Stanford University
EE 102A: Signal Processing and Linear Systems I
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Chapter 4: The Continuous-Time Fourier Transform

Motivations

- The Fourier series (FS) expresses a *periodic* signal as a *discrete sum* of imaginary exponentials.
  - In CT, these are $e^{jk\omega_0 t}$, ω_0 real, *k* integer.
  - This simplifies our analysis of LTI systems with periodic inputs.
    To compute the output signal, we multiply each $e^{jk\omega_0 t}$ by $H(jk\omega_0)$.
- $H(j\omega)$ is a *frequency response* that characterizes the input-output relation of a CT LTI system.
- The Fourier transform (FT) expresses an *aperiodic* signal as a *continuous integral* of imaginary exponentials.
  - In CT, these are $e^{j\omega t}$, ω real, $-\infty < \omega < \infty$.
  - This simplifies our analysis of LTI systems with aperiodic inputs.
    To compute the output signal, we multiply each $e^{j\omega t}$ by $H(j\omega)$.
- In Chapter 4, using the CTFT, we will:
  - Analyze aperiodic (and periodic) signals and compute the frequency response $H(j\omega)$ for various systems.
  - Study modulation and demodulation, which are essential for communications.

Major Topics in This Chapter

- Continuous-time Fourier transform
  - Derivation for aperiodic signals.
  - Fourier transforms in the limit.
  - Fourier transforms of periodic signals.
  - Properties of Fourier transforms.
- Convolution property and LTI system analysis
  - Frequency response as CTFT of impulse response.
  - LTI systems not described by finite-order differential equations:
    - Time shift, finite-time integrator, ideal lowpass filter.
  - LTI systems described by linear, constant-coefficient differential equations.
    - First-order: integrator, lowpass filter, highpass filter.
    - Second-order: lowpass filter, bandpass filter.
- Multiplication property
  - Amplitude modulation and demodulation.

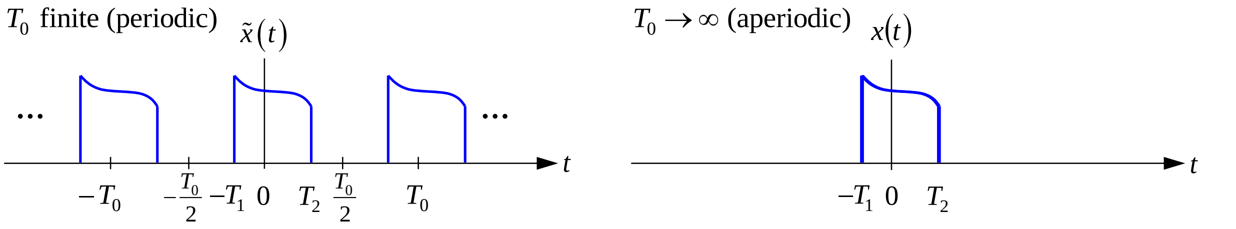
Continuous-Time Fourier Transform

Derivation of Continuous-Time Fourier Transform

- We are given an aperiodic CT signal $x(t)$, which is nonzero only over an interval $-T_1 \le t \le T_2$.

  Consider $x(t)$ to be a periodic CT signal $\tilde{x}(t)$, of period T_0, in the limit $T_0 \to \infty$.

  In that limit, $\tilde{x}(t)$ becomes $x(t)$, as shown.



- We start by representing the periodic signal $\tilde{x}(t)$ as a CT Fourier series (CTFS) with fundamental frequency $\omega_0 = \dfrac{2\pi}{T_0}$ and CTFS coefficients a_k, $-\infty < k < \infty$:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. \qquad \text{(CTFS synthesis)} \qquad (1)$$

- We can obtain the CTFS coefficients by performing analysis over any interval of duration T_0.

  We assume the analysis interval includes the interval $-T_1 \le t \le T_2$, over which $x(t)$ is nonzero.

  The CTFS coefficients of $\tilde{x}(t)$ are

$$a_k = \frac{1}{T_0} \int_{T_0} \tilde{x}(t) e^{-jk\omega_o t} dt . \qquad \text{(CTFS analysis)} \qquad (2)$$

- Note that $\tilde{x}(t) = x(t)$ within the analysis interval and $x(t) = 0$ outside the interval.

  So we can rewrite the CTFS analysis equation (2) as

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_o t} dt = \frac{1}{T_0} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_o t} dt . \qquad (3)$$

- Now we define $X(j\omega)$, a function of a continuous frequency variable ω.

  We compute $X(j\omega)$ from $x(t)$ using an integral

$$X(j\omega) \stackrel{d}{=} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt . \qquad \text{(CTFT or CTFT analysis)} \qquad (4)$$

  We refer to $X(j\omega)$ as the *CT Fourier transform* (CTFT) of the aperiodic signal $x(t)$.

  We refer to the integral (4) as the *CTFT analysis equation*, or simply the *CTFT*.

- Comparing (3) and (4), observe that we can obtain the a_k by sampling the $X(j\omega)$ at multiples of the fundamental frequency and scaling by $\frac{1}{T_0}$:

$$\frac{1}{T_0} X(j\omega)\Big|_{\omega=k\omega_0} = \frac{1}{T_0} X(jk\omega_0) = a_k \,. \tag{5}$$

- Using (5), we can rewrite the CTFS synthesis equation (1) for the periodic signal $\tilde{x}(t)$ as

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} X(jk\omega_0) e^{jk\omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0 \,. \tag{6}$$

We used $\frac{1}{T_0} = \frac{\omega_0}{2\pi}$ in the last step.

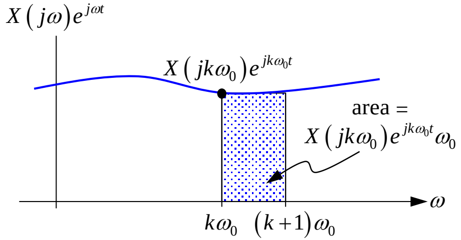
- Consider the limit $T_0 \to \infty$, in which the periodic signal $\tilde{x}(t)$ becomes the aperiodic signal $x(t)$:

$$k\omega_0 \to \omega \text{, a continuous variable}$$

$$\omega_0 \to d\omega \text{, an infinitesimal increment of } \omega$$

$$X(jk\omega_0) \to X(j\omega) \text{, a function of a continuous variable}$$

- This figure schematically shows $X(j\omega)e^{j\omega t}$ as a function of the continuous frequency variable ω.



- In the limit we are considering, the summation

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0 \,. \tag{6}$$

becomes a Riemann sum approximation of an integral

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \,. \qquad \text{(inverse CTFT or CTFT synthesis)} \tag{7}$$

- The integral (7) allows us to obtain the aperiodic signal $x(t)$ from $X(j\omega)$.
  We refer to (7) as the *inverse CTFT* or the *CTFT synthesis equation*.
  We refer to $x(t)$ as the *inverse CTFT* of $X(j\omega)$.

- In summary, we have derived the following two expressions.

$X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$	(CTFT or CTFT analysis)	(4)
$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$	(inverse CTFT or CTFT synthesis)	(7)

- The inverse CTFT integral (7) specifies how to *synthesize* an aperiodic signal $x(t)$ as a weighted sum of $e^{j\omega t}$, $-\infty<\omega<\infty$. The imaginary exponential $e^{j\omega t}$ at frequency ω is weighted by a factor $X(j\omega)$.

- The CTFT integral (4) specifies how, given an aperiodic signal $x(t)$, we can *analyze* $x(t)$ to obtain the weighting factor $X(j\omega)$.

- We may describe (4) and (7) in terms of a *CTFT operator F* and an *inverse CTFT operator* F^{-1}. Each operates on one function to produce the other:

$$F\left[x(t)\right] = X(j\omega), \tag{8}$$

and

$$F^{-1}\left[X(j\omega)\right] = x(t). \tag{9}$$

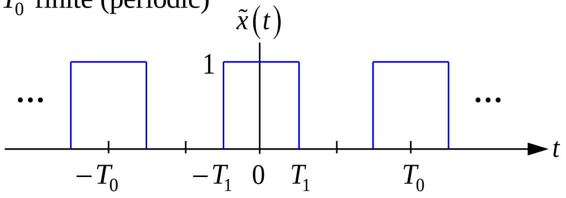
- We often denote a CT signal $x(t)$ and its CTFT $X(j\omega)$ as a *CTFT pair*:

$$x(t) \overset{F}{\leftrightarrow} X(j\omega). \tag{10}$$

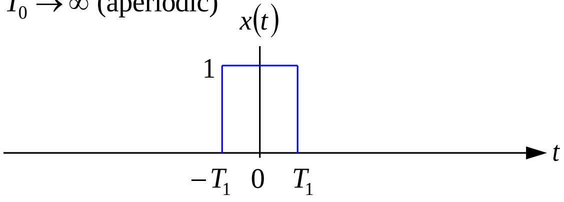
Fourier Series of a Rectangular Pulse Train in the Limit of a Long Period

- We illustrate our derivation of the CTFT by the example of a periodic rectangular pulse train.

T_0 finite (periodic)



$T_0 \to \infty$ (aperiodic)



- The periodic pulse train $\tilde{x}(t)$ shown above on the left has period $T_0 = \dfrac{2\pi}{\omega_0}$.

  We compute its CTFS coefficients using analysis equation (2) (see Chapter 3, slides 23-29 for details):

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x}(t) e^{-jk\omega_0 t} dt = \frac{1}{T_0}\int_{-T_1}^{T_1} e^{-jk\omega_0 t} dt = \frac{2T_1}{T_0}\operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\omega_0 T_1}{\pi}\operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right). \qquad (11)$$

- In the limit $T_0 \to \infty$, $\tilde{x}(t)$ becomes the single pulse $x(t)$ shown above on the right.

  Using (4), its CTFT is

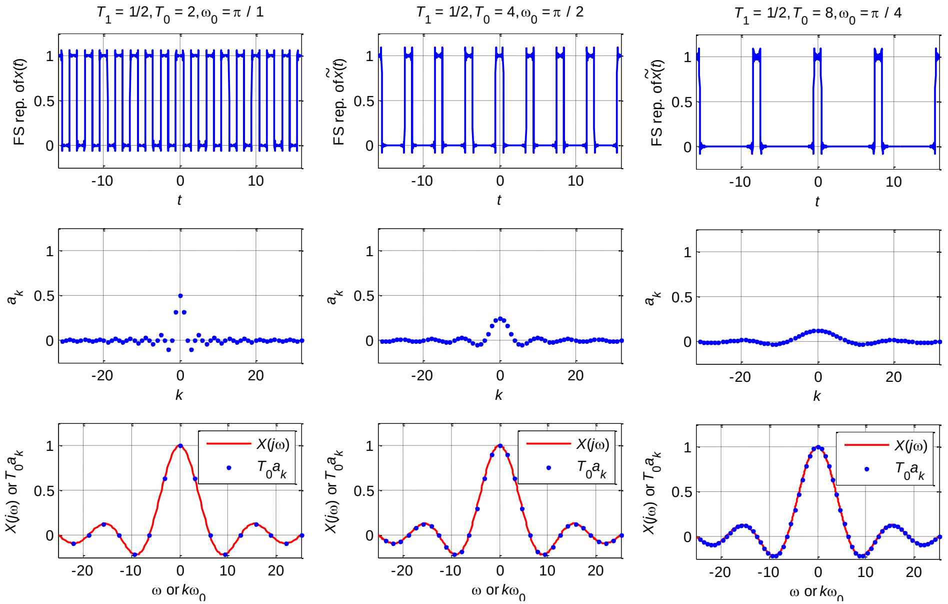
$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-T_1}^{T_1} e^{-j\omega t} dt = 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right). \qquad (12)$$

- By (5), sampling $X(j\omega)$ at $\omega = k\omega_0$ yields the CTFS coefficients of $\tilde{x}(t)$, the a_k, scaled by T_0:

$$X(j\omega)\Big|_{\omega=k\omega_0} = 2T_1 \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = T_0 a_k \,. \tag{13}$$

Using $\dfrac{2T_1}{T_0} = \dfrac{\omega_0 T_1}{\pi}$, we verify that (13) and (11) provide identical expressions for the a_k.

- The figure shows how the scaled CTFS coefficients $T_0 a_k$ approach the CTFT $X(j\omega)$ as we increase T_0. We hold the pulse width $2T_1$ fixed and increase the period T_0, decreasing the fundamental frequency ω_0.



- Top row: periodic pulse train $\tilde{x}(t)$.
- Middle row: its CTFS coefficients a_k vs. k.
- Bottom row: CTFT $X(j\omega)$ vs. ω and scaled CTFS coefficients $T_0 a_k$ vs. $k\omega_0$.
- As T_0 increases and ω_0 decreases, the samples $T_0 a_k = X(j\omega)\big|_{\omega = k\omega_0}$ become more closely spaced.
- In the limit $T_0 \to \infty$, $\omega_0 \to 0$, the samples reproduce the continuous curve $X(j\omega)$.

Alternate Analysis Method for Continuous-Time Fourier Series

- The CTFT derivation (slides 3-8) provides an alternate method to compute CTFS coefficients for periodic signals, which we summarize here.
- Suppose we are given a periodic signal $\tilde{x}(t)$ with period $T_0 = \dfrac{2\pi}{\omega_0}$.

  We can find its CTFS coefficients a_k, $-\infty < k < \infty$ by performing three steps:

1. Define an aperiodic signal $x(t)$ that represents one period of $\tilde{x}(t)$:

$$x(t) = \begin{cases} \tilde{x}(t) & t_1 \le t < t_1 + T_0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{for some } t_1 . \tag{14}$$

2. Compute the CTFT of the one-period signal $x(t)$ using

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt . \tag{4}$$

3. Sample the CTFT $X(j\omega)$ at integer multiples of the fundamental frequency to obtain the CTFS coefficients of the periodic signal $\tilde{x}(t)$:

$$a_k = \frac{1}{T_0} X(j\omega)\Big|_{\omega = k\omega_0} . \tag{5}$$

- This method may be easier to apply than the CTFS analysis equation (2), especially if you already know the CTFT of the one-period signal.
- This discussion provides a conceptual linkage between the CTFT and the CTFS:
  - According to (5), every set of CTFS coefficients represents samples of a CTFT.
  - All properties of the CTFS (Table 1, Appendix) are inherited from properties of the CTFT (Table 3, Appendix).

