**Stanford University**
**EE 102A: Signal Processing and Linear Systems I**
**Summer 2022**
**Instructor: Ethan Liang**

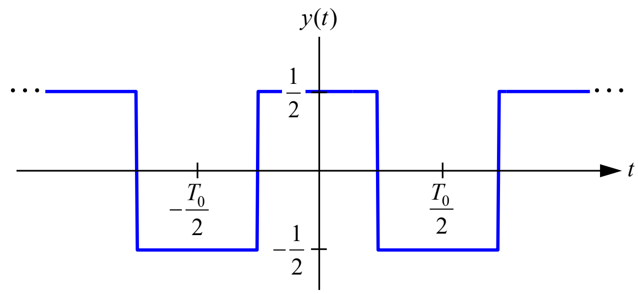
**Homework 4 Solutions**

*Not to be turned in for grading. Solutions distributed Tuesday, July 19.*
All references are to the *EE 102A Course Reader.*

*Using Properties to Compute CT FS Coefficients*

1. **(12 points)** *CT even and odd square waves.*

a. **(4 points)** An even square wave $y(t) \leftrightarrow b_k$ is shown.




Use the CT FS coefficients for a rectangular pulse train, with proper choice of pulse width $2T_1$, along with the linearity property of the CT FS, to find the CT FS coefficients $b_k$ of $y(t)$. Sketch the real or imaginary parts, whichever are nonzero. Comment on how any symmetry in $y(t)$ is reflected in the symmetry of $b_k$.

**Solution**

Consider $x(t) = y(t) + 1/2$, which is the even square wave shifted up by $1/2$. $x(t)$ is a rectangular pulse train with pulse width $T_1 = T_0/4$. Using the formula derived in class with $T_1 = T_0/4$, we can calculate the FS coefficients $a_k$ of $x(t)$ as

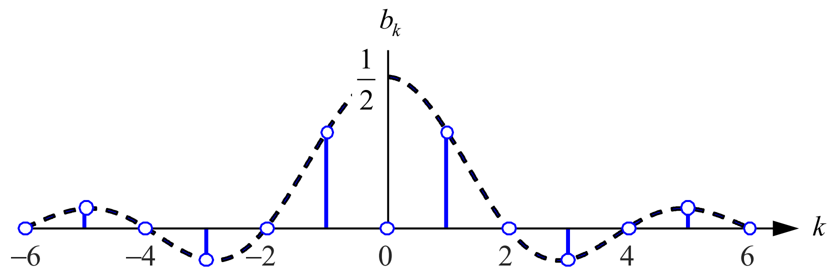
$$\begin{aligned} a_k &= \frac{\omega_0 T_0}{4\pi}\operatorname{sinc}\left(\frac{k\omega_0 T_0}{4\pi}\right) \\ &= \frac{1}{2}\operatorname{sinc}\left(\frac{k}{2}\right). \end{aligned}$$

Since $y(t) = x(t) - 1/2$, we can use linearity:

$$x(t) \xleftrightarrow{FS} \frac{1}{2}\text{sinc}\left(\frac{k}{2}\right)$$

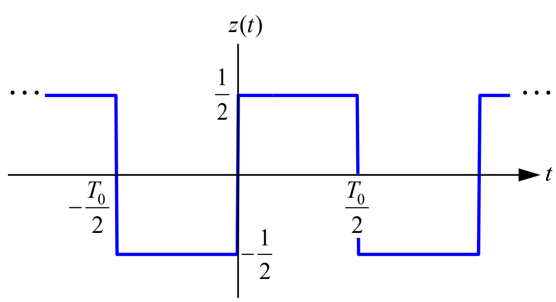
$$-\frac{1}{2} \xleftrightarrow{FS} \begin{cases} -\frac{1}{2} & k=0 \\ 0 & k \neq 0 \end{cases}$$

$$y(t) \xleftrightarrow{FS} b_k = \begin{cases} \frac{1}{2}\text{sinc}\left(\frac{k}{2}\right) & k \neq 0 \\ 0 & k=0 \end{cases}$$




$y(t)$ is real and even in $t$ $\leftrightarrow$ $b_k$ is real and even in $k$.

b. **(4 points)** An odd square wave $z(t) \leftrightarrow c_k$ is shown.

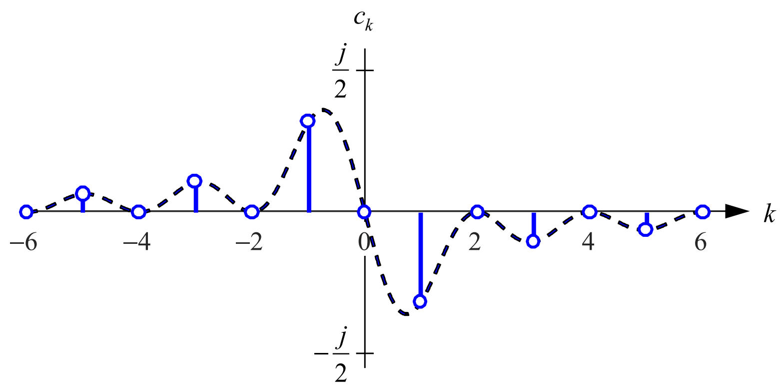



Using the CT FS coefficients found in part (a) and the time shift property of the CT FS, find the CT FS coefficients $c_k$ of $z(t)$. Sketch the real or imaginary parts of the $c_k$, whichever is nonzero. Comment on how any symmetry in $z(t)$ is reflected in symmetry of $c_k$.

