Stanford University
EE 102A: Signal Processing and Linear Systems I
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Chapter 3: Fourier Series

Motivations

- Many CT or DT signals may be expressed as a discrete sum or a continuous integral of imaginary or complex exponential signals at different frequencies.
- Expressing signals in terms of imaginary or complex exponentials simplifies LTI system analysis.

- This table summarizes the methods for so expressing signals that are used in EE 102A and 102B.

Method	Principal Signal Type	Expressed in Terms of	Course
CT Fourier series	Periodic power, CT	Imaginary exponentials $e^{jk\omega_0 t}$, ω_0 real, k integer	EE 102A
DT Fourier series	Periodic power, DT	Imaginary exponentials $e^{jk\Omega_0 n}$, Ω_0 real, k integer	
CT Fourier transform	Aperiodic energy, CT	Imaginary exponentials $e^{j\omega t}$, ω real and continuous	
DT Fourier transform	Aperiodic energy, DT	Imaginary exponentials $e^{j\Omega n}$, Ω real and continuous	
Laplace transform	Aperiodic, CT	Complex exponentials e^{st}, s complex and continuous	EE 102B
Z Transform	Aperiodic, DT	Complex exponentials z^n, z complex and continuous	

Major Topics in This Chapter (studied for both CT and DT unless noted otherwise)

- Complex exponentials as eigenfunctions. Transfer function of LTI system.
  Imaginary exponentials as eigenfunctions. Frequency response of LTI system.
- Fourier series
  - Trigonometric vs. exponential.
    Synthesis and analysis.
    Application to periodic or aperiodic signals (final section of chapter, skipped).
- Fourier series properties
  - Linearity, time-shift, multiplication, time reversal, conjugation. Parseval's identity.
  - Inner products.
- Response of LTI systems to periodic inputs
  - Computing the frequency response.
  - CT system examples: first-order lowpass and highpass filters.
  - DT system examples. Recursive: first-order. Non-recursive: moving average, edge detector.

Eigenfunctions of Continuous-Time Linear Time-Invariant Systems

General Case: Complex Exponentials

- Consider a CT LTI system H that has an impulse response $h(t)$. Given a general input signal $x(t)$, if we wish to predict the output $y(t)=H\{x(t)\}$, we perform a *convolution* between $x(t)$ and $h(t)$.

$$x(t) \longrightarrow \boxed{\quad h(t) \quad} \longrightarrow y(t)=h(t)*x(t)$$

- The *eigenfunctions* of CT LTI systems are *complex exponential* time signals

$$e^{st},\ s \text{ complex},\ -\infty < t < \infty .$$

- If we input one of these signals to an LTI system *H*, the output is the same signal e^{st}, multiplied by an *eigenvalue* denoted by $H(s)$, as shown.

$$x(t) = e^{st} \longrightarrow \boxed{h(t) \leftrightarrow H(s)} \longrightarrow y(t) = H(s) \cdot e^{st}$$

- The variable s is called *complex frequency*.
  $H(s)$ is called the *transfer function* of the LTI system *H*.

- Assuming we know the transfer function $H(s)$ as a function of *s*, then for an input e^{st}, we can predict the output by using *multiplication*, and do not need to use convolution.

- *Proof*: we input $x(t)=e^{st}$ to the system and compute the output $y(t)$ using convolution:

$$
\begin{aligned}
y(t) &= h(t)*x(t) \\
&= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\,d\tau \\
&= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau \\
&= e^{st}\cdot H(s)
\end{aligned}
\quad . \qquad (1)
$$

- In (1), we have defined the transfer function of the LTI system as

$$H(s) \stackrel{d}{=} \int_{-\infty}^{\infty} h(t)e^{-st}\,dt\,. \qquad (2)$$

  Given an impulse response $h(t)$, we compute the integral (2) to obtain the transfer function $H(s)$. The integral (2) defines $H(s)$ as the *bilateral Laplace transform* of the impulse response $h(t)$.

- The Laplace transform integral (2) converges for a large class of impulse responses $h(t)$, including some corresponding to unstable systems (see *EE 102B Course Reader*, Chapter 5).

Special Case: Imaginary Exponentials

- Now we consider the special case that s is purely imaginary, $s = j\omega$, where ω is real. We are considering a subset of the complex exponential signals, namely, the *imaginary exponential* signals

$$e^{j\omega t}, \ \omega \text{ real}, \ -\infty < t < \infty .$$

- These, too, are eigenfunctions of LTI systems. If we input an imaginary exponential $e^{j\omega t}$ to an LTI system, the output is the same signal, multiplied by an eigenvalue $H(j\omega)$, as shown.

$$x(t) = e^{j\omega t} \longrightarrow \boxed{h(t) \leftrightarrow H(j\omega)} \longrightarrow y(t) = H(j\omega) \cdot e^{j\omega t}$$

- The variable ω is simply called *frequency*.
  $H(j\omega)$ is called the *frequency response* of the LTI system *H*.

- As in the general case above, if we know $H(j\omega)$ as a function of ω, then for an imaginary exponential input signal $e^{j\omega t}$, we can predict the system output by using multiplication, and need not use convolution.

Proof

- We use an input $x(t)=e^{j\omega t}$ in (1). Then (1) yields an output

$$y(t)=e^{j\omega t}\cdot H(j\omega). \tag{3}$$

- The system frequency response appearing in (3) is defined by the following expression:

$$H(j\omega)\overset{d}{=}H(s)\big|_{s=j\omega}=\int_{-\infty}^{\infty}h(t)e^{-j\omega t}dt\ . \tag{4}$$

- Given an impulse response $h(t)$, we evaluate integral (4) to obtain the frequency response $H(j\omega)$. The integral (4) defines $H(j\omega)$ as the *CT Fourier transform* of the impulse response $h(t)$.

- The Fourier transform integral (4) converges for many impulse responses $h(t)$, but not in some cases (notably, some important unstable systems) for which Laplace transform (2) does converge. This motivates us to use the Laplace transform in studying feedback control and other applications involving potentially unstable systems (see *EE 102B Course Reader*, Chapters 5-6).

Application to LTI System Analysis

- Consider a CT LTI system *H*, and assume we know its transfer function $H(s)$.
- Suppose we are given an input signal that is expressed as a linear combination of complex exponential signals at *K* distinct values of *s*:

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + \cdots + a_K e^{s_K t} \, . \tag{5}$$

- Using the linearity of the system and the eigenfunction property (1), we can compute the output signal as

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + \cdots + a_K H(s_K) e^{s_K t} \, . \tag{6}$$

- We have computed each term in the output (6) by using only multiplication, not convolution.
- It is easy to apply (5) and (6) to the special case of imaginary exponential input signals.

Example

- An LTI system *H* with input $x(t)$ and output $y(t)$ has an input-output relation

$$y(t) = \frac{dx(t)}{dt} + x(t-1). \tag{7}$$

- We are given an input signal

$$x(t) = e^{jt} + e^{2t},$$

  which is in the form (5) with $s_1 = j$ and $s_2 = 2$.

- Using input-output relation (7), we can compute the output signal:

$$\begin{aligned} y(t) &= je^{jt} + e^{j(t-1)} + 2e^{2t} + e^{2(t-1)} \\ &= \left(j + e^{-j}\right)e^{jt} + \left(2 + e^{-2}\right)e^{2t} \\ &= H(j)e^{jt} + H(2)e^{2t} \end{aligned} \tag{8}$$

  Output (8) is consistent with the general form (6).

- We will learn in EE 102B that input-output relation (7) corresponds to a transfer function

$$H(s) = s + e^{-s}.$$

  Knowing this, we can use (6) to find output $y(t)$ without performing the computations in (8).

Types of Fourier Series Representations

- We are given a periodic CT signal that satisfies

$$x(t) = x(t + T_0) \quad \forall t\,, \tag{9}$$

  where T_0 is the *period* and the *fundamental frequency* is

$$\omega_0 = \frac{2\pi}{T_0}\,. \tag{10}$$

- *Fourier series* (FS) representation of periodic signal $x(t)$: a linear combination of sinusoidal or imaginary exponential basis signals, each at a frequency $k\omega_0$, where *k* is an integer.
- There are several ways to construct a FS representation, as shown in the table below.
  - Trigonometric FS (rows 1 and 2): using sinusoidal basis signals.
  - Exponential FS (row 3): using imaginary exponential basis signals.

Fourier Series Synthesis	**Frequencies**	**Pros**	**Cons**
$\hat{x}(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$	Positive	• Real coefficients for all real $x(t)$.	• Harder algebra. • Not eigenfunctions: cannot simply use multiplication.
$\hat{x}(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k \cos k\omega_0 t - C_k \sin k\omega_0 t]$	Positive		
$\hat{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	Positive and negative	• Easier algebra. • Eigenfunctions: use multiplication.	• Complex coefficients for many real $x(t)$.

- We let $\hat{x}(t)$ denote the FS representation of $x(t)$ for now, as $\hat{x}(t)$ is not generally identical to $x(t)$.
- In EE 102A and 102B, we use only exponential FS for CT signals (and DT signals):
  - Algebra is far easier on imaginary exponentials than on trigonometric functions.
  - Imaginary exponentials $e^{jk\omega_0 t}$ are eigenfunctions of LTI systems. If we input an exponential FS representation $\hat{x}(t)$ to an LTI system, we can use simple multiplication to compute the output $y(t)$, as in (6).

Continuous-Time Fourier Series

Synthesis Equation

- We are given a periodic CT signal $x(t)$ with period T_0 and fundamental frequency $\omega_0 = 2\pi / T_0$.
- We will represent $x(t)$ as an exponential continuous-time Fourier series (CTFS) in the form

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} . \tag{11}$$

  - We call (11) a *synthesis equation*: shows how to synthesize the periodic signal using imaginary exponentials $e^{jk\omega_0 t}$ at frequencies $k\omega_0$, $-\infty < k < \infty$.
  - We call the a_k, $-\infty < k < \infty$ the *CTFS coefficients* for the signal $x(t)$.
- We can verify that the CTFS synthesis (11) is periodic in time with period T_0:

$$\begin{aligned} \hat{x}(t+T_0) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (t+T_0)} \\ &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{j k \underbrace{\omega_0 T_0}_{=2\pi}} \\ &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \underbrace{e^{jk2\pi}}_{=1} \\ &= \hat{x}(t) \end{aligned} . \tag{12}$$

- Now we must address the question of how to perform *analysis*.
  Given a periodic signal $x(t)$, how can we determine the CTFS coefficients a_k, $-\infty < k < \infty$?

Obtaining the Fourier Series by Inspection

- In some simple cases, the CTFS coefficients can be found by inspection. For example, consider

$$x(t) = 1 + \sin 2\pi t + \cos 3\pi t \ .$$

- *Question*: how can we find the period T_0 and the fundamental frequency ω_0 of $x(t)$?

- The signal is periodic with period $T_0 = 2$ and fundamental frequency $\omega_0 = 2\pi / T_0 = \pi$.
  We can express it as a linear combination of imaginary exponentials with frequencies

$$k\omega_0 = k\frac{2\pi}{T_0} = k\pi \ .$$

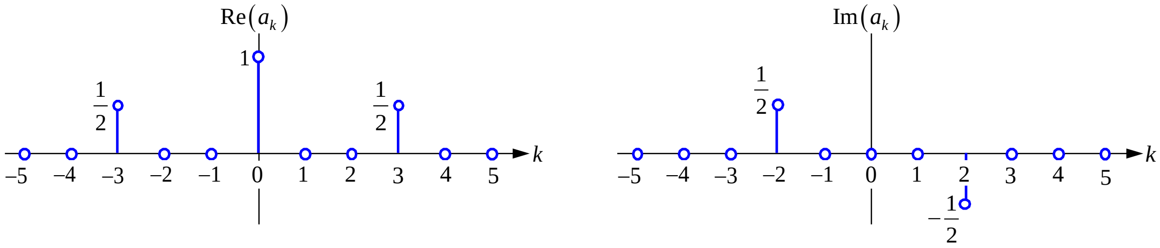
- Using Euler's relation (see Appendix, page 288), the representation is

$$x(t) = 1 + \sin 2\pi t + \cos 3\pi t = 1\left(e^{j0}\right) + \frac{1}{2j}\left(e^{j2\pi t} - e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j3\pi t} + e^{-j3\pi t}\right).$$

  Comparing this to the synthesis equation (11), we have found the CTFS coefficients as

$$a_0 = 1 \qquad a_{-2} = -a_2 = \frac{1}{2} j \qquad a_{-3} = a_3 = \frac{1}{2},$$

  while all other CTFS coefficients are zero. These CTFS coefficients are shown.



Imaginary Exponential Basis Signals

- In deriving general CTFS analysis method, we use *imaginary exponential basis signals*, defined as

$$\phi_k(t) = e^{jk\omega_0 t}, \; -\infty < k < \infty . \tag{13}$$

  Each basis signal is periodic and satisfies

$$\phi_k(t+T_0) = \phi_k(t) \quad \forall t .$$

  This can be proven as (12) was proven.

- We would like to show that the basis signals form an *orthogonal set*, so we compute an integral

$$\int_{T_0} \phi_k(t)\phi_m^*(t)\,dt = \int_{T_0} e^{j(k-m)\omega_0 t}\,dt . \tag{14}$$

  - Notation denoting an integral over an interval of length T_0 starting at arbitrary time t_1:

$$\int_{T_0} (\;)\,dt \quad \text{means} \quad \int_{t_1}^{t_1+T_0} (\;)\,dt, \; t_1 \text{ arbitrary.}$$

  - In linear algebra, (14) is called an *inner product* between $\phi_k(t)$ and $\phi_m(t)$.
    Inner products are discussed further on slides 38-42 below.

- We use Euler's relation (see Appendix, page 288) to express the imaginary exponential in (14) in terms of sinusoids.

  - Evaluating (14) for $k \neq m$, we find

$$\int_{T_0} e^{j(k-m)\omega_0 t} dt = \int_{T_0} \cos\left(\left(k-m\right)\omega_0 t\right) dt + j\int_{T_0} \sin\left(\left(k-m\right)\omega_0 t\right) dt = 0 .$$

    Each integral on the right-hand side vanishes because it includes an integer number of cycles of a sinusoid.

  - Evaluating (14) for $k = m$, we find

$$\int_{T_0} e^{j(k-m)\omega_0 t} dt = \int_{T_0} (1) dt = T_0 .$$

- In summary, the basis signals form an *orthogonal set* with pairwise integrals (inner products) given by

$$\int_{T_0} \phi_k(t)\phi_m^*(t) dt = \int_{T_0} e^{j(k-m)\omega_0 t} dt = \begin{cases} 0 & k \neq m \\ T_0 & k = m \end{cases} . \qquad (15)$$

Analysis Equation

- We are given a periodic signal $x(t)$ with period T_0 and fundamental frequency $\omega_0 = 2\pi / T_0$.
- We represent $x(t)$ by an exponential CTFS synthesis of the form (11).

  Equivalently, we approximate $x(t)$ as a linear combination of imaginary exponential basis signals:

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k \phi_k(t) . \qquad (11')$$

- To determine the CTFS coefficients a_k , $-\infty < k < \infty$ in (11′), assume for now the approximation is identical to the original signal:

$$\hat{x}(t) = x(t) . \qquad (16)$$

- We compute an inner product integral between both sides of (16) and the imaginary exponential basis signal $\phi_m(t) = e^{jm\omega_0 t}$:

$$\int_{T_0} x(t) e^{-jm\omega_o t} dt = \int_{T_0} \hat{x}(t) e^{-jm\omega_o t} dt . \qquad (17)$$

- Now substitute the synthesis equation (11) or (11′) for $\hat{x}(t)$ on the right-hand side of (17):

$$
\begin{aligned}
\int_{T_0} x(t) e^{-jm\omega_o t} dt &= \int_{T_0} \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jm\omega_o t} dt \\
&= \sum_{k=-\infty}^{\infty} a_k \underbrace{\int_{T_0} e^{j(k-m)\omega_0 t} dt}_{=\begin{cases} 0 & k\neq m \\ T_0 & k=m \end{cases}} \quad . \qquad (18) \\
&= T_0 a_m
\end{aligned}
$$

  - In the second line, we interchanged order of summation and integration, and evaluated the integral using pairwise orthogonality relation (15).
  - In the third line, we evaluated the sum, finding that only the term for $k = m$ is nonzero.
  - Rearranging (18) yields

$$a_m = \frac{1}{T_0} \int_{T_0} x(t) e^{-jm\omega_o t} dt \ . \qquad (19)$$

- Expression (19) is the *analysis equation* we sought. Given a periodic signal $x(t)$, (19) tells us how to obtain the CTFS coefficients a_m, $-\infty < m < \infty$ that are used in the synthesis equation (11).

Summary of Continuous-Time Fourier Series

- The synthesis and analysis equations are

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \text{(synthesis)} \qquad (11)$$

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_o t} dt \qquad \text{(analysis)} \qquad (19)$$

- We often denote a periodic CT signal $x(t)$ and its FS coefficients a_k as a *CTFS pair*:

$$x(t) \overset{FS}{\leftrightarrow} a_k \, . \qquad (20)$$

- See Table 4, Appendix for the CTFS coefficients for some important periodic CT signals.

Convergence of Continuous-Time Fourier Series ***(You may skip except for the "most important point".)***

- Suppose
  - A periodic signal $x(t)$, with period T_0, is used in analysis equation (19) to compute CTFS coefficients a_k, $-\infty < k < \infty$.
  - The coefficients a_k are used in synthesis equation (11) to form a CTFS approximation $\hat{x}(t)$.
- It can be shown that if $x(t)$ has finite power

$$\frac{1}{T_0}\int_{T_0} \left|x(t)\right|^2 dt < \infty ,$$

the power in the difference between $x(t)$ and $\hat{x}(t)$ vanishes:

$$\frac{1}{T_0}\int_{T_0} \left|x(t)-\hat{x}(t)\right|^2 dt = 0 .$$

- **Most important point:** this does not imply that $\hat{x}(t) = x(t)$ at all *t*. In fact, $\hat{x}(t)$ differs from $x(t)$ near any values of *t* where $x(t)$ has discontinuities. Near these values of *t*, $\hat{x}(t)$ exhibits ripples called the *Gibbs phenomenon*.

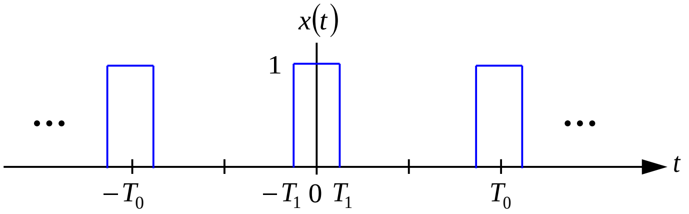
- It can be shown that $\hat{x}(t) = x(t)$ except near values of t where $x(t)$ has discontinuities if $x(t)$ satisfies the *Dirichlet conditions*:
  - $x(t)$ is absolutely integrable over any period:

$$\frac{1}{T_0}\int_{T_0} |x(t)| dt < \infty \, .$$

  - $x(t)$ has a finite number of local maxima and minima in each period.
  - $x(t)$ has a finite number of discontinuities in each period.
  - Any discontinuities of $x(t)$ are finite.
- From now on, we use $x(t)$ to represent both the original signal used in the analysis (19) and the signal formed by the synthesis (11), unless it is necessary to draw a distinction between them.
- For further discussion on the convergence of CTFS, see the textbook *OWN*, Section 3.4.

Example: Fourier Series of a Rectangular Pulse Train

- As an example of applying the CTFS analysis and synthesis equations, consider a periodic rectangular pulse train. The period is T_0 and the width of each pulse is $2T_1$.



- The pulse train can be described by a formula

$$x(t) = \begin{cases} 1 & |t| \le T_1 \\ 0 & T_1 < |t| \le \frac{T_0}{2} \end{cases}, \qquad x(t+T_0) = x(t).$$

It can alternatively be expressed as a sum of shifted rectangular pulse functions (see Appendix):

$$x(t) = \sum_{l=-\infty}^{\infty} \Pi\left(\frac{t - lT_0}{2T_1}\right).$$

Analysis

- We compute the CTFS coefficients using the analysis equation (19). Given the symmetry of $x(t)$, it is natural to choose a symmetric integration interval:

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T_0} \int_{-T_1}^{T_1} e^{-jk\omega_0 t} dt \; . \tag{21}$$

  - For $k \neq 0$, we evaluate (21) to obtain

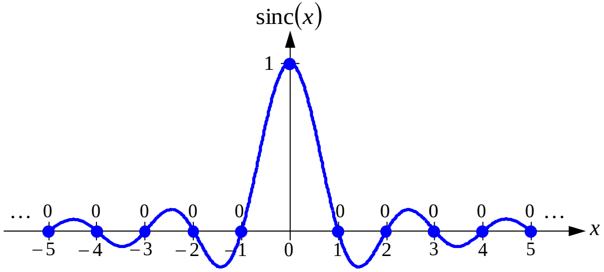
$$a_k = \frac{1}{-T_0 jk\omega_0} e^{-jk\omega_0 t} \Big|_{-T_1}^{T_1} = \frac{2\sin(k\omega_0 T_1)}{T_0 k\omega_0} = \frac{\omega_0 T_1}{\pi} \frac{\sin\left(\pi \frac{k\omega_0 T_1}{\pi}\right)}{\pi \frac{k\omega_0 T_1}{\pi}} \; . \tag{22}$$

  - For $k = 0$, we cannot apply (22), because that would entail division by zero. Instead, we set $e^{-jk\omega_0 t} = 1$, and evaluate integral (21) to obtain

$$a_0 = \frac{1}{T_0} \int_{-T_1}^{T_1} (1) dt = \frac{2T_1}{T_0} = \frac{\omega_0 T_1}{\pi} \; . \tag{23}$$

- We define the *sinc function* as shown below (see Appendix, page 290).

$$\text{sinc}(x) \stackrel{d}{=} \frac{\sin(\pi x)}{\pi x}$$

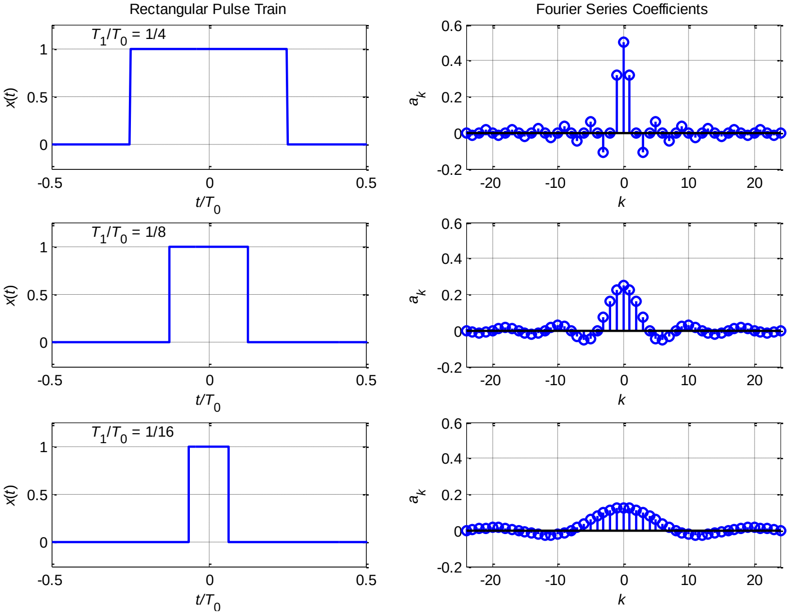


- For integer-valued arguments, the sinc function assumes values of zero or one.
  - For nonzero integer values of *x*, $\text{sinc}(x)$ is zero: $\text{sinc}(x)\big|_{x=\pm1,\pm2,\ldots} = 0$.
  - As *x* approaches zero, we use L'Hôpital's rule: $\lim_{x\to 0} \text{sinc}(x) = \lim_{x\to 0} \frac{\sin(\pi x)}{\pi x} = \lim_{x\to 0} \frac{\frac{d}{dx}\left[\sin(\pi x)\right]}{\frac{d}{dx}\left[\pi x\right]} = 1$.