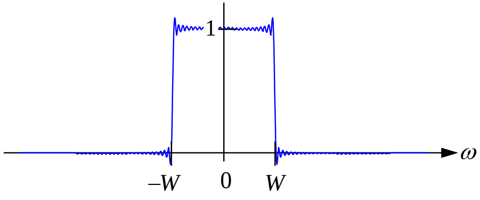
  These observations may simplify your learning of Fourier analysis.

- The CTFT integral (4) or inverse CTFT integral (7) may not converge for all values of ω or t.

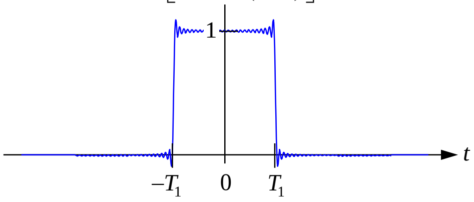
- If $X(j\omega)$ or $x(t)$ have discontinuities, their integral representations will exhibit nonuniform convergence, which will manifest as the Gibbs phenomenon.

- In the examples below, both the CTFT (left) and the inverse CTFT (right) are ideally rectangles. Both exhibit the Gibbs phenomenon.

$$F\left[\frac{W}{\pi}\mathrm{sinc}\left(\frac{Wt}{\pi}\right)\right]$$



$$F^{-1}\left[2T_1\mathrm{sinc}\left(\frac{\omega T_1}{\pi}\right)\right]$$



- We truncated the time signal or CTFT before applying the F or F^{-1} operator to make the Gibbs phenomenon more apparent in these figures.

(can skip slides 15-16.)

- We discuss the convergence of the inverse CTFT here. The convergence of the CTFT is analogous, owing to a duality between the CTFT integral (4) and the inverse CTFT integral (7). This duality is explained in Chapter 5.

- Given a signal $x(t)$ having a CTFT $X(j\omega)$, we denote the inverse CTFT representation of $x(t)$ by

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega .$$

- It can be shown that if $x(t)$ has finite energy (i.e., $x(t)$ is square integrable)

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty ,$$

  the energy in the difference between $x(t)$ and $\hat{x}(t)$ vanishes:

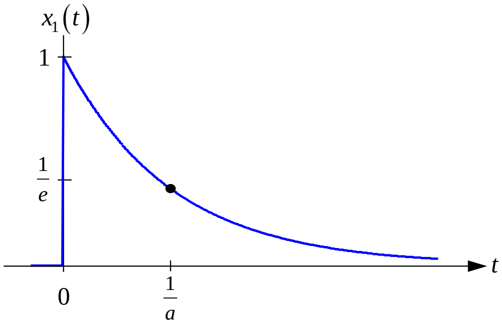
$$\int_{-\infty}^{\infty} |x(t) - \hat{x}(t)|^2 dt = 0 .$$

- This does not imply that $\hat{x}(t) = x(t)$ at all t. In fact, $\hat{x}(t)$ differs from $x(t)$ near values of t where $x(t)$ has discontinuities.

- It can be shown that $\hat{x}(t) = x(t)$ except near values of *t* where $x(t)$ has discontinuities if $x(t)$ satisfies the *Dirichlet conditions*:
  - $x(t)$ is absolutely integrable:
$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty .$$
  - $x(t)$ has a finite number of local maxima and minima in any finite interval.
  - $x(t)$ has a finite number of discontinuities in any finite interval.
  - Any discontinuities of $x(t)$ are finite.
- Many important functions (e.g., step functions, constants or sinusoids) are not square integrable or absolutely integrable. We are still able to compute their CTFTs (or inverse CTFTs) by taking a limit. As we will see shortly, the CTFTs (or inverse CTFTs) we obtain will contain impulse functions.

1. *Right-sided real exponential*. The signal and its CTFT are

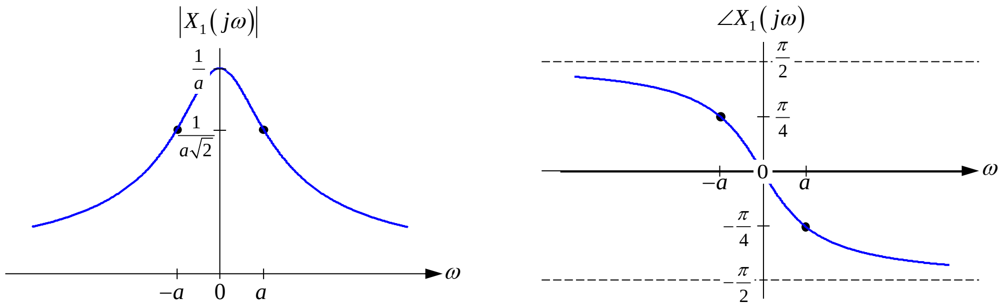
$$x_1(t) = e^{-at}u(t) \overset{F}{\leftrightarrow} X_1(j\omega) = \frac{1}{a+j\omega}, \ a \text{ real}, \ a > 0 .$$



- We compute $X_1(j\omega)$ using the CTFT (4):

$$\begin{aligned} X_1(j\omega) &= \int_0^\infty e^{-at}e^{-j\omega t}dt \\ &= -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^\infty . \\ &= \frac{1}{a+j\omega} \end{aligned}$$

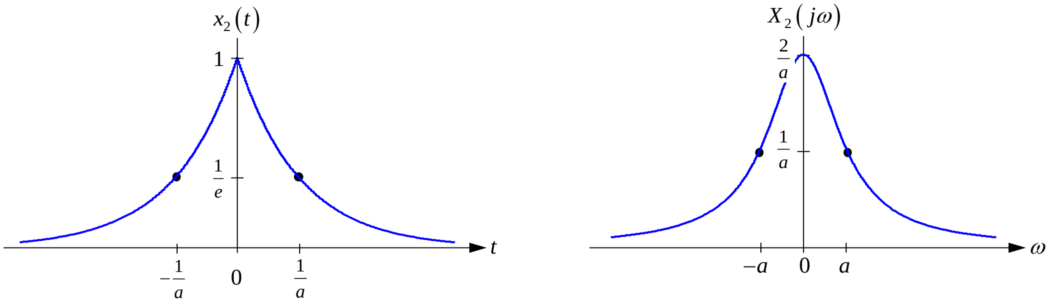
- This CTFT is complex-valued, so it is best visualized in magnitude and phase plots.



- $x_1(t)$ and $X_1(j\omega)$ are the impulse and frequency responses of a first-order lowpass filter with time constant $\tau = 1/a$ (see slides 79-81). Those responses are scaled by $\tau = 1/a$ in this example.
- Observe that:
  - Large a: the signal $x_1(t)$ is concentrated in time and the CTFT $X_1(j\omega)$ is spread out in frequency.
  - Small a: the signal $x_1(t)$ is spread out in time and the CTFT $X_1(j\omega)$ is concentrated in frequency.
- These observations illustrate the *inverse relationship between time and frequency*.
  All the following examples illustrate that principle.

2. *Two-sided real exponential*. The signal and its CTFT are given by

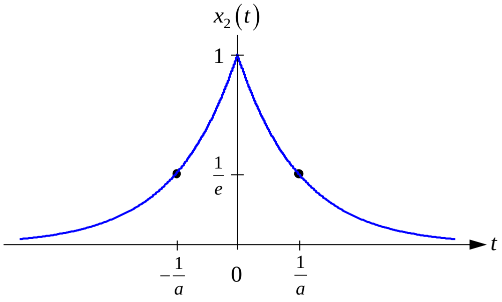
$$x_2(t) = e^{-a|t|} \overset{F}{\leftrightarrow} X_2(j\omega) = \frac{2a}{a^2+\omega^2}, \ a \text{ real}, \ a > 0 .$$

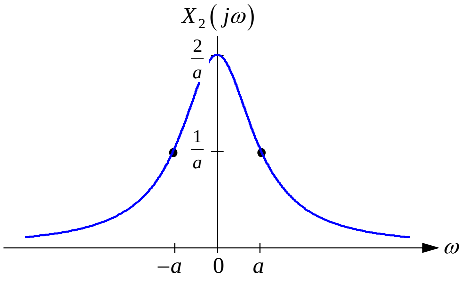


- We compute $X_2(j\omega)$ using (4). We divide the integral into two parts, each like the integral for $X_1(j\omega)$:

$$\begin{aligned} X_2(j\omega) &= \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt \\ &= \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt \\ &= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} \\ &= \frac{2a}{a^2+\omega^2} \end{aligned} .$$

- The CTFT $X_2(j\omega)$ is purely real, so is best visualized by plotting the real part.





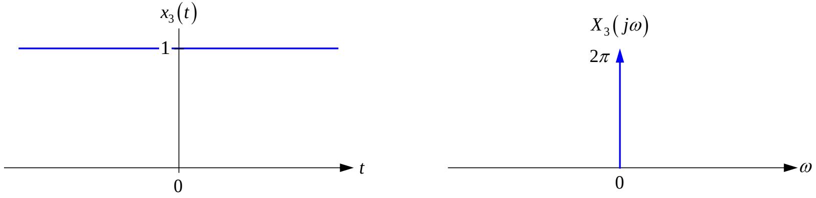
- As in Example 1, observe that:
  - Large *a*: the signal $x_2(t)$ is concentrated in time and the CTFT $X_2(j\omega)$ is spread out in frequency.
  - Small *a*: the signal $x_2(t)$ is spread out in time and the CTFT $X_2(j\omega)$ is concentrated in frequency.
- We saw a function similar to $X_2(j\omega)$, as a function of time, not frequency, in Chapter 1 (see slides 55-56).

  We used it there to represent an *impulse function of time*.

  In the following example, we use $X_2(j\omega)$ to represent an *impulse function of frequency*.

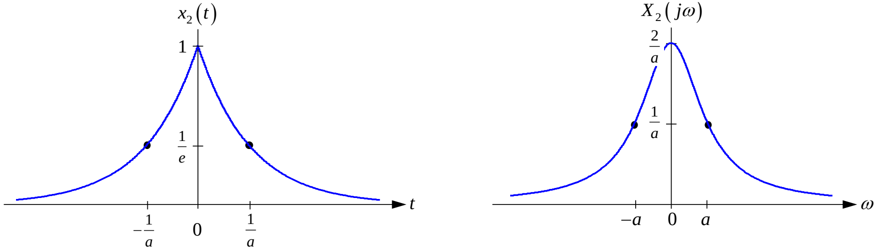
3. *Constant*. The signal and its CTFT are

$$x_3(t) = 1 \quad \forall t \quad \overset{F}{\leftrightarrow} \quad X_3(j\omega) = 2\pi\delta(\omega).$$



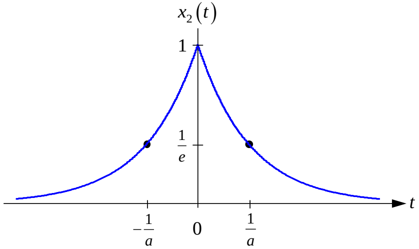
- To derive this, recall the CTFT pair from Example 2:

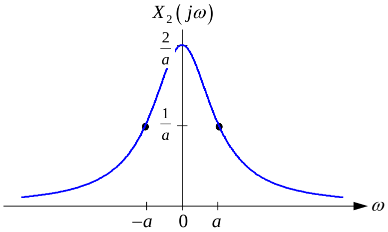
$$x_2(t) = e^{-a|t|} \quad \overset{F}{\leftrightarrow} \quad X_2(j\omega) = \frac{2a}{a^2+\omega^2}.$$



- In the limit $a \to 0$, the two-sided exponential $x_2(t) = e^{-a|t|}$ becomes the constant $x_3(t) = 1$.

$$x_3(t) = \lim_{a\to 0} x_2(t) = \lim_{a\to 0} e^{-a|t|} \quad \overset{F}{\leftrightarrow} \quad X_3(j\omega) = \lim_{a\to 0} X_2(j\omega) = \lim_{a\to 0} \frac{2a}{a^2+\omega^2}.$$





- Hence, in the limit $a \to 0$, the CTFT $X_2(j\omega)$ becomes the CTFT $X_3(j\omega)$.

  In this limit, the CTFT $X_2(j\omega) = \dfrac{2a}{a^2+\omega^2}$ has the following properties:

  - A peak of zero width and infinite height at $\omega = 0$.
  - A total area

$$\int_{-\infty}^{\infty} X_2(j\omega)d\omega = 2\pi\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(j\omega)e^{j\omega t}d\omega\right]_{t=0} = 2\pi x_2(0) = 2\pi\ .$$

- We conclude that $X_3(j\omega)$ is an impulse function of frequency with area 2π :

$$X_3(j\omega) = 2\pi\delta(\omega).$$

- The constant $x_3(t)$ is infinitely spread out in time, so its CTFT $X_3(j\omega)$ is infinitely narrow in frequency.

Continuous-Time Fourier Transform in the Limit

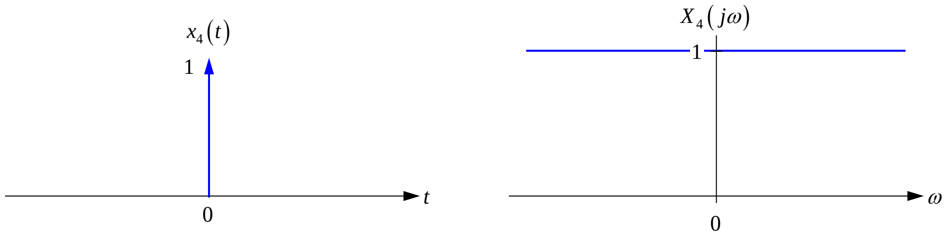
- The constant signal $x_3(t)$ is not absolutely or square integrable, so $X_3(j\omega)$ does not exist in a strict sense.
- We considered $x_3(t)$ as the limiting case of a signal whose CTFT does exist, and found an expression for $X_3(j\omega)$ that is not strictly defined at all values of ω.
- We say that the CTFT $X_3(j\omega)$ *exists in a generalized sense*.
- We use similar methods to compute generalized CTFTs of other important signals.

