**Stanford University**
**EE 102A: Signal Processing and Linear Systems I**
**Instructor: Ethan M. Liang**

# Chapter 5: The Discrete-Time Fourier Transform

## Motivations

- The Fourier series (FS) expresses a *periodic* signal as a *discrete sum* of imaginary exponentials.
  - In DT, these are $e^{jk\Omega_0 n}$, $\Omega_0 = 2\pi / N$ real, $k = \langle N \rangle$ (any $N$ consecutive values).
  - This simplifies our analysis of LTI systems with periodic inputs.
    To compute the output signal, we multiply each $e^{jk\Omega_0 n}$ by $H\left(e^{jk\Omega_0}\right)$.
- $H\left(e^{j\Omega}\right)$ is a *frequency response* that characterizes the input-output relation of a DT LTI system.
- The Fourier transform (FT) expresses an *aperiodic* signal as a *continuous integral* of imaginary exponentials.
  - In DT, these are $e^{j\Omega n}$, $\Omega$ real, $\Omega_1 \le \Omega < \Omega_1 + 2\pi$ (any interval of length $2\pi$ ).
  - This simplifies our analysis of LTI systems with aperiodic inputs.
    To compute the output signal, we multiply each $e^{j\Omega n}$ by $H\left(e^{j\Omega}\right)$.
- In Chapter 5, using the DTFT, we analyze aperiodic (and periodic) signals and compute the frequency response $H\left(e^{j\Omega}\right)$ for various systems.
- We also study the overall schema of Fourier series and transforms in CT and DT.
  This includes organizing principles and dualities within and between them.

## Major Topics in This Chapter

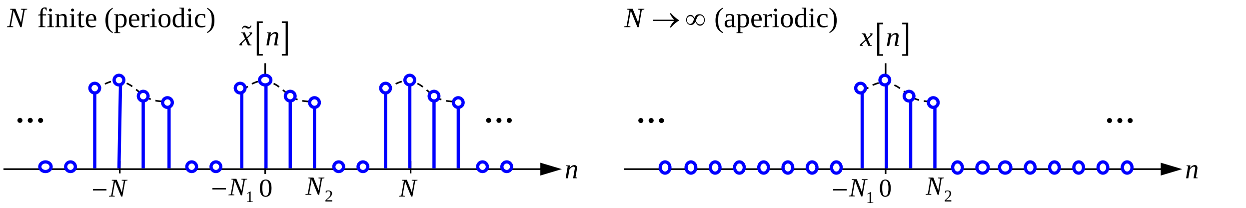
- Discrete-time Fourier transform
    - Derivation for aperiodic signals. Fourier transforms in the limit. Fourier transforms of periodic signals. Properties of Fourier transforms.
- Convolution property and LTI system analysis
    - Frequency response as DTFT of impulse response.
    - LTI systems not described by finite-order difference equations.
        - Ideal lowpass filter.
    - LTI systems described by linear, constant-coefficient difference equations.
        - Infinite impulse response: first-order system, second-order system.
        - Finite impulse response: moving average, approximation of ideal lowpass filter.
- Overview of CT and DT Fourier representations
    - Organizing principles: periodic vs. aperiodic, continuous vs. discrete.
    - Dualities: in CTFT, in DTFS, between CTFS and DTFT.

## Discrete-Time Fourier Transform

*Derivation of Discrete-Time Fourier Transform*

- Our derivation of the DTFT is similar to the derivation of the CTFT in Chapter 4 (see slides 3-8).
- We are given an aperiodic DT signal $x[n]$. We assume $x[n]$ is nonzero only for $-N_1 \leq n \leq N_2$.
- We consider the aperiodic signal $x[n]$ to be a periodic signal $\tilde{x}[n]$, with period $N$, in the limit $N \to \infty$.

  In that limit, the periodic signal $\tilde{x}[n]$ becomes the aperiodic signal $x[n]$.



- To derive the DTFT of the aperiodic signal $x[n]$, we start by representing the periodic signal $\tilde{x}[n]$ as a DTFS with fundamental frequency $\Omega_0 = 2\pi / N$ and DTFS coefficients $a_k$, $k = \langle N \rangle$:

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n} . \qquad \text{(DTFS synthesis)} \qquad (1)$$

(1) synthesizes the periodic signal $\tilde{x}[n]$ as a linear combination of $e^{jk\Omega_0 n}$ over *N* consecutive values of *k*.

- Recall that both the imaginary exponentials and the DTFS coefficients are periodic in *k* with period *N*:

$$e^{j(k+N)\Omega_0 n} = e^{jk\Omega_0 n} \quad \text{and} \quad a_{k+N} = a_k .$$

- We may obtain the DTFS coefficients by performing analysis over any *N* consecutive values of *n*:

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk\Omega_0 n} . \qquad \text{(DTFS analysis)} \qquad (2)$$

We choose the interval $n = \langle N \rangle$ to include $-N_1 \leq n \leq N_2$, over which the aperiodic signal $x[n]$ is nonzero.

- Because $\tilde{x}[n] = x[n]$ over $-N_1 \leq n \leq N_2$ and $x[n] = 0$ outside this interval, we may rewrite the DTFS analysis equation (2) as

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\Omega_0 n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\Omega_0 n} . \qquad (3)$$

- Now we define $X\left(e^{j\Omega}\right)$, a function of a continuous frequency $\Omega$. It is computed from $x[n]$ using:

$$X\left(e^{j\Omega}\right) \stackrel{d}{=} \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} . \qquad \text{(DTFT or DTFT analysis)} \tag{4}$$

- We refer to $X\left(e^{j\Omega}\right)$ as the *DT Fourier transform* (DTFT) of $x[n]$.
  We refer to (4) as the *DTFT analysis equation*, or simply the *DTFT*.

- Note that $X\left(e^{j\Omega}\right)$ is a periodic function of Ω with period 2π:

$$X\left(e^{j(\Omega+2\pi)}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\Omega+2\pi)n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \underbrace{e^{-j2\pi n}}_{=1} = X\left(e^{j\Omega}\right). \tag{5}$$

- Comparing (3) and (4), we can obtain the DTFS coefficients $a_k$ by sampling the DTFT $X\left(e^{j\Omega}\right)$ at integer multiples of the fundamental frequency and scaling by $1/N$:

$$\frac{1}{N} X\left(e^{j\Omega}\right)\Big|_{\Omega=k\Omega_0} = \frac{1}{N} X\left(e^{jk\Omega_0}\right) = a_k . \tag{6}$$

- Using (6), we can rewrite the DTFS synthesis equation (1) for the periodic signal $\tilde{x}[n]$ as

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} \frac{1}{N} X\left(e^{jk\Omega_0}\right) e^{jk\Omega_0 n} = \frac{1}{2\pi} \sum_{k=\langle N\rangle} X\left(e^{jk\Omega_0}\right) e^{jk\Omega_0 n} \Omega_0 . \tag{7}$$

  We used $1/N = \Omega_0/2\pi$ in the last step.

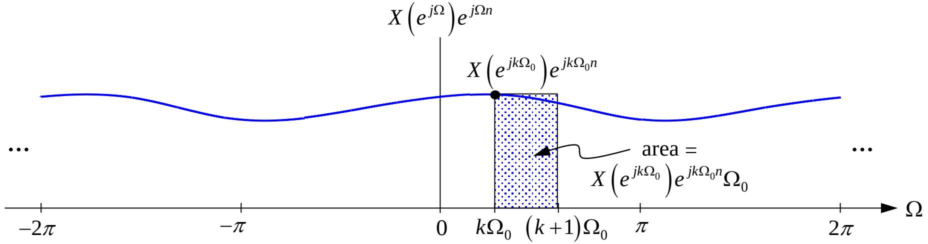
- Consider the limit $N \to \infty$, in which the periodic signal $\tilde{x}[n]$ becomes the aperiodic signal $x[n]$:

$$k\Omega_0 \to \Omega\text{, a continuous variable}$$

$$\Omega_0 \to d\Omega\text{, an infinitesimal increment of }\Omega$$

$$X\left(e^{j\Omega}\right)\Big|_{\Omega=k\Omega_0} \to X\left(e^{j\Omega}\right)\text{, a function of a continuous variable}$$

- This figure shows $X\left(e^{j\Omega}\right)e^{j\Omega n}$ as a function of the continuous frequency variable $\Omega$.




- In the limit we are considering, the summation

$$\tilde{x}[n] = \frac{1}{2\pi}\sum_{k=\langle N\rangle} X\left(e^{jk\Omega_0}\right)e^{jk\Omega_0 n}\Omega_0 \tag{7}$$

becomes a Riemann sum approximation of an integral

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\Omega}\right)e^{j\Omega n}d\Omega\,. \qquad \text{(inverse DTFT or DTFT synthesis)} \tag{8}$$

- Since (7) sums over any $N$ consecutive frequency intervals of length $\Omega_0 = 2\pi / N$, the integral (8) may be performed over any interval of length 2π.
- The integral (8) allows us to obtain the aperiodic signal $x[n]$ from $X\left(e^{j\Omega}\right)$.

  We refer to (8) as the *inverse DTFT* or the *DTFT synthesis equation*.

  We refer to $x[n]$ as the *inverse DTFT* of $X\left(e^{j\Omega}\right)$.
- In summary, we have derived two expressions:

| | | |
|---|---|---|
| $X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$ | (DTFT or DTFT analysis) | (4) |
| $x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\Omega}\right) e^{j\Omega n} d\Omega$ | (inverse DTFT or DTFT synthesis) | (8) |

- The inverse DTFT integral (8) specifies how to *synthesize* an aperiodic signal $x[n]$ as a weighted sum of $e^{j\Omega n}$. The continuous-valued frequency Ω spans any interval of length 2π.

  The imaginary exponential $e^{j\Omega n}$ at frequency Ω is weighted by a factor $X\left(e^{j\Omega}\right)$.
- The DTFT sum (4) specifies how, given an aperiodic signal $x[n]$, we may *analyze* $x[n]$ to obtain the weighting factor $X\left(e^{j\Omega}\right)$.

- We often describe (4) and (8) in terms of a *DTFT operator F* and an *inverse DTFT operator* $F^{-1}$. Each acts on one function to produce the other:

$$F\left[x[n]\right] = X\left(e^{j\Omega}\right), \tag{9}$$

and

$$F^{-1}\left[X\left(e^{j\Omega}\right)\right] = x[n]. \tag{10}$$

- We often denote a DT signal $x[n]$ and its DTFT $X\left(e^{j\Omega}\right)$ as a *DTFT pair*:

$$x[n] \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right). \tag{11}$$

*Alternate Analysis Method for Discrete-Time Fourier Series*

- The DTFT derivation (slides 3-8) provides an alternate method for computing the DTFS coefficients of periodic signals, which we summarize here.
- Suppose we are given a periodic signal $\tilde{x}[n]$ with period $N = \dfrac{2\pi}{\Omega_0}$.

  We can find its DTFS coefficients $a_k$ by performing three steps:

1. Define an aperiodic signal $x[n]$ that represents one period of $\tilde{x}[n]$:

$$x[n] = \begin{cases} \tilde{x}[n] & n_1 + 1 \le n \le n_1 + N \\ 0 & \text{otherwise} \end{cases} \quad \text{for some } n_1. \tag{12}$$

2. Compute the DTFT of the one-period signal $x[n]$ using

$$X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}. \tag{4}$$

3. Sample the DTFT $X\left(e^{j\Omega}\right)$ at integer multiples of the fundamental frequency to obtain the DTFS coefficients of the periodic signal $\tilde{x}[n]$:

$$a_k = \frac{1}{N} X\left(e^{j\Omega}\right)\Big|_{\Omega = k\Omega_0}. \tag{6}$$

- This method is sometimes easier than using the DTFS analysis equation (2), especially if you know the DTFT of the one-period signal.
- Even more importantly:
    - According to (6), every set of DTFS coefficients corresponds to the samples of a DTFT.
    - All the properties of the DTFS (Table 2, Appendix) are inherited from properties of the DTFT (Table 5, Appendix).

  Understanding this relationship can simplify your learning of DT Fourier analysis.

*Convergence of Discrete-Time Fourier Transform*

- Consider a signal and its DTFT

$$x[n] \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right).$$

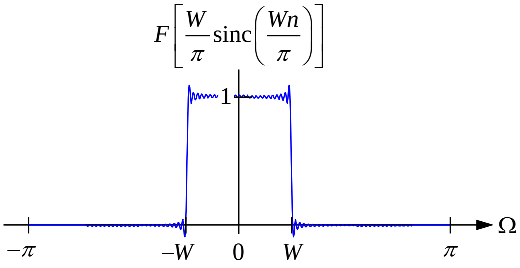
- If the DTFT $X\left(e^{j\Omega}\right)$ exists, the inverse DTFT $F^{-1}\left[X\left(e^{j\Omega}\right)\right]$ converges to the original signal $x[n]$.

- The DTFT

$$X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}. \tag{4}$$

  is a Fourier series synthesis of a periodic function of $\Omega$ with Fourier series coefficients given by $x[n]$.

- The convergence of the DTFT $X\left(e^{j\Omega}\right)$ is thus analogous to the convergence of a CTFS synthesis $\hat{x}(t)$. If the DTFT $X\left(e^{j\Omega}\right)$ has discontinuities, it will exhibit the Gibbs phenomenon.



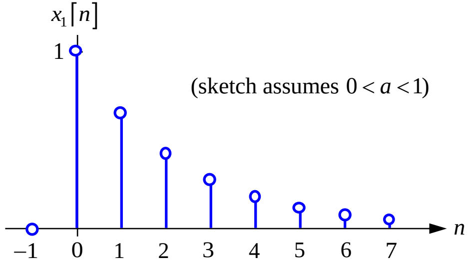
- The Gibbs phenomenon was exaggerated in the plot above by truncating the time signal before applying the *F* operator.
- As we will see later in this chapter (slides 96-104), the Gibbs phenomenon can negatively impact the performance of finite impulse response (FIR) DT filters.
- For example, if the DTFT shown represented the frequency response of an FIR lowpass filter, the Gibbs phenomenon would cause:
  - *Distortion* of desired signals in the passband, and
  - *Leakage* of undesired signals in the stopband.
- These effects can be mitigated by *windowing* the FIR filter's impulse response, as we will see in Chapter 6.

*Examples of Discrete-Time Fourier Transform*

- Since all DTFTs are periodic functions of $\Omega$ with period $2\pi$, we often plot them over a single period, such as $-\pi \le \Omega \le \pi$ .

1. *Right-sided real exponential*. The signal and its DTFT are given by

$$x_1[n] = a^n u[n] \overset{F}{\leftrightarrow} X_1\left(e^{j\Omega}\right) = \frac{1}{1-ae^{-j\Omega}}, \ a \text{ real}, \ |a|<1.$$



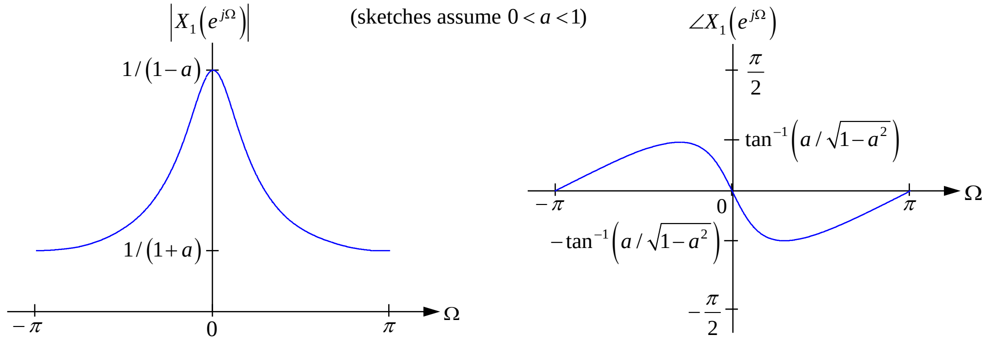

- To compute its DTFT, we evaluate the sum (4):

$$\begin{aligned} X_1\left(e^{j\Omega}\right) &= \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} \left(ae^{-j\Omega}\right)^n \\ &= \frac{1}{1-ae^{-j\Omega}} \end{aligned} .$$

Second line: express $X_1\left(e^{j\Omega}\right)$ as a geometric series.

Third line: sum the series, using $\left|ae^{-j\Omega}\right|<1$.

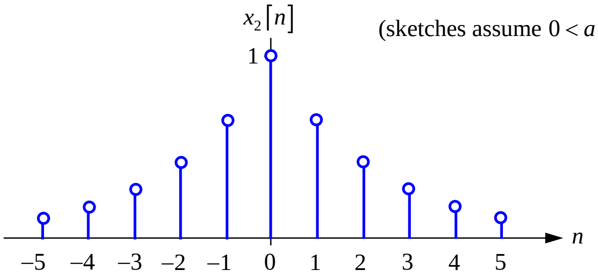
- We obtained this DTFT as the frequency response of a first-order system (Chapter 3, slides 126-131).
- The DTFT is complex-valued, so it is best visualized in terms of magnitude and phase plots. These plots assume $0 < a < 1$.



