Stanford University
EE 102A: Signal Processing and Linear Systems I
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Chapter 1: Signals and Systems

Major Topics in This Chapter

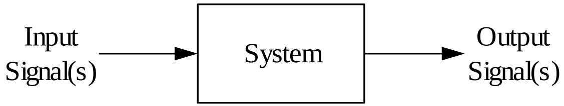
- Signals in continuous or discrete time
  - Classification: continuous- or discrete-time, even or odd, periodic or aperiodic
  - Energy and power
  - Operations on dependent or independent variables
- Elementary signals in continuous or discrete time
  - Exponentials: real, imaginary, complex. Sinusoids: steady, decaying, growing
  - Singularity function: step, impulse, ramp, etc.
- Systems in continuous or discrete time
  - Representation
  - Examples
  - Properties: stability, memory, invertibility, time-invariance, linearity, causality

Signals

- A *signal* is a function of one or more independent variables, and often represents a variable associated with a physical system.
    - A *one-dimensional signal* depends on one independent variable.
      Example: a voltage on a wire as a function of time.
    - A *multi-dimensional* signal depends on more than one independent variable.
      Example: a still image depends on spatial variables (x, y), so is two-dimensional.

Systems

- A *system* performs a mapping on one or more signal(s) to produce new signal(s).



- Examples
    - One-dimensional: a speaker converts a voltage on a wire to a sound wave.
    - Two-dimensional: an image processing program converts a blurry image to a sharper image.

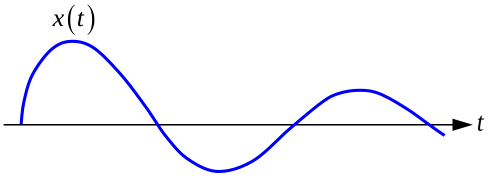
Classification of Signals

We mainly discuss one-dimensional signals, where the independent variable is time.

Continuous-Time vs. Discrete-Time Signals

Continuous-Time (CT) Signals

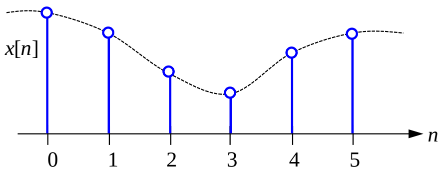
- A CT signal $x(t)$ is defined at all instants of time *t*, at least over a limited domain.
- A CT signal is also known as a *waveform*.



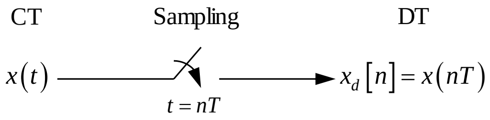
- *Example*: a CT signal $x(t)$ represents the voltage across a resistor in a circuit.

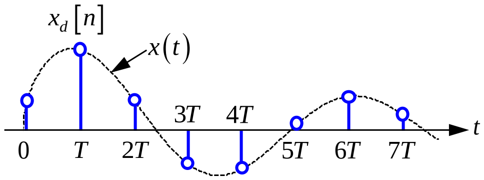
Discrete-Time (DT) Signals

- A DT signal $x[n]$ is defined only at integer values of *n*.
- A DT signal is also known as a *sequence*.



- *Example*: a DT signal $x[n]$ represents the closing price of a stock on trading days indexed by *n*.
- *Example*: a DT signal $x_d[n]$ corresponds to samples of a CT signal $x(t)$ taken at integer multiples of a *sampling interval T*. Music signals are often sampled at a sampling rate $1/T = 44.1\ \text{kHz}$.





Random vs. Deterministic Signals

- *Random signals* are not precisely predictable and are described by probability theory.
  For example, noise processes are random signals.

- *Deterministic signals* are completely specified functions of time.
  Almost all the signals studied in this course are deterministic.

How to Specify Deterministic Signals

- A deterministic signal may be specified by various means, depending on the nature of the signal.

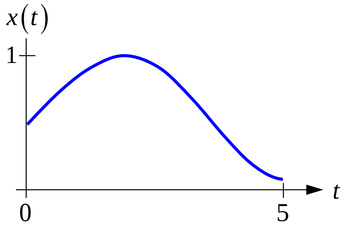
  - *Formula*: applicable to CT or DT.
    Examples:

$$x(t) = Ae^{2t} \quad \forall t \qquad\qquad x[n] = 3n \quad \forall n \, .$$

  - *Table*: applicable only to DT, finite domain.
    Example:

n	0	1	2	3
$x[n]$	2.6	1.4	3.7	9.3

- *Graph*: applicable only to CT, limited domain or DT, finite domain.
  Example:



- *Algorithm*: applicable only to DT.
  Example:

$$x[0]=1, \quad x[n+1]=x[n]-6 .$$

Even or Odd Signals

- These classifications are equally applicable to CT or DT signals.
  Described here for CT. Can extend to DT by changing (t) to $[n]$.

  - *Even and odd signals* satisfy

$$x_e(-t) = x_e(t) \quad \forall t \qquad\qquad x_o(-t) = -x_o(t) \quad \forall t$$

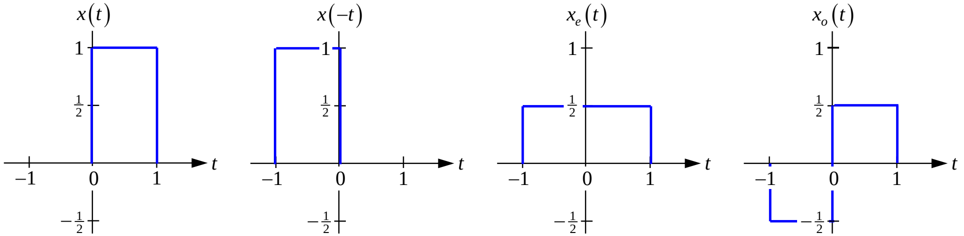
  - *Expressing an arbitrary signal as sum of even and odd parts*

$$x(t) = x_e(t) + x_o(t)$$

  - *Obtaining even and odd parts of an arbitrary signal*

$$x_e(t) = \frac{1}{2}\big(x(t) + x(-t)\big) \qquad\qquad x_o(t) = \frac{1}{2}\big(x(t) - x(-t)\big)$$

- Example for CT



Periodic vs. Aperiodic Signals

Periodic Continuous-Time Signals

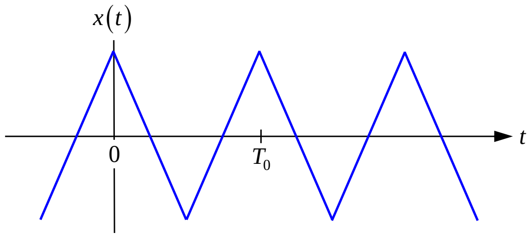
- A periodic CT signal satisfies

$$x(t) = x(t + T_0) \quad \forall t$$

  for some T_0. The *period* is the smallest positive value of T_0 satisfying the above equation.

  Note: in CT, we denote the period by T_0 to avoid confusion with a sampling interval T.

- *Example*: a triangle wave with period T_0 is shown.



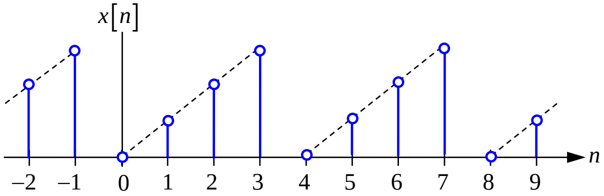
Periodic Discrete-Time Signals

- A periodic DT signal satisfies

$$x[n] = x[n+N] \quad \forall n$$

for some N. The *period* is the smallest positive value of N satisfying the above equation. *Note*: in DT, there is no confusion if we denote the period by N.

- *Example*: a sawtooth signal with period $N = 4$ is shown.

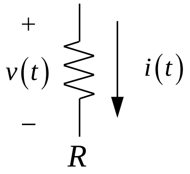


- *Question*: given a periodic CT signal with period T_0, if you sample it at rate $1/T$, will you get a periodic DT signal? What condition(s) must be satisfied?

Energy and Power of Signals

Continuous-Time Signals

- To motivate our general definitions of energy and power, consider a voltage $v(t)$ across a resistor R inducing a current $i(t)$.



- The instantaneous power dissipated is

$$p(t) = v(t)i(t) = i^2(t)R = \frac{v^2(t)}{R},$$

which has units of W. Setting $R = 1\,\Omega$, we obtain simpler expressions for the instantaneous power:

$$p(t) = i^2(t) = v^2(t).$$

- The total energy dissipated is

$$E = \int_{-\infty}^{\infty} v^2(t)dt \,,$$

while the average power dissipated is

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} v^2(t)dt \,.$$

After we set $R = 1\,\Omega$, these expressions yield values of E and P proportional to, but not generally equal to, the physical energy and physical average power, which have units of J and W, respectively.

- Given a general complex-valued CT signal $x(t)$, we define the signal energy and power as

$$E = \int_{-\infty}^{\infty} \left| x(t) \right|^2 dt \, ,$$

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| x(t) \right|^2 dt \, .$$

- These are mathematical definitions of signal energy and power. They are proportional to physical energy and power in most cases, but not necessarily equal to them.

- Given a *periodic CT signal*, we can compute the average power by averaging over one period of duration T_0:

$$P = \frac{1}{T_0} \int_{t_1}^{t_1 + T_0} \left| x(t) \right|^2 dt \, ,$$

where t_1, the start of the integration interval, may be chosen arbitrarily.

- Given a general complex-valued signal $x[n]$, we define the signal energy and power as

$$E = \sum_{n=-\infty}^{\infty} \left| x[n] \right|^2$$

$$P = \lim_{N \to \infty} \frac{1}{2N} \sum_{n=-N}^{N} \left| x[n] \right|^2 .$$

- Given a *periodic DT signal*, we can compute the average power by averaging over one period of N samples:

$$P = \frac{1}{N} \sum_{n=n_1}^{n_1+N-1} \left| x[n] \right|^2 ,$$

where n_1, the start of the summation interval, may be chosen arbitrarily.

Classification of Signals

- Many important CT or DT signals can be classified into one of the following two categories.

    - An *energy signal* has finite energy and zero average power:

$$0 \le E < \infty , \quad P = 0 .$$

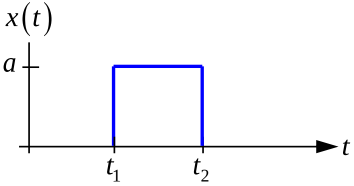
    - A *power signal* has finite average power and infinite energy:

$$0 < P < \infty , \quad E = \infty .$$

- Some signals cannot be classified as energy signals or power signals, as we may see in homework problems.