Signal *x*(*t*)	**CTFT *X*(*j*ω)**
1	$2\pi\delta(\omega)$
$\operatorname{sgn}(t)$	$\dfrac{2}{j\omega}$
$u(t)=\frac{1}{2}\left[1+\operatorname{sgn}(t)\right]$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$

$\sin \omega_0 t$	$\frac{\pi}{j}\delta\left(\omega-\omega_0\right)-\frac{\pi}{j}\delta\left(\omega+\omega_0\right)$

4. *Unit impulse*. This example is the dual of Example 3. The signal and its CTFT are

$$x_4(t) = \delta(t) \overset{F}{\leftrightarrow} X_4(j\omega) = 1 \ \forall \omega .$$



- We compute the CTFT of $x_4(t)$ using the CTFT (4):

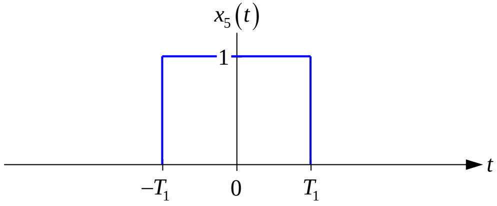
$$\begin{aligned} X_4(j\omega) &= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt \\ &= 1 \quad \forall \omega \end{aligned},$$

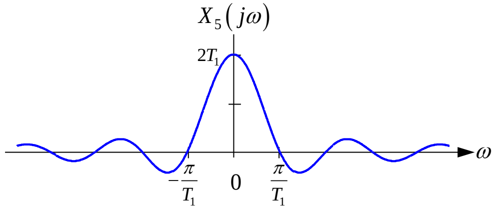
  We evaluated the integral using the sampling property of the CT impulse function.

- The signal $x_4(t)$ is infinitely narrow in time, so its CTFT $X_4(j\omega)$ is infinitely spread out in frequency.

5. *Rectangular pulse*. The signal and its CTFT are given by

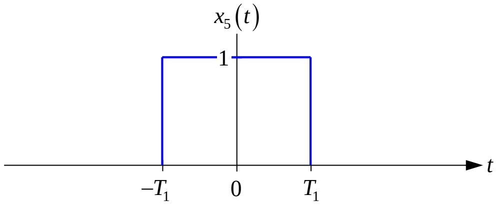
$$x_5(t) = \Pi\left(\frac{t}{2T_1}\right) \overset{F}{\leftrightarrow} X_5(j\omega) = 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right).$$

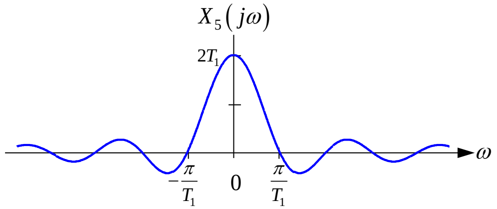




- We compute the CTFT of $x_5(t)$ using the (4):

$$\begin{aligned} X_5(j\omega) &= \int_{-\infty}^{\infty} \Pi\left(\frac{t}{2T_1}\right) e^{-j\omega t} dt \\ &= \int_{-T_1}^{T_1} e^{-j\omega t} dt \qquad . \qquad (12) \\ &= 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right) \end{aligned}$$





- Using the CTFT integral (4), the value of any CTFT at $\omega = 0$ equals the area under the corresponding time signal:

$$X(j0) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \bigg|_{\omega=0} = \int_{-\infty}^{\infty} x(t) dt .$$

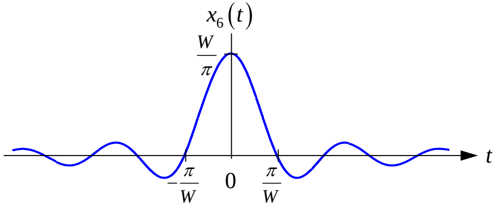
We can use this to check the CTFT:

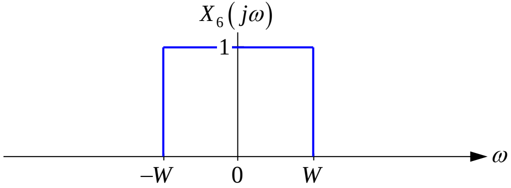
$$X_5(j0) = \int_{-\infty}^{\infty} x_5(t) dt \, .$$
$$= 2T_1$$

- We note that a small value of T_1 corresponds to a signal $x_5(t)$ concentrated in time and corresponds to a CTFT $X_5(j\omega)$ spread out in frequency (and vice versa).

6. *Sinc function*. This is the dual of Example 5. The signal and its CTFT are

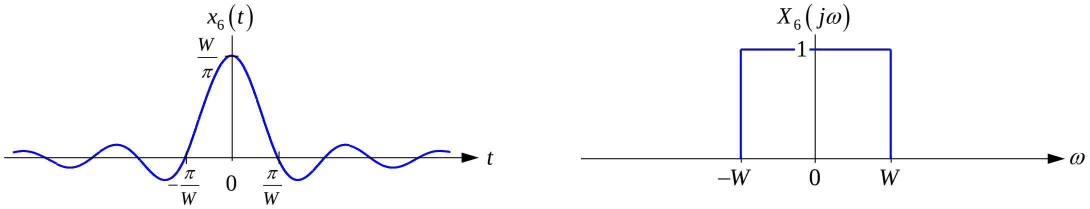
$$x_6(t) = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right) \overset{F}{\leftrightarrow} X_6(j\omega) = \Pi\left(\frac{\omega}{2W}\right)$$





- It is easiest to start with the CTFT $X_6(j\omega)$ and compute the time signal using the inverse CTFT (7):

$$\begin{aligned} x_6(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Pi\left(\frac{\omega}{2W}\right) e^{j\omega t} d\omega \\ &= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t} d\omega \\ &= \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right) \end{aligned} \quad . \qquad (16)$$



- Using the inverse CTFT integral (7), the value of a time signal at $t = 0$ equals $1/2\pi$ times the area under the corresponding CTFT:

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega\Big|_{t=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)d\omega\,.$$

We can use this to check the inverse CTFT:

$$\begin{aligned} x_6(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_6(j\omega)d\omega \\ &= \frac{2W}{2\pi} \\ &= \frac{W}{\pi} \end{aligned}\,.$$

- We note that a large value of W corresponds to a signal $x_6(t)$ concentrated in time and a CTFT $X_6(j\omega)$ spread out in frequency range (and vice versa).

Continuous-Time Fourier Transform of Periodic Signals

- In Chapter 3, we saw how to describe a periodic CT signal $x(t)$ by its CTFS coeffcients a_k.
  We now show it is also possible to describe a periodic CT signal $x(t)$ by a generalized CTFT $X(j\omega)$.

- The CTFT of a periodic signal is needed to analyze *multiplying* a periodic CT signal by another CT signal (typically aperiodic). For example:

  - *Modulating* a CT signal by a periodic sinusoid. We introduce modulation later in this chapter and discuss it further in Chapter 7.

  - *Sampling* a CT signal to obtain a DT signal. In Chapter 6, we sometimes model sampling as multiplying the CT signal by a periodic CT impulse train.

- The CTFT of a periodic signal is not needed to analyze *convolving* a periodic CT signal with another CT signal (typically aperiodic). We can analyze such problems using CTFS, as in Chapter 3.

- We start by studying an example.

7. *Imaginary exponential*. The signal and its CTFT are

$$x_7(t) = e^{j\omega_0 t} \overset{F}{\leftrightarrow} X_7(j\omega) = 2\pi\delta(\omega - \omega_0). \tag{17}$$

- To obtain (17), we start with the CTFT $X_7(j\omega)$ and use the inverse CTFT (7), as in Example 6:

$$\begin{aligned} x_7(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega \\ &= e^{j\omega_0 t} \end{aligned}.$$

We have used the sampling property of the impulse function to evaluate the integral.

General Periodic Signal

- Now we consider a signal $x(t)$ that is periodic with period $T_0 = \dfrac{2\pi}{\omega_0}$ and is synthesized by a CTFS:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. \tag{18}$$

The a_k, $-\infty < k < \infty$, are the CTFS coefficients for the signal $x(t)$.

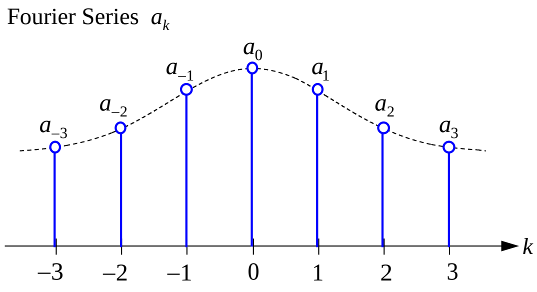
- We compute the CTFT of (18) term-by-term using (17) and the linearity of the CTFT:

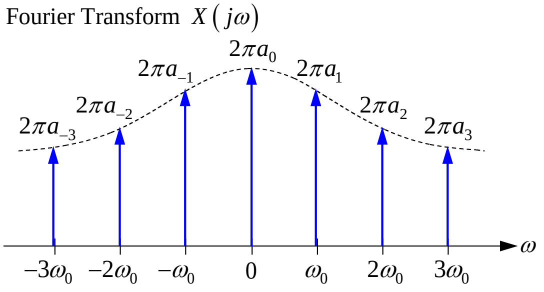
$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0). \tag{19}$$

- We can summarize our finding as a CTFT pair:

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \overset{F}{\leftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0). \tag{20}$$

- CTFT of a periodic signal:
  - A *train of impulses* at $\omega = k\omega_0$, which are integer multiples of the fundamental frequency ω_0.
  - The impulse at frequency $k\omega_0$ is scaled by 2π times the corresponding CTFS coefficient a_k.
  - The relationship between the CTFS coefficients a_k and the CTFT $X(j\omega)$ is shown.

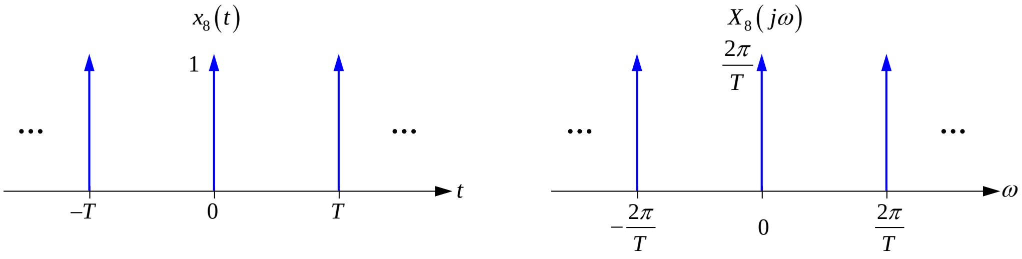




- The following example is used extensively in analyzing sampling and reconstruction (see Chapter 6).

8. *Periodic impulse train*. The signal and its CTFT are

$$x_8(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \overset{F}{\leftrightarrow} X_8(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right), \tag{21}$$



- The signal $x_8(t)$ is periodic with period $T_0 = T$ and fundamental frequency $\omega_0 = \frac{2\pi}{T}$.

- To obtain its CTFS coefficients, we integrate the CTFS analysis equation over $-\frac{T}{2} \le t < \frac{T}{2}$:

$$\begin{aligned} a_k &= \frac{1}{T} \int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} \delta(t - nT) e^{-jk\omega_o t} dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_o t} dt \\ &= \frac{1}{T} \end{aligned} \qquad . \qquad (22)$$

Second line of (22): only the impulse for $n = 0$ lies within the integration interval.

Third line of (22): we used the sampling property of the CT impulse function.

- Now we use (19) to obtain the CTFT of $x_8(t)$:

$$\begin{aligned} X_8(j\omega) &= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0) \\ &= \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right) \end{aligned} \qquad . \qquad (23)$$

- We have found that the *CTFT of a periodic impulse train is a periodic impulse train*.
- Note the inverse relationship between the time-domain spacing T and the frequency-domain spacing $\frac{2\pi}{T}$.

Properties of Continuous-Time Fourier Transform

- Like the CTFS and DTFS properties in Chapter 3, these CTFT properties are useful for:
  - Computing the CTFTs for new signals by using the CTFTs already known for other signals.
  - Checking the CTFTs we compute for new signals.
- For a complete list of CTFT properties, see Table 3, Appendix. We discuss only a few properties.
- We consider one or two signals and their CTFTs. Initially we denote these as

$$x(t) \overset{F}{\leftrightarrow} X(j\omega) \quad \text{and} \quad y(t) \overset{F}{\leftrightarrow} Y(j\omega).$$

Linearity

- A linear combination of $x(t)$ and $y(t)$ has a CTFT given by the corresponding linear combination of the CTFTs $X(j\omega)$ and $Y(j\omega)$:

$$ax(t)+by(t) \overset{F}{\leftrightarrow} aX(j\omega)+bY(j\omega).$$

Time Shift

- A signal time-shifted by t_0 has its CTFT multiplied by a factor $e^{-j\omega t_0}$:

$$x(t-t_0) \overset{F}{\leftrightarrow} e^{-j\omega t_0} X(j\omega). \tag{25}$$

- The magnitude and phase of $e^{-j\omega t_0} X(j\omega)$ are related to those of $X(j\omega)$ by

$$\begin{cases} \left|e^{-j\omega t_0} X(j\omega)\right| = \left|X(j\omega)\right| \\ \angle\left(e^{-j\omega t_0} X(j\omega)\right) = \angle X(j\omega) - \omega t_0 \end{cases}. \tag{25′}$$

- Time-shifting a signal by t_0 affects its CTFT by
  - Leaving the magnitude unchanged.
  - Adding a phase shift proportional to $-t_0$, which varies linearly with frequency ω.

Proof: given a time-shifted signal $x(t-t_0)$, we compute its CTFT $F\left[x(t-t_0)\right]$ using (4):

$$F\left[x(t-t_0)\right]=\int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}dt\ .$$

Changing the integration variable to $\tau = t - t_0$:

$$\begin{aligned} F\left[x(t-t_0)\right] &= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau+t_0)}d\tau \\ &= e^{-j\omega t_0}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau \\ &= e^{-j\omega t_0}X(j\omega) \end{aligned}$$

Time Scaling

$$x(at) \overset{F}{\leftrightarrow} \frac{1}{|a|}X\left(j\frac{\omega}{a}\right).$$

- Note the inverse relationship between time and frequency:
  - $|a|>1$ compresses time and expands frequency.
  - $|a|<1$ expands time and compresses frequency.

