Stanford University
EE 102A: Signal Processing and Linear Systems I
Summer 2022
Instructor: Ethan M. Liang

Homework 7 Solutions (122 points)

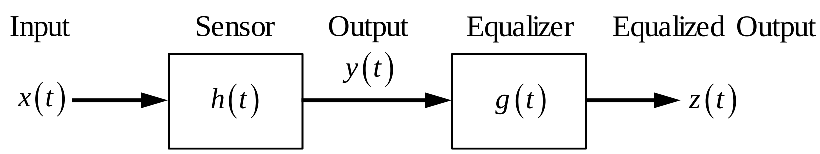
Not to be turned in for grading. Solutions distributed Tuesday, August 9.

Rem/unhighlight solns Check & unhighlight refs

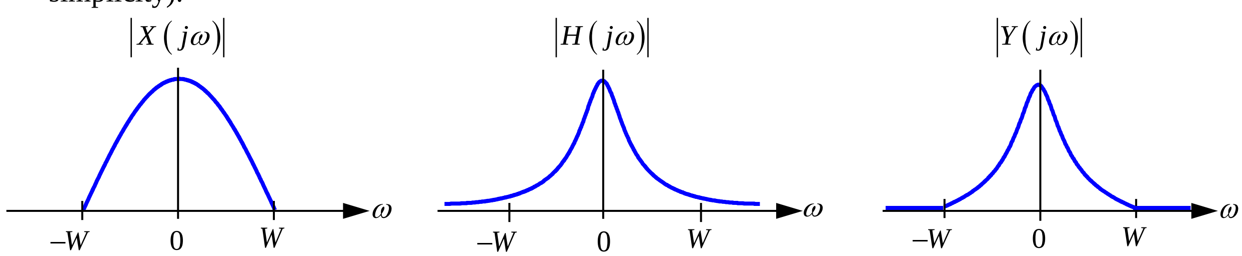
Unless noted otherwise, all references are to the *EE 102A Course Reader*.

CT LTI System Analysis using CT FT

1. **(6 points)** *Inverse of a lowpass filter*. An input signal $x(t)$ is measured by a sensor having an impulse response $h(t)$, yielding an output signal $y(t) = x(t) * h(t)$.



In terms of Fourier transforms, the corresponding relationship is $Y(j\omega) = X(j\omega)H(j\omega)$, where $H(j\omega)$ is the sensor frequency response. An example is shown here (only magnitudes are shown for simplicity).



The input signal spectrum is nonzero over a frequency range $|\omega| \le W$. Over that range, the sensor frequency response is not constant, so the measured output signal is a distorted version of the input signal. We would like to undo the distortion, so we implement an *equalizer*, which is an LTI filter with impulse response and frequency response $g(t) \leftrightarrow G(j\omega)$. We want the equalizer frequency response to be the inverse of the sensor frequency response, $G(j\omega) = 1/H(j\omega)$, at least over the frequency range $|\omega| \le W$, such that the equalizer output is the original input signal, $z(t) = x(t)$. Here we assume the sensor response is first-order lowpass filter:

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t).$$

a. **(3 points)** Obtain an expression for the equalizer frequency response $G(j\omega)$. Sketch the magnitude and phase, $|G(j\omega)|$ and $\angle G(j\omega)$. Use the same format as on page 181 and other pages in that section, plotting $20\log_{10}|G(j\omega)|$ (in dB) and $\angle G(j\omega)$ (in rad) vs. ω, with ω on a logarithmic scale. *Hint*: very little effort is needed to make the sketches if you consider how the magnitude and phase of $G(j\omega)$ relate to those of $H(j\omega)$.

Note: you may notice that the equalizer's magnitude response $|G(j\omega)|$ becomes infinite as $|\omega|$ becomes large. In part (b), you may observe that its impulse response $g(t)$ is not absolutely integrable. These are indicative of two problems with the equalizer $G(j\omega)$ as determined here. (i) It is not stable. In practice, we do not require the equalizer to invert the lowpass filter for $|\omega| > W$, so the equalizer can be realized as a stable system. (ii) If there is noise present in the sensor output $y(t)$, the equalizer will amplify the noise power at a frequency ω in proportion to $|G(j\omega)|^2$. In practice, to minimize noise enhancement, we want to design the equalizer so that $|G(j\omega)|^2$ decreases at high frequencies, $|\omega| > W$. This can be done by cascading a suitable lowpass filter with the response $G(j\omega)$ determined here.

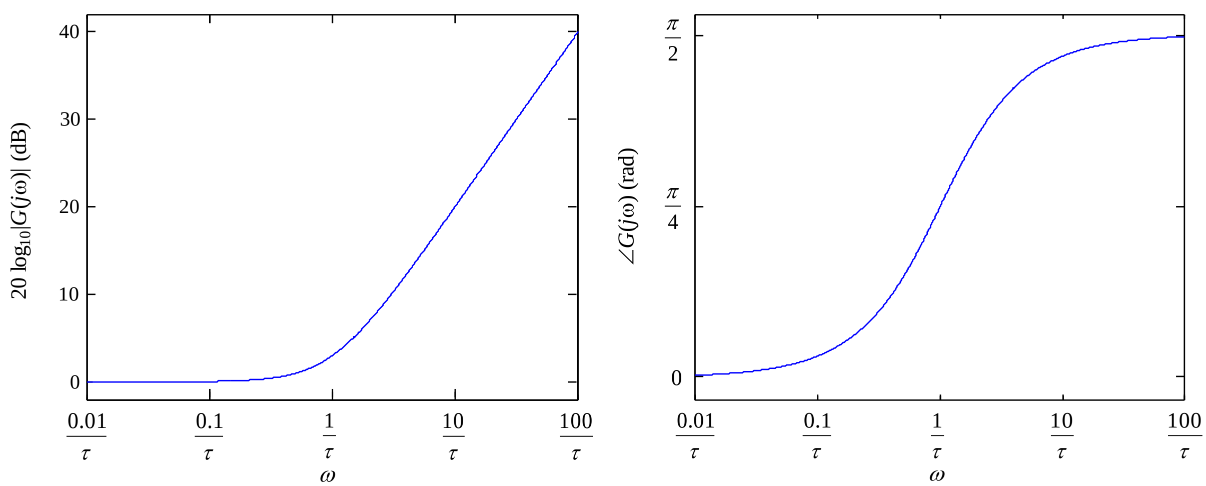
Solution The CT FT of $h(t)$ is given as

$$H(j\omega) = \frac{1}{\tau}\frac{1}{1/\tau + j\omega} = \frac{1}{1 + j\omega\tau}.$$

Therefore, the equalizer frequency response $G(j\omega)$ is

$$G(j\omega) = \frac{1}{H(j\omega)} = 1 + j\omega\tau.$$

As discussed and used in the Appendix (pages 289 and 298), the magnitude of a reciprocal is the reciprocal of the magnitude (the negative on a log scale), while the phase of a reciprocal is the negative of the phase. Since the magnitude and phase plots of $H(j\omega)$ are given on page 181, we can easily obtain the magnitude and phase plots of $G(j\omega)$, as shown below.



b. **(3 points)** Obtain an expression for the equalizer impulse response $g(t)$. Sketch a block diagram of a realization using simple components we have discussed in the course.

Solution Using Table 4, we

$$\delta(t) \leftrightarrow 1$$

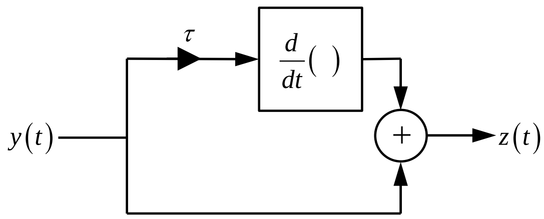
and

$$\tau\delta'(t) \quad \leftrightarrow \quad j\omega\tau\,.$$

Adding the two, we get

$$\delta(t) + \tau\delta'(t) \leftrightarrow 1 + j\omega\tau.$$

Therefore, the equalizer impulse response is $g(t) = \delta(t) + \tau\delta'(t)$. As shown in lecture and in Homework 3, the impulse response $\delta'(t)$ corresponds to a differentiator.



2. **(6 points)** *Communication channel with phase distortion*. Suppose a certain communication channel can be characterized as an LTI system with frequency response $H(j\omega) = e^{j\phi(\omega)}$, where $\phi(\omega)$ is real and $\phi(-\omega) = -\phi(\omega)$. Since $|H(j\omega)| = 1$, this channel causes no amplitude distortion. The channel does cause phase distortion, however. If we transmit a signal $x(t) \leftrightarrow X(j\omega)$ through the channel, the received signal is $y(t) \leftrightarrow Y(j\omega)$, where $Y(j\omega) = X(j\omega)H(j\omega)$. This problem considers an interesting but impractical scheme for overcoming the phase distortion. Suppose we transmit $x(t)$ through the channel and, at the receiver, record $y(t)$ on a first magnetic tape. Then we fly the tape back to the transmitting end and play it backwards, transmitting the time-reversed signal $y(-t)$ through the channel. At the receiving end, we receive a signal $z(t) \leftrightarrow Z(j\omega)$, which we record on a second magnetic tape. It is claimed that the second tape, if played backwards, would yield the original signal, i.e., $z(-t) = x(t)$. Would the scheme work? Justify your answer using Fourier transforms.

Solution The channel frequency response is

$$H(j\omega) = e^{j\phi(\omega)}\,. \tag{1}$$

We first transmit $x(t) \leftrightarrow X(j\omega)$ through the channel and receive

$$\begin{aligned} y(t) \leftrightarrow Y(j\omega) &= X(j\omega)H(j\omega) \\ &= X(j\omega)e^{j\phi(\omega)} \end{aligned}\,. \tag{2}$$

We record the signal $y(t)$ on a tape, and transmit the time-reversed signal $y(-t)$ through the channel. Using the FT time-reversal property and (2), it has a FT

$$\begin{aligned} y(-t) \leftrightarrow Y(-j\omega) &= X(-j\omega)e^{j\phi(-\omega)} \\ &= X(-j\omega)e^{-j\phi(\omega)} \end{aligned}\,. \tag{3}$$

We have used $\phi(-\omega) = -\phi(\omega)$ in the second line. When we transmit $y(-t) \leftrightarrow Y(-j\omega)$ through the channel, the received signal is $z(t)$. Its FT $Z(j\omega)$ is $Y(-j\omega)$ multiplied by $H(j\omega)$. Using (1) and (3), this is

$$\begin{aligned} z(t) \leftrightarrow Z(j\omega) &= Y(-j\omega)H(j\omega) \\ &= Y(-j\omega)e^{j\phi(\omega)} \\ &= X(-j\omega)e^{-j\phi(\omega)}e^{j\phi(\omega)} \\ &= X(-j\omega) \end{aligned}.$$

In the received signal recorded on the second tape, $z(t) \leftrightarrow Z(j\omega)$, the phase distortion has been cancelled. Since

$$z(t) \leftrightarrow Z(j\omega) = X(-j\omega),$$

we conclude that the signal recorded on the second tape is the time reversal of the original signal:

