Stanford University
EE 102A: Signal Processing and Linear Systems I
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Chapter 2: Linear Time-Invariant Systems

Linear Time-Invariant Systems

- Many important CT or DT systems are linear or may be modeled as approximately linear for small changes in the input that induce small changes in the output.
- Many important CT or DT systems are time-invariant or may be modeled as approximately time-invariant over relevant time scales.

Tools for Analyzing Linear Time-Invariant Systems

- *Impulse response*. (*Applicable to all LTI systems*.)
- *Frequency response*. (*Applicable to stable LTI systems and some unstable ones*.)
- *Linear, constant-coefficient differential equations* (in CT) or *difference equations* (in DT) describe many LTI systems. (*Applicable to finite-order, causal LTI systems.*)
- *Systematic realization*: many LTI systems can be realized using simple operations, including multiplication, addition and integration (in CT) or time-shifting (in DT). (*Applicable to finite-order causal LTI systems.*)

Major Topics in This Chapter (studied for both CT and DT unless noted otherwise)

- Impulse response and convolution for LTI systems
  - Determining the impulse response for a given system
  - Evaluating the convolution sum (in DT) or the convolution integral (in CT)
- Properties of convolution and of LTI systems
  - Distributive, associative, commutative
- Properties of an impulse response corresponding to properties of the LTI system it describes
  - Real, memoryless, causal, stable, invertible
- Systems described by finite-order, linear constant-coefficient differential equations (in CT) or difference equations (in DT)

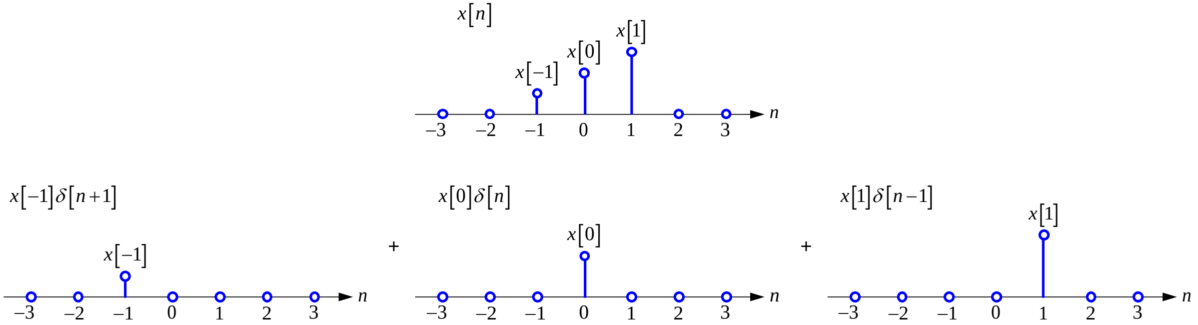
Impulse Response and Convolution Sum for Discrete-Time Linear Time-Invariant Systems

- Any DT signal $x[n]$ can be represented as a sum of scaled, shifted impulses:

$$\begin{aligned} x[n] &= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \\ &= \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots \end{aligned} \tag{1}$$

  - Expression (1) is the *sifting property* of the DT impulse function.
  - *Interpretation*: a DT signal is a function of time variable *n*. On the LHS of (1), $x[n]$ is a *signal*. On the RHS of (1), the $\delta[n-k]$ are *signals* and the $x[k]$ are *coefficients* that scale the signals.

- Example of a signal with three nonzero samples.



- Consider a DT LTI system *H*. Let the signal $x[n]$ be input to the system. We would like to compute the output signal $y[n]$, so we let the system act on the input, i.e., $y[n]=H\{x[n]\}$. Representing the input by (1), the output is

$$y[n]=H\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}.$$

The system *H* is linear and acts on signals but not on coefficients scaling the signals, so the output is

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]H\{\delta[n-k]\}. \qquad (2)$$

- As in Chapter 1, we define a *time-shift operator* D^k, which time shifts a signal by *k*:

$$D^k\{z[n]\}=z[n-k].$$

The signals appearing in each term of (2) can be represented as

$$\begin{aligned} H\{\delta[n-k]\} &= H\{D^k\{\delta[n]\}\} \\ &= D^k\{H\{\delta[n]\}\} \end{aligned}, \qquad (3)$$

where we exploit the time invariance of *H* to interchange the operators *H* and D^k.

- Let us define the *impulse response* $h[n]$ of the LTI system *H*, which is the output of the system when the input is a unit impulse $\delta[n]$:

$$h[n] \stackrel{d}{=} H\{\delta[n]\}. \tag{4}$$

Using definition (4) and the time-shift operator, we can rewrite (3) as

$$\begin{aligned} H\{\delta[n-k]\} &= D^k\{h[n]\} \\ &= h[n-k] \end{aligned}.$$

Substituting this into (2), we can express the system output as

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &\stackrel{d}{=} x[n] * h[n] \end{aligned}. \tag{5}$$

Expression (5) is called a *convolution sum*. It defines the mathematical operation of convolution between DT signals $x[n]$ and $h[n]$, denoted by the symbol "*", which yields a DT signal $y[n]$.

- We find another form of the convolution sum by changing the summation variable in (5) to $l = n - k$:

$$\begin{aligned} y[n] &= \sum_{l=-\infty}^{\infty} x[n-l]h[l] \\ &\stackrel{d}{=} h[n] * x[n] \end{aligned} . \tag{5'}$$

  Formula (5') defines the convolution between DT signals $h[n]$ and $x[n]$. Comparing (5) and (5'), we see that $y[n] = x[n] * h[n] = h[n] * x[n]$, i.e., *convolution is commutative*.

- In the convolution defined by (5) or (5'), values of the input signal $x[n]$ are redistributed in time, in a way that depends on the impulse response $h[n]$, to yield values of the output signal $y[n]$.

- Expressions (5) and (5') are *extremely important* in the analysis of DT LTI systems.
  Given a DT LTI system *H*, suppose we know its impulse response $h[n]$.
  Then, given any input signal $x[n]$, we can compute the output signal $y[n]$ using (5) or (5').

- We should choose whichever form, (5) or (5'), is easiest to evaluate in a given problem.

Step Response of a Discrete-Time Linear Time-Invariant Systems

- The *step response* $s[n]$ of a DT LTI system H is the output when the input is a unit step $u[n]$:

$$s[n] \stackrel{d}{=} H\{u[n]\}.$$

- Using (5'), the step response is given by the convolution

$$\begin{aligned} s[n] &= h[n] * u[n] \\ &= \sum_{k=-\infty}^{\infty} h[k]u[n-k] \end{aligned} . \tag{6}$$

The unit step function in (6) only assumes values of 0 or 1:

$$u[n-k] = \begin{cases} 1 & n-k \geq 0 \\ 0 & n-k < 0 \end{cases} = \begin{cases} 1 & k \leq n \\ 0 & k > n \end{cases},$$

so (6) can be rewritten without a unit step function as a summation over more restricted values of k:

$$s[n] = \sum_{k=-\infty}^{n} h[k]. \tag{7}$$

According to (7), the step response $s[n]$ is the *running sum* of the impulse response $h[n]$.

- *Aside*: since $h[n]$ is arbitrary, we note that convolving *any signal* $x[n]$ with a unit step function yields its running summation: $x[n]*u[n]=\sum_{k=-\infty}^{n} x[k]$.
- Let us evaluate the first difference of the step response using (7):

$$\begin{aligned} s[n]-s[n-1] &= \sum_{k=-\infty}^{n} h[k]-\sum_{k=-\infty}^{n-1} h[k] \\ &= h[n] \end{aligned} . \tag{8}$$

Conversely to (7), (8) says the impulse response $h[n]$ is the *first difference* of the step response $s[n]$.

- Since the impulse response $h[n]$ can be obtained from the step response $s[n]$, the step response also completely characterizes the input-output relationship of the system.
- There are reasons we sometimes study the step response instead of the impulse response:
  - The step response gives insight into key system properties, such as *rise time* or *overshoot/undershoot*. We study these properties later.
  - In practice, we may characterize a system by applying one or more input(s) $x[n]$ and measuring the resulting output(s) $y[n]$. If the device generating the $x[n]$ has a limited peak amplitude, it can be easier to measure the step response than the impulse response.

Computing the Impulse Response of a Discrete-Time Linear Time-Invariant System

- Suppose we have a description of the input-output relation of a DT LTI system H in some form and would like to obtain an expression for the impulse response $h[n]$. We only need to do this once. Then, given any input signal $x[n]$, we can predict the output signal $y[n]$.

Methods

- Here is a list of methods we will use. The list is not exhaustive, as other methods exist.

1. *Direct substitution* using $x[n]=\delta[n]$ and $y[n]=h[n]$. This is useful if the system is specified in terms of an explicit time-domain input-output relationship.
2. *Solution of difference equation*. This is useful if the system is specified in terms of a linear, constant-coefficient difference equation.
3. *Inverse Fourier transform*. This is useful if the system is specified in terms of a frequency response $H\left(e^{j\Omega}\right)$.

- We will study methods 1 and 2 now. We will study method 3 after developing the DT Fourier transform in Chapter 5.

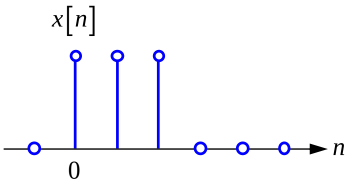
Examples

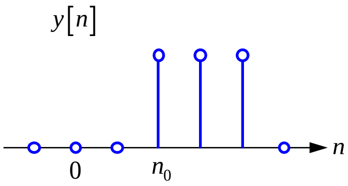
1. *Direct substitution*

- We are given a system D^{n_0} that time-shifts the input signal by n_0:

$$y[n] = D^{n_0}\{x[n]\} = x[n-n_0].$$

An example of input and output signals is shown.

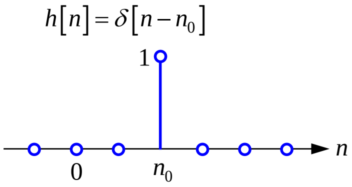




- Choosing $x[n] = \delta[n]$ as the input, we obtain the impulse response as the resulting output:

$$y[n] = h[n] = \delta[n-n_0].$$

This impulse response is shown.