**Solution**

Note that $z(t) = y(t - T_0/4)$. Using the time shift property of the CT FS, we obtain

$$
\begin{aligned}
c_k &= e^{-jk\omega_0 T_0/4} b_k \\
&= e^{-jk\pi/2} b_k \\
&= \left[ \underbrace{\cos\left(\frac{k\pi}{2}\right)}_{\text{zero for odd } k} - j\sin\left(\frac{k\pi}{2}\right) \right] \cdot \underbrace{b_k}_{\text{nonzero only for odd } k} \\
&= \underbrace{-j\sin\left(\frac{k\pi}{2}\right)}_{\text{odd in } k} \underbrace{b_k}_{\text{even in } k}
\end{aligned}
$$




$z(t)$ is real and odd in $t$ $\leftrightarrow$ $c_k$ is imaginary and odd in $k$.

c. **(4 points)** The even and odd square waves are obviously orthogonal over one period:

$$\int_{T_0} y(t) z^*(t) dt = 0.$$

Parseval's Identity (see pages 93-94) suggests that the CT FS coefficients must be orthogonal in frequency:

$$T_0 \sum_{k=-\infty}^{\infty} b_k c_k^* = 0.$$

Using the symmetries identified in parts (a) and (b), show this is true.

**Solution**

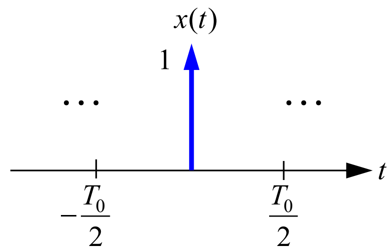
We know from parts (a) and (b) that $b_k$ is even in $k$ and $c_k$ is odd in $k$. This means that the sequence $b_k c_k^*$ is odd in $k$. The sum of any odd sequence over a symmetric interval is always $0$, so we have

$$T_0 \sum_{k=-\infty}^{\infty} b_k c_k^* = 0.$$

2. **(15 points)** *CT impulse train*. The periodic CT impulse train plays an important role in the analysis of sampling and reconstruction. An impulse train of period $T_0$ can be represented as

$$x(t) = \sum_{l=-\infty}^{\infty} \delta\left(t - lT_0\right)$$

and is shown here in one period




a. **(5 points)** Calculate its FS coefficients $a_k$.

**Solution**

Applying the general analysis equation and choosing a symmetric integration interval, we have

$$\begin{aligned}
a_k &= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt \\
&= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \sum_{l=-\infty}^{\infty} \delta(t - lT_0) e^{-jk\omega_0 t} dt \\
&= \frac{1}{T_0} \sum_{l=-\infty}^{\infty} \int_{-T_0/2}^{T_0/2} \delta(t - lT_0) e^{-jk\omega_0 t} dt \\
&= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-jk\omega_0 t} dt \\
&= \frac{1}{T_0} \qquad \forall k
\end{aligned}$$

b. **(5 points)** The FS synthesis of $x(t)$ is

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

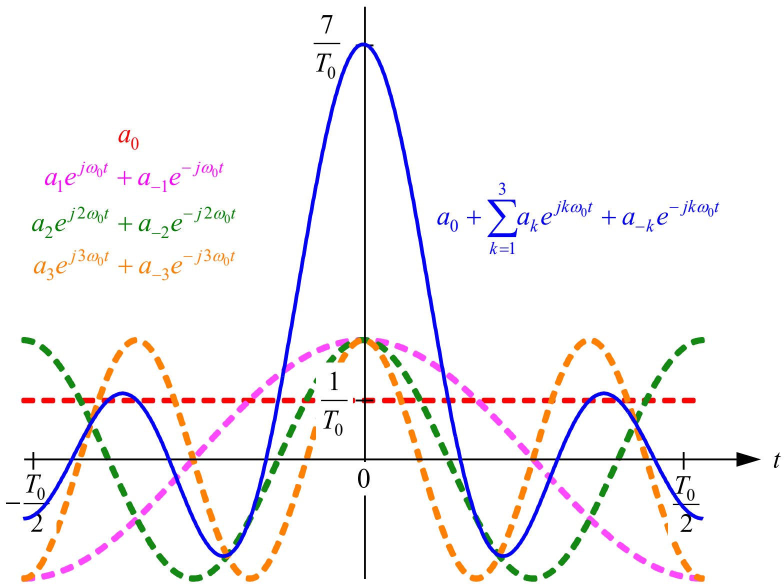
Since $x(t)$ is real, $a_{-k} = a_k^*$, so $a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}$ add up to give a purely real contribution for each $k \neq 0$. Sketch the first few terms in the synthesis, which are

$$\begin{gathered}
a_0 \\
a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t} \\
a_2 e^{j2\omega_0 t} + a_{-2} e^{-j2\omega_0 t} \\
a_3 e^{j3\omega_0 t} + a_{-3} e^{-j3\omega_0 t}
\end{gathered}$$

for $-T_0/2 \le t \le T_0/2$. These are a constant and three cosines. At most values of *t*, the cosines have different amplitudes, some positive and some negative, and tend to cancel out. At what value of *t* do they all add up constructively?