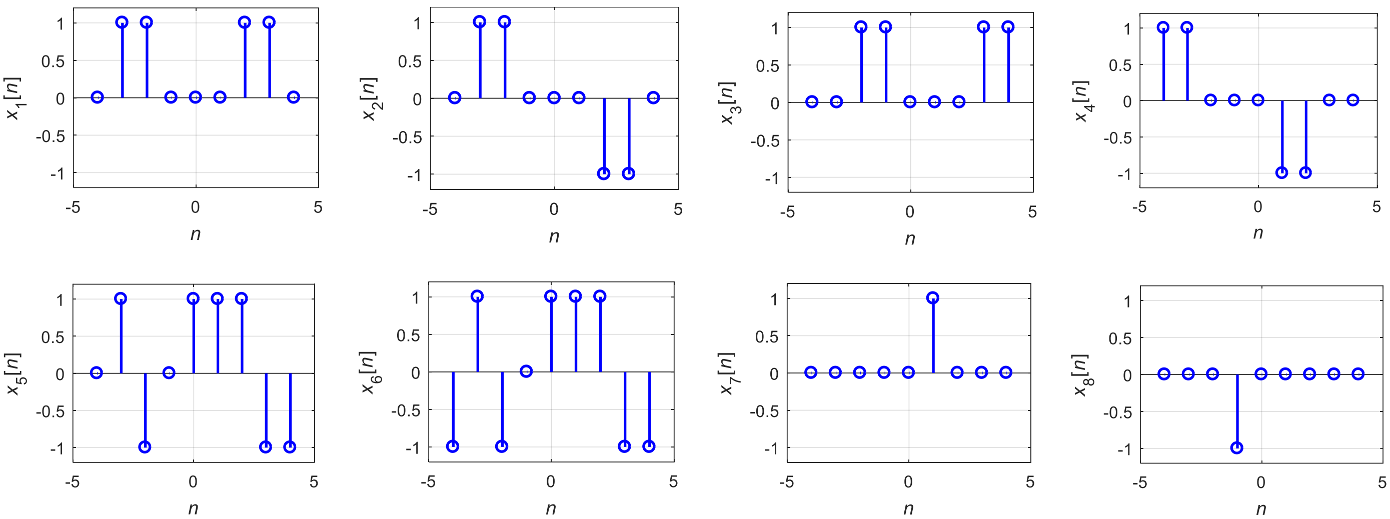
$$z(t) = x(-t).$$

If the signal recorded on the second tape is played backwards, the original signal is recovered:

$$z(-t) = x(t).$$

Discrete-Time Fourier Transforms

3. **(12 points)** *Discrete-time Fourier transforms*. Consider the eight real signals $x_1[n],\ldots,x_8[n]$ and the eight DTFTs $X_a\left(e^{j\Omega}\right),\ldots,X_h\left(e^{j\Omega}\right)$. Match each signal to its DTFT. Provide a table like the following, filling the appropriate letters in the second column. Provide a brief justification based on symmetry, slope of the phase $\angle X\left(e^{j\Omega}\right)$, d.c. value $X\left(e^{j0}\right)$, etc.


x1[n]
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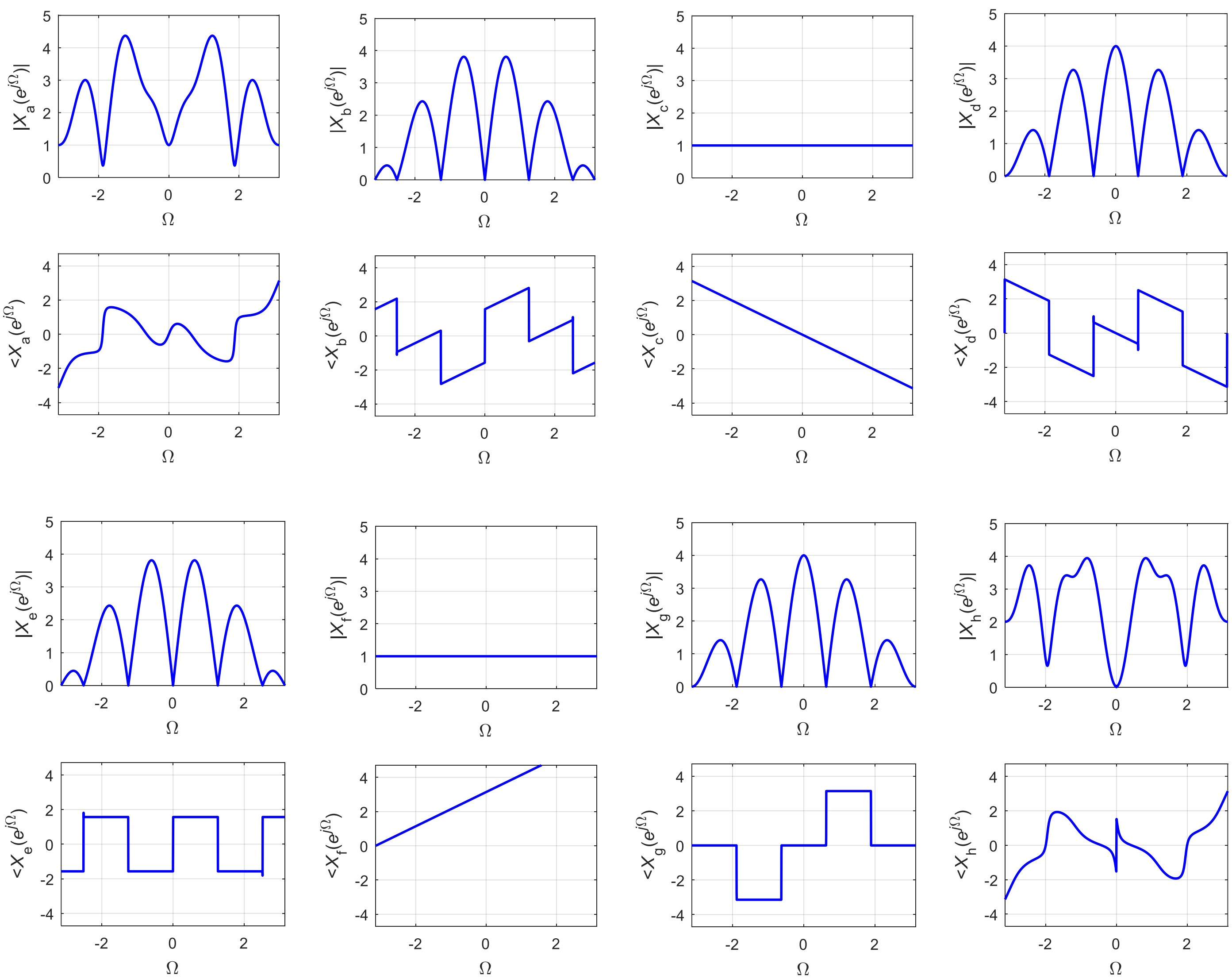
5.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Solution

Signal	DTFT	Explanation
1	g	$x[n]$ real, even $\leftrightarrow$ $X(e^{j\Omega})$ real, even, $\angle X(e^{j\Omega}) \in \{0,\pi\}$, $\sum_n x[n] = 4 \leftrightarrow X(e^{j0}) = 4$
2	e	$x[n]$ real, odd $\leftrightarrow$ $X(e^{j\Omega})$ imaginary, odd, $\angle X(e^{j\Omega}) \in \{-\pi/2, \pi/2\}$, $\sum_n x[n] = 0 \leftrightarrow X(e^{j0}) = 0$
3	d	$x[n]$ delayed real, even signal $\leftrightarrow$ $X(e^{j\Omega})$ real, even, but with added negative phase slope, $\sum_n x[n] = 4 \leftrightarrow X(e^{j0}) = 4$
4	b	$x[n]$ advanced real, odd signal $\leftrightarrow$ $X(e^{j\Omega})$ imaginary, odd, but with added positive phase slope, $\sum_n x[n] = 0 \leftrightarrow X(e^{j0}) = 0$
5	a	$\sum_n x[n] = 1 \leftrightarrow X(e^{j0}) = 1$
6	h	$\sum_n x[n] = 0 \leftrightarrow X(e^{j0}) = 0$
7	c	$x[n]$ delayed impulse $\leftrightarrow$ $\left\|X(e^{j\Omega})\right\| = 1$, negative phase slope, $\sum_n x[n] = 1 \leftrightarrow X(e^{j0}) = 1$
8	f	$x[n]$ advanced impulse $\leftrightarrow$ $\left\|X(e^{j\Omega})\right\| = 1$, positive phase slope, $\sum_n x[n] = -1 \leftrightarrow X(e^{j0}) = -1$

4. **(6 points)** *Discrete-time Fourier transforms*. Find expressions for the DTFTs of the following signals. You need not sketch them.

a. **(3 points)** $x[n] = u[n-1] - u[n-3]$.

Solution It is easy to compute the answer directly using the definition of the DTFT:

$$\begin{aligned} X(e^{j\Omega}) &= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n} \\ &= e^{-j\Omega} + e^{-j2\Omega} \\ &= 2e^{-j\frac{3}{2}\Omega} \cos(\Omega/2) \end{aligned}$$

Either of the last two lines is considered a correct answer.

b. **(3 points)** $x[n] = \exp\left(-\frac{|n|}{5}\right)$.

Solution Observe that $x[n]$ is a two-sided real exponential signal:

$$\exp\left(-\frac{|n|}{5}\right) = a^{|n|}, \text{ where } a = \exp\left(-\frac{1}{5}\right).$$

From Example 2, page 195, we know the DTFT pair

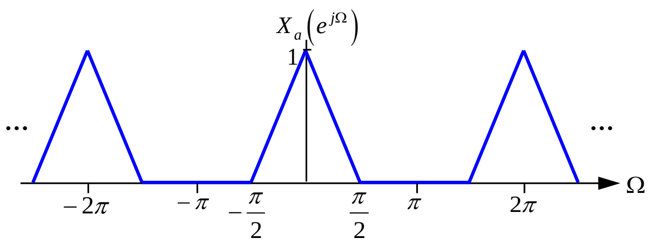
$$a^{|n|} \leftrightarrow \frac{1-a^2}{1-2a\cos\Omega + a^2}.$$

Thus we have

$$X\left(e^{j\Omega}\right) = \frac{1-e^{-2/5}}{1-2e^{-1/5}\cos\Omega + e^{-2/5}}.$$

5. (6 points) *Inverse discrete-time Fourier transforms*. Using known DTFT pairs and DTFT properties, find expressions for the DT signals that have the following DTFTs. You need not sketch them.

c. **(3 points)**



Solution From Example 12, pages 221-222, we know the DTFT pair

$$\frac{W}{\pi}\operatorname{sinc}^2\left(\frac{W}{\pi}n\right) \leftrightarrow \sum_{m=-\infty}^{\infty} \Lambda\left(\frac{\Omega - m2\pi}{2W}\right).$$

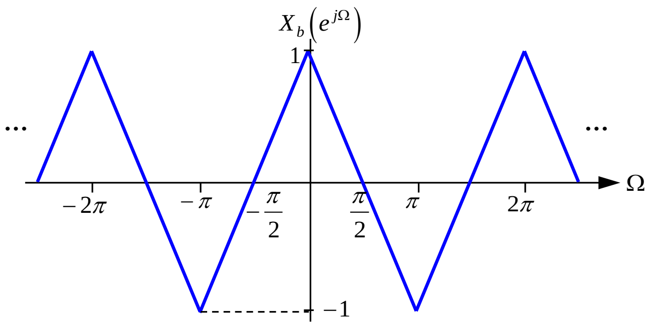
Our $X_a\left(e^{j\Omega}\right)$ is

$$X_a\left(e^{j\Omega}\right) = \sum_{m=-\infty}^{\infty} \Lambda\left(\frac{\Omega - m2\pi}{\pi/2}\right) = \sum_{m=-\infty}^{\infty} \Lambda\left(\frac{\Omega - m2\pi}{2(\pi/4)}\right),$$

so its inverse DTFT is

$$x_a[n] = \frac{1}{4}\operatorname{sinc}^2\left(\frac{n}{4}\right).$$

d. **(3 points)** *Hint*: express this DTFT $X_b\left(e^{j\Omega}\right)$ as a sum of scaled, frequency-shifted copies of the DTFT $X_a\left(e^{j\Omega}\right)$ appearing in part (a). Denote the inverse DTFT of $X_b\left(e^{j\Omega}\right)$ by $x_b[n]$. Demonstrate that $x_b[n] = 0$, n even, which is a consequence of $X_b\left(e^{j\Omega}\right)$ having half-wave-odd symmetry.