- We will often find it useful to represent time shifting by n_0 as a convolution with $\delta[n-n_0]$:

$$\begin{aligned} D^{n_0}\{x[n]\} &= x[n-n_0] \\ &= x[n]*\delta[n-n_0] \end{aligned}. \tag{9}$$

- *Question*: can you use the convolution sum (5) or (5') with $h[n] = \delta[n-n_0]$ and obtain $y[n] = x[n-n_0]$?

2. *Solution of difference equation*

- We are given a first-order system described by a linear, constant-coefficient linear difference equation

$$y[n] = x[n] + ay[n-1], \tag{10}$$

where *a* is a real constant. Assume we know the system is at initial rest, i.e., $y[n] = 0$ until $x[n]$ first becomes nonzero. Then we know the system is LTI (see Chapter 1).

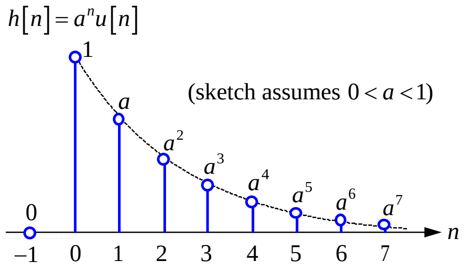
- To find the impulse response, we assume an input $x[n] = \delta[n]$ and solve for the output $y[n] = h[n]$.
  - The input is nonzero at $n = 0$, so the initial rest condition becomes $y[-1] = 0$.
  - We solve (10) by iteratively substituting for $x[n]$ and $y[n]$ in (10).
  - We create a table of inputs and outputs as a function of time *n*, starting at $n = -1$. At each *n*, knowing $x[n]$ and $y[n-1]$, we find $y[n]$. We can extend the procedure to arbitrary *n*.

n	-1	0	1	2	$\cdots$	n
$x[n] = \delta[n]$	0	1	0	0	$\cdots$	0
$y[n] = h[n]$	0	1	a	a^2	$\cdots$	a^n

- We have found the impulse response to be

$$h[n] = \begin{cases} 0 & n < 0 \\ a^n & n \geq 0 \end{cases} = a^n u[n]. \tag{11}$$

The impulse response is shown here, assuming $0 < a < 1$ (lowpass filter).



- Recall that difference equation (10) can describe a variety of DT LTI systems, depending on a.

System	**Value of *a***
Highpass filter	$-1 < a < 0$
Lowpass filter	$0 < a < 1$
Running summation (accumulation)	$a = 1$
Compound interest	$1 < a < \infty$ (typically)

- Z transforms (taught in EE 102B) are an efficient, systematic method for solving linear, constant-coefficient difference equations.

Evaluating Discrete-Time Convolution Sums

- Here we discuss how to evaluate a convolution sum in the form (5) or (5'):

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{5}$$

$$= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \tag{5'}$$

The two forms are equivalent. In solving a problem, we may choose the form we find easiest to evaluate.

Starting and Ending Times

- The following table summarizes how the starting time, ending time and length of $y[n]$ relate to those of $x[n]$ and $h[n]$. Knowing these can simplify evaluation of some convolutions and can help you check your results.

	$x[n]$	$h[n]$	$y[n]$
First nonzero sample	n_{1x}	n_{1h}	$n_{1y} = n_{1x} + n_{1h}$
Last nonzero sample	n_{2x}	n_{2h}	$n_{2y} = n_{2x} + n_{2h}$
Length (spanning first to last nonzero samples)	$L_x = n_{2x} - n_{1x} + 1$	$L_h = n_{2h} - n_{1h} + 1$	$L_y = L_x + L_h - 1$

- As we see from the table:
    - The starting time of $y[n]$ is the *sum of the starting times* of $x[n]$ and $h[n]$.
    - The ending time of $y[n]$ is the *sum of the ending times* of $x[n]$ and $h[n]$.
    - The length of $y[n]$ is *one less than the sum of the lengths* of $x[n]$ and $h[n]$.
- These formulas are applicable when the starting or ending times are infinite. For example, given any nonzero $x[n]$, if $h[n]$ ends at $n = \infty$ ($n_{2h} = \infty$), then $y[n]$ ends at $n = \infty$ ($n_{2y} = \infty$).

Methods for Evaluating DT Convolution Sums

- Here is a list of methods for evaluating convolution sums.
  The list is not exhaustive, as other methods exist.
- Given a pair of signals to be convolved, more than one technique may be applicable.
  Before computing a convolution, think about which technique is likely to be easiest.

1. *Add up scaled, shifted copies* of $x[n]$ (or $h[n]$). Applicable if $h[n]$ (or $x[n]$) has finite length.
2. *Flip and drag.* Always applicable, but not necessarily the easiest method.
3. *Symbolic.* Applicable if both $x[n]$ and $h[n]$ are specified as mathematical formulas.
4. *Numerical.* Applicable if both $x[n]$ and $h[n]$ can be represented as finite-length numerical vectors **`x`** and **`h`**. In MATLAB, we use the command **`y = conv(x,h)`**.

Discrete-Time Convolution Methods and Illustrative Examples

1. *Adding up scaled, shifted copies*

- This method is applicable if at least one of the signals has finite length. We consider the case that $h[n]$ has finite length. Suppose $h[n]$ has first and last nonzero samples n_{1h} and n_{2h} and length $L_h = n_{2h} - n_{1h} + 1$. The convolution sum (5') can be written

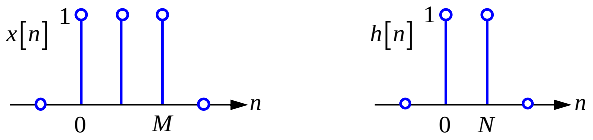
$$y[n] = \sum_{k=n_{1h}}^{n_{2h}} h[k]x[n-k]. \tag{12}$$

The convolution is a sum of L_h shifted copies $x[n-k]$, each scaled by a coefficient $h[k]$.

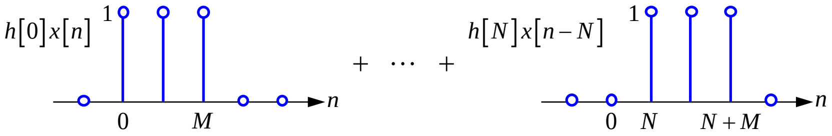
- If $x[n]$ has finite length, you may use the method described here with $h[n]$ and $x[n]$ interchanged. If both $h[n]$ and $x[n]$ have finite length, then treat the shorter signal as we treat $h[n]$ here.

Example: convolution of two rectangular pulses

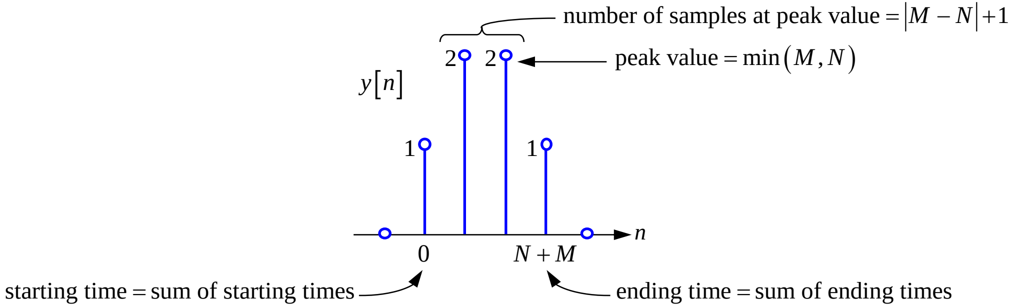
- Signals $x[n]$ and $h[n]$ are unit-amplitude rectangular pulses of length $M+1$ and $N+1$.



- Using (12), we obtain $y[n]$ by adding up the $N+1$ shifted copies $x[n],\ldots,x[n-N]$, each scaled by $h[0]=\cdots=h[N]=1$, as shown.



- The resulting convolution $y[n]$ is shown. When we convolve two rectangular pulses, we obtain a *trapezoid*, assuming the pulses have unequal lengths ($M \neq N$). In the special case of equal lengths ($M = N$), we obtain a *triangle*.



2. *Flip* and *drag*

- This method is applicable to any signals. It follows directly from the general convolution sum (5) or (5'). For concreteness, we focus on form (5'). To compute the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k], \tag{5'}$$

for $-\infty < n < \infty$, we perform the following steps:

1. Plot $h[k]$ vs. *k*.

   Start with *n* negative and large.

2. Plot $x[n-k]$ vs. *k*.

   To do this, reflect or *flip* $x[k]$ to obtain $x[-k]$, then shift or *drag* $x[-k]$ to obtain $x[n-k]$.

   When *n* is negative, shift it to the left.

3. At a given value of *n*, compute the convolution sum $y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$.

   To do this, multiply $h[k]$ by $x[n-k]$ at each *k*, then sum the product $x[n-k]h[k]$ over all *k*.

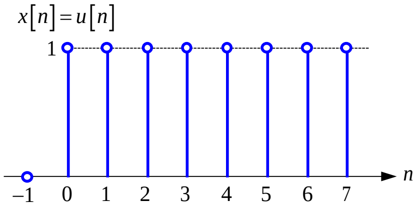
   Increase *n* by 1 and return to step 2.

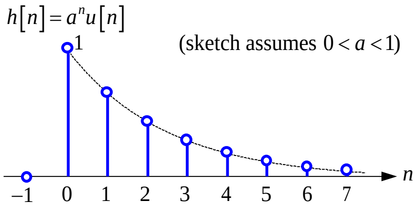
Example: step response of first-order system

- In this example, we compute the step response of a first-order system.

  The input is $x[n]=u[n]$. The impulse response (11) is $h[n]=a^n u[n]$, where a is real, $a \neq 1$.

  The output is $y[n]=s[n]$.