**Solution**

The first few terms in the synthesis and their sum are plotted below. Note that the terms add up constructively at $t = 0$ and destructively at $t \neq 0$.



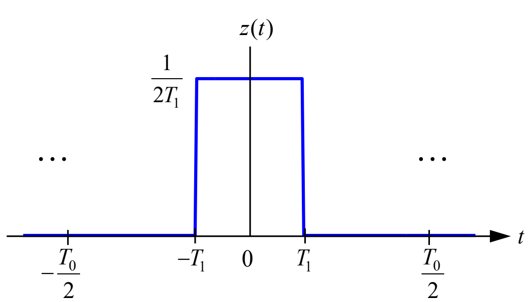

c. **(5 points)** In lecture we calculated the FS coefficients $b_k$ of a periodic rectangular pulse train $y(t)$:

$$y(t) = \sum_{l=-\infty}^{\infty} \Pi\left(\frac{t - lT_0}{2T_1}\right) \xleftrightarrow{FS} b_k = \frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right).$$

Here we scale it by $1/2T_1$ to obtain $z(t)$, which has FS coefficients $c_k$:

$$z(t) = \frac{1}{2T_1} \sum_{l=-\infty}^{\infty} \Pi\left(\frac{t - lT_0}{2T_1}\right) \xleftrightarrow{FS} c_k.$$

$z(t)$ is sketched in one period



Since an impulse can be defined as the limiting case of a narrow, high rectangular pulse of unit area, we note that

$$\lim_{T_1 \to 0} z(t) = x(t).$$

Argue that

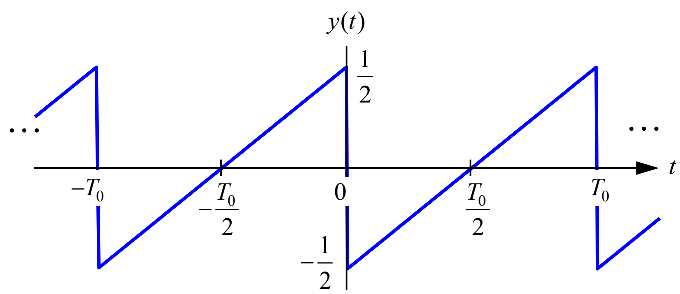
$$\lim_{T_1 \to 0} c_k = a_k.$$

**Solution**

$$z(t) = \frac{1}{2T_1} \sum_{l=-\infty}^{\infty} \Pi\left(\frac{t - lT_0}{2T_1}\right) \xleftrightarrow{FS} c_k = \frac{\omega_0}{2\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right)$$

$$\begin{aligned}
\lim_{T_1 \to 0} c_k &= \lim_{T_1 \to 0} \frac{\omega_0}{2\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) \\
&= \frac{\omega_0}{2\pi} \lim_{T_1 \to 0} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) \\
&= \frac{\omega_0}{2\pi} \cdot 1 \\
&= \frac{1}{T_0} \\
&= a_k \qquad \forall k
\end{aligned}$$

3. **(10 points)** *CT sawtooth wave*. Consider a periodic sawtooth signal $y(t) \xleftrightarrow{FS} b_k$




It can be expressed as

$$y(t) = \frac{t}{T_0} - \frac{1}{2} \qquad 0 \le t \le T_0$$

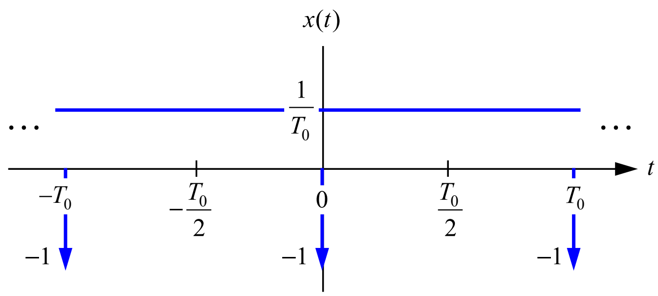
$$y(t + T_0) = y(t) \qquad \forall t$$

and its FS coefficients can be calculated by evaluating the analysis equation:

$$b_k = \frac{1}{T_0} \int_0^{T_0} \left(\frac{t}{t_0} - \frac{1}{2}\right) e^{-jk\omega_0 t} dt.$$

You need not evaluate this integral.

a. **(5 points)** Here we consider an alternate approach. Consider a related periodic signal $x(t) \xleftrightarrow{FS} a_k$ :




$x(t)$ can be expressed as the sum of a constant and a negative impulse train:

$$x(t) = \frac{1}{T_0} - \sum_{l=-\infty}^{\infty} \delta\left(t - lT_0\right).$$

Using linearity and the known FS coefficients of a constant and an impulse train, calculate the FS coefficients of $x(t)$ given by $a_k$.

