Stanford University
EE 102A: Signal Processing and Linear Systems I
Summer 2022
Instructor: Ethan M. Liang

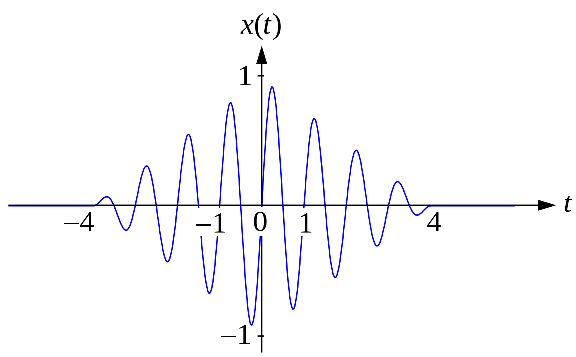
Homework 6 Solutions, due Friday, August 5

Unless noted otherwise, all references are to *EE 102A Course Reader*, Appendix, Tables 3 and 4.

1. **(8 points)** *Fourier transforms.* Sketch each signal $x(t)$. Using only tables and properties, obtain an expression for its Fourier transform $X(j\omega)$. Explain how the properties of $x(t)$ (real or imaginary, odd or even, etc.) are reflected in those of $X(j\omega)$.

   a. **(4 points)** $x(t)=\Lambda\left(\frac{t}{4}\right)\sin\left(2\pi t\right)$. Sketch the real or imaginary part of $X(j\omega)$, whichever is nonzero. *Hint*: use the known FT of the sine function and use the FT multiplication property.

   Solution The signal $x(t)$ is sketched below.



Using the CT FT pair derived in class

$$\Lambda\left(\frac{t}{2T_1}\right)\leftrightarrow 2T_1\text{sinc}^2\left(\frac{\omega T_1}{\pi}\right)$$

with $T_1=2$ gives

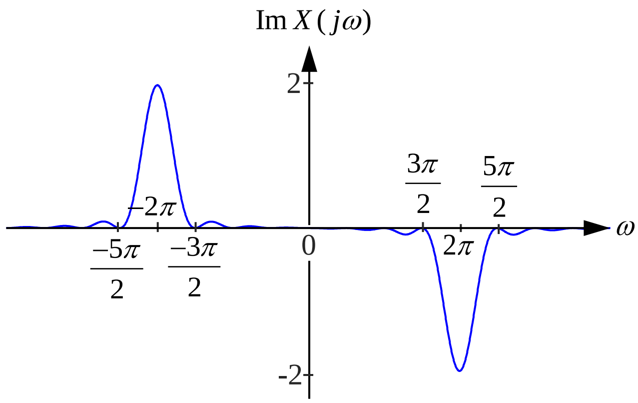
$$\Lambda\left(\frac{t}{4}\right)\leftrightarrow 4\text{sinc}^2\left(\frac{2\omega}{\pi}\right).$$

Also, the CT FT pair for the sine function is

$$\sin\left(2\pi t\right)\leftrightarrow\frac{\pi}{j}\left[\delta(\omega-2\pi)-\delta(\omega+2\pi)\right].$$

Using the CT FT multiplication property, we have

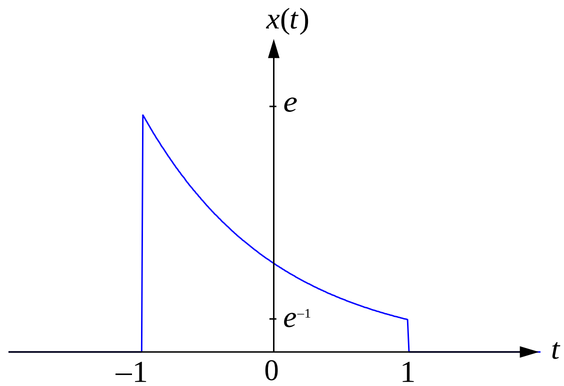
$$X(j\omega)=\frac{1}{2\pi}\left(4\operatorname{sinc}^2\left(\frac{2\omega}{\pi}\right)*\frac{\pi}{j}\left[\delta(\omega-2\pi)-\delta(\omega+2\pi)\right]\right)$$
$$=\frac{2}{j}\operatorname{sinc}^2\left(\frac{2(\omega-2\pi)}{\pi}\right)-\frac{2}{j}\operatorname{sinc}^2\left(\frac{2(\omega+2\pi)}{\pi}\right).$$



Since $x(t)$ is real and odd, $X(j\omega)$ is imaginary and odd.

b. **(4 points)** $x(t)=e^{-t}\left[u(t+1)-u(t-1)\right]$. You do not need to sketch $X(j\omega)$. *Hint*: write a term like $e^{-t}u(t+1)$ as $e\cdot e^{-(t+1)}u(t+1)$.

Solution The signal $x(t)$ is sketched here.



It can be expressed as

$$x(t)=e^{-t}\left[u(t+1)-u(t-1)\right]$$
$$=e\cdot e^{-(t+1)}u(t+1)-\frac{1}{e}\cdot e^{-(t-1)}u(t-1)$$
$$=e\cdot x_1(t+1)-\frac{1}{e}\cdot x_1(t-1)$$

where $x_1(t)=e^{-t}u(t)$. We showed in class that the right-sided real exponential with $a=1$ has CT FT

$$X_1(j\omega)=\frac{1}{1+j\omega}.$$

Therefore, using the time shifting property, we have

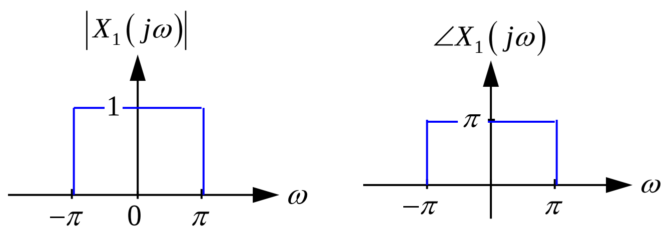
$$\begin{aligned} X(j\omega) &= e \cdot e^{j\omega} X_1(j\omega) - \frac{1}{e} e^{-j\omega} X_1(j\omega) \\ &= \frac{e^{j\omega+1} - e^{-j\omega-1}}{1+j\omega}. \end{aligned}$$

Since $x(t)$ is real, we expect that $X(j\omega)$ has conjugate symmetry. We can verify that:

$$X(-j\omega) = \frac{e^{-j\omega+1} - e^{j\omega-1}}{1-j\omega} = X^*(j\omega).$$

2. (**6 points)** *Inverse Fourier transforms.* The Fourier transforms given have identical magnitudes but different phases. Obtain an expression for each inverse Fourier transform using only tables and properties. Explain how the properties of each time signal (real or imaginary, odd or even, etc.) are reflected in those of its Fourier transform. *Hint*: express each transform in terms of simple functions, such as rectangular pulses, multiplied by constants or frequency-dependent phase factors.

   a. (**3 points)**

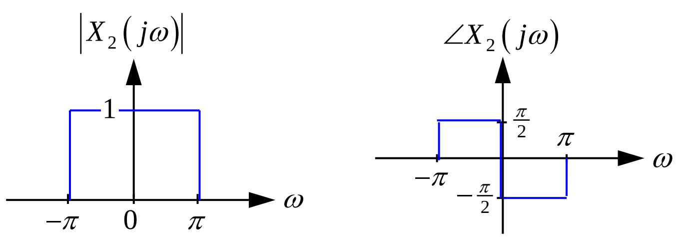


Solution Using Table 4, we have

$$X_1(j\omega) = -\Pi\left(\frac{\omega}{2\pi}\right) \leftrightarrow x_1(t) = -\text{sinc}(t).$$

$X_1(j\omega)$ is real and even, and so $x_1(t)$ is real and even.

   b. (**3 points)**



Solution Let $Y(j\omega) = \Pi\left(\frac{\omega}{\pi}\right) \leftrightarrow y(t) = \frac{1}{2}\text{sinc}\left(\frac{t}{2}\right)$. Then we have

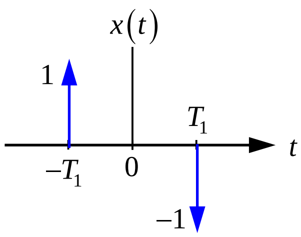
$$X_2(j\omega) = jY\left(j\left(\omega + \frac{\pi}{2}\right)\right) - jY\left(j\left(\omega - \frac{\pi}{2}\right)\right).$$

Using the linearity and frequency shifting properties, we have

$$\begin{aligned} x_2(t) &= je^{-j\frac{\pi}{2}t}y(t) - je^{j\frac{\pi}{2}t}y(t) \\ &= 2y(t)\left(\frac{e^{j\frac{\pi}{2}t} - e^{-j\frac{\pi}{2}t}}{2j}\right) \\ &= 2y(t)\sin\left(\frac{\pi}{2}t\right) \\ &= \operatorname{sinc}\left(\frac{t}{2}\right)\sin\left(\frac{\pi}{2}t\right). \end{aligned}$$

$X_2(j\omega)$ is imaginary and odd, and so $x_2(t)$ is real and odd.