Solution We can express this DTFT in terms of the DTFT appearing in part (a) as

$$X_b(e^{j\Omega}) = X_a(e^{j\Omega}) - X_a(e^{j(\Omega-\pi)}).$$

Using the frequency-shift property of the DTFT

$$x[n]e^{j\Omega_0 n} \leftrightarrow X(e^{j(\Omega-\Omega_0)})$$

with $\Omega_0 = \pi$, we obtain

$$\begin{aligned} x_b[n] &= x_a[n] - x_a[n]e^{j\pi n} \\ &= \left[1-(-1)^n\right]x_a[n] \\ &= \begin{cases} 2x_a[n] & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \\ &= \begin{cases} \frac{1}{2}\operatorname{sinc}^2\left(\frac{n}{4}\right) & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \end{aligned}.$$

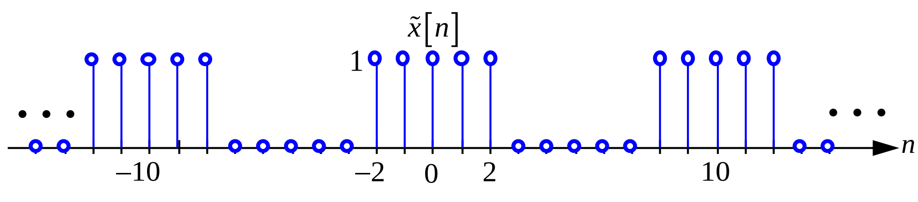
6. (9 points) *Relation between DTFS of periodic signal and DTFT of one period*. Consider a periodic rectangular pulse train

$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} \Pi\left(\frac{n-mN}{2N_1}\right).$$

Throughout this problem, assume that $N_1 = 2$ and $N = 10$.

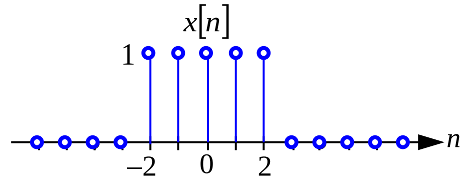
e. **(2 points) S**ketch two or three periods of $\tilde{x}[n]$ vs. *n*.

Solution The periodic signal $\tilde{x}[n]$ is shown.



f. **(3 points)** Specify an aperiodic signal $x[n]$ that is equal to $\tilde{x}[n]$ in one period, and is zero otherwise, and sketch that. Give an expression for $X\left(e^{j\Omega}\right)$, the DTFT of $x[n]$, and sketch two or three periods of $X\left(e^{j\Omega}\right)$ vs. Ω.

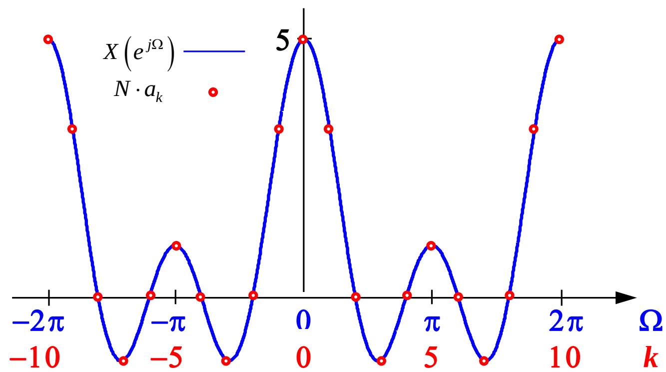
Solution The one-period signal is $x[n] = \Pi\left(\frac{n}{2N_1}\right) = \Pi\left(\frac{n}{4}\right)$, which is shown.



From Table 6, its DTFT is

$$X\left(e^{j\Omega}\right) = \frac{\sin\left[\Omega\left(N_1 + \frac{1}{2}\right)\right]}{\sin(\Omega/2)} = \frac{\sin[5\Omega/2]}{\sin(\Omega/2)},$$

which is shown in blue below.



g. **(4 points)** The periodic signal $\tilde{x}[n]$ has DTFS coefficients a_k. By sampling $X\left(e^{j\Omega}\right)$, obtain an expression for the a_k. Sketch $N \cdot a_k$ vs. k on your sketch of $X\left(e^{j\Omega}\right)$ vs. Ω for part (b), indicating the values of k.

Solution Since the period is $N = 10$, the fundamental frequency is $\Omega_0 = \pi/5$. We know that the DTFS coefficients of $\tilde{x}[n]$ are $1/N$ times the samples of $X\left(e^{j\Omega}\right)$ at $\Omega = k\Omega_0 = k\pi/5$:

$$\begin{aligned} a_k &= \frac{1}{N} X\left(e^{j\Omega}\right)\Big|_{\Omega = k\frac{\pi}{5}} \\ &= \frac{1}{10} \frac{\sin[5\Omega/2]}{\sin(\Omega/2)}\Bigg|_{\Omega = k\frac{\pi}{5}} . \\ &= \frac{1}{10} \frac{\sin[k\pi/2]}{\sin(k\pi/10)} \end{aligned}$$

We have sketched $N \cdot a_k$ in red on the figure above.

Analysis of DT LTI Systems using DTFT

7. **(12 points)** *Filtering sinusoids by LTI systems*. We are given a sum of two sinusoids:

$$x[n] = 2\cos\left(\frac{\pi}{8}n\right) + \sin\left(\frac{3\pi}{4}n\right).$$

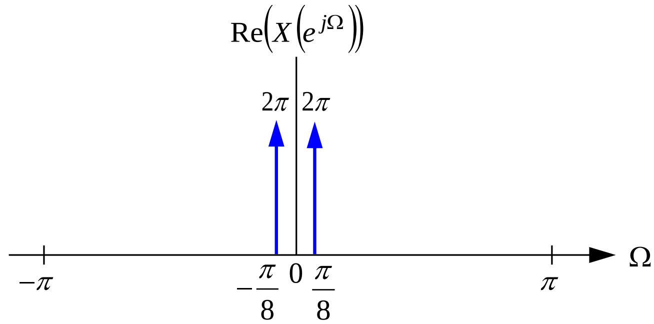
a. **(3 points)** Sketch the real and imaginary parts of its DTFT $X\left(e^{j\Omega}\right)$ for $-\pi \le \Omega \le \pi$.

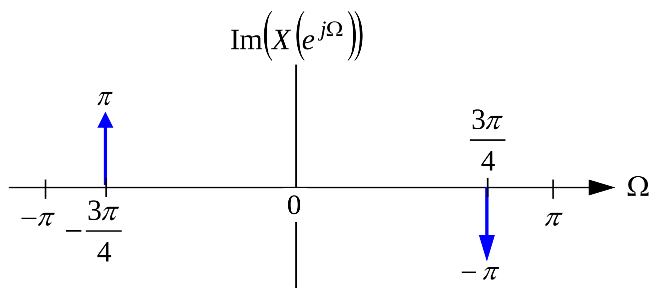
Solution We use the DTFT pairs from Table 6:

$$\cos\Omega_0 n \leftrightarrow \pi \sum_{l=-\infty}^{\infty}\left[\delta\left(\Omega-\Omega_0-2\pi l\right)+\delta\left(\Omega+\Omega_0-2\pi l\right)\right],\ \Omega_0 = \frac{\pi}{8}$$

$$\sin\Omega_0 n \leftrightarrow \frac{\pi}{j} \sum_{l=-\infty}^{\infty}\left[\delta\left(\Omega-\Omega_0-2\pi l\right)-\delta\left(\Omega+\Omega_0-2\pi l\right)\right],\ \Omega_0 = \frac{3\pi}{4}$$

to sketch the real and imaginary parts shown.



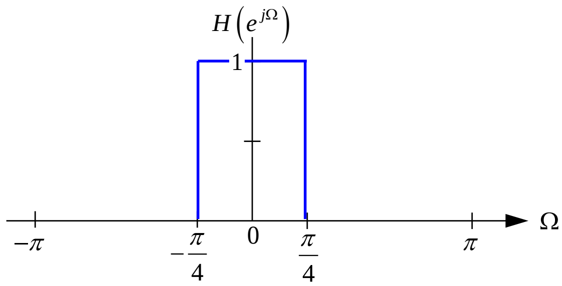


Now, for each of the following LTI systems with impulse response $h[n]$, sketch the real or imaginary part of the frequency response $H\left(e^{j\Omega}\right)$ (whichever is nonzero) for $-\pi \le \Omega \le \pi$. Then, assuming $x[n]$ is input to the system, give an expression for the output $y[n]$. You do not need to sketch the DTFT of the output, $Y\left(e^{j\Omega}\right)$. *Hint*: There is no need to perform convolution. Sketching $X\left(e^{j\Omega}\right)$ is not actually necessary for determining $y[n]$, but is a good exercise and is helpful in visualizing $y[n]$ and its DTFT $Y\left(e^{j\Omega}\right)$.

b. **(3 points)** $h[n] = \frac{1}{4}\text{sinc}\left(\frac{n}{4}\right)$.

Solution We use the DTFT pair from Table 6:

$$\frac{W}{\pi}\text{sinc}\left(\frac{Wn}{\pi}\right) \leftrightarrow \sum_{m=-\infty}^{\infty} \Pi\left(\frac{\Omega - m2\pi}{2W}\right),\ W = \frac{\pi}{4}.$$



The output is $y[n] = 2\cos\left(\frac{\pi}{8}n\right)$.

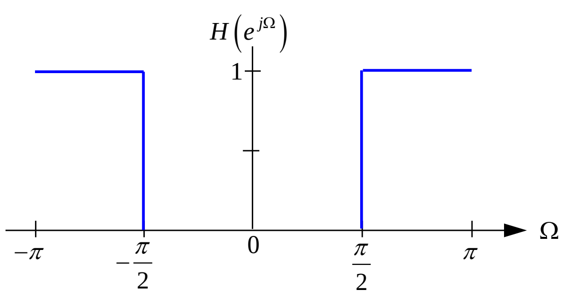
c. **(3 points)** $h[n] = (-1)^n \frac{1}{2}\text{sinc}\left(\frac{n}{2}\right)$.

Solution We use the DTFT pair from Table 6:

$$\frac{W}{\pi}\text{sinc}\left(\frac{Wn}{\pi}\right) \leftrightarrow \sum_{m=-\infty}^{\infty} \Pi\left(\frac{\Omega - m2\pi}{2W}\right),\ W = \frac{\pi}{2}$$

and the DTFT frequency-shift property

$$x[n]e^{j\Omega_0 n} \leftrightarrow X\left(e^{j(\Omega-\Omega_0)}\right) \text{ with } \Omega_0 = \pi \text{ .}$$



The output is $y[n] = \sin\left(\frac{3\pi}{4}n\right)$.