Time Reversal

- A special case of time scaling with $a=-1$. Reversal in time corresponds to reversal in frequency:

$$x(-t) \overset{F}{\leftrightarrow} X(-j\omega). \tag{26}$$

- If a signal is even in time, its CTFT is even in frequency:

$$x(-t)=x(t) \overset{F}{\leftrightarrow} X(-j\omega)=X(j\omega),$$

- If a signal is odd in time, its CTFT is odd in frequency:

$$x(-t)=-x(t) \overset{F}{\leftrightarrow} X(-j\omega)=-X(j\omega).$$

Conjugation

$$x^*(t) \overset{F}{\leftrightarrow} X^*(-j\omega). \tag{27}$$

Proof: using (4), the CTFT of $x^*(t)$ is

$$\begin{aligned} F\left[x^*(t)\right] &= \int_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt \\ &= \left(\int_{-\infty}^{\infty} x(t) e^{j\omega t} dt\right)^* . \\ &= X^*(-j\omega) \end{aligned}$$

Conjugate Symmetry for Real Signal

- A real signal $x(t)$ equals its complex conjugate $x^*(t)$.

  Combined with the conjugation property, this implies

$$x(t) = x^*(t) \overset{F}{\leftrightarrow} X(j\omega) = X^*(-j\omega). \tag{28}$$

- If a signal is real, its CTFT is *conjugate symmetric*. The CTFT at positive frequency equals the complex conjugate of the CTFT at negative frequency.

- We can restate conjugate symmetry in two ways:

  - The magnitude of the CTFT is even in frequency, while the phase of the CTFT is odd in frequency:

$$x(t) = x^*(t) \overset{F}{\leftrightarrow} \begin{cases} |X(j\omega)| = |X(-j\omega)| \\ \angle X(j\omega) = -\angle X(-j\omega) \end{cases}. \tag{28a}$$

  - The real part of the CTFT is even in frequency, while the imaginary part of the CTFT is odd in frequency:

$$x(t) = x^*(t) \overset{F}{\leftrightarrow} \begin{cases} \mathrm{Re}[X(j\omega)] = \mathrm{Re}[X(-j\omega)] \\ \mathrm{Im}[X(j\omega)] = -\mathrm{Im}[X(-j\omega)] \end{cases}. \tag{28b}$$

Real, Even or Real, Odd Signals

- Combining the time reversal and conjugation properties and, we find that

$$x(t) \text{ real and even in } t \overset{F}{\leftrightarrow} X(j\omega) \text{ real and even in } \omega$$

and

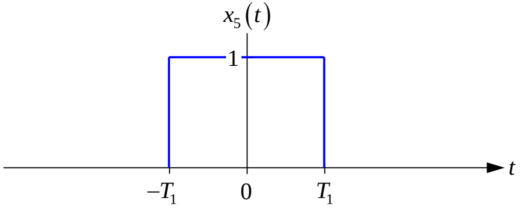
$$x(t) \text{ real and odd in } t \overset{F}{\leftrightarrow} X(j\omega) \text{ imaginary and odd in } \omega .$$

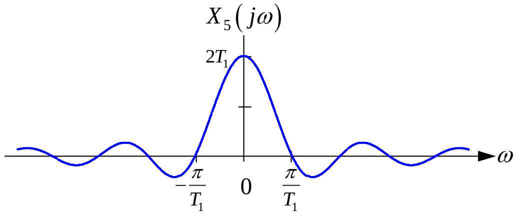
Examples of Symmetry Properties

5. *Real and even signal*. Recall Example 5:

$$x_5(t) = \Pi\left(\frac{t}{2T_1}\right) \overset{F}{\leftrightarrow} X_5(j\omega) = 2T_1 \text{sinc}\left(\frac{\omega T_1}{\pi}\right).$$

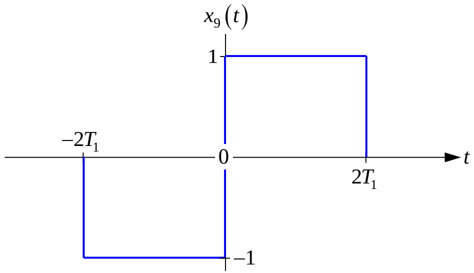
- The signal is real and even in t, and the CTFT is real and even in ω.

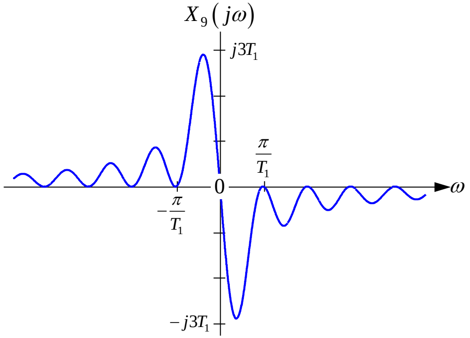




9. *Real and odd signal*.

$$x_9(t) = x_5(t - T_1) - x_5(t + T_1).$$





- We compute its CTFT using linearity and the time-shift property:

$$\begin{aligned} X_9(j\omega) &= e^{-j\omega T_1} X_5(j\omega) - e^{j\omega T_1} X_5(j\omega) \\ &= -2j\sin(\omega T_1) X_5(j\omega) \\ &= -4jT_1 \sin(\omega T_1)\operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right) \end{aligned}.$$

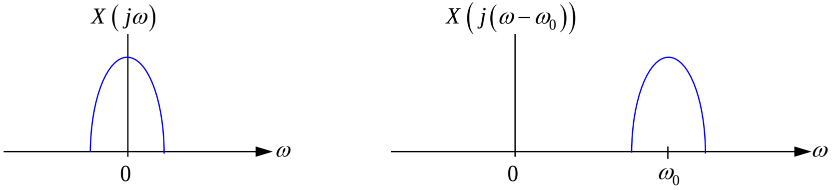
- The signal is real and odd in t, and the CTFT is imaginary and odd in ω.

Frequency Shift (dual of time-shift property)

- A signal multiplied by an imaginary exponential time signal $e^{j\omega_0 t}$ has its CTFT frequency-shifted by ω_0:

$$x(t)e^{j\omega_0 t} \overset{F}{\leftrightarrow} X\left(j(\omega-\omega_0)\right). \tag{29}$$

- A CTFT $X(j\omega)$ and the frequency-shifted CTFT $X\left(j(\omega-\omega_0)\right)$ are shown.



Proof. Using (4), the CTFT of $x(t)e^{j\omega_0 t}$ is

$$F\left[x(t)e^{j\omega_0 t}\right] = \int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt \ .$$

$$= X\left(j(\omega-\omega_0)\right)$$

- The frequency-shift property is the basis for amplitude modulation, which we discuss below.

Differentiation in Time

- *Differentiating* a signal in time corresponds to *multiplying* its CTFT by a factor $j\omega$:

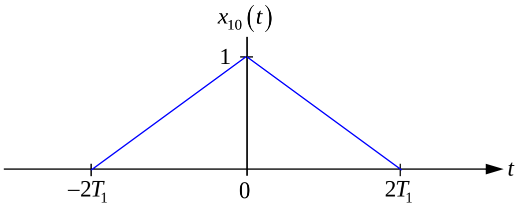
$$\frac{dx}{dt} \stackrel{F}{\leftrightarrow} j\omega X(j\omega). \tag{30}$$

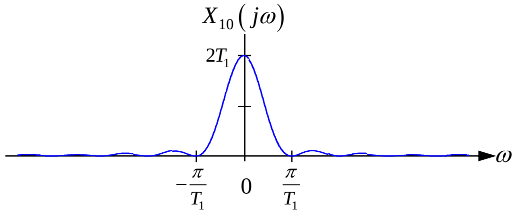
- To prove (30), we represent $x(t)$ by an inverse CTFT of $X(j\omega)$ (7) and differentiate with respect to *t*.

Example of Differentiation Property ***(We will skip but please read.)***

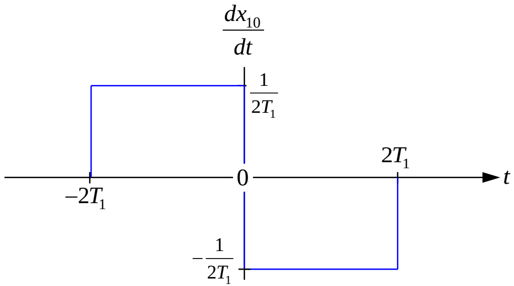
10. Triangular pulse. The signal and its CTFT are

$$x_{10}(t) = \Lambda\left(\frac{t}{2T_1}\right) \stackrel{F}{\leftrightarrow} X_{10}(j\omega) = 2T_1 \text{sinc}^2\left(\frac{\omega T_1}{\pi}\right)$$





- To compute the CTFT $X_{10}(j\omega)$, consider the time derivative of the triangular pulse, $\dfrac{dx_{10}}{dt}$.



- Referring to Example 9, observe that

$$\frac{dx_{10}}{dt} = -\frac{1}{2T_1} x_9(t),$$

from which it follows that

$$\frac{dx_{10}}{dt} \overset{F}{\leftrightarrow} -\frac{1}{2T_1} X_9(j\omega). \tag{31}$$

- The differentiation property (30) states that

$$\frac{dx_{10}}{dt} \overset{F}{\leftrightarrow} j\omega X_{10}(j\omega). \tag{32}$$

- Combining the right-hand sides of (31) and (32) yields

$$j\omega X_{10}(j\omega) = -\frac{1}{2T_1} X_9(j\omega).$$

- Solving for $X_{10}(j\omega)$, we find

$$\begin{aligned} X_{10}(j\omega) &= -\frac{1}{2j\omega T_1} X_9(j\omega) \\ &= 2\frac{\sin(\omega T_1)}{\omega}\operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right) \\ &= 2T_1\frac{\sin\left(\pi\frac{\omega T_1}{\pi}\right)}{\pi\frac{\omega T_1}{\pi}}\operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right) \\ &= 2T_1\operatorname{sinc}^2\left(\frac{\omega T_1}{\pi}\right) \end{aligned}. \qquad (33)$$

Integration in Time

- *Integrating* a signal in time corresponds to *dividing* its CTFT by a factor $j\omega$:

$$\begin{aligned}\int_{-\infty}^{t} x(\tau)\,d\tau &\overset{F}{\leftrightarrow} \frac{1}{j\omega}X(j\omega)+\pi X(j\omega)\delta(\omega)\\ &= \frac{1}{j\omega}X(j\omega)+\pi X(j0)\delta(\omega)\end{aligned}. \quad (35)$$

- This is intuitively appealing, since differentiating in time corresponds to multiplying the CTFT by $j\omega$.

- There is an additional term on the right-hand side of (35):

$$\pi X(j\omega)\delta(\omega)=\pi X(j0)\delta(\omega).$$

The equality follows from the sampling property of the impulse function.
This term is nonzero if the original time-domain signal $x(t)$ has a non-zero d.c. value:

$$X(j0)=\int_{-\infty}^{\infty} x(t)dt \neq 0 .$$

Example of Integration Property

11. Unit step function. The signal and its CTFT are

$$x_{11}(t)=u(t)\overset{F}{\leftrightarrow}X_{11}(j\omega)=\frac{1}{j\omega}+\pi\delta(\omega).$$

- Recall Example 4, where we found that the CTFT of an impulse function is a constant:

$$x_4(t)=\delta(t)\overset{F}{\leftrightarrow}X_4(j\omega)=1.$$

- Recall that the unit step function is the running integral of the impulse function.
  We obtain the DTFT using the integration property:

$$\begin{aligned} x_{11}(t)=\int_{-\infty}^{t}x_4(\tau)d\tau \overset{F}{\leftrightarrow} X_{11}(j\omega) &= \frac{1}{j\omega}X_4(j\omega)+\pi X_4(j0)\delta(\omega) \\ &= \frac{1}{j\omega}+\pi\delta(\omega) \end{aligned}.$$

Differentiation in Frequency ***(We will skip but please read.)***

- This is the dual of the differentiation-in-time property discussed earlier. It states

$$tx(t) \overset{F}{\leftrightarrow} j\frac{dX(j\omega)}{d\omega}.$$

- Multiplying a signal by time t corresponds to differentiating its CTFT with respect to frequency ω (and scaling by j).
- We prove this by differentiating the analysis equation (4) with respect to ω.

  This shows that $\frac{dX(j\omega)}{d\omega}$ is the CTFT of a signal $-jtx(t)$.

Parseval's Identity

- The importance of inner products between signals was explained in Chapter 3, slides 38-42.
- Parseval's identity for the CTFT helps us compute:
  - the inner product between two CT signals, or
  - the energy of one CT signal

  in either time or frequency.
- Depending on the signal(s), the computation is often easier in one domain than in the other.

Inner Product Between Signals

- The general form of Parseval's identity, for an *inner product between two CT signals*, states

$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x(t) y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) Y^*(j\omega) d\omega \ . \tag{36}$$

- Middle expression in (36): an inner product in the time domain between two signals, $x(t)$ and $y(t)$.
- Rightmost expression in (36): an inner product in the frequency domain between the corresponding CTFTs, $X(j\omega)$ and $Y(j\omega)$.
- *Example*: orthogonal multiplexing in communication systems (see slide 74).

Signal Energy

- The special case of (36) with $x(t) = y(t)$ and $X(j\omega) = Y(j\omega)$ yields an expression for the *energy of a CT signal*:

$$E = \int_{-\infty}^{\infty} \left|x(t)\right|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|X(j\omega)\right|^2 d\omega \ . \tag{37}$$

- Middle expression in (37): energy of signal computed in time domain.
- Rightmost expression in (37): energy of signal computed in frequency domain.
- Interpretation of rightmost expression: $\left|X(j\omega)\right|^2$ is the *energy density spectrum* of the signal $x(t)$.

  $\left|X(j\omega)\right|^2$ quantifies the energy contained in the component of the signal at a frequency ω.

  The rightmost expression in (37) is an integral of the energies contained in all frequencies ω, $-\infty < \omega < \infty$.