- The CTFS coefficients of the pulse train can be expressed for all values of *k* using the sinc function:

$$a_k = \frac{\omega_0 T_1}{\pi} \text{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right), \quad -\infty < k < \infty . \tag{24}$$

- Look at one period of $x(t)$ $(-\frac{T_0}{2} \le t \le \frac{T_0}{2})$ and CTFS coefficients a_k for various ratios T_1 / T_0.



Observations

- $x(t)$ is real and even in *t*, so the a_k are real and even in *k*. (See CTFS properties below.)
- We fix T_0, thus fixing $\omega_0 = 2\pi / T_0$.
- As we decrease T_1:
  - Each pulse becomes narrower in time.
  - The spectrum of CTFS coefficients a_k at frequencies $k\omega_0$, $-\infty < k < \infty$, spreads out in frequency.

- This example demonstrates the *inverse relationship between time and frequency*, a basic principle in Fourier analysis:
  - Narrower pulses, changing faster in time, are described using higher frequencies.
  - Wider pulses, changing more slowly in time, are described using lower frequencies.

Synthesis

- Now we study synthesis of the pulse train using CTFS coefficients (24) in synthesis equation (11). In the following figures:
  - The left column shows the contribution from the term(s) at $\pm k$:

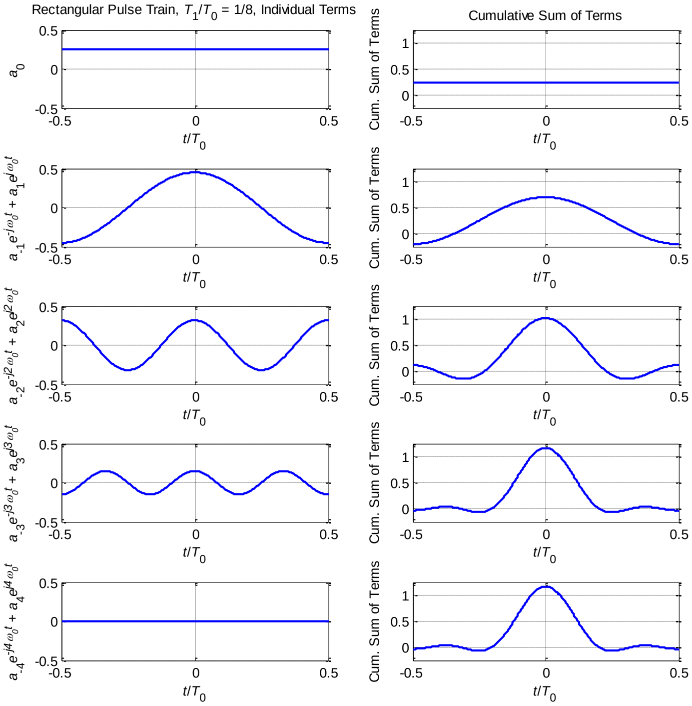
$$\begin{cases} a_0 & k=0 \\ a_{-k}e^{-jk\omega_0 t}+a_k e^{jk\omega_0 t} & k\neq 0 \end{cases}. \tag{25}$$

    Contribution (25) is a real constant ($k=0$) or a cosine function ($k\neq 0$).
    This is because the a_k are conjugate-symmetric, $a_{-k}=a_k^*$, since $x(t)$ is real (see slides 34-35).

  - The right column shows a synthesis using finite number of terms up to $\pm K$ in (11), which is

$$\hat{x}_K(t)=\sum_{k=-K}^{K} a_k e^{jk\omega_0 t} . \tag{26}$$

- All plots consider $T_1 / T_0 = 1/8$ and show the waveforms over one period, $-T_0/2 \le t \le T_0/2$.



Observations

- As K increases, $\hat{x}_K(t)$ better approximates $x(t)$. The leading and trailing edges of the pulses become more abrupt.
- The synthesis $\hat{x}_K(t)$ displays ripple near the discontinuities in $x(t)$, a manifestation of the Gibbs phenomenon.
- As K increases, the ripple is confined to a narrower time interval, but its peak amplitude does not diminish.

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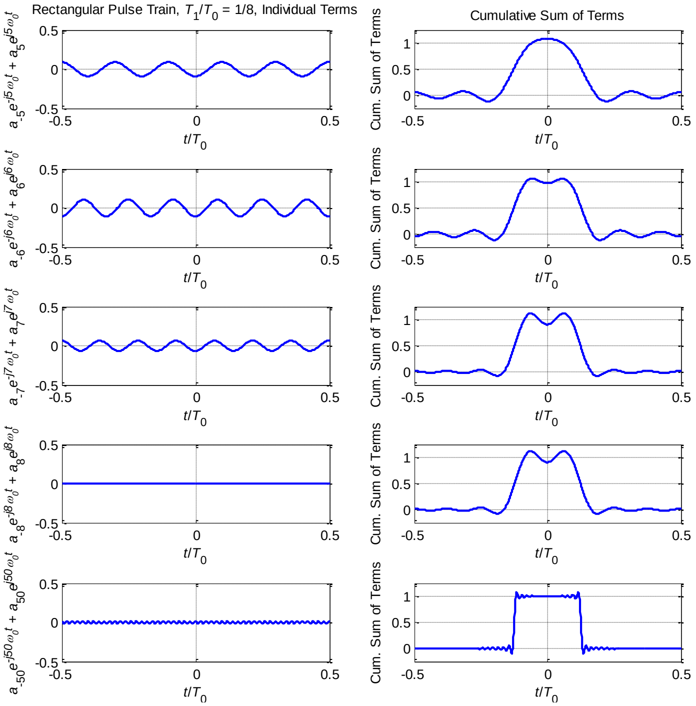

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Properties of the Continuous-Time Fourier Series

- These properties are useful for:
  - Computing the CTFS coefficients for new signals, with minimal effort, by using the CTFS coefficients already known for other signals.
  - Checking the CTFS coefficients we compute for new signals.
- We assume periodic signal(s) having a common period T_0 and fundamental frequency $\omega_0 = \frac{2\pi}{T_0}$.

  We consider one or two signals and their CTFS coefficients:

$$x(t) \overset{FS}{\leftrightarrow} a_k \quad \text{and} \quad y(t) \overset{FS}{\leftrightarrow} b_k .$$

Linearity

$$Ax(t) + By(t) \overset{FS}{\leftrightarrow} Aa_k + Bb_k .$$

- A linear combination of $x(t)$ and $y(t)$ is periodic with the same period T_0. It has CTFS coefficients given by the corresponding linear combination of the CTFS coefficients a_k and b_k.

Time Shift

$$x(t-t_0) \overset{FS}{\leftrightarrow} e^{-jk\omega_0 t_0} a_k \,. \tag{27}$$

- A signal time-shifted by t_0 has CTFS coefficients multiplied by a complex-valued factor $e^{-jk\omega_0 t_0}$. The magnitude and phase of $e^{-jk\omega_0 t_0} a_k$ are related to those of a_k by

$$\begin{cases} \left| e^{-jk\omega_0 t_0} a_k \right| = \left| a_k \right| \\ \angle\left(e^{-jk\omega_0 t_0} a_k \right) = \angle a_k - k\omega_0 t_0 \end{cases} . \tag{27'}$$

- Time-shifting a signal by t_0 affects its CTFS coefficients by
  - Leaving the magnitude unchanged.
  - Adding a phase shift proportional to the negative of the time shift, $-t_0$, which varies linearly with frequency $k\omega_0$.

Proof

- We define a time-shifted signal and its CTFS coefficients:

$$x(t-t_0) \overset{FS}{\leftrightarrow} b_k \, .$$

- We compute its CTFS coefficients using the analysis equation:

$$b_k = \frac{1}{T_0} \int_{T_0} x(t-t_0) e^{-jk\omega_o t} dt \, .$$

- We change the integration variable to $\tau = t - t_0$:

$$\begin{aligned} b_k &= \frac{1}{T_0} \int_{T_0} x(\tau) e^{-jk\omega_o(\tau+t_0)} d\tau \\ &= e^{-jk\omega_o t_0} \frac{1}{T_0} \int_{T_0} x(\tau) e^{-jk\omega_o \tau} d\tau \, . \\ &= e^{-jk\omega_o t_0} a_k \end{aligned}$$

We have used the analysis equation to substitute a_k in the third line.

QED

Multiplication

$$x(t)y(t) \overset{FS}{\leftrightarrow} \sum_{l=-\infty}^{\infty} a_l b_{k-l} \; .$$

- The product of $x(t)$ and $y(t)$ is periodic with the same period T_0.
- The CTFS coefficients of the product are obtained by a *convolution* between the sequences of CTFS coefficients a_k and b_k.

Time Reversal

$$x(-t) \overset{FS}{\leftrightarrow} a_{-k} \; . \tag{28}$$

- Reversal in time domain corresponds to reversal in the frequency domain.
- As a consequence, if a signal is even in time, its CTFS coefficients are even in frequency:

$$x(-t) = x(t) \overset{FS}{\leftrightarrow} a_{-k} = a_k \; ,$$

and if a signal is odd in time, its CTFS coefficients are odd in frequency:

$$x(-t) = -x(t) \overset{FS}{\leftrightarrow} a_{-k} = -a_k \; .$$

Conjugation

- Complex conjugation of a time signal corresponds to frequency reversal and complex conjugation of its CTFS coefficients:

$$x^*(t) \overset{FS}{\leftrightarrow} a_{-k}^* . \tag{29}$$

Conjugate Symmetry for Real Signal

- A real signal $x(t)$ is equal to its complex conjugate $x^*(t)$. This, in combination with (29), implies

$$x(t) = x^*(t) \overset{FS}{\leftrightarrow} a_k = a_{-k}^* . \tag{30}$$

  - If a signal is real, its CTFS coefficients at positive frequency equal the complex conjugates of its CTFS coefficients at negative frequency.
  - This property of the CTFS coefficients is called *conjugate symmetry*.

- This property can be restated in two alternate ways.
  - If a signal is real, the magnitudes of its CTFS coefficients are even in frequency, while the phases of its CTFS coefficients are odd in frequency:

$$x(t) = x^*(t) \overset{FS}{\leftrightarrow} \begin{cases} |a_k| = |a_{-k}| \\ \angle a_k = -\angle a_{-k} \end{cases}. \tag{30a}$$

  - If a signal is real, the real parts of the CTFS coefficients are even in frequency, while the imaginary parts of its CTFS coefficients are odd in frequency:

$$x(t) = x^*(t) \overset{FS}{\leftrightarrow} \begin{cases} \mathrm{Re}(a_k) = \mathrm{Re}(a_{-k}) \\ \mathrm{Im}(a_k) = -\mathrm{Im}(a_{-k}) \end{cases}. \tag{30b}$$

Real, Even or Real, Odd Signals

- Combining the time reversal and conjugation properties, we find that

$$x(t) \text{ real and even in } t \overset{FS}{\leftrightarrow} a_k \text{ real and even in } k \tag{31}$$

$$x(t) \text{ real and odd in } t \overset{FS}{\leftrightarrow} a_k \text{ imaginary and odd in } k. \tag{32}$$

- *Question*: can you provide examples of (31) and (32) using simple signals, such as sinusoids?

Fourier Series Synthesis of a Real Signal

- We clarify the roles of positive and negative frequencies and conjugate-symmetric CTFS coefficients. Consider the synthesis of a real signal:

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} , \qquad (11)$$

  where $a_k = a_{-k}^*$.

- The term in (11) at zero frequency satisfies $a_0 = a_{-0}^*$.

  The term is a real constant equal to the average value of the signal:

$$a_0 e^{j0\omega_0 t} = a_0 .$$

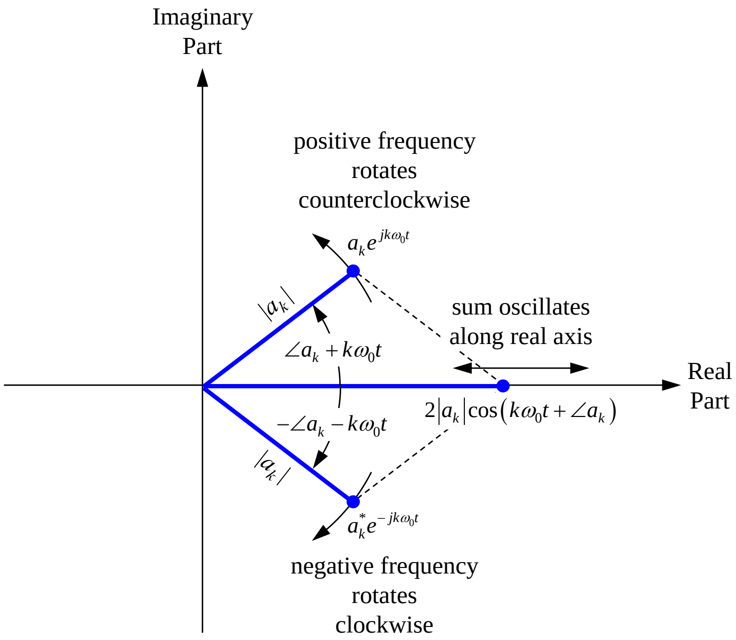
- Consider any pair of terms in (11) at positive and negative frequencies $\pm k\omega_0$, $k \neq 0$.

  The CTFS coefficients satisfy $a_k = a_{-k}^*$ (or $a_{-k} = a_k^*$).

  The two terms yield a real cosine at frequency $k\omega_0$:

$$\begin{aligned} a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t} &= a_k e^{jk\omega_0 t} + \left(a_k e^{jk\omega_0 t}\right)^* \\ &= 2\left|a_k\right| \cos\left(\angle a_k + k\omega_0 t\right) \end{aligned} . \qquad (33)$$

- This figure helps us interpret equation (33).



Observations

- Positive-frequency term $a_k e^{jk\omega_0 t}$:
  a vector rotating *counterclockwise*,
  with magnitude $|a_k|$ and
  phase $\angle a_k + k\omega_0 t$.

- Negative-frequency term
  $a_{-k} e^{-jk\omega_0 t} = a_k^* e^{-jk\omega_0 t} = \left(a_k e^{jk\omega_0 t}\right)^*$:
  a vector rotating *clockwise*,
  with magnitude $|a_k|$ and
  phase $-\angle a_k - k\omega_0 t$.

- The sum of these two vectors is always a real cosine that oscillates along the real axis.

Inner Products (you may choose to skip slides 38-42)

Dot Product

- The *dot product* is an important tool in the study of ordinary vectors, which are *N*-tuples with real- or complex-valued entries. The dot product is a function that maps a pair of vectors, **x** and **y**, to a scalar quantity denoted by $\mathbf{x} \cdot \mathbf{y}$.
- In the table below, the first column reviews key properties of the dot product.
    - Given a vector **x**, the *norm* or *length* of **x** is $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$, which is the square root of the dot product of **x** with itself.
    - Given nonzero vectors **x** and **y**, if their dot product is zero, $\mathbf{x} \cdot \mathbf{y} = 0$, the two vectors are *orthogonal*, i.e., they point along *perpendicular* directions.
    - The magnitude of the dot product between the two vectors cannot exceed the product of their norms, $|\mathbf{x} \cdot \mathbf{y}| \le \|\mathbf{x}\| \|\mathbf{y}\|$, with equality only when the two vectors are *parallel*, i.e., when one vector is a scalar multiple of the other.

Inner Product

- The *inner product* is a generalization of the dot product to more general types of vectors. The inner product is a function that maps a pair of vectors, *x* and *y*, to a scalar quantity denoted by $\langle x, y\rangle$.

- The second column of the table below presents key properties of the inner product.

  - In EE 102A and 102B, *x* and *y* may denote CT signals $x(t)$ and $y(t)$ or DT signals $x[n]$ and $y[n]$. The definition of the inner product $\langle x, y\rangle$ depends on whether the signals are CT or DT, and whether they are periodic or aperiodic.

  - Given a signal *x*, the *norm* of *x* is $\|x\| = (\langle x, x\rangle)^{1/2}$, which is the square root of the inner product of *x* with itself. The square of a signal's norm equals the signal's energy computed over one period (if the signal is periodic) or over all time (if it is aperiodic).

  - Given nonzero signals *x* and *y*, if their inner product is zero, $\langle x, y\rangle = 0$, the two signals are *orthogonal*.

  - The magnitude of the inner product between the two signals is less than or equal to the product of their norms, $|\langle x, y\rangle| \le \|x\|\|y\|$, with equality only when one signal is a scalar multiple of the other.

Comparison of Dot Product and Inner Product

	Dot Product	**Inner Product**
Applicable To	Vectors $\mathbf{x} = \{x_1, \ldots, x_N\}$ and $\mathbf{y} = \{y_1, \ldots, y_N\}$ Generally complex.	CT signals $x(t)$ and $y(t)$ or DT signals $x[n]$ and $y[n]$ Generally complex.
Denoted By	$\mathbf{x} \cdot \mathbf{y}$	$\langle x, y \rangle$
Definition	$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{N} x_i y_i^*$ Generally complex.	CT periodic: Chapter 3 (34) DT periodic: Chapter 3 (81) CT aperiodic: Chapter 4 (36) DT aperiodic: Chapter 5 (42) Generally complex.

Norm	$\|\mathbf{x}\| = (\mathbf{x}\cdot\mathbf{x})^{1/2}$ Always real. $\|\mathbf{x}\|$ is length of **x**.	$\|x\| = (\langle x, x\rangle)^{1/2}$ Always real. $\|x\|^2$ is energy of x $\begin{cases} \text{over one period} & x \text{ periodic} \\ \text{over all time} & x \text{ aperiodic} \end{cases}$.
Orthogonality	Assume **x**, **y** nonzero. If $\mathbf{x}\cdot\mathbf{y} = 0$, **x** and **y** are orthogonal, i.e., one is perpendicular to the other.	Assume x, y nonzero. If $\langle x, y\rangle = 0$, x and y are orthogonal.
Cauchy-Schwarz Inequality	Assume **x**, **y** nonzero. $\|\mathbf{x}\cdot\mathbf{y}\| \le \|\mathbf{x}\|\|\mathbf{y}\|$ Equality only if one is a multiple of the other, i.e., **x** and **y** are parallel.	Assume x, y nonzero. $\left\|\langle x, y\rangle\right\| \le \|x\|\|y\|$ Equality only if one is a multiple of the other.

Importance of Inner Products

- The inner products between signals are important in many applications. You will be better prepared to understand these after we study Parseval's identity for the CT Fourier transform (Chapter 4).
- For example, in digital communications, where the goal is to convey information bits:
  - We may use two different CT signals, $x(t)$ and $y(t)$, to encode the bits 0 and 1. The inner product between the two signals is a measure of how easily we can distinguish one signal from the other in the presence of noise. It determines the probability of mistaking one signal for the other. The two signals are most easily distinguished if one signal is the negative of the other, $y(t) = -x(t)$, in which case, $\langle x(t), y(t) \rangle = -\|x\|^2$ (see EE 279 or EE 379).
  - We often want to transmit several different signals simultaneously through a shared communication medium to maximize the rate at which information is conveyed. This is called *multiplexing* (see Chapter 7). It is desirable for a set of multiplexed signals to be *mutually orthogonal* so they do not interfere with each other. In other words, the inner product between any pair of signals should be zero.

Parseval's Identity

- *Parseval's identity* (or *Parseval's theorem*) is a property of Fourier representations.
- It exists in different forms for periodic or aperiodic signals in CT or DT.
- It helps us compute:
  - the inner product between two signals, or
  - the power or energy of one signal,

  either in time or frequency.
- Depending on the signal(s), the computation is often easier in one domain than in the other.

Parseval's Identify for Continuous-Time Fourier Series

- Parseval's identity for CTFS is relevant to periodic CT signals.
- We assume periodic signals having a common period T_0 and a common fundamental frequency $\omega_0 = 2\pi / T_0$. We consider one or two signals and their CTFS coefficients:

$$x(t) \overset{FS}{\leftrightarrow} a_k \quad \text{and} \quad y(t) \overset{FS}{\leftrightarrow} b_k .$$

General Case: Inner Product Between Signals

- The general form of Parseval's identity for an *inner product between two periodic CT signals* states:

$$\left\langle x(t), y(t)\right\rangle = \int_{T_0} x(t) y^*(t) dt = T_0 \sum_{k=-\infty}^{\infty} a_k b_k^* . \tag{34}$$

- Middle expression in (34): an inner product in the time domain between CT signals $x(t)$ and $y(t)$. Both are periodic signals, so we integrate over only a single period, of duration T_0.
- Rightmost expression in (34): an inner product in the frequency domain between the sequences of CTFS coefficients, a_k and b_k^*, which are infinite-length discrete vectors.

Example: Orthogonality of Even and Odd Square Waves

- In a homework problem, you will study the CTFS coefficients for an *even square wave* $y(t) \overset{FS}{\leftrightarrow} b_k$ and an *odd square wave* $z(t) \overset{FS}{\leftrightarrow} c_k$.
- By using symmetry properties, you will show that they are orthogonal in both the time and the frequency domains:

$$\int_{T_0} \underbrace{y(t)}_{\substack{\text{even}\\ \text{in } t}} \underbrace{z^*(t)}_{\substack{\text{odd}\\ \text{in } t}} dt = T_0 \sum_{k=-\infty}^{\infty} \underbrace{b_k}_{\substack{\text{even}\\ \text{in } k}} \underbrace{c_k^*}_{\substack{\text{odd}\\ \text{in } k}} = 0 .$$

Special Case: Signal Power

- Considering the special case of (34) with $x(t) = y(t)$ and $a_k = b_k$, we can compute the *power of a periodic CT signal*:

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 . \qquad (35)$$

- Middle expression in (35): power of signal computed in time domain.
It is the energy of $x(t)$ in one period, divided by the period T_0.

- Rightmost expression (35): power of signal computed in frequency domain.
It is the sum of the squared magnitudes of the CTFS coefficients a_k over $-\infty < k < \infty$.

- Interpretation of rightmost expression: $|a_k|^2$ is the *power density spectrum* of the periodic signal $x(t)$.
The term $|a_k|^2$ is the power contained in the component of the signal at frequency $k\omega_0$.
The rightmost expression in (35) is the sum of the powers at all frequencies $k\omega_0$, $-\infty < k < \infty$.