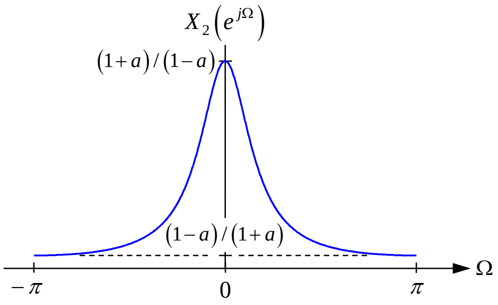

- Observe that:
  - $|a|$ close to 1: the signal $x_1[n]$ is spread out in time and the DTFT $X_1\left(e^{j\Omega}\right)$ is concentrated in frequency near $\Omega = 0$ (and $\Omega = \pm 2\pi, \pm 4\pi, \ldots$).
  - $|a|$ close to 0: the signal $x_1[n]$ is concentrated in time near $n = 0$ and the DTFT $X_1\left(e^{j\Omega}\right)$ is spread out in frequency.
- These observations illustrate an *inverse relationship between time and frequency* in the DTFT, similar to that noted for the CTFT in Chapter 4.

2. *Two-sided real exponential*. The signal and its DTFT are given by

$$x_2[n] = a^{|n|} \overset{F}{\leftrightarrow} X_2\left(e^{j\Omega}\right) = \frac{1-a^2}{1-2a\cos\Omega + a^2}, \; a \text{ real}, \; |a| < 1.$$




(sketches assume $0 < a < 1$)




- We compute its DTFT using (4). We divide the sum into two parts, each like that we computed for $X_1\left(e^{j\Omega}\right)$:

$$\begin{aligned} X_2\left(e^{j\Omega}\right) &= \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} a^n e^{-j\Omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\Omega n} \end{aligned}.$$

- To evaluate the second sum, we change the summation variable to $m=-1-n$, i.e., $n=-1-m$:

$$\begin{aligned} X_2\left(e^{j\Omega}\right) &= \sum_{n=0}^{\infty}\left(ae^{-j\Omega}\right)^n + ae^{j\Omega}\sum_{m=0}^{\infty}\left(ae^{j\Omega}\right)^m \\ &= \frac{1}{1-ae^{-j\Omega}} + \frac{ae^{j\Omega}}{1-ae^{j\Omega}} \\ &= \frac{1-a^2}{1-2a\cos\Omega+a^2} \end{aligned} .$$

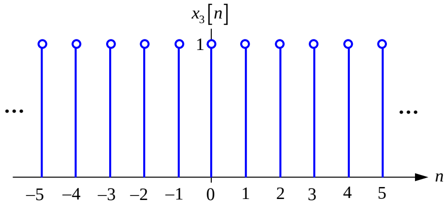
Second line: summed the two geometric series, which converge because $\left|ae^{-j\Omega}\right| = \left|ae^{j\Omega}\right| < 1$.

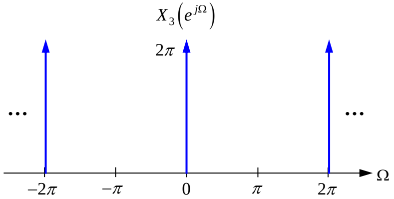
Third line, we added the two terms to obtain a real expression for $X_2\left(e^{j\Omega}\right)$.

- As in Example 1, observe that:
    - $|a|$ close to 1: signal $x_2[n]$ is spread out in time and DTFT $X_2\left(e^{j\Omega}\right)$ is concentrated in frequency near $\Omega=0$ (and $\Omega=\pm 2\pi,\pm 4\pi,\ldots$).
    - $|a|$ close to 0: signal $x_2[n]$ is concentrated in time and DTFT $X_2\left(e^{j\Omega}\right)$ is spread out over frequency.

3. *Constant*. The signal and its DTFT are given by

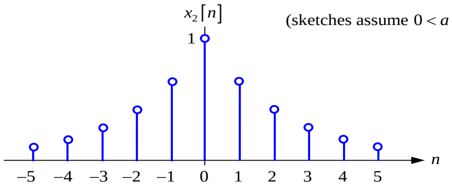
$$x_3[n]=1\ \forall n \overset{F}{\leftrightarrow} X_3(e^{j\Omega})=2\pi\sum_{l=-\infty}^{\infty}\delta(\Omega-l2\pi).$$



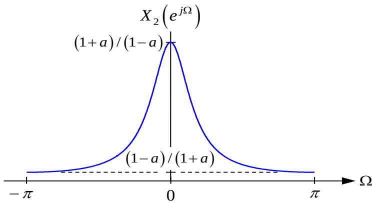




- To derive this, recall the DTFT pair from Example 2:

$$x_2[n]=a^{|n|} \overset{F}{\leftrightarrow} X_2(e^{j\Omega})=\frac{1-a^2}{1-2a\cos\Omega+a^2},\ a \text{ real},\ |a|<1.$$

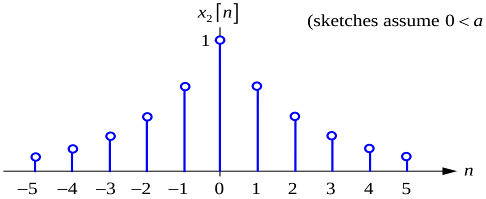



(sketches assume $0<a<1$)

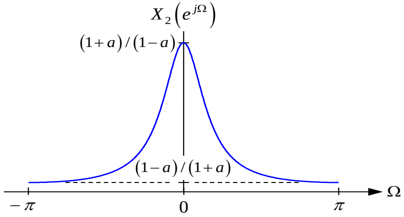



- In the limit $a\to 1$, the two-sided exponential $x_2[n]=a^{|n|}$ becomes the constant $x_3[n]=1$:

$$x_3[n]=\lim_{a\to 1}x_2[n]=\lim_{a\to 1}a^{|n|} \overset{F}{\leftrightarrow} X_3(e^{j\Omega})=\lim_{a\to 1}X_2(e^{j\Omega})=\lim_{a\to 1}\frac{1-a^2}{1-2a\cos\Omega+a^2}.$$




(sketches assume $0 < a < 1$)




- In the limit $a \to 1$, the DTFT $X_2\left(e^{j\Omega}\right)$ has the following properties:
  - It has peaks of zero width and infinite height at $\Omega = 0, \pm 2\pi, \pm 4\pi, \ldots$ .
  - Over any interval of length 2π, the area is

$$\int_{2\pi} X_2\left(e^{j\Omega}\right) d\Omega = \left[ \int_{2\pi} X_2\left(e^{j\Omega}\right) e^{j\Omega n} d\Omega \right]_{n=0} = 2\pi x_2[0] = 2\pi \, .$$

  - Each peak becomes an impulse function of frequency with area 2π.

- We conclude that the DTFT $X_3\left(e^{j\Omega}\right)$ becomes a periodic train of impulses at $\Omega = 0, \pm 2\pi, \pm 4\pi, \ldots$ :

$$X_3\left(e^{j\Omega}\right) = 2\pi \sum_{l=-\infty}^{\infty} \delta\left(\Omega - l2\pi\right).$$

- The signal $x_3[n]$ is maximally spread out in time, while the DTFT $X_3\left(e^{j\Omega}\right)$ is maximally concentrated in frequency.

*Discrete-Time Fourier Transform in the Limit*

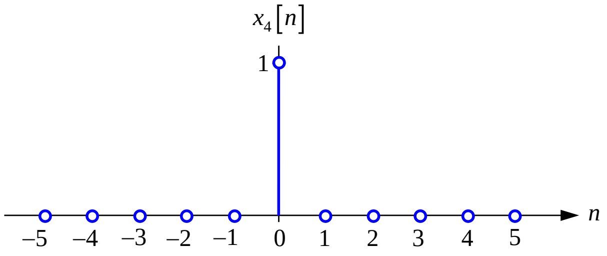
- The signal $x_3[n]$ is not absolutely or absolute-square summable, so $X_3\left(e^{j\Omega}\right)$ does not strictly exist.
- By considering $x_3[n]$ as the limiting case of a signal $x_2[n]$ whose DTFT does exist, we obtain an expression for $X_3\left(e^{j\Omega}\right)$ that includes impulse functions. We say $X_3\left(e^{j\Omega}\right)$ *exists in a generalized sense*.
- Using similar techniques, we obtain generalized DTFTs of other signals.

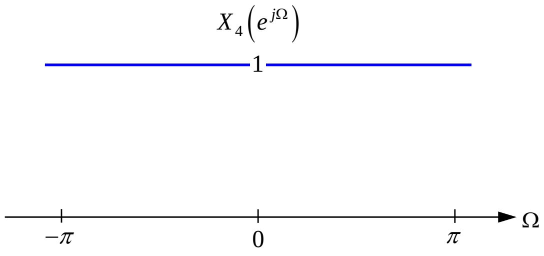
| **Signal $x[n]$** | **DTFT $X(e^{j\Omega})$** |
|---|---|
| $1$ | $2\pi \sum_{l=-\infty}^{\infty} \delta(\Omega - l2\pi)$ |
| $\mathrm{sgn}[n]$ | $\dfrac{1+e^{-j\Omega}}{1-e^{-j\Omega}}$ |
| $u[n] = \frac{1}{2}(1+\mathrm{sgn}[n]+\delta[n])$ | $\dfrac{1}{1-e^{-j\Omega}} + \pi \sum_{l=-\infty}^{\infty} \delta(\Omega - l2\pi)$ |
| $e^{j\Omega_0 n}$ | $2\pi \sum_{l=-\infty}^{\infty} \delta(\Omega - \Omega_0 - l2\pi)$ |
| $\cos\Omega_0 n$ | $\pi \sum_{l=-\infty}^{\infty} \left[\delta(\Omega - \Omega_0 - l2\pi) + \delta(\Omega + \Omega_0 - l2\pi)\right]$ |

| | |
|---|---|
| $\sin \Omega_0 n$ | $\frac{\pi}{j}\sum_{l=-\infty}^{\infty}\left[\delta\left(\Omega-\Omega_0-l2\pi\right)-\delta\left(\Omega+\Omega_0-l2\pi\right)\right]$ |

4. *Unit impulse*. This can be considered the dual of Example 3. The signal and its DTFT are given by

$$x_4[n] = \delta[n] \overset{F}{\leftrightarrow} X_4\left(e^{j\Omega}\right) = 1 \quad \forall \Omega .$$




- We compute the DTFT using (4):

$$\begin{aligned} X_4\left(e^{j\Omega}\right) &= \sum_{n=-\infty}^{\infty} \delta[n] e^{-jn\Omega} \\ &= 1 \quad \forall \Omega \end{aligned} .$$

  We have used the sampling property of the DT impulse to evaluate the sum.

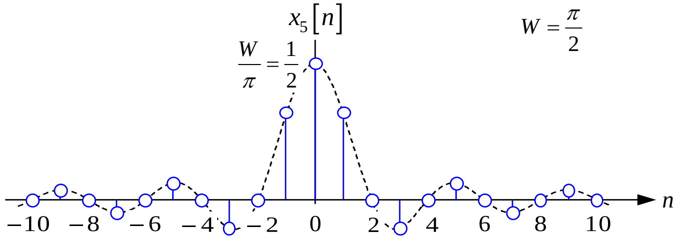
- As $x_4[n]$ is maximally concentrated in time, $X_4\left(e^{j\Omega}\right)$ is maximally spread out in frequency.

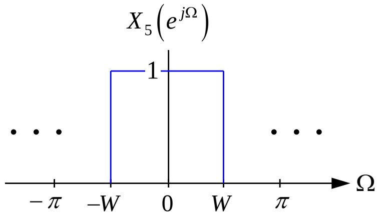
5. *Sinc function*. This describes an ideal lowpass filter with cutoff frequency $W$.

- The signal and its DTFT are

$$x_5[n] = \frac{W}{\pi}\operatorname{sinc}\left(\frac{W}{\pi}n\right) \stackrel{F}{\leftrightarrow} X_5\left(e^{j\Omega}\right) = \begin{cases} 1 & |\Omega| \le W < \pi \\ 0 & W < |\Omega| < \pi \end{cases}, \quad X_5\left(e^{j(\Omega+2\pi)}\right) = X_5\left(e^{j\Omega}\right).$$

These are shown for $W = \pi/2$.




- The DTFT can alternatively be expressed as a sum of shifted rectangular pulses:

$$X_5\left(e^{j\Omega}\right) = \sum_{l=-\infty}^{\infty} \Pi\left(\frac{\Omega - l2\pi}{2W}\right).$$

- To derive the DTFT pair (14), we start with the DTFT $X_5\left(e^{j\Omega}\right)$ and use the inverse DTFT (8). We choose an integration interval $-\pi \le \Omega < \pi$ :

$$
\begin{aligned}
x_5[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_5\left(e^{j\Omega}\right)e^{j\Omega n}d\Omega \\
&= \frac{1}{2\pi}\int_{-W}^{W} e^{j\Omega n}d\Omega \\
&= \frac{W}{\pi}\operatorname{sinc}\left(\frac{W}{\pi}n\right)
\end{aligned} \quad . \qquad (14)
$$

- Values of $W$ close to zero describe a signal spread out in time and a DTFT concentrated in frequency.
- Values of $W$ close to π describe a signal concentrated in time and a DTFT spread out in frequency. When $W = \pi$, $x_5[n] = \operatorname{sinc}(n) = \delta[n]$ and $X_5\left(e^{j\Omega}\right) = 1 \ \forall\Omega$, corresponding to Example 4.
- Using the inverse DTFT (8), the value of a time signal at $n = 0$ equals 1/2π times the area under one period of its DTFT. We can use this to check any DTFT we evaluate. In this example, we have

$$
\begin{aligned}
x_5[0] &= \left.\frac{1}{2\pi}\int_{-\pi}^{\pi} X_5\left(e^{j\Omega}\right)e^{j\Omega n}d\Omega\right|_{n=0} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_5\left(e^{j\Omega}\right)d\Omega \\
&= \frac{W}{\pi}
\end{aligned} \quad .
$$

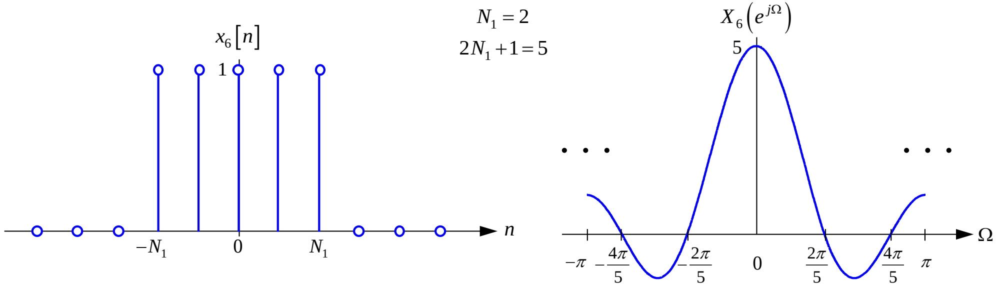
This agrees with (14).

6. *Rectangular pulse*. This can be considered the dual of Example 5.

- The signal and its DTFT are

$$x_6[n] = \Pi\left(\frac{n}{2N_1}\right) = \begin{cases} 1 & |n| \le N_1 \\ 0 & |n| > N_1 \end{cases} \stackrel{F}{\leftrightarrow} X_6\left(e^{j\Omega}\right) = \frac{\sin\left(\Omega\left(N_1 + \frac{1}{2}\right)\right)}{\sin\left(\frac{\Omega}{2}\right)}.$$

These are shown for $N_1 = 2$, corresponding to a pulse width $2N_1 + 1 = 5$.




- We compute the DTFT using (4):

$$X_6\left(e^{j\Omega}\right) = \sum_{n=-N_1}^{N_1} e^{-jn\Omega} = e^{j\Omega N_1} \sum_{l=0}^{2N_1} e^{-jl\Omega} . \tag{15}$$

We have changed the summation variable from $n$ to $l = n + N_1$.