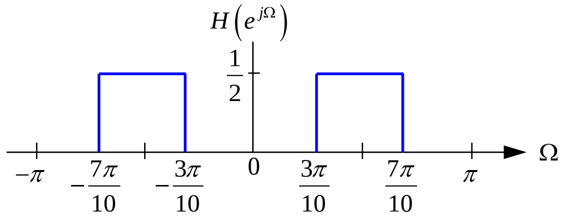
d. **(3 points)** $h[n] = \cos\left(\frac{\pi}{2}n\right)\frac{1}{5}\text{sinc}\left(\frac{n}{5}\right)$.

Solution We use the DTFT pair from Table 6:

$$\frac{W}{\pi}\operatorname{sinc}\left(\frac{Wn}{\pi}\right) \leftrightarrow \sum_{m=-\infty}^{\infty} \Pi\left(\frac{\Omega - m2\pi}{2W}\right),\ W = \frac{\pi}{5}$$

and the DTFT modulation property

$$x[n]\cos(\Omega_0 n) \leftrightarrow \frac{1}{2}\left[X\left(e^{j(\Omega-\Omega_0)}\right) + X\left(e^{j(\Omega+\Omega_0)}\right)\right] \text{ with } \Omega_0 = \frac{\pi}{2}.$$



The output is $y[n] = 0$.

DT LTI System Analysis using DTFT

8. (15 points) *Infinite impulse response DT system*. A first-order causal discrete-time LTI system has a frequency response

$$H\left(e^{j\Omega}\right) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}}.$$

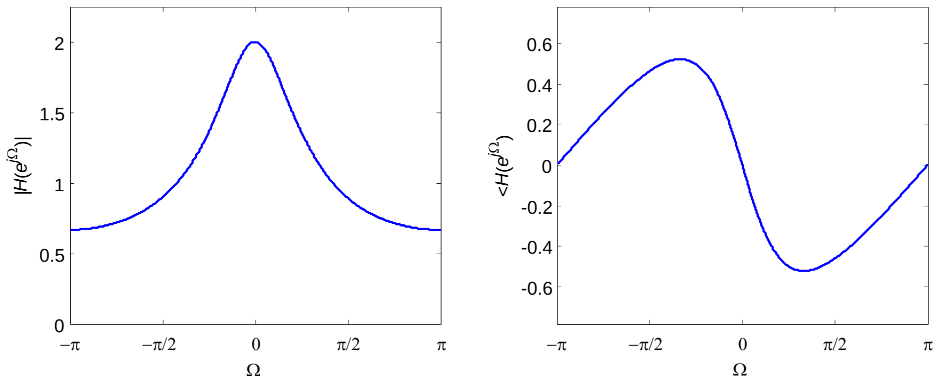
a. **(3 points)** Give expressions for the magnitude response $\left|H\left(e^{j\Omega}\right)\right|$ and the phase response $\angle H\left(e^{j\Omega}\right)$ and sketch them. They are smooth functions of Ω, so it is sufficient to evaluate them at a few values of Ω to make reasonable sketches.

Solution The magnitude and phase are

$$\left|H\left(e^{j\Omega}\right)\right| = \left|1 - \frac{1}{2}e^{-j\Omega}\right|^{-1} = \left[\left(1 - \frac{1}{2}\cos\Omega\right)^2 + \left(\frac{1}{2}\sin\Omega\right)^2\right]^{-\frac{1}{2}} = \left[\frac{5}{4} - \cos\Omega\right]^{-\frac{1}{2}},$$

$$\angle H\left(e^{j\Omega}\right) = -\angle\left(1 - \frac{1}{2}e^{-j\Omega}\right) = -\tan^{-1}\left(\frac{\frac{1}{2}\sin\Omega}{1 - \frac{1}{2}\cos\Omega}\right) = -\tan^{-1}\left(\frac{\sin\Omega}{2 - \cos\Omega}\right).$$

To make reasonable sketches, you just need to evaluate these expressions for a few values of Ω.



b. **(2 points)** Find a purely real expression for the output $y[n]$ if the input is

$$x[n] = \cos\left(\frac{\pi}{3}n\right).$$

It is fine to include trigonometric functions, such as $\tan^{-1}\left(\cos\frac{\pi}{3}\right)$, etc., in your answer without evaluating them to obtain specific numerical values.

Solution Using the result of Homework 5 Problem 1, the output is

$$\begin{aligned} y[n] &= \left|H\left(e^{j\frac{\pi}{3}}\right)\right| \cos\left(\frac{\pi}{3}n + \angle H\left(e^{j\frac{\pi}{3}}\right)\right) \\ &= \left[\frac{5}{4} - \cos\frac{\pi}{3}\right]^{-\frac{1}{2}} \cos\left(\frac{\pi}{3}n - \tan^{-1}\left(\frac{\sin\frac{\pi}{3}}{2-\cos\frac{\pi}{3}}\right)\right). \end{aligned}$$

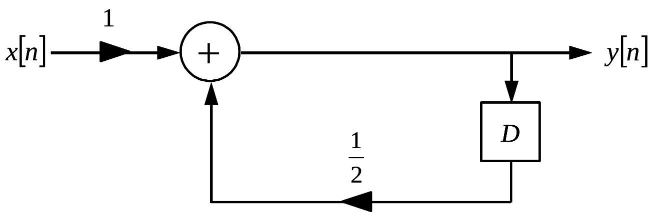
c. **(2 points)** Give the constant-coefficient difference equation that describes the system.

Solution The difference equation is

$$y[n] - \frac{1}{2}y[n-1] = x[n].$$

d. **(2 points)** Give a block diagram showing how to realize the system using delays and gain elements.

Solution



e. **(2 points)** Give an expression for the impulse response $h[n]$.

Solution The inverse DTFT of $H\left(e^{j\Omega}\right)$ is

$$h[n] = \left(\frac{1}{2}\right)^n u[n].$$

f. **(2 points)** Give an expression for the step response $s[n]$.

Solution The running summation of the impulse response is

$$\begin{aligned} s[n] &= \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m \\ &= \frac{(1/2)^{n+1} - 1}{(1/2) - 1} u[n] \\ &= \left[2 - \left(\frac{1}{2}\right)^n\right] u[n] \end{aligned}$$

Either of the last two lines is considered a correct answer.

g. **(2 points)** Evaluate the integral $\int_{-\pi}^{\pi} H\left(e^{j\Omega}\right) d\Omega$. *Hint*: this is related to the impulse response $h[n]$ at a particular time.

Solution The integral $\int_{-\pi}^{\pi} H\left(e^{j\Omega}\right) d\Omega$ is equal to 2π times the inverse DTFT evaluated at $n = 0$, so

$$\int_{-\pi}^{\pi} H\left(e^{j\Omega}\right) d\Omega = 2\pi\, h[0] = 2\pi .$$

9. (12 points) *Finite Impulse Response DT System*. The impulse response of a three-tap FIR filter can be expressed as

$$h[n] = h[-1]\delta[n+1] + h[0]\delta[n] + h[1]\delta[n-1].$$

By choosing the coefficients $h[-1]$, $h[0]$ and $h[1]$, we can obtain different types of filters. For each set of coefficients (a)-(d):

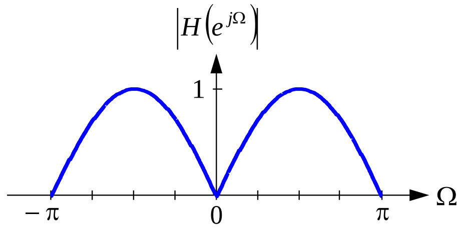
- Give an expression for the frequency response $H\left(e^{j\Omega}\right)$.
- Sketch the magnitude response $\left|H\left(e^{j\Omega}\right)\right|$ and phase response $\angle H\left(e^{j\Omega}\right)$.
- Classify the filter as lowpass, highpass, bandpass or bandstop. *Hint*: there is one of each.

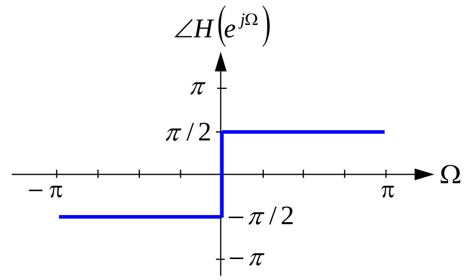
Note: in each case, $h[n]$ is real and even in n or real and odd in n. As a consequence, each $H\left(e^{j\Omega}\right)$ is either real and even in Ω or imaginary and odd in Ω. Hence, each phase response $\angle H\left(e^{j\Omega}\right)$ is piecewise constant with integer slope, so each filter causes no phase distortion.

a. **(3 points)** $h[-1]=\frac{1}{2}$, $h[0]=0$, $h[1]=-\frac{1}{2}$.

Solution

$$H\left(e^{j\Omega}\right)=\frac{1}{2}e^{j\Omega}-\frac{1}{2}e^{-j\Omega}=j\sin\Omega\,.$$ Bandpass filter.

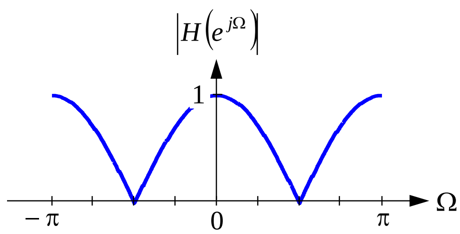


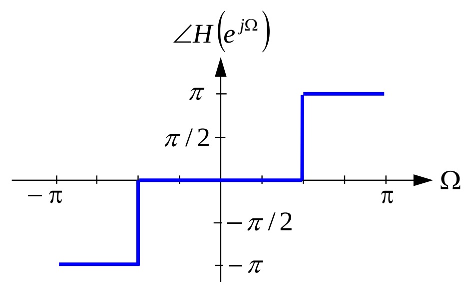


b. **(3 points)** $h[-1]=\frac{1}{2}$, $h[0]=0$, $h[1]=\frac{1}{2}$.

Solution Phases of $+\pi$ and $-\pi$ are equivalent and interchangeable.

$$H\left(e^{j\Omega}\right)=\frac{1}{2}e^{j\Omega}+\frac{1}{2}e^{-j\Omega}=\cos\Omega\,.$$ Bandstop filter.

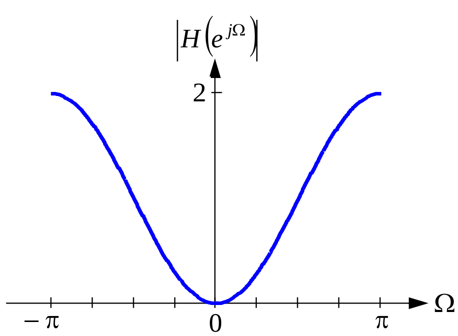


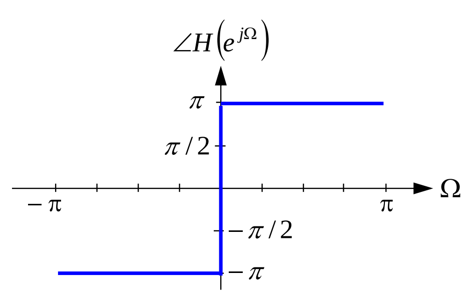


c. **(3 points)** $h[-1]=\frac{1}{2}$, $h[0]=-1$, $h[1]=\frac{1}{2}$.