3. (**9 points)** *Fourier transform integration property*. A signal $x(t)$ is a sum of two scaled, shifted impulses.

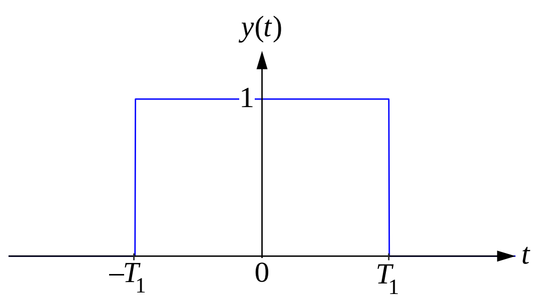


a. (**2 points)** Obtain an expression for $X(j\omega)$, the Fourier transform of $x(t)$.

Solution We have $x(t) = \delta(t+T_1) - \delta(t-T_1) \leftrightarrow X(j\omega) = e^{j\omega T_1} - e^{-j\omega T_1}$.

b. (**3 points)** Now consider $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$. Sketch $y(t)$.

Solution



c. (**4 points)** Using the Fourier transform integration property, obtain an expression for $Y(j\omega)$, the Fourier transform of $y(t)$. Put $Y(j\omega)$ in the standard form found in the table.

Solution

Using the Fourier transform integration property, we have

$$
\begin{aligned}
Y(j\omega) &= \frac{1}{j\omega} X(j\omega) \\
&= \frac{1}{j\omega}\left(e^{j\omega T_1} - e^{-j\omega T_1}\right) \\
&= 2T_1 \frac{1}{\omega T_1}\left(\frac{e^{j\omega T_1} - e^{-j\omega T_1}}{2j}\right) \\
&= 2T_1 \frac{\sin\left(\pi \frac{\omega T_1}{\pi}\right)}{\pi \frac{\omega T_1}{\pi}} \\
&= 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)
\end{aligned}
$$

as desired. This is the same form as the CT FT pair $\Pi\left(\frac{t}{2T_1}\right) \leftrightarrow 2T_1\operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)$ in Table 4.

4. **(14 points)** *Convolution property and relation between the Fourier series and the Fourier transform of one period.* We are given two rectangular pulses:

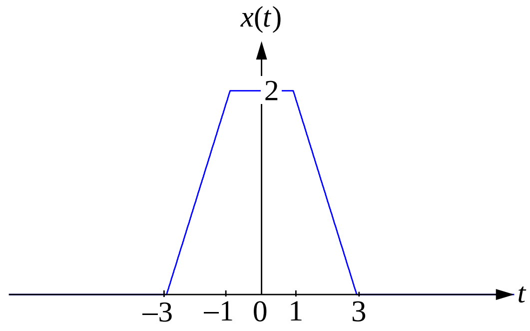
$$x_1(t) = \Pi\left(\frac{t}{2T_1}\right) \text{ and } x_2(t) = \Pi\left(\frac{t}{2T_2}\right).$$

Consider the convolution between them:

$$x(t) = x_1(t) * x_2(t).$$

a. **(3 points)** Sketch $x(t)$, assuming $T_1 = 1$ and $T_2 = 2$.

Solution



b. **(3 points)** Obtain an expression for $X(j\omega)$, the Fourier transform of $x(t)$, assuming general values of T_1 and T_2. Sketch $X(j\omega)$, assuming $T_1 = 1$ and $T_2 = 2$ (sketch the real or imaginary part, whichever is nonzero).

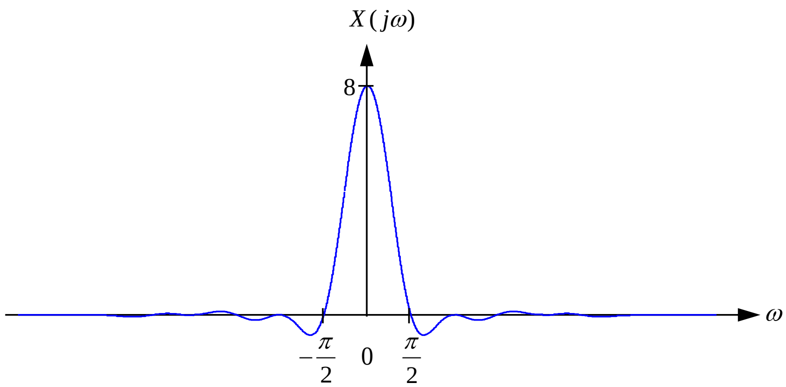
Solution We have

$$X_1(j\omega) = 2T_1 \text{sinc}\left(\frac{\omega T_1}{\pi}\right),$$

$$X_2(j\omega) = 2T_2 \text{sinc}\left(\frac{\omega T_2}{\pi}\right).$$

From the convolution property, we get

$$\begin{aligned} X(j\omega) &= X_1(j\omega) X_2(j\omega) \\ &= 4T_1T_2 \text{sinc}\left(\frac{\omega T_1}{\pi}\right) \text{sinc}\left(\frac{\omega T_2}{\pi}\right). \end{aligned}$$



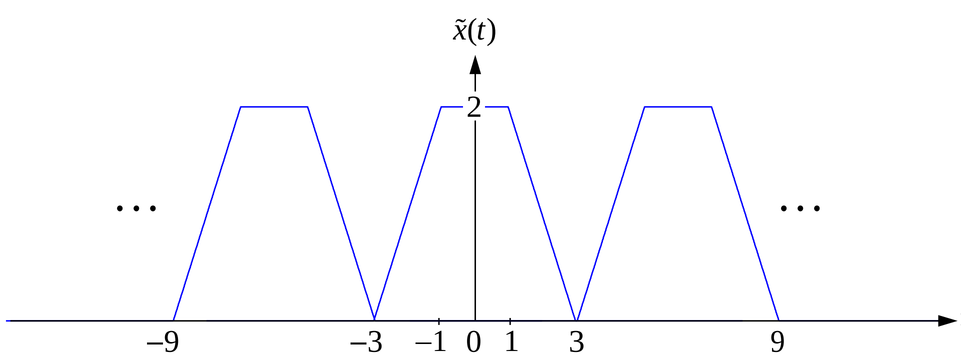
$x(t)$ is real and even, so $X(j\omega)$ is real and even.

c. **(4 points)** Now consider a periodic signal

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} x(t - nT_0),$$

assuming $T_0 \geq 2(T_1 + T_2)$. Sketch two or three periods of $\tilde{x}(t)$, assuming $T_1 = 1$ $T_2 = 2$ and $T_0 = 6$.

Solution



d. **(4 points)** By sampling $X(j\omega)$, obtain an expression for the Fourier series coefficients of $\tilde{x}(t)$, given by a_k. Sketch the a_k vs. k, assuming $T_1 = 1$ $T_2 = 2$ and $T_0 = 6$.

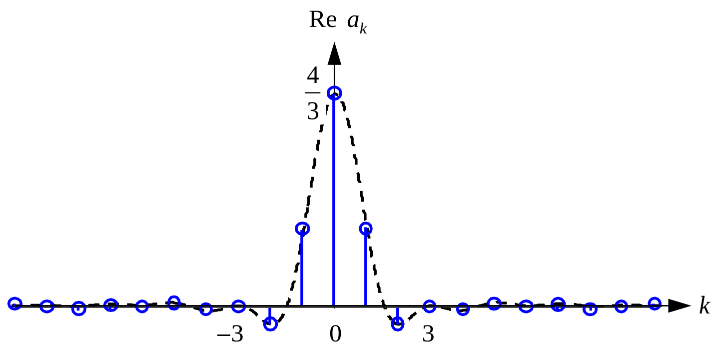
Solution

As shown in the lecture, sampling $X(j\omega)$ at $\omega = k\omega_0 = 2\pi k / T_0 = \pi k / 3$ yields the Fourier series coefficients of $\tilde{x}(t)$, the a_k, scaled by $T_0 = 6$:

$$\begin{aligned} X(j\omega)\big|_{\omega=k\omega_0} &= 4T_1T_2 \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right)\operatorname{sinc}\left(\frac{k\omega_0 T_2}{\pi}\right) \\ &= 8\operatorname{sinc}\left(\frac{k}{3}\right)\operatorname{sinc}\left(\frac{k2}{3}\right) \\ &= 6a_k. \end{aligned}$$

Therefore, we have

$$a_k = \frac{4}{3}\operatorname{sinc}\left(\frac{k}{3}\right)\operatorname{sinc}\left(\frac{k2}{3}\right).$$



Since $\tilde{x}(t)$ is real and even, the Fourier series coefficients are real and even.

5. (**11 points)** *Frequency-shift property and Parseval's Identity*. We are given two signals:

$$x_1(t) = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right) \text{ and } x_2(t) = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)\sin(\omega_0 t), \text{ assuming } \omega_0 \geq 2W .$$

a. (**4 points**) Sketch their Fourier transforms, $X_1(j\omega)$ and $X_2(j\omega)$, assuming $W = 2\pi$ and $\omega_0 = 4\pi$.