- In evaluating the sum (15), we consider two different cases:
  - When $\Omega = 0, \pm 2\pi, \pm 4\pi, \ldots$ , we have $e^{j\Omega N_1} = 1$ and $e^{-jl\Omega} = 1$, so

$$X_6\left(e^{j\Omega}\right) = e^{j\Omega N_1} \sum_{l=0}^{2N_1} e^{-jl\Omega} = 2N_1 + 1. \tag{16}$$

  - When $\Omega \neq 0, \pm 2\pi, \pm 4\pi, \ldots$, we sum the geometric series (15) to obtain

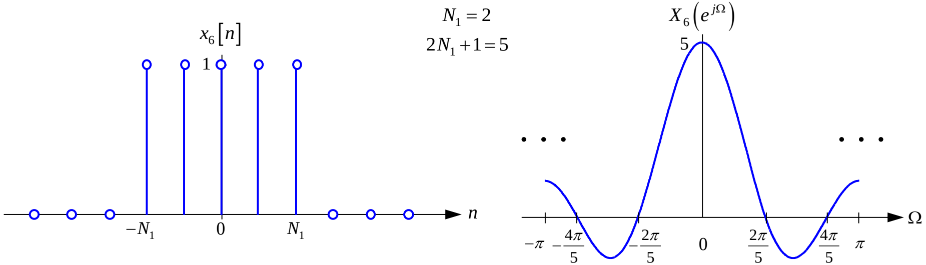
$$X_6\left(e^{j\Omega}\right) = e^{j\Omega N_1} \sum_{l=0}^{2N_1} e^{-jl\Omega} = e^{j\Omega N_1} \frac{1 - e^{-j\Omega(2N_1+1)}}{1 - e^{-j\Omega}}.$$

  Multiplying the numerator and denominator by $e^{j\Omega/2}$, we obtain

$$X_6\left(e^{j\Omega}\right) = \frac{e^{j\Omega\left(N_1 + \frac{1}{2}\right)} - e^{-j\Omega\left(N_1 + \frac{1}{2}\right)}}{e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}} = \frac{\sin\left(\Omega\left(N_1 + \frac{1}{2}\right)\right)}{\sin\left(\frac{\Omega}{2}\right)}. \tag{17}$$

- In the limit $\Omega \to 0, \pm 2\pi, \pm 4\pi, \ldots$, (17) approaches (16). We can express $X_6\left(e^{j\Omega}\right)$ for all $\Omega$ as

$$X_6\left(e^{j\Omega}\right) = \frac{\sin\left(\Omega\left(N_1 + \frac{1}{2}\right)\right)}{\sin\left(\frac{\Omega}{2}\right)} \quad \forall \Omega. \tag{17}$$




- Over the frequency range $-\pi \le \Omega \le \pi$, $X_6(e^{j\Omega})$ appears similar to a sinc function:
  - It peaks at $\Omega = 0$ and decays in amplitude and oscillating as $|\Omega|$ increases.
  - It has its first zeros at $|\Omega| = \dfrac{2\pi}{2N_1+1}$, which are closer to $\Omega = 0$ when $N_1$ is larger.

    A longer pulse $x_6[n]$ corresponds to a DTFT $X_6(e^{j\Omega})$ more concentrated in frequency.
- Unlike a sinc function, the DTFT $X_6(e^{j\Omega})$, given by (17), is a *periodic* function of $\Omega$ with period $2\pi$.
- Using the DTFT (4), for any signal and its DTFT, the sum of the samples of the time signal equals the value of the DTFT at $\Omega = 0, \pm 2\pi, \pm 4\pi, \ldots$. We can use this to verify the correctness of any DTFT we compute. In this example:

$$X_6(e^{j\Omega})\Big|_{\Omega=0,\pm 2\pi,\pm 4\pi,\ldots} = \sum_{n=-\infty}^{\infty} x_6[n] e^{-j\Omega n}\Big|_{\Omega=0,\pm 2\pi,\pm 4\pi,\ldots} = \sum_{n=-\infty}^{\infty} x_6[n] = 2N_1 + 1\,.$$

*Discrete-Time Fourier Transform of Periodic Signals*

- In Chapter 3, we studied how to describe a periodic DT signal $x[n]$ by its DTFS coefficients $a_k$.

  We can also describe a periodic DT signal $x[n]$ by a generalized DTFT $X\left(e^{j\Omega}\right)$, as we now show.

- As in the CT case (Chapter 4), the DTFT of a periodic DT signal is needed when *multiplying* a periodic DT signal by another DT signal (typically aperiodic). Examples include:
  - *Modulating* a DT signal by multiplying it by a DT sinusoid.
  - *Sampling* a DT signal by multiplying it by a DT impulse train.

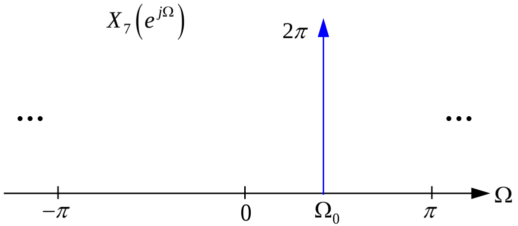
- As in CT, the DTFT of a periodic DT signal is not needed when *convolving* a periodic DT signal with another DT signal (typically aperiodic). We can perform Fourier analysis of such problems using the DTFS, as in Chapter 3.

- We begin the discussion by studying an example.

7. *Imaginary exponential*. The signal and its DTFT are given by

$$x_7[n] = e^{j\Omega_0 n} \overset{F}{\leftrightarrow} X_7\left(e^{j\Omega}\right) = 2\pi \sum_{l=-\infty}^{\infty} \delta\left(\Omega - \Omega_0 - l2\pi\right). \tag{18}$$

- The DTFT $X_7\left(e^{j\Omega}\right)$ is shown.




- We use the inverse DTFT (8) to find the corresponding time signal:

$$\begin{aligned} x_7[n] &= \frac{1}{2\pi} \int_{2\pi} X_7\left(e^{j\Omega}\right) e^{j\Omega n} d\Omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi\delta\left(\Omega - \Omega_0\right) e^{j\Omega n} d\Omega, \\ &= e^{j\Omega_0 n} \end{aligned}$$

- We have used the sampling property of the impulse function in evaluating the integral.

*General Periodic Signal*

- Now we consider a signal $x[n]$ that is periodic with period $N = \dfrac{2\pi}{\Omega_0}$ and is synthesized using a DTFS:

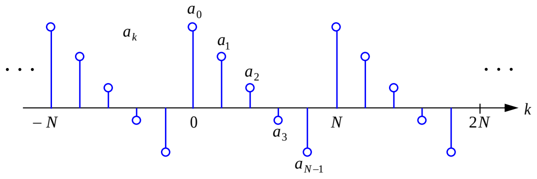
$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n} \,. \qquad (19)$$

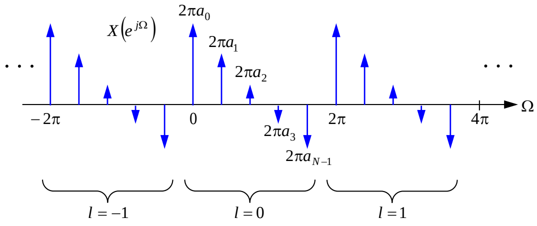
  The $a_k$ are the DTFS coefficients for the signal $x[n]$. They are periodic in $k$ with period $N$: $a_{k+N} = a_k$.

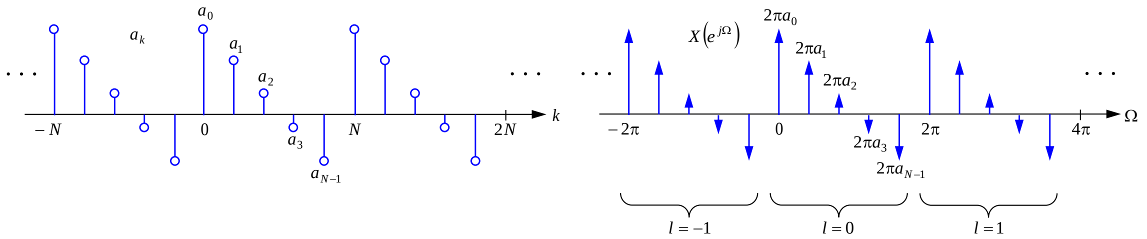
- We compute the DTFT of (19) term-by-term using the linearity of the DTFT:

$$X\left(e^{j\Omega}\right) = \sum_{k=\langle N \rangle} 2\pi a_k \sum_{l=-\infty}^{\infty} \delta\left(\Omega - k\Omega_0 - l2\pi\right). \qquad (20)$$

- We show some DTFS coefficients $a_k$ (not for a real signal $x[n]$) and the corresponding DTFT (20).
  - We choose the $N$ consecutive values of $k$, $k = \langle N \rangle$, to be $0 \le k \le N-1$.
  - Each value of $l$ in (20) contributes a set of $N$ impulses scaled by $2\pi a_k$, $0 \le k \le N-1$.






- We can replace the double summation over $k$ and $l$ in (20) by a single summation over $k$, $-\infty < k < \infty$. We express the result as a DTFT pair:

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n} \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - k\Omega_0\right). \tag{21}$$

- The DTFT of a periodic signal is a *train of impulses* at $\Omega = k\Omega_0$, integer multiples of $\Omega_0 = \dfrac{2\pi}{N}$.

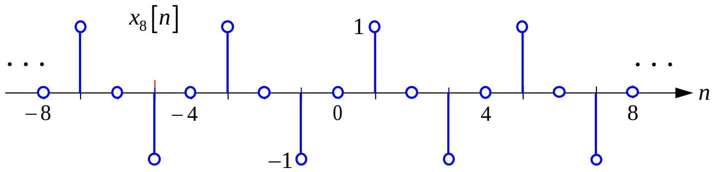
  Each impulse is scaled by $2\pi$ times the corresponding DTFS coefficient $a_k$.

- Now we present two more examples of DTFTs of periodic DT signals.

8. *Sine function*. The signal and its DTFT are given by

$$x_8[n] = \sin\left(\frac{\pi}{2}n\right) \overset{F}{\leftrightarrow} X_8\left(e^{j\Omega}\right) = \frac{\pi}{j}\sum_{l=-\infty}^{\infty}\left[\delta\left(\Omega - \frac{\pi}{2} - l2\pi\right) - \delta\left(\Omega + \frac{\pi}{2} - l2\pi\right)\right].$$

- The signal $x_8[n]$ is shown.




- We find the DTFS coefficients by inspection (see Chapter 3, slides 93-94). The signal has period $N = 4$.

We express it as a linear combination of imaginary exponentials with frequencies $k\Omega_0 = k\dfrac{2\pi}{N} = k\dfrac{\pi}{2}$:

$$x_8[n] = \frac{1}{2j}\left(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}\right).$$

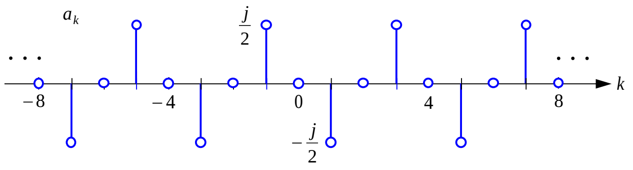
The DTFS coefficients are

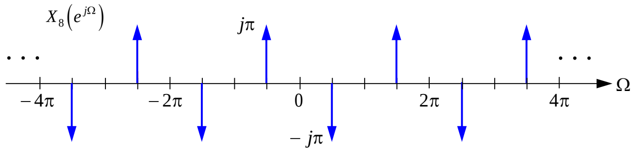
$$a_k = \begin{cases} \frac{j}{2} & k = -1 \\ -\frac{j}{2} & k = 1 \\ 0 & k = 0, 2 \end{cases}, \quad a_{k+4} = a_k .$$

- We obtain the DTFT $X_8\left(e^{j\Omega}\right)$ using (21):

$$X_8\left(e^{j\Omega}\right)=\frac{\pi}{j}\sum_{l=-\infty}^{\infty}\left[\delta\left(\Omega-\frac{\pi}{2}-l2\pi\right)-\delta\left(\Omega+\frac{\pi}{2}-l2\pi\right)\right].$$

- The figure shows the DTFS coefficients $a_k$ (left) and the DTFT $X_8\left(e^{j\Omega}\right)$ (right).

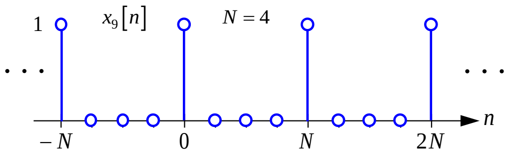


9. *Periodic impulse train*. The signal and its DTFT are given by

$$x_9[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN] \overset{F}{\leftrightarrow} X_9\left(e^{j\Omega}\right) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{N}\right), \tag{22}$$

- The signal $x_9[n]$ is periodic with period $N$ and fundamental frequency $\Omega_0 = \frac{2\pi}{N}$. It is shown for $N = 4$.




- We compute the DTFS coefficients for $x_9[n]$ using the DTFS analysis equation:

$$\begin{aligned} a_k &= \frac{1}{N} \sum_{n=\langle N \rangle} x_9[n] e^{-jk\Omega_0 n} \\ &= \frac{1}{N} \sum_{n=\langle N \rangle} \delta[n] e^{-jk\Omega_0 n} \,. \\ &= \frac{1}{N} \quad \forall k \end{aligned} \tag{23}$$

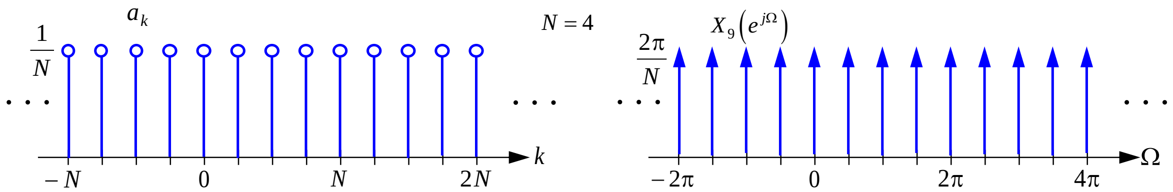
  - First line: we choose the summation interval $n = \langle N \rangle$ to include the origin $n = 0$.
  - Second line: $x_9[n]$ given by (22) is a sum over all $k$, but only $k = 0$ lies in the summation interval.
  - Third line: we use the sampling property of the DT impulse function.

- We use (21) to obtain the DTFT of $x_9[n]$:

$$X_9\left(e^{j\Omega}\right) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - k\Omega_0\right) \\ = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{N}\right) . \tag{24}$$

The *DTFT of a periodic impulse train is a periodic impulse train.*

- The DTFS coefficients $a_k$ and the DTFT $X_9\left(e^{j\Omega}\right)$ are shown in the figure below, assuming $N = 4$.




- Note the inverse relationship between the spacing of impulses in the time domain, $N$, and the spacing between the impulses in the frequency domain, $\frac{2\pi}{N}$.

- We observed a similar relationship for the CT impulse train and its CTFT.

## Properties of Discrete-Time Fourier Transform

- Like the CTFS, DTFS and CTFT properties we have studied, these DTFT properties are helpful for:
  - Computing DTFTs for new signals by using the DTFTs we already know for other signals.
  - Verifying DTFTs that we compute for new signals.
- A complete list of DTFT properties is given in Table 5 in the Appendix. We discuss some of the most important properties here.
- We consider one or two signals and their DTFTs. Initially we denote these as

$$x[n] \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right) \quad \text{and} \quad y[n] \overset{F}{\leftrightarrow} Y\left(e^{j\Omega}\right).$$

- Many properties of the DTFT are similar to CTFT properties, and we present those first.
- Then we discuss several DTFT properties that are significantly different from CTFT properties.

## Properties Similar to Continuous-Time Fourier Transform

*Linearity*

- A linear combination of $x[n]$ and $y[n]$ has a DTFT given by the corresponding linear combination of the DTFTs $X\left(e^{j\Omega}\right)$ and $Y\left(e^{j\Omega}\right)$:

$$ax[n]+by[n]\overset{F}{\leftrightarrow}aX\left(e^{j\Omega}\right)+bY\left(e^{j\Omega}\right).$$

*Time Shift*

- Time-shifting a signal by an integer $n_0$ corresponds to multiplication of its DTFT by a factor $e^{-j\Omega n_0}$:

$$x[n-n_0]\overset{F}{\leftrightarrow}e^{-j\Omega n_0}X\left(e^{j\Omega}\right). \tag{26}$$

The magnitude and phase of $e^{-j\Omega n_0}X\left(e^{j\Omega}\right)$ are related to those of $X\left(e^{j\Omega}\right)$ by

$$\begin{cases} \left|e^{-j\Omega n_0}X\left(e^{j\Omega}\right)\right| = \left|X\left(e^{j\Omega}\right)\right| \\ \angle\left(e^{-j\Omega n_0}X\left(e^{j\Omega}\right)\right) = \angle X\left(e^{j\Omega}\right)-\Omega n_0 \end{cases}. \tag{26$'$}$$

Time-shifting a signal by $n_0$ modifies its DTFT by

- Leaving the magnitude unchanged.
- Adding a phase shift proportional to $-n_0$, which varies linearly with frequency $\Omega$.

- *Proof*: given a time-shifted signal $x[n-n_0]$, we compute its DTFT $F\left[x[n-n_0]\right]$ using (4):

$$F\left[x[n-n_0]\right]=\sum_{n=-\infty}^{\infty}x[n-n_0]e^{-j\Omega n}.$$

We change the variable of summation to $m=n-n_0$, and the DTFT becomes

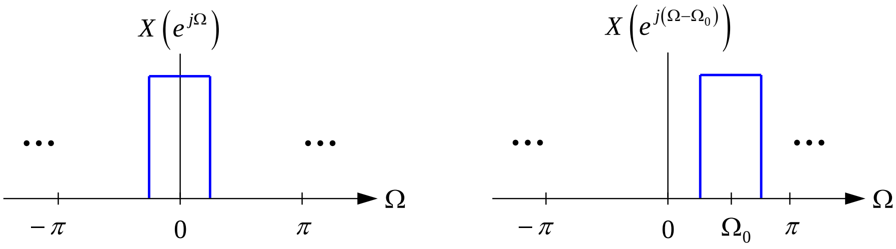
$$\begin{aligned}F\left[x[n-n_0]\right]&=\sum_{m=-\infty}^{\infty}x[m]e^{-j\Omega(m+n_0)}\\&=e^{-j\Omega n_0}\sum_{m=-\infty}^{\infty}x[m]e^{-j\Omega m}\\&=e^{-j\Omega n_0}X\left(e^{j\Omega}\right)\end{aligned}$$

*Frequency Shift*

- The frequency-shift property is the dual of the time-shift property.
- It states that multiplying a signal by an imaginary exponential time signal $e^{j\Omega_0 n}$ causes its DTFT to be frequency-shifted by $\Omega_0$:

$$x[n]e^{j\Omega_0 n}\overset{F}{\leftrightarrow}X\left(e^{j(\Omega-\Omega_0)}\right). \tag{27}$$

- A DTFT $X\left(e^{j\Omega}\right)$ and the frequency-shifted DTFT $X\left(e^{j(\Omega-\Omega_0)}\right)$ are shown.