Examples

1. A CT energy signal, as shown.



- Total energy:

$$E = \int_{t_1}^{t_2} a^2 dt = a^2 \left(t_2 - t_1\right).$$

- Average power:

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{t_1}^{t_2} a^2 dt = 0.$$

2. A periodic CT power signal:

$$x(t) = a\sin\omega_0 t \quad .$$

- We use a trigonometric identity

$$\sin^2\theta = \frac{1}{2}\left(1-\cos 2\theta\right).$$

- Total energy:

$$E = \int_{-\infty}^{\infty} a^2 \sin^2 \omega_0 t dt = \frac{a^2}{2}\left[\underbrace{\int_{-\infty}^{\infty}(1)\,dt}_{\infty} - \underbrace{\int_{-\infty}^{\infty}\cos 2\omega_0 t dt}_{\text{bounded}}\right] = \infty\ .$$

- Average power:
    - Using the general definition, which is applicable to either periodic or aperiodic signals:

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} a^2 \sin^2 \omega_0 t dt = \frac{a^2}{2} \lim_{T\to\infty} \frac{1}{2T} \left[\int_{-T}^{T} (1) dt - \int_{-T}^{T} \cos 2\omega_0 t dt \right]$$

$$= \frac{a^2}{2} \lim_{T\to\infty} \frac{1}{2T} \left[\underset{\text{grows large}}{2T} - \underbrace{\tfrac{1}{\omega_0} \sin 2\omega_0 T}_{\text{bounded}} \right]$$

$$= \frac{a^2}{2}$$

- Since the signal is periodic, we may compute the average power by averaging over just one period of duration $T_0 = 2\pi / \omega_0$:

$$P = \frac{\omega_0}{2\pi} \int_{-\pi/\omega_0}^{\pi/\omega_0} a^2 \sin^2 \omega_0 t dt = \frac{a^2 \omega_0}{4\pi} \left[\underbrace{\int_{-\pi/\omega_0}^{\pi/\omega_0} (1) dt}_{\frac{2\pi}{\omega_0}} - \underbrace{\int_{-\pi/\omega_0}^{\pi/\omega_0} \cos 2\omega_0 t dt}_{\frac{1}{\omega_0} \sin\left(2\omega_0 \frac{\pi}{\omega_0}\right) = 0} \right] = \frac{a^2}{2} .$$

This agrees with the first approach.

Operating on Signals to Produce New Signals

- These operations are often used in practical applications.
- The analysis of signals and systems is often made easier if complicated signals are expressed in terms of simpler signals that have been modified by various operations.

Operations on the Dependent Variable

1. *Amplitude Scaling*

$$y(t) = Cx(t) \qquad y[n] = Cx[n].$$

2. *Addition*

$$y(t) = x_1(t) + x_2(t) \qquad y[n] = x_1[n] + x_2[n].$$

3. *Multiplication*

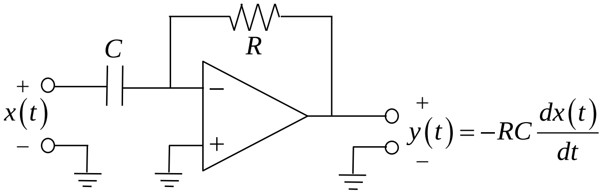
$$y(t) = x_1(t) \cdot x_2(t) \qquad y[n] = x_1[n] \cdot x_2[n].$$

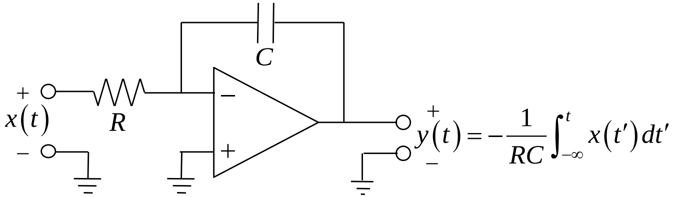
4a. Differentiation and Running Integration, CT Signals Only

- These operations are defined by

$$y(t) = \frac{dx(t)}{dt} \quad \text{and} \quad y(t) = \int_{-\infty}^{t} x(t')dt'$$

- These operations can be implemented on real-valued signals using the circuits shown, assuming the operational amplifiers are ideal (see *EE 102B Course Reader*, Chapter 5).





4b. First Difference and Running Summation (Accumulation), DT Signals Only

- These operations are defined by

$$y[n] = x[n] - x[n-1]$$

$$y[n] = \sum_{k=-\infty}^{n} x[k].$$

- These are not unique operations but are derivable from addition and scaling.
- The first difference and running summation of DT signals are somewhat analogous to the differentiation and running integration of CT signals.

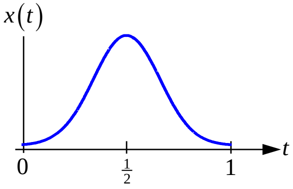
Operations on the Independent Variable

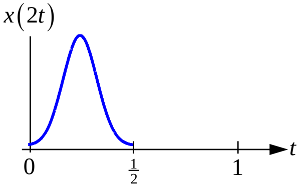
1a. CT Time Scaling

- Consider a real, positive constant $a > 0$. Given a CT signal $x(t)$, the *time-scaled signal* is

$$y(t) = x(at).$$

  - Time is *compressed* for $a > 1$. An example is shown for $a = 2$.





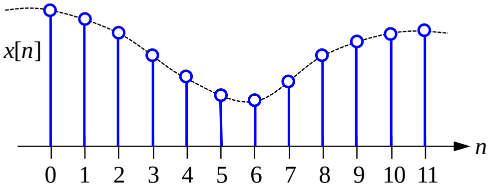
  - Time is *expanded* for $0 < a < 1$.

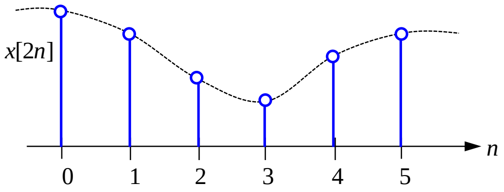
1b. DT Time Compression ***(skip)***

- Consider a positive integer $k \geq 1$. Given a DT signal $x[n]$, the *compressed signal* is

$$y[n] = x[kn].$$

- If $k > 1$, samples of the signal are lost. An example is shown for $k = 2$.



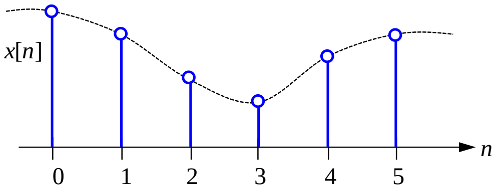


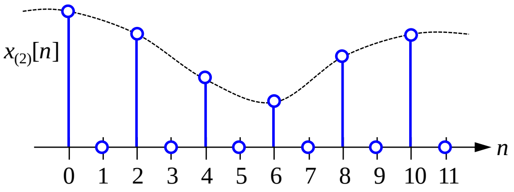
1c. DT Time Expansion ***(skip)***

- Consider a positive integer $m \geq 1$. Given a DT signal $x[n]$, the *expanded signal* is

$$y[n] = x_{(m)}[n] = \begin{cases} x\left[\dfrac{n}{m}\right] & \dfrac{n}{m} \text{ integer} \\ 0 & \text{otherwise} \end{cases}.$$

- For any positive integer *m*, no samples of the signal are lost. An example is shown for $m = 2$.



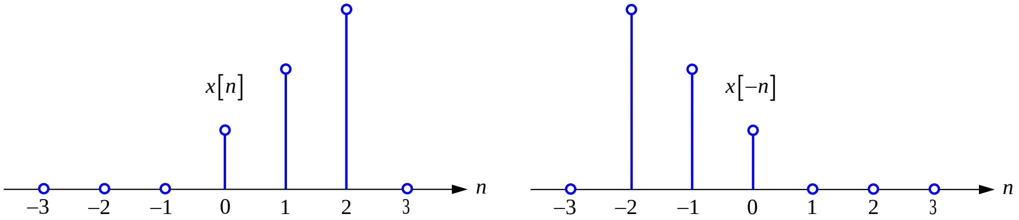


2. Time Reversal (Reflection)

- This operation is defined for CT and DT signals by

$$y(t) = x(-t) \qquad y[n] = x[-n].$$

A DT example is shown.

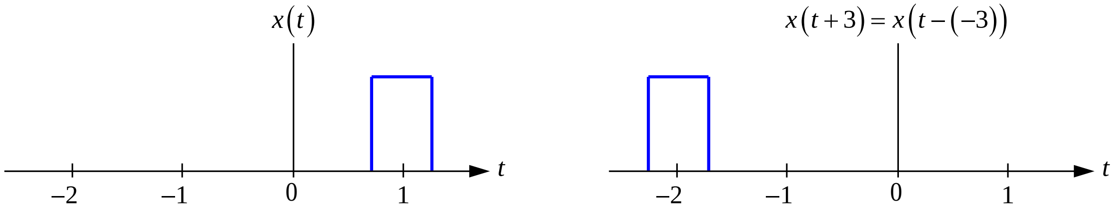


3a. CT Time Shifting

- Consider a real constant t_0. Given a CT signal $x(t)$, the *time-shifted signal* is

$$y(t) = x(t - t_0).$$

  - For $t_0 > 0$, the signal is *delayed* (shifted right).
  - For $t_0 < 0$, the signal is *advanced* (shifted left). An example is shown with $t_0 = -3$.



3b. DT Time Shifting

- Consider an integer n_0. Given a signal $x[n]$, the *time-shifted signal* is

$$y[n] = x[n - n_0].$$

Analogous to the CT case:

- For $n_0 > 0$, the signal is *delayed* (shifted right).
- For $n_0 < 0$, the signal is *advanced* (shifted left).

Combining Time Shifting and Time Scaling or Reversal

- We consider only CT signals. Given a signal $x(t)$, we consider

$$y(t) = x\big(a(t - t_0)\big),$$

  where t_0 and a are real constants (a is not necessarily positive).

  - For $t_0 \neq 0$, the signal is *time-shifted*.
  - For $|a| > 1$, the signal is *compressed*, while for $|a| < 1$, the signal is *expanded*.
  - For $a < 0$, the signal is *time-reversed*.

- To sketch $y(t)$, we perform three steps:

1. Express $y(t)$ in the form $y(t) = x\big(a(t - t_0)\big)$.
2. Form a time-scaled and/or time-reversed signal:

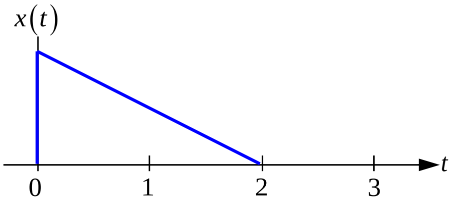
$$v(t) = x(at).$$

3. Time shift $v(t)$ it to obtain the final signal:

$$y(t) = v(t - t_0) = x\big(a(t - t_0)\big).$$

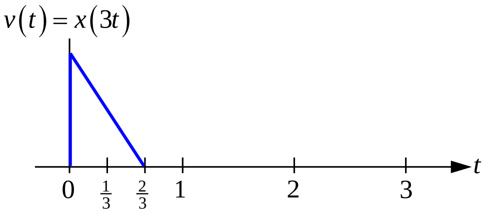
- Example: time is shifted and compressed.

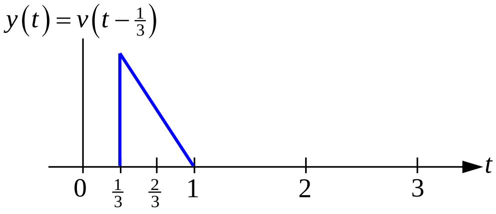
Given



We are asked to sketch $y(t) = x(3t - 1)$,

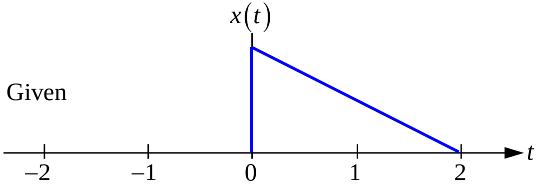
which we express as $y(t) = x\left(3\left(t - \tfrac{1}{3}\right)\right)$.