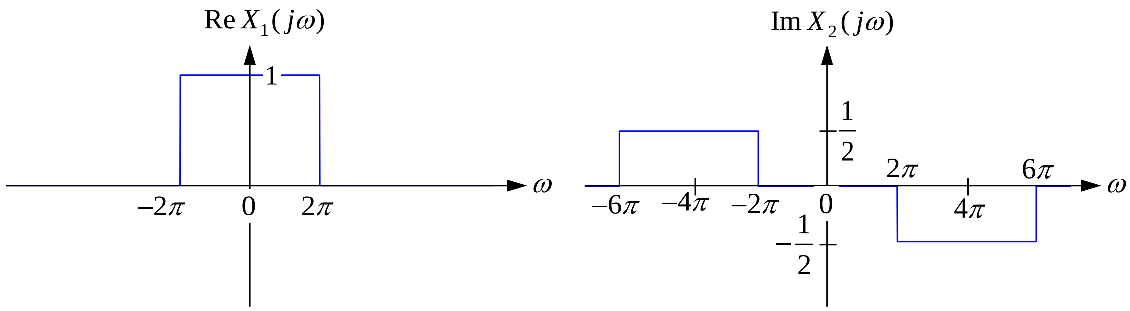
Solution From Table 4, we have

$$X_1(j\omega) = \Pi\left(\frac{\omega}{2W}\right).$$

Since $x_2(t) = x_1(t)\sin(\omega_0 t)$, and $\sin(\omega_0 t) \leftrightarrow -j\pi\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$, from the multiplication property of the CT FS, we have

$$\begin{aligned} X_2(j\omega) &= \frac{-j}{2}X_1\left(j(\omega-\omega_0)\right) + \frac{j}{2}X_1\left(j(\omega+\omega_0)\right) \\ &= \frac{-j}{2}\Pi\left(\frac{\omega-\omega_0}{2W}\right) + \frac{j}{2}\Pi\left(\frac{\omega+\omega_0}{2W}\right). \end{aligned}$$

The plots below assume $W = 2\pi$ and $\omega_0 = 4\pi$.



b. **(4 points)** Compute the energies $E_{x_1} = \int_{-\infty}^{\infty} |x_1(t)|^2 dt$ and $E_{x_2} = \int_{-\infty}^{\infty} |x_2(t)|^2 dt$.

Solution Using Parseval's identity for the energy, we have

$$\begin{aligned} E_{x_1} = \int_{-\infty}^{\infty} |x_1(t)|^2 dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_1(j\omega)|^2 d\omega \\ &= \frac{1}{2\pi} \cdot 1^2 \cdot 2W \\ &= \frac{W}{\pi}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} E_{x_2} = \int_{-\infty}^{\infty} |x_2(t)|^2 dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_2(j\omega)|^2 d\omega \\ &= \frac{1}{2\pi} \cdot \left[\left(\frac{1}{2}\right)^2 \cdot 2W + \left(\frac{1}{2}\right)^2 \cdot 2W \right] \\ &= \frac{W}{2\pi}. \end{aligned}$$

c. **(3 points)** Compute the inner product $\langle x_1(t), x_2(t) \rangle = \int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt$.

Solution Using Parseval's identity for the inner product, we have

$$\begin{aligned} \int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(j\omega) X_2^*(j\omega) d\omega \\ &= 0, \end{aligned}$$

where the last equality holds because $X_1(j\omega)$ and $X_2(j\omega)$ do not overlap in frequency when $\omega_0 > 2W$.

CT LTI System Analysis

6. (**9 points)** *Ideal filter*. A filter has an impulse response

$$h(t)=\frac{W}{\pi}\operatorname{sinc}\left(\frac{W(t-t_0)}{\pi}\right)\cos\left(\omega_0(t-t_0)\right), \text{ assuming } \omega_0 \geq W.$$

a. (**4 points)** Obtain an expression for the frequency response $H(j\omega)$. Sketch the magnitude and phase, $|H(j\omega)|$ and $\angle H(j\omega)$, assuming $W=\frac{3}{4}\pi$, $\omega_0=\pi$ and $t_0=1$.

Solution First consider the signal

$$\begin{aligned} g(t) &= \underbrace{\frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)}_{g_1(t)}\underbrace{\cos(\omega_0 t)}_{g_2(t)} \\ &= g_1(t)g_2(t), \end{aligned}$$

where

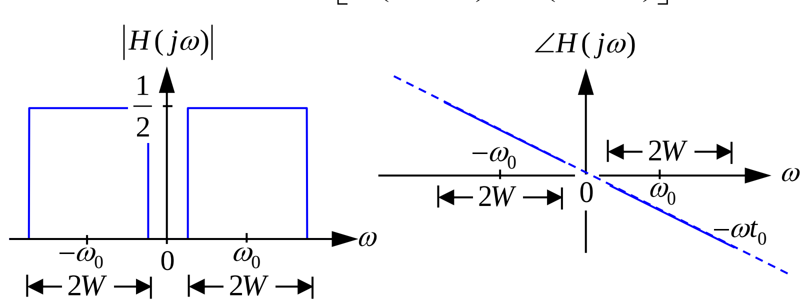
$$g_1(t)=\frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right) \leftrightarrow G_1(j\omega)=\Pi\left(\frac{\omega}{2W}\right),$$
$$g_2(t)=\cos(\omega_0 t) \leftrightarrow G_2(j\omega)=\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right].$$

Then, using the CT FT multiplication property, we have

$$\begin{aligned} G(j\omega) &= \frac{1}{2\pi}G_1(j\omega)*G_2(j\omega) \\ &= \frac{1}{2}\left[\Pi\left(\frac{\omega-\omega_0}{2W}\right)+\Pi\left(\frac{\omega+\omega_0}{2W}\right)\right]. \end{aligned}$$

Since $h(t)=g(t-t_0)$, the time-shifting property of the CT FT gives us

$$\begin{aligned} H(j\omega) &= e^{-j\omega t_0}G(j\omega) \\ &= \frac{1}{2}e^{-j\omega t_0}\left[\Pi\left(\frac{\omega-\omega_0}{2W}\right)+\Pi\left(\frac{\omega+\omega_0}{2W}\right)\right]. \end{aligned}$$



b. (**1 point)** What kind of filter is this (lowpass, highpass, etc.)?

Solution This is a bandpass filter.

c. (**4 points)** We input a signal

$$x(t)=1+2\sin(t)+\cos(2\pi t).$$

Obtain an expression for the output signal $y(t)$, assuming the same filter parameters as in part (a).

Solution We can use the result derived in HW 5, Problem 1: If we input a cosine $x(t) = \cos(\omega_0 t)$ into the filter, then the output is a cosine at the same frequency:

$$y(t) = |H(j\omega_0)| \cos(\omega_0 t + \angle H(j\omega_0)).$$

If we input a sine $x(t) = \sin(\omega_0 t)$ into the filter, the output is a sine at the same frequency:

$$y(t) = |H(j\omega_0)| \sin(\omega_0 t + \angle H(j\omega_0)).$$

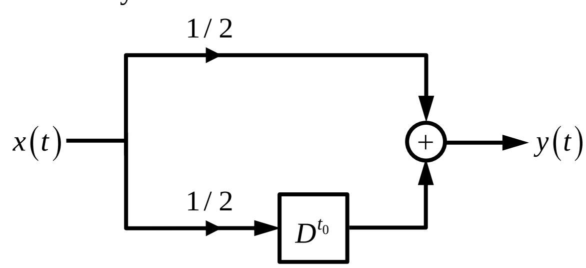
Therefore, using linearity, we get

$$y(t) = |H(j0)| + 2|H(1j)| \sin(t - t_0) + |H(2\pi j)| \cos(2\pi(t - t_0)).$$

Note that the d.c., sine or cosine terms disappear from the output if their respective frequencies do not lie in the passband of the filter. For $W = \frac{3}{4}\pi$, $\omega_0 = \pi$ and $t_0 = 1$, the filter passband is the range $\omega = \left(\frac{-7\pi}{4}, \frac{-\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{7\pi}{4}\right)$. The DC component $|H(j0)|$ disappears from the output because $\omega = 0$ is not contained in the passband. The cosine component $|H(2\pi j)| \cos(2\pi(t - t_0))$ also disappears from the output because $\omega = 2\pi$ lies outside the passband. Only the sine component $2|H(1j)| \sin(t - t_0)$ remains because $\omega = 1$ lies inside the passband. Since $|H(1j)| = \frac{1}{2}$, we have

$$y(t) = \sin(t - 1).$$

7. (**10 points**) *Delay-and-add system*. A system splits an input signal $x(t)$ into two copies, each scaled by 1/2, delays one copy by t_0, and combines the two copies to obtain an output signal $y(t)$. *Hint*: this is the CT analogue of a DT system we studied in lecture.



a. (**2 points**) Obtain an expression for the impulse response $h(t)$.