- *Proof. Using* (4), the DTFT of $x[n]e^{j\Omega_0 n}$ is

$$F\left[x[n]e^{j\Omega_0 n}\right] = \sum_{n=-\infty}^{\infty} x[n]e^{j\Omega_0 n}e^{-j\Omega n}$$
$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\Omega-\Omega_0)n} \quad .$$
$$= X\left(e^{j(\Omega-\Omega_0)}\right)$$

- Frequency shifting:
  - Is the basis for amplitude modulation of DT signals.
  - Can transform one type of DT filter into another, e.g., transform a lowpass filter into a bandpass filter or a highpass filter.

*Time Reversal*

- Reversal in time corresponds to reversal in frequency:

$$x[-n] \overset{F}{\leftrightarrow} X\left(e^{-j\Omega}\right). \tag{28}$$

- If a signal is even in time, its DTFT is even in frequency:

$$x[-n] = x[n] \overset{F}{\leftrightarrow} X\left(e^{-j\Omega}\right) = X\left(e^{j\Omega}\right).$$

  If a signal is odd in time, its DTFT is odd in frequency:

$$x[-n] = -x[n] \overset{F}{\leftrightarrow} X\left(e^{-j\Omega}\right) = -X\left(e^{j\Omega}\right).$$

*Conjugation*

$$x^*[n] \overset{F}{\leftrightarrow} X^*\left(e^{-j\Omega}\right). \tag{29}$$

- *Proof*: using (4), the DTFT of $x^*[n]$ is

$$\begin{aligned} F\left[x^*[n]\right] &= \sum_{n=-\infty}^{\infty} x^*[n] e^{-j\Omega n} \\ &= \left(\sum_{n=-\infty}^{\infty} x[n] e^{j\Omega n}\right)^* . \\ &= X^*\left(e^{-j\Omega}\right) \end{aligned}$$

*Conjugate Symmetry for Real Signal*

- A real signal $x[n]$ is equal to its complex conjugate $x^*[n]$. In combination with the conjugation property, this implies that

$$x[n]=x^*[n] \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right)=X^*\left(e^{-j\Omega}\right). \qquad (30)$$

  If a signal is real, its DTFT is *conjugate symmetric*: the DTFT at positive frequency is equal to the complex conjugate of the DTFT at negative frequency.

- The conjugate symmetry property can be restated in two ways. If a signal is real, then:

  - The magnitude of its DTFT is even in frequency, while the phase of its DTFT is odd in frequency:

$$x[n]=x^*[n] \overset{F}{\leftrightarrow} \begin{cases} \left|X\left(e^{j\Omega}\right)\right|=\left|X\left(e^{-j\Omega}\right)\right| \\ \angle X\left(e^{j\Omega}\right)=-\angle X\left(e^{-j\Omega}\right) \end{cases}. \qquad (30a)$$

  - The real part of its DTFT is even in frequency, while the imaginary part of its DTFT is odd in frequency:

$$x[n]=x^*[n] \overset{F}{\leftrightarrow} \begin{cases} \mathrm{Re}\left[X\left(e^{j\Omega}\right)\right]=\mathrm{Re}\left[X\left(e^{-j\Omega}\right)\right] \\ \mathrm{Im}\left[X\left(e^{j\Omega}\right)\right]=-\mathrm{Im}\left[X\left(e^{-j\Omega}\right)\right] \end{cases}. \qquad (30b)$$

*Real, Even or Real, Odd Signals*

- Combining the time reversal and conjugation properties, we find that

$$x[n] \text{ real and even in } n \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right) \text{ real and even in } \Omega$$

and

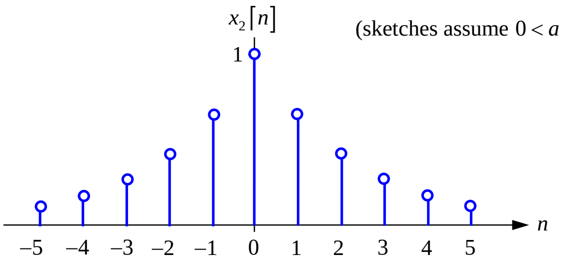
$$x[n] \text{ real and odd in } n \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right) \text{ imaginary and odd in } \Omega\,.$$

*Examples of Symmetry Properties*

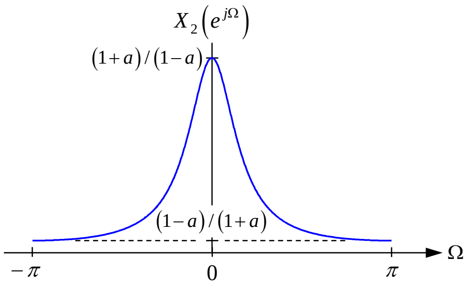
2. *Real and even signal*. Recall Example 2:

$$x_2[n] = a^{|n|} \overset{F}{\leftrightarrow} X_2\left(e^{j\Omega}\right) = \frac{1-a^2}{1-2a\cos\Omega + a^2}\,, \quad |a|<1\,.$$

- The signal is real and even in $n$, so the DTFT is real and even in $\Omega$.



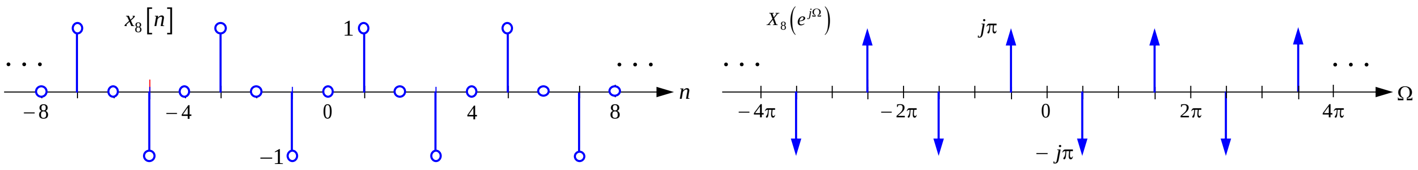

(sketches assume $0 < a < 1$)



8. *Real and odd signal*. Recall Example 8:

$$x_8[n] = \sin\left(\frac{\pi}{2}n\right) \overset{F}{\leftrightarrow} X_8\left(e^{j\Omega}\right) = \frac{\pi}{j}\sum_{l=-\infty}^{\infty}\left[\delta\left(\Omega - \frac{\pi}{2} - l2\pi\right) - \delta\left(\Omega + \frac{\pi}{2} - l2\pi\right)\right].$$

- The signal is real and odd in $n$, so the DTFT is imaginary and odd in $\Omega$.




*Differentiation in Frequency*

- This property states that

$$nx[n] \overset{F}{\leftrightarrow} j\frac{dX\left(e^{j\Omega}\right)}{d\Omega}. \qquad (31)$$

- Multiplying a signal by time $n$ corresponds to differentiating its DTFT with respect to frequency $\Omega$ (and scaling by a factor $j$).

- To prove this property, we differentiate the analysis equation (4) with respect to $\Omega$, finding that $dX\left(e^{j\Omega}\right)/d\Omega$ is the DTFT of a signal $-jnx[n]$.

*Example of Differentiation-in-Frequency Property*

10. *Impulse response of second-order system*. In this example, we derive the DTFT pair

$$x_{10}[n]=(n+1)a^n u[n] \overset{F}{\leftrightarrow} X_{10}\left(e^{j\Omega}\right)=\frac{1}{\left(1-ae^{-j\Omega}\right)^2}, \quad a \text{ real}, \quad |a|<1. \qquad (32)$$

- We will use (32) to describe second-order systems with $0 \le r < 1$ and $\theta = 0$ or $\pi$ (see slides 83-89).
- To prove (32), we start with the DTFT pair derived in Example 1:

$$x_1[n]=a^n u[n] \overset{F}{\leftrightarrow} X_1\left(e^{j\Omega}\right)=\frac{1}{1-ae^{-j\Omega}}.$$

- Applying the differentiation-in-frequency property:

$$nx_1[n]=na^n u[n] \overset{F}{\leftrightarrow} j\frac{dX_1\left(e^{j\Omega}\right)}{d\Omega}=\frac{ae^{-j\Omega}}{\left(1-ae^{-j\Omega}\right)^2}.$$

- Using the time-shifting property with $n_0=-1$:

$$(n+1)a^{n+1}u[n+1] \overset{F}{\leftrightarrow} \frac{a}{\left(1-ae^{-j\Omega}\right)^2}.$$

- At time $n=-1$, the signal $(n+1)a^{n+1}u[n+1]$ vanishes, so it can be rewritten as $(n+1)a^{n+1}u[n]$. Dividing both sides of the last expression by *a*, we obtain (32).

**Convolution Property**

- This is identical in form to the convolution property for the CTFT. Consider two DT signals and their DTFTs

$$p[n] \overset{F}{\leftrightarrow} P\left(e^{j\Omega}\right) \quad \text{and} \quad q[n] \overset{F}{\leftrightarrow} Q\left(e^{j\Omega}\right).$$

- The convolution property states that

$$p[n]*q[n] \overset{F}{\leftrightarrow} P\left(e^{j\Omega}\right)Q\left(e^{j\Omega}\right). \tag{33}$$

- *Convolution in the time domain* corresponds to *multiplication in the frequency domain*.

*Proof of Convolution Property*

- We express the convolution as a sum and compute its DTFT using (4):

$$F\left[p[n]*q[n]\right]=\sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty}p[k]q[n-k]\right]e^{-jn\Omega} .$$

- Now we interchange the order of summation:

$$F\left[p[n]*q[n]\right]=\sum_{k=-\infty}^{\infty}p[k]\left[\sum_{n=-\infty}^{\infty}q[n-k]e^{-jn\Omega}\right].$$

- The quantity in square brackets is the DTFT of $q[n-k]$.

  By the time-shift property (26), this is $Q\left(e^{j\Omega}\right)e^{-jk\Omega}$.

- Thus, we have

$$F\left[p[n]*q[n]\right]=Q\left(e^{j\Omega}\right)\sum_{k=-\infty}^{\infty}p[k]e^{-jk\Omega}.$$

- We recognize the sum as $P\left(e^{j\Omega}\right)$, the DTFT of $p[n]$. We have proven (33).

*Example of Convolution Property*

- In a homework problem on cascading two first-order DT systems, you derived the convolution

$$a^n u[n] * b^n u[n] = \frac{b^{n+1} - a^{n+1}}{b-a} u[n], \qquad a \neq b .$$

- Here we study the case in which $a = b$. We start with the result of Example 1:

$$x_1[n] = a^n u[n] \overset{F}{\leftrightarrow} X_1\left(e^{j\Omega}\right) = \frac{1}{1 - ae^{-j\Omega}} .$$

- Applying the convolution property, we have $x_1[n] * x_1[n] \overset{F}{\leftrightarrow} X_1^2\left(e^{j\Omega}\right)$, or

$$a^n u[n] * a^n u[n] \overset{F}{\leftrightarrow} \frac{1}{\left(1 - ae^{-j\Omega}\right)^2} . \tag{34}$$

- But we know from Example 10 that

$$(n+1) a^n u[n] \overset{F}{\leftrightarrow} \frac{1}{\left(1 - ae^{-j\Omega}\right)^2} . \tag{32}$$

- Since the right-hand sides of (34) and (32) are equal, the left-hand sides must be equal:

$$a^n u[n] * a^n u[n] = (n+1) a^n u[n] . \tag{34.1}$$

- We have shown that the second-order system with impulse and frequency responses given by (32) is equivalent to a cascade of two identical first-order systems.

*Question*: can you derive expression (34.1) by using convolution? This is more straightforward than the DTFT method we used above.

**Frequency Response of Discrete-Time Linear Time-Invariant Systems**

- The most important application of the DTFT convolution property is filtering of signals by LTI systems.
- Consider a DT LTI system with impulse response $h[n]$. The frequency response $H\left(e^{j\Omega}\right)$ is the DTFT of $h[n]$, assuming it exists. The impulse response and frequency response *form a DTFT pair*:

$$h[n] \overset{F}{\leftrightarrow} H\left(e^{j\Omega}\right). \tag{35}$$

- Suppose an input signal $x[n] \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right)$ is fed into the system, yielding an output signal $y[n] \overset{F}{\leftrightarrow} Y\left(e^{j\Omega}\right)$.

$$x[n] \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right) \longrightarrow \boxed{h[n] \overset{F}{\leftrightarrow} H\left(e^{j\Omega}\right)} \longrightarrow y[n] \overset{F}{\leftrightarrow} Y\left(e^{j\Omega}\right)$$

- In the *time domain*, the output signal is obtained by *convolving* the input signal and the impulse response:

$$y[n] = h[n] * x[n]. \tag{36}$$

- Using the convolution property (33), the DTFT of the output signal (36) is the right-hand side of

$$y[n] = h[n] * x[n] \overset{F}{\leftrightarrow} Y\left(e^{j\Omega}\right) = H\left(e^{j\Omega}\right) X\left(e^{j\Omega}\right). \tag{37}$$

  In the *frequency domain*, the DTFT of the output signal is found by *multiplying* the DTFT of the input signal by the frequency response of the system.

- Viewing DT LTI filtering as frequency-domain multiplication is intuitively appealing, as in CT. In many problems, it provides an easier method of solution than time-domain convolution.

- All the properties of the frequency response studied in Chapter 3 can be understood in terms of DTFT properties studied in this chapter.

- For example, the frequency response is always periodic in frequency $\Omega$ with period $2\pi$:

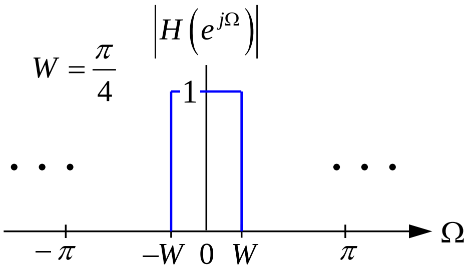
$$H\left(e^{j(\Omega+2\pi)}\right)=H\left(e^{j\Omega}\right). \tag{5′}$$

- Also, if the impulse response $h[n]$ is real, by (30), the frequency response has conjugate symmetry:

$$h[n]=h^*[n] \overset{F}{\leftrightarrow} H\left(e^{j\Omega}\right)=H^*\left(e^{-j\Omega}\right). \tag{30c}$$

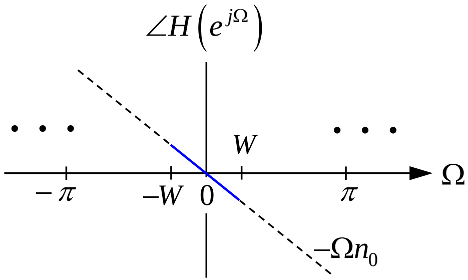
*Example: Ideal Lowpass Filter*

- We study an *ideal lowpass filter* with cutoff frequency $W$ and group delay $n_0$ .
- It is similar to an ideal CT lowpass filter (Chapter 4, slides 64-65), except its frequency response $H\left(e^{j\Omega}\right)$ must be periodic in Ω. We describe $H\left(e^{j\Omega}\right)$ over a single period, $-\pi \le \Omega \le \pi$ .




*Magnitude* $\left|H\left(e^{j\Omega}\right)\right|$

- Constant in the *passband* $|\Omega| < W$ .
- Abrupt *cutoff* or *transition* at $|\Omega| = W$ .
- Zero in the *stopband* $W < |\Omega| < \pi$ .




*Phase* $\angle H\left(e^{j\Omega}\right)$

- Linear in the passband $|\Omega| < W$ .

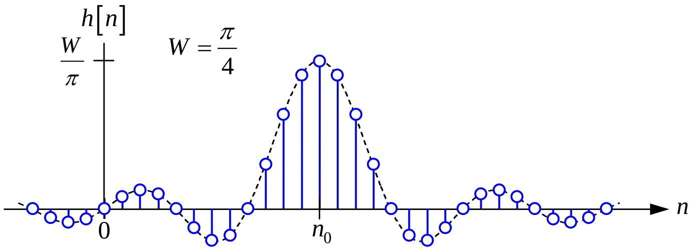
  Slope corresponds to constant integer-valued group delay $-d\angle H\left(e^{j\Omega}\right)/d\Omega = n_0$ .

- Can assume any value in the stopband $W < |\Omega| < \pi$ , as indicated by the dashed lines.