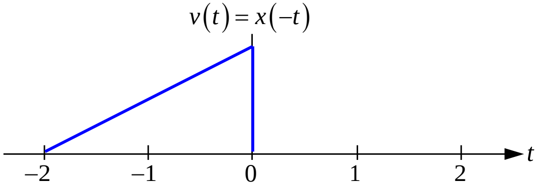
- Example: time is shifted and reversed.

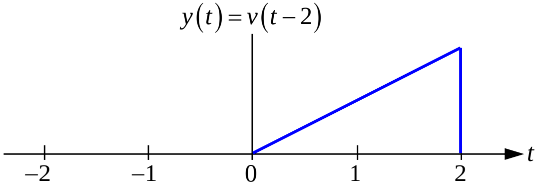
Given



We are asked to sketch $y(t) = x(-t+2)$,

which we express as $y(t) = x(-(t-2))$.





Elementary Signals

- We study two important families, each in both CT and DT:

1. *Exponential functions* (real, imaginary or complex) and *sinusoids* (steady, decaying or growing).

    - These are solutions to common first- or second-order differential or difference equations.
    - The exponential signals are eigenfunctions of linear time-invariant systems.

2. *Singularity functions*: impulse, step, ramp, etc.

    - These can be used as building blocks for other signals.
    - These are important in the analysis of linear time-invariant systems.

Continuous-Time Exponentials and Sinusoids

- CT exponential signals are of the general form

$$x(t) = Ce^{at}, \tag{1}$$

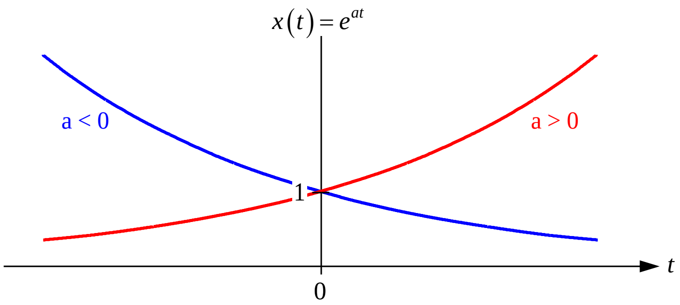
where a and C are complex constants.

Real Exponentials

- Let $C = 1$ and choose a real. These signals are of the form

$$x(t) = e^{at}.$$

- A CT real exponential is a purely decaying or growing function, depending on the sign of a, as shown.



Imaginary Exponentials

- Starting with the general form of a CT exponential (1), we choose the constants as follows:
  - Let $a = j\omega_0$, where ω_0 is a real constant called the *fundamental frequency* (rad/s).
  - Let $C = e^{j\phi}$, where ϕ is a real constant called the *initial phase* (rad).
- A CT imaginary exponential is of the form

$$\begin{aligned} x(t) &= e^{j\phi} e^{j\omega_0 t} = e^{j(\omega_0 t + \phi)} \\ &= \cos(\omega_0 t + \phi) + j\sin(\omega_0 t + \phi) \end{aligned}$$

  We used Euler's relation to express its real and imaginary parts in terms of the cosine and sine.

- Two characteristics distinguish a CT imaginary exponential from a DT imaginary exponential:

1. It is periodic for any fundamental frequency ω_0 with period $T_0 = \dfrac{2\pi}{\omega_0}$:

$$e^{j\omega_0(t+T_0)} = e^{j\omega_0\left(t+\frac{2\pi}{\omega_0}\right)} = e^{j\omega_0 t} \underbrace{e^{j2\pi}}_{1} = e^{j\omega_0 t}.$$

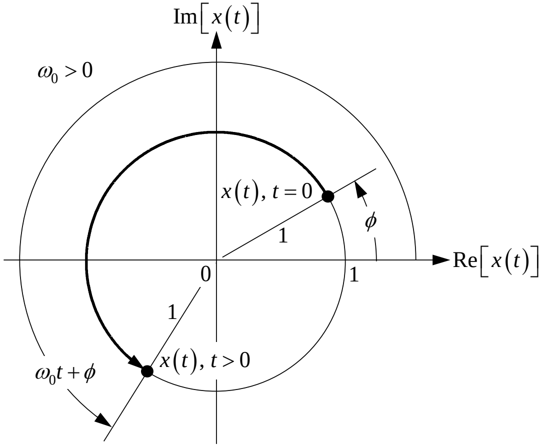
2. The signals for any distinct values of the fundamental frequency ω_0 are distinct.

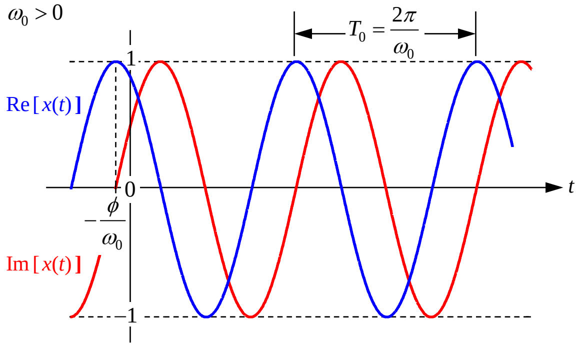
- We can graph a CT imaginary exponential signal

$$x(t) = e^{j(\omega_0 t+\phi)} = \cos(\omega_0 t+\phi) + j\sin(\omega_0 t+\phi)$$

in two ways:

- Left: as a point in a complex plane. It lies on a circle of radius $\left|e^{j(\omega_0 t+\phi)}\right| = 1$ and has a phase $\angle e^{j(\omega_0 t+\phi)} = \omega_0 t+\phi$. Assuming $\omega_0 > 0$, the point rotates counterclockwise as time t increases.
- Right: in terms of its real and imaginary signal components, $\cos(\omega_0 t+\phi)$ and $\sin(\omega_0 t+\phi)$.





Complex Exponentials

- Starting with the general form of a CT exponential (1), we choose the constants as follows:
  - Let $a = \sigma + j\omega_0$, where σ and ω_0 are real constants.
  - Let $C = e^{j\phi}$, where ϕ is a real constant.
- A CT complex exponential is of the form

$$x(t) = e^{j\phi} e^{(\sigma + j\omega_0)t} = e^{\sigma t} e^{j(\omega_0 t + \phi)}$$
$$= e^{\sigma t}\left[\cos(\omega_0 t + \phi) + j\sin(\omega_0 t + \phi)\right]$$

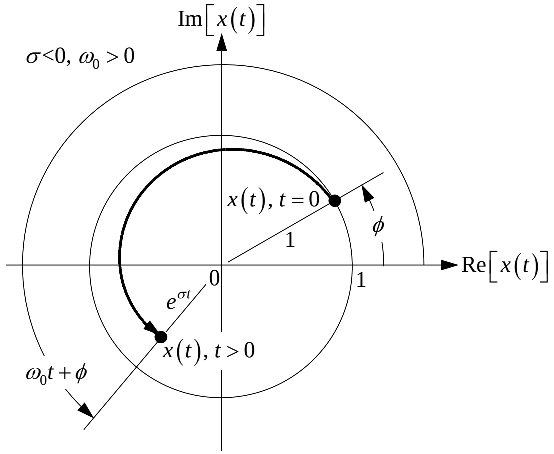
  We used Euler's relation to express its real and imaginary parts in terms of the cosine and sine.

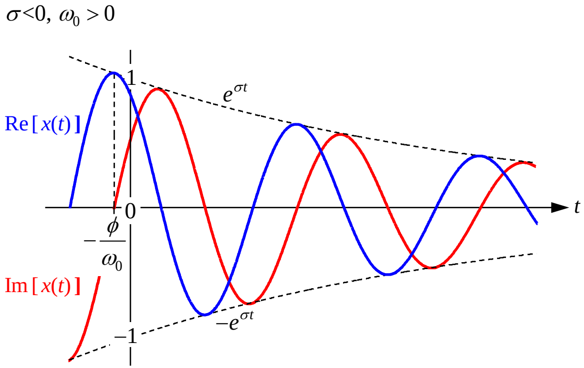
- We can graph a CT complex exponential signal

$$x(t) = e^{\sigma t} e^{j(\omega_0 t + \phi)} = e^{\sigma t}\left[\cos(\omega_0 t + \phi) + j\sin(\omega_0 t + \phi)\right]$$

  in two ways:

- Left: as a point in a complex plane. It lies at a radius $\left|e^{\sigma t}e^{j(\omega_0 t+\phi)}\right| = e^{\sigma t}$ and has a phase $\angle e^{\sigma t}e^{j(\omega_0 t+\phi)} = \omega_0 t + \phi$. Assuming $\omega_0 > 0$ and $\sigma < 0$, it rotates counterclockwise and spirals inward as t increases.
- Right: in terms of its real and imaginary components, $e^{\sigma t}\cos(\omega_0 t+\phi)$ and $e^{\sigma t}\sin(\omega_0 t+\phi)$. They oscillate within the envelope defined by $\pm e^{\sigma t}$. Assuming $\sigma < 0$, they are decaying (damped) sinusoids.





Discrete-Time Exponentials and Sinusoids

- DT exponential signals can be written in the general form

$$x[n] = Ce^{\beta n}, \tag{2}$$

  where β and C are complex constants.

- We usually define a complex constant $\alpha = e^{\beta}$ and write a DT exponential signal in a form that looks different but is equivalent to (2):

$$x[n] = C\alpha^{n}. \tag{2'}$$

Real Exponentials

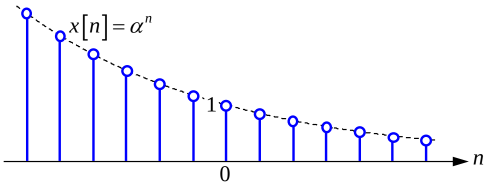
- These signals are of the form

$$x[n] = \alpha^{n},$$

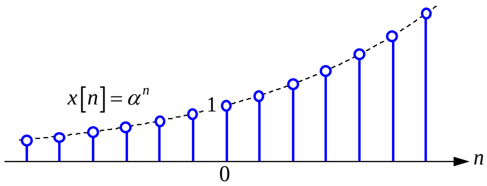
  where α is a real constant (β need not be real). We have set the constant C in (2′) to unity.

- When $\alpha > 0$, a DT real exponential is a purely decaying function or growing function.