Solution The impulse response is

$$h(t) = \frac{1}{2}\left(\delta(t) + \delta(t - t_0)\right).$$

This system is the CT analogue of the DT two-sample moving average filter (see *EE 102A Course Reader*, Appendix, pages 73-74, 131-132, 301-302).

b. (**4 points)** Obtain an expression for the frequency response $H(j\omega)$. Sketch the magnitude and phase $|H(j\omega)|$ and $\angle H(j\omega)$, assuming a general value of the delay t_0.

Solution We have

$$H(j\omega)=\frac{1}{2}\left(1+e^{-j\omega t_0}\right).$$

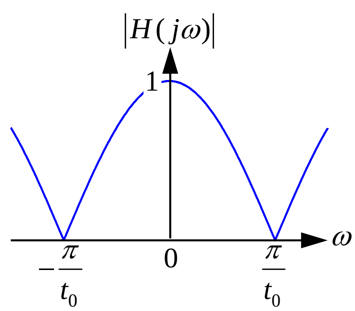
The magnitude response is

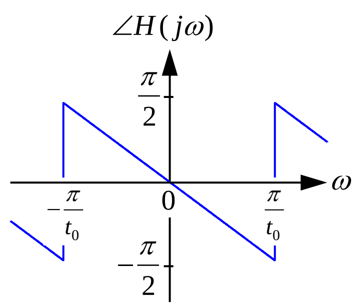
$$\begin{aligned}|H(j\omega)| &= \frac{1}{2}\left|1+e^{-j\omega t_0}\right| \\ &= \left|e^{-j\omega t_0/2}\left(\frac{e^{j\omega t_0/2}+e^{-j\omega t_0/2}}{2}\right)\right| \\ &= \left|\cos\left(\frac{\omega t_0}{2}\right)\right|.\end{aligned}$$

The phase response is

$$\begin{aligned}\angle H(j\omega) &= \angle e^{-j\omega t_0/2}+\angle\cos\left(\frac{\omega t_0}{2}\right) \\ &= \begin{cases}-\omega t_0/2 & \cos(\omega t_0/2)\geq 0 \\ -\omega t_0/2+\pi & \cos(\omega t_0/2)<0\end{cases}.\end{aligned}$$

The plots below assume $t_0>0$.





Note the similarity of the magnitude and phase plots of this CT system to those of the DT two-sample moving average.

c. (**4 points)** An input signal is $x(t)=\sin(\omega_0 t)$. Specify all values of ω_0 such that the output is $y(t)=0$, assuming a general value of the delay t_0.

Solution As derived in Homework 5 Problem 1, we know that a cosine or sine input at frequency ω_0 gets scaled by the magnitude $|H(j\omega_0)|$ and phase shifted by the phase $\angle H(j\omega_0)$:

$$\begin{aligned}y(t) &= |H(j\omega_0)|\sin\left(\omega_0 t+\angle H(j\omega_0)\right) \\ &= \left|\cos\left(\frac{\omega_0 t_0}{2}\right)\right|\sin\left(\omega_0 t+\angle H(j\omega_0)\right).\end{aligned}$$

Therefore, for the output $y(t)=0$, we require

$$\cos\left(\frac{\omega_0 t_0}{2}\right)=0,$$

which yields

$$\omega_0 = \frac{(2k+1)\pi}{t_0}\text{, } k \text{ integer.}$$

Note that the input signal destructively interferes with its delayed version at the output for these values of ω_0:

$$\frac{1}{2}\Big(\sin(\omega_0 t) + \sin(\omega_0 (t - t_0))\Big) = \frac{1}{2}\Big(\sin(\omega_0 t) + \sin\big(\omega_0 t - (2k+1)\pi\big)\Big) = \frac{1}{2}\Big(\sin(\omega_0 t) - \sin(\omega_0 t)\Big) = 0.$$

8. (**9 points)** *Critically damped second-order lowpass filter*. We discuss a second-order lowpass filter in the *EE 102A Course Reader*, Chapter 4, pages 183-186. Assuming a natural frequency ω_n and a damping constant $\zeta = 1$ (critical damping), the impulse and frequency responses are given by

$$h(t) = \omega_n^2 t e^{-\omega_n t} u(t) \overset{F}{\leftrightarrow} H(j\omega) = \frac{1}{\left(1 + j\dfrac{\omega}{\omega_n}\right)^2} = \frac{\omega_n^2}{(j\omega)^2 + 2\omega_n(j\omega) + \omega_n^2}. \tag{1}$$

We derive the CTFT pair (1) in this problem. To facilitate using earlier results, we make a substitution $\omega_n \to 1/\tau$, so (1) becomes

$$h(t) = \frac{t}{\tau^2} e^{-\frac{t}{\tau}} u(t) \overset{F}{\leftrightarrow} H(j\omega) = \frac{1}{(1 + j\omega\tau)^2}. \tag{1'}$$

a. (**4 points)** Use the CTFT pair

$$x(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t) \overset{F}{\leftrightarrow} X(j\omega) = \frac{1}{1 + j\omega\tau} \tag{2}$$

and the CTFT differentiation-in-frequency property to prove (1').

Solution Applying the differentiation-in-frequency property

$$t x(t) \overset{F}{\leftrightarrow} j\frac{dX(j\omega)}{d\omega}$$

to (2) yields

$$\frac{t}{\tau} e^{-\frac{t}{\tau}} u(t) \overset{F}{\leftrightarrow} j\frac{-j\tau}{(1 + j\omega\tau)^2} = \frac{\tau}{(1 + j\omega\tau)^2},$$

and dividing both sides by τ yields the desired result:

$$\frac{t}{\tau^2} e^{-\frac{t}{\tau}} u(t) \overset{F}{\leftrightarrow} \frac{1}{(1 + j\omega\tau)^2}. \qquad \text{(1').}$$

b. (**5 points)** In Homework 3, we studied the cascade of two CT first-order lowpass filters with time constants τ_1 and τ_2, $\tau_1 \neq \tau_2$, and derived the resulting impulse response

$$\frac{1}{\tau_1}e^{-\frac{t}{\tau_1}}u(t)*\frac{1}{\tau_2}e^{-\frac{t}{\tau_2}}u(t)=\frac{1}{\tau_1-\tau_2}\left(e^{-\frac{t}{\tau_1}}-e^{-\frac{t}{\tau_2}}\right)u(t)\,.$$

Here we study the case $\tau_1=\tau_2=\tau$. Use (1'), (2) and the CTFT convolution property to show that the cascade has an impulse response

$$\frac{1}{\tau}e^{-\frac{t}{\tau}}u(t)*\frac{1}{\tau}e^{-\frac{t}{\tau}}u(t)=\frac{t}{\tau^2}e^{-\frac{t}{\tau}}u(t). \tag{3}$$

This shows that a critically damped second-order lowpass filter is equivalent to a cascade of two identical first-order lowpass filters.

Solution Applying the CTFT convolution property and (2), we have

$$x(t)*x(t)\overset{F}{\leftrightarrow}X(j\omega)\cdot X(j\omega)$$

or

$$\frac{1}{\tau}e^{-\frac{t}{\tau}}u(t)*\frac{1}{\tau}e^{-\frac{t}{\tau}}u(t)\overset{F}{\leftrightarrow}\frac{1}{(1+j\omega\tau)^2}\,. \tag{4}$$

Recall from above

$$\frac{t}{\tau^2}e^{-\frac{t}{\tau}}u(t)\overset{F}{\leftrightarrow}\frac{1}{(1+j\omega\tau)^2}\,. \tag{1'}$$

Since the right-hand sides of (1') and (4) are equal, the left-hand sides must be equal, which yields the result we were asked to prove:

$$\frac{1}{\tau}e^{-\frac{t}{\tau}}u(t)*\frac{1}{\tau}e^{-\frac{t}{\tau}}u(t)=\frac{t}{\tau^2}e^{-\frac{t}{\tau}}u(t). \tag{3}$$

9. (6 points) *CT FT symmetry properties*. Consider a real, right-sided signal $x(t)$, which satisfies $x(t)=0$, $t \le 0$ (note that $x(t)=0$ at $t=0$). It has a Fourier transform $X(j\omega)$. We are given the real part of its Fourier transform, $\mathrm{Re}\big(X(j\omega)\big)$. Explain how we can compute $x(t)$. *Hint*: consider decomposing $x(t)$ into even and odd parts.