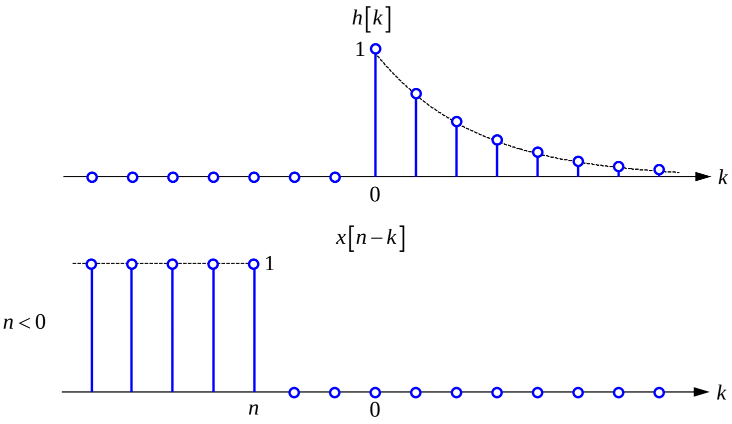
- To compute the convolution sum

$$y[n]=\sum_{k=-\infty}^{\infty} x[n-k]h[k] \tag{5'}$$

  using the flip and drag method, we plot $h[k]$ and $x[n-k]$ vs. k.

- We show $x[n-k]$ for two cases: $n<0$ and $n \geq 0$.

- Case 1: $n < 0$



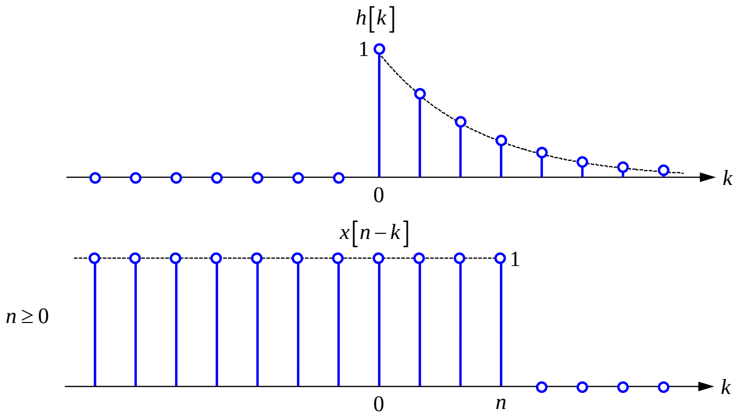
- For $n < 0$, multiplying $h[k]$ by $x[n-k]$, we find

$$x[n-k]h[k] = 0 \quad \forall k$$

so

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] = 0 .$$

- Case 2: $n \geq 0$



- For $n \geq 0$, multiplying $h[k]$ by $x[n-k]$, we find

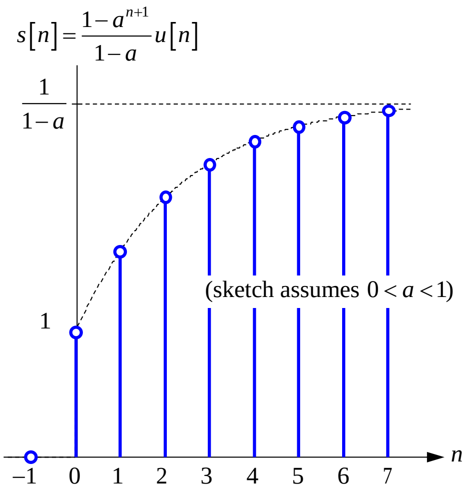
$$x[n-k]h[k] = \begin{cases} 0 & k < 0 \\ a^k & 0 \leq k \leq n \\ 0 & k > n \end{cases} \qquad \text{so} \qquad \begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\ &= \sum_{k=0}^{n} a^k \\ &= \frac{1-a^{n+1}}{1-a} \qquad a \neq 1 \end{aligned} .$$

- In the final step, we have summed a finite-length geometric series (see Appendix).

- Combining the cases $n<0$ and $n \geq 0$, the step response of the first-order system, assuming $a \neq 1$, is

$$y[n] = s[n] = \begin{cases} 0 & n<0 \\ \dfrac{1-a^{n+1}}{1-a} & n \geq 0 \end{cases} = \frac{1-a^{n+1}}{1-a} u[n]. \tag{13}$$

The step response (13) is shown below.



3. *Symbolic*

- Symbolic convolution is applicable when both the input signal $x[n]$ and the impulse response $h[n]$ are specified as mathematical formulas. Instead of trying to explain it in general terms, we will provide two examples.

Example: step response of first-order system

- In this example, we compute the step response of a first-order system with impulse response $h[n]=a^n u[n]$, where a is real, assuming $a \neq 1$. We computed this using the flip and drag method in the preceding example.

- Earlier in this chapter, we showed that for any DT LTI system, the step response $s[n]$ is the running sum of the impulse response $h[n]$:

$$s[n]=\sum_{k=-\infty}^{n} h[k]. \tag{7}$$

- For the impulse response $h[n]=a^n u[n]$, the step response is the running sum

$$s[n]=\sum_{k=-\infty}^{n} a^k u[k].$$

- Now we use the fact that $u[k]=0$, $k<0$.
- When $n<0$, the summation yields $s[n]=0$.
- When $n \geq 0$, the summation becomes

$$s[n]=\sum_{k=0}^{n} a^k \ .$$

This is the finite sum of a geometric series (see Appendix):

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}\ , \quad a \neq 1 .$$

- Combining the cases $n<0$ and $n \geq 0$, the step response can be written as

$$\begin{aligned} s[n] &= \begin{cases} 0 & n<0 \\ \dfrac{1-a^{n+1}}{1-a} & n \geq 0 \end{cases} . \\ &= \frac{1-a^{n+1}}{1-a} u[n] \end{aligned} \qquad (13)$$

This agrees with the result obtained using the flip and drag method in the previous example.

Example: output of first-order system given arbitrary input

- In this example, given an arbitrary input signal $x[n]$, we compute the output from a first-order system with impulse response $h[n]=a^n u[n]$, *a* real. Using general expression (5) for the convolution sum, the output is

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} x[k]a^{n-k}u[n-k] \end{aligned}.$$

- Using the fact that

$$u[n-k] = \begin{cases} 0 & n-k<0 \\ 1 & n-k\geq 0 \end{cases} = \begin{cases} 0 & k>n \\ 1 & k\leq n \end{cases},$$

  the output is given by

$$y[n] = \sum_{k=-\infty}^{n} x[k]a^{n-k} .$$

- This agrees with (10) given in Chapter 1.
- At time *n*, the output $y[n]$ is a weighted sum of inputs $x[k]$ at all past and present times $k \leq n$, with weighting factor a^{n-k}.

Impulse Response and Convolution Integral for Continuous-Time Linear Time-Invariant Systems

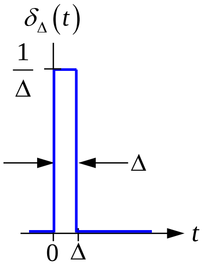
- Any sufficiently smooth CT signal $x(t)$ can be represented as a weighted sum of shifted impulses:

$$x(t) = \int_{-\infty}^{\infty} x(t')\delta(t-t')dt' . \tag{14}$$

Expression (14) is called the *sifting property* of the CT impulse function.

Proof

- As in Chapter 1, we approximate the CT impulse function $\delta(t)$ as $\delta_\Delta(t)$, a rectangular pulse of width Δ and height $1/\Delta$, as shown.

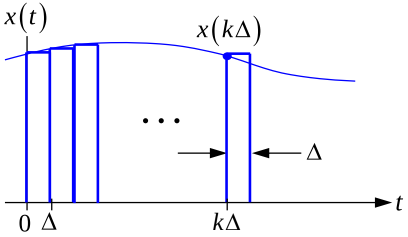


Note that $\Delta \cdot \delta_\Delta(t)$ is a rectangular pulse of width Δ and unit height.

- Given a smooth signal $x(t)$, we approximate it as a weighted sum of shifted rectangular pulses:

$$x(t) \approx \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta . \tag{15}$$

An example is shown below.



- Now we consider the limit $\Delta \to 0$. In this limit:

$$\delta_\Delta(t) \to \delta(t)$$

$$k\Delta \to t'$$

$$\Delta \to dt'$$

and the sum (15) approaches a Riemann sum representation of the integral

$$x(t) = \int_{-\infty}^{\infty} x(t')\delta(t-t')dt' . \tag{14}$$

QED

- Now we consider a CT LTI system *H*. Let the signal $x(t)$ be input to the system.

  To compute the output signal $y(t)$, we let the system act on the input: $y(t) = H\{x(t)\}$.

  Representing the input signal using the sifting property (14), the output is

$$y(t) = H\left\{\int_{-\infty}^{\infty} x(t')\delta(t-t')dt'\right\}.$$

- Since the system *H* is linear and acts on signals (functions of time *t*) but not on coefficients scaling the signals, the output is

$$y(t) = \int_{-\infty}^{\infty} x(t')H\{\delta(t-t')\}dt'. \tag{16}$$

- As in Chapter 1, we define a *time-shift operator* $D^{t'}$, which time shifts a signal by t':

$$D^{t'}\{z(t)\} = z(t-t').$$

  This allows us to represent the signal appearing in the integrand of (16) as

$$\begin{aligned} H\{\delta(t-t')\} &= H\left\{D^{t'}\{\delta(t)\}\right\} \\ &= D^{t'}\left\{H\{\delta(t)\}\right\} \end{aligned}. \tag{17}$$

  We have exploited the time invariance of *H* to interchange the operators *H* and $D^{t'}$.

- We define the *impulse response* $h(t)$ of the LTI system *H*, which is the output of the system when the input is a unit impulse $\delta(t)$:

$$h(t) \stackrel{d}{=} H\left\{\delta(t)\right\}. \tag{18}$$

Using definition (18) and the time-shift operator, we can rewrite (17) as

$$\begin{aligned} H\left\{\delta(t-t')\right\} &= D^{t'}\left\{H\left\{\delta(t)\right\}\right\} \\ &= h(t-t') \end{aligned}.$$

- Substituting this into (16), we can express the system output as

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(t')h(t-t')dt' \\ &\stackrel{d}{=} x(t) * h(t) \end{aligned}. \tag{19}$$

Expression (19) is called a *convolution integral*. It defines the mathematical operation of convolution ("*") between CT signals $x(t)$ and $h(t)$,which yields a signal $y(t)$.

- We obtain another form of the convolution integral by changing the integration variable in (19) to $\mu = t - t'$:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(t-\mu) h(\mu) d\mu \\ &\overset{d}{=} h(t) * x(t) \end{aligned} \quad . \tag{19'}$$

  We see that $y(t) = x(t) * h(t) = h(t) * x(t)$, i.e., *convolution is commutative*.