$0 < \alpha < 1$



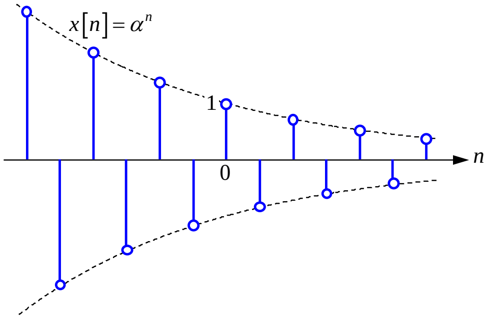
$\alpha > 1$



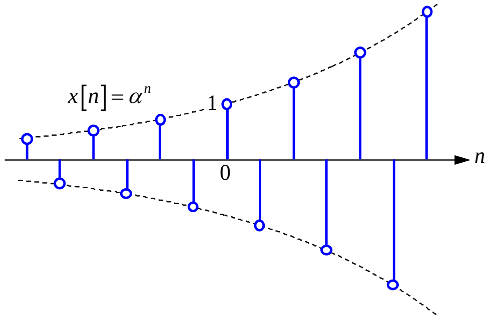
- When $\alpha < 0$, a DT real exponential alternates sign as a function of *n*:

$$x[n] = (-1)^n |\alpha|^n .$$

$-1 < \alpha < 0$



$\alpha < -1$



Imaginary Exponentials

- Starting with the general form of a DT exponential, we choose the constants as follows:
  - Let $\beta = j\Omega_0$, where Ω_0 is a real constant called the *fundamental frequency* (rad).

    It follows that $\alpha = e^{\beta} = e^{j\Omega_0}$.
  - Let $C = e^{j\phi}$, where ϕ is a real constant called the *initial phase* (rad).
- A DT imaginary exponential is of the form

$$\begin{aligned} x[n] &= e^{j\phi}e^{j\Omega_0 n} = e^{j(\Omega_0 n+\phi)} \\ &= \cos(\Omega_0 n + \phi) + j\sin(\Omega_0 n + \phi) \end{aligned}.$$

- Two characteristics distinguish a DT imaginary exponential from a CT imaginary exponential:

1. It is periodic only if the fundamental frequency Ω_0 is a rational multiple of 2π.

    - Periodicity requires that

    $$e^{j\Omega_0(n+N)} = e^{j\Omega_0 n} \quad \forall n \quad \text{for some integer } N > 0 .$$

    This can be simplified to

    $$e^{j\Omega_0 N} = 1 ,$$

    which requires that

    $$\Omega_0 N = m 2\pi \quad \text{for some integer } m,$$

    or

    $$\Omega_0 = \frac{m}{N} 2\pi .$$

    - We conclude that a DT imaginary exponential is periodic only if the fundamental frequency Ω_0 is a rational multiple m/N of 2π.

    - The period is the value of N when the ratio m/N is expressed in lowest terms.

2. The signals for distinct values of the fundamental frequency Ω_0 are distinct only if the values of Ω_0 do not differ by an integer multiple of 2π.

- Consider fundamental frequencies differing by an integer k times 2π, Ω_0 and $\Omega_0 + k2\pi$. The signal at frequency $\Omega_0 + k2\pi$ is

$$e^{j(\Omega_0 + k2\pi)n} = e^{j\Omega_0 n} \underbrace{e^{jk2\pi n}}_{1} = e^{j\Omega_0 n},$$

which is identical to the signal at frequency Ω_0.

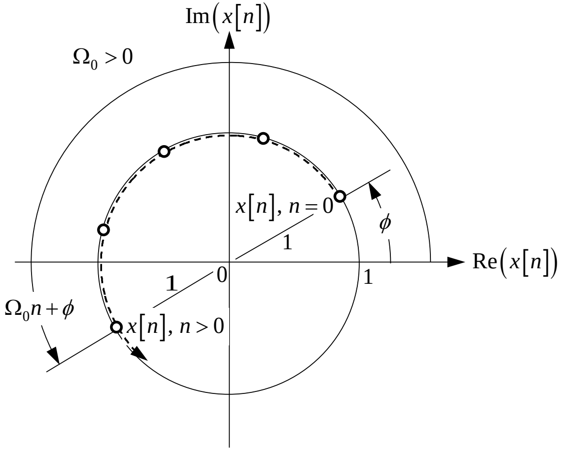
- To avoid ambiguity, we often choose the fundamental frequency Ω_0 to lie only within an interval of length 2π, such as $-\pi \le \Omega_0 < \pi$ or $0 \le \Omega_0 < 2\pi$.

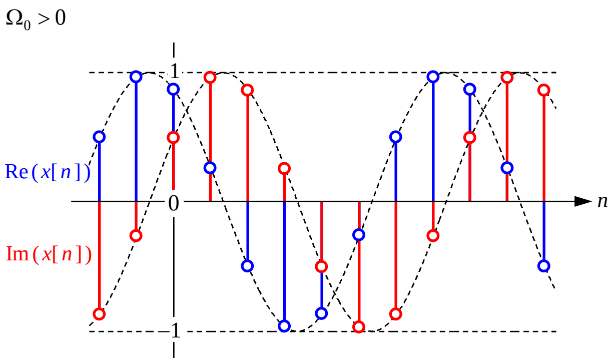
- We can graph a DT imaginary exponential signal

$$x[n] = e^{j(\Omega_0 n+\phi)} = \cos(\Omega_0 n+\phi) + j\sin(\Omega_0 n+\phi)$$

in two ways:

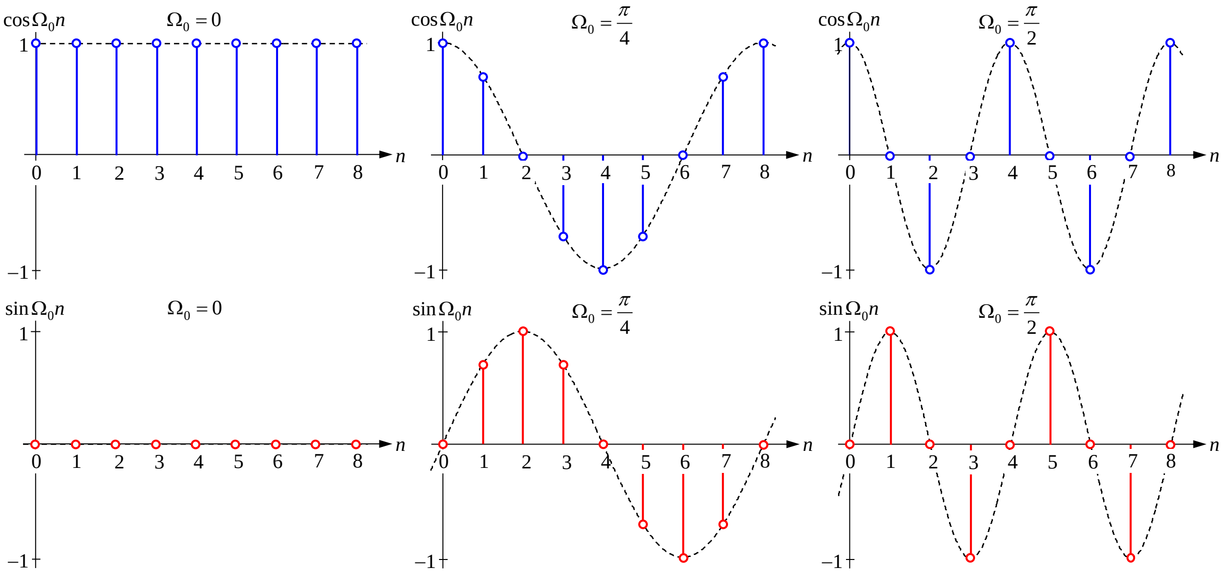
- Left: as a point on a unit circle in a complex plane. It lies on a circle of radius $\left|e^{j(\Omega_0 n+\phi)}\right| = 1$ and has a phase $\angle e^{j(\Omega_0 n+\phi)} = \Omega_0 n+\phi$. For $\Omega_0 > 0$, the point rotates counterclockwise as *n* increases.
- Right: in terms of its real and imaginary signal components, $\cos(\Omega_0 n+\phi)$ and $\sin(\Omega_0 n+\phi)$.



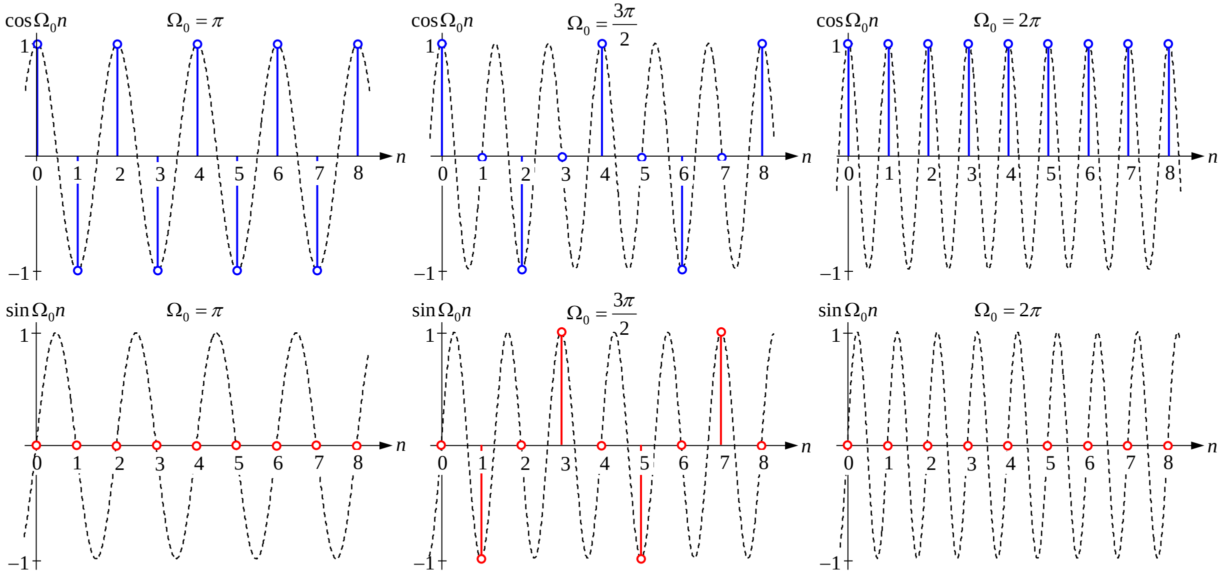


- The figure shows the DT sinusoids $\cos\Omega_0 n$ and $\sin\Omega_0 n$ for several values of Ω_0 between 0 and 2π. The initial phase ϕ has been set to zero. All the DT sinusoids shown are periodic. The dashed lines show the signal values if n were a continuous variable.

- As Ω_0 increases toward π, the DT sinusoids oscillate between –1 and +1 at an *increasing* rate.



- At $\Omega_0 = \pi$, $\cos\Omega_0 n = \cos\pi n = (-1)^n$, which represents the *highest rate of oscillation possible*. As Ω_0 exceeds π , the DT sinusoids oscillate between -1 and $+1$ at a *decreasing* rate. At $\Omega_0 = 2\pi$, the DT sinusoids are identical to those for $\Omega_0 = 0$. This illustrates that DT sinusoids whose fundamental frequencies differ by 2π (or a multiple of 2π) are identical.