Proof of General Case ***(You may skip this.)***

- We prove the general form

$$\left\langle x(t), y(t) \right\rangle = \int_{T_0} x(t) y^*(t) dt = T_0 \sum_{k=-\infty}^{\infty} a_k b_k^* . \qquad (34)$$

- In the middle expression, we represent $y^*(t)$ by the complex conjugate of the CTFS synthesis of $y(t)$:

$$y^*(t) = \left(\sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} \right)^* .$$

- We substitute this into the integral to obtain

$$\int_{T_0} x(t) y^*(t) dt = \int_{T_0} x(t) \left(\sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} \right)^* dt .$$

- We interchange the order of summation and integration to obtain

$$\int_{T_0} x(t) y^*(t) dt = \sum_{k=-\infty}^{\infty} \left(\int_{T_0} x(t) e^{-jk\omega_0 t} dt \right) b_k^* = T_0 \sum_{k=-\infty}^{\infty} a_k b_k^* .$$

We have used the analysis equation (19) to substitute $T_0 a_k$ for the integral in parentheses.

QED

- *Note*: all other forms of Parseval's identity (for DTFS, CTFT, DTFT) are proven using similar steps.

Response of Continuous-Time Linear Time-Invariant Systems to Periodic Inputs

- We are given an LTI system with impulse response $h(t)$. We assume the integral defining the frequency response converges:

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \,. \tag{4}$$

- Then the imaginary exponentials

$$e^{j\omega t}, \ \omega \text{ real}, \ -\infty < t < \infty \,.$$

are eigenfunctions of the system, with eigenvalues given by $H(j\omega)$.

$$x(t) = e^{j\omega t} \longrightarrow \boxed{h(t) \leftrightarrow H(j\omega)} \longrightarrow y(t) = H(j\omega) \cdot e^{j\omega t}$$

- We input a signal $x(t)$, which is periodic with period $T_0 = 2\pi / \omega_0$ and can be expressed by a CTFS with coefficients a_k, $-\infty < k < \infty$:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \,. \tag{11}$$

- The output $y(t)$ is also periodic with period T_0. Using linearity of the system and the eigenfunction property (3) of the imaginary exponentials, the output $y(t)$ can be expressed by a CTFS

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t} \ . \tag{36}$$

- The output $y(t)$ has CTFS coefficients

$$b_k = a_k H(jk\omega_0), \ -\infty < k < \infty \ . \tag{37}$$

These are the CTFS coefficients of the input $x(t)$, scaled by values of the frequency response $H(j\omega)$ evaluated at $\omega = k\omega_0$, as shown.

$$x(t) \overset{FS}{\leftrightarrow} a_k \longrightarrow \boxed{h(t) \leftrightarrow H(j\omega)} \longrightarrow y(t) \overset{FS}{\leftrightarrow} b_k = a_k H(jk\omega_0)$$

- We can rewrite (37) to relate the magnitudes and phases of the input and output CTFS coefficients:

$$\begin{cases} |b_k| = |a_k| |H(jk\omega_0)| \\ \angle b_k = \angle a_k + \angle H(jk\omega_0) \end{cases} . \tag{37'}$$

Frequency Response of Continuous-Time Linear Time-Invariant Systems

- The frequency response $H(j\omega)$ of a CT LTI system is:
  - The Fourier transform of the impulse response $h(t)$:

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \ . \tag{4}$$

  - The eigenvalue for an imaginary exponential input signal $e^{j\omega t}$:

$$H\left\{e^{j\omega t}\right\} = H(j\omega) e^{j\omega t} \ .$$

- We study several important aspects of the frequency response.

Frequency Response of a Real System

- Consider an LTI system whose impulse response $h(t)$ is real:

$$h(t) = h^*(t). \tag{38}$$

- The frequency response $H(j\omega)$ at frequency ω is given by (4). To compute the frequency response at frequency $-\omega$, we evaluate (4) with the substitution $\omega \to -\omega$, and use (38):

$$\begin{aligned} H(-j\omega) &= \int_{-\infty}^{\infty} h(t) e^{j\omega t} dt \\ &= \int_{-\infty}^{\infty} h^*(t) e^{j\omega t} dt \\ &= \left(\int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \right)^* \\ &= H^*(j\omega) \end{aligned}. \tag{39}$$

- When the impulse response is real, the frequency response at negative frequency equals the complex conjugate of the frequency response at positive frequency.
- This property of the frequency response is called *conjugate symmetry*.

- We can summarize our finding succinctly as

$$h(t) = h^*(t) \leftrightarrow H(-j\omega) = H^*(j\omega). \tag{40}$$

We can restate (40) in two alternate ways. If the impulse response is real, then:

- The magnitude of the frequency response is even in frequency, while the phase of the frequency response is odd in frequency:

$$h(t) = h^*(t) \leftrightarrow \begin{cases} |H(-j\omega)| = |H(j\omega)| \\ \angle H(-j\omega) = -\angle H(j\omega) \end{cases}. \tag{40a}$$

- The real part of the frequency response is even in frequency, while the imaginary part of the frequency response is odd in frequency:

$$h(t) = h^*(t) \leftrightarrow \begin{cases} \mathrm{Re}\left[H(-j\omega)\right] = \mathrm{Re}\left[H(j\omega)\right] \\ \mathrm{Im}\left[H(-j\omega)\right] = -\mathrm{Im}\left[H(j\omega)\right] \end{cases}. \tag{40b}$$

Verifying a Real Input Leads to a Real Output **(We will skip but please read.)**

- Suppose we have a real system satisfying

$$h(t) = h^*(t) \leftrightarrow H(-j\omega) = H^*(j\omega). \tag{40}$$

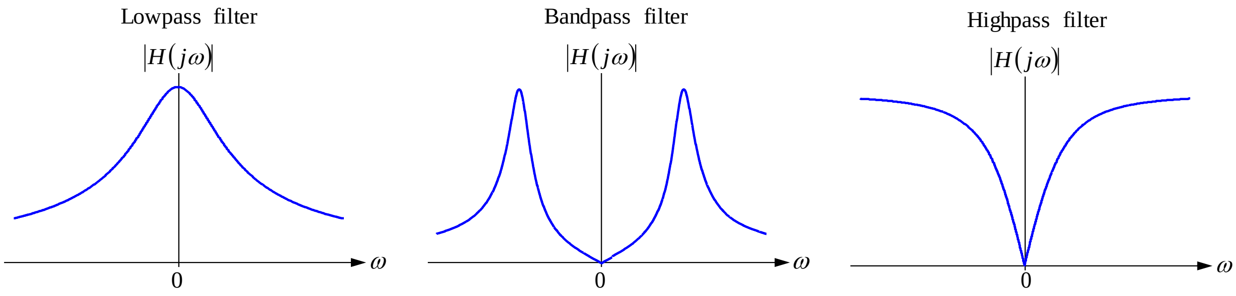
- Suppose we input a real, periodic signal $x(t)$, which has CTFS coefficients a_k.
  The CTFS coefficients of the input are conjugate-symmetric: $a_k = a_{-k}^*$, by (30).
- We obtain a periodic output $y(t)$ with CTFS coefficients $b_k = a_k H(jk\omega_0)$, by (37).
- The frequency response of the system is conjugate-symmetric: $H(jk\omega_0) = H^*(-jk\omega_0)$, by (40).
- Hence, the CTFS coefficients of the output $y(t)$ satisfy

$$\begin{aligned} b_k &= a_k H(jk\omega_0) \\ &= a_{-k}^* H^*(-jk\omega_0). \\ &= b_{-k}^* \end{aligned}$$

- In summary, the CTFS coefficients of the output $y(t)$ are conjugate symmetric: $b_k = b_{-k}^*$.
  Hence, the output $y(t)$ is real, as expected.

Types of Linear Distortion and Filters

- The *magnitude response* $\left|H(j\omega)\right|$ determines the scaling of different frequency components appearing at the output of an LTI system. We may refer to $\left|H(j\omega)\right|$ as the *amplitude response*.
    - *Magnitude distortion* (also *amplitude distortion*) occurs if $\left|H(j\omega)\right|$ is frequency-dependent.
    - Filters with a frequency-dependent magnitude response $\left|H(j\omega)\right|$ are often classified as *lowpass*, *bandpass*, or *highpass*, as illustrated here by typical examples.



- *Question*: do you know of some applications in which some of these types of filters are used?

- The *phase response* $\angle H(j\omega)$ determines the phase shifts of different frequency components appearing at the output of an LTI system.
  - If the phase is a linear function of frequency

$$\angle H(j\omega) = -t_0\omega,$$

    then all frequency components are subject to an equal time shift t_0.
  - Given any phase response $\angle H(j\omega)$, we define the *group delay* as $-\dfrac{d\angle H(j\omega)}{d\omega}$.
    The group delay describes how different frequency components of a signal are time-shifted.
  - *Phase distortion* occurs when $\angle H(j\omega)$ is not a linear function of frequency or, equivalently, when the group delay $-d\angle H(j\omega)/d\omega$ is frequency-dependent.

Distortionless System

- A *distortionless system* may scale and time-shift signals but causes no magnitude or phase distortion.

  Its impulse response is of the form (see Chapter 2, slides 35-36):

$$h(t) = C\,\delta(t - t_0)\,, \tag{41}$$

  where C is a constant, which we typically assume is real.

- Given any input $x(t)$, the output is

$$y(t) = C\,x(t - t_0)\,.$$

- We can obtain the frequency response of a distortionless system using the impulse response (41) in (4), which computes its CT Fourier transform:

$$\begin{aligned} H(j\omega) &= \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} C\,\delta(t - t_0) e^{-j\omega t} dt\,. \\ &= C e^{-j\omega t_0} \end{aligned} \tag{42}$$

- *Question*: how did we evaluate the integral in the second line of (42)?

- We have found that for a distortionless system

$$H(j\omega) = Ce^{-j\omega t_0} \tag{42}$$

  - The magnitude response $|H(j\omega)| = |C|$ is *constant*.
  - The phase response $\angle H(j\omega) = \angle C - \omega t_0$ varies *linearly with frequency*.

    The slope $-t_0$ is proportional to the negative of the time shift.

Verifying Consistency with CTFS Time-Shift Property **(We will skip but please read.)**

- We can verify that (42) is consistent with the CTFS time-shift property (27).
- We input a periodic signal

$$x(t) \overset{FS}{\leftrightarrow} a_k.$$

- We obtain a periodic output signal $y(t) = C\,x(t - t_0)$. We find its CTFS coefficients using (37) with frequency response (42):

$$y(t) \overset{FS}{\leftrightarrow} b_k = H(jk\omega_0)a_k = Ce^{-jk\omega_0 t_0}a_k.$$

  This is consistent with the CTFS time-shift property

$$x(t-t_0) \overset{FS}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k \,. \tag{27}$$

Methods for Evaluating the Frequency Response

- Here we describe two methods for computing the frequency response of a given LTI system. The list is not exhaustive, as other methods exist.

1. *Fourier transform of impulse response*

- Suppose an LTI system is specified in terms of an *impulse response* $h(t)$.

  We can find the frequency response by evaluating the CT Fourier transform integral

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \ . \tag{4}$$

- We just used this procedure to find the frequency response (42) of a distortionless system from its impulse response (41).

- We will be able to evaluate the CT Fourier transform integral (4) for more complicated impulse responses after we study the CT Fourier transform in Chapter 4.

2. *Substitution in differential equation*

- Suppose an LTI system is specified by a *linear, constant-coefficient differential equation* of the form (35) in Chapter 2, slide 69:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} .$$

- Suppose we know the system's frequency response exists, i.e., that the integral (4) converges for the system's impulse response.
- Then we can compute the frequency response directly from the differential equation without using (4). We use the following simple procedure:

1. Substitute the following input and output signals in the differential equation:

$$x(t) = e^{j\omega t} \quad \text{and} \quad y(t) = H(j\omega)e^{j\omega t} .$$

   Evaluate any time derivatives using

$$\frac{d}{dt} e^{j\omega t} = j\omega e^{j\omega t} .$$

2. Cancel all factors of $e^{j\omega t}$ and solve for $H(j\omega)$.

- We will learn how to determine when (4) converges when we study the CT Fourier transform in Chapter 4. Until then, we will apply this method only to carefully chosen examples.

Examples

- Here we apply Method 2 to three examples.

1. Differentiator. This is described by a differential equation

$$y(t) = \frac{dx}{dt} \quad .$$

- Substituting $x(t) = e^{j\omega t}$ and $y(t) = H(j\omega)e^{j\omega t}$ and evaluating the derivative:

$$H(j\omega)e^{j\omega t} = j\omega e^{j\omega t} \, .$$

- Cancelling factors of $e^{j\omega t}$ and solving for the frequency response:

$$H(j\omega) = j\omega \, .$$

- You will be asked in a homework problem to sketch the magnitude and phase of the differentiator frequency response.

2. *First-Order Lowpass Filter.* This is described by a differential equation

$$\tau \frac{dy}{dt} + y(t) = x(t) .$$

- Substituting $x(t) = e^{j\omega t}$ and $y(t) = H(j\omega)e^{j\omega t}$ and evaluating the derivative:

$$j\omega\tau H(j\omega)e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t} .$$

- Cancelling factors of $e^{j\omega t}$ and solving for the frequency response:

$$H(j\omega) = \frac{1}{1 + j\omega\tau} .$$

- In computing the magnitude and phase, we use the *reciprocal property* (see Appendix, page 289). Given complex-valued $z = |z|e^{j\angle z}$, its reciprocal is

$$\frac{1}{z} = \frac{1}{|z|e^{j\angle z}} = \frac{1}{|z|}e^{-j\angle z} .$$

  The magnitude and phase of the reciprocal are

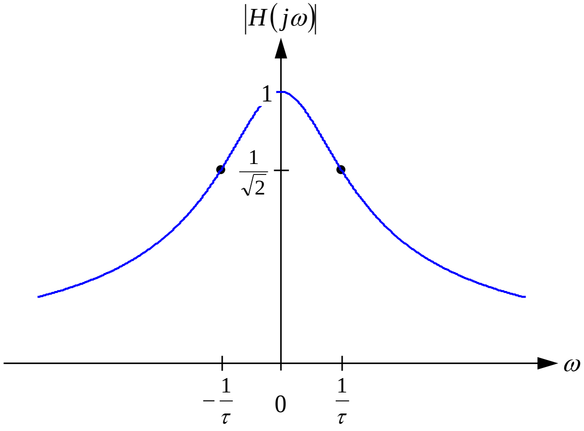
$$\left|\frac{1}{z}\right| = \frac{1}{|z|} \quad \text{and} \quad \angle\left(\frac{1}{z}\right) = -\angle z .$$

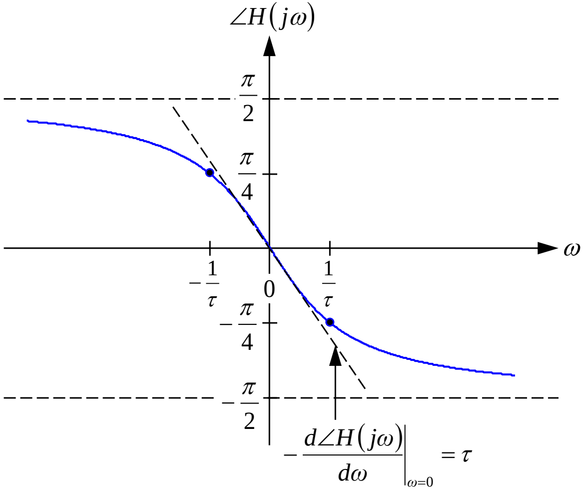
- Using the reciprocal property with $z = 1 + j\omega\tau$, we write the magnitude and phase responses of the lowpass filter as

$$\left|H(j\omega)\right| = \frac{1}{\left|1 + j\tau\omega\right|} = \frac{1}{\sqrt{1 + (\tau\omega)^2}}$$

$$\angle H(j\omega) = -\angle(1 + j\tau\omega) = -\tan^{-1}(\tau\omega).$$

- These are plotted below. This filter causes both magnitude distortion and phase distortion.





- Magnitude and phase values at some key frequencies are:

$$\left|H(j\omega)\right| = \begin{cases} 1 & \omega = 0 \\ \dfrac{1}{\sqrt{2}} & \omega = \pm\dfrac{1}{\tau} \\ 0 & \omega \to \pm\infty \end{cases} \quad \text{and} \quad \angle H(j\omega) = \begin{cases} 0 & \omega = 0 \\ \mp\dfrac{\pi}{4} & \omega = \pm\dfrac{1}{\tau} \\ \mp\dfrac{\pi}{2} & \omega \to \pm\infty \end{cases}.$$

- A first-order lowpass filter is often characterized by a *cutoff frequency* at $\omega = 1/\tau$, at which $\left|H(j\omega)\right|^2$ has half the value it has for $\omega = 0$, corresponding to a decrease of $10\log_{10}(1/2) \approx -3 \text{ dB}$. The nominal passband is often considered to be the frequency range $|\omega| \leq 1/\tau$.

- The group delay is obtained by differentiating the phase:

$$-\frac{d\angle H(j\omega)}{d\omega} = -\frac{d}{d\omega}\left[-\tan^{-1}(\tau\omega)\right] = \frac{\tau}{1+(\tau\omega)^2}.$$

  Near ω = 0, where the magnitude is largest, the phase $\angle H(j\omega)$ has a slope –τ, corresponding to a group delay $-d\angle H(j\omega)/d\omega = \tau$, as indicated on the plot above.

3. *First-Order Highpass Filter.* This is described by a differential equation

$$\frac{dy}{dt}+\frac{1}{\tau}y(t)=\frac{dx}{dt}\,.$$

- Substituting $x(t)=e^{j\omega t}$ and $y(t)=H(j\omega)e^{j\omega t}$ and evaluating the derivatives:

$$j\omega H(j\omega)e^{j\omega t}+\frac{1}{\tau}H(j\omega)e^{j\omega t}=j\omega e^{j\omega t}\,.$$

- Cancelling factors of $e^{j\omega t}$ and solving for the frequency response:

$$H(j\omega)=\frac{j\omega\tau}{1+j\omega\tau}\,.$$

- In computing the magnitude and phase, we use the *quotient property* (see Appendix, page 289). Given complex-valued $z_1=|z_1|e^{j\angle z_1}$ and $z_2=|z_2|e^{j\angle z_2}$, the quotient is

$$\frac{z_1}{z_2}=\frac{|z_1|e^{j\angle z_1}}{|z_2|e^{j\angle z_2}}=\frac{|z_1|}{|z_2|}e^{j(\angle z_1-\angle z_2)}\,.$$

  The magnitude and phase of the quotient are

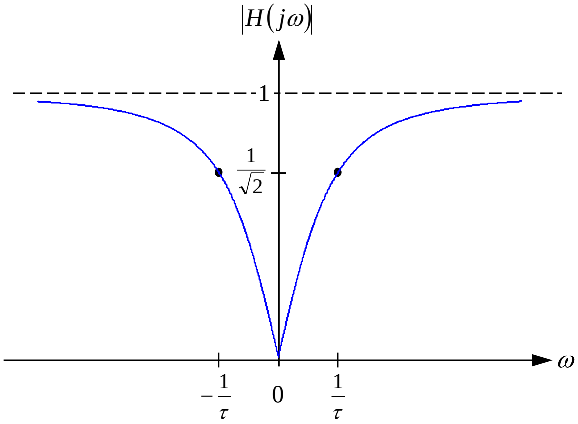
$$\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}\quad\text{and}\quad\angle\left(\frac{z_1}{z_2}\right)=\angle z_1-\angle z_2\,.$$

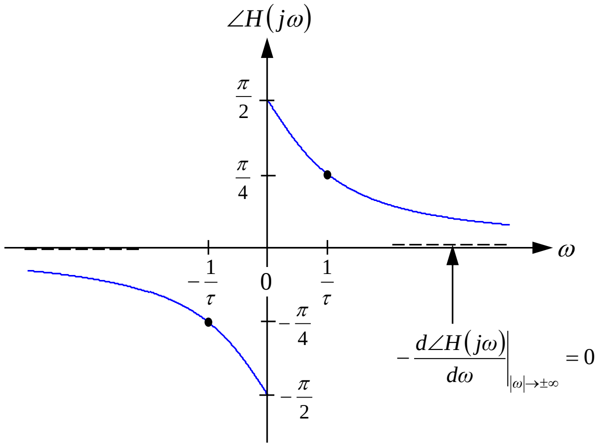
- Using the quotient property with $z_1 = j\omega\tau$ and $z_2 = 1 + j\omega\tau$, we write the magnitude and phase responses of the highpass filter as

$$\left|H(j\omega)\right| = \frac{\left|j\tau\omega\right|}{\left|1+j\tau\omega\right|} = \frac{\tau\left|\omega\right|}{\sqrt{1+(\tau\omega)^2}}$$

$$\angle H(j\omega) = \angle j\tau\omega - \angle(1+j\tau\omega) = \frac{\pi}{2}\operatorname{sgn}\omega - \tan^{-1}(\tau\omega)\,.$$

These are plotted below. This filter causes both magnitude distortion and phase distortion.





- Magnitude and phase values at some key frequencies are as follows:

$$\left|H\left(j\omega\right)\right| = \begin{cases} 0 & \omega = 0 \\ \dfrac{1}{\sqrt{2}} & \omega = \pm\dfrac{1}{\tau} \\ 1 & \omega \to \pm\infty \end{cases} \quad \text{and} \quad \angle H\left(j\omega\right) = \begin{cases} \pm\dfrac{\pi}{2} & \omega = 0 \\ \pm\dfrac{\pi}{4} & \omega = \pm\dfrac{1}{\tau} \\ 0 & \omega \to \pm\infty \end{cases}.$$

A first-order highpass filter is characterized by a *cutoff frequency* at $\omega = 1/\tau$, at which $\left|H\left(j\omega\right)\right|^2$ has half the value it has for $|\omega| \to \infty$. The nominal passband may be considered to be the range $|\omega| \geq 1/\tau$.

- The group delay is computed by differentiating the phase:

$$-\frac{d\angle H\left(j\omega\right)}{d\omega} = \frac{\tau}{1+\left(\tau\omega\right)^2}, \ \omega \neq 0.$$

As $\omega \to \pm\infty$, where the magnitude response is largest, the phase $\angle H\left(j\omega\right)$ has a zero slope, corresponding to a group delay $-d\angle H\left(j\omega\right)/d\omega = 0$, as indicated on the plot above.

Examples of Filtering Periodic Continuous-Time Signals by Linear Time-Invariant Systems

Method of Analysis

- We use the analysis presented on slides 47-48 above, which is summarized in the figure.

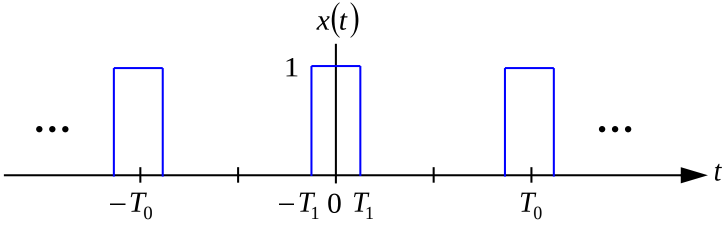
Periodic Input

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow \boxed{\begin{array}{c}\text{LTI System} \\ h(t) \leftrightarrow H(j\omega)\end{array}} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

Periodic Output

Input Signal

- The input signal $x(t)$ is a rectangular pulse train with period $T_0 = 2\pi / \omega_0$ and pulse width $2T_1$. We studied this signal on slides 23-29 above.