Solution Following the hint, we can express any signal $x(t)$ as a sum of even and odd parts:

$$x(t) = x_e(t) + x_o(t),$$

where

$$x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right] \quad \text{and} \quad x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right].$$

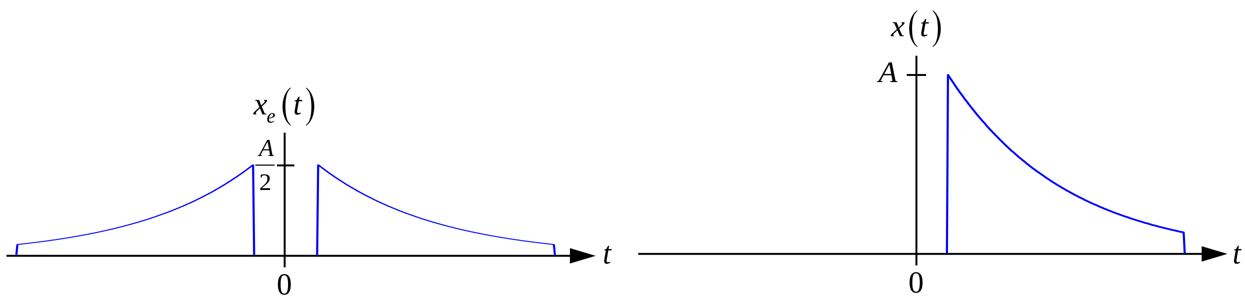
Note that for a real $x(t)$ with conjugate-symmetric Fourier transform $X(j\omega)$, the Fourier transform of the even part $x_e(t)$ is the real part of $X(j\omega)$:

$$\begin{aligned} x_e(t) &\leftrightarrow \frac{1}{2}\left[X(j\omega) + X(-j\omega)\right] \\ &= \frac{1}{2}\left[X(j\omega) + X^*(j\omega)\right]. \\ &= \mathrm{Re}\left[X(j\omega)\right] \end{aligned}$$

Thus, the inverse Fourier transform of $\mathrm{Re}\left[X(j\omega)\right]$ yields $x_e(t)$:

$$x_e(t) = F^{-1}\left\{\mathrm{Re}\left[X(j\omega)\right]\right\},$$

which is the signal shown on the left in the figure below.



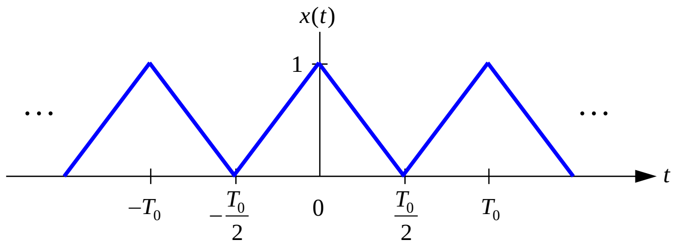
Since we know that $x(t)=0$, $t \le 0$, we can obtain $x(t)$ by taking twice the portion of $x_e(t)$ over positive time *t*:

$$\begin{aligned} x(t) &= \begin{cases} 2F^{-1}\left\{\mathrm{Re}\left[X(j\omega)\right]\right\} & t > 0 \\ 0 & t < 0. \end{cases} \\ &= 2F^{-1}\left\{\mathrm{Re}\left[X(j\omega)\right]\right\} \cdot u(t) \end{aligned}$$

Either answer given above is considered correct.

10. (**11 points**) *Convolution or multiplication of periodic and aperiodic signals*.

a. **(3 points)** $x(t)$ is the periodic triangular pulse train shown.



Sketch $X(j\omega)$, the CT FT of $x(t)$. *Hint*: you computed the CT FS coefficients of a scaled version of $x(t)$ in Homework 4.

Solution We can express $x(t)$ as a CT FS:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} ,$$

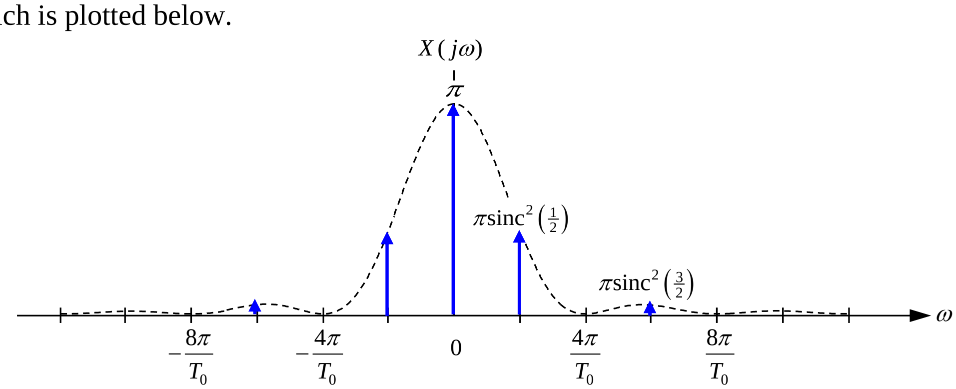
where $\omega_0 = 2\pi / T_0$. We derived the CT FS coefficients for a triangular pulse train in Homework 4, Problem 4. Scaling those coefficients by $1/2T_1$ and choosing $T_1 = T_0/4$, the CT FS coefficients are

$$a_k = \frac{2T_1}{T_0}\text{sinc}^2\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{1}{2}\text{sinc}^2\left(\frac{k}{2}\right).$$

Using the formula for the CT FT of a periodic signal (expression (19) on page 157), the CT FT of $x(t)$ is

$$\begin{aligned} X(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - k\omega_0\right) \\ &= \pi \sum_{k=-\infty}^{\infty} \text{sinc}^2\left(\frac{k}{2}\right)\delta\left(\omega - k\frac{2\pi}{T_0}\right), \end{aligned}$$

which is plotted below.



b. **(2 points)** $h(t)$ is an aperiodic signal

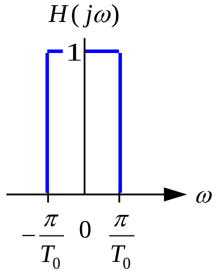
$$h(t) = \frac{1}{T_0}\operatorname{sinc}\left(\frac{t}{T_0}\right) .$$

Sketch $H(j\omega)$, the CT FT of $h(t)$.

Solution Referring to Table 4, Appendix, we use the CT FT pair

$$\frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right) \overset{F}{\leftrightarrow} \Pi\left(\frac{\omega}{2W}\right)$$

with $W = \pi / T_0$ to obtain $H(j\omega) = \Pi\left(\frac{\omega}{2\pi / T_0}\right)$, as plotted below. It is an ideal lowpass response.



c. **(3 points)** Let $y(t) = x(t) * h(t)$. Is $y(t)$ periodic or aperiodic? Sketch $Y(j\omega)$, the CT FT of $y(t)$.

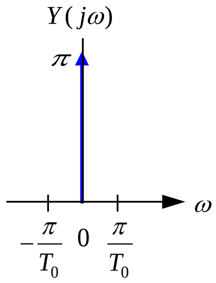
Solution The output signal $y(t)$ is *periodic* (see page 95). Using the CT FT convolution property, the CT FT of the output signal is

$$Y(j\omega) = X(j\omega) H(j\omega) .$$

Among all the impulses in $X(j\omega)$, the lowpass filter $H(j\omega)$ passes only the one for $k = 0$:

$$Y(j\omega) = \pi\delta(\omega) .$$

This is plotted below.

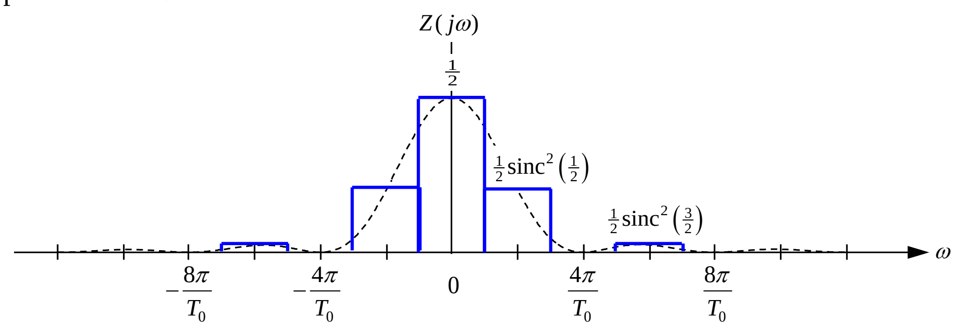


d. **(3 points)** Let $z(t) = x(t) \times h(t)$. Is $z(t)$ periodic or aperiodic? Sketch $Z(j\omega)$, the CT FT of $z(t)$.

Solution The output signal $z(t)$ is *aperiodic*. Using the CT FT multiplication property, the CT FT of the output signal is

$$\begin{aligned} Z(j\omega) &= \frac{1}{2\pi} X(j\omega) * H(j\omega) \\ &= \frac{1}{2\pi} \Pi\left(\frac{\omega}{2\pi / T_0}\right) * \pi \sum_{k=-\infty}^{\infty} \operatorname{sinc}^2\left(\frac{k}{2}\right) \delta\left(\omega - k\frac{2\pi}{T_0}\right). \\ &= \frac{1}{2} \sum_{k=-\infty}^{\infty} \operatorname{sinc}^2\left(\frac{k}{2}\right) \Pi\left(\frac{\omega - k2\pi / T_0}{2\pi / T_0}\right) \end{aligned}$$

This is plotted below.



