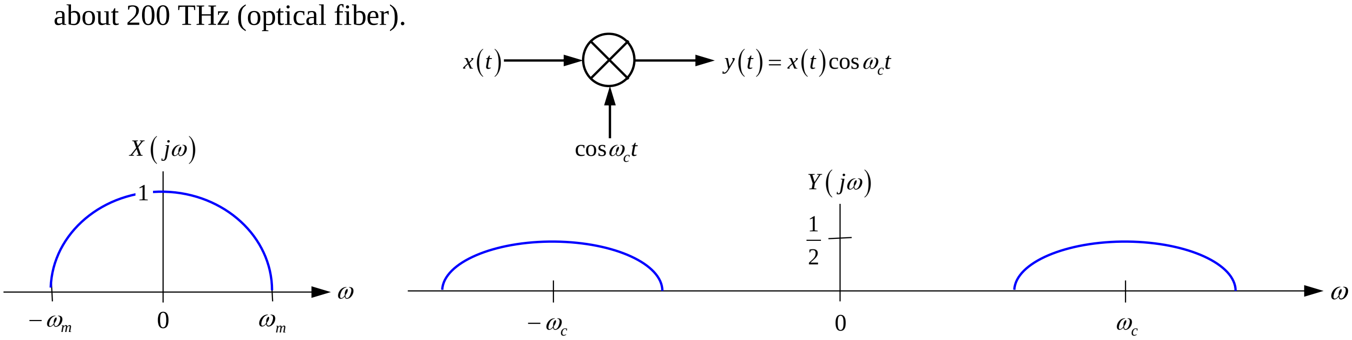
This week's laboratory demonstrates *modulation* and *demodulation*, in this case, double-sideband amplitude modulation with synchronous demodulation. For additional discussion, refer to the *EE 102A Course Reader*, Chapter 4 (pages 174-177) and Chapter 7 (pages 268-272).

Modulation and demodulation are at the heart of any communication system. You know that a radio station is identified by its frequency, and you can listen to a station by tuning a receiver to that frequency. For example, the California Department of Transportation uses the AM station 1610 kHz. Here, "AM" refers to the type of modulation used, in this case amplitude modulation, which we implement in this lab. 1610 kHz is the *carrier frequency* of this station. Note that the demodulation we implement here (synchronous) is different from what is used for demodulating broadcast AM radio (asynchronous).

In *amplitude modulation*, a message signal $x(t)$, whose spectrum $X(j\omega)$ is near d.c., is multiplied by a carrier signal $\cos\omega_c t$, yielding a modulated signal $y(t)=x(t)\cos\omega_c t$. The modulated signal spectrum is

$$Y(j\omega)=\frac{1}{2}\left[X\left(j(\omega-\omega_c)\right)+X\left(j(\omega+\omega_c)\right)\right], \tag{1}$$

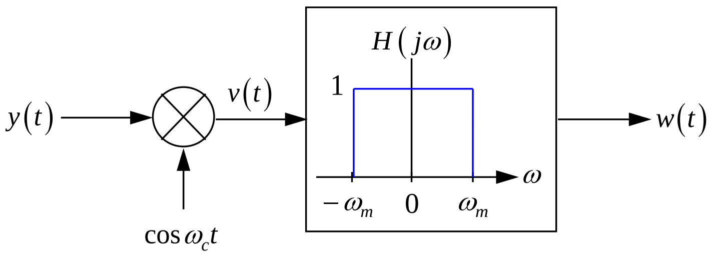
which is the spectrum $X(j\omega)$ replicated at $\pm\omega_c$. The carrier frequency ω_c is chosen to allow the signal to propagate through a medium as an electromagnetic wave. Depending on the system, the carrier frequency $\omega_c/2\pi$ may range from about 1 MHz (broadcast AM radio) to about 1-5 GHz (WiFi or cellular radio) to about 200 THz (optical fiber).