- Its CTFS coefficients are given by

$$a_k = \frac{\omega_0 T_1}{\pi} \mathrm{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right). \tag{24}$$

We choose $\frac{T_1}{T_0} = \frac{1}{8}$, so the CTFS coefficients become

$$a_k = \frac{1}{4} \mathrm{sinc}\left(\frac{k}{4}\right),$$

and the CTFS representation of the input becomes

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{1}{4} \mathrm{sinc}\left(\frac{k}{4}\right) e^{jk\omega_0 t}.$$

- Because $x(t)$ is real and even in *t*, its CTFS coefficients a_k are real and even in *k* (recall (31)). Since the a_k are purely real, their phases can only be an integer multiple of π, and are typically chosen as

$$\angle a_k = \begin{cases} 0 & a_k > 0 \\ \pm\pi & a_k < 0 \end{cases}.$$

In the figures below, when $a_k < 0$, we make the specific choices $\angle a_k = +\pi$ for $k < 0$ and $\angle a_k = -\pi$ for $k > 0$ so the phase appears with the odd symmetry expected, but this is not necessary. (See Appendix, pages 300-301, for further explanation.)

Linear Time-Invariant Systems

- We consider the first-order lowpass and first-order highpass filters, whose frequency responses were analyzed on slides 60-65 above.

- For each filter, we consider two choices of the time constant τ, which determines the filter cutoff frequency at $\omega = 1/\tau$:
    - High cutoff frequency: $\tau / T_0 = 0.03$.
    - Low cutoff frequency: $\tau / T_0 = 0.3$.

Output Signal

- Given an LTI system with frequency response $H\left(j\omega\right)$, using (36), the CTFS representation of the output is

$$y\left(t\right) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k H\left(jk\omega_0\right) e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{1}{4} \operatorname{sinc}\left(\frac{k}{4}\right) H\left(jk\omega_0\right) e^{jk\omega_0 t} .$$

- Recall that the output CTFS coefficients b_k are given by the input CTFS coefficients a_k, scaled by values of the frequency response $H(j\omega)$ evaluated at $\omega = k\omega_0$:

$$b_k = a_k H(jk\omega_0). \tag{37}$$

As a result, the magnitudes and phases of the input and output CTFS coefficients are related by

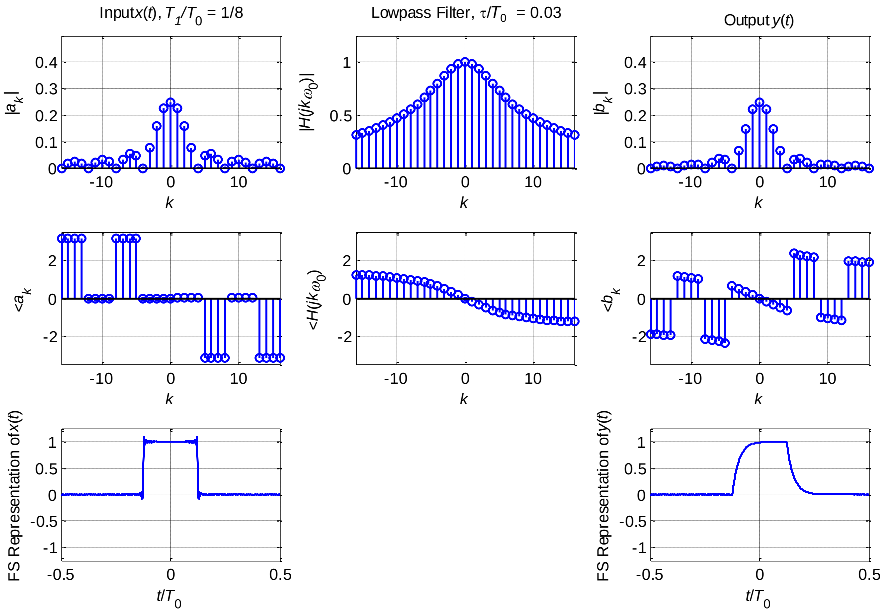
$$\begin{cases} |b_k| = |a_k||H(jk\omega_0)| \\ \angle b_k = \angle a_k + \angle H(jk\omega_0) \end{cases}. \tag{37'}$$

In each figure below, the relationship (37') should be evident in the first row (which shows $|a_k|$, $|H(jk\omega_0)|$ and $|b_k|$) and in the second row (which shows $\angle a_k$, $\angle H(jk\omega_0)$ and $\angle b_k$).

Generating Plots

- The plots of $x(t)$ and $y(t)$ shown are computed using CTFS terms for $-128 \le k \le 128$, a total of 257 terms.

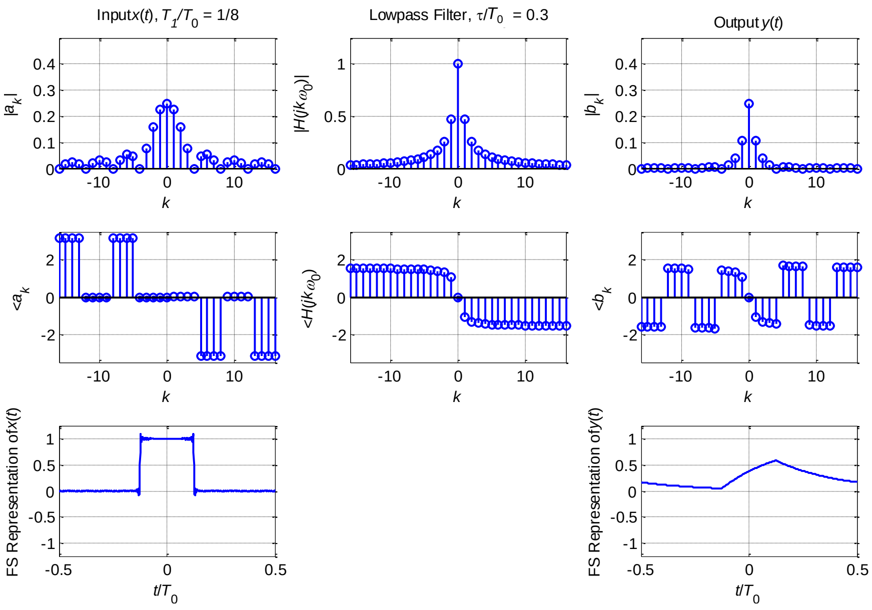
Filtering by First-Order Lowpass Filter



Observations

- The d.c. level (average value) is preserved, since $H(j0)=1$.
- As τ increases, the rise and fall times increase.
- The centroid of each pulse, determined mainly by low-frequency components, is delayed noticeably, and is consistent with $-d\angle H(j\omega)/d\omega\big|_{\omega=0}=\tau$.

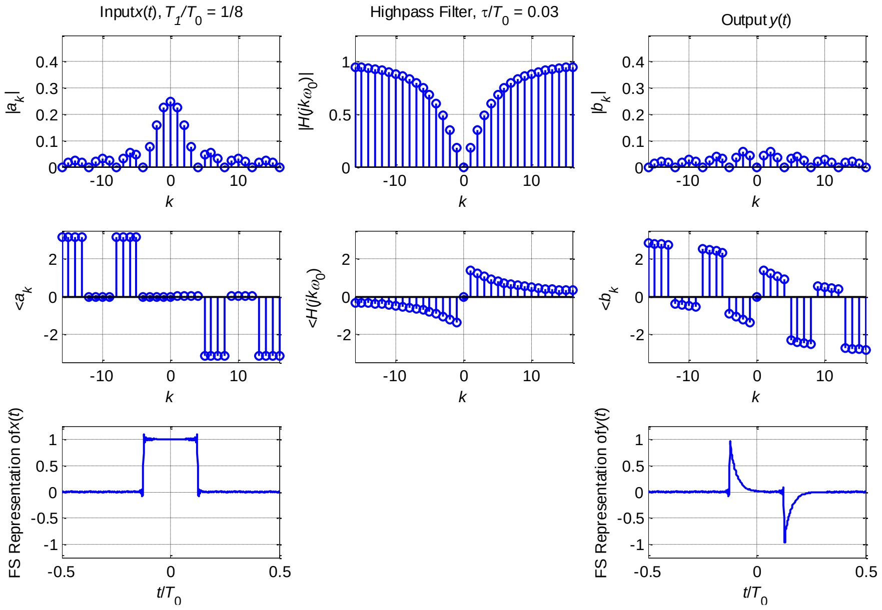
Filtering by First-Order Lowpass Filter (cont.)



Observations

- The d.c. level (average value) is preserved, since $H(j0)=1$.
- As τ increases, the rise and fall times increase.
- The centroid of each pulse, determined mainly by low-frequency components, is delayed noticeably, and is consistent with $-d\angle H(j\omega)/d\omega\big|_{\omega=0}=\tau$.

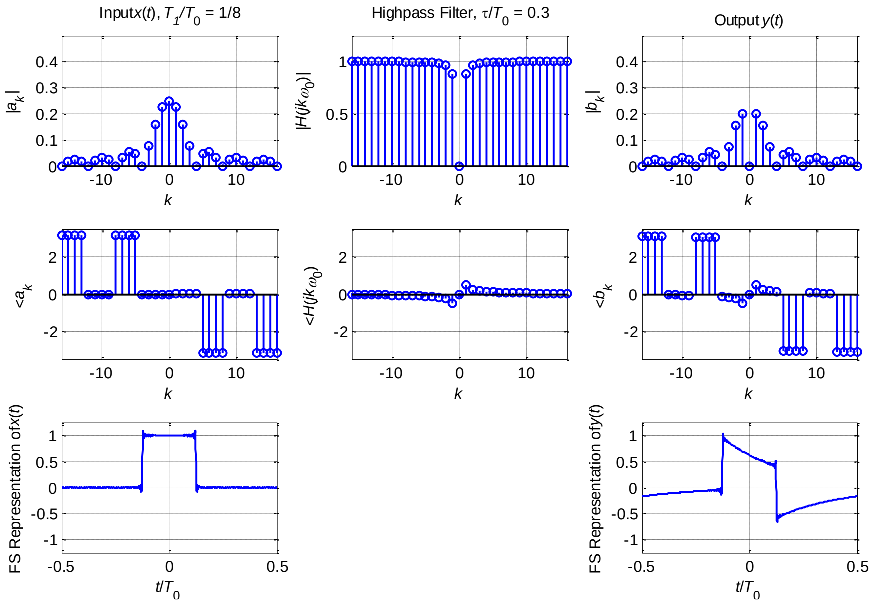
Filtering by First-Order Highpass Filter



Observations

- The d.c. level (average value) is removed, since $H(j0)=0$.
- At small τ, only the leading and trailing edges of each pulse remain. At larger τ, more of each pulse remains, but the baseline (formerly the zero level) wanders up and down.
- The leading and trailing edges, determined mainly by high-frequency components, are delayed little, consistent with $-d\angle H(j\omega)/d\omega\big|_{\omega\to\pm\infty}=0$.

Filtering by First-Order Highpass Filter (cont.)



Observations

- The d.c. level (average value) is removed, since $H(j0) = 0$.
- At small τ, only the leading and trailing edges of each pulse remain. At larger τ, more of each pulse remains, but the baseline (formerly the zero level) wanders up and down.
- The leading and trailing edges, determined mainly by high-frequency components, are delayed little, consistent with $-d\angle H(j\omega)/d\omega\big|_{\omega\to\pm\infty} = 0$.

Comment on Method of Analysis ***(We will skip but please read.)***

- We have analyzed these examples using CTFS.
- We could analyze them using convolution methods from Chapter 2. In that case, we would:
  - Represent the periodic rectangular pulse train input $x(t)$ as an infinite sum of scaled and shifted step functions.
  - Represent the periodic output $y(t)$ as a corresponding sum of scaled and shifted step responses.

  The periodic outputs $y(t)$ we obtain here can be understood using this approach.
- The FS method we used offers important advantages:
  - It takes account of the overlap between all the scaled, shifted step responses.
  - It can be applied to *any* periodic input signal $x(t)$ with finite power, even if that signal cannot be represented easily in terms of simple functions, such as step functions.

First-Order and Higher-Order Systems ***(We will skip but please read.)***

- These simple first-order filters are easy to analyze. But they are not suitable for many applications, since they have only one energy storage element, and are described by only a single parameter τ that governs both their time-domain response and frequency response.
- Important time-domain properties include:
  - Impulse response, step response, rise time and overshoot.
- Important frequency-domain properties include:
  - Abruptness of the transition from the passband to the stopband.
  - Group delay $-d\angle H(j\omega)/d\omega$.
  - Variations of the magnitude and group delay within the passband.
- If we introduce more energy storage elements, such as inductors and capacitors in electrical circuits, or kinetic and potential energy in mechanical systems, we obtain systems described by *higher-order differential equations*.
- Such *higher-order systems* offer more flexibility in their characteristics. They can achieve a sharper passband-stopband transition and offer more control over tradeoffs between time-domain response and frequency response. In studying the CT Fourier transform (Chapter 4), we will learn more about the time- and frequency-domain properties of systems and will study second-order continuous-time systems.

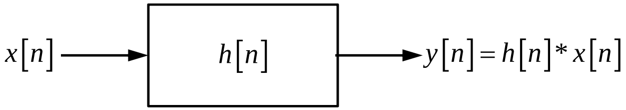
Eigenfunctions of Discrete-Time Linear Time-Invariant Systems

- We have discussed the concepts of eigenfunctions, FS and frequency response for CT.
  We have applied them to analyzing the response of CT LTI systems to periodic inputs.
- Now we will present analogous concepts for DT.
  Much of the development is similar to CT, but there are important differences.

General Case: Complex Exponentials

- Consider a DT LTI system *H*, which has an impulse response $h[n]$.
- If we are given a general input signal $x[n]$ and wish to predict the output $y[n] = H\{x[n]\}$,

  we must perform a *convolution* between $x[n]$ and $h[n]$.



- The *eigenfunctions* of DT LTI systems are *complex exponential* time signals

$$z^n, \; z \text{ complex}, \; -\infty < n < \infty .$$

- If we input one of these signals to an LTI system *H*, the output is the same signal z^n, multiplied by an *eigenvalue* denoted by $H(z)$, as shown below.

$$x[n] = z^n \longrightarrow \boxed{h[n] \leftrightarrow H(z)} \longrightarrow y[n] = H(z)\cdot z^n$$

- The variable z is called *complex frequency*.
  $H(z)$ is called the *transfer function* of the LTI system *H*.

- Assuming we know the transfer function $H(z)$ as a function of *z*, then for an input z^n, we can predict the output by using *multiplication*, and need not use convolution.

- *Proof*: we input $x[n]=z^n$ to the system and compute the output $y[n]$ using the convolution sum

$$
\begin{aligned}
y[n] &= h[n]*x[n] \\
&= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]z^{n-k} \qquad . \\
&= z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} \\
&= z^n \cdot H(z)
\end{aligned}
\tag{43}
$$

- We have defined the transfer function in (43) as

$$
H(z) \stackrel{d}{=} \sum_{n=-\infty}^{\infty} h[n]z^{-n} \ . \tag{44}
$$

Given an impulse response $h[n]$, we compute the sum (44) to obtain the transfer function $H(z)$. The sum (44) defines $H(z)$ as the *bilateral Z transform* of the impulse response $h[n]$.

- The Z transform sum (44) converges for a many impulse responses $h[n]$, including some describing unstable systems (see *EE 102B Course Reader*, Chapter 7).

Special Case: Imaginary Exponentials

- Now consider the special case that $z = e^{j\Omega}$, where Ω is real. We are considering a subset of the complex exponential signals, which are the *imaginary exponential* signals

$$e^{j\Omega n}, \ \Omega \text{ real}, \ -\infty < n < \infty .$$

- These, too, are eigenfunctions of DT LTI systems. If we input an imaginary exponential $e^{j\Omega n}$ to an LTI system, the output is the same signal, multiplied by an eigenvalue $H\left(e^{j\Omega}\right)$, as shown below.

$$x[n] = e^{j\Omega n} \longrightarrow \boxed{h[n] \leftrightarrow H\left(e^{j\Omega}\right)} \longrightarrow y[n] = H\left(e^{j\Omega}\right) \cdot e^{j\Omega n}$$

- The variable Ω is simply called *frequency*.
  $H\left(e^{j\Omega}\right)$ is called the *frequency response* of the LTI system *H*.

- As in the general case above, if we know $H\left(e^{j\Omega}\right)$ as a function of Ω, then for an imaginary exponential input signal $e^{j\Omega n}$, we can predict the system output by using multiplication, and do need to use convolution.

Proof

- We use an input $x[n]=e^{j\Omega n}$ in (43). Then (43) yields an output

$$y[n]=e^{j\Omega n}\cdot H\left(e^{j\Omega}\right). \tag{45}$$

- The system frequency response appearing in (45) is defined by the following expression:

$$H\left(e^{j\Omega}\right)\stackrel{d}{=}H(z)\Big|_{z=e^{j\Omega}}=\sum_{n=-\infty}^{\infty}h[n]e^{-jn\Omega}. \tag{46}$$

- Given an impulse response $h[n]$, we evaluate the sum (46) to find the frequency response $H\left(e^{j\Omega}\right)$. The sum (46) defines $H\left(e^{j\Omega}\right)$ as the *DT Fourier transform* of $h[n]$.

- The DT Fourier transform sum (46) converges for many impulse responses $h[n]$, but not in some cases (notably, some important unstable systems) for which the Z transform (44) converges. This motivates us to study the Z transform in EE 102B (see *EE 102B Course Reader*, Chapter 7).

Application to LTI System Analysis

- Consider a DT LTI system *H*. Assume we know its transfer function $H(z)$.

- Suppose we are given an input signal expressed as a linear combination of complex exponential signals at *K* distinct values of *z*:

$$x[n] = a_1 z_1^n + a_2 z_2^n + \cdots + a_K z_K^n \,. \tag{47}$$

- Using the linearity of the system and the eigenfunction property (43), we can compute the output signal as

$$y[n] = a_1 H(z_1) z_1^n + a_2 H(z_2) z_2^n + \cdots + a_K H(z_K) z_K^n \,. \tag{48}$$

- We have computed each term in the output (48) by using multiplication, without using convolution.

- We can easily apply (47) and (48) in the special case of imaginary exponential signals.

Example

- An LTI system *H* with input $x[n]$ and output $y[n]$ has an input-output relation

$$y[n] = x[n] - x[n-2]. \tag{49}$$

- We are given an input signal

$$x[n] = e^{j\frac{\pi}{4}n} + 2^n,$$

which is in the form of (47) with $z_1 = e^{j\pi/4}$ and $z_2 = 2$.

- Using input-output relation (49), we can compute the output signal:

$$\begin{aligned} y[n] &= e^{j\frac{\pi}{4}n} - e^{j\frac{\pi}{4}(n-2)} + 2^n - 2^{(n-2)} \\ &= \left(1 - e^{-j\frac{\pi}{2}}\right) e^{j\frac{\pi}{4}n} + \left(1 - 2^{-2}\right) 2^n . \\ &= H\left(e^{j\frac{\pi}{4}}\right) e^{j\frac{\pi}{4}n} + H(2) 2^n \end{aligned} \tag{50}$$

The output signal (50) is consistent with the general form (48).

- In EE 102B, we will learn that input-output relation (49) corresponds to a transfer function

$$H(z) = 1 - z^{-2}.$$

Knowing this, we can use (48) to obtain the output $y[n]$ without doing the computations in (50).

Discrete-Time Fourier Series

- We are given a periodic DT signal that satisfies

$$x[n] = x[n+N] \quad \forall n, \tag{51}$$

  where N is the *period* and the *fundamental frequency* is

$$\Omega_0 = \frac{2\pi}{N}. \tag{52}$$

- *Discrete-time Fourier series* (DTFS) representation of periodic signal $x[n]$: a linear combination of imaginary exponential basis sequences, each at a frequency $k\Omega_0$, k is an integer.
- We use only exponential FS for DT signals, for the same reasons we use only exponential FS for CT signals.
- We will show that for some simple periodic DT signals, it is possible to determine the FS coefficients by inspection, as in CT.
- We will first explain the properties of the basis sequences and derive the synthesis and analysis equations.

Imaginary Exponential Basis Sequences

- In the exponential DTFS, we use *imaginary exponential basis sequences*, which are defined as

$$\phi_k[n] = e^{jk\left(\frac{2\pi}{N}\right)n},\ k = 0,\ \pm 1,\ \pm 2, \ldots . \tag{53}$$

- Each of these sequences is periodic with period *N*:

$$\phi_k[n+N] = e^{jk\left(\frac{2\pi}{N}\right)(n+N)} = e^{jk\left(\frac{2\pi}{N}\right)n} \underbrace{e^{jk2\pi}}_{=1} = \phi_k[n] \quad \forall n . \tag{54}$$

- Recall that any two DT imaginary exponential sequences are identical if their frequencies differ by an integer multiple of 2π (Chapter 1, slide 40). As a consequence, $\phi_k[n]$ and $\phi_{k+N}[n]$ are identical:

$$\phi_{k+N}[n] = e^{j(k+N)\left(\frac{2\pi}{N}\right)n} = e^{jk\left(\frac{2\pi}{N}\right)n} \underbrace{e^{j2\pi n}}_{=1} = \phi_k[n] \quad \forall n . \tag{55}$$

- We only need to consider the *N* distinct sequences

$$\phi_k[n],\ k = \langle N \rangle . \tag{56}$$

- The following notation denotes any *N* consecutive values of a discrete variable *k* or *n*:

$$k = \langle N \rangle \quad \text{means} \quad k_1 + 1 \le k \le k_1 + N, \quad k_1 \text{ arbitrary}$$

$$n = \langle N \rangle \quad \text{means} \quad n_1 + 1 \le n \le n_1 + N, \quad n_1 \text{ arbitrary} .$$

- We will show that the basis sequences form an orthogonal set over any time interval of length *N*.
  We compute the following summation, which is an inner product between $\phi_k[n]$ and $\phi_l[n]$:

$$\sum_{n=\langle N\rangle} \phi_k[n]\phi_l^*[n] = \sum_{n=\langle N\rangle} e^{j(k-l)\left(\frac{2\pi}{N}\right)n} \ . \tag{57}$$

- We use Euler's relation (see Appendix, page 288) to express the imaginary exponentials in (57) in terms of sinusoids.
  - Evaluating the right-hand side of (57) for $k \neq l$, we obtain

$$\sum_{n=\langle N\rangle} \left[\cos\left((k-l)\left(\frac{2\pi}{N}\right)n \right) + j\sin\left((k-l)\left(\frac{2\pi}{N}\right)n \right) \right] = 0 \ .$$

    Each of the two sums (real and imaginary) is over an integer number of cycles.
    Each contains positive and negative contributions that cancel precisely.
  - Evaluating the right-hand side of (57) for $k = l$, we set the summand to unity and obtain

$$\sum_{n=\langle N\rangle} (1) = N \ .$$

- In summary, the basis sequences form an orthogonal set, with pairwise inner product sums given by

$$\sum_{n=\langle N\rangle} \phi_k[n]\phi_l^*[n] = \sum_{n=\langle N\rangle} e^{j(k-l)\left(\frac{2\pi}{N}\right)n} = \begin{cases} 0 & k \neq l \\ N & k = l \end{cases} \ . \tag{58}$$

Synthesis Equation

- We are given a periodic DT signal $x[n]$ with period *N* and fundamental frequency $\Omega_0 = 2\pi / N$.
- We will represent $x[n]$ as an exponential DTFS in the form

$$\hat{x}[n] = \sum_{k=\langle N \rangle} a_k \phi_k[n] = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} . \tag{59}$$

  - We call the coefficients a_k, $k = \langle N \rangle$ the *DTFS coefficients* for the DT signal $x[n]$.
  - Expression (59) is the *synthesis equation* for the DTFS. It shows how to synthesize a periodic signal using *N* distinct imaginary exponentials $e^{jk\Omega_0 n}$ at frequencies $k\Omega_0 = k(2\pi / N)$, $k = \langle N \rangle$.
  - By contrast, the CTFS synthesis (11) is, in general, a summation of terms at an infinite number of frequencies $k\omega_0$, $-\infty < k < \infty$. This is a major difference between the CTFS and the DTFS.
- We can verify that the DTFS synthesis (59) describes a DT signal that is *periodic in time n* with period *N*, as desired:

$$\hat{x}[n+N] = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)(n+N)} = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \underbrace{e^{jk2\pi}}_{=1} = \hat{x}[n]. \tag{60}$$

- The DTFS synthesis (59) is *identical* to the original periodic signal at all times n:

$$\hat{x}[n] = x[n]. \tag{61}$$

  This is because:

  - A periodic DT signal $x[n]$ is defined *entirely* by its values over any one period, $n = \langle N \rangle$.
    It can be considered a vector in an N-dimensional space.