Laboratory 6

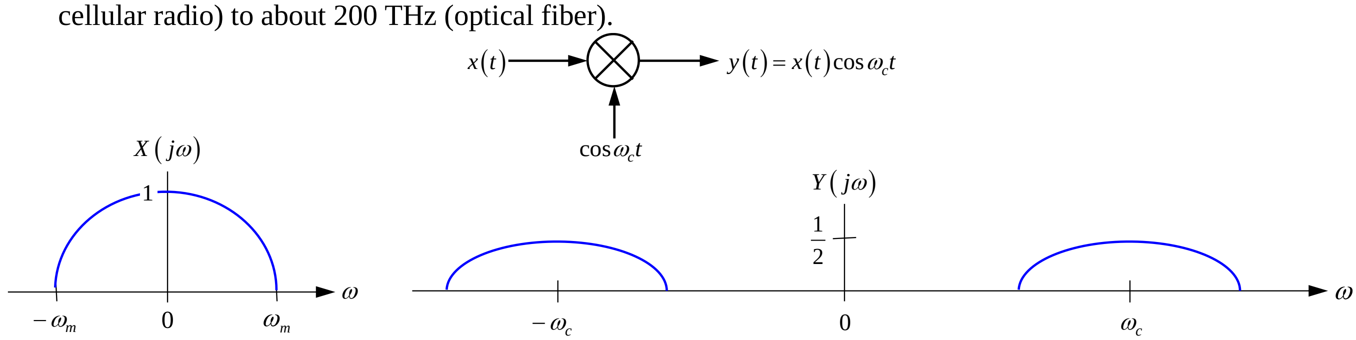
This week's laboratory demonstrates *modulation* and *demodulation*, in this case, double-sideband amplitude modulation with synchronous demodulation. For additional discussion, refer to the *EE 102A Course Reader*, Chapter 4 (pages 174-177) and Chapter 7 (pages 268-272).

Modulation and demodulation are at the heart of any communication system. You know that a radio station is identified by its frequency, and you can listen to a station by tuning a receiver to that frequency. For example, the California Department of Transportation uses the AM station 1610 kHz. Here, "AM" refers to the type of modulation used, in this case amplitude modulation, which we implement in this lab. 1610 kHz is the *carrier frequency* of this station. Note that the demodulation we implement here (synchronous) is different from what is used for demodulating broadcast AM radio (asynchronous).

In *amplitude modulation*, a message signal $x(t)$, whose spectrum $X(j\omega)$ is near d.c., is multiplied by a carrier signal $\cos\omega_c t$, yielding a modulated signal $y(t)=x(t)\cos\omega_c t$. The modulated signal spectrum is

$$Y(j\omega)=\frac{1}{2}\Big[X\big(j(\omega-\omega_c)\big)+X\big(j(\omega+\omega_c)\big)\Big], \tag{1}$$

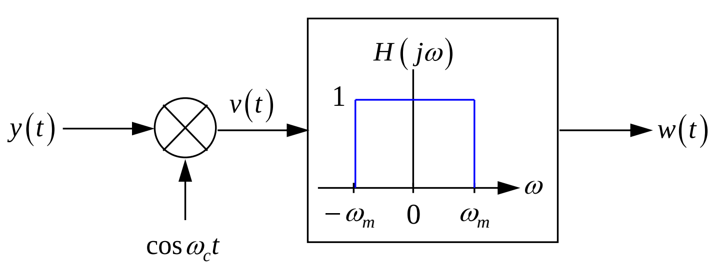
which is the spectrum $X(j\omega)$ replicated at $\pm\omega_c$. The carrier frequency ω_c is chosen to allow the signal to propagate through a medium as an electromagnetic wave. Depending on the system, the carrier frequency $\omega_c/2\pi$ may range from about 1 MHz (broadcast AM radio) to about 1-5 GHz (WiFi or cellular radio) to about 200 THz (optical fiber).



In *synchronous demodulation*, we multiply the modulated signal $y(t)$ by a replica of the carrier signal, $\cos\omega_c t$, obtaining a signal $v(t)$. In the time domain, we have

$$v(t)=y(t)\cos\omega_c t=x(t)\cos^2\omega_c t=\frac{1}{2}x(t)+\frac{1}{2}x(t)\cos 2\omega_c t, \tag{2}$$

which includes a term $(1/2)x(t)$ near d.c. and a term $(1/2)x(t)\cos 2\omega_c t$ near $\pm 2\omega_c$. Then we pass $v(t)$ into a lowpass filter with impulse response $h(t)$ and frequency response $H(j\omega)$. The term near $\pm 2\omega_c$ is blocked by the lowpass filter, so the filter output is $w(t)=(1/2)x(t)$, a scaled copy of the message signal.



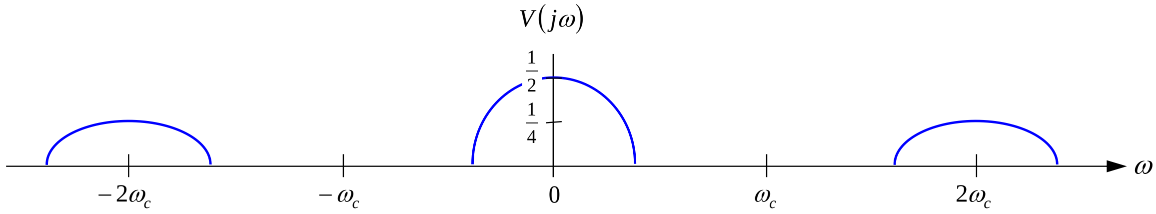
In the frequency domain, since $v(t) = y(t)\cos\omega_c t$, we have

$$V(j\omega) = \frac{1}{2}\left[Y\left(j(\omega - \omega_c)\right) + Y\left(j(\omega + \omega_c)\right)\right]. \tag{3}$$

Using (1), this becomes

$$V(j\omega) = \frac{1}{4}X\left(j(\omega - 2\omega_c)\right) + \frac{1}{2}X(j\omega) + \frac{1}{4}X\left(j(\omega + 2\omega_c)\right), \tag{4}$$

as shown below.



The lowpass filter passes only the term $\frac{1}{2}X(j\omega)$, yielding an output $w(t) = \frac{1}{2}x(t)$.

Notes on MATLAB Implementation

Throughout these simulations, time is discretized by a discretization interval Δt, represented in MATLAB by **`deltat`**.

In order to help you visualize the results in the frequency domain, we provide a MATLAB function **`CTFT_approx.m`**., which numerically approximates the continuous time Fourier transform. Given a signal $x(t)$ represented by discretized time vector **`t`** and signal **`x`**, it returns a numerical approximation of the continuous time Fourier transform $X(j\omega)$, represented by discretized frequency vector **`omega`** and Fourier transform **`X`**. It is called using the syntax

```
[X,omega] = CTFT_approx(x,t).
```

The frequency vector **`omega`** covers a span corresponding very nearly to $-\pi/\Delta t \le \omega \le \pi/\Delta t$ or $-1/2\Delta t \le \omega/2\pi \le 1/2\Delta t$.

1. Triangular pulse function

In this lab, the modulated signal $x(t)$ will be a triangular pulse. Write and turn in a function that returns a unit triangular pulse, as defined in lecture:

$$\Lambda(t) = \begin{cases} 1 - |t| & |t| \le 1 \\ 0 & |t| > 1 \end{cases}.$$

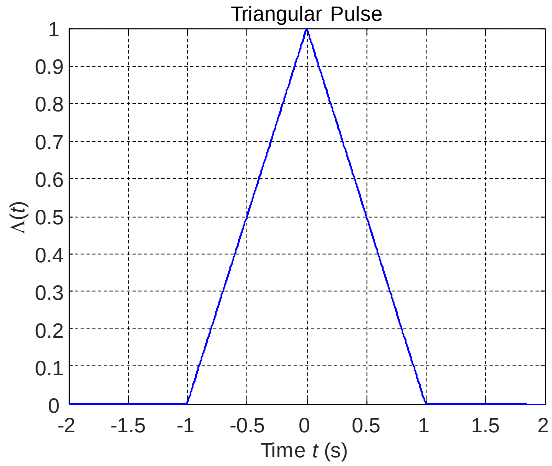
Implement the triangular pulse function described by the header below:

```
function y = Lambda(t)
% Unit triangular pulse
```

You can test your function with the following code in the Command Window:

```
>> t = -2:.01:2;
>> figure; plot(t,Lambda(t), 'LineWidth', 1.5);
>> set(gca,'FontName','arial','FontSize',14);
>> xlabel('Time \itt\rm (s)'); ylabel('\it\Pi(t)');
>> title('Triangular Pulse');
```

You should see the following output.