- In convolution, values of the input signal $x(t)$ are redistributed in time, in a way that depends on the impulse response $h(t)$, to yield the output signal $y(t)$.

- The convolution integrals are *very useful* in analyzing CT LTI systems. Given a CT LTI system *H* and its impulse response $h(t)$, we can compute the output signal $y(t)$ for any input signal $x(t)$.

- In solving a given problem, we should choose whichever form, (19) or (19'), is easiest to evaluate.

Step Response of a Continuous-Time Linear Time-Invariant Systems

- The *step response* $s(t)$ of a CT LTI system *H* is the output when the input is a unit step $u(t)$:

$$s(t) \stackrel{d}{=} H\left\{u(t)\right\}.$$

Using (19'), the step response can be computed as a convolution integral

$$\begin{aligned} s(t) &= h(t) * u(t) \\ &= \int_{-\infty}^{\infty} h(t')u(t-t')dt' \end{aligned}. \tag{20}$$

The unit step function in (20) only assumes values of 0 or 1:

$$u(t-t') = \begin{cases} 1 & t-t' \geq 0 \\ 0 & t-t' < 0 \end{cases} = \begin{cases} 1 & t' \leq t \\ 0 & t' > t \end{cases}.$$

It restricts the region of integration but does not otherwise change the integrand. The convolution (20) can be rewritten without a unit step function as

$$s(t) = \int_{-\infty}^{t} h(t')dt'. \tag{21}$$

The step response $s(t)$ is the *running integral* of the impulse response $h(t)$.

- *Aside*: since $h(t)$ is arbitrary, we note that convolving *any signal* $x(t)$ with a unit step function yields its running integral: $x(t) * u(t) = \int_{-\infty}^{t} x(t')dt'$.
- Conversely, using (21), we can evaluate the derivative of the step response, which is

$$\frac{ds(t)}{dt} = h(t). \tag{22}$$

The impulse response $h(t)$ is the *derivative* of the step response $s(t)$.

- Since the impulse response $h(t)$ can be obtained from the step response $s(t)$, the step response also fully characterizes the input-output relationship of the system.
- There are reasons we sometimes study the step response instead of the impulse response for CT systems (as for DT systems):
    - The step response gives insight into important system properties, such as *rise time* or *overshoot/undershoot.*
    - In practice, we may characterize a system by applying one or more input(s) $x(t)$ and measuring the resulting output(s) $y(t)$. It can be easier to measure the step response than the impulse response, particularly if the device generating the $x(t)$ has a limited peak amplitude.

Computing the Impulse Response of a Continuous-Time Linear Time-Invariant System

- Suppose we have a description of the input-output relation of a CT LTI system *H* and want to obtain an expression for the impulse response $h(t)$. We need only compute $h(t)$ once. Then, given any input signal $x(t)$, we can compute the output signal $y(t)$.

Methods

- Here is a list of several methods we will employ. Other methods exist, so the list is not exhaustive.

  1. *Direct substitution* using $x(t)=\delta(t)$ and $y(t)=h(t)$. This is applicable if we have an explicit input-output relation in the time domain.

  2. *Solution of differential equation*. This is applicable if the system is specified in terms of a linear, constant-coefficient differential equation.

  3. *Inverse Fourier transform*. This is applicable if the system is specified in terms of a frequency response $H(j\omega)$.

- We will study methods 1 and 2 now. We will study method 3 after developing the CT Fourier transform in Chapter 4.

Examples

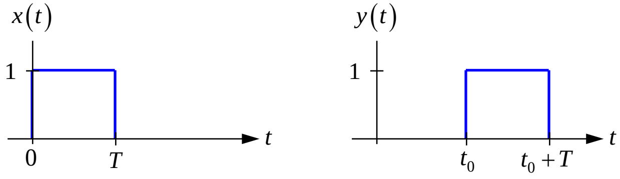
- These examples are closely analogous to those we studied for DT LTI systems.

1. *Direct substitution*

- Consider a system D^{t_0} that time-shifts the input signal by t_0:

$$\begin{aligned} y(t) &= D^{t_0}\{x(t)\} \\ &= x(t-t_0) \end{aligned}.$$

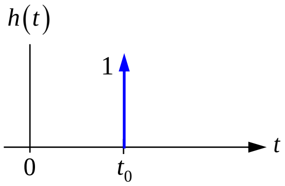
Exemplary input and output signals are shown below.



- Choosing the input to be $x(t)=\delta(t)$, the resulting output is the impulse response:

$$\begin{aligned} y(t) &= h(t) \\ &= \delta(t-t_0) \end{aligned}.$$

This impulse response is shown.



- We often represent time shifting by t_0 as a convolution with $\delta(t-t_0)$:

$$\begin{aligned} D^{t_0}\{x(t)\} &= x(t-t_0) \\ &= x(t)*\delta(t-t_0) \end{aligned}. \tag{23}$$

- *Question*: can you use the convolution integral (19) or (19') with $h(t) = \delta(t - t_0)$ and obtain $y(t) = x(t - t_0)$?

2. *Solution of differential equation*.

- We are given a *first-order lowpass filter* described by constant-coefficient linear differential equation

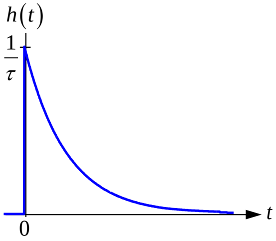
$$\tau \frac{dy}{dt} + y(t) = x(t), \tag{24}$$

where τ is real and $\tau > 0$. Assume we know the system is at initial rest, i.e., $y(t) = 0$ until $x(t)$ first becomes nonzero. We know the system is LTI (see Chapter 1).

- To compute the impulse response, assume an input $x(t) = \delta(t)$ and solve for the output $y(t) = h(t)$.
  - This input is first nonzero at $t = 0$, so the initial rest condition becomes $y(t) = 0$, $t < 0$.
  - We can solve the differential equation using the Laplace transform (EE 102B). Here, we guess the solution and verify it satisfies the differential equation and initial condition. We guess the impulse response is

$$h(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t), \tag{25}$$

as shown below.



- This satisfies the initial rest condition $h(t)=0$, $t<0$.

- Now we verify (25) satisfies differential equation (24) with $x(t)=\delta(t)$ and $y(t)=h(t)$, i.e.,

$$\tau\frac{dh}{dt}+h(t)=\delta(t). \tag{26}$$

Differentiating the product of two functions in (25) and using $\frac{du(t)}{dt}=\delta(t)$, we find

$$\begin{aligned}\frac{dh}{dt}&=-\frac{1}{\tau^2}e^{-\frac{t}{\tau}}u(t)+\frac{1}{\tau}e^{-\frac{t}{\tau}}\delta(t)\\&=-\frac{1}{\tau^2}e^{-\frac{t}{\tau}}u(t)+\frac{1}{\tau}\delta(t)\end{aligned}.$$

Substituting this in (26), we verify that (26) is satisfied.

Evaluating Continuous-Time Convolution Integrals

- In this section, we discuss how to evaluate a convolution integral:

$$y(t)=\int_{-\infty}^{\infty} x(t')h(t-t')dt' \tag{19}$$

$$=\int_{-\infty}^{\infty} x(t-t')h(t')dt' \tag{19'}$$

The two forms are equivalent. In solving a problem, choose whichever form you find easiest to evaluate.

Starting and Ending Times

- The following table shows how the starting time (first nonzero value), ending time (last nonzero value), and duration (interval between starting and ending times) of $y(t)$ are related to those of $x(t)$ and $h(t)$. Knowing the starting and ending times can simplify the evaluation of some convolutions and can help you check your results.

	$x(t)$	$h(t)$	$y(t)$
First nonzero value	t_{1x}	t_{1h}	$t_{1y} = t_{1x} + t_{1h}$
Last nonzero value	t_{2x}	t_{2h}	$t_{2y} = t_{2x} + t_{2h}$
Duration (spanning first to last nonzero values)	$T_x = t_{2x} - t_{1x}$	$T_h = t_{2h} - t_{1h}$	$T_y = T_x + T_h$

- As we see from the table:
  - The starting time of $y(t)$ is the *sum of the starting times* of $x(t)$ and $h(t)$.
  - The ending time of $y(t)$ is the *sum of the ending times* of $x(t)$ and $h(t)$.
  - The duration of $y(t)$ is the *sum of the durations* of $x(t)$ and $h(t)$.
- The formulas are applicable when the starting or ending times are infinite. As an example, given any nonzero $x(t)$, if $h(t)$ ends at $t=\infty$ ($t_{2h}=\infty$), then $y(t)$ ends at $t=\infty$ ($t_{2y}=\infty$).

Methods for Evaluating CT Convolution Integrals

- Here are several methods for evaluating convolution integrals. The list is not exhaustive, as other methods exist.
- Given two signals to be convolved, more than one technique may be applicable.

  Think about which technique is likely to be easiest before computing a convolution.

1. *Add up scaled, shifted copies* of $x(t)$ (or $h(t)$).

   Applicable if $h(t)$ (or $x(t)$) is comprised only of one or more scaled, shifted impulses.

2. *Flip and drag.* Always applicable, but not necessarily the easiest method to use.

3. *Symbolic.* Applicable if both $x(t)$ and $h(t)$ are given as mathematical formulas.

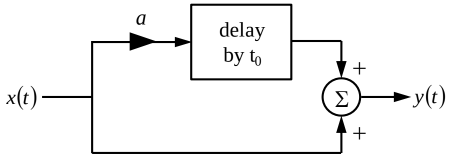
4. *Numerical.* CT convolution can be approximated by numerical computation, like any integration of a function of a continuous variable. We study this in homework MATLAB exercises.

Continuous-Time Convolution Methods and Illustrative Examples

1. *Adding up scaled, shifted copies*. This method is applicable when either $x(t)$ or $h(t)$ comprises only a sum of scaled, shifted impulses.

Example: delay-and-add system

- A *delay-and-add* system describes propagation of signals along two paths with different delays.
- Examples arise in acoustics (some types of echoes), electromagnetic waves (multipath propagation) and circuits (where the two paths may be intended or unintended).
- *Question*: can you think of some real-life examples?
- A block diagram representation of a delay-and-add system is shown.