Solution Phases of $+\pi$ and $-\pi$ are equivalent and interchangeable.

$$H\left(e^{j\Omega}\right)=\frac{1}{2}e^{j\Omega}-1+\frac{1}{2}e^{-j\Omega}=\cos\Omega-1\,.$$ Highpass filter.

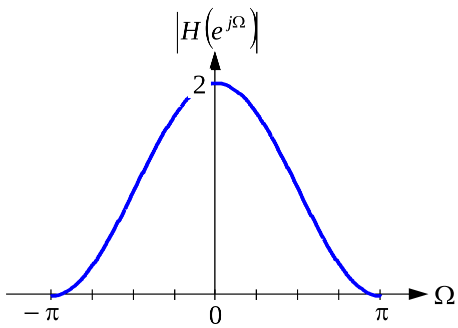


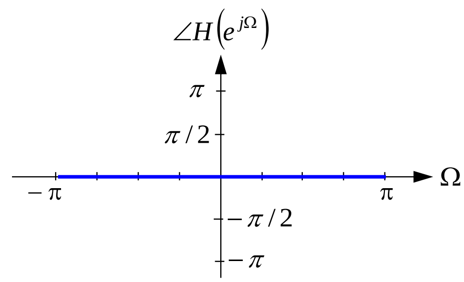


d. **(3 points)** $h[-1]=\frac{1}{2}$, $h[0]=1$, $h[1]=\frac{1}{2}$.

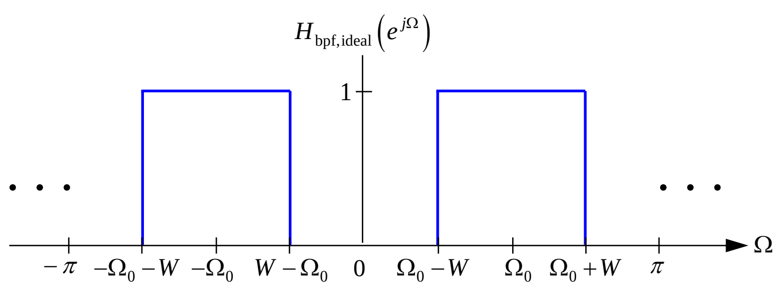
Solution

$$H\left(e^{j\Omega}\right)=\frac{1}{2}e^{j\Omega}+1+\frac{1}{2}e^{-j\Omega}=\cos\Omega+1.$$ Lowpass filter.





10. (12 points) *DT bandpass filter*. An ideal bandpass filter with zero delay has the purely real frequency response for $-\pi \le \Omega \le \pi$ shown below. The frequency response is periodic with period 2π. The filter has passbands centered at $\pm\Omega_0$, each with bandwidth $2W$. You may assume $0<W<\pi/2$ and $W<\Omega_0<\pi-W$.



a. **(4 points)** Using known DTFT pairs and DTFT properties, find an expression for the impulse response $h_{\text{bpf,ideal}}[n]$ of the bandpass filter. *Hint*: start with an ideal lowpass filter and modulate its impulse response by a sinusoid.

Solution We start with an ideal lowpass filter with cutoff frequency W, which has impulse response and frequency response

$$h_{\text{lpf,ideal}}[n]=\frac{W}{\pi}\text{sinc}\left(\frac{W}{\pi}n\right)\leftrightarrow H_{\text{lpf,ideal}}\left(e^{j\Omega}\right)=\Pi\left(\frac{\Omega}{2W}\right),\ H_{\text{lpf,ideal}}\left(e^{j(\Omega+2\pi)}\right)=H_{\text{lpf,ideal}}\left(e^{j\Omega}\right).$$

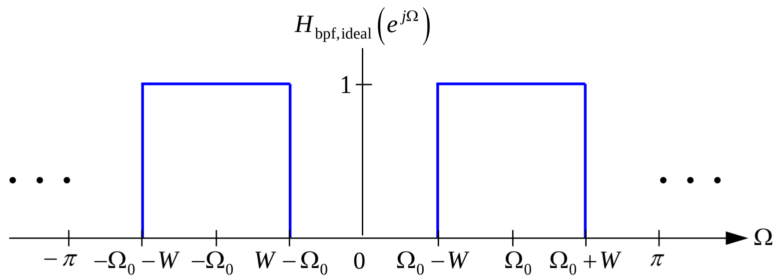
To obtain the impulse response of the bandpass filter, we multiply this by twice a cosine signal:

$$\begin{aligned}h_{\text{bpf,ideal}}[n]&=h_{\text{lpf,ideal}}[n]\cdot 2\cos(\Omega_0 n)\\&=\frac{2W}{\pi}\text{sinc}\left(\frac{W}{\pi}n\right)\cos(\Omega_0 n)\end{aligned}.$$

Since $2\cos(\Omega_0 n)=e^{j\Omega_0 n}+e^{-j\Omega_0 n}$, using the DTFT frequency shift property, this corresponds to a frequency response

$$H_{\text{bpf,ideal}}\left(e^{j\Omega}\right) = H_{\text{lpf,ideal}}\left(e^{j(\Omega-\Omega_0)}\right) + H_{\text{lpf,ideal}}\left(e^{j(\Omega+\Omega_0)}\right).$$

This is sketched here.



b. **(4 points)** Now we truncate the impulse response to be nonzero only for $-N_1 \le n \le N_1$, and delay it to make it causal. Give an expression for the causal impulse response $h_{\text{bpf,causal}}[n]$.

Solution First we truncate to $-N_1 \le n \le N_1$, obtaining an impulse response

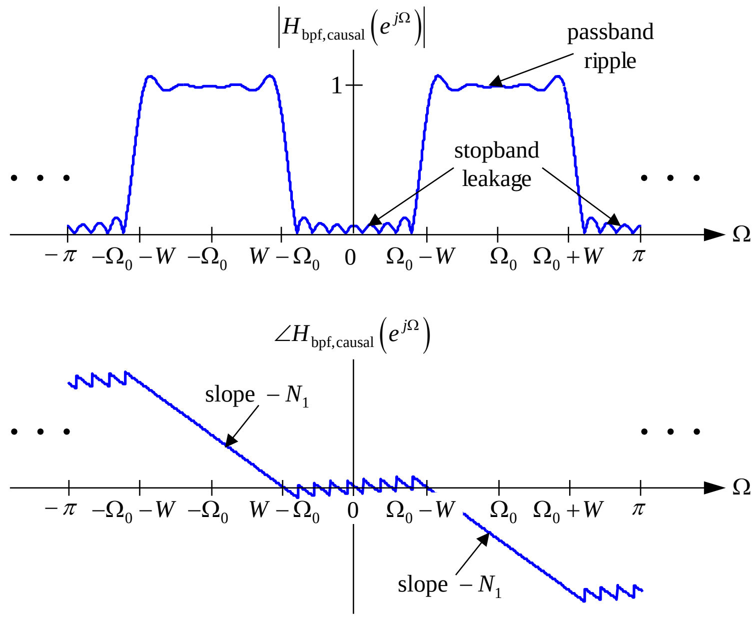
$$\begin{aligned} h_{\text{bpf,trunc}}[n] &= h_{\text{bpf,ideal}}[n]\Pi\left(\frac{n}{2N_1}\right) \\ &= \frac{2W}{\pi}\text{sinc}\left(\frac{W}{\pi}n\right)\cos\left(\Omega_0 n\right)\Pi\left(\frac{n}{2N_1}\right) \\ &= \begin{cases} \dfrac{2W}{\pi}\text{sinc}\left(\dfrac{W}{\pi}n\right)\cos\left(\Omega_0 n\right) & |n| \le N_1 \\ 0 & |n| > N_1 \end{cases} \end{aligned}.$$

Then we delay it by N_1 samples to make it causal:

$$\begin{aligned} h_{\text{bpf,causal}}[n] &= h_{\text{bpf,trunc}}[n-N_1] \\ &= \frac{2W}{\pi}\text{sinc}\left(\frac{W}{\pi}(n-N_1)\right)\cos\left(\Omega_0(n-N_1)\right)\Pi\left(\frac{n-N_1}{2N_1}\right) \\ &= \begin{cases} \dfrac{2W}{\pi}\text{sinc}\left(\dfrac{W}{\pi}(n-N_1)\right)\cos\left(\Omega_0(n-N_1)\right) & 0 \le n \le 2N_1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}.$$

c. **(4 points)** Sketch the magnitude response $\left|H_{\text{bpf,causal}}\left(e^{j\Omega}\right)\right|$ (on a linear scale, not a decibel scale), and indicate key features, including passband ripple and stopband leakage. Sketch the phase response $\angle H_{\text{bpf,causal}}\left(e^{j\Omega}\right)$, and indicate the phase slope in the passband. You needn't be concerned about the values of the phase response in the stopband. You can assume a small value of N_1, for example $N_1 = 16$.

Solution The magnitude and phase responses are shown below. Passband ripple and stopband leakage are indicated. In the passbands, the phase slope is $-N_1$, corresponding to a group delay $-d\angle H_{\text{bpf,causal}}\left(e^{j\Omega}\right)/d\Omega = N_1$. At any frequency Ω, any integer multiple of 2π can be added to the phase.



$\left|H_{\text{bpf,causal}}\left(e^{j\Omega}\right)\right|$
1
passband ripple
stopband leakage
Ω
$-\pi$
$-\Omega_0 - W$
$-\Omega_0$
$W - \Omega_0$
0
$\Omega_0 - W$
Ω_0
$\Omega_0 + W$
π
$\angle H_{\text{bpf,causal}}\left(e^{j\Omega}\right)$
slope $-N_1$
slope $-N_1$

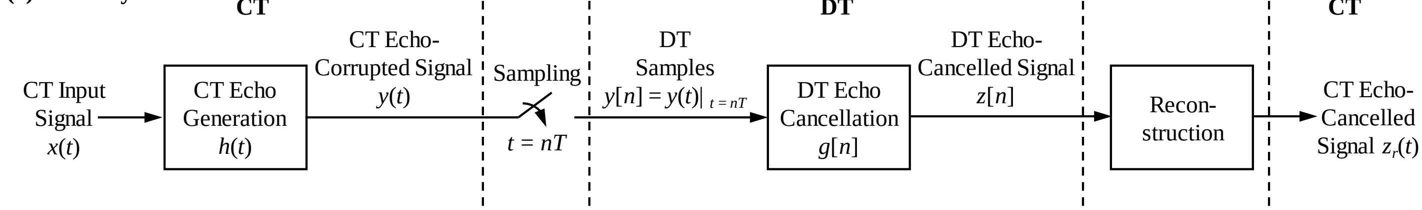
Laboratory 7

This week we study *echo cancellation*, an application of signal processing that arises frequently in practice. The basic idea is that a signal of interest is corrupted by one or more echoes, which are delayed and scaled copies of the original signal. The goal of echo cancellation is to process the corrupted signal so as to recover the original signal.