Proof of General Case ***(You may skip this.)***

- We start with the middle expression in (36) and represent $x(t)$ by the inverse CTFT of $X(j\omega)$:

$$\int_{-\infty}^{\infty} x(t)y^*(t)\,dt = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega\right]y^*(t)\,dt\,.$$

- We interchange the order of integration and recognize the quantity in square brackets as $Y(j\omega)$:

$$\begin{aligned}\int_{-\infty}^{\infty} x(t)y^*(t)\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\left[\int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt\right]^* d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)Y^*(j\omega)\,d\omega\end{aligned}.$$

We have proven (36).

Example: Energy of Sinc Function

- We would like to determine the energy of the signal studied in Example 6:

$$\frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right) \overset{F}{\leftrightarrow} \Pi\left(\frac{\omega}{2W}\right).$$

- To compute the energy in the time domain, we must evaluate the integral

$$E = \left(\frac{W}{\pi}\right)^2 \int_{-\infty}^{\infty} \operatorname{sinc}^2\left(\frac{Wt}{\pi}\right) dt \,,$$

  which is difficult.

- Using Parseval's identity (37), it is easy to compute the energy in the frequency domain:

$$\begin{aligned} E &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Pi^2\left(\frac{\omega}{2W}\right) d\omega \\ &= \frac{1}{2\pi}\int_{-W}^{W} d\omega \\ &= \frac{W}{\pi} \end{aligned} \quad .$$

Convolution Property

- Given two signals and their CTFTs

$$p(t) \overset{F}{\leftrightarrow} P(j\omega) \text{ and } q(t) \overset{F}{\leftrightarrow} Q(j\omega),$$

  the convolution property states

$$p(t) * q(t) \overset{F}{\leftrightarrow} P(j\omega)Q(j\omega). \tag{38}$$

- *Convolution in the time domain* corresponds to *multiplication in the frequency domain*.

Proof of Convolution Property

- We express the convolution as an integral and compute its CTFT using (4):

$$F\left[p(t)*q(t)\right]=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}p(\tau)q(t-\tau)d\tau\right]e^{-j\omega t}dt\ .$$

- Now we interchange the order of integration:

$$F\left[p(t)*q(t)\right]=\int_{-\infty}^{\infty}p(\tau)\left[\int_{-\infty}^{\infty}q(t-\tau)e^{-j\omega t}dt\right]d\tau\ .$$

- We recognize the quantity in square brackets as the CTFT of $q(t-\tau)$.

  Using the time-shift property (25), this is $Q(j\omega)e^{-j\omega\tau}$.

- Hence, the CTFT of the convolution is

$$F\left[p(t)*q(t)\right]=Q(j\omega)\int_{-\infty}^{\infty}p(\tau)e^{-j\omega\tau}d\tau\ .$$

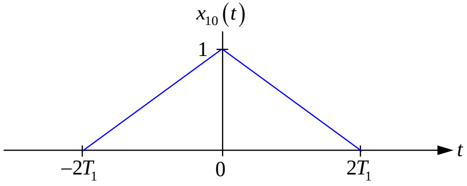
  We recognize the integral as $P(j\omega)$, the CTFT of $p(t)$.

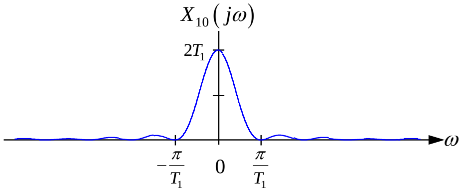
- We have proven (38).

Example of Convolution Property

10. Triangular pulse (alternate approach). The signal and its CTFT are

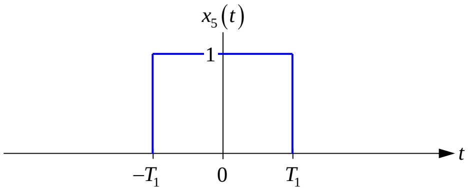
$$x_{10}(t) = \Lambda\left(\frac{t}{2T_1}\right) \overset{F}{\leftrightarrow} X_{10}(j\omega) = 2T_1 \mathrm{sinc}^2\left(\frac{\omega T_1}{\pi}\right).$$

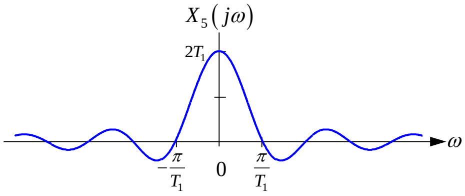




- Recall Example 5:

$$x_5(t) = \Pi\left(\frac{t}{2T_1}\right) \overset{F}{\leftrightarrow} X_5(j\omega) = 2T_1 \mathrm{sinc}\left(\frac{\omega T_1}{\pi}\right).$$





- Observe that

$$x_{10}(t) = \frac{1}{2T_1} x_5(t) * x_5(t) \ .$$

- Using the convolution property (38), we find

$$\begin{aligned} X_{10}(j\omega) &= \frac{1}{2T_1} X_5(j\omega) X_5(j\omega) \\ &= 2T_1 \operatorname{sinc}^2\left(\frac{\omega T_1}{\pi}\right) \end{aligned} .$$

- This agrees with (33) and (34), which we found using the differentiation property (see slides 42-44). The convolution property provides an easier method for computing this CTFT.

Frequency Response of Continuous-Time Linear Time-Invariant Systems

- The most important application of the convolution property is to filtering signals by LTI systems.
- Consider a CT LTI system having impulse response $h(t)$.

  The frequency response $H(j\omega)$, assuming it exists, is the CTFT of $h(t)$ (see Chapter 3, slides 7-8).
- The impulse and frequency responses of a CT LTI system *form a CTFT pair*:

$$h(t) \overset{F}{\leftrightarrow} H(j\omega). \tag{39}$$

- Suppose a signal $x(t) \overset{F}{\leftrightarrow} X(j\omega)$ is input, yielding an output signal $y(t) \overset{F}{\leftrightarrow} Y(j\omega)$.

$$x(t) \overset{F}{\leftrightarrow} X(j\omega) \longrightarrow \boxed{h(t) \overset{F}{\leftrightarrow} H(j\omega)} \longrightarrow y(t) \overset{F}{\leftrightarrow} Y(j\omega)$$

- In the *time domain*, the output is obtained by *convolving* the input and the impulse response:

$$y(t) = h(t) * x(t). \tag{40}$$

- By the convolution property, in the *frequency domain*, the output is obtained by *multiplying* their CTFTs:

$$Y(j\omega) = H(j\omega)X(j\omega). \tag{41}$$

- Expressions (40) and (41) provide two different ways to compute the output of an LTI system:

$$y(t) = h(t) * x(t) \overset{F}{\leftrightarrow} Y(j\omega) = H(j\omega) X(j\omega) .$$

- Viewing LTI filtering as frequency-domain multiplication is intuitively appealing.

  In many problems, frequency-domain multiplication is easier than time-domain convolution.

- The frequency response $H(j\omega)$ is the CTFT of the impulse response $h(t)$.

  All the properties of the frequency response can be understood as properties of the CTFT.

  For example, if the impulse response is real, by (28), the frequency response is conjugate symmetric:

$$h(t) = h^*(t) \overset{F}{\leftrightarrow} H(j\omega) = H^*(-j\omega) .$$

Examples of Frequency Responses

1. *Time shift*. Consider an LTI system such that the output is a time-shifted version of the input:

$$y(t) = x(t - t_0) = h(t) * x(t) . \qquad (42)$$

- We expressed the input-output relation as a convolution with an impulse response $h(t)$, although we need not consider an explicit formula for $h(t)$.

- We compute the CTFT of (42) using time-shift property (25):

$$Y(j\omega) = e^{-j\omega t_0} X(j\omega) = H(j\omega) X(j\omega) .$$

We find the frequency response is

$$H(j\omega) = e^{-j\omega t_0} . \qquad (43)$$

This confirms the result found in Chapter 3 (slides 55-56).

- Alternatively, we could use the impulse response given in Chapter 2 (slides 35-36):

$$h(t) = \delta(t - t_0) .$$

Evaluating its CTFT using the sampling property of the impulse function, we obtain (43).

2. *Integrator*. Consider a causal LTI system whose output is a running integral of the input:

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau = h(t) * x(t). \tag{44}$$

- We wrote it as convolution with an impulse response $h(t)$, although we need not consider $h(t)$ explicitly.

- We compute the CTFT of (44) using the integration-in-time property (35) and (41):

$$Y(j\omega) = \frac{1}{j\omega} X(j\omega) + \pi X(j\omega)\delta(\omega) = H(j\omega) X(j\omega).$$

  We have found the frequency response of the integrator is

$$H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega). \tag{45}$$

- Alternatively, we could use the impulse response of the integrator (Homework 3):

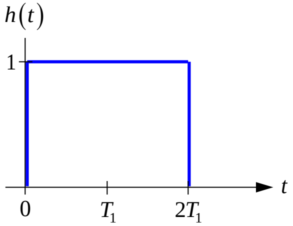
$$h(t) = u(t). \tag{46}$$

  Evaluating its CTFT (Example 11, slide 46), we obtain the frequency response (45).

- Because the impulse response (46) is not absolutely integrable, $\int_{-\infty}^{\infty} |h(t)|\, dt = \infty$, the frequency response (45) contains an impulse function and exists only in the generalized sense.

3. *Finite-time integrator*. Consider a causal LTI system that has an impulse response

$$h(t)=u(t)-u(t-2T_1)=\Pi\left(\frac{t-T_1}{2T_1}\right). \tag{47}$$



- Given an input $x(t)$, the output is

$$\begin{aligned} y(t) &= x(t)*h(t) \\ &= x(t)*u(t)-x(t)*u(t-2T_1) \\ &= \int_{-\infty}^{t} x(\tau)d\tau - \int_{-\infty}^{t-2T_1} x(\tau)d\tau \\ &= \int_{t-2T_1}^{t} x(\tau)d\tau \end{aligned} \tag{48}$$

- *Question*: how do we obtain the second term in the third line in (48)?

- To obtain the frequency response, we could compute the CTFT of the input-output relation (48), using the integration-in-time property (35) and the time-shift property (25).

  Question: can you do this calculation and get the same result as (49) below?

- It is easier to get the frequency response by computing the CTFT of the impulse response (47).

  Using the CTFT of the rectangular pulse (12) and the time-shift property (25), we get

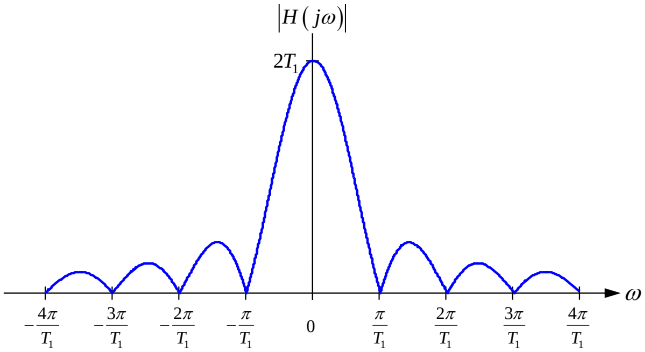
$$h(t) = \Pi\left(\frac{t-T_1}{2T_1}\right) \overset{F}{\leftrightarrow} H(j\omega) = 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right) e^{-j\omega T_1}. \tag{49}$$

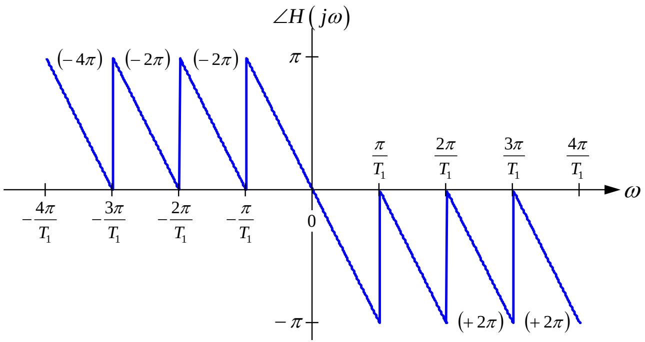
- The frequency response (49) is complex, so we plot its magnitude and phase.

  Using the product rule (Appendix, page 289), we find

$$\left|H(j\omega)\right| = \left|2T_1\right| \left|\operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)\right| \left|e^{-j\omega T_1}\right| = 2T_1 \left|\operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)\right|.$$

$$\angle H(j\omega) = \angle 2T_1 + \angle \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right) + \angle e^{-j\omega T_1} = \begin{cases} 0 + k2\pi - \omega T_1 & \operatorname{sinc}(\omega T_1/\pi) > 0 \\ \pi + k2\pi - \omega T_1 & \operatorname{sinc}(\omega T_1/\pi) < 0 \end{cases}.$$





Magnitude $|H(j\omega)|$

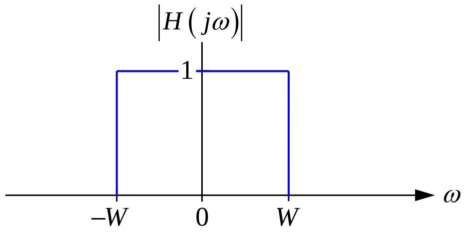
- This is a lowpass filter. It is not ideal, as $|H(j\omega)|$ falls off slowly as $|\omega|$ increases.

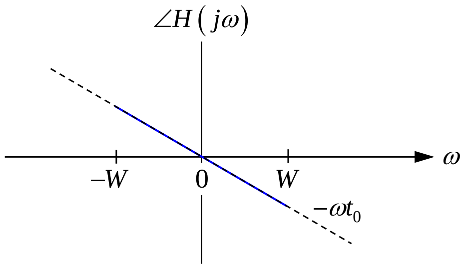
Phase $\angle H(j\omega)$

- Jumps by π radians at values of ω where the sinc function passes through zero and changes sign.
- Away from the zeros of the sinc function, the phase is linear, with a slope corresponding to a group delay $-d\angle H(j\omega)/d\omega = T_1$, the average delay of the impulse response (47).
- The magnitude and phase were plotted by MATLAB, which added multiples of 2π (in parentheses) to keep the phase in the interval $[-\pi,\pi]$ (see Appendix, pages 300-301).

4. *Ideal lowpass filter*.