The impulse response is

$$h(t) = \delta(t) + a\delta(t - t_0).$$

where a and t_0 are real and $t_0 \geq 0$.

- Given any input signal $x(t)$, the output is the sum of $x(t)$ and a scaled, delayed version of it:

$$y(t) = x(t) + ax(t - t_0).$$

We can verify this using the linearity of convolution and expression (23) from above:

$$\begin{aligned} D^{t_0}\{x(t)\} &= x(t - t_0) \\ &= x(t) * \delta(t - t_0) \end{aligned}. \qquad (23)$$

2. *Flip and drag*

- This method is applicable to any signals. It follows from the general convolution integral (19) or (19'). For concreteness, we focus on form (19'). To compute the convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(t-t')h(t')dt' \tag{19'}$$

for $-\infty < t < \infty$, we perform the following steps:

1. Plot $h(t')$ vs. t'.

   Start with t negative and large.

2. Plot $x(t-t')$ vs. t'.

   To do this, reflect or *flip* $x(t')$ to obtain $x(-t')$, then shift or *drag* $x(-t')$ to obtain $x(t-t')$.

   When t is negative, shift it to the left.

3. At a given value of t, compute the value of the convolution integral $y(t) = \int_{-\infty}^{\infty} x(t-t')h(t')dt'$.

   To do this, multiply $h(t')$ by $x(t-t')$ at each t', then integrate the product $x(t-t')h(t')$ over all t'.

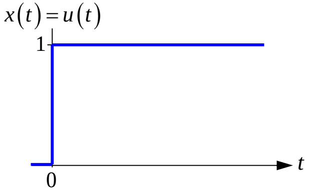
   Increase t and return to step 2.

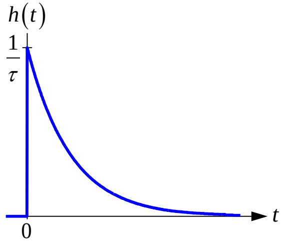
Example: step response of first-order lowpass filter

- In this example, we compute the step response of a first-order lowpass filter.

  The input is $x(t) = u(t)$. The impulse response is $h(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t)$, where τ is real and $\tau > 0$.

  The output is $y(t) = s(t)$.





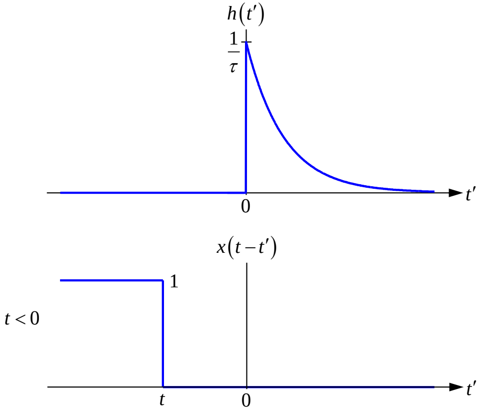
- To compute the convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(t - t') h(t') dt' \qquad (19')$$

  using the flip and drag method, we plot $h(t')$ and $x(t - t')$ vs. t'.

- We show $x(t - t')$ for two cases: $t < 0$ and $t \geq 0$.

- Case 1: $t < 0$



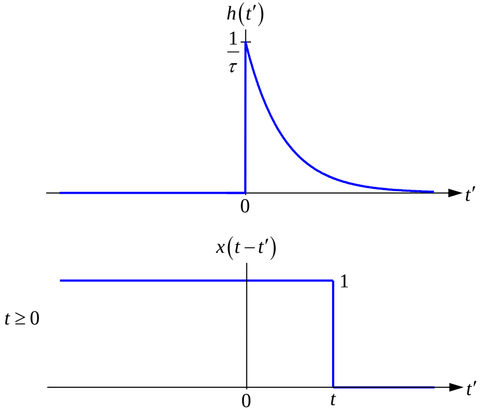
- For $t < 0$, multiplying $h(t')$ by $x(t-t')$, we find

$$x(t-t')h(t') = 0 \quad \forall t'$$

so

$$y(t) = \int_{-\infty}^{\infty} x(t-t')h(t')dt' = 0 \,.$$

- Case 2: $t \geq 0$



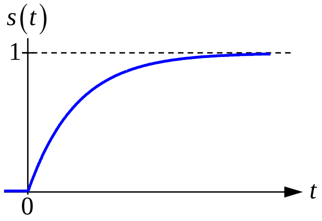
- For $t \geq 0$, multiplying $h(t')$ by $x(t-t')$, we find

$$x(t-t')h(t') = \begin{cases} 0 & t' < 0 \\ \frac{1}{\tau} e^{-\frac{t'}{\tau}} & 0 \leq t' \leq t \\ 0 & t' > t \end{cases} \quad \text{so} \quad \begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(t-t')h(t')dt' \\ &= \frac{1}{\tau}\int_0^t e^{-\frac{t'}{\tau}} dt' \\ &= 1 - e^{-\frac{t}{\tau}} \end{aligned} .$$

- Combining the cases $t<0$ and $t \geq 0$, the step response of the first-order lowpass filter is

$$y(t) = s(t) = \begin{cases} 0 & t<0 \\ 1-e^{-\frac{t}{\tau}} & t \geq 0 \end{cases} = \left(1-e^{-\frac{t}{\tau}}\right)u(t). \tag{27}$$

The step response (27) is shown below.

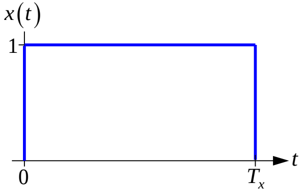


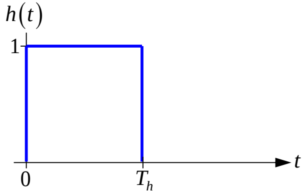
3. *Symbolic*

- This method is applicable when both the input $x(t)$ and the impulse response $h(t)$ are described by mathematical formulas.

Example: convolution of two rectangular pulses

- The signals $x(t)$ and $h(t)$ are unit-amplitude rectangular pulses of duration T_x and T_h, respectively, as shown.





- We express each signal as a sum of scaled, shifted step functions:

$$x(t) = u(t) - u(t - T_x) \quad \text{and} \quad h(t) = u(t) - u(t - T_h),$$

and represent the convolution as

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \left[u(t) - u(t - T_x)\right] * \left[u(t) - u(t - T_h)\right] \end{aligned}. \tag{28}$$

- Recall that:
  - Convolving any signal with a unit step yields its running integral (Chapter 1).
  - The running integral of the unit step is the unit ramp (Chapter 1).

  Combining these, we find that convolving a unit step with itself yields a unit ramp:

$$u(t)*u(t)=\int_{-\infty}^{t}u(t')dt'=r(t)\,. \tag{29}$$

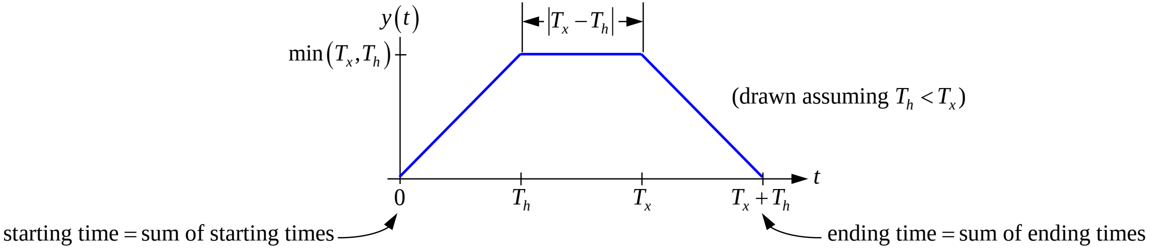
- Using (29), we can rewrite the convolution between two rectangular pulses (28) as

$$y(t)=r(t)-r(t-T_x)-r(t-T_h)+r\left(t-(T_x+T_h)\right). \tag{30}$$

- Important properties of CT convolution are used here. For example, in the third term:

$$\begin{aligned}
u(t)*u(t-T_h) &= u(t)*\left[u(t)*\delta(t-T_h)\right] && \textit{time shifting as convolution with shifted impulse}\\
&= \left[u(t)*u(t)\right]*\delta(t-T_h) && \textit{associative property of convolution}\\
&= r(t)*\delta(t-T_h) && \textit{convolving step with step yields ramp}\\
&= r(t-T_h) && \textit{time shifting as convolution with shifted impulse}
\end{aligned}$$

- The convolution $y(t)$ is shown below. This example demonstrates that when we convolve two rectangular pulses, we obtain a *trapezoid*, assuming the pulses have unequal durations ($T_x \neq T_h$). In the special case of equal-duration pulses ($T_x = T_h$), we obtain a *triangle*.



Example: step response of first-order lowpass filter

- We compute the step response of a first-order lowpass filter by convolving an input $x(t) = u(t)$ with the impulse response $h(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t)$, where τ is real and $\tau > 0$.

- Recall that the step response is the running integral of the impulse response:

$$s(t) = \int_{-\infty}^{t} h(t') dt' . \tag{21}$$

- For the impulse response $h(t)=\frac{1}{\tau}e^{-\frac{t}{\tau}}u(t)$, the step response is the running integral

$$s(t)=\frac{1}{\tau}\int_{-\infty}^{t}e^{-\frac{t'}{\tau}}u(t')\,dt' .$$

- Now we use the fact that $u(t')=0,\ t'<0$.

- When $t<0$, the integral yields $s(t)=0$.

- When $t\geq 0$, the integral becomes

$$s(t)=\frac{1}{\tau}\int_{0}^{t}e^{-\frac{t'}{\tau}}dt' ,$$

which can be evaluated to yield $s(t)\ =1-e^{-\frac{t}{\tau}}$.

- Combining the two cases, we obtain

$$s(t)\ =\ \begin{cases} 0 & t<0 \\ 1-e^{-\frac{t}{\tau}} & t\geq 0 \end{cases}\ =\ \left(1-e^{-\frac{t}{\tau}}\right)u(t), \tag{27}$$

which agrees with the result obtained using the flip and drag method.

Example: output of first-order lowpass filter given arbitrary input

- Given an arbitrary input signal $x(t)$, we compute the output from a first-order lowpass filter with impulse response $h(t)=\frac{1}{\tau}e^{-\frac{t}{\tau}}u(t)$, where τ is real and $\tau > 0$.