- Using the sinc time signal's DTFT (14) and the time-shift property (26), the impulse and frequency responses are

$$h[n]=\frac{W}{\pi}\operatorname{sinc}\left(\frac{W(n-n_0)}{\pi}\right) \overset{F}{\leftrightarrow} H\left(e^{j\Omega}\right)=e^{-j\Omega n_0}\sum_{l=-\infty}^{\infty}\Pi\left(\frac{\Omega-l2\pi}{2W}\right). \tag{38}$$

- On the right-hand side of (38):
  - The summation makes $H\left(e^{j\Omega}\right)$ periodic in Ω with period 2π.
  - The linear phase factor need not be inside the summation, since $e^{-j(\Omega-l2\pi)n_0}=e^{-j\Omega n_0}\underbrace{e^{jl2\pi n_0}}_{=1}=e^{-j\Omega n_0}$.
- The impulse response (shown for $W=\pi/4$) peaks at $n=n_0$, but extends to $n=\pm\infty$.
  An ideal DT lowpass filter cannot be causal, except in the limit of infinite group delay, $n_0\to\infty$.




- We will often set the group delay $n_0$ to zero to simplify our analyses. Nevertheless, a causal filter that approximates the abrupt transition of an ideal filter must have a long group delay.

*Example: Ideal Lowpass Filter with Sinc Function Input*

- This example shows how frequency-domain multiplication can provide an easier solution than time-domain convolution.
- We are given:
  - Input signal:

$$x[n] = \frac{1}{2}\operatorname{sinc}\left(\frac{n}{2}\right).$$

  - System impulse response:

$$h[n] = \frac{1}{4}\operatorname{sinc}\left(\frac{n}{4}\right).$$

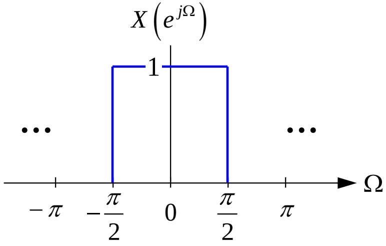
    Corresponds to ideal lowpass filter (38) with $W = \pi / 4$ and $n_0 = 0$.

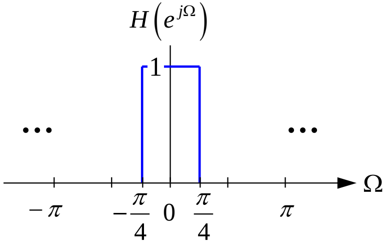
- We wish to compute the output signal.
- To use time-domain convolution, we must compute

$$y[n] = x[n] * h[n] = \frac{1}{2}\operatorname{sinc}\left(\frac{n}{2}\right) * \frac{1}{4}\operatorname{sinc}\left(\frac{n}{4}\right).$$

  This is difficult to evaluate.

- To use frequency-domain multiplication, we compute the DTFT of the input $X\left(e^{j\Omega}\right)$ and the system frequency response $H\left(e^{j\Omega}\right)$.




- The DTFT of the output is their product:

$$Y\left(e^{j\Omega}\right) = X\left(e^{j\Omega}\right)H\left(e^{j\Omega}\right) = H\left(e^{j\Omega}\right).$$

- The output signal is

$$y[n] = h[n] = \frac{1}{4}\text{sinc}\left(\frac{n}{4}\right).$$

**Frequency Response of Cascaded Linear Time-Invariant Systems**

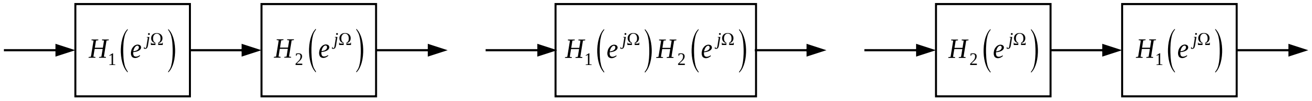
- Consider two LTI systems

$$h_1[n] \overset{F}{\leftrightarrow} H_1\left(e^{j\Omega}\right) \text{ and } h_2[n] \overset{F}{\leftrightarrow} H_2\left(e^{j\Omega}\right).$$

- Recall that when two LTI systems are cascaded, the overall impulse response is the convolution of their impulse responses. It does not depend on the order in which the two systems are cascaded (see Chapter 2, slides 55-56).

- By the convolution property of the CTFT, the overall frequency response of the cascade is the product of the frequency responses of the two systems. It does not depend on the order in which the two systems are cascaded.

- The following three LTI systems yield identical input-output relationships.



**Properties Different from Continuous-Time Fourier Transform**

- These DTFT properties differ significantly from the corresponding CTFT properties because:
    - DT signals are functions of a *discrete* time variable *n*.
    - DTFTs are *periodic* functions of frequency Ω.
- We consider one or two signals and their DTFTs. We denote these for now as

$$x[n] \overset{F}{\leftrightarrow} X\left(e^{j\Omega}\right) \quad \text{and} \quad y[n] \overset{F}{\leftrightarrow} Y\left(e^{j\Omega}\right).$$

*Periodicity*

- Any DTFT must be a periodic function of frequency Ω with period 2π:

$$X\left(e^{j(\Omega+2\pi)}\right) = X\left(e^{j\Omega}\right).$$

*First Difference*

- Taking the first difference of a DT signal corresponds to *multiplying* its DTFT by a factor $1-e^{-j\Omega}$:

$$x[n]-x[n-1] \overset{F}{\leftrightarrow} \left(1-e^{-j\Omega}\right)X\left(e^{j\Omega}\right).$$

- This is not a separate property. It is a consequence of linearity and the time-shift property (26).
- The first difference of a DT signal may seem somewhat analogous to the time derivative of a CT signal.
- We can design FIR DT filters that approximate CT differentiation better than a DT first difference system (Chapter 6).

*Running Summation (Accumulation)*

- The running summation of a time-domain signal corresponds to *dividing* its DTFT by a factor $1-e^{-j\Omega}$:

$$\begin{aligned}\sum_{m=-\infty}^{n} x[m] &\overset{F}{\leftrightarrow} \frac{1}{1-e^{-j\Omega}} X\left(e^{j\Omega}\right) + \pi X\left(e^{j\Omega}\right) \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k2\pi\right) \\ &= \frac{1}{1-e^{-j\Omega}} X\left(e^{j\Omega}\right) + \pi X\left(e^{j0}\right) \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k2\pi\right)\end{aligned} . \qquad (39)$$

- This is expected because the time-domain first difference corresponds to *multiplying* the DTFT by $1-e^{-j\Omega}$.

- As in the CTFT integration property, there is an additional term on the right-hand side of (39):

$$\pi X\left(e^{j\Omega}\right) \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k2\pi\right) = \pi X\left(e^{j0}\right) \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k2\pi\right).$$

This is nonzero if the original time-domain signal $x[n]$ has a non-zero d.c. value:

$$X\left(e^{j0}\right) = \sum_{n=-\infty}^{\infty} x[n] \neq 0 .$$

- *Question*: what property of the unit impulse and what property of the DTFT did we use in replacing $X(e^{j\Omega})$ by $X(e^{j0})$ above?

*Example of Accumulation Property*

*11. Unit step function*. The signal and its DTFT are given by

$$x_{11}[n]=u[n] \overset{F}{\leftrightarrow} X_{11}\left(e^{j\Omega}\right)=\frac{1}{1-e^{-j\Omega}}+\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-k2\pi).$$

- Recall that the unit step is the running summation of the unit impulse:

$$u[n]=\sum_{m=-\infty}^{n}\delta[m].$$

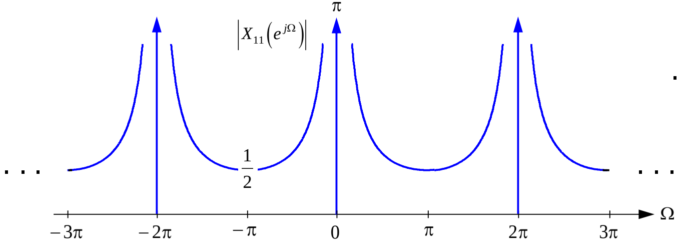
- Recall the DTFT of the unit impulse:

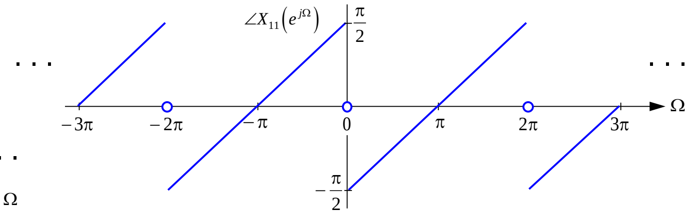
$$x_4[n]=\delta[n] \overset{F}{\leftrightarrow} X_4\left(e^{j\Omega}\right)=1 \quad \forall\Omega.$$

- Using the accumulation property (39), we find the DTFT of the unit step is

$$X_{11}\left(e^{j\Omega}\right)=\frac{1}{1-e^{-j\Omega}}+\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-k2\pi).$$

The DTFT $X_{11}\left(e^{j\Omega}\right)$ is complex-valued, so we plot its magnitude and phase.



**Time Scaling** **(Skip slides 57-60.)**

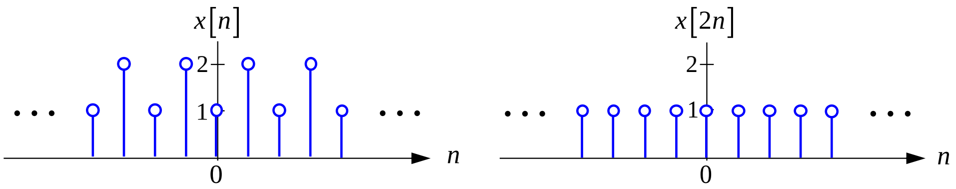
- We briefly review time compression and time expansion for DT signals (see Chapter 1, slides 23-24).

*Time Compression*

- Consider a positive integer $k \geq 1$ . Given a DT signal $x[n]$, the *compressed signal* is

$$x[kn].$$

- If $k > 1$, samples of the signal are lost. An example is shown for $k = 2$ .



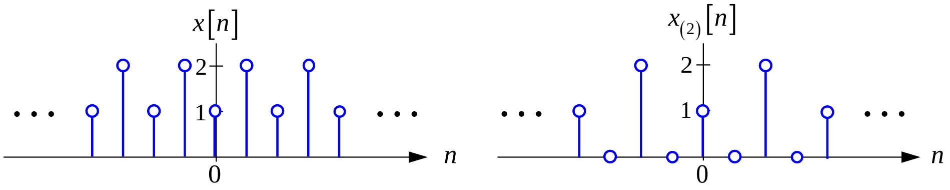

- We will not consider time compression further in EE 102A.

*Time Expansion*

- Consider a positive integer $m \ge 1$. Given a DT signal $x[n]$, the *expanded signal* is

$$x_{(m)}[n] = \begin{cases} x\left[\dfrac{n}{m}\right] & \dfrac{n}{m} \text{ integer} \\ 0 & \text{otherwise} \end{cases}. \tag{40}$$

- For any positive integer *m*, no samples of the signal are lost. An example is shown for $m = 2$.



*DTFT of Time-Expanded Signal*

- Now we compute the DTFT of the time-expanded signal (40), which we denote by $X_{(m)}\left(e^{j\Omega}\right)$.

  Using the DTFT analysis equation (4), we have

$$X_{(m)}\left(e^{j\Omega}\right)=\sum_{n=-\infty}^{\infty} x_{(m)}[n]e^{-j\Omega n} .$$

- Note that $x_{(m)}[n]=0$ unless $\frac{n}{m}$ is an integer. We change the summation index to $l=\frac{n}{m}$ (so that $n=lm$):

$$X_{(m)}\left(e^{j\Omega}\right)=\sum_{l=-\infty}^{\infty} x_{(m)}[lm]e^{-j\Omega lm} .$$

- Now we use the fact that by (40), $x_{(m)}[lm]=x[l]$ to write

$$\begin{aligned} X_{(m)}\left(e^{j\Omega}\right) &= \sum_{l=-\infty}^{\infty} x[l]e^{-j(m\Omega)l} \\ &= X\left(e^{jm\Omega}\right) \end{aligned} .$$

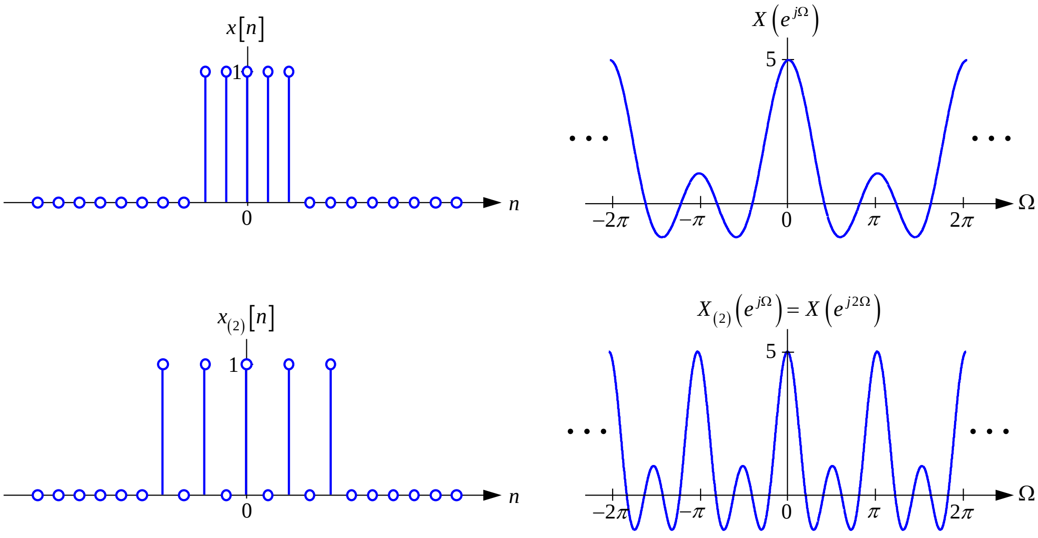
- In summary, we have found that

$$x_{(m)}[n]\overset{F}{\leftrightarrow} X\left(e^{jm\Omega}\right). \tag{41}$$

  In other words, *expanding time* by a factor $m \geq 1$ *compresses frequency* by a factor $m$ in the DTFT.

- Time expansion is used in *digital upsampling*. We discuss this briefly in Chapter 6, and in depth in the *EE 102B Course Reader*, Chapter 2.

- In the figures below, we use the rectangular pulse signal from Example 6 to illustrate time expansion and frequency compression by a factor $m = 2$.



**Parseval’s Identity**

- Parseval’s identity for the DTFT helps us compute
  - the inner product between two DT signals, or
  - the energy of one DT signal

  in either time or frequency.
- Depending on the signal(s), the computation is often easier in one domain or the other.

*Inner Product Between Signals*

- The general form of Parseval’s identity, for an *inner product between two DT signals*, states

$$\left\langle x[n], y[n] \right\rangle = \sum_{n=-\infty}^{\infty} x[n] y^*[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\Omega}\right) Y^*\left(e^{j\Omega}\right) d\Omega \,. \tag{42}$$

- Middle expression in (42): an inner product in the time domain between the signals $x[n]$ and $y[n]$.
- Rightmost expression in (42): an inner product in the frequency domain between the corresponding DTFTs, $X\left(e^{j\Omega}\right)$ and $Y\left(e^{j\Omega}\right)$.

*Signal Energy*

- The special case of (42) with $x[n]=y[n]$ and $X\left(e^{j\Omega}\right)=Y\left(e^{j\Omega}\right)$, provides an expression for the *energy of a DT signal*:

$$E=\sum_{n=-\infty}^{\infty}\left|x[n]\right|^{2}=\frac{1}{2\pi}\int_{2\pi}\left|X\left(e^{j\Omega}\right)\right|^{2}d\Omega\ . \tag{43}$$

- Middle expression in (43): energy of signal computed in time domain.
- Rightmost expression in (43): energy of signal computed in frequency domain.
- Interpretation of rightmost expression: $\left|X\left(e^{j\Omega}\right)\right|^{2}$ is the *energy density spectrum* of the signal $x[n]$. $\left|X\left(e^{j\Omega}\right)\right|^{2}$ measures the energy in the Fourier component of the signal at a frequency Ω.
  The rightmost expression in (43) is an integral of the energies in all the frequencies over an interval of length 2π.

*Proof of General Case* **(You may skip this.)**

- We start with the middle expression in (42) and represent $x[n]$ by the inverse DTFT of $X\left(e^{j\Omega}\right)$:

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{2\pi} X\left(e^{j\Omega}\right)e^{j\Omega n}d\Omega\right]y^*[n].$$

- Now we interchange the order of summation and integration, and recognize the quantity in square brackets as $Y\left(e^{j\Omega}\right)$:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x[n]y^*[n] &= \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\Omega}\right)\left[\sum_{n=-\infty}^{\infty} y[n]e^{-j\Omega n}\right]^* d\Omega \\ &= \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\Omega}\right)Y^*\left(e^{j\Omega}\right)d\Omega\end{aligned}.$$

- We have proven (42).