- We study an *ideal lowpass filter* with cutoff frequency W and group delay t_0.
  It is described most easily in the frequency domain.





Magnitude $|H(j\omega)|$

- Constant in the *passband* $|\omega| < W$.
- Abrupt *cutoff* or *transition* at $|\omega| = W$.
- Zero in the *stopband* $|\omega| > W$.

Phase $\angle H(j\omega)$

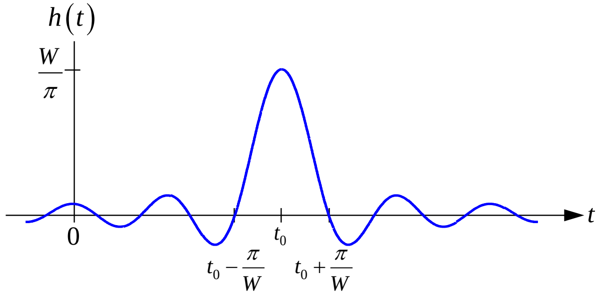
- Linear in the passband $|\omega| < W$. Slope corresponds to a constant group delay $-d\angle H(j\omega)/d\omega = t_0$.
- Can assume any value in the stopband $|\omega| > W$, as indicated by the dashed lines.
- No multiples of 2π were added to keep the phase in the interval $[-\pi, \pi]$.

- Using the sinc time signal's CTFT (16) and the time-shift property (25), the impulse and frequency responses are

$$h(t)=\frac{W}{\pi}\operatorname{sinc}\left(\frac{W(t-t_0)}{\pi}\right)\overset{F}{\leftrightarrow} H(j\omega)=\Pi\left(\frac{\omega}{2W}\right)e^{-j\omega t_0}\,. \tag{50}$$

- The impulse response $h(t)$ is peaked at $t=t_0$, but has tails that extend to $t=\pm\infty$.

  An ideal lowpass filter cannot be causal, except if the group delay tends to infinity, $t_0\to\infty$.



- We will use ideal lowpass, bandpass and highpass filters throughout EE 102A and 102B.

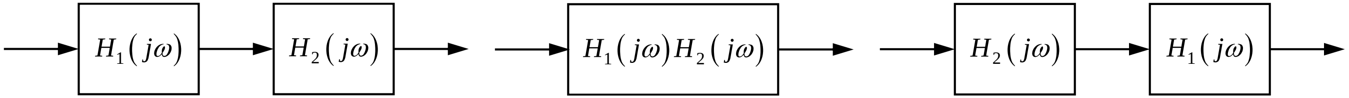
  To simplify the analyses, we often set the group delay t_0 to zero. Nevertheless, a causal filter that approximates the abrupt transition of an ideal filter must have a substantial group delay.

Frequency Response of Cascaded Linear Time-Invariant Systems

- Consider two LTI systems

$$h_1(t) \overset{F}{\leftrightarrow} H_1(j\omega) \quad \text{and} \quad h_2(t) \overset{F}{\leftrightarrow} H_2(j\omega).$$

- Recall that when two LTI systems are cascaded, the overall impulse response is the convolution of their impulse responses. It does not depend on the order in which the two systems are cascaded (see Chapter 2, slides 55-56).
- By the convolution property of the CTFT, the overall frequency response of the cascade is the product of the frequency responses of the two systems. It does not depend on the order in which the two systems are cascaded.
- The following three LTI systems yield identical input-output relationships.



Multiplication Property

- This property is the dual of the convolution property. We are given two signals and their CTFTs

$$p(t) \overset{F}{\leftrightarrow} P(j\omega) \quad \text{and} \quad q(t) \overset{F}{\leftrightarrow} Q(j\omega).$$

  The multiplication property states

$$p(t)q(t) \overset{F}{\leftrightarrow} \frac{1}{2\pi} P(j\omega) * Q(j\omega). \tag{51}$$

- *Multiplication in the time domain* corresponds to *convolution in the frequency domain*.

- Note the factor $\frac{1}{2\pi}$ on the right-hand side of (51), which is not present in the convolution property (38).

- The proof of (51) is similar to the proof of (38) with *t* and ω interchanged.

- Important applications of the multiplication property:
  - Modulation and demodulation (Chapters 4 and 7).
  - Sampling and reconstruction (Chapter 6).
  - Windowing (using the DTFT multiplication property, Chapter 6).

Modulation

- *Modulation* embeds an information-bearing *message signal* into a *carrier signal* to form a *modulated signal*.
- A system that performs modulation is called a *modulator*.
- The carrier signal is usually a sinusoid at a *carrier frequency* ω_c.
- The carrier frequency ω_c is chosen so the modulated signal can propagate as a wave through a communication medium.
- Exemplary values of $\frac{\omega_c}{2\pi}$:
  - Low-frequency submarine links: 3 – 300 Hz
  - Broadcast AM radio: 550 – 1610 kHz
  - Broadcast FM radio: 88 – 108 MHz
  - Cellular telephony and data: 700 – 2500 MHz
  - Wireless local data networks: 2.5 – 60 GHz
  - Optical fibers: 185 – 350 THz
- We will study various forms of modulation in Chapter 7.

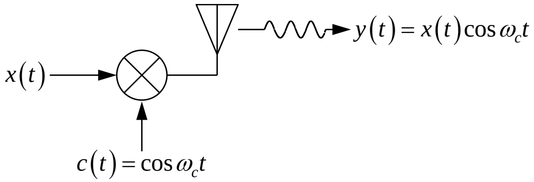
Amplitude Modulation

- *Amplitude modulation* (AM) is the simplest modulation method. Here we study the simplest form of AM. In Chapter 7, we call it *double-sideband amplitude modulation with suppressed carrier* (DSB-AM-SC).

- We multiply a message signal $x(t)$ by a sinusoidal carrier $c(t)=\cos\omega_c t$ to form a modulated signal

$$y(t) = x(t)c(t) = x(t)\cos\omega_c t\,. \qquad (52)$$

- The system for performing AM is called an *amplitude modulator*. The modulated signal $y(t)$ is shown being radiated by an antenna.



- We would like to compute the spectrum $Y(j\omega)$ of the modulated signal using the CTFT. We could write the carrier signal as

$$c(t) = \cos\omega_c t = \frac{1}{2}e^{j\omega_c t} + \frac{1}{2}e^{-j\omega_c t} \qquad (53)$$

and use the frequency-shifting property (29) for each of the two imaginary exponential signals in (53).

- *Question*: what expression for $Y(j\omega)$ do you obtain?

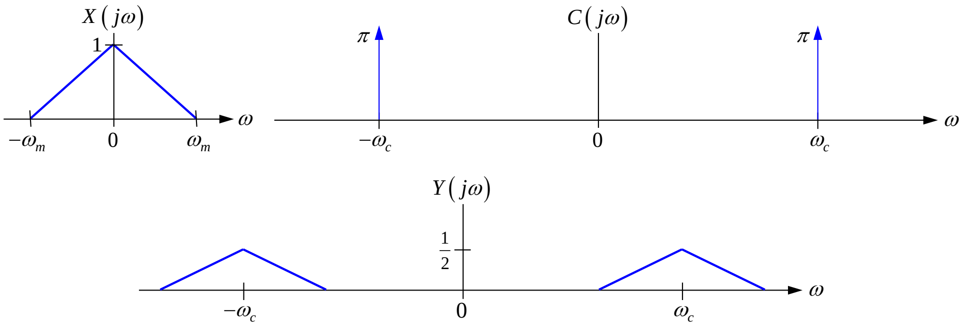
- Instead, we use the CTFT of the carrier signal

$$C(j\omega) = \pi\delta(\omega-\omega_c)+\pi\delta(\omega+\omega_c)\ . \tag{54}$$

We use the multiplication property (51) to obtain the spectrum of the modulated signal

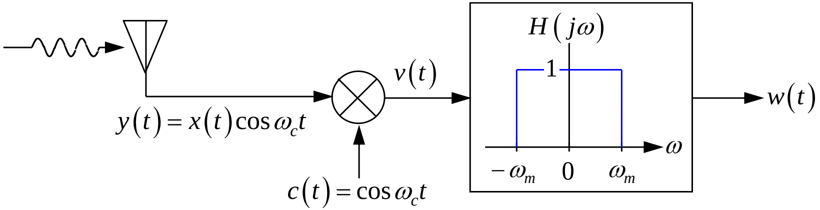
$$\begin{aligned} y(t)=x(t)c(t) \stackrel{F}{\leftrightarrow} Y(j\omega) &= \frac{1}{2\pi}X(j\omega)*C(j\omega) \\ &= \frac{1}{2}X\big(j(\omega-\omega_c)\big)+\frac{1}{2}X\big(j(\omega+\omega_c)\big) \end{aligned}\ . \tag{55}$$

- The modulated signal spectrum $Y(j\omega)$ (55) comprises copies of the message signal spectrum $X(j\omega)$ frequency-shifted to $\pm\omega_c$ and scaled by $1/2$.
- We assume the message signal $x(t)$ is bandlimited. Its spectrum $X(j\omega)$ is nonzero only for $|\omega|<\omega_m$.



Synchronous Demodulation

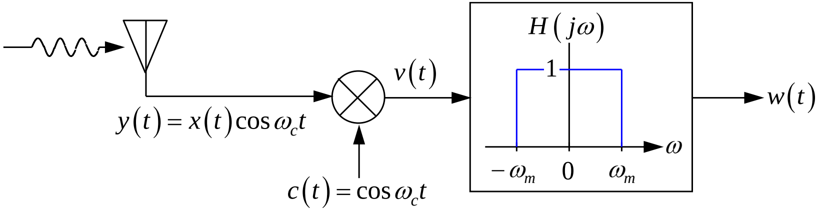
- A system that receives a modulated signal and recovers the message signal is called a *demodulator*.
- To demodulate this form of AM, we must use a *synchronous demodulator*. It requires a replica carrier signal $c(t)$ whose frequency and phase are synchronized to the carrier used in the modulator.



- The first step in synchronous demodulation is to multiply the modulated signal $y(t)$ by the replica carrier signal $c(t)$. The signal obtained at the multiplier output is

$$\begin{aligned} v(t) &= y(t)c(t) \\ &= x(t)\cos^2\omega_c t \\ &= \frac{1}{2}x(t) + \frac{1}{2}x(t)\cos 2\omega_c t \end{aligned} \quad . \qquad (56)$$

We used the trigonometric identity $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ to obtain the third line of (56).



- The multiplier output

$$v(t) = y(t)c(t) = \frac{1}{2}x(t) + \frac{1}{2}x(t)\cos 2\omega_c t \, . \tag{56}$$

comprises two terms:

  - First term: $\frac{1}{2}x(t)$ is a scaled replica of the message signal.

  - Second term: $\frac{1}{2}x(t)\cos 2\omega_c t$ is the scaled message modulated onto a carrier at a frequency $2\omega_c$.

- An ideal lowpass filter with unity passband gain and cutoff frequency ω_m passes the first term and blocks the second term.

- The lowpass filter output signal is a scaled replica of the message signal:

$$w(t) = \frac{1}{2}x(t) \, . \tag{57}$$

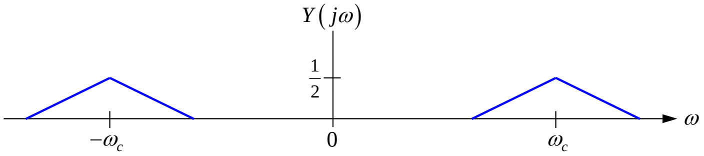
- We now study demodulation in the frequency domain using the multiplication property (51).
  At the multiplier output, we have

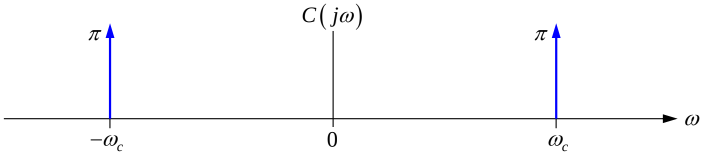
$$v(t)=y(t)c(t) \overset{F}{\leftrightarrow} V(j\omega)=\frac{1}{2\pi}Y(j\omega)*C(j\omega).$$

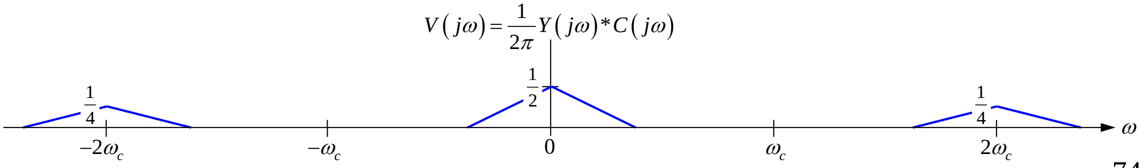
- Using the CTFTs of the modulated signal (55) and the carrier (54), the CTFT of the multiplier output is

$$\begin{aligned} V(j\omega) &= \frac{1}{2\pi}\underbrace{\left[\frac{1}{2}X\left(j(\omega-\omega_c)\right)+\frac{1}{2}X\left(j(\omega+\omega_c)\right)\right]}_{Y(j\omega)} * \underbrace{\left[\pi\delta(\omega-\omega_c)+\pi\delta(\omega+\omega_c)\right]}_{C(j\omega)} . \\ &= \frac{1}{4}X\left(j(\omega-2\omega_c)\right)+\frac{1}{2}X(j\omega)+\frac{1}{4}X\left(j(\omega+2\omega_c)\right) \end{aligned} \tag{58}$$

- $V(j\omega)$ contains scaled copies of the message spectrum $X(j\omega)$ at frequencies $\omega=0$ and $\omega=\pm 2\omega_c$.