  - Hence, it can be represented *exactly* using a linear combination of N orthogonal sequences.

- By contrast, a CTFS synthesis $\hat{x}(t)$ is not necessarily identical to the original signal $x(t)$ at all t.
  For example, they differ near values of t where $x(t)$ has discontinuities.

- This is another major difference between the CTFS and the DTFS.

Analysis Equation

- We are given a periodic DT signal $x[n]$ with period N and fundamental frequency $\Omega_0 = 2\pi / N$.
- We will represent $x[n]$ as an exponential DTFS in the form

$$\hat{x}[n] = \sum_{k=\langle N \rangle} a_k \phi_k[n] = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} . \tag{59}$$

- We will derive an analysis equation to obtain the DTFS coefficients a_k in (59).

  We start with (61):

$$x[n] = \hat{x}[n] . \tag{61}$$

- We compute an inner product sum between both sides of (61) and the imaginary exponential basis signal $\phi_l[n] = e^{jl\left(\frac{2\pi}{N}\right)n}$:

$$\sum_{n=\langle N \rangle} x[n] e^{-jl\left(\frac{2\pi}{N}\right)n} = \sum_{n=\langle N \rangle} \hat{x}[n] e^{-jl\left(\frac{2\pi}{N}\right)n} . \tag{62}$$

- Now we substitute synthesis equation (59) for $\hat{x}[n]$ on the right-hand side of (62):

$$
\begin{aligned}
\sum_{n=\langle N\rangle} x[n]e^{-jl\left(\frac{2\pi}{N}\right)n} &= \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} e^{-jl\left(\frac{2\pi}{N}\right)n} \\
&= \sum_{k=\langle N\rangle} a_k \underbrace{\sum_{n=\langle N\rangle} e^{j(k-l)\left(\frac{2\pi}{N}\right)n}}_{=\begin{cases}0 & k\neq l\\ N & k=l\end{cases}} \quad . \qquad (63) \\
&= Na_l
\end{aligned}
$$

  - In the second line, we interchanged the order of summation and evaluated the sum over *n* using (58).
  - In the third line, we evaluated the sum over *k*. We found that only the term for $k=l$ is nonzero.
  - Rearranging (63), we obtain

$$
a_l = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jl\left(\frac{2\pi}{N}\right)n} \quad . \qquad (64)
$$

- Equation (64) is the *analysis equation* for the DTFS. Given a periodic signal $x[n]$, it tells us how to obtain the DTFS coefficients a_k , $k=\langle N\rangle$ that are used in the synthesis equation (59).

- Observe that the DTFS coefficients given by analysis equation (64) are *periodic in frequency k* with period *N*:

$$a_{k+N} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-j(k+N)\left(\frac{2\pi}{N}\right)n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n}\underbrace{e^{-jk2\pi n}}_{=1} = a_k \quad \forall k\,. \tag{65}$$

  This periodicity of the DTFS coefficients is equivalent to the fact that the synthesis (59) only needs to use terms at *N* different frequencies, $k = \langle N\rangle$.

- By contrast, the CTFS coefficients are not generally periodic in frequency *k*.
  The CTFS synthesis requires, in general, an infinite number of terms, $-\infty < k < \infty$.

- We noted this above as a major difference between the CTFS and the DTFS.

Summary of Discrete-Time Fourier Series

- The synthesis and analysis equations are

$$\hat{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \qquad \text{(synthesis)} \qquad (59)$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n} \qquad \text{(analysis)} \qquad (64)$$

- We often denote a signal $x[n]$ and its DTFS coefficients a_k as a *DTFS pair*:

$$x[n] \overset{FS}{\leftrightarrow} a_k \ . \qquad (66)$$

- The DTFS coefficients for some important periodic DT signals are given in Table 6, Appendix.

Choice of Summation Interval

- In the synthesis and analysis equations (59) and (64), we can sum over any N consecutive values of k or n, denoted by $k = \langle N \rangle$ or $n = \langle N \rangle$.
- In some cases, it is natural to sum over an asymmetric interval, such as $0 \le k \le N-1$ or $0 \le n \le N-1$.
- In many cases, it is desirable to sum over an interval that is approximately symmetric about $k = 0$ or $n = 0$. In such cases, we can choose the interval as shown in the following table.

<table>
<tr><th>N</th><th>Minimum n or k</th><th>Maximum n or k</th></tr>
<tr><td>Odd</td><td>$-\frac{N-1}{2}$</td><td>$\frac{N-1}{2}$</td></tr>
<tr><td rowspan="3">Even</td><td>$-\frac{N}{2}+1$</td><td>$\frac{N}{2}$</td></tr>
<tr><td colspan="2">or</td></tr>
<tr><td>$-\frac{N}{2}$</td><td>$\frac{N}{2}-1$</td></tr>
</table>

Obtaining the Fourier Series by Inspection

- The DTFS coefficients of some simple periodic DT signals can be found by inspection.
  We consider an example:

$$x[n] = \cos\left(\frac{3\pi}{5}n\right) - \sin\left(\frac{4\pi}{5}n\right).$$

- *Question*: how can we find the period N and the fundamental frequency Ω_0 of $x[n]$?

- This signal is periodic with period $N = 10$ and fundamental frequency $\Omega_0 = 2\pi/10 = \pi/5$.
  We express it as a linear combination of imaginary exponentials with frequencies $k\Omega_0 = k\pi/5$.
- By inspection, we obtain

$$x[n] = \frac{1}{2}\left[e^{j3\left(\frac{\pi}{5}\right)n} + e^{-j3\left(\frac{\pi}{5}\right)n}\right] - \frac{1}{2j}\left[e^{j4\left(\frac{\pi}{5}\right)n} - e^{-j4\left(\frac{\pi}{5}\right)n}\right].$$

- By inspection, we obtained

$$x[n] = \frac{1}{2}\left[e^{j3\left(\frac{\pi}{5}\right)n} + e^{-j3\left(\frac{\pi}{5}\right)n}\right] - \frac{1}{2j}\left[e^{j4\left(\frac{\pi}{5}\right)n} - e^{-j4\left(\frac{\pi}{5}\right)n}\right].$$

- We choose a summation interval $k = \langle N \rangle$ as $-4 \le k \le 5$. The DTFS synthesis equation is

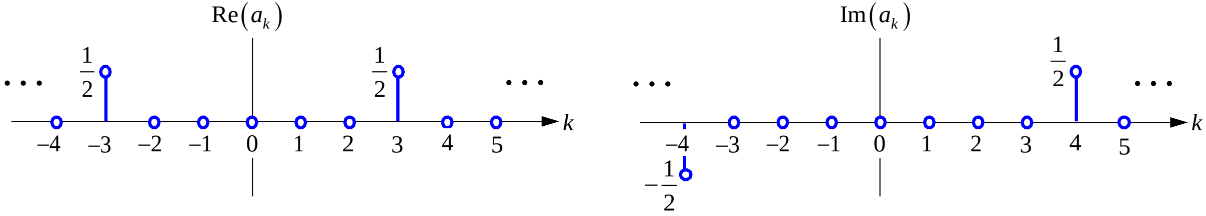
$$x[n] = \sum_{k=\langle 10 \rangle} a_k e^{jk\left(\frac{\pi}{5}\right)n} = \sum_{k=-4}^{5} a_k e^{jk\left(\frac{\pi}{5}\right)n}$$

- Comparing these two expressions for $x[n]$, we obtain non-zero DTFS coefficients:

$$a_{-3} = a_3 = \tfrac{1}{2} \quad \text{and} \quad -a_{-4} = a_4 = \tfrac{1}{2} j.$$

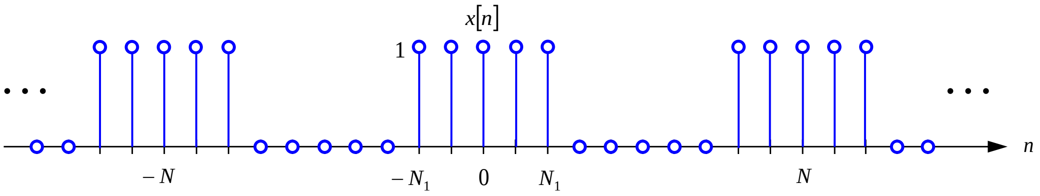
All other a_k are zero for $-4 \le k \le 5$. The a_k are periodic with period $N = 10$: $a_{k+10} = a_k \quad \forall k$.

- The DTFS coefficients are plotted here over one period, $-4 \le k \le 5$.



Fourier Series of a Rectangular Pulse Train

- As an example of using the analysis and synthesis equations, we consider a periodic rectangular pulse train. The period is N and the width of each pulse is $2N_1+1$.



- The pulse train can be described by the formula

$$x[n]=\begin{cases}1 & |n|\le N_1\\ 0 & N_1<|n|\le \dfrac{N}{2}\end{cases},\qquad x[n+N]=x[n].$$

Alternatively, it can be expressed as a series of shifted rectangular pulses (see Appendix, page 290) as:

$$x[n]=\sum_{l=-\infty}^{\infty}\Pi\left(\frac{n-lN}{2N_1}\right).$$

Analysis

- We use analysis equation (64) to compute the DTFS coefficients.

  $x[n]$ is symmetric, so we choose the summation interval, of length N, to include $-N_1 \le n \le N_1$.

  The analysis equation can be expressed as

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} (1) e^{-jk\left(\frac{2\pi}{N}\right)n} \ . \tag{67}$$

  We change the summation variable to $m = n + N_1$, so (67) becomes

$$a_k = \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)(m-N_1)} = \frac{1}{N} e^{jk\left(\frac{2\pi}{N}\right)N_1} \sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)m} \ . \tag{68}$$

  The summation in (68) is a sum of the first $2N_1 + 1$ terms of a geometric series.

  - For $k \neq 0, \pm N, \pm 2N, \ldots$ evaluating the summation (68) yields

$$a_k = \frac{1}{N} e^{jk\left(\frac{2\pi}{N}\right)N_1} \left(\frac{1 - e^{-jk(2N_1+1)\left(\frac{2\pi}{N}\right)}}{1 - e^{-jk\left(\frac{2\pi}{N}\right)}} \right), \ k \neq 0, \pm N, \pm 2N, \ldots$$

Factoring out $e^{-jk\left(\frac{2\pi}{2N}\right)}$ from both the numerator and denominator yields

$$a_k = \frac{1}{N}\frac{e^{-jk\left(\frac{2\pi}{2N}\right)}\left[e^{jk\left(N_1+\frac{1}{2}\right)\left(\frac{2\pi}{N}\right)} - e^{-jk\left(N_1+\frac{1}{2}\right)\left(\frac{2\pi}{N}\right)}\right]}{e^{-jk\left(\frac{2\pi}{2N}\right)}\left[e^{jk\left(\frac{2\pi}{2N}\right)} - e^{-jk\left(\frac{2\pi}{2N}\right)}\right]},$$

This can be simplified to

$$a_k = \frac{1}{N}\frac{\sin\left(2\pi k\left(N_1+1/2\right)/N\right)}{\sin\left(\pi k/N\right)} \qquad k \neq 0, \pm N, \pm 2N, \ldots \tag{69}$$

- For $k = 0, \pm N, \pm 2N, \ldots$, expression (69) entails division by zero. We return to

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1}(1)e^{-jk\left(\frac{2\pi}{N}\right)n}. \tag{67}$$

We set $e^{-jk\left(\frac{2\pi}{N}\right)n} = 1$ and perform the summation to obtain

$$a_k = \frac{2N_1+1}{N} \qquad k = 0, \pm N, \pm 2N, \ldots \tag{70}$$

Note that (69) converges to (70) in the limit $k \to 0, \pm N, \pm 2N, \ldots$, which we can show using L'Hôpital’s rule.

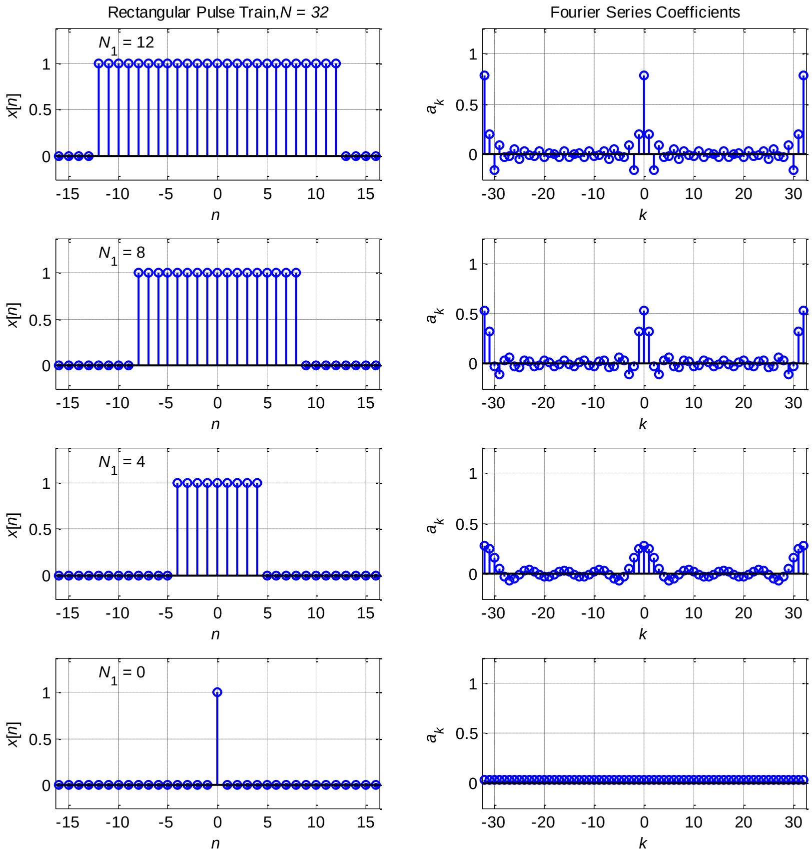
- We can express the DTFS coefficients for all k as

$$a_k = \frac{1}{N}\frac{\sin\left(2\pi k\left(N_1 + 1/2\right)/N\right)}{\sin\left(\pi k/N\right)} \quad \forall k\ . \tag{71}$$

In evaluating (71) for $k = 0, \pm N, \pm 2N, \ldots$, it is understood that we take the limit $k \to 0, \pm N, \pm 2N, \ldots$

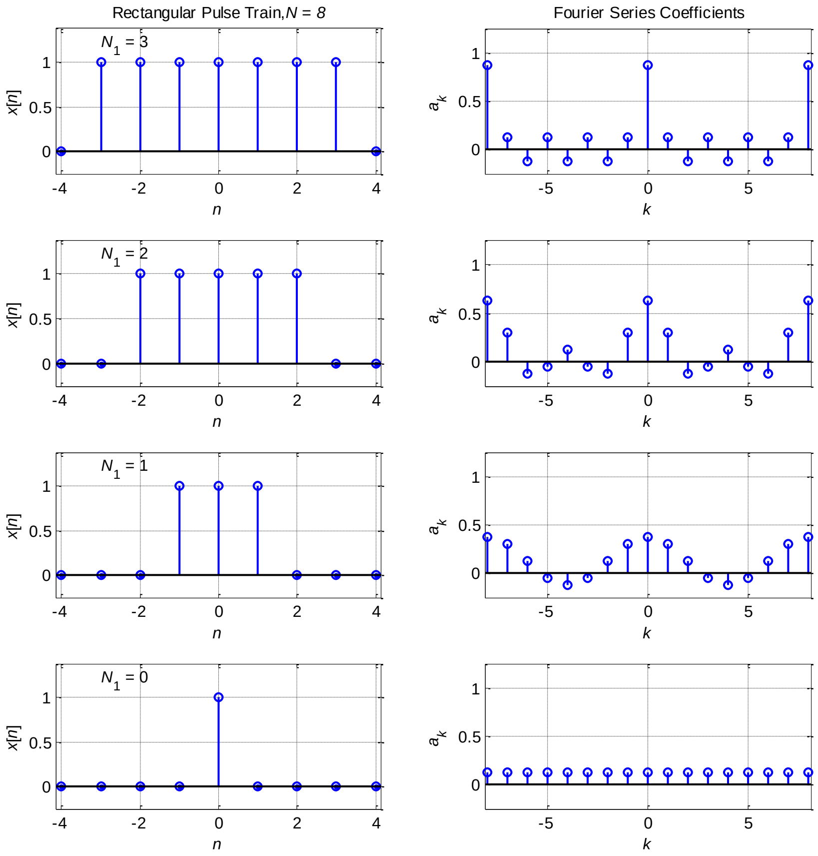
- This discussion is similar to that for the CT rectangular pulse train (see slides 23-29 above).
  - The function (71) is the DT counterpart of the sinc function.
    We will learn more about it in studying DT Fourier transforms (see Chapter 5).
  - The sinc function (24) is aperiodic, and has just one peak, at $k = 0$.
  - The function in (71) is periodic, and has peaks at $k = 0, \pm N, \pm 2N, \ldots$
    We may informally call it a "periodic sinc function".
  - In both functions, the peaks occur at values of k where the denominator vanishes.

- The following two figures show the rectangular pulse train $x[n]$ (over one period) and the DTFS coefficients a_k given by (71) (over two periods) for N = 32 and N = 8, for various values of N_1.



Observations

- $x[n]$ is real and even in *n*.
  The a_k are real and even in *k*.
  (See DTFS properties.)
- $x[n]$ and a_k are periodic with period *N*.
- Suppose we fix N, fixing $\Omega_0 = 2\pi / N$.
  As we decrease N_1:
  - Each pulse narrows in time.
  - The DTFS coefficients a_k at frequencies $k\Omega_0$, $k = \langle N \rangle$, spread out in frequency.
- In the limiting case $N_1 = 0$:
  - Pulse width is $2N_1 + 1 = 1$.
  - $x[n]$ is a periodic impulse train.
  - The DTFS coefficients (71) are
    $a_k = \frac{1}{N}$, $\forall k$.



Observations

- $x[n]$ is real and even in *n*.
  The a_k are real and even in *k*.
  (See DTFS properties.)
- $x[n]$ and a_k are periodic with period *N*.
- Suppose we fix N, fixing $\Omega_0 = 2\pi / N$.
  As we decrease N_1:
  - Each pulse narrows in time.
  - The DTFS coefficients a_k at frequencies $k\Omega_0$, $k = \langle N \rangle$, spread out in frequency.
- In the limiting case $N_1 = 0$:
  - Pulse width is $2N_1 + 1 = 1$.
  - $x[n]$ is a periodic impulse train.
  - The DTFS coefficients (71) are
    $a_k = \frac{1}{N}, \ \forall k$.

Synthesis

- Now we synthesize the pulse train using DTFS coefficients a_k (71) in synthesis equation (59). We consider $N=8$ ($\Omega_0=\pi/4$) and $N_1=2$ (pulse width $2N_1+1=5$).
- We choose *N* consecutive values of *k* as $k=\langle N\rangle=-3,\ldots,4$, so synthesis equation (59) becomes

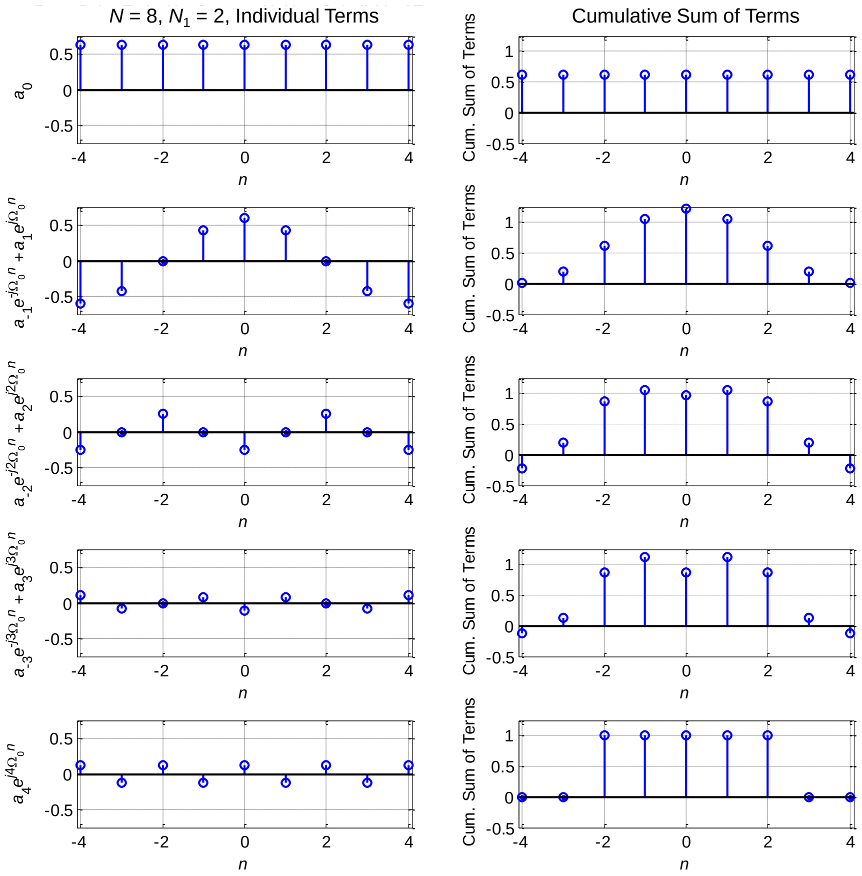
$$x[n]=\sum_{k=\langle 8\rangle} a_k e^{jk\Omega_0 n}=\sum_{k=-3}^{4} a_k e^{jk\Omega_0 n} .$$

- The left column shows the contribution from the term(s) at ±*k*, at frequencies $\pm k\Omega_0=\pm k\pi/4$:

$$\begin{cases} a_0 & k=0 \\ a_{-k}e^{-jk\Omega_0 n}+a_k e^{jk\Omega_0 n} & k=1,\,2,\,3 \\ a_4 e^{j4\Omega_0 n} & k=4 \end{cases} . \tag{72}$$

- Each contribution is real. As $x[n]$ is real, a_k are conjugate-symmetric, $a_{-k}=a_k^*$, (see slide 106).
- To show that the contribution for $k=4$ is real, we also use the periodicity of the a_k to note that $a_4=a_{-4}=a_4^*$, and observe that $e^{j4\Omega_0 n}=e^{j\pi n}=(-1)^n$ is real.
- The right column shows a synthesis using a cumulative sum of the terms for $k\le K$, which is

$$\hat{x}_K[n]=\sum_{|k|\le K} a_k e^{jk\Omega_0 n} . \tag{73}$$



Observations

- As we increase K, the partial synthesis $\hat{x}_K[n]$ better approximates $\hat{x}[n]$.
- Once we include N terms, $\hat{x}_K[n] = x[n]$ *exactly*, as expected from (61).