**Solution**

$$\frac{1}{T_0} \xleftrightarrow{FS} \begin{cases} \frac{1}{T_0} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

$$-\sum_{l=-\infty}^{\infty} \delta\left(t - lT_0\right) \xleftrightarrow{FS} \frac{-1}{T_0} \qquad \forall k$$

Adding the two yields

$$x(t) \xleftrightarrow{FS} a_k = \begin{cases} 0 & k = 0 \\ -\frac{1}{T_0} & k \neq 0 \end{cases}$$

b. **(5 points)** Observe that the two waveforms are related by

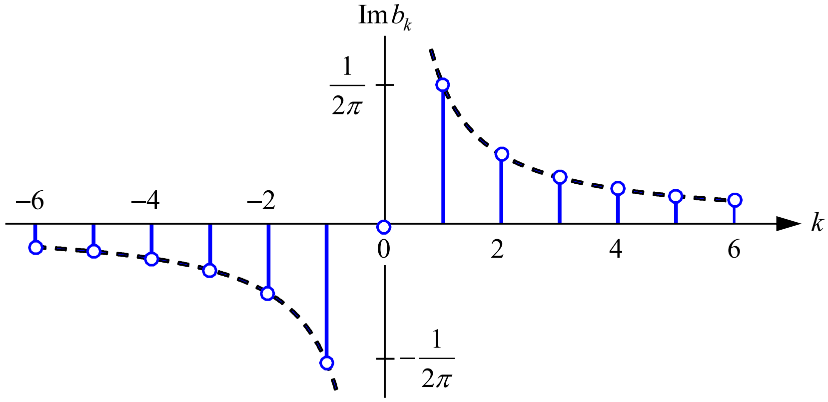
$$y(t) = \int_{-\infty}^{t} x(t')dt'.$$

Refer to the running integration property in the Appendix, Table 1. Is the condition on the FS coefficients of $x(t)$, $a_0 = 0$, satisfied? If so, use the property to compute the FS coefficients of $y(t)$, given by $b_k$. Sketch the real or imaginary part of $b_k$, whichever is nonzero. Identify any symmetry in $y(t)$ and the corresponding symmetry in $b_k$.

**Solution**

Yes, $a_0 = 0$. Note that $b_0 = 0$, since the average value of $y(t)$ is zero. Using the running integration property, for $k \neq 0$ we obtain

$$
\begin{aligned}
b_k &= \frac{1}{jk\omega_0} a_k \qquad k \neq 0 \\
&= -\frac{1}{jk\omega_0 T_0} \qquad k \neq 0 \\
&= \frac{j}{2\pi k} \qquad k \neq 0
\end{aligned}
$$



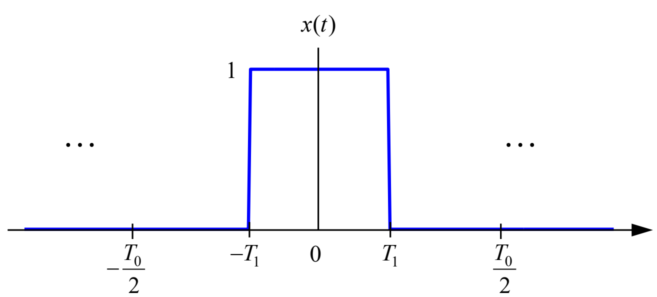

$y(t)$ is real and odd in $t$ $\leftrightarrow$ $b_k$ is imaginary and odd in $k$.

4. **(10 points)** *CT triangular pulse train.* Given two periodic signals $x(t)$ and $y(t)$ with a common period $T_0$, the periodic convolution is defined as

$$z(t) = \int_{T_0} x(t')y(t-t')dt'. \tag{1}$$

This looks like a regular convolution, but the integration over $t'$ is performed over any interval of length $T_0$, instead of over $-\infty < t' < \infty$. Note that the periodic convolution $z(t)$ is periodic in $t$ with period $T_0$, since the only place $t$ appears on the right-hand side of (1) is in $y(t-t')$, which is periodic in $t$.

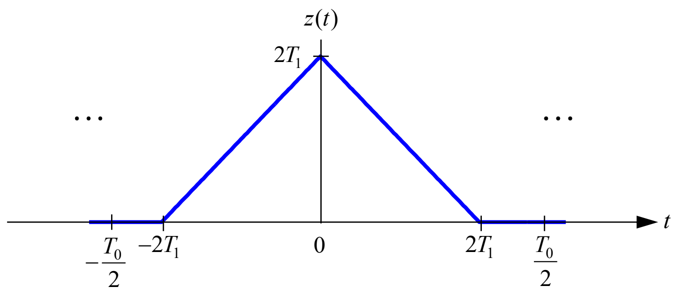
a. **(4 points)** Suppose $x(t) = y(t)$ is the periodic rectangular pulse train shown




Assume $T_1 \leq T_0 / 4$. Sketch the periodic convolution $z(t)$.