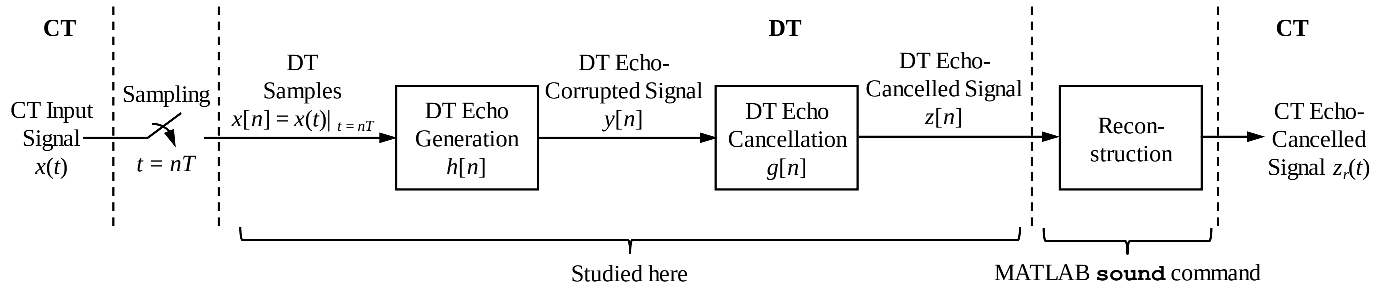
In most applications, the echoes are generated by a continuous-time (CT) process, and we sample the echo-corrupted signal and cancel the echoes using discrete-time (DT) signal processing. The DT processing is usually implemented numerically, in which case, it is called *digital signal processing* (DSP). DSP of sampled CT signals is used in a wide range of applications, including speech, music and video processing, sensing and control systems, and communication systems.

The setup for echo cancellation is shown below.

(a) Actual system



(b) Equivalent system (assuming sufficient sampling rate 1/*T*)



The actual system is shown in (a). In echo generation, the CT input signal $x(t)$ is convolved with an impulse response $h(t)$ to produce an echo-corrupted signal $y(t)$. The echo-corrupted signal $y(t)$ is sampled at a rate $1/T$ sufficient to fully characterize it.[1] The echo-corrupted samples $y[n]$ are convolved with a DT echo canceller, which has impulse response $g[n]$, yielding a DT echo-cancelled signal $z[n]$. Using the samples $z[n]$, a reconstruction system reconstructs a CT echo-cancelled signal $z_r(t)$.

In this lab, we study an equivalent system, which is shown in (b). The CT input signal $x(t)$ is sampled at a rate $1/T$ sufficient to fully characterize it,[2] yielding samples $x[n]$. The echo-generation process is

---

[1] As we will learn in lecture, we can identify conditions under which the samples $y[n]$ are sufficient to reconstruct $y(t)$ perfectly. For now, we simply assume these conditions are satisfied.

[2] That rate is the same as that required to characterize $y(t)$, since the echo generator $h(t)$ is a linear system.

described by an equivalent DT system with impulse response $h[n]$.[3] The echo-corrupted samples $y[n]$ are obtained by convolving the input samples $x[n]$ with the echo-generation impulse response $h[n]$. The remainder of system (b) is identical to system (a). In this lab, we let MATLAB take care of reconstruction for us when we listen to signals using the **`sound`** command. In Laboratory 8, we will study reconstruction in detail.

There are two common models for echo generation.

In the first echo-generation model, there are only a few echoes. An example is propagation of radio signals outdoors, where there are a few different transmission paths between the transmitting antenna and receiving antenna, each with a different path gain and path delay. This is called *multipath propagation*. In the simplest instance, there is a primary path and just one echo. In discrete time, the system has an impulse response

$$h[n] = \delta[n] + \alpha\delta[n-N],$$

where N is the delay between the primary path and the echo, and α is the echo amplitude. This system is described by a difference equation

$$y[n] = x[n] + \alpha x[n-N]. \tag{1}$$

The first echo-generation system is *nonrecursive*, and has a *finite impulse response*.

In the second echo-generation model, there is an infinite sequence of echoes with decreasing amplitudes. This may occur in a concert hall, where sound is reflected off multiple surfaces. In discrete time, such a system has an impulse response

$$\begin{aligned} h[n] &= \delta[n] + \alpha\delta[n-N] + \alpha^2\delta[n-2N] + \alpha^3\delta[n-3N] + \ldots \\ &= \sum_{m=0}^{\infty} \alpha^m \delta[n-mN] \end{aligned},$$

and is described by an Nth-order difference equation

$$y[n] = x[n] + \alpha y[n-N]. \tag{2}$$

The second echo-generation system is *recursive*, and has an *infinite impulse response*. This is the echo generation system we will study in this lab. The corresponding echo-cancellation system is described by a difference equation

$$z[n] = y[n] - \alpha y[n-N], \tag{3}$$

where $y[n]$ and $z[n]$ are the input and output of the echo-cancellation system. Notice that in order to suppress the infinite train of echoes, the echo canceler simply needs to subtract from $y[n]$ a scaled and delayed copy of itself. The echo-cancellation system is *nonrecursive*, and has a *finite impulse response*. In the ideal case that the scale factor and delay match those in the echo-generation system, it is the *inverse* of the echo-generation system.

[3] As you will learn in EE 102B, in general, $h[n]$ is not accurately modeled by simply sampling $h(t)$. Nevertheless, if the sampling rate $1/T$ is sufficiently high, $h[n]$ can be approximated well by sampling $h(t)$.

Task 1: Verifying the echo-cancellation system. We will first look at the cascade of the echo-generation and echo-cancellation systems to verify that the echoes are cancelled.

1. Draw a block diagram of the echo-generation system (2) followed by the echo-cancellation system (3). You can represent an N-sample delay by a box labeled by D^N.

2. Derive the overall difference equation for the cascade of the echo-generation system (2) followed by the echo-cancellation system (3). Verify that the echoes are cancelled.

(3 points) Task 1 Solution The block diagram of the echo-generation system followed by the echo-cancellation system is shown below in Figure L1:

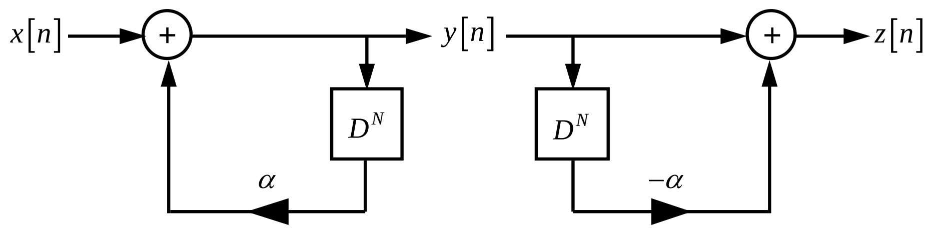


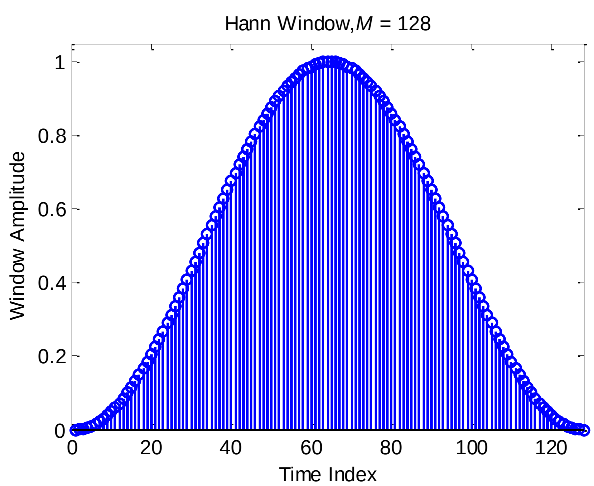
Figure L1: Block diagram of echo-cancellation system cascaded with echo-generation system.

Before we obtain the overall output $z[n]$, we note that $y[n]$ is given as $y[n] = x[n] + \alpha y[n-N]$. Thus, $z[n]$ can be written as:

$$\begin{aligned} z[n] &= y[n] - \alpha y[n-N] \\ &= \left(x[n] + \alpha y[n-N]\right) - \alpha y[n-N] \\ &= x[n] \end{aligned}$$

We see that the echo-cancellation system does restore the original signal perfectly.

Task 2: Echo generation. Our input signal $x[n]$ will be a short pulse at a single frequency, generated by multiplying a sinusoid by a Hann window. A Hann window of length M represents M samples of a single cycle of a cosine, which is raised up by adding 1, then scaled by 1/2. A Hann window of length 128 is shown here. Notice how it goes smoothly to zero at the end points. Window functions are widely used to truncate a signal or impulse response to a finite length while tapering it smoothly to zero at the end points. For further discussion of window functions, see pages 257-260.



The Hann window is built into the MATLAB Signal Processing Toolbox. If you do not have that toolbox, you can use the following code.

```
function w = hann(M)
% Symmetric Hann window
w = 0.5*(1-cos(2*pi*(0:M-1)'/(M-1)));
end
```

In our test signal, we will choose the sinusoid to have a frequency in the audio range, so we can hear the results of echo generation and cancellation. The following **m-file** generates the pulse.

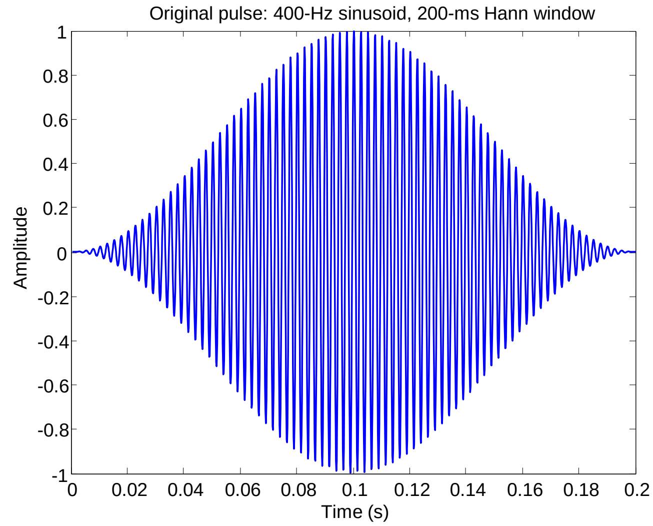
```
function x = pulse(f,Tdur,fs)
% Sinusoid multiplied by a Hann window function.
% f sinsusoid frequency (Hz)
% Tdur duration (s)
% fs sampling frequency (Hz)
nmax = floor(Tdur*fs);                  % duration (samples)
n = 1:nmax;                             % discrete time index
x = hann(nmax).'.* cos(2*pi*f*n/fs);    % waveform samples
end
```

We specify the frequency, duration and sampling frequency, and the **m-file** outputs the pulse waveform **x** and a vector **n** defining the time scale. You can test this function using the following script. Throughout this lab, we will use the same values of the sampling frequency (8192 Hz), pulse frequency (400 Hz) and pulse duration (0.2 s).

```
fs = 8192;                      % sampling frequency (Hz)
f = 400;                        % sinusoid frequency (Hz)
Tdur = 0.2;                     % pulse duration (s)
n = 1:floor(fs*Tdur);
x = pulse(f,Tdur,fs);
plot(n/fs,x); xlabel('Time (s)'); ylabel('Amplitude');
title('Original pulse: 400-Hz sinusoid, 200-ms Hann window')
pause; sound(x);
```

Although **x** represents samples of a sinusoidal CT signal, and is therefore a DT signal, the sampling frequency is very high relative to the CT sinusoid's frequency, and it is easier to visualize it using the **plot** command than using the **stem** command.