(2 points) Solution

```
function y = Lambda(t)
% Unit triangular pulse
y = (1-abs(t)).*double(abs(t)<=1);
end
```

2. Message signal and its spectrum

Create a new script and define your time vector:

```
% time and frequency
deltat = 1/300;              % discretization interval (s)
tmax = 1.5;                  % time vector runs from -tmax to tmax (s)
t = -tmax:deltat:tmax;       % time vector for x(t),y(t),v(t),h(t) (s)
```

Use your **Lambda** function to create a triangular pulse of unit peak value and half-width $T_x = 1/2$:

$$x(t) = \Lambda\left(\frac{t}{T_x}\right),$$

represented by a vector **x**. Its spectrum can be understood from the CT FT pair derived in lecture:

$$\Lambda\left(\frac{t}{T_x}\right) \leftrightarrow T_x \mathrm{sinc}^2\left(\frac{\omega T_x}{2\pi}\right).$$

Use the provided function **CTFT_approx** to compute the spectrum of **x**, and call it **X**. You should verify that its d.c. value is $X(j0) = T_x = 1/2$, and that the first zeros of $X(j\omega)$ occur when $\omega T_x / 2\pi = \pm 1$, corresponding to $\omega = \pm 2\pi / T_x = \pm 4\pi$ rad/s or $\omega / 2\pi = \pm 1/T_x = \pm 2$ Hz.

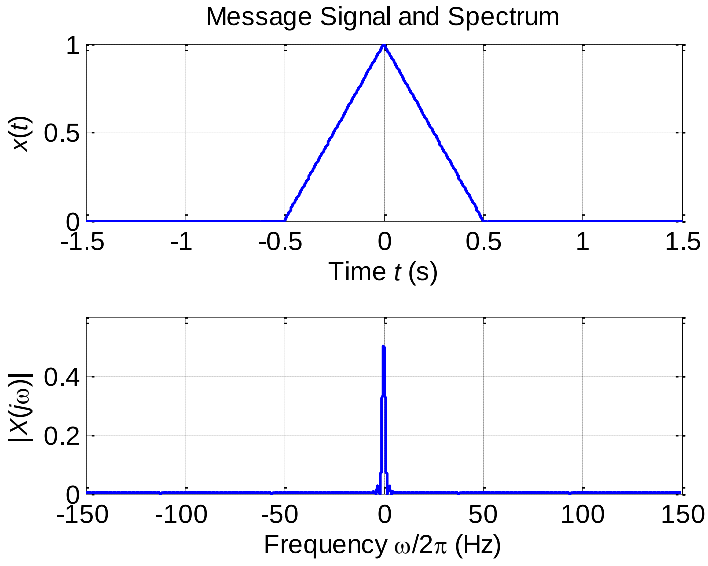
(2 points) Solution

```
% message signal
Tx = 0.5;                       % message signal nonzero for -Tx to Tx (s)

% time domain
x = Lambda(t/Tx);               % message, triangular pulse

% freq domain
[X,omega] = CTFT_approx(x,t);
```

Plot the original signal and the magnitude of its spectrum using the code provided. Note that the frequency axis is in Hz.

```
figure (1);

subplot(211)
plot(t,x)
l=get(gca,'children'); set(l,'linewidth',1.5)
set(gca,'FontName','arial'); set(gca,'FontSize',14);
xlabel('Time \itt\rm (s)');
ylabel('\itx\rm(\itt\rm)');
grid
title('Message Signal and Spectrum');

subplot(212)
plot(omega/(2*pi),abs(X));
l=get(gca,'children'); set(l,'linewidth',1.5);
set(gca,'FontName','arial'); set(gca,'FontSize',14);
xlabel('Frequency \omega/2\pi (Hz)');
ylabel('|\itX\rm(\itj\rm\omega)|');
grid
```

Your plot should look like the one below. Compare this with the analytical result above.



3. Modulation

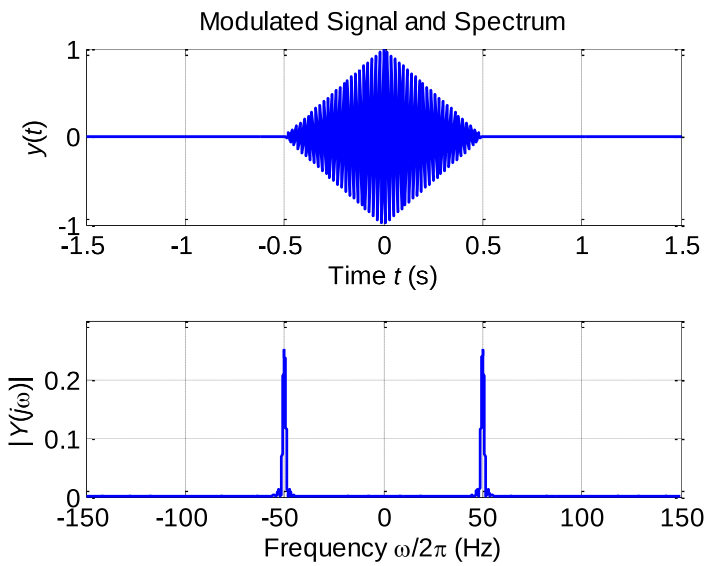
Here we choose the carrier frequency $\omega_c / 2\pi$ very low to make it easy to visualize the results. Use the carrier frequency defined below:

```
% modulation
fc = 50; omegac = 2*pi*fc;      % carrier frequency (Hz and rad/s)
```

Define $y(t) = x(t)\cos\omega_c t$. Plot $y(t)$ and its magnitude spectrum $|Y(j\omega)|$.

(3 points) Solution

```
%% Modulation

% time domain
y = x.*cos(omegac*t);    % modulated signal

% freq domain
[Y,omega] = CTFT_approx(y,t);

% display results
figure(2);

subplot(211)
plot(t,y)
l=get(gca,'children'); set(l,'linewidth',1.5);
set(gca,'FontName','arial'); set(gca,'FontSize',14);
xlabel('Time \itt\rm (s)');
ylabel('\ity\rm(\itt\rm)');
grid
title('Modulated Signal and Spectrum');

subplot(212)
plot(omega/(2*pi),abs(Y));
l=get(gca,'children'); set(l,'linewidth',1.5);
set(gca,'FontName','arial'); set(gca,'FontSize',14);
xlabel('Frequency \omega/2\pi (Hz)');
ylabel('|\itY\rm(\itj\rm\omega)|');
grid
```



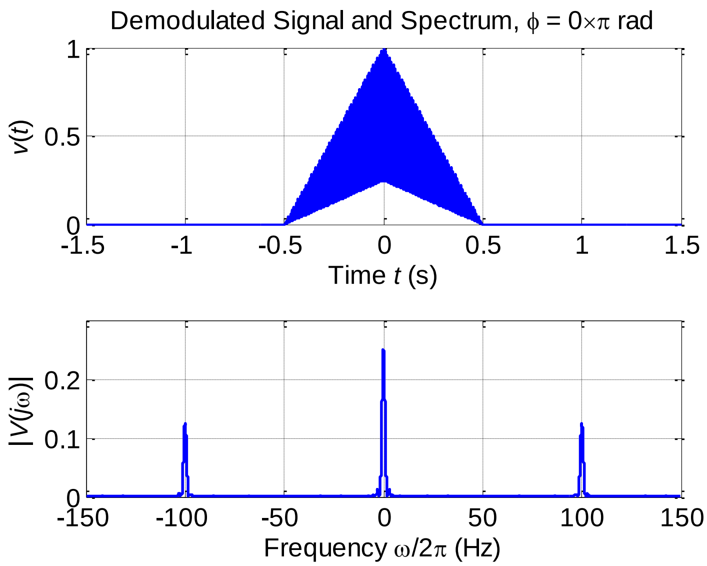
4. Demodulation

Define $v(t)$, the demodulated, unfiltered signal. Plot $v(t)$ and its magnitude spectrum $|V(j\omega)|$. Comment on the spectrum of $v(t)$, comparing it to those of $y(t)$ and $x(t)$.

(3 points) Solution

```
%% Demodulation

% time domain
v = y.*cos(omegac*t);      % demodulated signal

% freq domain
[V,omega] = CTFT_approx(v,t);

% display results

figure(3)

subplot(211)
plot(t,v)
l=get(gca,'children'); set(l,'linewidth',1.5);
set(gca,'FontName','arial'); set(gca,'FontSize',14);
xlabel('Time \itt\rm (s)');
ylabel('\itv\rm(\itt\rm)');
grid
title('Demodulated Signal and Spectrum');

subplot(212)
plot(omega/(2*pi),abs(V));
l=get(gca,'children'); set(l,'linewidth',1.5);
set(gca,'FontName','arial'); set(gca,'FontSize',14);
xlabel('Frequency \omega/2\pi (Hz)');
```

```
ylabel('|\itV\rm(\itj\rm\omega)|');
grid
```



5. Lowpass filtering to recover message signal

We will use a second-order lowpass filter, as described in *EE 102A Course Reader*, Chapter 4, pages 183-186. It has a cutoff frequency ω_n, and is chosen to be critically damped, with damping constant $\varsigma = 1$. We have provided a MATLAB function **hsolpfcd.m**, which is called as **hsolpfcd(t,omegan)**.