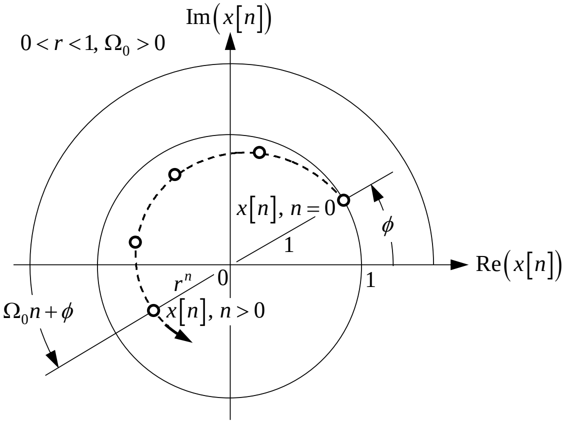


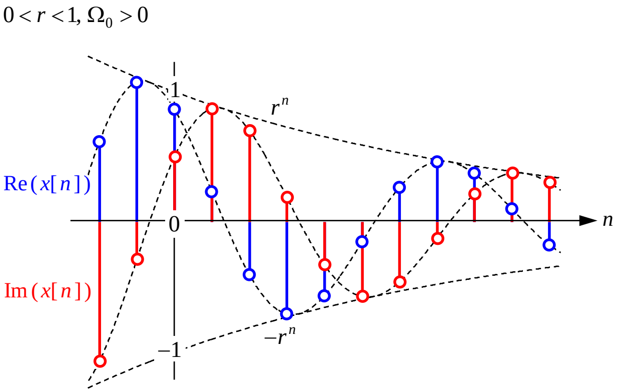
Complex Exponentials

- Starting with the general form of the DT exponential, we choose the constants as follows:
  - Let $\beta = \ln r + j\Omega_0$, where Ω_0 and r are real constants and $r > 0$. It follows that $\alpha = e^{\beta} = re^{j\Omega_0}$.
  - Let $C = e^{j\phi}$, where ϕ is a real constant.
- A DT complex exponential is of the form

$$\begin{aligned} x[n] &= e^{j\phi}\left(re^{j\Omega_0}\right)^n = r^n e^{j(\Omega_0 n + \phi)} \\ &= r^n\left[\cos\left(\Omega_0 n + \phi\right) + j\sin\left(\Omega_0 n + \phi\right)\right] \end{aligned}$$

- We can graph it one of two ways:
  - Left: as a point in a complex plane. It lies at a radius $\left|r^n e^{j(\Omega_0 n+\phi)}\right| = r^n$ and has a phase $\angle r^n e^{j(\Omega_0 n+\phi)} = \Omega_0 n + \phi$. Assuming $\Omega_0 > 0$ and $0 < r < 1$, it rotates counterclockwise and spirals inward as n increases.
  - Right: in terms of its real and imaginary components, $r^n \cos(\Omega_0 n+\phi)$ and $r^n \sin(\Omega_0 n+\phi)$. They oscillate within the envelope defined by $\pm r^n$. Assuming $0 < r < 1$, they are decaying (damped) sinusoids.





Continuous-Time Singularity Functions

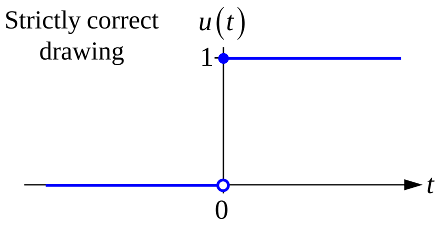
- Include impulse, step, ramp, etc.
- "Singularity" refers to value(s) of t at which the function or its derivatives are not defined.

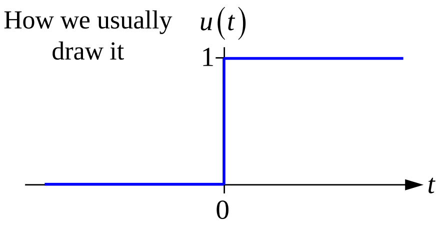
Unit Step Function

- The CT unit step function is defined as

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}.$$

- A strictly correct drawing is shown on the left. We usually draw it as shown on the right.
  (We are not concerned about values assumed by CT signals precisely at finite discontinuities.)

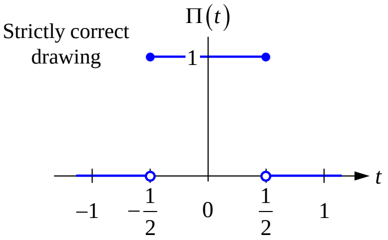


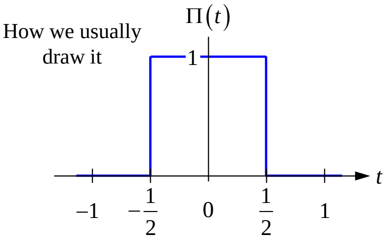


- The CT unit step function is often used as a building block to represent more complicated signals.

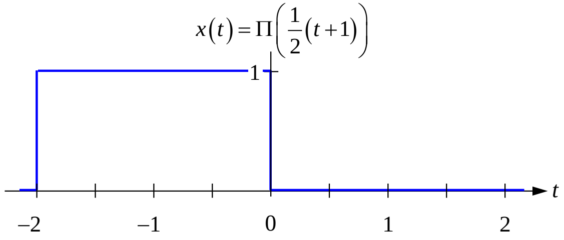
- *Example*: the unit rectangular pulse is defined and drawn as shown (see Appendix).

$$\Pi(t) \stackrel{d}{=} \begin{cases} 1 & |t| \le \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases}$$





- We are given a rectangular pulse of width 2 centered at $t = -1$:



- We can express it as a sum of scaled and shifted step functions:

$$x(t) = u(t+2) - u(t).$$

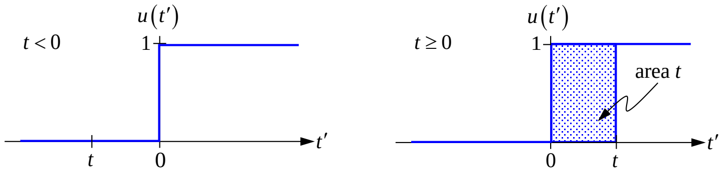
Expressing $x(t)$ in this form is useful if we input $x(t)$ to a linear, time-invariant system and wish to compute the output as a sum of scaled and shifted step responses (see Chapter 2).

Unit Ramp Function

- The CT unit ramp function is the running integral of the unit step function:

$$r(t)=\int_{-\infty}^{t} u(t')\,dt' .$$

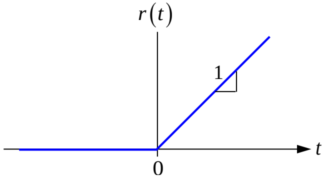
- We perform the integration considering the cases $t<0$ and $t\geq 0$.



We obtain

$$r(t)=\begin{cases}0 & t<0\\ t & t\geq 0\end{cases}=t\cdot u(t).$$

- This is shown below.



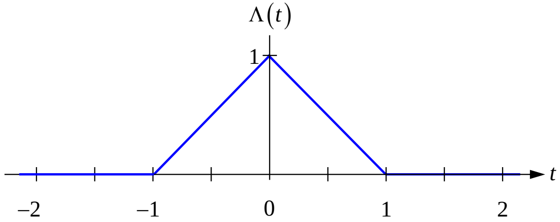
- Conversely, the unit step function is the derivative of the unit ramp function:

$$u(t)=\frac{d}{dt}r(t).$$

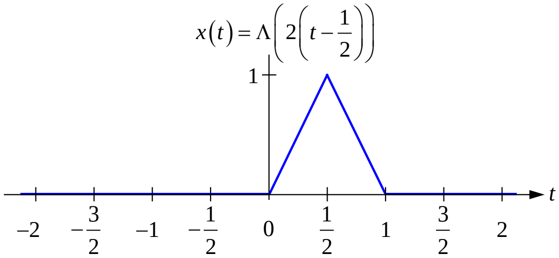
- The CT unit ramp function may be used to synthesize more complicated signals.

  Example: the unit triangular pulse, which has width 2, is defined and drawn as shown (see Appendix).

$$\Lambda(t) \overset{d}{=} \begin{cases} 1-|t| & |t| \le 1 \\ 0 & |t| > 1 \end{cases}$$



- We are given a triangular pulse of width 1 centered at $t = 1/2$:



- We can express it as a sum of scaled and shifted ramp functions:

$$x(t) = 2r(t) - 4r\left(t - \frac{1}{2}\right) + 2r(t-1).$$

Unit Impulse Function (Dirac Delta Function)

- The CT unit impulse function is a tool for analyzing systems. It is an idealization of a pulse whose duration is very short in comparison to the response time of the system.

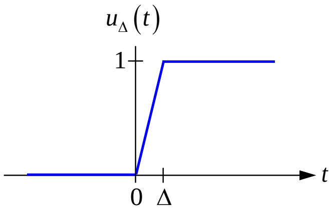
- The unit impulse function $\delta(t)$ is formally defined as the derivative of the unit step function:

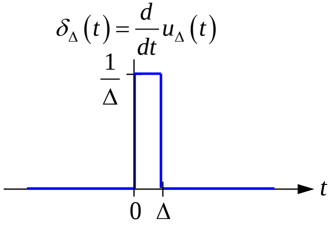
$$\delta(t) \stackrel{d}{=} \frac{d}{dt} u(t).$$

  Since $u(t)$ is discontinuous, it is not straightforward to evaluate its derivative.

- We approximate $u(t)$ by $u_\Delta(t)$, which is continuous at all t.

  We approximate $\delta(t)$ by $\delta_\Delta(t)$, the derivative of $u_\Delta(t)$.





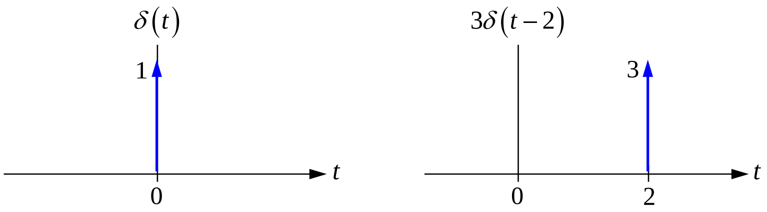
- In the limit $\Delta \to 0$, $u_\Delta(t)$ becomes $u(t)$:

$$u(t) = \lim_{\Delta \to 0} u_\Delta(t).$$

In that limit, $\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t)$ becomes $\delta(t)$:

$$\delta(t) = \lim_{\Delta \to 0} \frac{d}{dt} u_\Delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t).$$

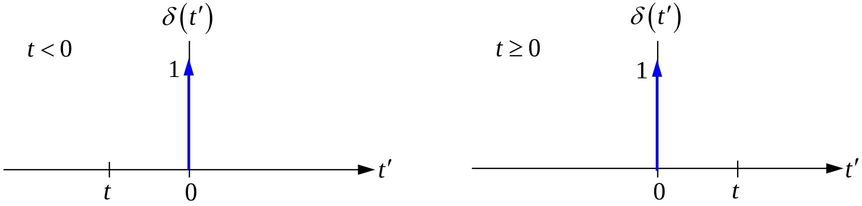
- A unit impulse function $\delta(t)$ is a rectangular pulse $\delta_\Delta(t)$ at $t = 0$ that has *zero width*, *infinite height* and *unit area*.
- We draw the unit impulse function as an arrow, labeled by the area under the function.