Your plot should look like the one shown. Do not turn in this plot. Listen to the waveform. You should hear a single short pulse of sound.



In order to generate the echo-corrupted signal, we will implement equation (2) using the MATLAB **`filter`** command:

```
>> y = filter(b,a,x);.
```

This recursively evaluates an *N*th-order difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \ . \tag{4}$$

The **`filter`** command assumes that the b_k and a_k coefficients in (4) are described as vectors $\mathbf{b} = (b_0, ..., b_M)$ and $\mathbf{a} = (a_0, ..., a_N)$.

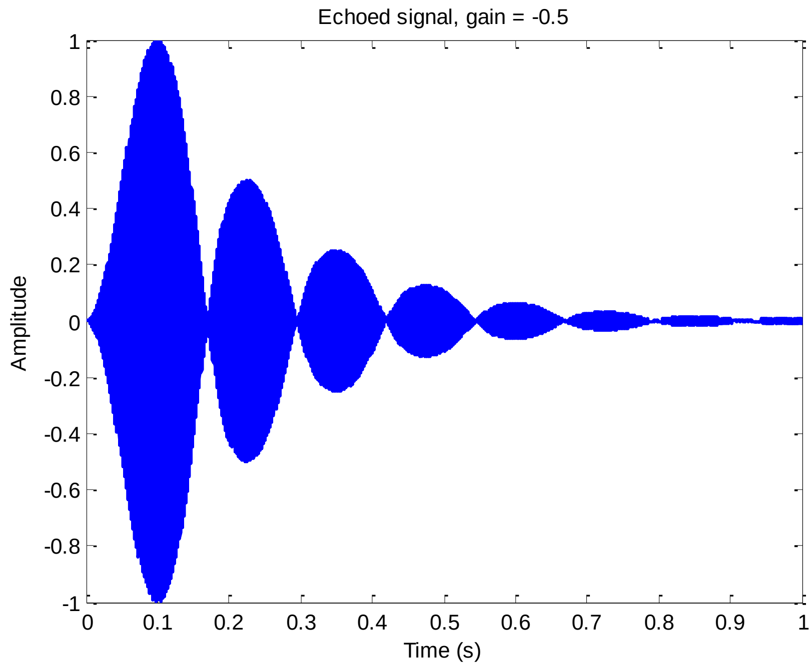
When applying the **`filter`** command to a signal **`x`**, the result **`y`** has the same duration as **`x`**. In this lab, the echoes will potentially extend the length of the signal significantly, so we will first zero-pad the signal to a duration of 1 s using:

```
Tzp = 1;                          % zero-padded duration (s)
Nzp = floor(fs*Tzp);              % zero-padded duration (samp.)
xzp = [x zeros(1,Nzp-length(x))]; % zero-padded signal
```

Then we can generate the echoes using the following commands. Throughout this lab, we will use the same nominal values of the echo delay (0.125 s) and the echo gain (–0.5).

```
Tdel = 0.125;                          % echo delay (s)
alpha = -0.5;                          % echo gain
Ndel = floor(Tdel*fs);                 % echo delay (samples)
a = [1 zeros(1,Ndel-1) -alpha]; b = 1; % filter coefficients
y = filter(b,a,xzp);
```

The echo-corrupted waveform should look like the plot below. Turn in a plot of your echo-corrupted waveform. Listen to your echo-corrupted waveform. Can you hear the echoes?



(8 points) Task 2 Solution The MATLAB code and plots (L2 and L3) are shown below. We can easily hear the echoes because (i) the signal is in the audible frequency and, (ii) the echo delay of 0.125 seconds is perceptible by the human ear.

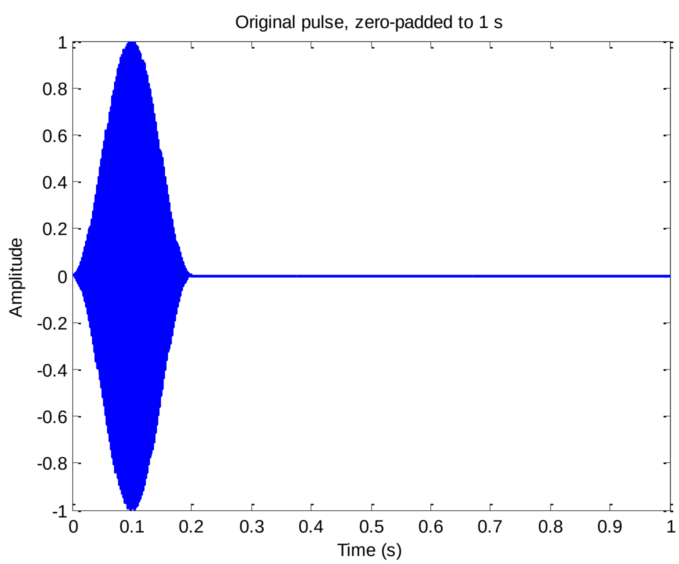


Figure L2: Zero-padded original pulse.

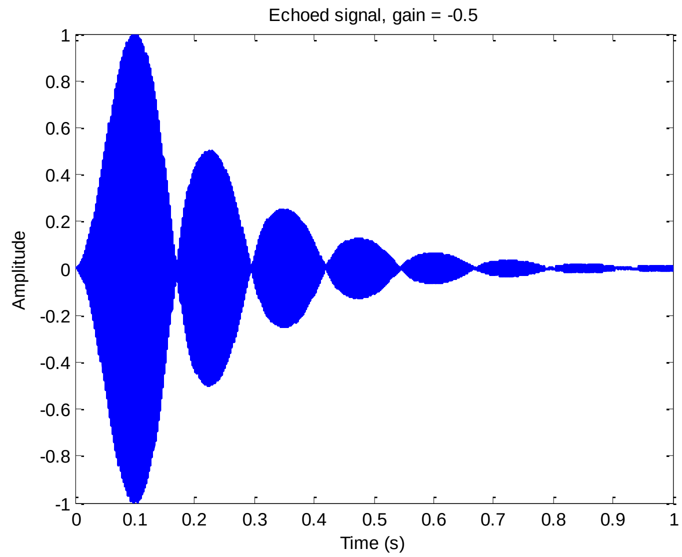


Figure L3: Echoed pulse with α = −0.5 and 0.125-second echo delay.

```
%% Task 2 - Generating echoes
clear
lw = 1.5;
% Parameters of original pulse
fs = 8192;                % sampling frequency (Hz)
f = 400;                  % sinusoid frequency (Hz)
Tdur = 0.2;               % pulse duration (s)
Tzp = 1;                 % zero-padded duration (s)
% Parameters of echoes
Tdel = 0.125;             % echo delay (s)
alpha = -0.5;             % echo gain

% Generate original pulse
n = 1:floor(fs*Tdur);
x = pulse(f,Tdur,fs);
figure(1)
plot(n/fs,x); l=get(gca,'children'); set(l,'linewidth',lw);
xlabel('Time (s)'); ylabel('Amplitude');
title('Original pulse: 400-Hz sinusoid, 200-ms Hann window')
pause; sound(x);

% Zero-pad original pulse
Nzp = floor(fs*Tzp);
nzp = 1:Nzp;
xzp = [x zeros(1,Nzp-length(x))];
figure(2)
plot(nzp/fs,xzp); l=get(gca,'children'); set(l,'linewidth',lw);
```

```
xlabel('Time (s)'); ylabel('Amplitude');
title(['Original pulse, zero-padded to ' num2str(Tzp,3) ' s'])
pause; sound(xzp);

% Generate echoes
Ndel = floor(Tdel*fs);                   % echo delay (samples)
a = [1 zeros(1,Ndel-1) -alpha]; b = 1;   % filter coefficients
y = filter(b,a,xzp);
figure(3)
plot(nzp/fs,y); l=get(gca,'children'); set(l,'linewidth',lw);
xlabel('Time (s)'); ylabel('Amplitude');
title(['Echoed signal, gain = ' num2str(alpha,3)]);
pause; sound(y);
```

Task 3: Echo cancellation. Now we are ready to peform echo cancellation.

First, write a MATLAB script that directly implements equation (3) by subtracting a scaled and shifted copy of **y** from **y**. Turn in a plot of the echo-cancelled signal. Listen to the sound. Are the echoes cancelled effectively?

An even easier way to implement the echo-cancellation system (3) is to use the **filter()** command and interchange the coefficient vectors **a** and **b**:

```
>> z = filter(a,b,y);
```

This can be understood in the context of pages 222-224 of the *Course Reader*. Let $H\left(e^{j\Omega}\right)$ denote the frequency response of the echo-generation system. Given an input with DTFT $X\left(e^{j\Omega}\right)$, the DTFT of the echo-corrupted output $Y\left(e^{j\Omega}\right)$ is

$$Y\left(e^{j\Omega}\right) = H\left(e^{j\Omega}\right) X\left(e^{j\Omega}\right).$$

Now let us try to use an echo canceller whose frequency response is the reciprocal of the frequency response of the echo-generation system:

$$H^{-1}\left(e^{j\Omega}\right) = \frac{1}{H\left(e^{j\Omega}\right)}.$$

If we input an echo-corrupted signal with DTFT $Y\left(e^{j\Omega}\right)$ to the echo canceller, its output has a DTFT

$$Z\left(e^{j\Omega}\right) = H^{-1}\left(e^{j\Omega}\right) Y\left(e^{j\Omega}\right) = H^{-1}\left(e^{j\Omega}\right) H\left(e^{j\Omega}\right) X\left(e^{j\Omega}\right) = X\left(e^{j\Omega}\right).$$

In other words, the echo canceler, which is the inverse system for the echo-generation system, has a frequency response that is its reciprocal. On page 224 of the *Course Reader*, equation (53) shows that the frequency response of the echo-generation system is a rational function of $e^{-j\Omega}$ with numerator coefficients given by vector **b** and denominator coefficients given by vector **a**. We conclude that the inverse system – the echo canceler – has a frequency response whose numerator coefficients are given by vector **a** and whose denominator coefficients are given by vector **b**.

Turn in a plot of the echo-cancelled signal. Listen to the sound. Are your results equivalent to those obtained using the delay-and-subtract method?

(7 points) Task 3 Solution By interchanging the coefficients **a** and **b** in the **filter()** command, we are able to cancel the echoes perfectly. Likewise, by implementing equation (3) directly, we are able to cancel the echoes perfectly.