Properties of the Discrete-Time Fourier Series

- These properties are useful for:
  - Computing the DTFS coefficients for new signals, with minimal effort, by using the DTFS coefficients already known for other signals.
  - Checking the DTFS coefficients we compute for new signals.
- We assume periodic signals having a common period N and a common fundamental frequency $\Omega_0 = 2\pi / N$. We consider one or two signals and their DTFS coefficients:

$$x[n] \overset{FS}{\leftrightarrow} a_k \quad \text{and} \quad y[n] \overset{FS}{\leftrightarrow} b_k \ .$$

- Many DTFS properties are similar to CTFS properties, and we discuss those first.
  Then we discuss key DTFS properties that are different from CTFS properties.

Properties Similar to Continuous-Time Fourier Series

Linearity

- A linear combination of $x[n]$ and $y[n]$ is periodic with the same period N, and has DTFS coefficients given by the same linear combination of DTFS coefficients a_k and b_k:

$$Ax[n]+By[n]\overset{FS}{\leftrightarrow}Aa_k+Bb_k\,.$$

Time Shift

- A signal time-shifted by n_0 has DTFS coefficients multiplied by a complex-valued factor $e^{-jk\Omega_0 n_0}$:

$$x[n-n_0]\overset{FS}{\leftrightarrow}e^{-jk\Omega_0 n_0}a_k\,. \tag{74}$$

The magnitude and phase of $e^{-jk\Omega_0 n_0}a_k$ related to those of a_k as follows:

$$\begin{cases} \left|e^{-jk\Omega_0 n_0}a_k\right|=\left|a_k\right| \\ \angle\left(e^{-jk\Omega_0 n_0}a_k\right)=\angle a_k-k\Omega_0 n_0 \end{cases}. \tag{74'}$$

Time-shifting a signal by n_0 leaves the magnitude unchanged and adds a phase shift proportional to the negative of the time shift, $-n_0$, which varies linearly with frequency $k\Omega_0$.

Time Reversal

- Time-domain reversal corresponds to frequency-domain reversal:

$$x[-n] \overset{FS}{\leftrightarrow} a_{-k} \,. \tag{75}$$

As a consequence, an even signal has even DTFS coefficients:

$$x[-n] = x[n] \overset{FS}{\leftrightarrow} a_{-k} = a_k \,,$$

while an odd signal has odd DTFS coefficients:

$$x[-n] = -x[n] \overset{FS}{\leftrightarrow} a_{-k} = -a_k \,.$$

Conjugation

- Complex conjugation of a time signal corresponds to frequency reversal and complex conjugation of its DTFS coefficients:

$$x^*[n] \overset{FS}{\leftrightarrow} a_{-k}^* \,. \tag{76}$$

Conjugate Symmetry for Real Signal

- The DTFS coefficients of a real signal have conjugate symmetry:

$$x[n]=x^*[n]\overset{FS}{\leftrightarrow}a_k=a_{-k}^* . \tag{77}$$

Stated alternatively, the DTFS coefficients have even magnitudes and odd phases:

$$x[n]=x^*[n]\overset{FS}{\leftrightarrow}\begin{cases}|a_k|=|a_{-k}| \\ \angle a_k=-\angle a_{-k}\end{cases},$$

and the DTFS coefficients have even real parts and odd imaginary parts:

$$x[n]=x^*[n]\overset{FS}{\leftrightarrow}\begin{cases}\mathrm{Re}(a_k)=\mathrm{Re}(a_{-k}) \\ \mathrm{Im}(a_k)=-\mathrm{Im}(a_{-k})\end{cases}.$$

Real, Even or Real, Odd Signals

- Combining the time reversal and conjugation properties, we find

$$x[n] \text{ real and even in } n \overset{FS}{\leftrightarrow} a_k \text{ real and even in } k \tag{78}$$

$$x[n] \text{ real and odd in } n \overset{FS}{\leftrightarrow} a_k \text{ imaginary and odd in } k . \tag{79}$$

Fourier Series Synthesis of a Real Signal

- We clarify the significance of positive and negative frequencies and conjugate-symmetric DTFS coefficients. Consider the synthesis of a real signal:

$$\hat{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n} \ , \tag{59}$$

  where $a_k = a_{-k}^*$.

- The term in (59) at zero frequency satisfies $a_0 = a_{-0}^*$.

  The term is a real constant equal to the average value of the signal:

$$a_0 e^{j0\Omega_0 n} = a_0 \ .$$

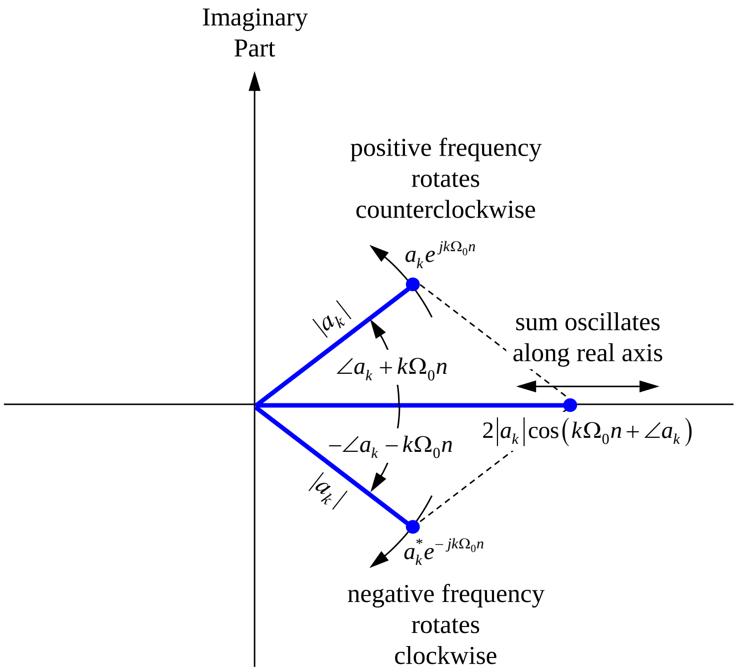
- Consider a pair of terms in (59) at positive and negative frequencies $\pm k\Omega_0$, $k \neq 0$.

  The DTFS coefficients satisfy $a_k = a_{-k}^*$ (or $a_{-k} = a_k^*$).

  The two terms add up to form a real cosine at frequency $k\Omega_0$:

$$\begin{aligned} a_k e^{jk\Omega_0 n} + a_{-k} e^{-jk\Omega_0 n} &= a_k e^{jk\Omega_0 n} + \left(a_k e^{jk\Omega_0 n}\right)^* \\ &= 2|a_k|\cos\left(\angle a_k + k\Omega_0 n\right) \end{aligned} \ . \tag{80}$$

- This figure helps us interpret equation (80).



Observations

- Positive-frequency term $a_k e^{jk\Omega_0 n}$:
  a vector rotating *counterclockwise*,
  with magnitude $|a_k|$ and
  phase $\angle a_k + k\Omega_0 n$.

- Negative-frequency term
  $a_{-k} e^{-jk\Omega_0 n} = a_k^* e^{-jk\Omega_0 n} = \left(a_k e^{jk\Omega_0 n}\right)^*$:
  a vector rotating *clockwise*,
  with magnitude $|a_k|$ and
  phase $-\angle a_k - k\Omega_0 n$.

- The sum of these two vectors is always a real cosine that oscillates along the real axis.

Properties Different from Continuous-Time Fourier Series

Periodicity

- The DTFS coefficients are periodic in the frequency index *k* with period *N*:

$$a_{k+N} = a_k \quad \forall k . \tag{65}$$

Multiplication

- The product of $x[n]$ and $y[n]$ is periodic with the same period N .

  Its DTFS coefficients are a *periodic convolution* between the DTFS coefficients a_k and b_k :

$$x[n] \cdot y[n] \overset{FS}{\leftrightarrow} \sum_{l=\langle N \rangle} a_l b_{k-l} \ .$$

- Periodic convolution between periodic discrete sequences is similar to ordinary (linear) convolution (see Chapter 2).
- The only difference: the summation is performed only over *N* consecutive values of *l* (one period).
- The resulting sequence of DTFS coefficients is periodic in *k* because b_{k-l} is periodic in *k*.

Parseval's Identity for Discrete-Time Fourier Series

- The utility of Parseval's identity for the DTFS is similar to that for the CTFS (see slides 43-46).
- It enables us to compute
    - the inner product between two periodic DT signals, or
    - the power of one periodic DT signal

  either in the time domain or in the frequency domain.
- Depending on the signal(s), the calculation is often easier in one domain or the other.

General Case: Inner Product Between Signals ***(You may skip.)***

- The general form of Parseval's identity, for an *inner product between two periodic DT signals*, states

$$\left\langle x[n], y[n] \right\rangle = \sum_{n=\langle N \rangle} x[n] y^*[n] = N \sum_{k=\langle N \rangle} a_k b_k^* . \tag{81}$$

- Middle expression: an inner product between signals $x[n]$ and $y[n]$ computed in time domain. Both are periodic, so we sum over only a single period, consisting of any N consecutive values of n.
- Rightmost expression: an inner product between DTFS coefficients a_k and b_k^* in frequency domain. Both are periodic, so we sum over only a single period, consisting of any N consecutive values of k.

Special Case: Signal Power **(Please read.)**

- Considering the special case of (81) with $x[n] = y[n]$ and $a_k = b_k$, we can compute the *power of a periodic DT signal*:

$$P = \frac{1}{N} \sum_{n=\langle N \rangle} \left| x[n] \right|^2 = \sum_{k=\langle N \rangle} \left| a_k \right|^2 . \tag{82}$$

- Middle expression: the power of $x[n]$ computed in the time domain.

  It is the energy of $x[n]$ in one period divided by the period.

- Rightmost expression: the power of $x[n]$ computed in the frequency domain.

  It is the sum of the squared magnitudes of the DTFS coefficients a_k over one period.

- Interpretation of rightmost expression: $\left| a_k \right|^2$ is the *power density spectrum* of the periodic signal $x[n]$.

  The term $\left| a_k \right|^2$ is the power contained in the signal component at frequency $k\Omega_0$.

  The rightmost expression in (82) is the sum of the powers in N frequency components $k\Omega_0$, $k = \langle N \rangle$.

Response of Discrete-Time Linear Time-Invariant Systems to Periodic Inputs

- Suppose we are given an LTI system whose impulse response is $h[n]$. We assume the sum defining the system frequency response converges

$$H\left(e^{j\Omega}\right)=\sum_{n=-\infty}^{\infty} h[n]e^{-jn\Omega} \ . \tag{46}$$

- Then the imaginary exponential signals

$$e^{j\Omega n}, \ \Omega \text{ real}, \ -\infty < n < \infty \ .$$

  are eigenfunctions of the system, with eigenvalues given by $H\left(e^{j\Omega}\right)$.

$$x[n]=e^{j\Omega n} \longrightarrow \boxed{h[n] \leftrightarrow H\left(e^{j\Omega}\right)} \longrightarrow y[n]=H\left(e^{j\Omega}\right)\cdot e^{j\Omega n}$$

- We input a signal $x[n]$, which is periodic with period $N = 2\pi / \Omega_0$, and can be represented by a DTFS with coefficients a_k, $k = \langle N \rangle$:

$$x[n]=\sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n} \ . \tag{59}$$

- The output $y[n]$ is also periodic with period N. Using linearity of the system and the eigenfunction property (45) of the imaginary exponentials, the output $y[n]$ can be expressed by a DTFS

$$y[n] = \sum_{k=\langle N\rangle} b_k e^{jk\Omega_0 n} = \sum_{k=\langle N\rangle} a_k H\left(e^{jk\Omega_0}\right) e^{jk\Omega_0 n} . \tag{83}$$

- The output $y[n]$ has DTFS coefficients

$$b_k = a_k H\left(e^{jk\Omega_0}\right), \tag{84}$$

These are the DTFS coefficients of the input $x[n]$, scaled by values of the frequency response $H\left(e^{j\Omega}\right)$ at frequencies $\Omega = k\Omega_0$, as shown.

$$x[n] \overset{FS}{\leftrightarrow} a_k \longrightarrow \boxed{h[n] \leftrightarrow H\left(e^{j\Omega}\right)} \longrightarrow y[n] \overset{FS}{\leftrightarrow} b_k = a_k H\left(e^{jk\Omega_0}\right)$$

- We can rewrite (84) to relate the magnitudes and phases of the input and output DTFS coefficients:

$$\begin{cases} |b_k| = |a_k| \left|H\left(e^{jk\Omega_0}\right)\right| \\ \angle b_k = \angle a_k + \angle H\left(e^{jk\Omega_0}\right) \end{cases} . \tag{84'}$$

Frequency Response of Discrete-Time Linear Time-Invariant Systems

- The frequency response $H\left(e^{j\Omega}\right)$ of a DT LTI system is:
  - The DT Fourier transform of the impulse response $h[n]$:

$$H\left(e^{j\Omega}\right)=\sum_{n=-\infty}^{\infty} h[n]e^{-jn\Omega} \ . \tag{46}$$

  - The eigenvalue for an imaginary exponential input signal $e^{j\Omega n}$:

$$H\left\{e^{j\Omega n}\right\}=H\left(e^{j\Omega}\right)e^{j\Omega n} \ .$$

- We study several important aspects of the frequency response.
  Most are similar to those for CT LTI systems, but there are important differences.

Frequency Response of a Real System

- Consider a DT LTI system whose impulse response $h[n]$ is real:

$$h[n] = h^*[n]. \tag{85}$$

- The frequency response $H\left(e^{j\Omega}\right)$ at frequency Ω is given by (46). To compute the frequency response at frequency $-\Omega$, we evaluate (46) with the substitution $\Omega \to -\Omega$, and use (85):

$$\begin{aligned} H\left(e^{-j\Omega}\right) &= \sum_{n=-\infty}^{\infty} h[n] e^{jn\Omega} \\ &= \sum_{n=-\infty}^{\infty} h^*[n] e^{jn\Omega} \\ &= \left(\sum_{n=-\infty}^{\infty} h[n] e^{-jn\Omega} \right)^* \\ &= H^*\left(e^{j\Omega}\right) \end{aligned}. \tag{86}$$

- As in CT, when the impulse response is real, the frequency response at negative frequency equals the complex conjugate of the frequency response at positive frequency.
- This property of the frequency response is called *conjugate symmetry*.

- Our finding can be summarized as

$$h[n] = h^*[n] \leftrightarrow H\left(e^{-j\Omega}\right) = H^*\left(e^{j\Omega}\right). \tag{87}$$

Like its counterpart in CT, (87) can be expressed in two alternate ways. If the impulse response is real:

- The magnitude of the frequency response is even in frequency, while the phase of the frequency response is odd in frequency:

$$h[n] = h^*[n] \leftrightarrow \begin{cases} \left|H\left(e^{-j\Omega}\right)\right| = \left|H\left(e^{j\Omega}\right)\right| \\ \angle H\left(e^{-j\Omega}\right) = -\angle H\left(e^{j\Omega}\right) \end{cases}. \tag{87a}$$

- The real part of the frequency response is even in frequency, while the imaginary part of the frequency response is odd in frequency:

$$h[n] = h^*[n] \leftrightarrow \begin{cases} \mathrm{Re}\left[H\left(e^{-j\Omega}\right)\right] = \mathrm{Re}\left[H\left(e^{j\Omega}\right)\right] \\ \mathrm{Im}\left[H\left(e^{-j\Omega}\right)\right] = -\mathrm{Im}\left[H\left(e^{j\Omega}\right)\right] \end{cases}. \tag{87b}$$

Verifying a Real Input Leads to a Real Output ***(Analogous to CT on slide 52. We will skip but please read.)***

- Suppose we have a real system satisfying

$$h[n]=h^*[n] \leftrightarrow H\left(e^{-j\Omega}\right)=H^*\left(e^{j\Omega}\right). \quad (87)$$

- Suppose we input a real, periodic signal $x[n]$, which has DTFS coefficients a_k.
  The DTFS coefficients of the input are conjugate-symmetric: $a_k = a_{-k}^*$, by (77).

- We obtain a periodic output $y[n]$ with DTFS coefficients $b_k = a_k H\left(e^{jk\Omega_0}\right)$, by (84).

- The frequency response of the system is conjugate-symmetric: $H\left(e^{jk\Omega_0}\right) = H^*\left(e^{-jk\Omega_0}\right)$, by (87).

- Hence, the DTFS coefficients of the output $y[n]$ satisfy

$$\begin{aligned} b_k &= a_k H\left(e^{jk\Omega_0}\right) \\ &= a_{-k}^* H^*\left(e^{-jk\Omega_0}\right). \\ &= b_{-k}^* \end{aligned}$$

- We conclude that the DTFS coefficients of the output $y[n]$ are conjugate symmetric: $b_k = b_{-k}^*$.
  Hence, the output $y[n]$ is real, as expected.

Periodicity of the Frequency Response

- The frequency response $H\left(e^{j\Omega}\right)$ of a DT LTI system is periodic in frequency Ω with period 2π .
  - This is a major difference between DT LTI systems and CT LTI systems.
  - Its origin is that DT imaginary exponentials are identical if their frequencies differ by a multiple of 2π (see Chapter 1, slide 40).
- To prove this, we evaluate (46) at a frequency $\Omega+2\pi$:

$$H\left(e^{j(\Omega+2\pi)}\right) = \sum_{n=-\infty}^{\infty} h[n]e^{-jn(\Omega+2\pi)} = \sum_{n=-\infty}^{\infty} h[n]e^{-jn\Omega}\underbrace{e^{-jn2\pi}}_{=1} = H\left(e^{j\Omega}\right) \tag{88}$$

- We saw above that given a periodic input signal with period $N=2\pi/\Omega_0$ and DTFS coefficients a_k , the output signal is periodic with DTFS coefficients (84), $b_k = a_k H\left(e^{jk\Omega_0}\right)$ (see slides 113-114).
- Both the input and output DTFS coefficients must be periodic in *k* with period *N*, which requires that $H\left(e^{jk\Omega_0}\right)$ be periodic in *k* with period *N*.
- We can verify this by substituting $k\Omega_0 = k2\pi/N$ for Ω in (88), or by simply writing

$$H\left(e^{j(k+N)\Omega_0}\right) = H\left(e^{jk\Omega_0}e^{jN(2\pi/N)}\right) = H\left(e^{jk\Omega_0}e^{j2\pi}\right) = H\left(e^{jk\Omega_0}\right).$$

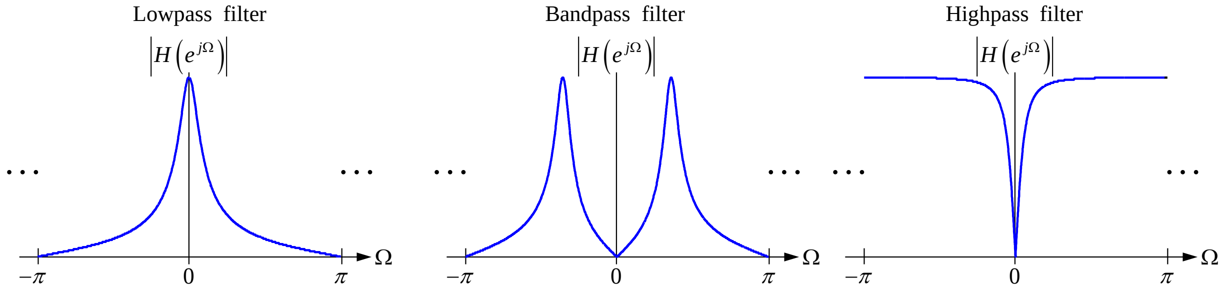
Types of Linear Distortion and Filters

- The classification of filters and linear distortion in DT LTI systems is similar to that for CT (see slides 53-56). There are some important differences, which we note here.

- The *magnitude (or amplitude) response* $\left|H\left(e^{j\Omega}\right)\right|$ determines the scaling of different frequency components appearing at the output of a DT LTI system, as in CT.

  - *Magnitude (or amplitude) distortion* occurs if $\left|H\left(e^{j\Omega}\right)\right|$ is frequency-dependent.

  - Filters with a frequency-dependent magnitude response $\left|H\left(e^{j\Omega}\right)\right|$ are often classified as *lowpass*, *bandpass*, or *highpass*, as illustrated here by typical examples.



  - $\left|H\left(e^{j\Omega}\right)\right|$ is periodic in Ω with period 2π, so we show only one period, $-\pi \leq \Omega \leq \pi$.

- In DT, "low", "medium" and "high" frequencies, which are relevant to classifying filters as lowpass, bandpass and highpass, are interpreted relative to the frequency range $-\pi \le \Omega \le \pi$.
- By contrast, in CT, "low", "medium" and "high" frequencies are interpreted relative to the frequency range $-\infty < \omega < \infty$.

- The *phase response* $\angle H\left(e^{j\Omega}\right)$ determines the phase shifts of different frequency components appearing at the output of a DT LTI system.
  - If the phase is a linear function of frequency with an integer slope

$$\angle H\left(e^{j\Omega}\right) = -n_0\Omega \,,$$

    where n_0 is an integer, then all frequency components are subject to an equal time shift n_0.
  - Given a phase response $\angle H\left(e^{j\Omega}\right)$, we define the *group delay* as $-\dfrac{d\angle H\left(e^{j\Omega}\right)}{d\Omega}$.
    The group delay describes how different frequency components are time-shifted.
    If the group delay has a constant integer value, the system causes no phase distortion.
  - *Phase distortion* occurs when $\angle H\left(e^{j\Omega}\right)$ is not a linear function of frequency with an integer slope, $-n_0$.