**Solution**

$z(t) = \int_{-T_0/2}^{T_0/2} x(t')x(t-t')dt'$ is the periodic triangular pulse train shown.




b. **(6 points)** $z(t)$ has CT FS coefficients $c_k$, which we want to compute. Note that

$$z(t) = \begin{cases} 2T_1\left(1 - \frac{|t|}{2T_1}\right) & |t| < 2T_1 \\ 0 & 2T_1 \le |t| \le \frac{T_0}{2} \end{cases}$$

$$z(t+T_0) = z(t) \qquad \forall t$$

We could compute the $c_k$ using the analysis equation:

$$\begin{aligned} c_k &= \frac{1}{T_0}\int_{T_0} z(t)e^{-jk\omega_0 t}dt \\ &= \frac{2T_1}{T_0}\int_{-2T_1}^{2T_1}\left(1 - \frac{|t|}{2T_1}\right)e^{-jk\omega_0 t}dt. \end{aligned}$$

You do not need to evaluate this integral. Instead, use the periodic convolution property of the CT FS to determine $c_k$.

**Solution**

We know that

$$z(t) = \int_{T_0} x(t')x(t-t')dt'$$

$$x(t) \overset{FS}{\longleftrightarrow} a_k = \frac{\omega_0 T_1}{\pi}\text{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right)$$

Using the periodic convolution property:

$$\begin{aligned} z(t) \overset{FS}{\longleftrightarrow} c_k &= T_0 a_k \cdot a_k \\ &= T_0\left(\frac{\omega_0 T_1}{\pi}\right)^2 \text{sinc}^2\left(\frac{k\omega_0 T_1}{\pi}\right) \\ &= \frac{2\omega_0 T_1^2}{\pi}\text{sinc}^2\left(\frac{k\omega_0 T_1}{\pi}\right) \\ &= \frac{4T_1^2}{T_0}\text{sinc}^2\left(\frac{k\omega_0 T_1}{\pi}\right). \end{aligned}$$

*CT or DT LTI System Analysis*

5. **(8 points)** *Time shift and differentiator*. Consider these LTI systems with input $x(t)$ and output $y(t)$. For each system, by setting $x(t) = e^{j\omega t}$ and $y(t) = H(j\omega)e^{j\omega t}$, find an expression for the frequency response $H(j\omega)$, and sketch the magnitude $|H(j\omega)|$ and phase $\angle H(j\omega)$.

   *a.* **(4 points)** *Time shift*

$$y(t) = x(t - t_0).$$

**Solution**

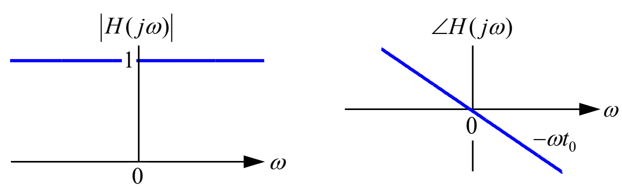
With the substitutions suggested, the input-output relation becomes

$$H(j\omega)e^{j\omega t} = e^{j\omega(t-t_0)},$$

so the frequency response is

$$H(j\omega) = e^{-j\omega t_0},$$

which has magnitude $|H(j\omega)| = 1$ and phase $\angle H(j\omega) = -\omega t_0$. The sketch here assumes $t_0 > 0$ .




   *b.* **(4 points)** *Differentiator*

$$y(t) = \frac{dx}{dt}.$$

**Solution**

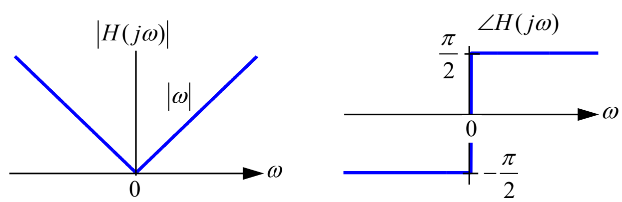
With the substitutions suggested, the input-output relation becomes

$$H(j\omega)e^{j\omega t} = j\omega e^{j\omega t},$$

so the frequency response is

$$H(j\omega) = j\omega,$$

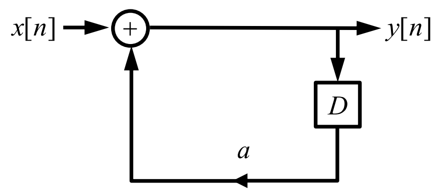
which has magnitude $|H(j\omega)| = |\omega|$ and phase $\angle H(j\omega) = \frac{\pi}{2}\text{sgn}(\omega)$.