- Using the general convolution integral (19) with this particular impulse response, the output is

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(t')h(t-t')dt' \\
&= \frac{1}{\tau}\int_{-\infty}^{\infty} x(t')e^{-\frac{t-t'}{\tau}}u(t-t')dt'
\end{aligned}
.
$$

  Using the fact that

$$
u(t-t') \ = \ \begin{cases} 0 & t-t'<0 \\ 1 & t-t' \geq 0 \end{cases} \ = \ \begin{cases} 0 & t'>t \\ 1 & t' \leq t \end{cases},
$$

  The output is given by

$$
y(t)=\frac{1}{\tau}\int_{-\infty}^{t} x(t')e^{-\frac{t-t'}{\tau}}dt' .
$$

- This agrees with the result obtained in Chapter 1.

Properties of Convolution and of Linear Time-Invariant Systems

- DT and CT convolution satisfy *distributive*, *associative* and *commutative* properties. These are proven by expressing convolution as a sum (5) or (5') or as an integral (19) or (19').

- All DT and CT LTI systems satisfy the distributive, associative and commutative properties. This follows from the fact that their input-output relationship can be described by a convolution.

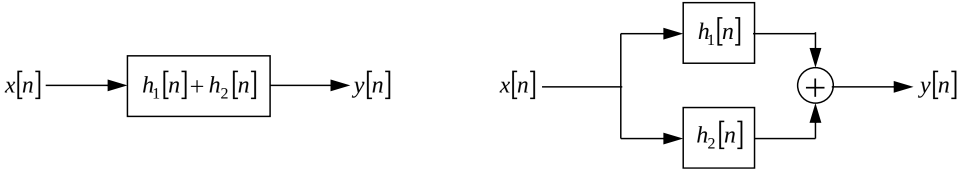
- These properties are stated here only for DT. Their extension to CT is straightforward.

Distributive Property

- Convolution is a *distributive* operation:

$$x[n]*\left(h_1[n]+h_2[n]\right)=x[n]*h_1[n]+x[n]*h_2[n].$$

This corresponds to the distributive property of LTI systems shown below.



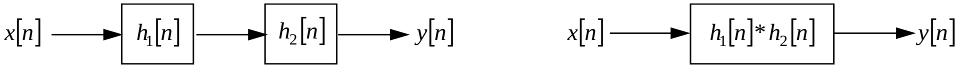
- We will use the notation implied by these block diagrams. For example, passing a signal $x[n]$ into a block labeled $h_1[n]$ implies the output is $x[n]*h_1[n]$.

Associative Property

- Convolution is an *associative* operation:

$$\left(x[n]*h_1[n]\right)*h_2[n]=x[n]*\left(h_1[n]*h_2[n]\right).$$

This corresponds to the associative property of LTI systems shown below.



Commutative Property

- Convolution is a *commutative* operation:

$$h_1[n]*h_2[n]=h_2[n]*h_1[n].$$

This corresponds to the commutative property of LTI systems shown below.

- *Question*: how would you prove these properties?

Properties of Impulse Responses and Corresponding LTI Systems

- We study how various properties of an LTI system – real, memoryless, causal, stable or invertible – are reflected in the system's impulse response.

Real Systems

- If a DT or CT LTI system maps all real input signals to real output signals, then its impulse response, $h[n]$ or $h(t)$, must be a real-valued function.

Memoryless Systems

- Recall that a system is memoryless if, at any given time, the value of the output depends only on the present value of the input, and not on past or future values of the input.
- The impulse response of a memoryless DT LTI system must be of the form

$$h[n] = C \cdot \delta[n].$$

Likewise, the impulse response of a memoryless CT LTI system must be of the form

$$h(t) = C \cdot \delta(t).$$

In both cases, C is an arbitrary constant.

Causal Systems

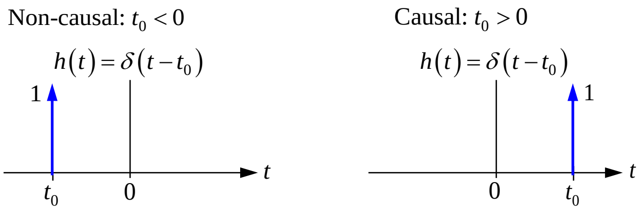
- Recall that a system is causal if, at any given time, the value of the output depends only on the present and past values of the input, and not on future values of the input.
- The impulse response of a causal DT LTI system must satisfy

$$h[n] = 0 \,,\ n < 0 \,.$$

Similarly, the impulse response of a causal CT LTI system must satisfy

$$h(t) = 0 \,,\ t < 0 \,.$$

- As an example, we compare the impulse responses of non-causal and causal CT time-shifting systems.



- Here we justify the condition for the CT case. Given an LTI system with impulse response $h(t)$ and input $x(t)$, the output is given by

$$y(t) = \int_{-\infty}^{\infty} x(t-t')h(t')dt' \ . \qquad (19')$$

  - In order to satisfy causality, $y(t)$ should depend only on $x(t-t')$ at past and present times $t-t'$, $0 \le t' < \infty$.
  - In order for the impulse response to enforce causality, it must exclude the influence of inputs $x(t-t')$ at future times $t-t'$, $-\infty < t' < 0$.
  - Therefore we require $h(t') = 0$, $t' < 0$.
  - The explanation for the DT case is entirely analogous.

Stable Systems

- Recall that a system is bounded-input, bounded-output stable (BIBO stable) if and only if every bounded input induces a bounded output.

- A DT LTI system is stable if and only if its impulse response is *absolutely summable*:

$$\sum_{n=-\infty}^{\infty} \left|h[n]\right| < \infty . \tag{31}$$

A CT LTI system is stable if and only if its impulse response is *absolutely integrable*:

$$\int_{-\infty}^{\infty} \left|h(t)\right| dt < \infty . \tag{32}$$

- Here we prove that for a CT LTI system *H* with input $x(t)$, impulse response $h(t)$ and output $y(t)$, and for positive real constants M_x and M_y, the BIBO stability condition

$$\left|x(t)\right| \le M_x < \infty \ \forall t \ \Rightarrow \ \left|H\left\{x(t)\right\}\right| = \left|y(t)\right| \le M_y < \infty \ \forall t$$

is satisfied if and only if $\int_{-\infty}^{\infty} \left|h(t)\right| dt < \infty$.

Proof ***(Skip)***

- Show that $\int_{-\infty}^{\infty} |h(t)| dt < \infty$ is sufficient. The magnitude of the output can be expressed as

$$\begin{aligned} |y(t)| &= |x(t) * h(t)| \\ &= \left| \int_{-\infty}^{\infty} h(t') x(t-t') dt' \right| \\ &\le \int_{-\infty}^{\infty} |h(t') x(t-t')| dt' \quad \text{(Riemann sum representation of integral and } |a+b| \le |a|+|b|) \\ &= \int_{-\infty}^{\infty} |h(t')| |x(t-t')| dt' \quad (|ab| = |a| \cdot |b|) \end{aligned}$$

Assume that $|x(t)| \le M_x < \infty \ \forall t$. Then

$$|y(t)| \le M_x \int_{-\infty}^{\infty} |h(t')| dt'.$$

We conclude that $|y(t)| \le M_y < \infty \ \forall t$ if $\int_{-\infty}^{\infty} |h(t)| dt < \infty$.

- Show that $\int_{-\infty}^{\infty} |h(t)| dt < \infty$ is necessary. Given $h(t)$, we choose an input

$$x(t) = \begin{cases} \dfrac{h^*(-t)}{|h(-t)|} & h(-t) \neq 0 \\ 0 & h(-t) = 0 \end{cases}.$$

The resulting output is

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(t') x(t-t') dt' \\ &= \int_{-\infty}^{\infty} \frac{h(t') h^*(t'-t)}{|h(t'-t)|} dt' \end{aligned}.$$

$$\begin{aligned} y(0) &= \int_{-\infty}^{\infty} \frac{h(t') h^*(t')}{|h(t')|} dt' \\ &= \int_{-\infty}^{\infty} |h(t')| dt' \end{aligned}.$$

Satisfying $|y(t)| \leq M_y < \infty \ \forall t$ requires $\int_{-\infty}^{\infty} |h(t)| dt < \infty$.

QED

Invertible Systems

- Recall that a system is invertible if the input can always be recovered from the output.
- A DT LTI system H with impulse response $h[n]$ is invertible if and only if there exists a stable inverse system H^{-1} with impulse response $h^{-1}[n]$ such that

$$h^{-1}[n]*h[n]=\delta[n].$$

- Similarly, a CT LTI system H with impulse response $h(t)$ is invertible if and only if there exists a stable inverse system H^{-1} with impulse response $h^{-1}(t)$ such that

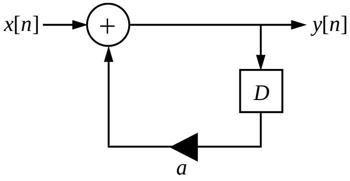
$$h^{-1}(t)*h(t)=\delta(t).$$

Example: Discrete-Time First-Order System and its Inverse

- A DT first-order system H is described by

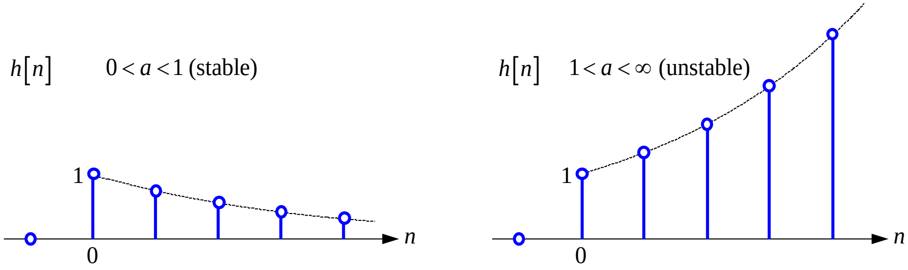
$$y[n] = x[n] + ay[n-1], \tag{10}$$

and can be realized by the block diagram shown. The constant a is assumed real.