In *synchronous demodulation*, we multiply the modulated signal $y(t)$ by a replica of the carrier signal, $\cos\omega_c t$, obtaining a signal $v(t)$. In the time domain, we have

$$v(t)=y(t)\cos\omega_c t=x(t)\cos^2\omega_c t=\frac{1}{2}x(t)+\frac{1}{2}x(t)\cos 2\omega_c t, \tag{2}$$

which includes a term $(1/2)x(t)$ near d.c. and a term $(1/2)x(t)\cos 2\omega_c t$ near $\pm 2\omega_c$. Then we pass $v(t)$ into a lowpass filter with impulse response $h(t)$ and frequency response $H(j\omega)$. The term near $\pm 2\omega_c$ is blocked by the lowpass filter, so the filter output is $w(t)=(1/2)x(t)$, a scaled copy of the message signal.



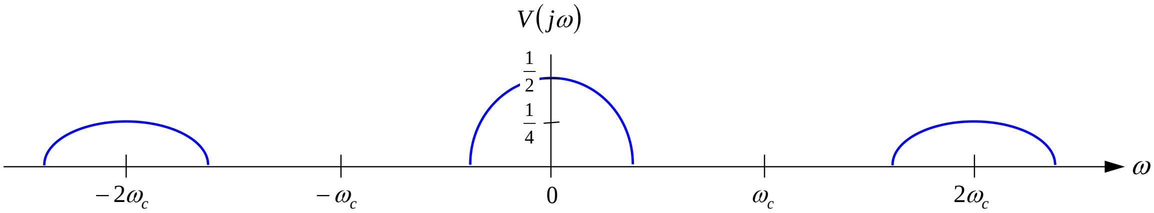
In the frequency domain, since $v(t) = y(t)\cos\omega_c t$, we have

$$V(j\omega) = \frac{1}{2}\left[Y\left(j(\omega-\omega_c)\right) + Y\left(j(\omega+\omega_c)\right)\right]. \tag{3}$$

Using (1), this becomes

$$V(j\omega) = \frac{1}{4}X\left(j(\omega-2\omega_c)\right) + \frac{1}{2}X(j\omega) + \frac{1}{4}X\left(j(\omega+2\omega_c)\right), \tag{4}$$

as shown below.



The lowpass filter passes only the term $\frac{1}{2}X(j\omega)$, yielding an output $w(t) = \frac{1}{2}x(t)$.

Notes on MATLAB Implementation

Throughout these simulations, time is discretized by a discretization interval Δt, represented in MATLAB by **`deltat`**.

In order to help you visualize the results in the frequency domain, we provide a MATLAB function **`CTFT_approx.m`**., which numerically approximates the continuous-time Fourier transform. Given a signal $x(t)$ represented by discretized time vector **`t`** and signal **`x`**, it returns a numerical approximation of the continuous-time Fourier transform $X(j\omega)$, represented by discretized frequency vector **`omega`** and Fourier transform **`X`**. It is called using the syntax

```
[X,omega] = CTFT_approx(x,t).
```

The frequency vector **`omega`** covers a span corresponding very nearly to $-\pi/\Delta t \le \omega \le \pi/\Delta t$ or $-1/2\Delta t \le \omega/2\pi \le 1/2\Delta t$.

1. Triangular pulse function

In this lab, the modulated signal $x(t)$ will be a triangular pulse. Write and turn in a function that returns a unit triangular pulse, as defined in lecture:

$$\Lambda(t) = \begin{cases} 1-|t| & |t| \le 1 \\ 0 & |t| > 1 \end{cases}.$$

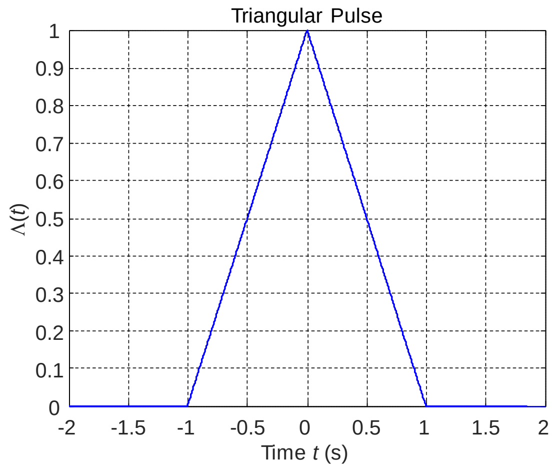
Implement the triangular pulse function described by the header below:

```
function y = Lambda(t)
% Unit triangular pulse
```