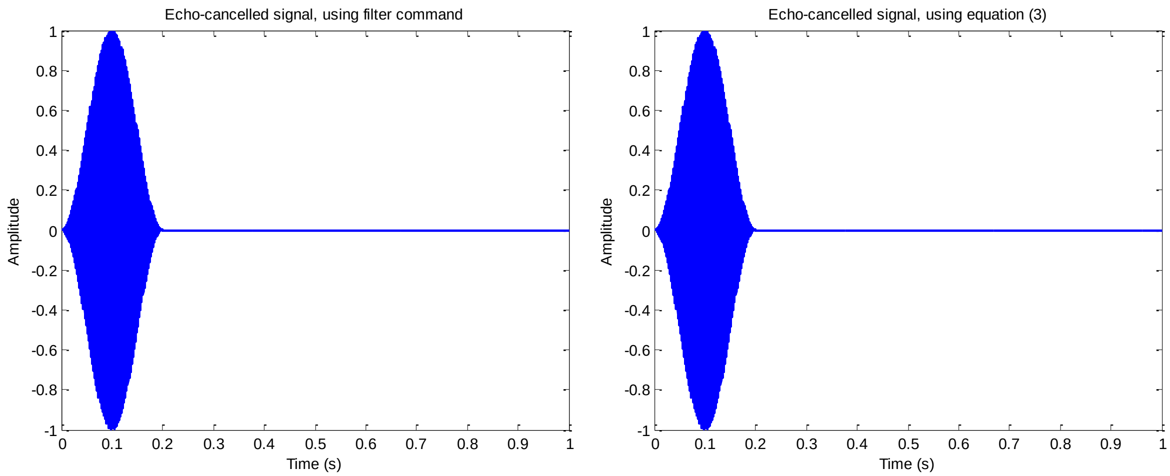


Figure L4: After echo cancellation using filter command (left) or equation (3) (right).

```
%% Task 3 - Cancelling echoes

% Computing z1 using filter command
zfilt = filter(a,b,y);
figure(4)
plot(nzp/fs,zfilt); l=get(gca,'children'); set(l,'linewidth',lw);
xlabel('Time (s)'); ylabel('Amplitude');
title('Echo-cancelled signal, using filter command');
pause; sound(zfilt);

% Computing z directly using Equation 3
z(1:Ndel) = y(1:Ndel);
for i = Ndel + 1:Nzp
    z(i) = y(i) - alpha*y(i-Ndel);
end
figure(5)
plot(nzp/fs,z); l=get(gca,'children'); set(l,'linewidth',lw);
xlabel('Time (s)'); ylabel('Amplitude');
title('Echo-cancelled signal, using equation (3)');
pause; sound(z);
```

Task 4: Effect of errors in gain or delay. Performing echo cancellation effectively requires having an accurate model for the echo-generation system which, in this case, corresponds to knowing the gain and delay parameters α and N. In practice, these are not known in advance, and must be estimated. Here we look at the sensitivity of echo cancellation to errors in those parameters. Instead of using the correct parameters α and N, the echo-cancellation system uses estimate parameters $\hat{\alpha}$ and $\hat{N}$:

$$z[n] = y[n] - \hat{\alpha}\, y[n - \hat{N}], \tag{3'}$$

First, we assume perfect knowledge of the delay, $\hat{N} = N$, and study the effect of errors in the gain, $\hat{\alpha} \neq \alpha$. Derive an expression for the echo-cancelled signal $z[n]$ as $x[n]$ plus an error term proportional to $y[n - N]$. Using the gain value for our simulations, $\alpha = -0.5$, and assuming $\hat{\alpha} = -0.45$, a 10% error, how large is the peak of the error term relative to the desired signal $x[n]$? Use MATLAB to compute the echo-cancelled signal for these parameters, and check your prediction. Does the first echo at approximately sample 2000 have the amplitude you expect? Listen to the sound. Can you hear the echo? We conclude that the echo canceller is not very sensitive to the exact value of $\hat{\alpha}$, since a 10% error still significantly suppresses the echo.

Now we assume perfect knowledge of the gain, $\hat{\alpha} = \alpha$, and study the effect of errors in the delay, $\hat{N} \neq N$. Assume errors in the delay, $\hat{N} - N$, of 1, 2, 10 and 20 samples. For each value of the error, plot the results and listen to the sound. Explain why, as the error is increased, the echo suppression first degrades, then improves. We conclude that the echo canceller is very sensitive to errors in the delay, since even a one-sample error leads to an audible echo.

(8 points) Task 4 Solution With an imperfect channel gain estimate $\hat{\alpha}$, using equation (3), the echo-cancelled signal can be written as

$$\begin{aligned} z[n] &= y[n] - \hat{\alpha} y[n-N] \\ &= \left(x[n] + \alpha y[n-N]\right) - \hat{\alpha} y[n-N] \\ &= x[n] + (\alpha - \hat{\alpha}) y[n-N] \end{aligned}$$

When $\hat{\alpha} - \alpha = 0.05$, the peak of $y[n-N]$ is only 5% of the peak value of the desired signal. This can be verified from the MATLAB plot shown in Figure L5. The amplitude of the sample 2000 is about 5% of the peak amplitude. Although the echo is not cancelled completely, we do not hear the residual echo when we play the sound. This leads us to the conclusion that we can achieve good echo cancellation even with imperfect knowledge of the echo gain.

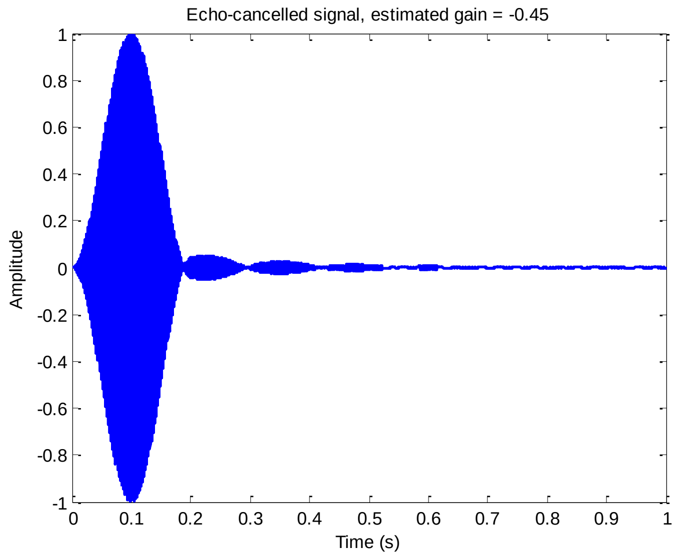


Figure L5: Echo cancellation with imperfect gain estimate, $\hat{\alpha} - \alpha = 0.05$.

We now consider the effect of an imperfect delay estimate. Plots are shown below in Figure L6. As we increase the delay error from 1 to 2 to 10 samples, we see and hear an increasingly strong residual echo. The residual echo is audible even for an error of 1 sample, which corresponds to $1/1024 \approx 0.1\%$ of the delay N. We thus conclude that for good echo cancellation, accurate delay estimation can be more important than accurate echo gain estimation.

When we increase the error to 20 samples, however, we see and hear only a weak residual echo. The reason for this is the periodic nature of the original pulse. The sinusoid has a period of 1/400 = 2.5 ms. A delay error $\hat{N} - N$ = 20 samples corresponds to a delay of 20/8192 = 2.44 ms. The difference between 2.5 ms and 2.44 ms corresponds to $(2.5\text{–}2.44)\times10^{-3} \times 8192 \approx 0.5$ sample, so the residual error is small.

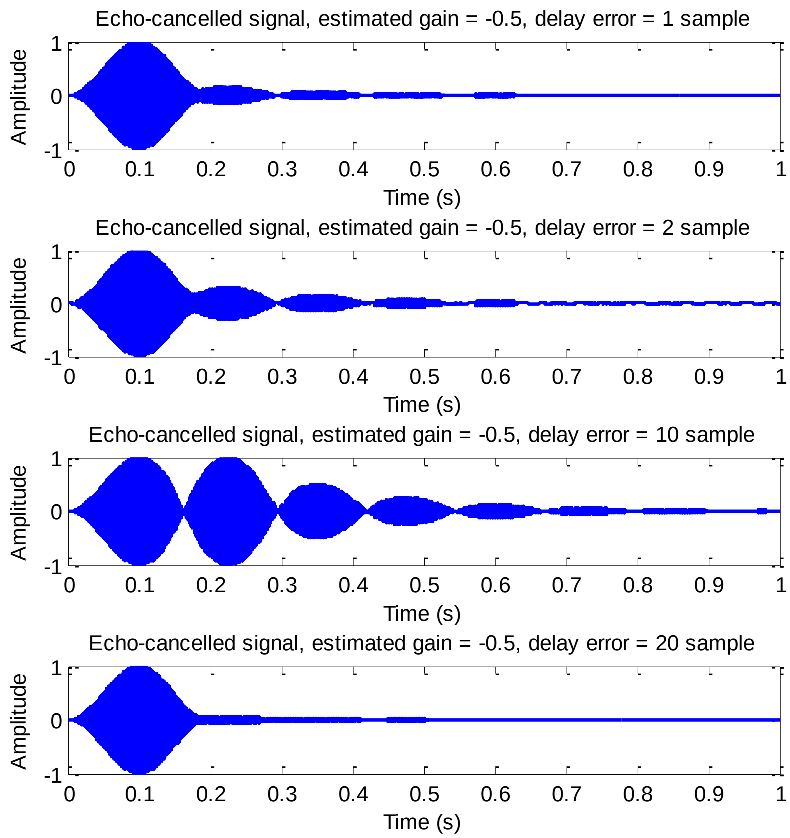


Figure L6: Echo cancellation with imperfect delay estimate, $\hat{N} - N$ = 1, 2, 10 and 20 samples.

```
%% Task 4 - Studying sensitivity to errors

% Gain  error
alpha_hat = -0.45;  % estimated echo gain
z(1:Ndel) = y(1:Ndel);
for i = Ndel + 1:Nzp
    z(i) = y(i) - alpha_hat*y(i-Ndel);
end
figure(6)
plot(nzp/fs,z); l=get(gca,'children'); set(l,'linewidth',lw);
xlabel('Time (s)'); ylabel('Amplitude');
title(['Echo-cancelled signal, estimated gain = '
num2str(alpha_hat,3)]);
pause; sound(z);

% Delay (and possibly gain) error
alpha_hat = alpha;              % estimated echo gain
Ndel_err = [1 2 10 20];         % delay errors (samples)
Ndel_hat = Ndel + Ndel_err;     % estimated delays (samples)
```

```
for j = 1:4
    z(j,1:Ndel_hat(j)) = y(1:Ndel_hat(j));
    for i = Ndel_hat(j)+1:Nzp
        z(j,i) = y(i) - alpha_hat*y(i-Ndel_hat(j));
    end
end
figure(7)
clf
subplot(4,1,1); plot(nzp/fs,z(1,:)); l=get(gca,'children');
set(l,'linewidth',lw);
xlabel('Time (s)'); ylabel('Amplitude');
title(['Echo-cancelled signal, estimated gain = ' ...
    num2str(alpha_hat,3) ', delay error = ' num2str(Ndel_err(1),3) '
sample']);
pause; sound(z(1,:));
subplot(4,1,2); plot(nzp/fs,z(2,:)); l=get(gca,'children');
set(l,'linewidth',lw);
xlabel('Time (s)'); ylabel('Amplitude');
title(['Echo-cancelled signal, estimated gain = ' ...
    num2str(alpha_hat,3) ', delay error = ' num2str(Ndel_err(2),3) '
sample']);
pause; sound(z(2,:));
subplot(4,1,3); plot(nzp/fs,z(3,:)); l=get(gca,'children');
set(l,'linewidth',lw);
xlabel('Time (s)'); ylabel('Amplitude');
title(['Echo-cancelled signal, estimated gain = ' ...
    num2str(alpha_hat,3) ', delay error = ' num2str(Ndel_err(3),3) '
sample']);
pause; sound(z(3,:));
subplot(4,1,4); plot(nzp/fs,z(4,:)); l=get(gca,'children');
set(l,'linewidth',lw);
xlabel('Time (s)'); ylabel('Amplitude');
title(['Echo-cancelled signal, estimated gain = ' ...
    num2str(alpha_hat,3) ', delay error = ' num2str(Ndel_err(4),3) '
sample']);
pause; sound(z(4,:));
```