- We found earlier it has an impulse response

$$h[n] = a^n u[n]. \tag{11}$$

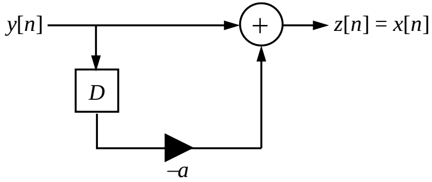


- The system is stable only for $|a|<1$, in which case, the impulse response is absolutely summable, $\sum_{n=-\infty}^{\infty}|h[n]|<\infty$, and thus satisfies the condition for stability.

- The inverse of the first-order system, denoted by H^{-1}, with input $y[n]$ and output $z[n]$, is described by a constant-coefficient linear difference equation

$$z[n]=y[n]-ay[n-1]=x[n], \tag{33}$$

and can be realized by the block diagram shown.



- By direct substitution, with input $y[n]=\delta[n]$ and output $z[n]=h^{-1}[n]$, we find the inverse system has an impulse response

$$h^{-1}[n]=\delta[n]-a\delta[n-1]. \tag{34}$$

- We can verify through two different methods that this is the inverse of the first-order system.

- First, we start with the difference equation for the inverse system:

$$z[n] = y[n] - ay[n-1] = x[n]. \tag{33}$$

Then we use the difference equation for the original system

$$y[n] = x[n] + ay[n-1], \tag{10}$$

and substitute the right-hand side of (10) for $y[n]$ into (33), obtaining

$$z[n] = x[n] + ay[n-1] - ay[n-1] = x[n].$$

This shows that the inverse system allows us to recover $x[n]$ from $y[n]$.

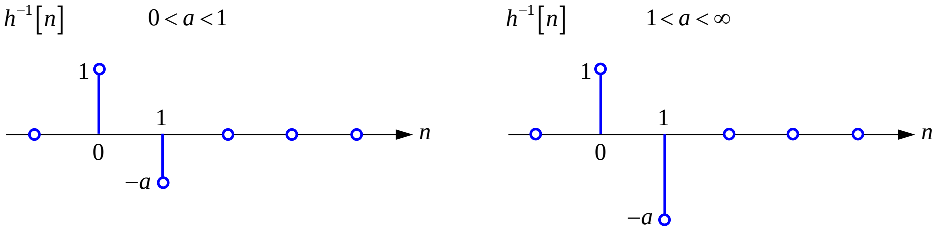
- Second, we can convolve the impulse responses of the first-order system and its inverse:

$$\begin{aligned} h[n] * h^{-1}[n] &= a^n u[n] * \{\delta[n] - a\delta[n-1]\} \\ &= a^n \{u[n] - u[n-1]\} \\ &= a^n \delta[n] \\ &= \delta[n] \end{aligned},$$

verifying that we obtain $\delta[n]$.

- *Question*: which property of the unit impulse did we use in the last line above?

- The inverse system impulse response is shown below for positive a, including $0 < a < 1$ (on the left) and $1 < a < \infty$ (on the right).



- *Question*: we know the first-order system is stable only for $|a| < 1$. For what values of a is the inverse system stable?

Continuous-Time Systems Described by Linear, Constant-Coefficient Differential Equations

- A general linear, constant-coefficient differential equation, describing a CT system with input $x(t)$ and output $y(t)$, is of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \qquad (35)$$

  - The constants a_k, $k = 0,\ldots,N$ and b_k, $k = 0,\ldots,M$ are real in systems that map real inputs to real outputs.
  - The right-hand side is a linear combination of the input $x(t)$ and its first *M* derivatives.
  - The left-hand side is a linear combination of the output $y(t)$ and its first *N* derivatives.

    The parameter *N*, denoting the highest derivative of $y(t)$, is called the *order* of the equation and of the system.
  - Typically, *N* corresponds to the number of independent energy storage elements in the system. In a circuit, *N* typically equals the number of inductors and capacitors. In a mechanical system, *N* typically equals the number of elements storing potential and kinetic energies.
  - By construction, (35) describes a causal system.
  - EE 102B teaches how to solve equations of the form (35) using Laplace transforms.

General Case $N \neq 0$

- In the general case $N \neq 0$, (35) is an *implicit* description for the output $y(t)$ given an input $x(t)$.

  Suppose we specify:

  - N initial conditions $y(t_0),\ldots,y^{(N-1)}(t_0)$, values of the output and its $N-1$ derivatives at time t_0.
  - An input $x(t)$, $t > t_0$.

  Then we can solve for the output $y(t)$, $t > t_0$ *uniquely*.

- If the system is *at initial rest* (the output is zero until the input becomes nonzero)

$$x(t) = 0,\ t < t_0 \ \Rightarrow \ y(t) = \cdots = y^{(N-1)}(t) = 0,\ t < t_0,$$

  then equation (35) describes a *causal*, *LTI system*. We can calculate its impulse response $h(t)$ assuming input, output and initial conditions

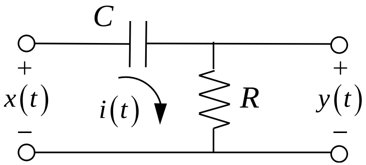
$$x(t) = \delta(t)$$

$$y(t) = h(t)$$

$$y(t) = \cdots = y^{(N-1)}(t) = 0,\ t < 0.$$

Example: First-Order Highpass Filter

- A first-order *highpass filter*, shown below, may be used to remove the d.c. component or low-frequency components from signals.



We can relate the input voltage $x(t)$ and output voltage $y(t)$ to the current $i(t)$:

$$x(t) - y(t) = \frac{1}{C}\int_{-\infty}^{t} i(t')dt'$$

$$y(t) = i(t)R$$

From these, we can obtain a first-order differential equation

$$\frac{dy}{dt} + \frac{1}{\tau} y(t) = \frac{dx}{dt}, \qquad (36)$$

where $\tau = RC$.

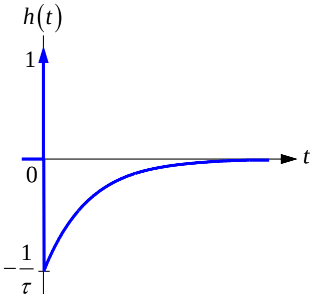
- We can compute the impulse response by solving (36), assuming input, output and initial condition

$$x(t) = \delta(t)$$
$$y(t) = h(t)$$
$$y(t) = 0,\ t < 0.$$

We obtain

$$h(t) = \delta(t) - \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t),$$

which is shown here.



You will be asked in a homework problem to verify that this impulse response satisfies (36) and the initial condition stated.

Special Case $N = 0$

- In the special case $N = 0$, the differential equation (35) provides an *explicit* description for the output $y(t)$ given an input $x(t)$. We can immediately solve (35) to obtain the output:

$$y(t) = \frac{1}{a_0} \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} . \tag{37}$$

The system described by (37) is LTI. We can obtain its impulse response by directly substituting $x(t) = \delta(t)$ and $y(t) = h(t)$ into (37):

$$h(t) = \frac{1}{a_0} \sum_{k=0}^{M} b_k \frac{d^k \delta(t)}{dt^k} . \tag{38}$$

In this special case, the impulse response is a scaled sum of a CT impulse function and its derivatives.

Example: Differentiator

- The CT differentiator has input-output relation

$$y(t) = \frac{dx(t)}{dt}.$$

This input-output relation is already in the form (37). Using (38), the impulse response of the differentiator is

$$h(t) = \frac{d\delta(t)}{dt}. \tag{39}$$

The differentiator's impulse response is the *derivative of the unit impulse function*, often called a *unit doublet function*.

Unit Doublet Function ***(Skip)***

- The *unit doublet function* is the derivative of the CT unit impulse function:

$$\delta'(t) = \frac{d\delta(t)}{dt}.$$

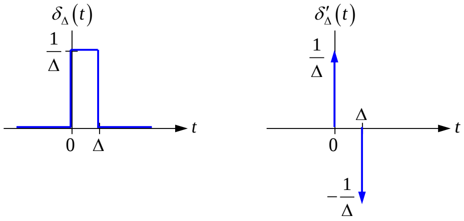
- Recall that the unit impulse is represented as the limiting case of a narrow, tall rectangular pulse that has unit area:

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t).$$

Differentiating this representation of the unit impulse yields

$$\delta'(t) = \lim_{\Delta \to 0} \delta'_\Delta(t) = \lim_{\Delta \to 0} \frac{1}{\Delta}\left[\delta(t) - \delta(t-\Delta)\right]. \tag{40}$$

The rectangular pulse $\delta_\Delta(t)$ and its derivative $\delta'_\Delta(t)$ are shown, assuming nonzero Δ.



We can think of $\delta'_\Delta(t)$ as a pair of unit impulses, scaled by $\pm 1/\Delta$ and offset by Δ, in the limit $\Delta \to 0$.

- We can use this description of the unit doublet to verify that this function is the impulse response of a differentiator. Consider a signal $x(t)$ that is differentiable near $t=0$. Convolving it with the differentiator impulse response (39) and representing the doublet using (40) yields

$$\begin{aligned} x(t)*\delta'(t) &= \lim_{\Delta\to 0} x(t)*\delta'_{\Delta}(t) \\ &= \lim_{\Delta\to 0} x(t)*\left[\frac{1}{\Delta}\delta(t)-\frac{1}{\Delta}\delta(t-\Delta)\right] \\ &= \lim_{\Delta\to 0} \frac{x(t)-x(t-\Delta)}{\Delta} \\ &= x'(t) \end{aligned}.$$

In the next-to-last line, we used (23). In the last line, we used the fact that in the limit, a finite difference becomes a derivative.

Block Diagram Realizations ***(Skip)***

- Any CT system described by a finite-order, linear, constant-coefficient differential equation can be realized using differentiators (or integrators), scale factors and adders. Such realizations were used historically in analog computers to solve differential equations before the advent of digital computers. Such realizations are still used in feedback control systems.

Example: First-Order Equation

- Consider a first-order constant-coefficient equation

$$\frac{dy}{dt} + ay(t) = bx(t) \; . \qquad (41)$$

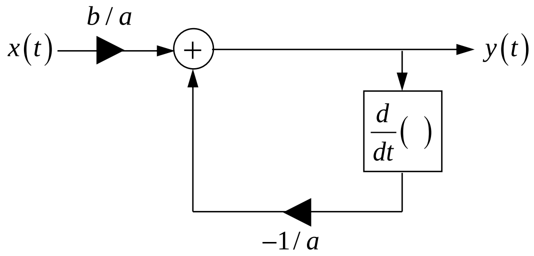
The first-order lowpass filter equation (24) is an example of (41) with $a = b = \frac{1}{\tau}$.