You can test your function with the following code in the Command Window:

```
>> t = -2:.01:2;
>> figure; plot(t,Lambda(t), 'LineWidth', 1.5);
>> set(gca,'FontName','arial','FontSize',14);
>> xlabel('Time \itt\rm (s)'); ylabel('\it\Pi(t)');
>> title('Triangular Pulse');
```

You should see the following output.



2. Message signal and its spectrum

Create a new script and define your time vector:

```
% time and frequency
deltat = 1/300;              % discretization interval (s)
tmax = 1.5;                  % time vector runs from -tmax to tmax (s)
t = -tmax:deltat:tmax;       % time vector for x(t),y(t),v(t),h(t) (s)
```

Use your **Lambda** function to create a triangular pulse of unit peak value and half-width $T_x = 1/2$:

$$x(t) = \Lambda\left(\frac{t}{T_x}\right),$$

represented by a vector **x**. Its spectrum can be understood from the CT FT pair derived in lecture:

$$\Lambda\left(\frac{t}{T_x}\right) \leftrightarrow T_x \mathrm{sinc}^2\left(\frac{\omega T_x}{2\pi}\right).$$

Use the provided function **CTFT_approx** to compute the spectrum of **x**, and call it **X**. You should verify that its d.c. value is $X(j0) = T_x = 1/2$, and that the first zeros of $X(j\omega)$ occur when $\omega T_x / 2\pi = \pm 1$, corresponding to $\omega = \pm 2\pi / T_x = \pm 4\pi$ rad/s or $\omega / 2\pi = \pm 1 / T_x = \pm 2$ Hz.

Plot the original signal and the magnitude of its spectrum using the code provided. Note that the frequency axis is in Hz.

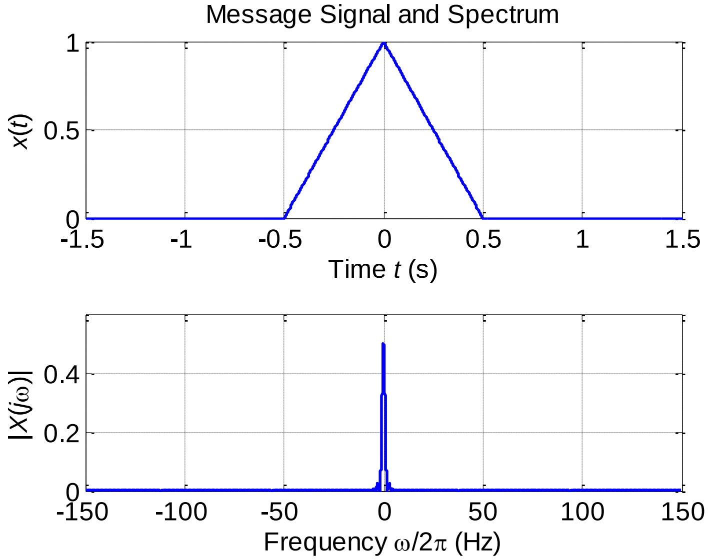
```
figure (1);

subplot(211)
plot(t,x)
```

```
l=get(gca,'children'); set(l,'linewidth',1.5)
set(gca,'FontName','arial'); set(gca,'FontSize',14);
xlabel('Time \itt\rm (s)');
ylabel('\itx\rm(\itt\rm)');
grid
title('Message Signal and Spectrum');

subplot(212)
plot(omega/(2*pi),abs(X));
l=get(gca,'children'); set(l,'linewidth',1.5);
set(gca,'FontName','arial'); set(gca,'FontSize',14);
xlabel('Frequency \omega/2\pi (Hz)');
ylabel('|\itX\rm(\itj\rm\omega)|');
grid
```

Your plot should look like the one below. Compare this with the analytical result above.