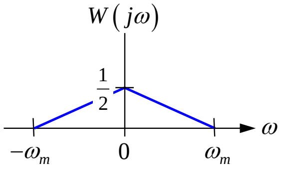






- The lowpass filter blocks the copies of $X(j\omega)$ shifted to $\omega = \pm 2\omega_c$ and passes only the copy of $X(j\omega)$ at $\omega = 0$. The lowpass filter output spectrum contains only the scaled message spectrum:

$$\begin{aligned} W(j\omega) &= H(j\omega)V(j\omega) \\ &= \frac{1}{2}X(j\omega) \end{aligned} . \tag{59}$$

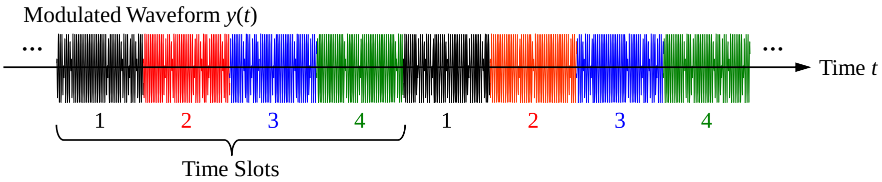


- Expression (59) agrees with the result of the time-domain analysis:

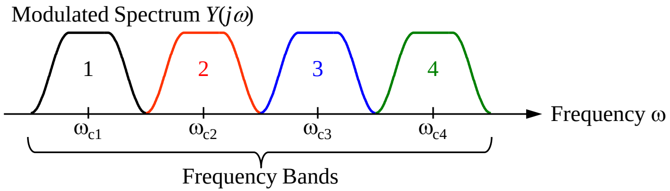
$$w(t) = \frac{1}{2}x(t). \tag{57}$$

Orthogonal Multiplexing (Recall discussion of inner products, Chapter 3, slide 42.)

- In communications, we often want to transmit several different signals through a shared medium to maximize the rate at which information is conveyed. This is called *multiplexing*.
- We want multiplexed signals to be *mutually orthogonal* so they do not interfere with each other. In other words, the inner product between any pair of signals should be zero (see (36) on slide 48). Two methods that are popular in wireless and optical fiber systems:
- *Time-division multiplexing*: all the signals are transmitted at the same carrier frequency, but in different time slots. Since they do not overlap in time, they are mutually orthogonal.



- *Frequency-division multiplexing*: all the signals are transmitted at the same time, but on different carrier frequencies. Since they do not overlap in frequency, they are mutually orthogonal.



Linear Time-Invariant Systems Governed by Constant-Coefficient Differential Equations

- We consider causal LTI CT systems described by a constant-coefficient linear differential equations (see Chapter 2, slides 69-74):

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} . \tag{60}$$

  The $x(t)$ and $y(t)$ are the input and output signals. The a_k, $k = 0,\ldots,N$ and b_k, $k = 0,\ldots,M$ are constants, which are real in systems that map real inputs to real outputs.

- We study a method for computing the frequency response of a system described by (60).
  It is equivalent to one introduced in Chapter 3 (see Method 2, slides 58-59 in Chapter 3).

- The method is applicable only if the impulse response is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty , \tag{61}$$

  so the system is BIBO stable and the frequency response $H(j\omega)$ exists in the strict sense.

- For example, it is not applicable to an integrator (slide 59). Although the integrator can be described by (60) with $a_1 = b_0 = 1$, its impulse response (46) does not satisfy (61), and its frequency response (45) exists only in a generalized sense.

- Assuming (61) is satisfied, the system input-output relation can be described in time or frequency by

$$y(t) = h(t) * x(t) \overset{F}{\leftrightarrow} Y(j\omega) = H(j\omega) X(j\omega) . \tag{41}$$

- We can solve (41) to obtain an expression for the frequency response

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} . \tag{62}$$

- Given an input signal $x(t) \overset{F}{\leftrightarrow} X(j\omega)$, suppose we can determine the output signal $y(t) \overset{F}{\leftrightarrow} Y(j\omega)$ generated by that input. Then we can use (62) to determine the frequency response $H(j\omega)$ at all frequencies at which $X(j\omega) \neq 0$.
- Now we consider the differential equation (60). We compute its CTFT term-by-term, exploiting linearity. We compute the CTFT of the k^{th} derivative using the differentiation property (30) k times:

$$\frac{d^k x}{dt^k} \overset{F}{\leftrightarrow} (j\omega)^k X(j\omega) . \tag{63}$$

- We find the CTFT of the differential equation (60) is

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega) . \tag{64}$$

- We factor out the $Y(j\omega)$ and $X(j\omega)$ in (64) and solve for $Y(j\omega)/X(j\omega)$.

  Using (62), we obtain an expression for the frequency response:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}. \qquad (65)$$

- We found that for any LTI system described by a differential equation of the form (60), the frequency response (if it exists in the strict sense) is a ratio of two polynomials in powers of $j\omega$.

- Such a ratio of two polynomials is called a *rational function* of $j\omega$.

- The coefficients in the numerator and denominator polynomials in (65) are the same coefficients b_k, $k = 0,\ldots,M$ and a_k, $k = 0,\ldots,N$ that appear in the differential equation (60).

- In other words, given a differential equation in the form (60), we can find the frequency response (65) *by inspection*.

- Conversely, given a frequency response in rational form (65), we can find the corresponding differential equation *by inspection*.

- This method is useful in analyzing LTI systems, as we demonstrate through several examples.

Examples of LTI Systems Governed by Constant-Coefficient Differential Equations

- In these examples, we compute the frequency response $H(j\omega)$ directly from the differential equation. The frequency response could be obtained by computing the CTFT of the impulse response $h(t)$, but that requires more work than the approach we use here.

- In some examples, we also state the impulse response $h(t)$ and step response $s(t)$ without derivation. These are derived using Laplace transforms in *EE 102B Course Reader*, Chapter 5.

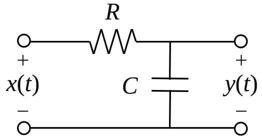
- Our previous plots used linear scales for $|H(j\omega)|$, $\angle H(j\omega)$ and ω.

- Here we use logarithmic scales for $|H(j\omega)|$ and ω, while using a linear scale for $\angle H(j\omega)$. These highlight the asymptotic behavior of $|H(j\omega)|$ and $\angle H(j\omega)$ at low and high frequencies.

- We show only positive ω, since $|H(j\omega)|$ and $\angle H(j\omega)$ are even and odd functions of ω, respectively, for these systems with real impulse responses.

- We also describe the group delay $-d\angle H(j\omega)/d\omega$. Because of the logarithmic frequency scale, the group delay does not correspond to the slope of the plots, which is $-d\angle H(j\omega)/d\log(\omega)$.

First-Order Lowpass Filter. This is realized by the circuit shown.

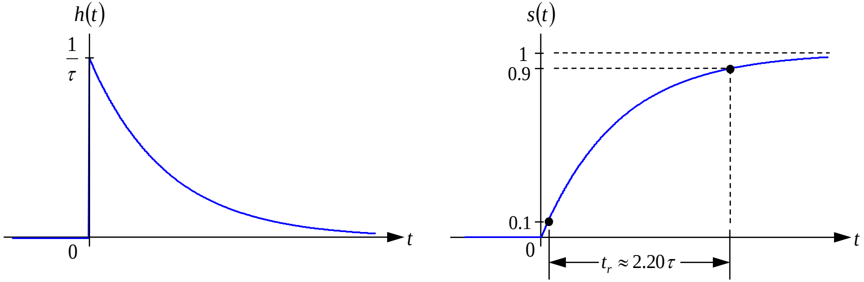


- Defining $RC = \tau$, the circuit is described by a first-order differential equation

$$\frac{dy}{dt} + \frac{1}{\tau} y(t) = \frac{1}{\tau} x(t).$$

- Its impulse and step responses are

$$h(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t) \quad \text{and} \quad s(t) = \left(1 - e^{-\frac{t}{\tau}}\right) u(t).$$



- The *rise time*, in which the step response rises from 10% to 90% of its maximum value, is

$$t_r = (\ln 0.9 - \ln 0.1)\tau \approx 2.20\tau.$$

- To find the frequency response, we take the Fourier transform of the differential equation:

$$j\omega Y(j\omega) + \frac{1}{\tau} Y(j\omega) = \frac{1}{\tau} X(j\omega).$$

- Solving for $H(j\omega) = Y(j\omega) / X(j\omega)$, we obtain

$$H(j\omega) = \frac{1}{1 + j\omega\tau}.$$

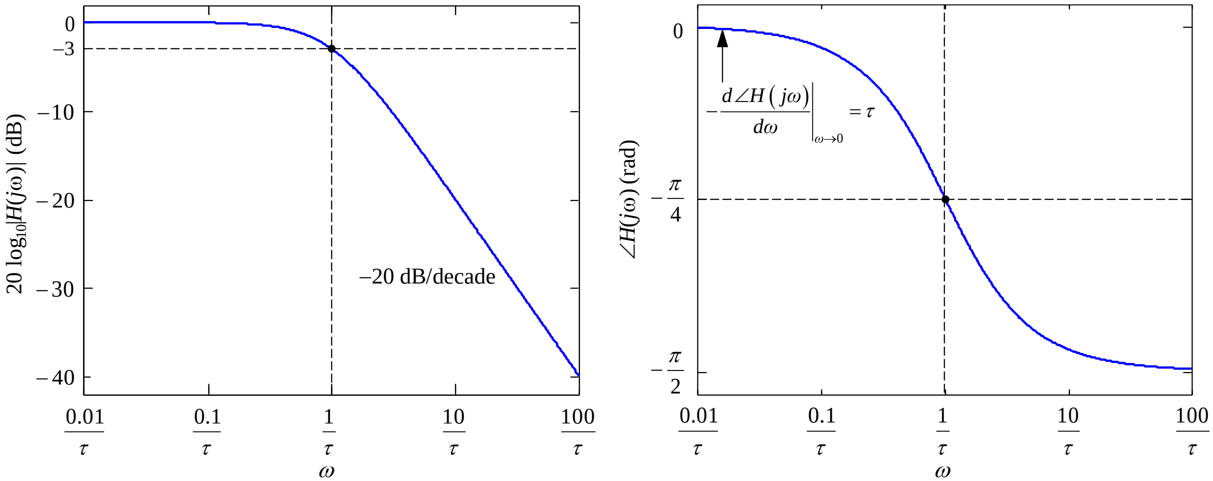
- The magnitude and phase responses are

$$|H(j\omega)| = \frac{1}{\sqrt{1 + (\tau\omega)^2}} \quad \text{and} \quad \angle H(j\omega) = -\tan^{-1}(\tau\omega).$$

- The group delay is

$$-\frac{d\angle H(j\omega)}{d\omega} = \frac{\tau}{1 + (\tau\omega)^2}.$$

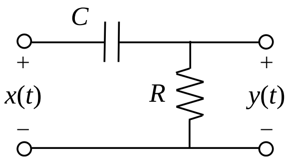
  Near $\omega = 0$, where the magnitude response is largest, the group delay is τ.



- Observations about the magnitude and phase plots (for $\omega > 0$ only).

Region	Frequencies	$\lvert H(j\omega)\rvert$	$20\log_{10}\lvert H(j\omega)\rvert$	$\angle H(j\omega)$	$-\dfrac{d\angle H(j\omega)}{d\omega}$
Passband	$\omega << 1/\tau$	1	0 dB	0	τ
Cutoff	$\omega = 1/\tau$	$1/\sqrt{2} \approx 0.707$	≈ -3 dB	$-\pi/4$	–
Stopband	$\omega >> 1/\tau$	$\propto 1/\omega$	–20 dB/decade	$-\pi/2$	–

First-Order Highpass Filter. This is realized by the circuit shown.

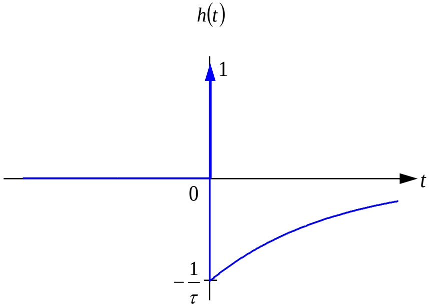


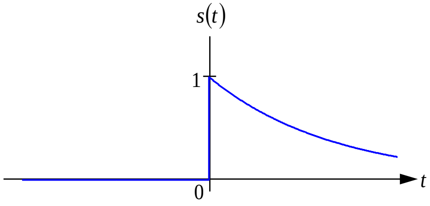
- Defining τ = *RC*, the circuit is described by a first-order differential equation

$$\frac{dy}{dt} + \frac{1}{\tau} y(t) = \frac{dx}{dt} .$$

- Its impulse and step responses are

$$h(t) = \delta(t) - \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t) \quad \text{and} \quad s(t) = e^{-\frac{t}{\tau}} u(t) .$$





- To find the frequency response, we take the Fourier transform of the differential equation:

$$j\omega Y(j\omega) + \frac{1}{\tau} Y(j\omega) = j\omega X(j\omega) \ .$$

- Solving for $H(j\omega) = Y(j\omega) / X(j\omega)$, we obtain

$$H(j\omega) = \frac{j\omega\tau}{j\omega\tau + 1} \, .$$

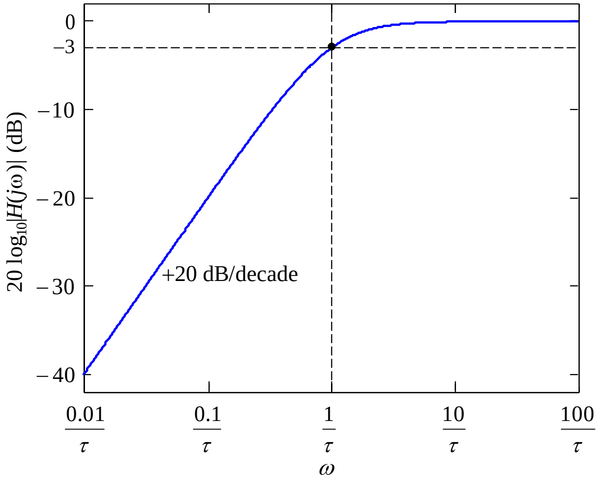
- The magnitude and phase responses are

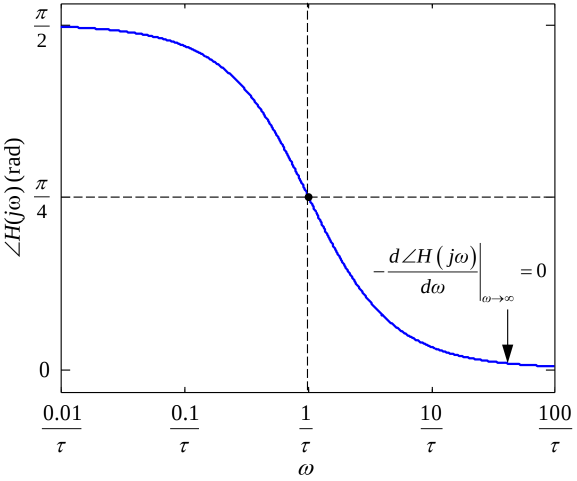
$$\left| H(j\omega) \right| = \frac{|\omega|\tau}{\sqrt{1+(\omega\tau)^2}} \quad \text{and} \quad \angle H(j\omega) = \frac{\pi}{2} \operatorname{sgn}\omega - \tan^{-1}\omega\tau \ .$$

- The group delay is

$$-\frac{d\angle H(j\omega)}{d\omega} = \frac{\tau}{1+(\tau\omega)^2} \, , \ \ \omega \neq 0 \ \ .$$

For $\omega \to \pm\infty$, where the magnitude response is largest, the group delay is 0.