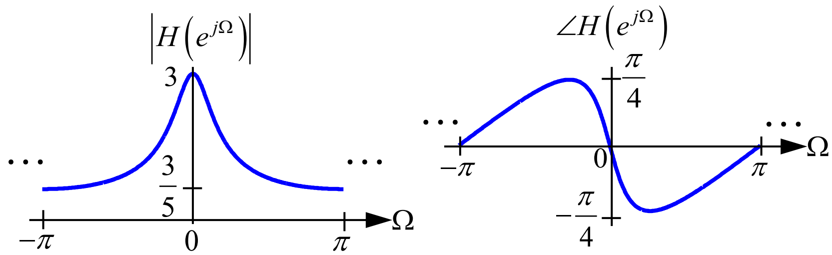
6. *Improving DT infinite impulse response lowpass filter*. In lecture we discussed a first-order DT system described by a difference equation

$$y[n]-ay[n-1]=x[n].$$

and realized as shown.




Here we consider $0<a<1$, so it describes a lowpass filter. Because this IIR filter uses just one shift register with feedback, if we choose $a$ close to 1, it can realize a lowpass filter with a low cutoff frequency using very little hardware. For example, here is a plot of its frequency response magnitude and phase for $a=2/3$.

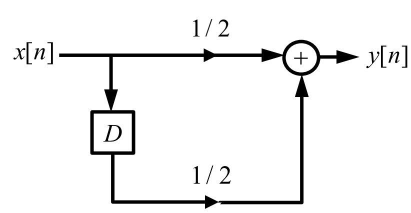



This filter has two obvious deficiencies, however. First, its magnitude response does not go to zero at $\Omega=\pm\pi$. Second, its phase response is not a linear function of frequency, so it causes phase distortion. Here we address the first deficiency.

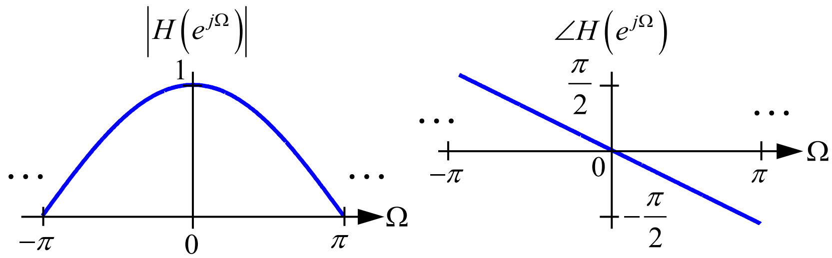
A two-sample moving average filter is described by a difference equation

$$y[n]=\frac{1}{2}\left(x[n]+x[n-1]\right)$$

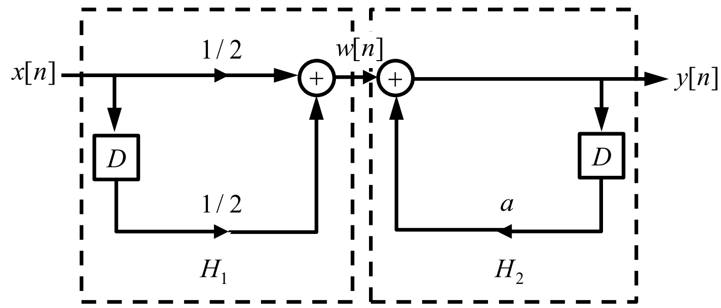
and is realized as shown




Its magnitude response goes to zero at $\Omega=\pm\pi$, as shown. Since its phase is a linear function of $\Omega$, it causes no phase distortion.




But since this FIR filter does not use feedback, achieving a low cutoff frequency would require many more shift registers. We can cascade the two systems as shown.




a. What is the difference equation relating $x[n]$ to $y[n]$? *Hint*: express the intermediate signal $w[n]$ in terms of $x[n]$, express the output $y[n]$ in terms of $w[n]$, and substitute the first relation into the second.

**Solution** The difference equation relating $x[n]$ to $w[n]$ is

$$w[n] = \frac{1}{2}\left(x[n] + x[n-1]\right).$$

The difference equation relating $w[n]$ to $y[n]$ is

$$y[n] - ay[n-1] = w[n].$$

Therefore, the difference equation relating $x[n]$ to $y[n]$ is

$$y[n] - ay[n-1] = \frac{1}{2}\left(x[n] + x[n-1]\right).$$

b. What is the impulse response $h[n]$ of the overall system with input $x[n]$ and output $y[n]$? *Hint*: find $h_1[n]$ such that $w[n] = x[n] * h_1[n]$ and $h_2[n]$ such that $y[n] = w[n] * h_2[n]$. Then $h[n] = h_1[n] * h_2[n]$.