*Example: Energy of Sinc Function*

- We would like to compute the energy of a signal

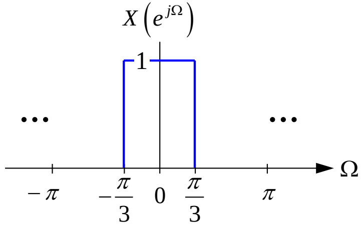
$$x[n] = \frac{1}{3}\mathrm{sinc}\left(\frac{n}{3}\right).$$

- To compute the energy in the time domain, we must evaluate the following sum, which is difficult.

$$E = \sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 = \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^2 \mathrm{sinc}^2\left(\frac{n}{3}\right)$$

- It is easy to compute this in the frequency domain. Using Example 5 with $W = \frac{\pi}{3}$, the DTFT of $x[n]$ is

$$X\left(e^{j\Omega}\right) = \sum_{l=-\infty}^{\infty} \Pi\left(\frac{\Omega - l2\pi}{2\pi/3}\right)$$




- Using the rightmost expression in (43) and choosing an integration interval $-\pi \le \Omega \le \pi$, the energy is

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X\left(e^{j\Omega}\right)\right|^2 d\Omega = \frac{1}{2\pi}\int_{-\pi/3}^{\pi/3}(1)^2 d\Omega = \frac{1}{3}.$$

*Example: Inner Product Between Two Signals*

- We would like to compute the inner product between two signals

$$x[n]=\frac{1}{4}\mathrm{sinc}\left(\frac{n}{4}\right)\cos\left(\frac{\pi}{2}n\right) \quad \text{and} \quad y[n]=\frac{1}{4}\mathrm{sinc}\left(\frac{n}{4}\right)\cos(\pi n).$$

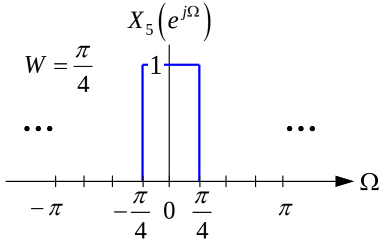
- To compute the inner product in the time domain, we need to evaluate the sum

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n]=\sum_{n=-\infty}^{\infty}\left(\frac{1}{4}\right)^2 \mathrm{sinc}^2\left(\frac{n}{4}\right)\cos\left(\frac{\pi}{2}n\right)\cos(\pi n).$$

- It is easier to evaluate the inner product in the frequency domain.

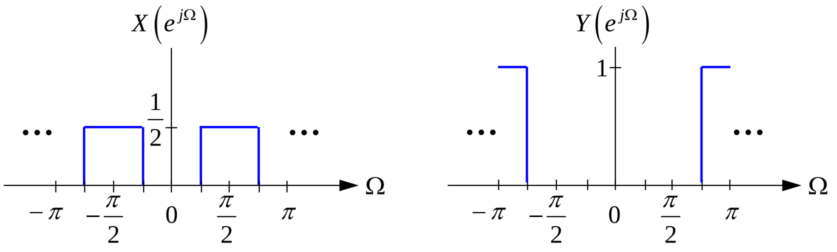
- We first use Example 5 with $W=\pi/4$ to obtain the DTFT pair

$$x_5[n]=\frac{1}{4}\mathrm{sinc}\left(\frac{n}{4}\right) \overset{F}{\leftrightarrow} X_5\left(e^{j\Omega}\right)=\sum_{l=-\infty}^{\infty}\Pi\left(\frac{\Omega-l2\pi}{\pi/2}\right).$$



- We compute the DTFTs of $x[n]$ and $y[n]$ using the frequency-shift property (27):

$$x_5[n]\cos(\Omega_0 n) = \frac{1}{2}x_5[n]\left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right) \overset{F}{\leftrightarrow} \frac{1}{2}\left[X_5\left(e^{j(\Omega-\Omega_0)}\right) + X_5\left(e^{j(\Omega+\Omega_0)}\right)\right].$$




- In $Y\left(e^{j\Omega}\right)$, the scaled copies of $X_5\left(e^{j\Omega}\right)$ shifted to $\pm\Omega_0 = \pm\pi$ overlap and add to yield a height of 1.

- We compute the inner product using Parseval's identity (42):

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\Omega}\right)Y^*\left(e^{j\Omega}\right)d\Omega = 0\,.$$

- $X\left(e^{j\Omega}\right)$ and $Y\left(e^{j\Omega}\right)$ do not overlap in frequency.

  The inner product between $x[n]$ and $y[n]$ vanishes.

  The two signals are mutually orthogonal.

**Multiplication Property**

- The DTFT multiplication property has important applications, like the corresponding CTFT property:
  - Modulation and demodulation of DT signals.
  - Sampling of DT signals (see EE 102B).
  - Windowing (see Chapter 6 and EE 102B).
- Consider two DT signals and their DTFTs

$$p[n] \overset{F}{\leftrightarrow} P\left(e^{j\Omega}\right) \text{ and } q[n] \overset{F}{\leftrightarrow} Q\left(e^{j\Omega}\right)$$

- The multiplication property states

$$p[n]q[n] \overset{F}{\leftrightarrow} \frac{1}{2\pi}\int_{2\pi} P\left(e^{j\theta}\right)Q\left(e^{j(\Omega-\theta)}\right)d\theta \,. \tag{44}$$

- *Multiplication in time* corresponds to *convolution in frequency*, as expected.
- The DTFTs $P\left(e^{j\Omega}\right)$ and $Q\left(e^{j\Omega}\right)$ are periodic in Ω, so the right-hand side of (44) is a *periodic convolution*.

  In periodic convolution:
  - We *integrate over one period* – any interval of length 2π – instead of from $-\infty$ to $\infty$.
  - The result obtained is periodic in Ω because the factor $Q\left(e^{j(\Omega-\theta)}\right)$ is periodic in Ω.

*Proof* ***(Skip)***

- Using the DTFT analysis equation, the DTFT of $p[n]q[n]$ is

$$F\left[p[n]q[n]\right]=\sum_{n=-\infty}^{\infty}p[n]q[n]e^{-jn\Omega}.$$

- Using (8), we express $p[n]$ as the inverse DTFT of $P\left(e^{j\Omega}\right)$, employing an integration variable $\theta$:

$$F\left[p[n]q[n]\right]=\sum_{n=-\infty}^{\infty}q[n]\left[\frac{1}{2\pi}\int_{2\pi}P\left(e^{j\theta}\right)e^{j\theta n}d\theta\right]e^{-jn\Omega}.$$

- We interchange the order of summation and integration:

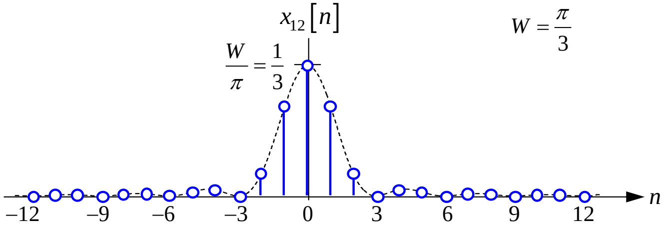
$$F\left[p[n]q[n]\right]=\frac{1}{2\pi}\int_{2\pi}P\left(e^{j\theta}\right)\left[\sum_{n=-\infty}^{\infty}q[n]e^{-jn(\Omega-\theta)}\right]d\theta.$$

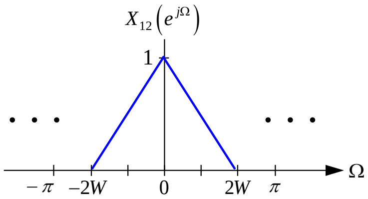
- We recognize the sum as $Q\left(e^{j(\Omega-\theta)}\right)$, the DTFT of $q[n]$ at frequency $\Omega-\theta$. We have proven (44).

*Example of Multiplication Property*

*12. Sinc-squared function*. The signal and its DTFT, which we derive here, are given by

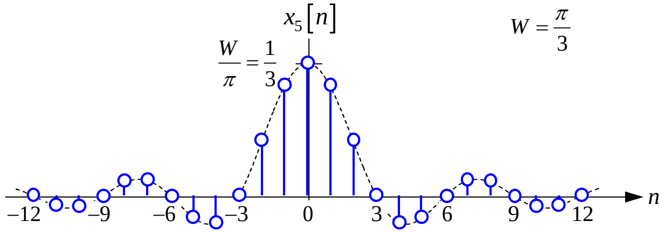
$$x_{12}[n]=\frac{W}{\pi}\operatorname{sinc}^2\left(\frac{W}{\pi}n\right) \overset{F}{\leftrightarrow} X_{12}\left(e^{j\Omega}\right)=\sum_{l=-\infty}^{\infty}\Lambda\left(\frac{\Omega-l2\pi}{2W}\right). \tag{45}$$

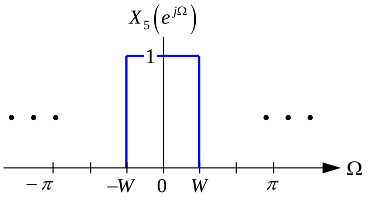



- Recall Example 5:

$$x_5[n]=\frac{W}{\pi}\operatorname{sinc}\left(\frac{W}{\pi}n\right) \overset{F}{\leftrightarrow} X_5\left(e^{j\Omega}\right)=\sum_{l=-\infty}^{\infty}\Pi\left(\frac{\Omega-l2\pi}{2W}\right). \tag{14}$$



- Note that signal $x_{12}[n]$ is $\pi / W$ times the square of signal $x_5[n]$:

$$x_{12}[n] = \frac{\pi}{W} x_5^2[n].$$

- Using the multiplication property (44), the $X_{12}\left(e^{j\Omega}\right)$ is a scaled periodic convolution between $X_5\left(e^{j\Omega}\right)$ and itself:

$$X_{12}\left(e^{j\Omega}\right) = \frac{\pi}{W} \times \frac{1}{2\pi} \int_{2\pi} X_5\left(e^{j\theta}\right) X_5\left(e^{j(\Omega-\theta)}\right) d\theta.$$

- $X_5\left(e^{j\Omega}\right)$ is a rectangular pulse train, so the periodic convolution yields a triangular pulse train $X_{12}\left(e^{j\Omega}\right)$.

## Linear Time-Invariant Systems Governed by Constant-Coefficient Difference Equations

- We discuss causal LTI DT systems described by constant-coefficient linear difference equations (see Chapter 2, slides 79-84)

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \tag{46}$$

The $x[n]$ and $y[n]$ are the input and output signals. The $a_k$, $k = 0,\ldots,N$ and $b_k$, $k = 0,\ldots,M$ are constants, which are real in systems that map real inputs to real outputs.

- We study a method for computing the frequency response of a system described by (46). It is equivalent to one introduced in Chapter 3 (see Method 2, slide 125),

- The method is applicable only if the impulse response is absolutely summable:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty, \tag{47}$$

so the system is BIBO stable and the frequency response $H\left(e^{j\Omega}\right)$ exists in a strict sense.

*Aside: example in which this technique is not applicable*

- Consider an accumulator, whose input-output relation is

$$y[n] = \sum_{m=-\infty}^{n} x[m]. \tag{48}$$

- It can be described by a difference equation

$$y[n] - y[n-1] = x[n]. \tag{10}$$

  This is of the general form (46) with three nonzero coefficients: $a_0 = 1$, $a_1 = -1$ and $b_0 = 1$.

- Recall that the accumulator's impulse response is

$$h[n] = u[n], \tag{49}$$

  which does not satisfy the absolute summability condition (47).

- The frequency response is the DTFT of the impulse response (49):

$$H\left(e^{j\Omega}\right) = \frac{1}{1-e^{-j\Omega}} + \pi \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k2\pi\right). \tag{50}$$

  This exists only in a generalized sense. We used Example 11 to obtain (50).

- Assuming the system's frequency response $H\left(e^{j\Omega}\right)$ exists, its input-output relation can be described in time or frequency by

$$y[n]=h[n]*x[n] \overset{F}{\leftrightarrow} Y\left(e^{j\Omega}\right)=H\left(e^{j\Omega}\right)X\left(e^{j\Omega}\right). \tag{37}$$

- We can solve the right-hand side of (37) to obtain an expression for the frequency response:

$$H\left(e^{j\Omega}\right)=\frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)}. \tag{51}$$

- Suppose that given an input $x[n]\overset{F}{\leftrightarrow}X\left(e^{j\Omega}\right)$, we can determine the output $y[n]\overset{F}{\leftrightarrow}Y\left(e^{j\Omega}\right)$ induced. Then we can use (51) to compute the frequency response $H\left(e^{j\Omega}\right)$ at all frequencies at which $X\left(e^{j\Omega}\right)\neq 0$.

- Now we compute the DTFT of the difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \tag{46}$$

- Using linearity and the time-shift property

$$x[n-k] \overset{F}{\leftrightarrow} e^{-jk\Omega} X\left(e^{j\Omega}\right), \tag{26'}$$

the DTFT of (46) is

$$\sum_{k=0}^{N} a_k e^{-jk\Omega} Y\left(e^{j\Omega}\right) = \sum_{k=0}^{M} b_k e^{-jk\Omega} X\left(e^{j\Omega}\right). \tag{52}$$

- We factor out the $Y\left(e^{j\Omega}\right)$ and $X\left(e^{j\Omega}\right)$ in (52) and solve for $Y\left(e^{j\Omega}\right) / X\left(e^{j\Omega}\right)$. Using (51), the frequency response is

$$H\left(e^{j\Omega}\right) = \frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)} = \frac{\sum_{k=0}^{M} b_k e^{-jk\Omega}}{\sum_{k=0}^{N} a_k e^{-jk\Omega}}. \tag{53}$$

- We found that for any LTI system described by a difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \tag{46}$$

the frequency response (if it exists strictly) if of the form

$$H\left(e^{j\Omega}\right) = \frac{\sum_{k=0}^{M} b_k e^{-jk\Omega}}{\sum_{k=0}^{N} a_k e^{-jk\Omega}}. \tag{53}$$

- This is a ratio between two polynomials in powers of $e^{-j\Omega}$, and is called a *rational function* of $e^{-j\Omega}$.
- The coefficients $b_k$, $k = 0,\ldots,M$ and $a_k$, $k = 0,\ldots,N$ in (53) are the same as those in (46).

  Hence:

  - Given a difference equation in the form (46), we can find the frequency response (53) *by inspection*.
  - Given a frequency response in the form (53), we can find the difference equation *by inspection*.
- The technique derived here is useful in analyzing LTI systems, as the following examples show.

## Examples of LTI Systems Governed by Constant-Coefficient Difference Equations

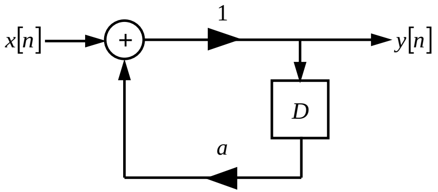
- In plotting the frequency response $H\left(e^{j\Omega}\right)$ in these examples:
  - We use a linear scale for frequency Ω, since $H\left(e^{j\Omega}\right)$ is a periodic function of Ω.
  - We use a logarithmic scale for the magnitude $\left|H\left(e^{j\Omega}\right)\right|$ and a linear scale for the phase $\angle H\left(e^{j\Omega}\right)$.

## Infinite Impulse Response Systems

- These systems are recursive. They are described by constant-coefficient linear difference equations of order $N$ = 1 or 2.
- We state the impulse and step responses without derivation. They are derived using Z transforms in the *EE 102B Course Reader*, Chapter 7.
- We obtain the frequency responses directly from the difference equations, as in deriving (53). Alternatively, we could compute the DTFTs of the impulse responses.

*First-Order System*

- A simple first-order system can be realized by the recursive system shown.




- The system is described by a first-order difference equation

$$y[n]-ay[n-1]=x[n].$$

- Its impulse response is

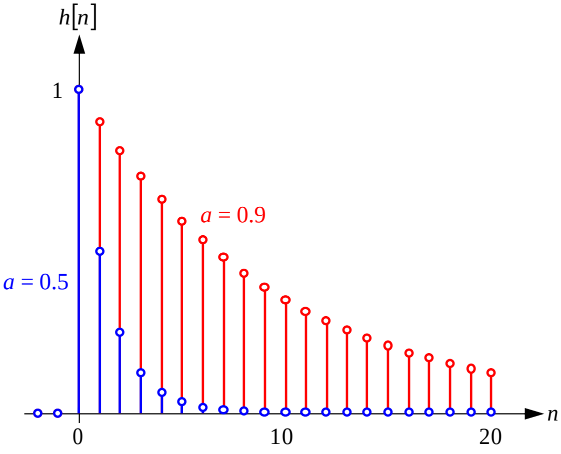
$$h[n]=a^n u[n],$$

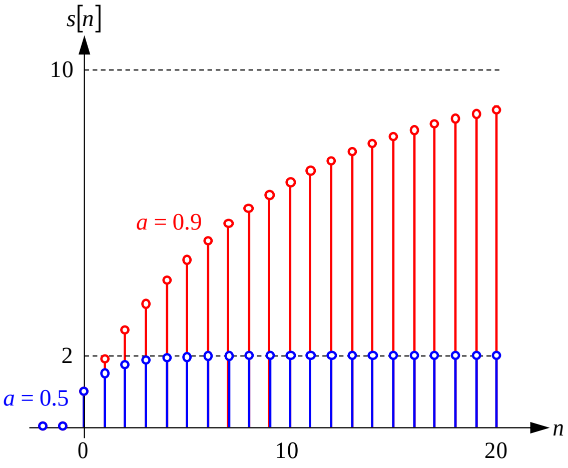
and is step response is

$$s[n]=\begin{cases}\dfrac{1-a^{n+1}}{1-a}u[n] & a\neq 1\\ (n+1)u[n] & a=1\end{cases}.$$

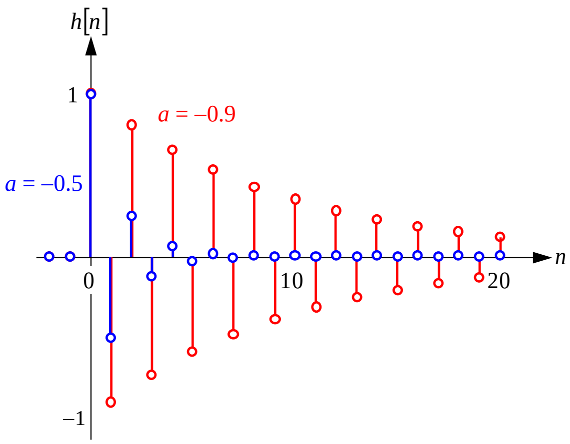
- The system is stable, and the frequency response exists, only for $|a|<1$. We assume $|a|<1$ below.