- Since the unit impulse is the derivative of the unit step, conversely, the unit step is the running integral of the unit impulse:

$$u(t) = \int_{-\infty}^{t} \delta(t')dt' .$$

- We perform the integration considering the cases $t < 0$ and $t \geq 0$.



We obtain

$$\int_{-\infty}^{t} \delta(t')dt' = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} = u(t) .$$

Properties of the Unit Impulse Function

1. *Evenness*

$$\delta(-t)=\delta(t)\ .$$

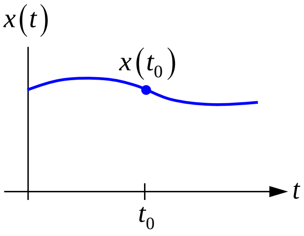
2. *Time Scaling*. Let a be a real constant.

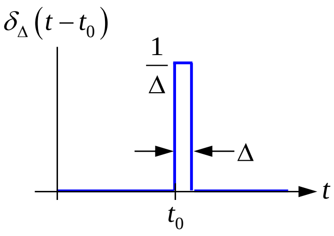
$$\delta(at)=\frac{1}{|a|}\delta(t).$$

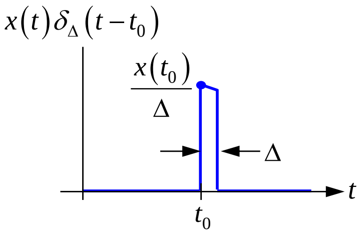
3a. *Sampling*. Assume $x(t)$ is continuous at $t=t_0$. Then

$$x(t)\delta(t-t_0)=x(t_0)\delta(t-t_0).$$

- To prove it, represent $\delta(t-t_0)$ by $\delta_\Delta(t-t_0)=\frac{d}{dt}u_\Delta(t-t_0)$. Then take the limit $\Delta\rightarrow 0$.







3b. Sampling (Alternate Statement)

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_0)\,.$$

- This form is obtained by integrating property 3a over time.

3c. Sifting ***(defer to Chapter 2)***

$$x(t) = \int_{-\infty}^{\infty} x(t')\delta(t-t')\,dt'\,.$$

- The sifting property is mathematically equivalent to the sampling property 3b.

  It is obtained from property 3b by using evenness property 1 and changing variables.

- The sifting property is used in Chapter 2 to analyze linear time-invariant systems.

  Its interpretation there is entirely different from that of the sampling property.

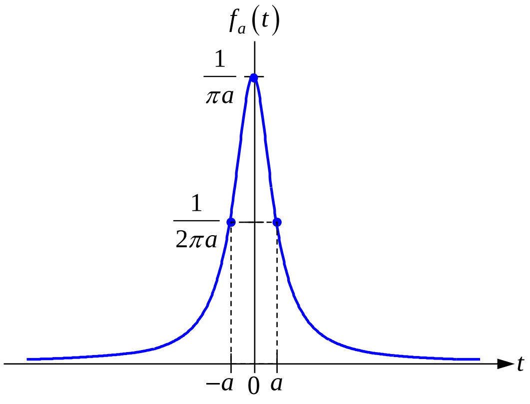
Alternate Representation of the Unit Impulse Function ***(defer to Chapter 4)***

- It is often useful to represent $\delta(t)$ using a smooth (differentiable) function that is peaked near $t = 0$, in the limit that the function has zero width and infinite height, while maintaining unit area.

- As an example, consider the function

$$f_a(t) = \frac{a}{\pi} \frac{1}{a^2 + t^2} .$$

This is sometimes called the *Lorentzian line shape function*. It is plotted here.



- The real parameter a governs the height and width of $f_a(t)$:
  - Height $f_a(0) = \frac{1}{\pi a}$.
  - Full width at half-height $2a$, since $\frac{f_a(\pm a)}{f_a(0)} = \frac{1}{2}$.
  - Unit area $\int_{-\infty}^{\infty} f_a(t)\,dt = 1$ for any value of a.
- Hence, we can represent the unit impulse $\delta(t)$ as $f_a(t)$ in the limit that a becomes small:

$$\delta(t) = \lim_{a \to 0} f_a(t).$$

This representation of the unit impulse is used in deriving some CT Fourier transforms in Chapter 4.

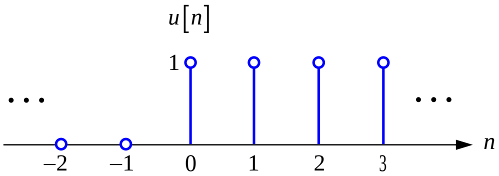
Discrete-Time Singularity Functions

- They are not actually singular but are named by analogy to their CT counterparts.

Unit Step Function

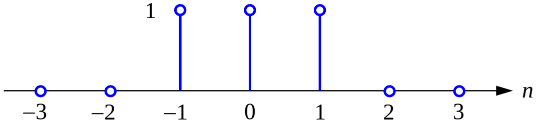
- The DT unit step function is defined as

$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}.$$



- The DT step function is often used as a building block to construct other DT signals.
  Example: we are given a rectangular pulse of width 2 centered at $n = 0$:

$$x[n] = \Pi\left(\frac{n}{2}\right)$$



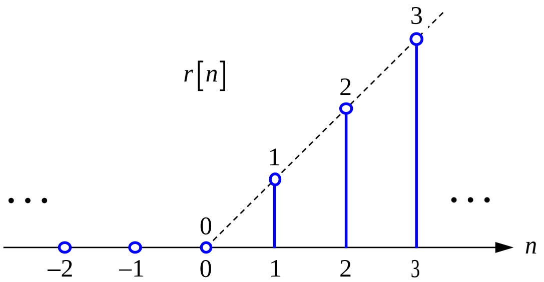
- We can express it as a sum of scaled and shifted step functions:

$$x[n] = u[n+1] - u[n-2].$$

Unit Ramp Function

- The DT unit ramp function is defined as

$$r[n] = \begin{cases} 0 & n < 0 \\ n & n \geq 0 \end{cases} = n \cdot u[n].$$



- It is related to the DT unit step function through running summation and first difference:

$$r[n] = \sum_{k=-\infty}^{n-1} u[k]$$

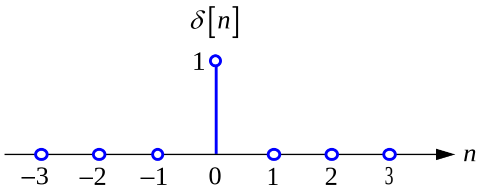
$$u[n] = r[n+1] - r[n].$$

Unit Impulse Function

- The DT unit impulse function is defined as

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}.$$

- It is graphed as shown.



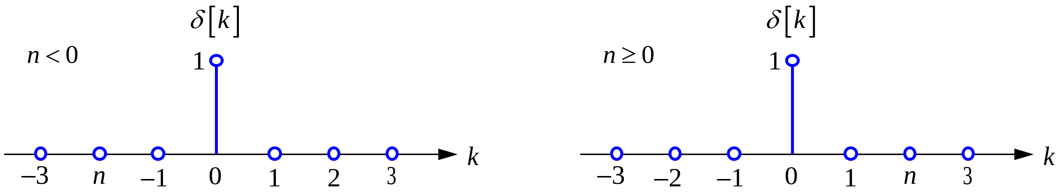
- The unit impulse function $\delta[n]$ is the first difference of the unit step function $u[n]$:

$$\delta[n] = u[n] - u[n-1].$$

- The unit step function $u[n]$ is also the running summation of the unit impulse function:

$$u[n]=\sum_{k=-\infty}^{n}\delta[k].$$

To perform the summation, we consider the cases $n<0$ and $n\geq 0$, as shown below.



We obtain

$$\sum_{k=-\infty}^{n}\delta[k]=\begin{cases}0 & n<0\\ 1 & n\geq 0\end{cases}=u[n].$$

Properties of the Unit Impulse Function

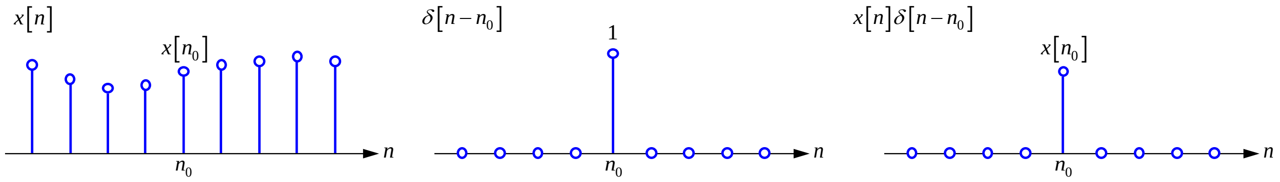
1. *Evenness*

$$\delta[-n]=\delta[n] .$$

2a. *Sampling*. Given a signal $x[n]$,

$$x[n]\delta[n-n_0]=x[n_0]\delta[n-n_0].$$

- This property follows from the fact that $\delta[n-n_0]$ is nonzero only for $n=n_0$, as illustrated below.



2b. Sampling (Alternate Statement)

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n-n_0] = x[n_0].$$

- This form of the sampling property is obtained by summing sampling property 2a over time.

2c. Sifting ***(defer to Chapter 2)***

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

- The sifting property is mathematically equivalent to the sampling property 2b.

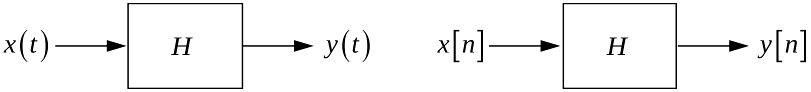
  It is obtained from property 2b by using evenness property 1 and changing variables.

- The sifting property is used in Chapter 2 in the analysis of linear time-invariant systems.

Representing Systems

Block Diagram

- A CT or DT system *H*, having input *x* and output *y*, can be represented as a block diagram, as shown.



Symbolic

- A CT or DT system *H*, having input *x* and output *y*, can be represented with the following notation:

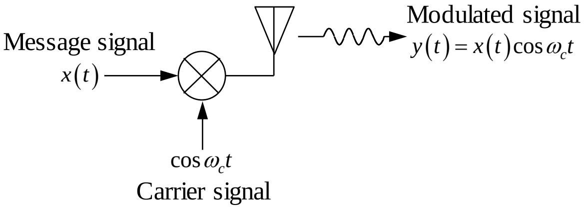
$$y(t) = H\{x(t)\} \qquad\qquad y[n] = H\{x[n]\}.$$

We may read such an expression as "the system *H* acts on input *x* to yield output *y*".

System Examples

1. CT Amplitude Modulation

- *Amplitude modulation* (AM) is used to shift a signal from low frequencies to high frequencies, often so it can be transmitted as an electromagnetic wave.



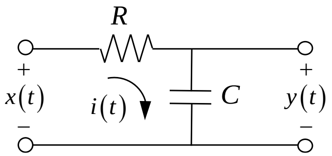
The $\otimes$ denotes a multiplier.