3. Modulation

Here we choose the carrier frequency $\omega_c / 2\pi$ very low to make it easy to visualize the results. Use the carrier frequency defined below:

```
% modulation
fc = 50; omegac = 2*pi*fc;      % carrier frequency (Hz and rad/s)
```

Define $y(t) = x(t)\cos\omega_c t$. Plot $y(t)$ and its magnitude spectrum $|Y(j\omega)|$.

4. Demodulation

Define $v(t)$, the demodulated, unfiltered signal. Plot $v(t)$ and its magnitude spectrum $|V(j\omega)|$. Comment on the spectrum of $v(t)$, comparing it to those of $y(t)$ and $x(t)$.

5. Lowpass filtering to recover message signal

We will use a second-order lowpass filter, as described in *EE 102A Course Reader*, Chapter 4, pages 183-186. It has a cutoff frequency ω_n, and is chosen to be critically damped, with damping constant $\varsigma = 1$. We have provided a MATLAB function **hsolpfcd.m**, which is called as **hsolpfcd(t,omegan)**.

Define the cutoff frequency for the lowpass filter and call the provided function:

```
%% Filtering
fn = 5; omegan = 2*pi*fn;         % LPF cutoff frequency (Hz and rad/s)

% time domain
hlpf = hsolpfcd(t,omegan);  % lowpass filter
```

Plot the filter's impulse response $h(t)$ and its magnitude and phase responses $|H(j\omega)|$ and $\angle H(j\omega)$.

Finally, convolve $v(t)$ with the lowpass filter impulse response $h(t)$ to obtain $w(t)$, represented by a vector **w**. Refer to Laboratory 3 for the proper way to approximate CT convolution using discretized signals. You will need to define a time vector **tw** that has the proper starting and ending times for **w**. You did something similar in Laboratory 3, part 1(a).

Plot the lowpass filter output $w(t)$ and its magnitude spectrum $|W(j\omega)|$. Explain how the filtered signal differs from the original message and suggest ways in which the demodulation system could be improved.

6. Effect of phase offset: analytical

In demodulation, we multiply the modulated signal $y(t)$ by a replica of the carrier. Until now, we have assumed the carrier replica is synchronized perfectly. What would occur if this is not the case? Let us assume the carrier replica is $\cos(\omega_c t + \phi)$, where ϕ is a phase offset. When a phase offset is present, after multiplying $y(t)$ by the phase-offset carrier, we obtain

$$v(t) = y(t) \cdot \cos(\omega_c t + \phi) = x(t)\cos(\omega_c t) \cdot \cos(\omega_c t + \phi) = \frac{1}{2} x(t)\cos(\phi) + \frac{1}{2} x(t)\cos(2\omega_c t + \phi),$$

where we have used the trigonometric identity $\cos(A) \cdot \cos(B) = (1/2)\cos(A - B) + (1/2)\cos(A + B)$. Given the above expression for $v(t)$, write an expression for the lowpass filter output $w(t)$, assuming an ideal lowpass filter, as in *EE 102A Course Reader*, pages 270-271 (not the second-order filter used in the present MATLAB simulations). Evaluate your expression for phase offsets of $\phi = 0, \pi/2$ or π. Which of the three phase offsets is worst?

7. Effect of phase offset: verification using MATLAB

Verify your analytical results using MATLAB. Submit plots of $v(t)$, $|V(j\omega)|$, $w(t)$ and $|W(j\omega)|$ for $\phi = \pi/2$ and $\phi = \pi$. Use the same second-order lowpass filter as in Part 5.