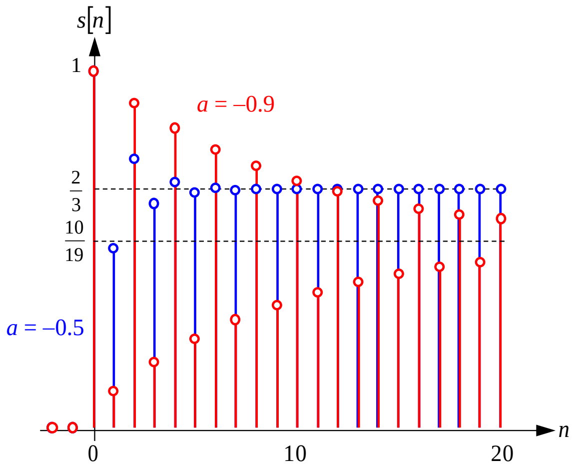
- When $0 < a < 1$, the system is a *lowpass filter*.
- The impulse response $h[n]$ decays monotonically to zero.
- The step response $s[n]$ asymptotically approaches a limiting value $(1-a)^{-1} > 1$.



- When $-1 < a < 0$, as shown below, the system is a *highpass filter*.
- The impulse response $h[n]$ alternates sign as it decays to zero.
- The step response $s[n]$ oscillates as it approaches a limiting value $(1-a)^{-1} < 1$.



- To find the frequency response, we start with the difference equation

$$y[n]-ay[n-1]=x[n]$$

and take the DTFT of each term using the time-shift property:

$$Y\left(e^{j\Omega}\right)-aY\left(e^{j\Omega}\right)e^{-j\Omega}=X\left(e^{j\Omega}\right).$$

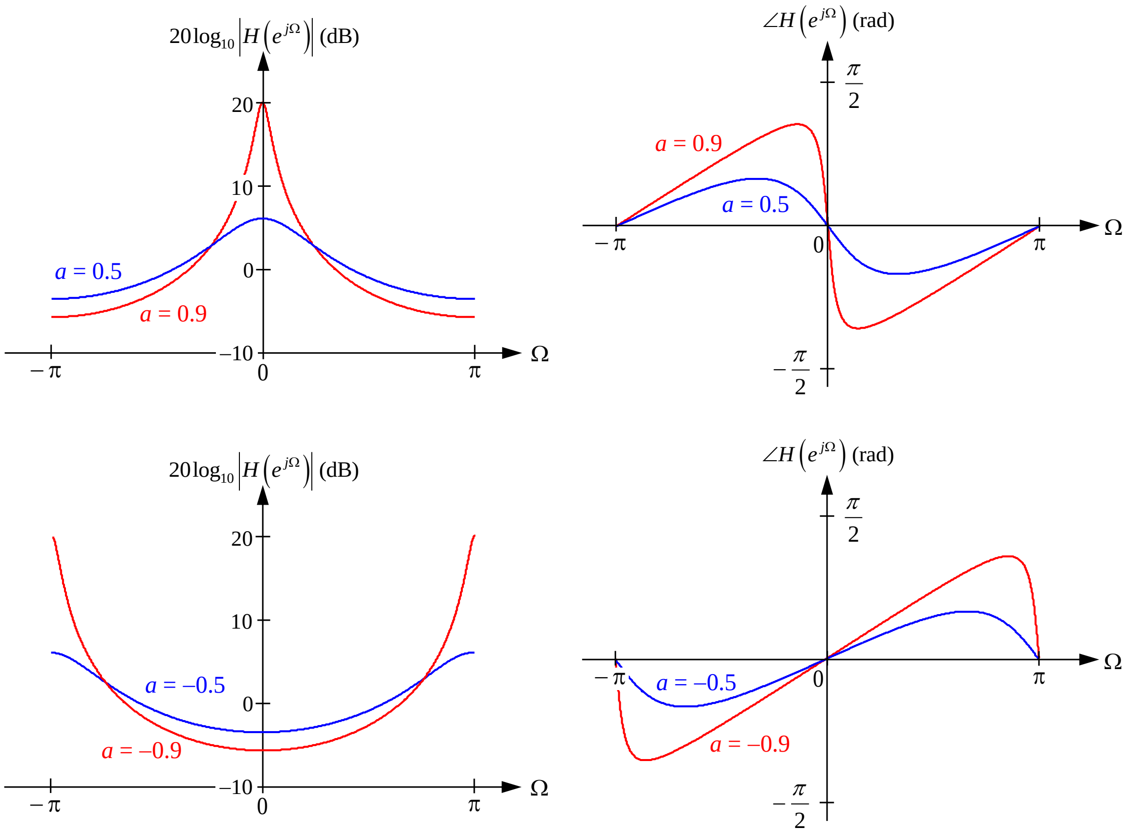
- Solving for $H\left(e^{j\Omega}\right)=Y\left(e^{j\Omega}\right)/X\left(e^{j\Omega}\right)$, we obtain

$$H\left(e^{j\Omega}\right)=\frac{1}{1-ae^{-j\Omega}}.$$

- The magnitude and phase responses are

$$\left|H\left(e^{j\Omega}\right)\right|=\frac{1}{\sqrt{\left(1-a\cos\Omega\right)^2+\left(a\sin\Omega\right)^2}} \quad \text{and} \quad \angle H\left(e^{j\Omega}\right)=-\tan^{-1}\left(\frac{a\sin\Omega}{1-a\cos\Omega}\right).$$

We computed these as in Chapter 3 (see slides 127-128).


$20\log_{10}\left|H\left(e^{j\Omega}\right)\right|$ (dB)
20
10
0
–10
$-\pi$
0
$\pi$
$\Omega$
$a$ = 0.5
$a$ = 0.9
$\angle H\left(e^{j\Omega}\right)$ (rad)
$\frac{\pi}{2}$
$-\frac{\pi}{2}$
$a$ = 0.9
$a$ = 0.5
$-\pi$
0
$\pi$
$\Omega$
$20\log_{10}\left|H\left(e^{j\Omega}\right)\right|$ (dB)
20
10
0
–10
$-\pi$
0
$\pi$
$\Omega$
$a$ = –0.5
$a$ = –0.9
$\angle H\left(e^{j\Omega}\right)$ (rad)
$\frac{\pi}{2}$
$-\frac{\pi}{2}$
$-\pi$
0
$\pi$
$\Omega$
$a$ = –0.5
$a$ = –0.9

*Observations*

- For a given $|a|$, the frequency responses for $a = |a|$ (lowpass) and $a = -|a|$ (highpass) are identical except for a frequency shift of π. The impulse responses are identical except for a factor $(-1)^n = e^{j\pi n}$. By the DTFT frequency-shift property with $\Omega_0 = \pi$, their frequency responses are related by a frequency shift of $\Omega_0 = \pi$:

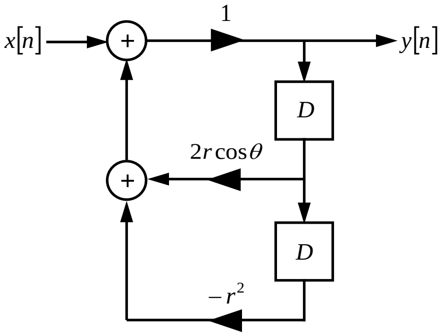
$$h_{-|a|}[n] = (-1)^n h_{|a|}[n] = e^{j\pi n} h_{|a|}[n] \overset{F}{\leftrightarrow} H_{-|a|}\left(e^{j\Omega}\right) = H_{|a|}\left(e^{j(\Omega-\pi)}\right).$$

- For any *a*, $\left|H\left(e^{j\Omega}\right)\right|$ is not constant, so the system causes amplitude distortion. The peaking of $\left|H\left(e^{j\Omega}\right)\right|$ becomes more pronounced as $|a| \to 1$, consistent with the increased duration of the impulse response.

- For any *a*, $\angle H\left(e^{j\Omega}\right)$ is not a linear function of Ω, so the system causes phase distortion. At values of Ω near the peak of $\left|H\left(e^{j\Omega}\right)\right|$, $\angle H\left(e^{j\Omega}\right)$ is approximately linear in Ω. At these frequencies, the group delay $-d\angle H\left(e^{j\Omega}\right)/d\Omega$ increases as $|a| \to 1$, consistent with the increased duration of the impulse response.

*Second-Order System*

- A simple second-order system can be realized by the recursive system shown.




- The system is described by a difference equation

$$y[n]-2r\cos\theta\, y[n-1]+r^2 y[n-2]=x[n],$$

where $r$ and θ are real. All cases of interest can be described by considering $0 \le r < \infty$ and $0 \le \theta \le \pi$ .

- Similar to the CT second-order lowpass filter, the system has three regimes, depending on the value of θ.
  The impulse and step responses have different forms in each regime.
  We provide expressions for the impulse responses here.

  - $\theta = 0$ (critically damped):

$$h[n] = \left[(n+1)r^n\right]u[n].$$

  - $0 < \theta < \pi$ (underdamped):

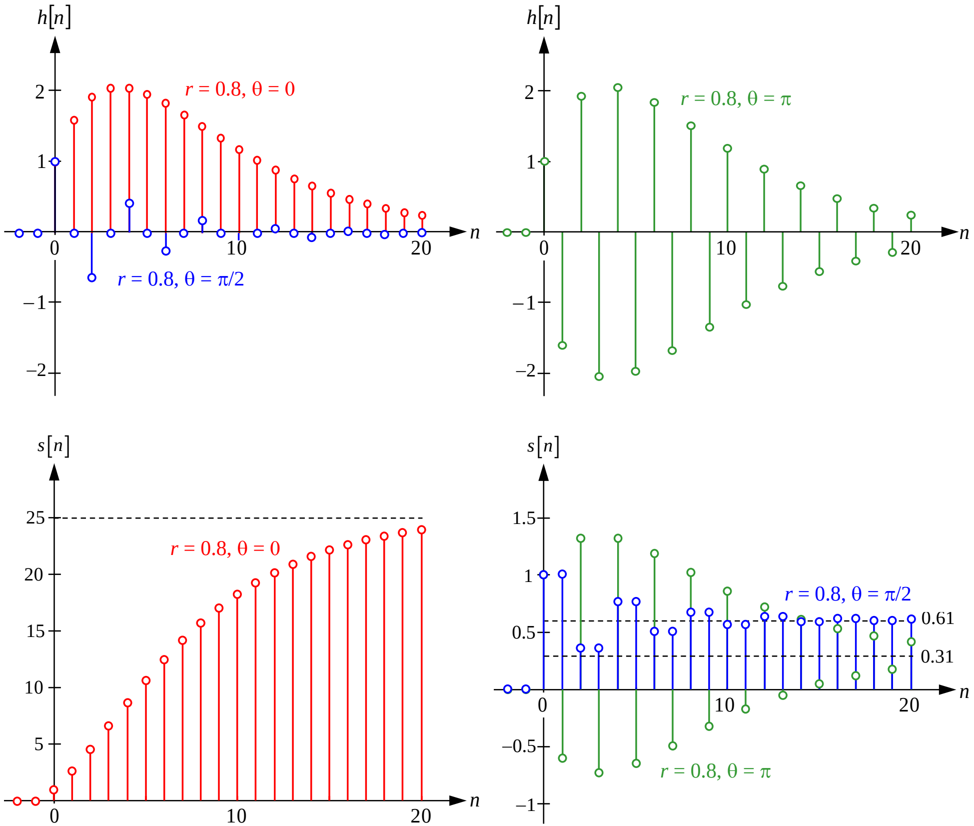
$$h[n] = \frac{1}{\sin\theta}\left[r^n \sin\left((n+1)\theta\right)\right]u[n].$$

  - $\theta = \pi$ (underdamped):

$$h[n] = \left[(n+1)(-r)^n\right]u[n].$$

- The system is stable, and the frequency response exists, only for $|r| < 1$.
  We consider only $0 \leq r < 1$ below.

- The impulse response $h[n]$ and step response $s[n]$ are shown for $r = 0.8$, $\theta = 0$, $\pi/2$ and $\pi$.



*Observations*

- $\theta = 0$: *lowpass filter*
  $h[n]$ is a decaying exponential multiplied by a factor $n + 1$.

- $\theta = \pi / 2$: *bandpass filter*
  $h[n]$ is a decaying exponential multiplied by a sinusoidal function.

- $\theta = \pi$ : *highpass filter*
  $h[n]$ is a decaying exponential multiplied by a factor $n + 1$ and an alternating sequence $(-1)^n = e^{j\pi n}$.

- For other values of $r$, the impulse response duration decreases as $r \to 0$ and increases as $r \to 1$.

- To find the frequency response, we start with the difference equation

$$y[n]-2r\cos\theta\, y[n-1]+r^2 y[n-2]=x[n]$$

and take the DTFT of each term using the DTFT time-shift property:

$$Y\left(e^{j\Omega}\right)-2r\cos\theta\cdot Y\left(e^{j\Omega}\right)e^{-j\Omega}+r^2\cdot Y\left(e^{j\Omega}\right)e^{-j2\Omega}=X\left(e^{j\Omega}\right).$$

- Solving for $H\left(e^{j\Omega}\right)=Y\left(e^{j\Omega}\right)/X\left(e^{j\Omega}\right)$, we obtain

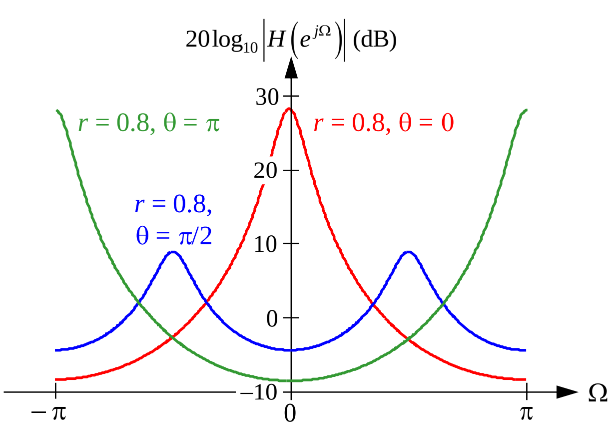
$$H\left(e^{j\Omega}\right)=\frac{1}{1-2r\cos\theta e^{-j\Omega}+r^2 e^{-j2\Omega}}.$$

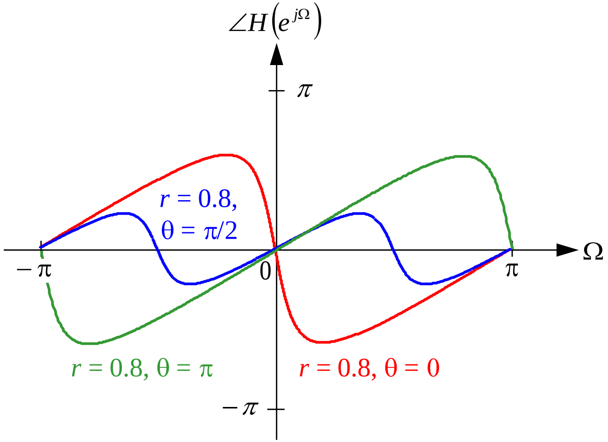
We do not compute expressions for the magnitude and phase responses here.

- Note that in the special cases $\theta=\begin{cases}0\\ \pi\end{cases}$, the frequency response becomes $H\left(e^{j\Omega}\right)=\dfrac{1}{\left(1\mp re^{-j\Omega}\right)^2}$.

The impulse and frequency responses $h[n]\overset{F}{\leftrightarrow}H\left(e^{j\Omega}\right)$ correspond to (32) with a substitution $a\rightarrow\pm r$ (see slide 42).

- The magnitude and phase responses are shown for $r = 0.8$ for $\theta = 0$ (lowpass), $\theta = \pi / 2$ (bandpass) and $\theta = \pi$ (highpass).