- In this form of AM, an information-bearing *message signal* $x(t)$ is multiplied by a sinusoidal carrier $\cos\omega_c t$, where ω_c is a *carrier frequency*, yielding a *modulated signal*

$$y(t)=x(t)\cos\omega_c t \ . \tag{3}$$

2. CT First-Order Lowpass Filter

- A CT first-order lowpass filter may be used intentionally to smooth out unwanted noise or high-frequency components appearing in CT signals.
- In circuits, resistance, inductance and capacitance are unavoidable, causing lowpass filtering even when it is not desired, potentially attenuating or distorting high-frequency signals.



- Using elementary circuit analysis, we can relate the input voltage $x(t)$ and output voltage $y(t)$ to the current $i(t)$:

$$x(t) = i(t)R + y(t)$$

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} i(t')dt' .$$

We obtain a first-order differential equation relating $x(t)$ and $y(t)$:

$$\tau\frac{dy}{dt} + y(t) = x(t), \qquad (4)$$

where $\tau = RC$.

- We will assume that the system is at *initial rest*. In general, an initial rest condition means that the *output is zero before any input is applied*. Here we assume that the input $x(t)$ is specified explicitly for all time starting at $t = -\infty$, so the initial rest condition is expressed as

$$y(t) = 0 \text{ at } t = -\infty . \tag{5}$$

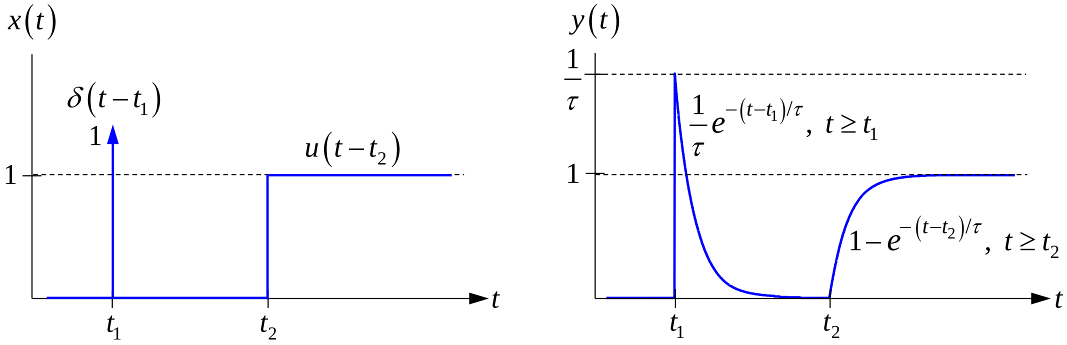
The unique solution to differential equation (4) satisfying initial condition (5) is

$$y(t) = \frac{1}{\tau} \int_{-\infty}^{t} e^{-\frac{t-t'}{\tau}} x(t') dt' . \tag{6}$$

- Some comments about the solution (6):
  - The output $y(t)$ at time t is a weighted sum of inputs $x(t')$ at all past and present times $t' \leq t$, with weighting factor $(1/\tau)e^{-\frac{t-t'}{\tau}}$.
  - This weighting factor gives more weight to recent inputs (small $t - t'$) and less weight to older inputs (large $t - t'$).
  - Expression (6) is in the form of a *convolution integral* between the input signal $x(t)$ and an impulse response $h(t)$ describing the system. This will be explained in Chapter 2.

- EE 102A does not teach how to solve differential equations or difference equations.
  EE 102B teaches how to solve them using Laplace and Z transforms.

- A representative input signal $x(t)$ and the resulting output signal $y(t)$ are shown below.
  These will be explained in Chapter 2.



3. DT Squarer

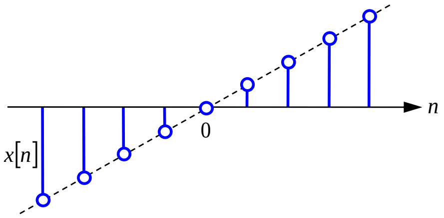
- A DT squaring system is shown.

$$x[n] \longrightarrow \boxed{(\ \ )^2} \longrightarrow y[n] = x^2[n]$$

Given an input signal $x[n]$, the output signal is

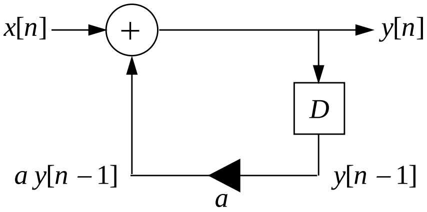
$$y[n] = x^2[n]. \tag{7}$$

- *Question*: here is an exemplary input signal. Can you sketch the corresponding output signal?

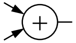


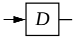
4. DT First-Order Lowpass Filter

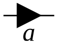
- Can be realized by the block diagram shown.



Symbols:

 Addition.

 Delay by a time step. May be realized in hardware by a shift register (digital or analog).

 Multiplication by a constant a.

- The input $x[n]$ and output $y[n]$ are related by a difference equation

$$y[n] = x[n] + ay[n-1]. \tag{8}$$

This is a *first-order* difference equation because at time n, the output $y[n]$ is dependent on past outputs back to $y[n-1]$.

- Assuming a is real and $0 \le a \le 1$, this system is a *lowpass filter*.
  May be used to smooth out unwanted noise or high-frequency components of DT signals.
  Using other values of a, this system is a *highpass filter* or *accumulator* (see below).

- Assume the system is at *initial rest*, so the output is zero before any input is applied:

$$y[n] = 0 \text{ at } n = -\infty . \tag{9}$$

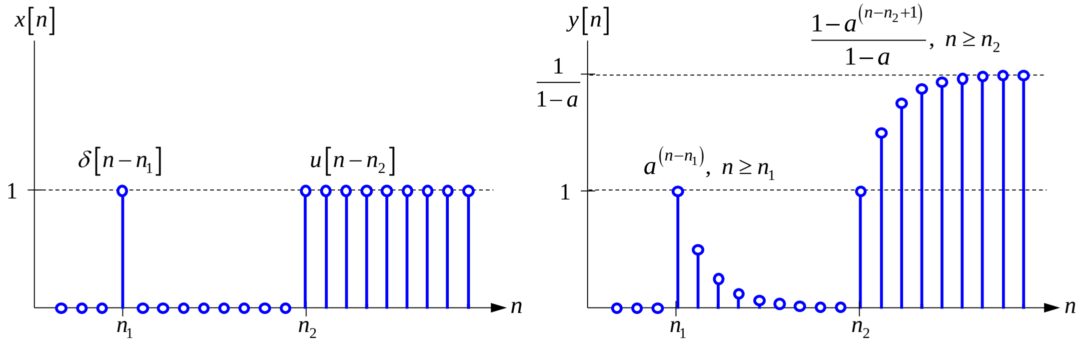
The unique solution to difference equation (8) satisfying initial condition (9) is

$$y[n] = \sum_{k=-\infty}^{n} a^{n-k} x[k] . \tag{10}$$

Some comments about the solution (10):

- It is a weighted sum of inputs at all past and present times $k \le n$, with weighting factor a^{n-k}.
- If $0 \le a \le 1$, the filter gives more weight to recent inputs (small $n-k$) and less weight to older inputs (large $n-k$).
- It is a *convolution sum* between the input signal $x[n]$ and an impulse response $h[n]$ describing the system. This will be explained in Chapter 2.

- A representative input signal $x[n]$ and the resulting output signal $y[n]$ are shown below. These will be explained fully in Chapter 2.



- The real constant *a* determines the type of filter.

System	**Value of *a***	**Stable**
Highpass filter	$-1 < a < 0$	Yes
Lowpass filter	$0 < a < 1$	Yes
Running summation (accumulation)	$a = 1$	No
Compound interest	$1 < a < \infty$ (typically)	No

  - In compound interest applications, $a > 1$ provided the interest rate is positive.
  - The property of *stability* is discussed shortly.

Properties of Systems

- It is useful to classify a system according to which properties it satisfies. We consider:
    - Stability
    - Memory
    - Invertibility
    - Time invariance
    - Linearity
    - Causality
- A system's properties determine:
    - The tools that can be used to analyze it.
    - How it can be realized in hardware or software.
    - Linear, time-invariant systems can be analyzed and realized using several powerful methods (see Chapters 2, 3, 4, 5).
- The definition stated here for each property is relevant for both CT and DT systems unless noted otherwise. If it is given for only one of the two types, extension to the other type is considered obvious.

Stability

- An unstable system may be difficult to realize or use (depending on how it is used).
- *DT definition*: a DT system is *bounded-input, bounded-output stable* (BIBO stable) if and only if every bounded input induces a bounded output. Mathematically, the system is BIBO stable if and only if

$$\left|x[n]\right| \le M_x < \infty \ \ \forall n \ \ \Rightarrow \ \ \left|y[n]\right| \le M_y < \infty \ \ \forall n ,$$

  where M_x and M_y are positive real constants.

- The definition for CT is analogous.

Examples

1. *CT Amplitude Modulation.* Assume the input $x(t)$ is bounded:

$$\left|x(t)\right| \le M_x < \infty \ \ \forall t .$$

   Using the input-output relation $y(t) = x(t)\cos\omega_c t$, the magnitude of the output $y(t)$ satisfies

$$\left|y(t)\right| = \left|x(t)\right|\left|\cos\omega_c t\right| \le M_x < \infty \ \ \forall t ,$$

   and the system is *stable*.

2. *CT First-Order Lowpass Filter*. Assume the input $x(t)$ is bounded:

$$\left|x(t)\right| \leq M_x < \infty \quad \forall t \, .$$

Using the input-output relation

$$y(t) = \frac{1}{\tau} \int_{-\infty}^{t} e^{-\frac{t-t'}{\tau}} x(t') dt' \, , \tag{6}$$

the magnitude of the output $y(t)$ satisfies

$$\left|y(t)\right| \leq \frac{1}{\tau} \int_{-\infty}^{t} \left| \underbrace{e^{-\frac{t-t'}{\tau}}}_{\text{positive}} x(t') \right| dt' \leq \frac{1}{\tau} \int_{-\infty}^{t} e^{-\frac{t-t'}{\tau}} M_x \, dt' = M_x < \infty \quad \forall t \, ,$$

and the system is *stable*.

3. *DT Squarer*. Assume the input $x[n]$ is bounded:

$$\left|x[n]\right| \leq M_x < \infty \quad \forall n \, .$$

Using the input-output relation $y[n] = x^2[n]$, the magnitude of the output $y[n]$ satisfies

$$\left|y[n]\right| \leq M_x^2 < \infty \quad \forall n \, ,$$

and the system is *stable*.

4. *DT First-Order Lowpass Filter*. Assume the input $x[n]$ is bounded:

$$\left|x[n]\right| \leq M_x < \infty \quad \forall n .$$

Using the input-output relation

$$y[n] = \sum_{k=-\infty}^{n} a^{n-k} x[k], \tag{10}$$

and recalling $0 < a < 1$, the magnitude of the output $y[n]$ satisfies

$$\left|y[n]\right| \leq \sum_{k=-\infty}^{n} \left| \underbrace{a^{n-k}}_{\text{positive}} x[k] \right| \leq \sum_{k=-\infty}^{n} a^{n-k} M_x .$$

Changing the variable of summation to $m = n - k$, this becomes

$$\left|y[n]\right| \leq M_x \sum_{m=\infty}^{0} a^m = M_x \sum_{m=0}^{\infty} a^m = \frac{M_x}{1-a} < \infty .$$

We have summed the geometric series using (see Appendix)

$$\sum_{m=0}^{\infty} a^m = \frac{1}{1-a}, \ |a| < 1 .$$

The system is *stable*. We will see that the system is unstable for some other choices of *a*.