**Solution** We have

$$h_1[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]$$

and

$$h_2[n] = a^n u[n].$$

Therefore, the impulse response $h[n]$ of the overall system is

$$
\begin{aligned}
h[n] &= h_1[n] * h_2[n] \\
&= \left(\frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]\right) * a^n u[n]. \\
&= \frac{1}{2}a^n u[n] + \frac{1}{2}a^{n-1}u[n-1]
\end{aligned}
$$

c. Give an expression for the frequency response $H\left(e^{j\Omega}\right)$ of the overall system. *Hint*: use the difference equation you found in part (a). You can express $H\left(e^{j\Omega}\right)$ in terms of the frequency responses $H_1\left(e^{j\Omega}\right)$ and $H_2\left(e^{j\Omega}\right)$ of the two constituent systems.

**Solution** The frequency response $H_1$ of the two-sample moving average filter is

$$H_1\left(e^{j\Omega}\right) = \frac{1}{2}\left(1 + e^{-j\Omega}\right).$$

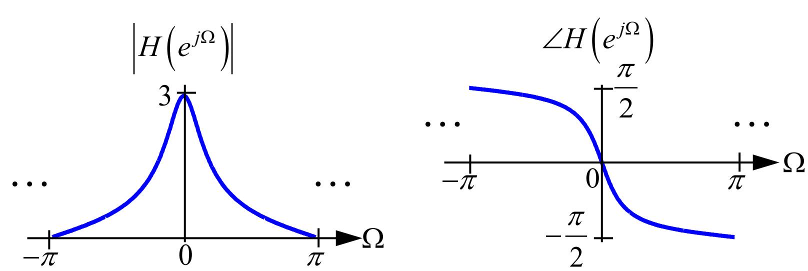
The frequency response $H_2$ of the lowpass filter is

$$H_2\left(e^{j\Omega}\right) = \frac{1}{1 - ae^{-j\Omega}}.$$

The overall frequency response $H\left(e^{j\Omega}\right)$ of the cascaded system is

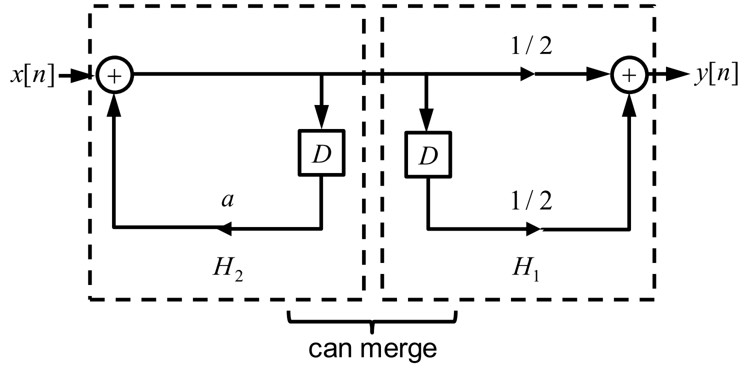
$$
\begin{aligned}
H\left(e^{j\Omega}\right) &= H_1\left(e^{j\Omega}\right) \cdot H_2\left(e^{j\Omega}\right) \\
&= \frac{1}{2}\left(\frac{1 + e^{-j\Omega}}{1 - ae^{-j\Omega}}\right).
\end{aligned}
$$

The overall magnitude response is $\left|H\left(e^{j\Omega}\right)\right| = \left|H_1\left(e^{j\Omega}\right)\right| \cdot \left|H_2\left(e^{j\Omega}\right)\right|$ and the overall phase response is $\angle H\left(e^{j\Omega}\right) = \angle H_1\left(e^{j\Omega}\right) + \angle H_2\left(e^{j\Omega}\right)$.

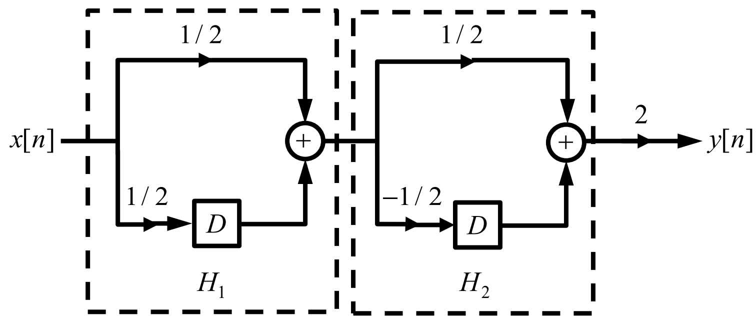



Note that the cascaded system is an improved lowpass IIR filter because its magnitude response goes to zero at $\Omega = \pm\pi$.

*Note*: The constituent systems $H_1$ and $H_2$ are LTI, and therefore commute. By reversing their order and merging the two shift registers, one can realize the overall system using one shift register.




7. *DT finite impulse response filter; filtering periodic DT signals.* We cascade a two-sample moving average filter $H_1$, an edge detector $H_2$, and a constant gain of 2:




a. What is the impulse response of the overall system?