- A *distortionless DT system* may scale and time-shift signals but causes no magnitude or phase distortion. Its impulse response is of the form (see Chapter 2, slides 10-11):

$$h[n] = C\delta[n-n_0], \tag{89}$$

where n_0 is an integer and C is a constant, which we will typically assume is real.

- Given any input $x[n]$, the output is

$$y[n] = Cx[n-n_0].$$

- We can obtain the frequency response of a distortionless system using the impulse response (89) in (46), which computes its DT Fourier transform:

$$\begin{aligned} H\left(e^{j\Omega}\right) &= \sum_{n=-\infty}^{\infty} h[n]e^{-jn\Omega} \\ &= \sum_{n=-\infty}^{\infty} C\delta[n-n_0]e^{-jn\Omega} \\ &= Ce^{-jn_0\Omega} \end{aligned} \tag{90}$$

- *Question*: how did we evaluate the sum in the second line of (90)?

- We have found that for a distortionless system

$$H\left(e^{j\Omega}\right)=Ce^{-jn_0\Omega}, \tag{90}$$

where n_0 is an integer.

  - The magnitude response $\left|H\left(e^{j\Omega}\right)\right|=|C|$ is *constant*.
  - The phase response $\angle H\left(e^{j\Omega}\right)=\angle C-\Omega n_0$ varies *linearly with frequency*.

    The slope $-n_0$ is proportional to the negative of the time shift.

Verifying Consistency with DTFS Time-Shift Property

(Analogous to CT, slide 56. We will skip but please read.)

- We can verify that (90) is consistent with the DTFS time-shift property (74).
- We input a periodic signal

$$x[n]\overset{FS}{\leftrightarrow}a_k.$$

- We obtain a periodic output signal . We find its DTFS coefficients using (84) with frequency response (90):

$$y[n]\overset{FS}{\leftrightarrow}b_k=b_k=a_kH\left(e^{jk\Omega_0}\right)=Ce^{-jk\Omega_0 n_0}a_k.y[n]=C\,x[n-n_0]$$

  This is consistent with the CTFS time-shift property

$$x[n-n_0] \overset{FS}{\longleftrightarrow} e^{-jk\Omega_0 n_0} a_k . \tag{74}$$

Methods for Evaluating the Frequency Response

- Here we describe two methods for computing the frequency response of a given LTI system. The list is not exhaustive, as other methods exist.

1. *Fourier transform of impulse response*

- Suppose a DT LTI system is specified in terms of an *impulse response* $h[n]$.

  We can find the frequency response by evaluating the DT Fourier transform sum:

$$H\left(e^{j\Omega}\right)=\sum_{n=-\infty}^{\infty} h[n]e^{-jn\Omega} \ . \tag{46}$$

- We just used this procedure to find the frequency response of a distortionless system (90) from its impulse response (89).

- We will be able to evaluate the DT Fourier transform sum (46) for more complicated impulse responses after we study the DT Fourier transform in Chapter 5.

2. *Substitution in difference equation*

- Suppose an LTI system is specified by a *linear, constant-coefficient difference equation* of the form (42) in Chapter 2, slide 79:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

- Suppose we know the system's frequency response exists, i.e., that the sum (46) converges for the system's impulse response.

- Then we can compute the frequency response directly from the difference equation without using (46). We use the following simple procedure:

  1. Substitute the following input and output signals in the difference equation:

$$x[n] = e^{j\Omega n} \quad \text{and} \quad y[n] = H\left(e^{j\Omega}\right)e^{j\Omega n}.$$

  2. Cancel all factors of $e^{j\Omega n}$ and solve for $H\left(e^{j\Omega}\right)$.

- We will learn how to determine when (46) converges when we study the DT Fourier transform in Chapter 5. Until then, we will apply this method only to judiciously chosen examples.

Examples

- Here we apply Method 2 to three examples.

1. *First-Order System*

- We studied a simple first-order DT system in Chapters 1 and 2. It is described by a difference equation

$$y[n] - ay[n-1] = x[n], \tag{91}$$

where a is a real constant. *Question*: is this an FIR or an IIR system? Why?

- The choice of a determines the type of system, whether it is stable, and whether the frequency response exists. (The existence of the frequency response will be explained in Chapter 5.)

System	**Value of *a***	**Stable**	**Frequency Response Exists (in Strict Sense)**
Highpass filter	$-1 < a < 0$	Yes	Yes
Lowpass filter	$0 < a < 1$	Yes	Yes
Running summation (accumulation)	$a = 1$	No	No
Compound interest	$1 < a < \infty$ (typically)	No	No

- Here we assume $|a|<1$ and compute the frequency response from difference equation (91).

  Substituting for $x[n]$ and $y[n]$:

$$H\left(e^{j\Omega}\right)e^{j\Omega n}-aH\left(e^{j\Omega}\right)e^{j\Omega(n-1)}=e^{j\Omega n}\,.$$

  Cancelling factors of $e^{j\Omega n}$ and solving for the frequency response:

$$H\left(e^{j\Omega}\right)=\frac{1}{1-ae^{-j\Omega}}\,.$$

- In computing the magnitude and phase, we use the *reciprocal property* (see Appendix, page 289).

  Given complex-valued $z=|z|e^{j\angle z}$, its reciprocal is

$$\frac{1}{z}=\frac{1}{|z|e^{j\angle z}}=\frac{1}{|z|}e^{-j\angle z}\,.$$

  The magnitude and phase of the reciprocal are

$$\left|\frac{1}{z}\right|=\frac{1}{|z|}\quad\text{and}\quad\angle\left(\frac{1}{z}\right)=-\angle z\,.$$

- Using the reciprocal property with

$$z = 1 - ae^{-j\Omega} = 1 - a\cos\Omega + ja\sin\Omega\,,$$

the magnitude and phase responses of the first-order system are

$$\left|H\left(e^{j\Omega}\right)\right| = \frac{1}{\left|1 - ae^{-j\Omega}\right|} = \frac{1}{\left|1 - a\cos\Omega + ja\sin\Omega\right|} = \frac{1}{\sqrt{\left(1 - a\cos\Omega\right)^2 + \left(a\sin\Omega\right)^2}}$$

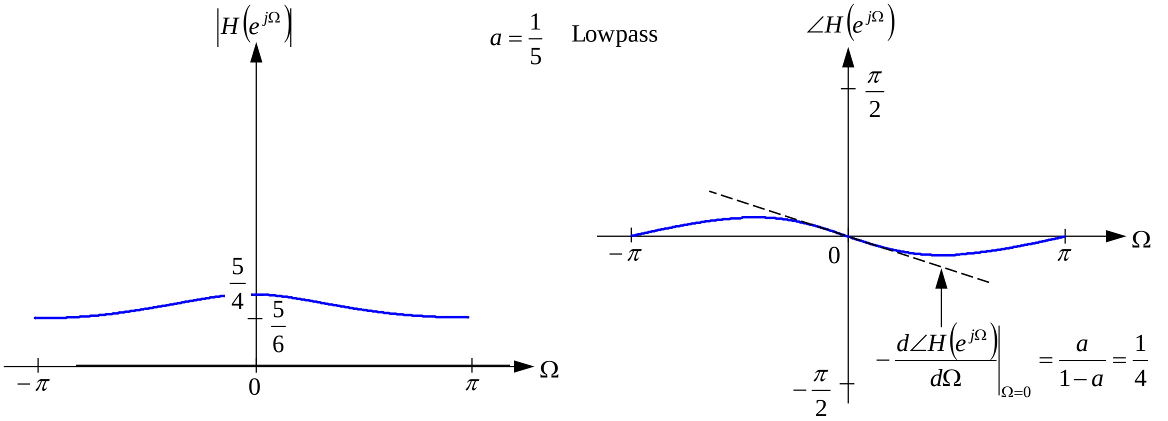
$$\angle H\left(e^{j\Omega}\right) = -\angle\left(1 - ae^{-j\Omega}\right) = -\angle\left(1 - a\cos\Omega + ja\sin\Omega\right) = -\tan^{-1}\left(\frac{a\sin\Omega}{1 - a\cos\Omega}\right).$$

From their mathematical forms, we see that the system causes both magnitude and phase distortion.

- Here we assume $0 < a < 1$, describing lowpass filters.
  We also consider $-1 < a < 0$, describing highpass filters in the Appendix, pages 303-304,

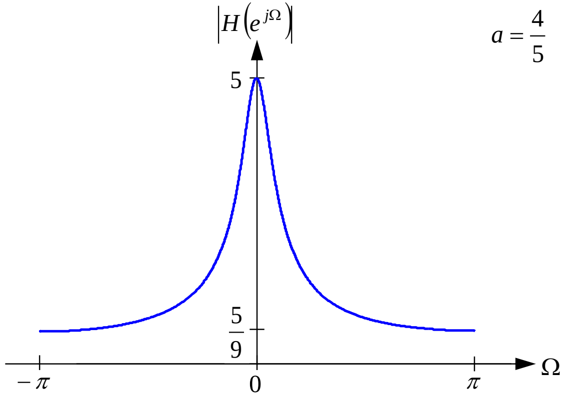
- a = 1/5 is a weak lowpass filter.
  - The ratio between the magnitude responses at $\Omega = 0$ and $\Omega = \pi$ is only $3/2$.
  - Near $\Omega = 0$, where the magnitude response is largest, the group delay is less than one sample:

$$-d\angle H\left(e^{j\Omega}\right)/d\Omega\Big|_{\Omega=0} = 1/4.$$

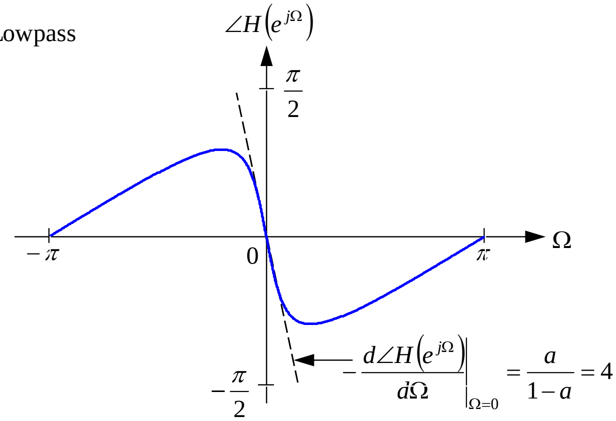


- a = 4/5 is a stronger lowpass filter.
  - The ratio between the magnitude responses at $\Omega = 0$ and $\Omega = \pi$ is 9.
  - Near $\Omega = 0$, where the magnitude response is largest, the group delay is four samples:

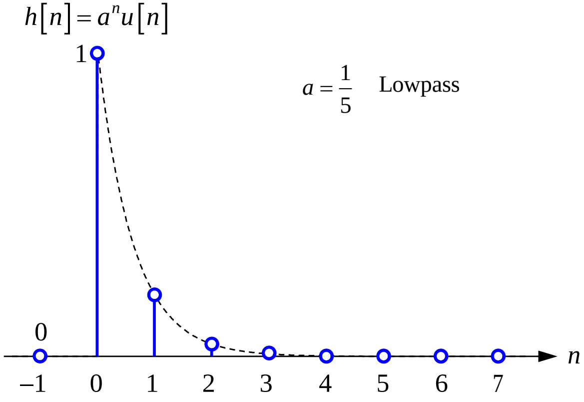
$$-d\angle H\left(e^{j\Omega}\right)/d\Omega\Big|_{\Omega=0} = 4 .$$

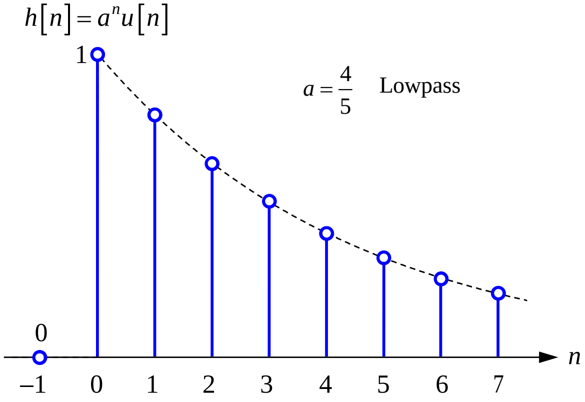


$a = \frac{4}{5}$ Lowpass



- As a increases from 1/5 to 4/5, the group delay near $\Omega = 0$ increases from 1/4 to 4. This can be understood intuitively by comparing the impulse responses.





Two-Point Moving Average

- We studied a two-point moving average in Chapter 2 (slides 85-86). It is described by difference equation

$$y[n] = \frac{1}{2}\left(x[n] + x[n-1]\right). \qquad (92)$$

- *Question*: is this an FIR or an IIR system? Why?

- To compute the frequency response, we substitute $x[n] = e^{j\Omega n}$ and $y[n] = H\left(e^{j\Omega}\right)e^{j\Omega n}$ in (92):

$$H\left(e^{j\Omega}\right)e^{j\Omega n} = \frac{1}{2}\left[e^{j\Omega n} + e^{j\Omega(n-1)}\right].$$

  Cancelling factors of $e^{j\Omega n}$, we obtain the frequency response

$$H\left(e^{j\Omega}\right) = \frac{1}{2}\left[1 + e^{-j\Omega}\right].$$

- To make it easier to compute the magnitude and phase, we factor out $e^{-j\Omega/2}$:

$$H\left(e^{j\Omega}\right) = e^{-j\frac{\Omega}{2}} \frac{e^{j\frac{\Omega}{2}} + e^{-j\frac{\Omega}{2}}}{2} = e^{-j\frac{\Omega}{2}} \cos\left(\frac{\Omega}{2}\right).$$

- In computing the magnitude and phase, we use the *product property* (see Appendix, page 289). Given complex-valued $z_1 = |z_1| e^{j\angle z_1}$ and $z_2 = |z_2| e^{j\angle z_2}$, the product is

$$z_1 z_2 = |z_1| e^{j\angle z_1} |z_2| e^{j\angle z_2} = |z_1||z_2| e^{j(\angle z_1 + \angle z_2)}.$$

The magnitude and phase of the product are

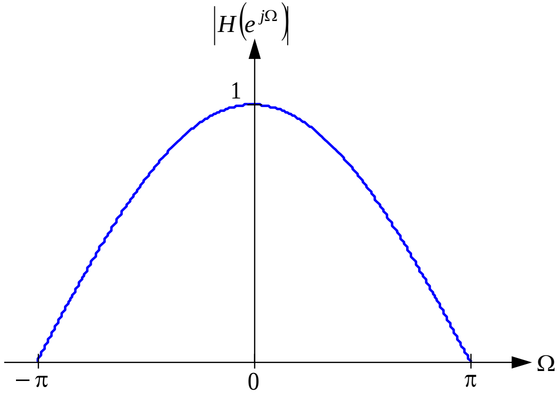
$$|z_1 z_2| = |z_1||z_2| \quad \text{and} \quad \angle z_1 z_2 = \angle z_1 + \angle z_2.$$

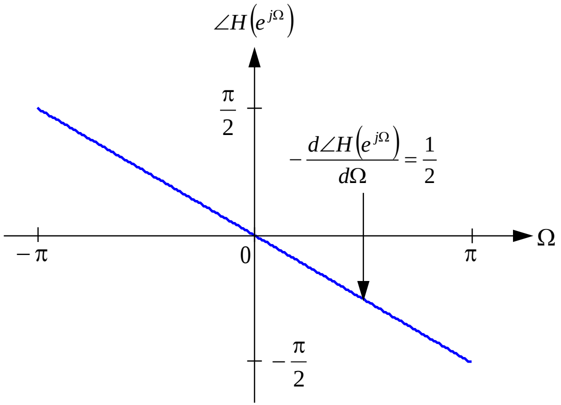
- Using the product property with $z_1 = e^{-j\frac{\Omega}{2}}$ and $z_2 = \cos\left(\frac{\Omega}{2}\right)$, the magnitude and phase are

$$\left|H\left(e^{j\Omega}\right)\right| = \left|e^{-j\frac{\Omega}{2}}\right| \left|\cos\left(\frac{\Omega}{2}\right)\right| = \left|\cos\left(\frac{\Omega}{2}\right)\right|$$

$$\angle H\left(e^{j\Omega}\right) = \angle e^{-j\frac{\Omega}{2}} + \angle \cos\left(\frac{\Omega}{2}\right) = \begin{cases} -\Omega/2 & \cos(\Omega/2) > 0 \\ -\Omega/2 - \pi & \cos(\Omega/2) < 0 \end{cases}.$$

- The magnitude and phase responses are shown.
  - This lowpass filter causes magnitude distortion. The highest frequencies are completely rejected: $\left|H\left(e^{\pm j\pi}\right)\right| = 0$.
  - The group delay is less than one sample at all frequencies: $-d\angle H\left(e^{j\Omega}\right)/d\Omega = 1/2$. Since the group delay is not integer-valued, the filter causes phase distortion.





Edge Detector ***(Very similar to moving average just above. We skip some of the details, but please read.)***

- We studied an edge detector in Chapter 2 (slides 87-88). It is described by a difference equation

$$y[n]=\frac{1}{2}\left(x[n]-x[n-1]\right). \qquad (93)$$

- The difference equation (93) is non-recursive, describes an FIR system, and provides an explicit input-output relation for the system, like the two-point moving average described by (92).

- To compute the frequency response, we substitute $x[n]=e^{j\Omega n}$ and $y[n]=H\left(e^{j\Omega}\right)e^{j\Omega n}$ in (93):

$$H\left(e^{j\Omega}\right)\cdot e^{j\Omega n}=\frac{1}{2}\left[e^{j\Omega n}-e^{j\Omega(n-1)}\right].$$

  Cancelling factors of $e^{j\Omega n}$, we obtain the frequency response:

$$H\left(e^{j\Omega}\right)=\frac{1}{2}\left[1-e^{-j\Omega}\right].$$

- To help compute the magnitude and phase, we factor out $e^{-j\Omega/2}$ and multiply and divide by *j*:

$$H\left(e^{j\Omega}\right)=je^{-j\frac{\Omega}{2}}\frac{e^{j\frac{\Omega}{2}}-e^{-j\frac{\Omega}{2}}}{2j}=je^{-j\frac{\Omega}{2}}\sin\left(\frac{\Omega}{2}\right).$$

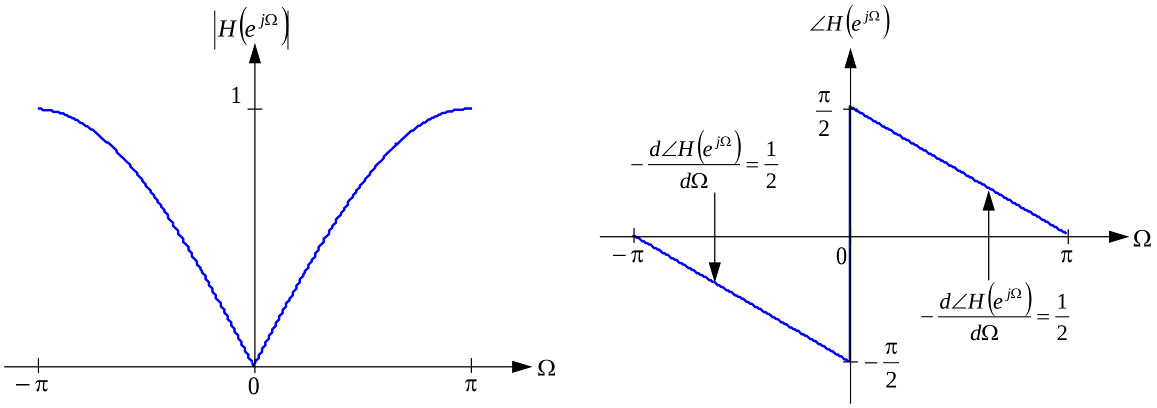
- Using the product property with $z_1 = j$, $z_2 = e^{-j\frac{\Omega}{2}}$ and $z_3 = \sin\left(\frac{\Omega}{2}\right)$, the magnitude and phase are

$$\left|H\left(e^{j\Omega}\right)\right| = |j|\left|e^{-j\frac{\Omega}{2}}\right|\left|\sin\left(\frac{\Omega}{2}\right)\right| = \left|\sin\left(\frac{\Omega}{2}\right)\right|$$

$$\angle H\left(e^{j\Omega}\right) = \angle j + \angle e^{-j\frac{\Omega}{2}} + \angle\sin\left(\frac{\Omega}{2}\right) = \frac{\pi}{2} - \frac{\Omega}{2} + \begin{cases} 0 & \sin\left(\Omega/2\right) > 0 \\ -\pi & \sin\left(\Omega/2\right) < 0 \end{cases} = \begin{cases} -\Omega/2 + \pi/2 & \sin\left(\Omega/2\right) > 0 \\ -\Omega/2 - \pi/2 & \sin\left(\Omega/2\right) < 0 \end{cases}.$$

- The magnitude and phase responses are shown.
- This highpass filter causes magnitude distortion. The filter completely rejects d.c.: $\left|H\left(e^{j0}\right)\right| = 0$.
- The group delay is less than one sample at all frequencies: $-d\angle H\left(e^{j\Omega}\right)/d\Omega = 1/2$.

  Since the group delay is not integer-valued, the filter causes phase distortion.



Examples of Filtering Periodic Discrete-Time Signals by Linear Time-Invariant Systems

Method of Analysis

- We use the analysis method presented on slides 113-114, which is summarized in the figure below.

Periodic Input

$$x[n]=\sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n} \longrightarrow \boxed{\begin{array}{c}\text{LTI System}\\ h[n]\leftrightarrow H\left(e^{j\Omega}\right)\end{array}} \longrightarrow y[n]=\sum_{k=\langle N\rangle} b_k e^{jk\Omega_0 n}=\sum_{k=\langle N\rangle} a_k H\left(e^{jk\Omega_0}\right)e^{jk\Omega_0 n}$$

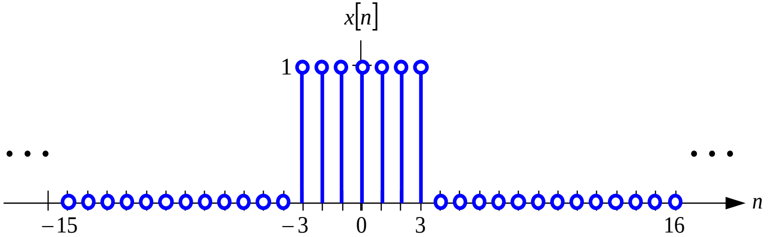
Periodic Output

Input Signal

- The input signal $x[n]$ is a rectangular pulse train, which we studied on slides 95-102.

  We assume a period $N=32$ and $N_1=3$, so the pulse width is $2N_1+1=7$.

  The fundamental frequency is $\Omega_0=\dfrac{2\pi}{N}=\dfrac{\pi}{16}$.