Memory

- A system with memory store value(s) of physical or numerical variable(s) internally.
- *Definition*: a system is *memoryless* if, at any time, the value of the output depends only on the present value of the input. If the output depends on past or future values of the input, the system has *memory*.

Examples

1. *CT Amplitude Modulation*

$$y(t) = x(t)\cos\omega_c t\ . \tag{3}$$

The output $y(t)$ depends only on the present input $x(t)$. The system is *memoryless*.

2. *CT First-Order Lowpass Filter*

$$y(t) = \frac{1}{\tau}\int_{-\infty}^{t} e^{-\frac{t-t'}{\tau}} x(t')\,dt'\ . \tag{6}$$

- *Question*: does the system have memory? If so, where is the storage?

3. *DT Squarer*

$$y[n] = x^2[n]. \tag{7}$$

- *Question*: does the system have memory? If so, where is the storage?

4. DT First-Order Lowpass Filter

$$y[n]=\sum_{k=-\infty}^{n} a^{n-k}x[k]. \tag{10}$$

- *Question*: does the system have memory? If so, where is the storage?

Invertibility

- *Example*: signals fed back from a speaker to a microphone may cause *echoes*. If the echo-generation system is invertible, the echoes can be removed using an *echo canceler* (see Homework 7).

- *General definition*: A system is *invertible* if the input can always be recovered from the output.

- *DT Definition*. A DT system H with input $x[n]$ and output $y[n]=H\{x[n]\}$ is invertible if and only if there exists a stable *inverse system* H^{-1} such that

$$x[n]=H^{-1}\{y[n]\} \quad \forall x[n],\, y[n].$$

The system and inverse system satisfy $H^{-1}H=I$, where I is an *identity operator*.

- The definition for a CT system is analogous.

Examples

1. *CT Amplitude Modulation*

$$y(t) = x(t)\cos\omega_c t \ . \tag{3}$$

- The modulator system is *not invertible*. For example, if the input $x(t)$ includes impulses at zero crossings of $\cos\omega_c t$, the impulses will not appear in the output $y(t)$.
- Although the system is not invertible, if $x(t)$ is suitably bandlimited, then we can recover $x(t)$ from $y(t)$ using a system called a *demodulator* (see Chapters 4 and 7).

2. *CT First-Order Lowpass Filter*

- The system is *not invertible*. The lowpass filter attenuates high frequencies. Its inverse system would need to have unbounded gain at very high frequencies and would be unstable.
- Although the lowpass filter is not invertible, if the input $x(t)$ is bandlimited to a finite bandwidth, we can recover it from the output $y(t)$ using a stable inverse system (see Homework 7).

3. *DT Squarer*

$$y[n]=x^2[n]. \tag{7}$$

- *Question*: is the system invertible? Why or why not? If it is, describe the inverse system.

4. *DT First-Order Lowpass Filter*

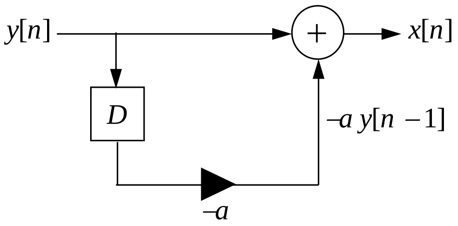
- The input $x[n]$ and output $y[n]$ are related by the difference equation

$$y[n]=x[n]+ay[n-1]. \tag{8}$$

We can solve this for a difference equation allowing us to obtain $x[n]$ from $y[n]$:

$$x[n]=y[n]-ay[n-1].$$

The inverse system, which realizes this difference equation, is shown.



This inverse system is stable for any $|a| < \infty$ (see Chapter 2). The system is *invertible*.

Time Invariance

- A system is *time-invariant* if any shift of the input signal leads only to an identical shift of the output signal.

- *CT Definition*. Given a system *H*, if

$$H\{x(t)\} = y(t),$$

then H is time-invariant if

$$H\{x(t-t_0)\} = y(t-t_0)$$

for any $x(t)$ and any t_0.

- Equivalently, we can define a *time-shift operator* D^{t_0}:

$$D^{t_0}\{x(t)\} = x(t-t_0).$$

A system *H* is time-invariant if

$$D^{t_0}\{H\{x(t)\}\} = H\{D^{t_0}\{x(t)\}\}$$

for any $x(t)$ and any t_0. The system H is time-invariant if these two block diagrams are equivalent.

$$\longrightarrow \boxed{D^{t_0}} \longrightarrow \boxed{H} \longrightarrow \qquad \longrightarrow \boxed{H} \longrightarrow \boxed{D^{t_0}} \longrightarrow$$

- The definition of time-invariance for a DT system is analogous.

Examples

1. *CT Amplitude Modulation.* The input-output relation (3) states

$$H\left\{x(t)\right\} = y(t) = x(t)\cos\omega_c t \; .$$

Time-shifting the input, the output is

$$H\left\{x(t-t_0)\right\} = x(t-t_0)\cos\omega_c t \; ,$$

while modulating the input and then time-shifting the output yields

$$y(t-t_0) = x(t-t_0)\cos\omega_c(t-t_0) \, .$$

These are different, and the system is *not time-invariant*.

2. *CT First-Order Lowpass Filter.* The input-output relation (6) states

$$H\left\{x(t)\right\} = y(t) = \frac{1}{\tau}\int_{-\infty}^{t} e^{-\frac{t-t'}{\tau}} x(t')dt' .$$

Time-shifting the input and then filtering it, the output is

$$H\left\{x(t-t_0)\right\} = \frac{1}{\tau}\int_{-\infty}^{t} e^{-\frac{t-t'}{\tau}} x(t'-t_0)dt' .$$

Changing the variable of integration to $t'' = t' - t_0$, so that $t' = t'' + t_0$, yields

$$H\left\{x(t-t_0)\right\} = \frac{1}{\tau}\int_{-\infty}^{t-t_0} e^{-\frac{t-t_0-t''}{\tau}} x(t'')dt'' .$$

Filtering the input and then time-shifting the output yields

$$y(t-t_0) = \frac{1}{\tau}\int_{-\infty}^{t-t_0} e^{-\frac{t-t_0-t'}{\tau}} x(t')dt'$$

The two previous expressions are equivalent. The system is *time-invariant*.

- Note, however, that if the system were not assumed to be at initial rest, it would not be time invariant (see Homework 2).

3. *DT Squarer*. The input-output relation (7) states

$$H\{x[n]\} = y[n] = x^2[n]$$

Time-shifting the input, the output is

$$H\{x[n-n_0]\} = x^2[n-n_0].$$

Squaring the input and then time-shifting the output yields

$$y[n-n_0] = x^2[n-n_0].$$

These are equivalent, and the system is *time-invariant*.

4. *DT First-Order Lowpass Filter*. The input-output relation (10) states

$$H\left\{x[n]\right\} = y[n] = \sum_{k=-\infty}^{n} a^{n-k} x[k]$$

Time-shifting the input and then filtering it, the output is

$$H\left\{x[n-n_0]\right\} = \sum_{k=-\infty}^{n} a^{n-k} x[k-n_0].$$

Changing the variable of summation to $m = k - n_0$, so that $k = m + n_0$, yields

$$H\left\{x[n-n_0]\right\} = \sum_{m=-\infty}^{n-n_0} a^{n-n_0-m} x[m].$$

Filtering the input and then time-shifting the output yields

$$y[n-n_0] = \sum_{k=-\infty}^{n-n_0} a^{n-n_0-k} x[k].$$

The two previous expressions are equivalent. The system is *time-invariant*.

- Like the CT lowpass filter, if the system were not assumed to be at initial rest, it would not be time invariant.

Linearity

- A system is *linear* if, given an input that is a weighted sum of several signals, the output is the weighted sum of the responses of the system to each of the signals.

- *DT Definition*. A DT system H is *linear* if

$$H\left\{\sum_{i=1}^{N} a_i x_i[n]\right\} = \sum_{i=1}^{N} a_i H\left\{x_i[n]\right\}$$

for any constants a_i and signals $x_i[n]$, $i = 1, \ldots, N$.

- The definition of linearity for a CT system is analogous.

Examples

1. *CT Amplitude Modulation.* The system is *linear*:

$$H\left\{a_1 x_1(t) + a_2 x_2(t)\right\} = a_1 H\left\{x_1(t)\right\} + a_2 H\left\{x_2(t)\right\} = a_1 x_1(t)\cos\omega_c t + a_2 x_2(t)\cos\omega_c t \ .$$

2. *CT First-Order Lowpass Filter*. The system is *linear*:

$$H\left\{a_1x_1(t)+a_2x_2(t)\right\}=a_1H\left\{x_1(t)\right\}+a_2H\left\{x_2(t)\right\}=\frac{a_1}{\tau}\int_{-\infty}^{t}e^{-\frac{t-t'}{\tau}}x_1(t')dt'+\frac{a_2}{\tau}\int_{-\infty}^{t}e^{-\frac{t-t'}{\tau}}x_2(t')dt' .$$

- If the system were not assumed to be at initial rest, it would not be linear (see Homework 2).

3. *DT Squarer*. The system is *nonlinear*, owing to the squaring operation:

$$H\left\{a_1x_1[n]+a_2x_2[n]\right\}=a_1^2x_1^2[n]+a_2^2x_2^2[n]+2a_1a_2x_1[n]x_2[n],$$

whereas

$$a_1H\left\{x_1[n]\right\}+a_2H\left\{x_2[n]\right\}=a_1x_1^2[n]+a_2x_2^2[n].$$

4. *DT First-Order Lowpass Filter*. The system is *linear*:

$$H\left\{a_1x_1[n]+a_2x_2[n]\right\}=a_1H\left\{x_1[n]\right\}+a_2H\left\{x_2[n]\right\}=a_1\sum_{k=-\infty}^{n}a^{n-k}x_1[k]+a_2\sum_{k=-\infty}^{n}a^{n-k}x_2[k].$$

- Like the CT lowpass filter, if the system were not assumed to be at initial rest, it would not be linear.

Causality

- A system is *causal* if, at any given time, the value of the output depends only on present and past values of the input, not on future values of the input.

- *CT Definition*. A CT system with input $x(t)$ and output $y(t)$ is *causal* if $y(t)$ depends only upon $x(t-t')$, $t' \geq 0$. The definition for a DT system is analogous.

- Non-causal systems:

  - Cannot operate on signals in real time.
    Can operate on signals that have been stored or time-delayed.

  - Are useful analytical tools.
    Example: a system to advance DT signals by three time steps is non-causal:

$$D^{-3}\{x[n]\} = x[n+3].$$

Examples

1. *CT Amplitude Modulation*. The system is *causal*.

2. *CT First-Order Lowpass Filter*. The system is *causal*.

3. *DT Squarer*. The system is *causal*.

4. *DT First-Order Lowpass Filter*. The system is *causal*.