**Solution** The impulse response of the two-sample moving average filter is

$$h_1[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1],$$

and the impulse response of the edge detector is

$$h_2[n] = \frac{1}{2}\delta[n] - \frac{1}{2}\delta[n-1].$$

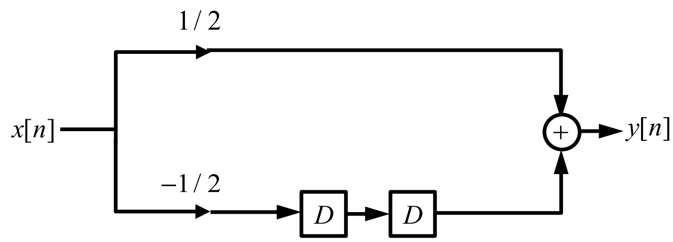
The impulse response of the overall system is

$$\begin{aligned} h[n] &= 2 \cdot h_1[n] * h_2[n] \\ &= 2\left(\frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]\right) * \left(\frac{1}{2}\delta[n] - \frac{1}{2}\delta[n-1]\right) \\ &= \frac{1}{2}\delta[n] - \frac{1}{2}\delta[n-2]. \end{aligned}$$

b. Give an explicit expression for $y[n]$ in terms of $x[n]$.

**Solution** We have $y[n] = h[n] * x[n] = \frac{1}{2}x[n] - \frac{1}{2}x[n-2].$

c. Show that this system gives the same impulse response $h[n]$.




**Solution** The difference equation relating $x[n]$ and $y[n]$ is

$$y[n] = \frac{1}{2}x[n] - \frac{1}{2}x[n-2],$$

which means that the impulse response $h[n]$ is given by

$$h[n] = \frac{1}{2}\delta[n] - \frac{1}{2}\delta[n-2].$$

This is the same impulse response as in part (a).

d. Give an expression for the frequency response of the system, $H\left(e^{j\Omega}\right)$. Sketch its magnitude and phase. Is it a lowpass, highpass or bandpass filter?

**Solution** The frequency response of this system is

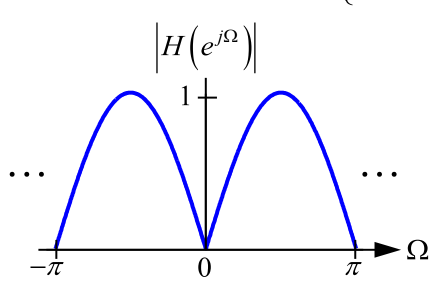
$$\begin{aligned} H\left(e^{j\Omega}\right) &= \frac{1}{2}\left(1 - e^{-j2\Omega}\right) \\ &= je^{-j\Omega}\left(\frac{e^{j\Omega} - e^{-j\Omega}}{2j}\right) \\ &= je^{-j\Omega}\sin(\Omega). \end{aligned}$$

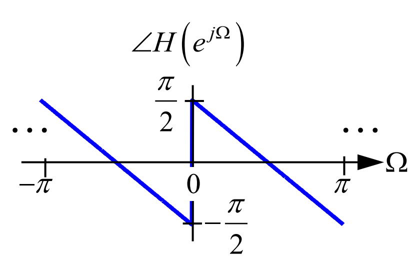
Therefore, its magnitude response is

$$\begin{aligned} \left|H\left(e^{j\Omega}\right)\right| &= |j|\left|e^{-j\Omega}\right|\left|\sin(\Omega)\right| \\ &= \left|\sin(\Omega)\right|, \end{aligned}$$

and its phase response is

$$\begin{aligned} \angle H\left(e^{j\Omega}\right) &= \angle j + \angle e^{-j\Omega} + \angle \sin(\Omega) \\ &= \begin{cases} \pi/2 - \Omega & \sin(\Omega) \geq 0 \\ -\pi/2 - \Omega & \sin(\Omega) < 0 \end{cases} \end{aligned}$$



This is a *bandpass filter*.

As a check, we consider the frequency responses of the constituent systems: a gain of 2, the moving average $H_1$, and the edge detector $H_2$ (for the latter two, see *EE 102A Course Reader*, Appendix, pages 297-299). The frequency response of the overall system should be the product:

$$\begin{aligned} H\left(e^{j\Omega}\right) &= 2 \cdot H_1\left(e^{j\Omega}\right) \cdot H_2\left(e^{j\Omega}\right) \\ &= 2 \cdot e^{-j\frac{\Omega}{2}} \cos\left(\frac{\Omega}{2}\right) \cdot je^{-j\frac{\Omega}{2}} \sin\left(\frac{\Omega}{2}\right). \\ &= je^{-j\Omega} \sin\left(\Omega\right) \end{aligned}$$

This agrees with the result calculated just above. Note also that the magnitude and phase of the overall system should be related to those of the constituent systems by

$$\left|H\left(e^{j\Omega}\right)\right| = |2| \cdot \left|H_1\left(e^{j\Omega}\right)\right| \cdot \left|H_2\left(e^{j\Omega}\right)\right|$$

$$\angle H\left(e^{j\Omega}\right) = \angle 2 + \angle H_1\left(e^{j\Omega}\right) + \angle H_2\left(e^{j\Omega}\right),$$

which we can verify by comparing the graphs just above to those for $H_1$ and $H_2$ given in the Appendix.