- Its DTFS coefficients are given by (71). With $N=32$ and $N_1=3$ in (71), the DTFS coefficients are

$$a_k = \frac{1}{32}\frac{\sin(7\pi k/32)}{\sin(\pi k/32)}.$$

- Throughout this example, we choose $N=32$ consecutive values of *n* and *k* as $n=\langle N\rangle=-15,\ldots,16$ and $k=\langle N\rangle=-15,\ldots,16$. The DTFS representation of the input signal is

$$x[n]=\sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n} = \sum_{k=-15}^{16}\frac{1}{32}\frac{\sin(7\pi k/32)}{\sin(\pi k/32)}e^{jk\left(\frac{\pi}{16}\right)n}.$$

- Because $x[n]$ is real and even in *n*, its DTFS coefficients a_k are real and even in *k* (see (78)). Since the a_k are purely real, their phases are an integer multiple of π, and we choose them as

$$\angle a_k = \begin{cases} 0 & a_k > 0 \\ \pm\pi & a_k < 0 \end{cases}.$$

In the plots below, when $a_k < 0$, we make the specific choices $\angle a_k = -\pi$ for $k<0$ and $\angle a_k = +\pi$ for $k>0$ so the phase appears with the odd symmetry expected, but this is not necessary. (See Appendix, pages 300-301, for further explanation.)

Linear Time-Invariant Systems

- We consider the three systems whose frequency responses were analyzed on slides 126-137:
  - First-order system. This is an IIR system. We choose two values of the parameter a:
    - $a = 1/5$: weak lowpass filter
    - $a = 4/5$: stronger lowpass filter
  - Two-point moving average: FIR lowpass filter.
  - Edge detector: FIR highpass filter.

Output Signal

- Given an LTI system with frequency response $H\left(e^{j\Omega}\right)$, using (84), the output is represented by a DTFS

$$y[n] = \sum_{k=\langle N\rangle} b_k e^{jk\Omega_0 n} = \sum_{k=\langle N\rangle} a_k H\left(e^{jk\Omega_0}\right) e^{jk\Omega_0 n} = \sum_{k=-15}^{16} \frac{1}{32} \frac{\sin\left(7\pi k/32\right)}{\sin\left(\pi k/32\right)} H\left(e^{jk\left(\frac{\pi}{16}\right)}\right) e^{jk\left(\frac{\pi}{16}\right)n} .$$

- The output DTFS coefficients b_k are given by the input DTFS coefficients a_k, scaled by values of the frequency response $H\left(e^{j\Omega}\right)$ evaluated at $\Omega = k\Omega_0$:

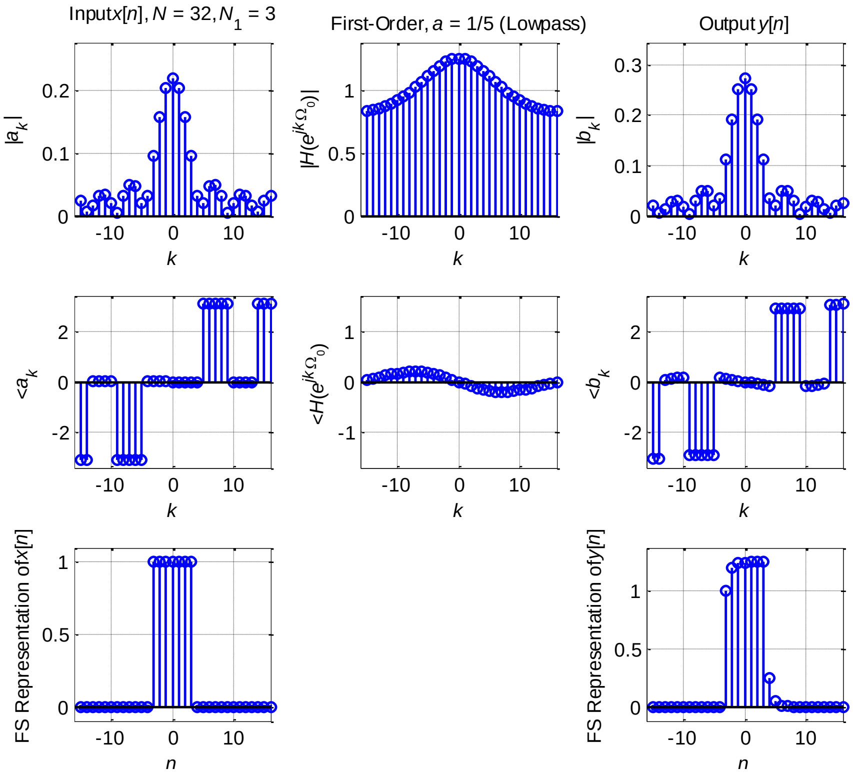
$$b_k = a_k H\left(e^{jk\Omega_0}\right). \tag{84}$$

As a result, the magnitudes and phases of the input and output DTFS coefficients are related by

$$\begin{cases} |b_k| = |a_k|\left|H\left(e^{jk\Omega_0}\right)\right| \\ \angle b_k = \angle a_k + \angle H\left(e^{jk\Omega_0}\right) \end{cases}. \tag{84'}$$

In each figure below, the relationship (84') should be evident in the first row (which shows $|a_k|$, $\left|H\left(e^{jk\Omega_0}\right)\right|$ and $|b_k|$) and in the second row (which shows $\angle a_k$, $\angle H\left(e^{jk\Omega_0}\right)$ and $\angle b_k$).

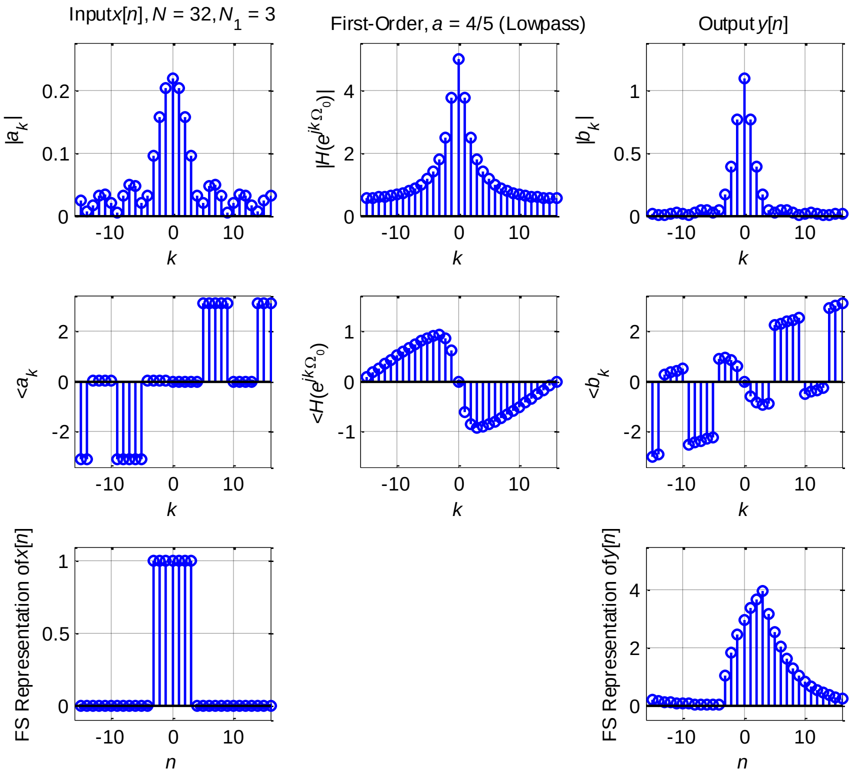
Filtering by First-Order System, a = 1/5 (Infinite Impulse Response, Weak Lowpass Filter)



Observations

- The d.c. level (average value) is scaled by $H\left(e^{j0}\right)=5/4$.
- The rise and fall times are about one sample.
- The pulse centroid, determined mainly by low frequencies, is delayed less than one sample. This is consistent with the low-frequency group delay $-d\angle H\left(e^{j\Omega}\right)/d\Omega\Big|_{\Omega=0}=1/4$.

Filtering by First-Order System, a = 4/5 (Infinite Impulse Response, Stronger Lowpass Filter)



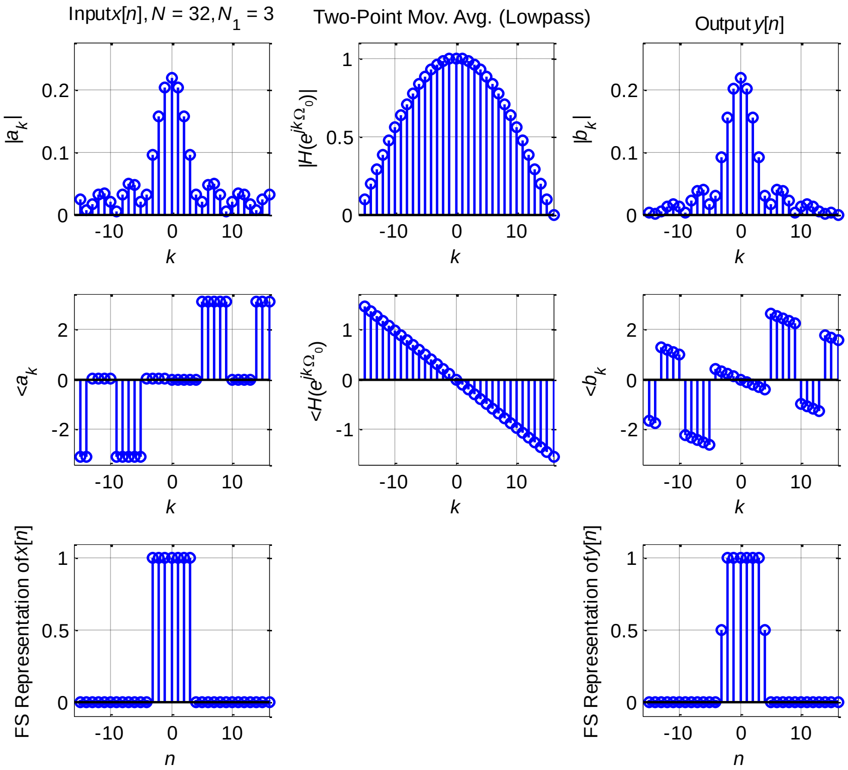
Observations

- The d.c. level (average value) is scaled by $H\left(e^{j0}\right)=5$.

- The rise and fall times are about five samples.

- The pulse centroid, determined mainly by low frequencies, is delayed by several samples. This is consistent with the low-frequency group delay $-d\angle H\left(e^{j\Omega}\right)/\,d\Omega\Big|_{\Omega=0}=4$.

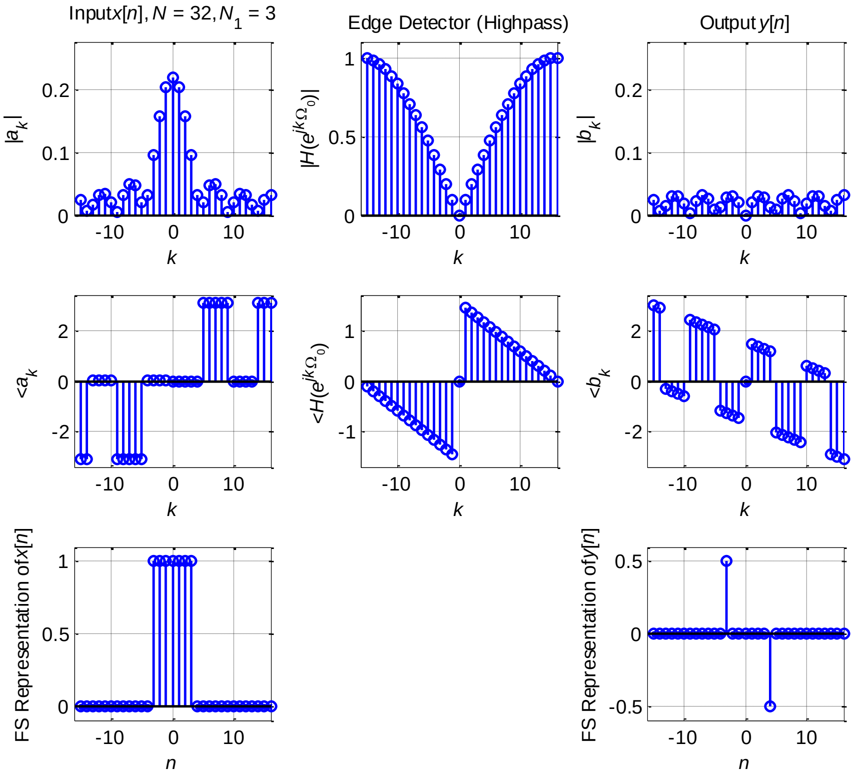
Filtering by Two-Point Moving Average, (Finite Impulse Response, Lowpass Filter)



Observations

- The d.c. level (average value) is preserved, since $H\left(e^{j0}\right)=1$.
- The highest frequencies are attenuated completely, i.e., $H\left(e^{\pm j\pi}\right)=0$.
- The rise and fall times are two samples.
- The pulse centroid, determined mainly by low frequencies, is delayed less than one sample. This is consistent with the group delay $-d\angle H\left(e^{j\Omega}\right)/d\Omega=1/2$.

Filtering by Edge Detector, (Finite Impulse Response, Highpass Filter)



Observations

- The d.c. level (average value) is removed, since $H\left(e^{j0}\right)=0$.
- The leading and trailing edges remain, since $H\left(e^{\pm j\pi}\right)=1$.
- The leading and trailing edges are, on average, delayed less than one sample. This is consistent with the group delay $-d\angle H\left(e^{j\Omega}\right)/d\Omega=1/2$.

Comment on Method of Analysis ***(We will skip but please read. Very similar to CT case on slide 74.)***

- We have analyzed these examples using DTFS.
- We could analyze them using convolution methods from Chapter 2. To do that, we would:
  - Represent the rectangular pulse train input $x[n]$ as an infinite sum of scaled and shifted step functions.
  - Represent the periodic output $y[n]$ could as a corresponding sum of scaled and shifted step responses.
  - The periodic outputs $y[n]$ obtained here can be understood using this approach.
- The Fourier series method we have used offers important advantages.
  - It takes account of the overlap between all the scaled, shifted step responses.
  - It is applicable to *any* periodic input $x[n]$ with finite power, even if that signal cannot be represented by simple functions, such as step functions.

First-Order and Higher-Order Infinite Impulse Response Systems ***(We will skip but please read.)***

- First-order systems, in which a single past output $y[n-1]$ is fed back, are easy to analyze and understand. But they are not suitable for many applications. In the first-order system considered here, the single parameter *a* governs both the time-domain and frequency-domain responses.
- If we feed back additional delayed outputs, such as $y[n-2]$, etc., we obtain systems described by *higher-order difference equations*. Such *higher-order systems* offer more flexibility in their characteristics. For example, they can achieve a sharper passband-stopband transition, and offer more control over tradeoffs between time-domain response and frequency response.
- We will learn about second-order IIR systems in studying the DT Fourier transform in Chapter 5. These are covered in more depth in EE 102B.

Finite Impulse Response Systems

- FIR filters, in which the output is formed by a linear combination of delayed inputs without feedback of past outputs, offer many options for performing filtering functions:
  - The two-point moving average can be extended to average over any desired number of samples.
  - The two-point edge detector computes a first difference. Discrete-time filters approximating important continuous-time functions, such as the first derivative or a higher derivative, can be realized.

- Nearly ideal lowpass, bandpass or highpass filters can be realized, which have a nearly flat magnitude response in the passband and a nearly zero response in the stopband, with an abrupt transition between passband and stopband.
- Filters whose phase response is a linear function of frequency, such that the group delay $-d\angle H\left(e^{j\Omega}\right)/d\Omega$ assumes a constant value in the passband, can be realized.

- FIR approximations of a differentiator and of an ideal lowpass filter are discussed briefly in Chapter 6. FIR filters are studied in more depth in EE 102B.

Fourier Series Representation of Different Signal Types (You may skip slides 149-153.)

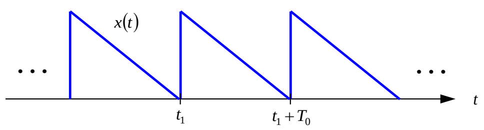
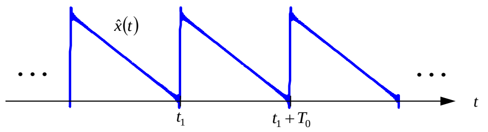
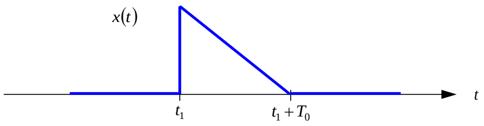
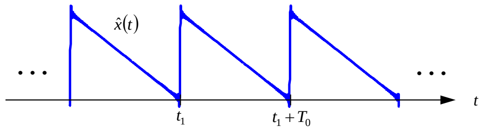
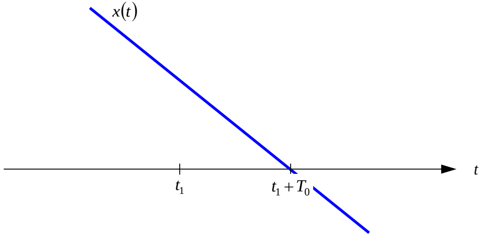
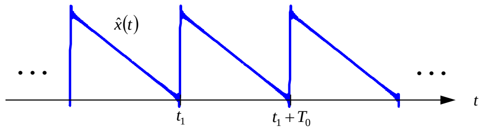
- In studying the FS for CT or DT signals, we have thus far assumed that the original signal used in analysis, $x(t)$ or $x[n]$, is *periodic*. We found that FS synthesis yields a signal, $\hat{x}(t)$ or $\hat{x}[n]$, that is *periodic* like the original $x(t)$ or $x[n]$.

- Here we consider what happens if we perform FS analysis on a signal, $x(t)$ or $x[n]$, that is *not periodic*. We will see that FS synthesis still yields a *periodic* $\hat{x}(t)$ or $\hat{x}[n]$. In other words, the FS can be used to obtain a periodic signal from an aperiodic signal.

- A similar method is used in starting with a CT filter (whose frequency response is not generally periodic in frequency) and using it to obtain a DT filter (whose frequency response must be periodic in frequency). The method is described briefly in Chapter 6, and in more detail in *EE 102B Course Reader*, Chapter 3.

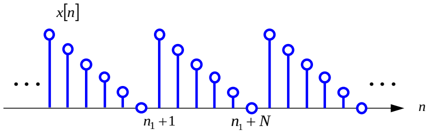
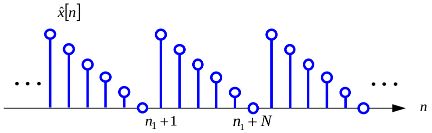
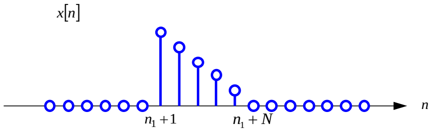
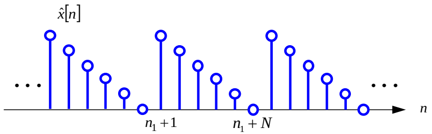
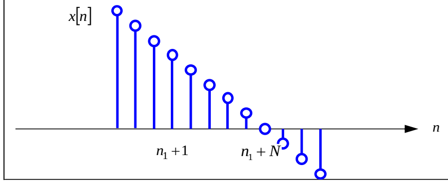
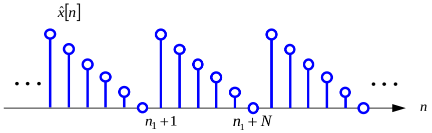
Continuous-Time Case

- Given a signal $x(t)$, not necessarily periodic, we choose an interval $t_1 \le t \le t_1 + T_0$, and use the analysis equation (19) to compute CTFS coefficients a_k, $-\infty < k < \infty$. We use these CTFS coefficients in the synthesis equation (11) to form a CTFS representation $\hat{x}(t)$, $-\infty < t < \infty$. Assuming the Dirichlet conditions (slides 21-22) are satisfied over the interval $t_1 \le t \le t_1 + T_0$, we know that:
    - $\hat{x}(t)$ reproduces $x(t)$ for $t_1 \le t \le t_1 + T_0$, except for differences associated with the Gibbs phenomenon.
    - $\hat{x}(t)$ is periodic with period T_0, i.e., $\hat{x}(t+T_0) = \hat{x}(t) \;\; \forall t$.
- The figure below shows the CTFS representation $\hat{x}(t)$ that results if the starting signal $x(t)$ is of three different types. The three different starting signals $x(t)$ are all *different*. Only one of the three is periodic. However, the three starting signals $x(t)$ are *identical* over the interval $t_1 \le t \le t_1 + T_0$ used in the analysis equation (19). As a result, their CTFS representations $\hat{x}(t)$ are all identical. All three CTFS representations $\hat{x}(t)$ are *periodic*.

Signal $x(t)$	Signal approximated by Fourier series representation $\hat{x}(t)$
Periodic: $x(t+T_0)=x(t)\ \ \forall t$ 	Periodic $x(t)$ 
Time-limited: $x(t)=0$ except for $t_1 \le t \le t_1+T_0$ 	Periodic extension of $x(t)$: $\sum_{l=-\infty}^{\infty} x(t-lT_0)$ 
General 	Periodic extension of time-limited version of $x(t)$: $\sum_{l=-\infty}^{\infty} \tilde{x}(t-lT_0)$, where $\tilde{x}(t)=\begin{cases} x(t) & t_1 \le t \le t_1+T_0 \\ 0 & \text{otherwise} \end{cases}$ 

Discrete-Time Case

- Given a signal $x[n]$, not necessarily periodic, we choose an interval $n_1+1 \le n \le n_1+N$, and use the analysis equation (64) to compute DTFS coefficients a_k, $k=\langle N \rangle$. We use these DTFS coefficients in the synthesis equation (59) to form a DTFS representation $\hat{x}[n]$, $-\infty < n < \infty$. We know that:
    - $\hat{x}[n]=x[n]$ exactly for $n_1+1 \le n \le n_1+N$.
    - $\hat{x}[n]$ is periodic with period N, i.e., $x[n]=x[n+N] \quad \forall n$, .
- The figure below shows the DTFS representation $\hat{x}[n]$ that results if the starting signal $x[n]$ is of three different types. The three different starting signals $x[n]$ are all *different*. Only one of the three is periodic. The three starting signals $x[n]$ are, however, *identical* over the interval $n_1+1 \le n \le n_1+N$ used in the analysis equation (64). As a result, their DTFS representations $\hat{x}[n]$ are all identical. All three DTFS representations $\hat{x}[n]$ are *periodic*.

<table>
<tr><th>Signal $x[n]$</th><th>Fourier series representation $\hat{x}[n]$</th></tr>
<tr><td>Periodic: $x[n+N]=x[n] \quad \forall n$

</td><td>Identical: $\hat{x}[n]=x[n] \quad \forall n$

</td></tr>
<tr><td>Time-limited: $x[n]=0$ except for $n_1+1 \le n \le n_1+N$

</td><td>Periodic extension: $\hat{x}[n]=\sum_{l=-\infty}^{\infty} x[n-lN]$

</td></tr>
<tr><td>General

</td><td>Periodic extension of time-limited version:
$$\tilde{x}[n]=\begin{cases} x[n] & n_1+1 \le n \le n_1+N \\ 0 & \text{otherwise} \end{cases}$$
$$\hat{x}[n]=\sum_{l=-\infty}^{\infty} \tilde{x}[n-lN]$$

</td></tr>
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