Define the cutoff frequency for the lowpass filter and call the provided function:

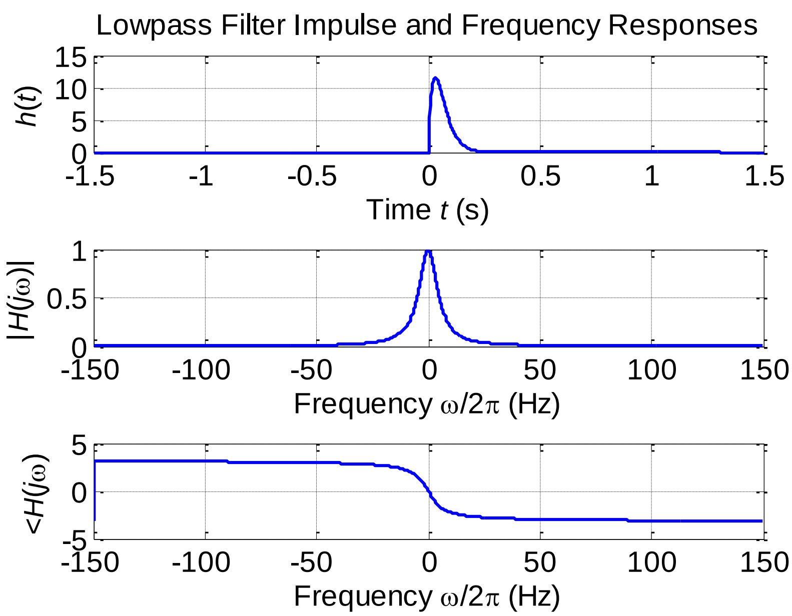
```
%% Filtering
fn = 5; omegan = 2*pi*fn;        % LPF cutoff frequency (Hz and rad/s)
% time domain
hlpf = hsolpfcd(t,omegan);  % lowpass filter
```

Plot the filter's impulse response $h(t)$ and its magnitude and phase responses $|H(j\omega)|$ and $\angle H(j\omega)$.

(3 points) Solution

```
% freq domain
[Hlpf,omega] = CTFT_approx(hlpf,t);
% display results
figure(4)
subplot(311)
plot(t,hlpf)
l=get(gca,'children'); set(l,'linewidth',1.5);
set(gca,'FontName','arial'); set(gca,'FontSize',14);
xlabel('Time \itt\rm (s)');
```

```
ylabel('\ith\rm(\itt\rm)');
grid
title('Lowpass Filter Impulse and Magnitude Responses');

subplot(312)
plot(omega/(2*pi),abs(Hlpf));
l=get(gca,’children’); set(l,’linewidth’,1.5);
set(gca,'FontName','arial'); set(gca,'FontSize',14);
xlabel('Frequency \omega/2\pi (Hz)');
ylabel('|\itH\rm(\itj\rm\omega)|');
grid

subplot(313)
plot(omega/(2*pi),angle(Hlpf));
l=get(gca,’children’); set(l,’linewidth’,1.5);
set(gca,'FontName','arial'); set(gca,'FontSize',14);
xlabel('Frequency \omega/2\pi (Hz)');
ylabel('<\itH\rm(\itj\rm\omega)');
grid
```



Finally, convolve $v(t)$ with the lowpass filter impulse response $h(t)$ to obtain $w(t)$, represented by a vector **`w`**. Refer to Laboratory 3 for the proper way to approximate CT convolution using discretized signals. You will need to define a time vector **`tw`** that has the proper starting and ending times for **`w`**. You did something similar in Laboratory 3, part 1(a).

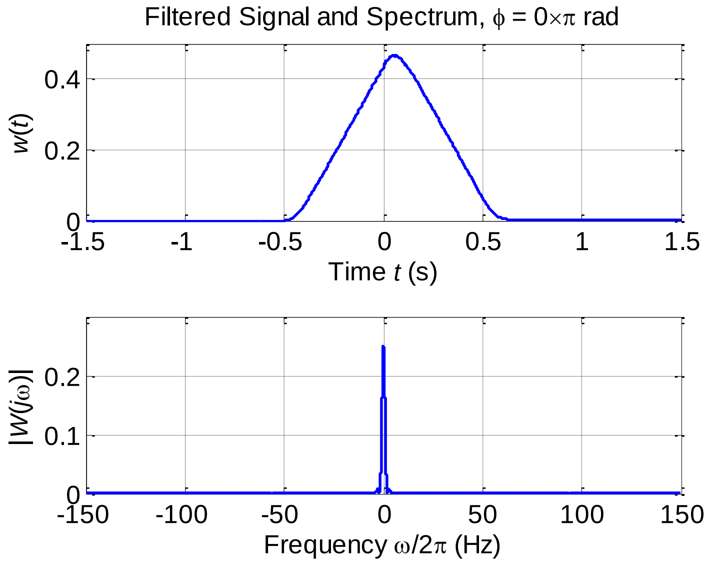
(3 points) Solution

```
%% Filtering

% time domain
tw = -2*tmax:deltat:2*tmax; % time vector for w(t) = v(t)*h(t) (s)
hlpf = hsolpfcd(t,omegan);  % lowpass filter
w = deltat*conv(v,hlpf);    % filtered signal

% freq domain
[W,omega] = CTFT_approx(w,tw);
```

Plot the lowpass filter output $w(t)$ and its magnitude spectrum $|W(j\omega)|$. Explain how the filtered signal differs from the original message and suggest ways in which the demodulation system could be improved.

(3 points) Solution

```
% display results

figure(5)

subplot(211)
plot(tw,w)
l=get(gca,'children'); set(l,'linewidth',1.5);
set(gca,'FontName','arial'); set(gca,'FontSize',14);
xlabel('Time \itt\rm (s)');
ylabel('\itw\rm(\itt\rm)');
grid
title(['Filtered Signal and Spectrum, \phi = ' num2str(phi/pi,3)
'\times\pi rad']);

subplot(212)
plot(omega/(2*pi),abs(W));
l=get(gca,'children'); set(l,'linewidth',1.5);
set(gca,'FontName','arial'); set(gca,'FontSize',14);
xlabel('Frequency \omega/2\pi (Hz)');
ylabel('|\itW\rm(\itj\rm\omega)|');
grid
```



Improvements: the filtered signal $w(t)$ differs from the original message signal $x(t)$ in three ways: (a) a small leakage of the $2\omega_c$ component, (b) a smoothing of sharp features because of amplitude distortion, and (c) a time delay because of the lowpass filter's group delay. We can improve (a) by reducing the lowpass filter cutoff frequency ω_n, but that will exacerbate (b). We cannot escape this trade off using only a second-order filter. A possible solution is to use a higher-order filter and choose its cutoff frequency somewhat higher than we have chosen here. It is impossible to avoid the time delay (c). In fact, a higher-order filter will have a larger delay than this second-order filter.

6. Effect of phase offset: analytical

In demodulation, we multiply the modulated signal $y(t)$ by a replica of the carrier. Until now, we have assumed the carrier replica is synchronized perfectly. What would occur if this is not the case? Let us assume the carrier replica is $\cos(\omega_c t+\phi)$, where ϕ is a phase offset. When a phase offset is present, after multiplying $y(t)$ by the phase-offset carrier, we obtain

$$v(t)=y(t)\cdot\cos(\omega_c t+\phi)=x(t)\cos(\omega_c t)\cdot\cos(\omega_c t+\phi)=\frac{1}{2}x(t)\cos(\phi)+\frac{1}{2}x(t)\cos(2\omega_c t+\phi),$$

where we have used the trigonometric identity $\cos(A)\cdot\cos(B)=(1/2)\cos(A-B)+(1/2)\cos(A+B)$. Given the above expression for $v(t)$, write an expression for the lowpass filter output $w(t)$, assuming an ideal lowpass filter, as in *EE 102A Course Reader*, pages 270-271 (not the second-order filter used in the present MATLAB simulations). Evaluate your expression for phase offsets of $\phi=0,\pi/2$ or π. Which of the three phase offsets is worst?

(2 points) Solution Phase offsets of $\phi=0,\pi/2$ and π yield lowpass filter outputs $w(t)=\frac{1}{2}x(t)$, 0 and $-\frac{1}{2}x(t)$, respectively. A phase offset of $\phi=\pi/2$ is the worst, since no signal whatsoever is recovered.

7. Effect of phase offset: verification using MATLAB

Verify your analytical results using MATLAB. Submit plots of $v(t)$, $|V(j\omega)|$, $w(t)$ and $|W(j\omega)|$ for $\phi = \pi/2$ and $\phi = \pi$. Use the same second-order lowpass filter as in Part 5.

(3 points) Solution You can reuse most of the code from parts 1-4 above. The only notable changes are to define ϕ:

```
phi = pi;                          % phase error (rad)
```

and compute $v(t)$ with a phase offset:

```
% time domain
v = s.*cos(omegac*t + phi);        % demodulated signal
```

Note the different vertical scales in the plots below.