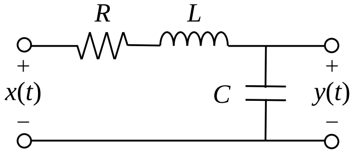
- Observations about the magnitude and phase plots (for $\omega > 0$ only).

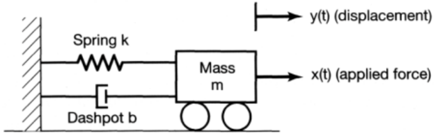
Region	Frequencies	$\lvert H(j\omega)\rvert$	$20\log_{10}\lvert H(j\omega)\rvert$	$\angle H(j\omega)$	$-\frac{d\angle H(j\omega)}{d\omega}$
Stopband	$\omega << 1/\tau$	$\propto \omega$	+20 dB/decade	$+\pi/2$	–
Cutoff	$\omega = 1/\tau$	$1/\sqrt{2} \approx 0.707$	≈ -3 dB	$+\pi/4$	–
Passband	$\omega >> 1/\tau$	1	0 dB	0	0

Second-Order Lowpass Filter. This is described by a second-order differential equation

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t).$$

- Two parameters in the differential equation: the *natural frequency* ω_n and the *damping coefficient* ζ.
- Two systems described by the equation are shown, and the values of ω_n and ζ are indicated.





$$\omega_n = \frac{1}{\sqrt{LC}} \qquad \omega_n = \sqrt{\frac{k}{m}}$$

$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}} \qquad \zeta = \frac{b}{2\sqrt{km}}$$

- The natural frequency ω_n is the frequency at which the system exhibits a resonant response in the absence of damping.
- The damping coefficient ζ is proportional to the physical quantity causing energy dissipation, *R* or *b*, respectively. When damping is present, the system exhibits a lowpass response, and ω_n represents a cutoff frequency, above which the magnitude response decreases.

- The second-order lowpass filter has three regimes, distinguished based on the damping constant value:
  - Underdamped $0<\zeta<1$
  - Critically damped $\zeta=1$
  - Overdamped $1<\zeta<\infty$
- The impulse and step responses have different forms in each regime. Here we give the impulse responses.
  - Overdamped, $1<\zeta<\infty$:

$$h(t)=\frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(e^{-\omega_n\left(\zeta-\sqrt{\zeta^2-1}\right)t}-e^{-\omega_n\left(\zeta+\sqrt{\zeta^2-1}\right)t}\right)u(t)\,.$$

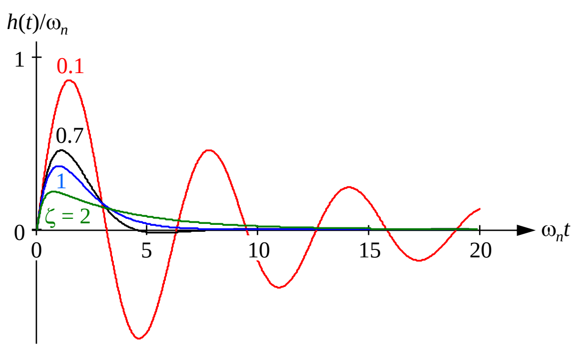
  - Underdamped, $0<\zeta<1$:

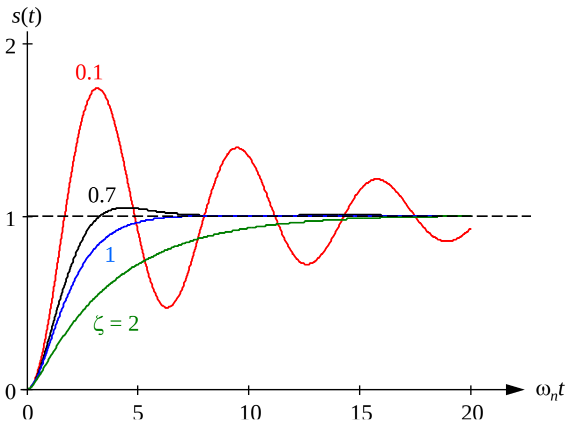
$$h(t)=\frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}\cdot t\right)u(t)\,.$$

  - Critically damped, $\zeta=1$:

$$h(t)=\omega_n^2 t e^{-\omega_n t}u(t)\,.$$

- In these plots, the time scale is $\omega_n t$: time is normalized by the natural response time of the system.





- The step response $s(t)$ is especially informative:
    - For all damping values, $0 < \zeta < \infty$, the step response approaches unity, $s(t) \to 1$, as $t \to \infty$.
    - For a small damping coefficient, $\zeta << 1$, the step response exhibits a short rise time but exhibits significant *overshoot* and *ringing*.
    - For a damping coefficient $\zeta > 1$, there is no overshoot or ringing, but as ζ increases, the rise time becomes longer. We observe below that the group delay increases with increasing ζ.
- Many second-order systems are designed for ζ ~ 0.7-1.0, a good compromise between overshoot and response time.

- To find the frequency response, we take the Fourier transform of the differential equation:

$$(j\omega)^2 Y(j\omega) + 2\zeta\omega_n (j\omega) Y(j\omega) + \omega_n^2 Y(j\omega) = \omega_n^2 X(j\omega) \ .$$

- Solving for $H(j\omega) = Y(j\omega)/X(j\omega)$, we obtain

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n (j\omega) + \omega_n^2} \ .$$

- The magnitude and phase responses are

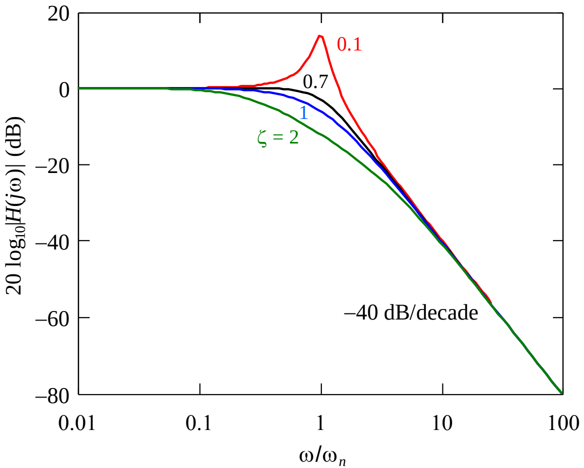
$$\left|H(j\omega)\right| = \frac{\omega_n^2}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2}} \qquad \text{and} \qquad \angle H(j\omega) = -\tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right).$$

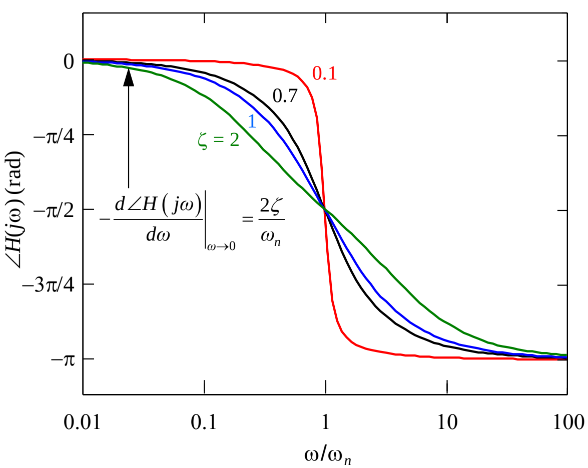
- In the passband, $\omega << \omega_n$, the group delay is

$$-\frac{d\angle H(j\omega)}{d\omega} = \frac{2\zeta}{\omega_n}, \ \omega << \omega_n \ .$$

  This increases with increasing ζ, since damping slows down the system's response.

- Near the cutoff frequency $\omega \approx \omega_n$, when $\zeta < 0.7$ (in the underdamped regime), $\left|H(j\omega)\right| > 1$ ($20\log_{10}\left|H(j\omega)\right| > 0 \text{ dB}$). This is called *peaking*, and becomes more pronounced as $\zeta \to 0$.



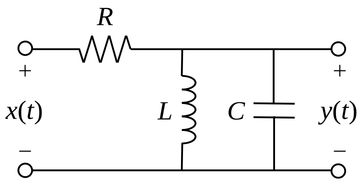


- Observations about the magnitude and phase plots (for $\omega > 0$ only).

Region	Frequencies	$\lvert H(j\omega)\rvert$	$20\log_{10}\lvert H(j\omega)\rvert$	$\angle H(j\omega)$	$-\frac{d\angle H(j\omega)}{d\omega}$
Passband	$\omega << \omega_n$	1	0 dB	0	$2\zeta / \omega_n$
Cutoff	$\omega = \omega_n$	depends on ζ		$-\pi / 2$	–
Stopband	$\omega >> \omega_n$	$\propto 1/\omega^2$	–40 dB/decade	$-\pi$	–

Second-Order Bandpass Filter

- Bandpass filters are used in radio-frequency circuits to control the frequency of an oscillator or select a channel at a desired frequency while rejecting channels at undesired frequencies.
- This is one of several different passive *RLC* networks that realize a second-order bandpass filter.



- It is described by a differential equation

$$\frac{d^2 y}{dt^2} + \frac{1}{RC}\frac{dy}{dt} + \frac{1}{LC} y(t) = \frac{1}{RC}\frac{dx}{dt}.$$

- We define two parameters to describe the circuit: the *resonance frequency*

$$\omega_n = \frac{1}{\sqrt{LC}}$$

and the *damping coefficient*

$$\zeta = \frac{1}{2R}\sqrt{\frac{L}{C}}.$$

- Similar parameters were defined for the second-order lowpass filter. Here, they have a similar relationship to the frequency response, but a different dependence on *R*, *L* and *C*.

- The amount of damping may alternatively be described by a *quality factor*

$$Q = \frac{1}{2\zeta} = R\sqrt{\frac{C}{L}} \ .$$

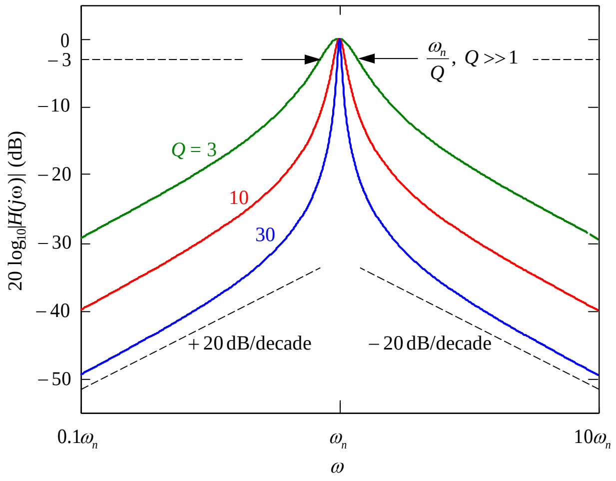
A small value of ζ corresponds to a large value of *Q*.

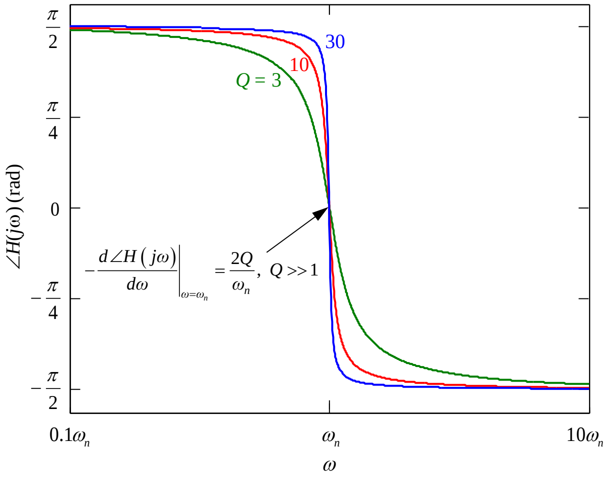
- Using the parameters defined above, the differential equation can be rewritten

$$\frac{d^2 y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2 y(t) = 2\zeta\omega_n \frac{dx}{dt} \qquad \text{or} \qquad \frac{d^2 y}{dt^2} + \frac{\omega_n}{Q}\frac{dy}{dt} + \omega_n^2 y(t) = \frac{\omega_n}{Q}\frac{dx}{dt} \ .$$

- To find the frequency response, we take the Fourier transform of the differential equation:

$$H(j\omega) = \frac{2\zeta\omega_n j\omega}{(j\omega)^2 + 2\zeta\omega_n (j\omega) + \omega_n^2} = \frac{\frac{\omega_n}{Q} j\omega}{(j\omega)^2 + \frac{\omega_n}{Q}(j\omega) + \omega_n^2} \ .$$





- A few observations, assuming the filter is highly underdamped, $\zeta << 1$ or $Q >> 1$, (for $\omega > 0$ only):
  - $\left|H\left(j\omega\right)\right| = 1/\sqrt{2}$ at $\omega = \omega_n \pm \dfrac{\omega_n}{2Q}$, so the bandpass filter has a 3-dB bandwidth $\dfrac{\omega_n}{Q}$, which decreases as Q increases.
  - At $\omega = \omega_n$, the group delay is $-\dfrac{d\angle H\left(j\omega\right)}{d\omega}\bigg|_{\omega=\omega_n} = \dfrac{2Q}{\omega_n}$, which increases as Q increases. The input signal is stored in the L-C resonator, delaying the output relative to the input.