Differentiator-Based Realization

- We rewrite (41) as

$$y(t) = \frac{b}{a}x(t) - \frac{1}{a}\frac{dy}{dt} ,$$

which is realized by the system shown.

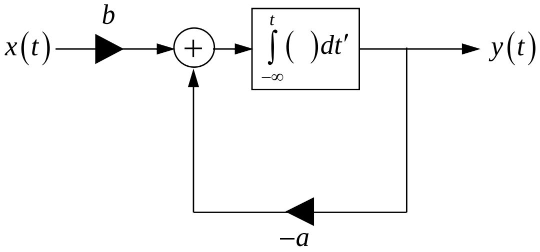


Integrator-Based Realization

- We rearrange (41) as $\frac{dy}{dt} = bx(t) - ay(t)$ and integrate from time $-\infty$ to time *t*, obtaining

$$y(t) = \int_{-\infty}^{t} \left[bx(t') - ay(t') \right] dt' .$$

This is realized by the system shown.



Discrete-Time Systems Described by Linear, Constant-Coefficient Difference Equations

- A general linear, constant-coefficient difference equation, describing a DT system with input $x[n]$ and output $y[n]$, is of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \tag{42}$$

  - The constants a_k, $k = 0,\ldots,N$ and b_k, $k = 0,\ldots,M$ are real in systems that map real inputs to real outputs.
  - The right-hand side is a linear combination of the present and *M* past values of the input $x[n]$.
  - The left-hand side is a linear combination of the present and *N* past values of the output $y[n]$. The parameter *N*, which specifies the oldest output values that contribute to the present output $y[n]$, is called the *order* of the equation and of the system.
  - We can draw the block diagram of a realization of the general DT system (42) using shift registers (discrete-time delays), scale factors and adders. These are used extensively in digital signal processing (see EE 102B).
  - By construction, (42) describes a causal system (provided $a_0 \neq 0$).
  - EE 102B teaches how to solve equations of the form (42) using Z transforms.

General Case $N \neq 0$

- In the general case $N \neq 0$, difference equation (42) describes a system in which N past output values are fed back and contribute to the present output $y[n]$. We say that the system is *recursive*.

- The difference equation (42) provides an *implicit* description of output $y[n]$ given input $x[n]$.

  Suppose we specify:

  - N initial conditions $y[n_0 - 1],\ldots,y[n_0 - N]$, the N most recent past output values at time n_0.

  - An input $x[n]$, $n \geq n_0$.

  Then we can solve for the output $y[n]$, $n \geq n_0$ *uniquely*.

- If we assume the system is *at initial rest* (the output is zero until the input becomes nonzero)

$$x[n]=0\,,\; n<n_0 \quad \Rightarrow \quad y[n]=0\,,\; n<n_0\,,$$

then the linear, constant-coefficient difference equation (42) describes a *causal, LTI system*. We can calculate its impulse response $h[n]$ by solving (42) assuming input, output and initial conditions

$$x[n]=\delta[n]$$

$$y[n]=h[n]$$

$$y[n]=0\,,\; n<0\,.$$

- In the general case $N \neq 0$, such a DT LTI system is called an *infinite impulse response* (IIR) system. The impulse response $h[n]$ never becomes identically zero for any finite time $n>0$.

- Intuitively, the impulse response has infinite duration because *N* past values of the output are fed back and contribute to the present output $y[n]$.

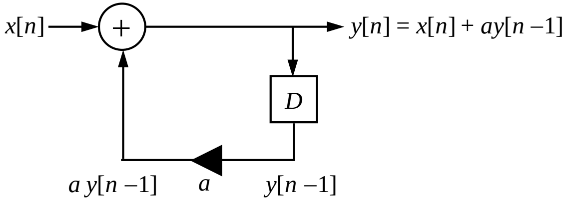
Example: First-Order System

- Recall the first-order system described by a constant-coefficient linear difference equation

$$y[n] = x[n] + ay[n-1], \tag{10}$$

where a is a real constant.

- It can be realized by the block diagram shown.



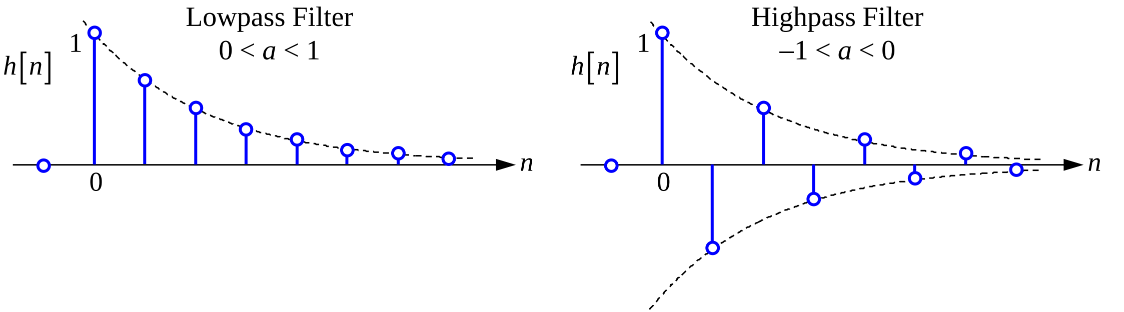
The previous output $y[n-1]$ is fed back and contributes to the present output $y[n]$.

- As a result of this feedback or recursion, the first-order system impulse response

$$h[n] = a^n u[n] \tag{11}$$

has infinite duration.

- The impulse response (11) is shown below for
  - Lowpass filter: $0 < a < 1$.
  - Highpass filter: $-1 < a < 0$.



- For both filters, the values of a satisfy $|a| < 1$, so the impulse response (11) is absolutely summable:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty,$$

and both filters are stable systems.

Special Case $N = 0$

- In this special case, the difference equation (42) provides an *explicit* description for the output $y[n]$ given an input $x[n]$. We can immediately solve (42) to obtain an expression for the output:

$$y[n] = \frac{1}{a_0} \sum_{k=0}^{M} b_k x[n-k]. \tag{43}$$

The system described by (43) is LTI. We can obtain its impulse response by directly substituting $x[n] = \delta[n]$ and $y[n] = h[n]$ into (43):

$$h[n] = \frac{1}{a_0} \sum_{k=0}^{M} b_k \delta[n-k]. \tag{44}$$

In this special case, the impulse response is a scaled sum of delayed DT impulse functions.

- Such a DT LTI system is called a *finite impulse response* (FIR) system because the impulse response $h[n]$ becomes identically zero for $n > M$, and thus has finite length.

- FIR DT filters are used extensively in practical digital signal processing.

- We study two simple FIR filters here, and will study more examples in Chapters 3, 5 and 6.

Example: Two-Point Moving Average (Lowpass Filter)

- A two-point *moving average filter* is described by an input-output relation

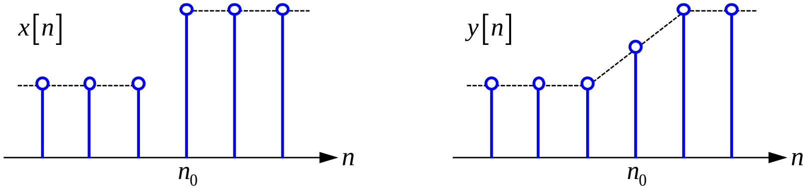
$$y[n] = \frac{1}{2}\left(x[n] + x[n-1]\right). \tag{45}$$

  Expression (45) is an instance of (43) with $M = 1$.

- As the term "moving average" implies, at time *n*, the output $y[n]$ is an average of the present and preceding input values, $x[n]$ and $x[n-1]$.

- Moving average filters are a type of lowpass filter.
  They are often used to smooth out fluctuations or noise appearing in signals or data.
  Two-dimensional versions of such filters may be used in smoothing images.

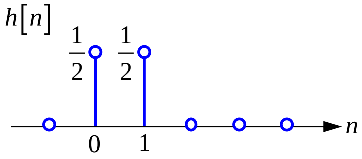
- An input and the resulting output are shown.

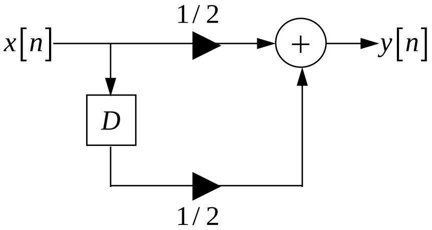


- The impulse response is found by substituting $x[n]=\delta[n]$ and $y[n]=h[n]$ into (45):

$$h[n]=\frac{1}{2}\left(\delta[n]+\delta[n-1]\right). \tag{46}$$

The impulse response and a block diagram of a realization are shown.





Example: Edge Detector (Highpass Filter)

- An *edge detector filter* is described by an input-output relation

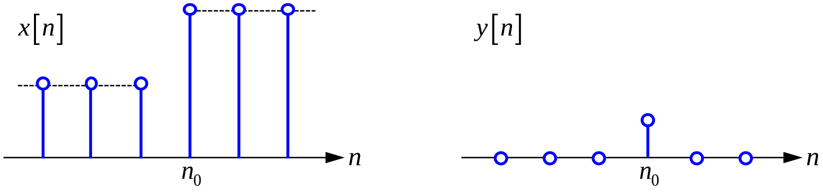
$$y[n]=\frac{1}{2}\left(x[n]-x[n-1]\right), \tag{47}$$

  and is another instance of (43) with $M=1$.

- At time *n*, the output $y[n]$ is half the difference between the present and preceding input value, $x[n]$ and $x[n-1]$.

- Edge detectors are a type of highpass filter.
  They are often used to accentuate changes appearing in signals or data.
  Two-dimensional versions of such filters may be used to accentuate edges in images.

- An input and the resulting output are shown.



- The impulse response is found by substituting $x[n]=\delta[n]$ and $y[n]=h[n]$ into (47):

$$h[n]=\frac{1}{2}\left(\delta[n]-\delta[n-1]\right). \tag{48}$$

The impulse response and a block diagram of a realization are shown.