*Observations*

- These observations are very similar to those for the first-order system (see slide 82).
- For a given *r*, the frequency responses for $\theta = 0$ (lowpass) and $\theta = \pi$ (highpass) are identical, except for a frequency shift of π. The impulse responses are identical except for a factor $(-1)^n = e^{j\pi n}$.
  By the DTFT frequency-shift property with $\Omega_0 = \pi$, their frequency responses are related by a frequency shift of $\Omega_0 = \pi$:

$$h_{\theta=\pi}[n] = (-1)^n h_{\theta=0}[n] = e^{j\pi n} h_{\theta=0}[n] \overset{F}{\leftrightarrow} H_{\theta=\pi}\left(e^{j\Omega}\right) = H_{\theta=0}\left(e^{j(\Omega-\pi)}\right).$$

- For any *r* and θ, $\left|H\left(e^{j\Omega}\right)\right|$ is not constant, so the system causes amplitude distortion. The peaking of $\left|H\left(e^{j\Omega}\right)\right|$ becomes more pronounced as $r \to 1$, consistent with the increased duration of the impulse response.
- For any *r* and θ, $\angle H\left(e^{j\Omega}\right)$ is not a linear function of Ω, so the system causes phase distortion. At values of Ω near the peak of $\left|H\left(e^{j\Omega}\right)\right|$, the phase is approximately linear in Ω. At these frequencies, the group delay $-d\angle H\left(e^{j\Omega}\right)/d\Omega$ increases as $r \to 1$, consistent with the increased duration of the impulse response.

## Finite Impulse Response Systems

- These are non-recursive systems described by constant-coefficient linear difference equations of order $N = 0$.

*Symmetric Moving Average*

- A system averaging the input $x[n]$ over $2N_1+1$ times centered at present time $n$ is described by a difference equation

$$y[n] = \frac{1}{2N_1+1}\sum_{k=-N_1}^{N_1} x[n-k].$$

- It has an impulse response

$$\begin{aligned} h_{\text{ma, sym}}[n] &= \frac{1}{2N_1+1}\sum_{k=-N_1}^{N_1} \delta[n-k] \\ &= \frac{1}{2N_1+1}\Pi\left(\frac{n}{2N_1}\right) \end{aligned}. \tag{54}$$

- This system is not causal, so it cannot be used to average real-time signals.
  It can be used to average signals that have been recorded previously.

- To compute the frequency response, it is easiest to use the known DTFT of the rectangular pulse function $\Pi\left(n/2N_1\right)$, given by (17), scaling it by $1/\left(2N_1+1\right)$:

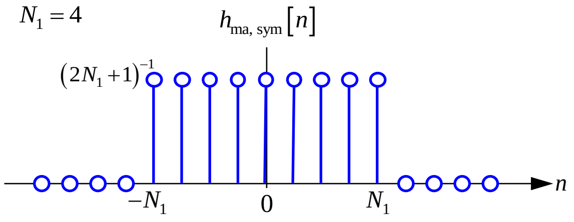
$$H_{\text{ma, sym}}\left(e^{j\Omega}\right)=\frac{1}{2N_1+1}\frac{\sin\left(\Omega\left(N_1+\frac{1}{2}\right)\right)}{\sin\left(\Omega/2\right)} \quad . \tag{55}$$

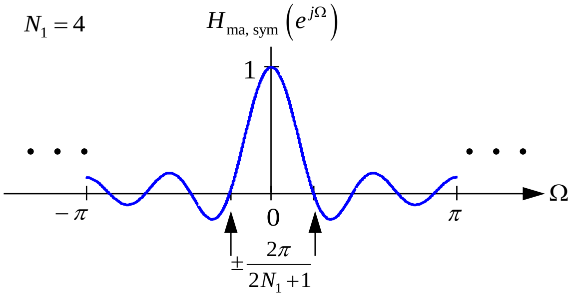
- An alternate way to find the frequency response is to take the DTFT of the difference equation term-by-term using the DTFT time-shift property. Solving for $H\left(e^{j\Omega}\right)=Y\left(e^{j\Omega}\right)/\,X\left(e^{j\Omega}\right)$, we obtain

$$H_{\text{ma, sym}}\left(e^{j\Omega}\right)=\frac{1}{2N_1+1}\sum_{k=-N_1}^{N_1}e^{-jk\Omega} \quad . \tag{55'}$$

Then we must sum the geometric series in (55') to obtain (55).

- The impulse response $h_{\text{ma, sym}}[n]$ and frequency response $H_{\text{ma, sym}}\left(e^{j\Omega}\right)$ are shown.




*Observations*

- $h_{\text{ma, sym}}[n]$ is real and even in $n$, so $H_{\text{ma, sym}}\left(e^{j\Omega}\right)$ is real and even in $\Omega$.

- $H_{\text{ma, sym}}\left(e^{j\Omega}\right)$ has a peak value of 1 at $\Omega = 0$, and decreases to zero at $\Omega = \pm 2\pi / (2N_1+1)$.

- As we increase the averaging window length $2N_1+1$, the width of the passband decreases.

- $H_{\text{ma, sym}}\left(e^{j\Omega}\right)$ is periodic in $\Omega$ with period $2\pi$, owing to the sinusoidal functions appearing in (55).

*Causal Moving Average*

- To compute a moving average in real time, we want a causal system that averages over the present and $2N_1$ past samples of the input. It is described by a difference equation

$$y[n]=\frac{1}{2N_1+1}\sum_{k=0}^{2N_1}x[n-k].$$

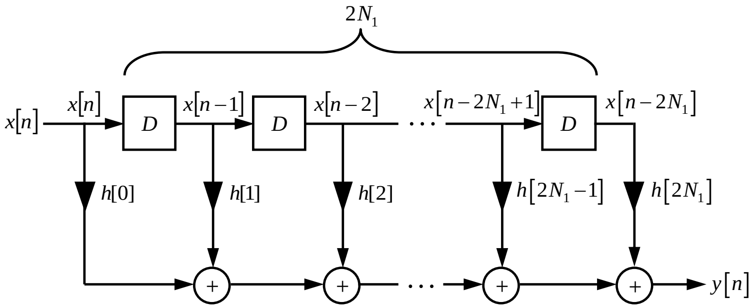
- It has an impulse response

$$\begin{aligned} h_{\text{ma, causal}}[n] &= \frac{1}{2N_1+1}\sum_{k=0}^{2N_1}\delta[n-k] \\ &= \frac{1}{2N_1+1}\Pi\left(\frac{n-N_1}{2N_1}\right) \end{aligned}. \tag{56}$$

- The causal impulse response (56) is the symmetric impulse response (54) delayed by $N_1$ samples:

$$h_{\text{ma, causal}}[n]=h_{\text{ma, sym}}[n-N_1]. \tag{57}$$

- The causal moving average system can be realized by the system shown.
  The coefficients $h[0], h[1], h[2], \ldots$ correspond to the samples of $h_{\text{ma, causal}}[n]$.



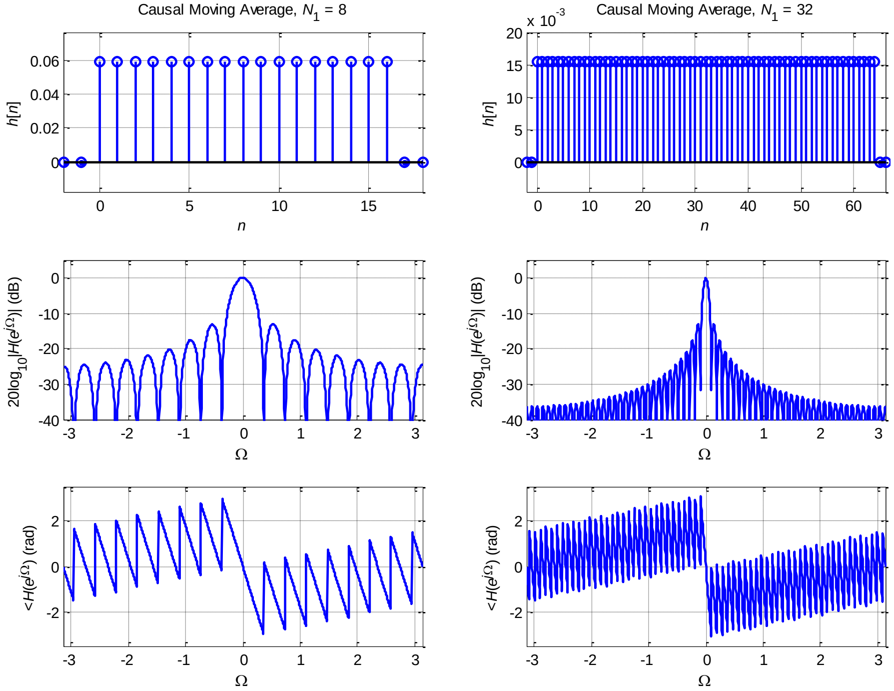

- The causal impulse response (56) is the symmetric impulse response (54) delayed by $N_1$:

$$h_{\text{ma, causal}}[n] = h_{\text{ma, sym}}[n - N_1]. \tag{57}$$

- By the DTFT time-shift property, the causal frequency response is the same as the symmetric frequency response (55), but with a linear phase factor arising from the $N_1$-sample delay:

$$\begin{aligned} H_{\text{ma, causal}}\left(e^{j\Omega}\right) &= H_{\text{ma, sym}}\left(e^{j\Omega}\right) e^{-jN_1\Omega} \\ &= \frac{1}{2N_1+1} \frac{\sin\left(\Omega\left(N_1 + \frac{1}{2}\right)\right)}{\sin\left(\Omega/2\right)} e^{-jN_1\Omega} \end{aligned} . \tag{58}$$

- The impulse response $h_{\text{ma, causal}}[n]$ and frequency response $H_{\text{ma, causal}}\left(e^{j\Omega}\right)$ are shown for $N_1 = 8$ and $N_1 = 32$.




*Observations*

- As $N_1$ increases:
  - The passband width, proportional to $1/(2N_1+1)$, decreases.
  - The passband group delay $N_1$ increases.
- The moving average filter induces significant amplitude distortion.
- Its phase is linear with integer slope, so it induces no phase distortion.

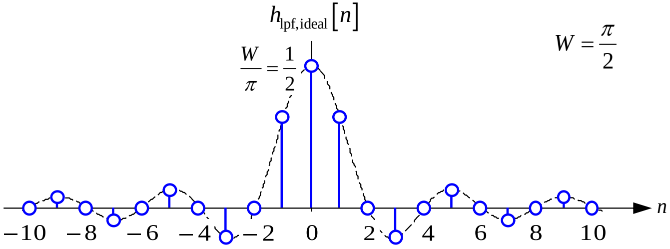
*Symmetric Finite Approximation of Ideal Lowpass Filter*

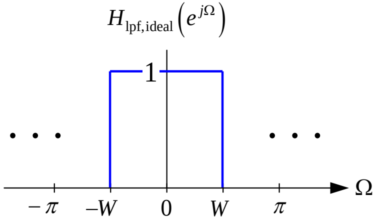
- We would like to realize an ideal lowpass filter. An ideal filter with cutoff frequency $W < \pi$ has an impulse response and frequency response given by (38) with $n_0 = 0$:

$$h_{\text{lpf,ideal}}\left[n\right] = \frac{W}{\pi}\operatorname{sinc}\left(\frac{W}{\pi}n\right) \overset{F}{\leftrightarrow} H_{\text{lpf,ideal}}\left(e^{j\Omega}\right) = \sum_{l=-\infty}^{\infty}\Pi\left(\frac{\Omega - l2\pi}{2W}\right). \tag{59}$$

- Recall that the summation on the right-hand side of (59) makes the frequency response $H_{\text{lpf,ideal}}\left(e^{j\Omega}\right)$ periodic in Ω with period 2π.

- Here we show impulse and frequency responses (59) for an ideal filter with cutoff frequency $W = \pi/2$. Observe that $h_{\text{lpf,ideal}}\left[n\right]$ is real and even in *n*, so $H_{\text{lpf,ideal}}\left(e^{j\Omega}\right)$ is real and even in Ω.




- The ideal lowpass filter cannot be implemented. It is non-causal and its impulse response has infinite extent over both positive and negative time *n*.

- As a first step toward a realizable filter, we truncate the impulse response to finite duration. To do this, we multiply $h_{\text{lpf,ideal}}[n]$ by a rectangular function $\Pi(n/2N_1)$.

- The truncated impulse response is nonzero over a symmetric interval $-N_1 \le n \le N_1$, a total of $2N_1+1$ samples:

$$\begin{aligned} h_{\text{lpf, trunc}}[n] &= \Pi\left(\frac{n}{2N_1}\right)\cdot h_{\text{lpf, ideal}}[n] \\ &= \begin{cases} \dfrac{W}{\pi}\operatorname{sinc}\left(\dfrac{W}{\pi}n\right) & -N_1 \le n \le N_1 \\ 0 & \text{otherwise} \end{cases}. \end{aligned} \tag{60}$$

- The truncated lowpass filter is described by a difference equation

$$y[n] = \frac{W}{\pi}\sum_{k=-N_1}^{N_1}\operatorname{sinc}\left(\frac{W}{\pi}k\right)x[n-k]. \tag{61}$$

- This difference equation generates the same output as a convolution of the input $x[n]$ with the impulse response (60):

$$\begin{aligned} y[n] &= h_{\text{lpf, trunc}}[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} h_{\text{lpf, trunc}}[k]\,x[n-k] \end{aligned}.$$

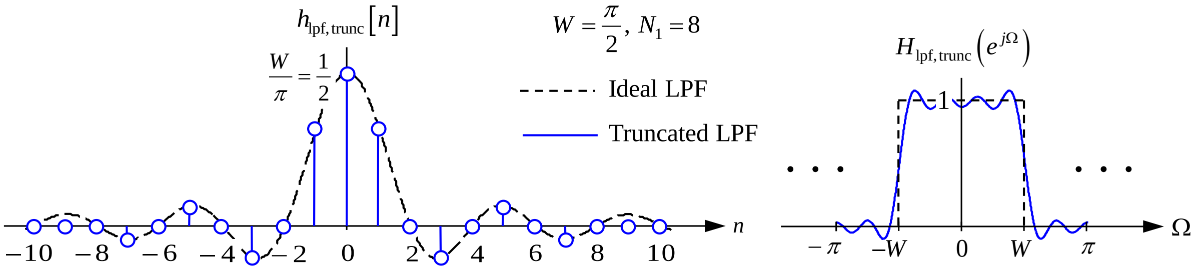
- To obtain the frequency response $H_{\text{lpf, trunc}}\left(e^{j\Omega}\right)$, we can compute the DTFT of impulse response (60):

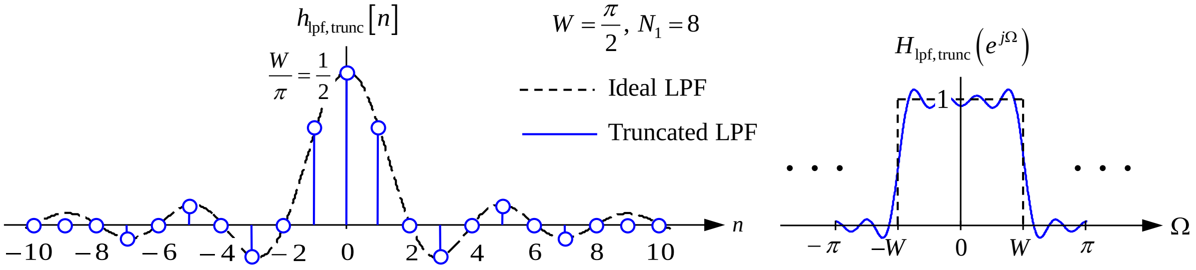
$$\begin{aligned} H_{\text{lpf, trunc}}\left(e^{j\Omega}\right) &= \sum_{n=-\infty}^{\infty} h_{\text{lpf, trunc}}[n] e^{-jn\Omega} \\ &= \frac{W}{\pi} \sum_{n=-N_1}^{N_1} \operatorname{sinc}\left(\frac{W}{\pi} n\right) e^{-jn\Omega} \end{aligned} . \qquad (62)$$

- Alternatively, we can take the DTFT of the difference equation (61) term-by-term using the DTFT time-shift property. Solving for $H\left(e^{j\Omega}\right) = Y\left(e^{j\Omega}\right) / X\left(e^{j\Omega}\right)$, we will obtain the same expression (62).

- The frequency response expression (62) cannot be simplified into a closed-form analytical expression, so we evaluate it numerically.

- The impulse and frequency responses $h_{\text{lpf, trunc}}[n] \overset{F}{\leftrightarrow} H_{\text{lpf, trunc}}\left(e^{j\Omega}\right)$ are shown for $W = \pi/2$ and $N_1 = 8$, corresponding to $2N_1 + 1 = 17$. Note that $h_{\text{lpf, trunc}}[n]$ real, even in $n$ $\overset{F}{\leftrightarrow}$ $H_{\text{lpf, trunc}}\left(e^{j\Omega}\right)$ real, even in $\Omega$.






- Unlike the ideal response $H_{\text{lpf,ideal}}\left(e^{j\Omega}\right)$, the truncated response $H_{\text{lpf, trunc}}\left(e^{j\Omega}\right)$ exhibits:
  - A gradual transition between passband and stopband at $\Omega = \pm W$ .
  - Ripple in the passband and leakage in the stopband.
- These non-ideal characteristics of $H_{\text{lpf, trunc}}\left(e^{j\Omega}\right)$ can be understood in two different ways.
- First, the frequency response

$$H_{\text{lpf, trunc}}\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} h_{\text{lpf, trunc}}[n] e^{-jn\Omega} \tag{62}$$

is a FS synthesis of $H_{\text{lpf, trunc}}\left(e^{j\Omega}\right)$, a periodic function of Ω, with FS coefficients given by $h_{\text{lpf, trunc}}[n]$.

- The FS synthesis is intended to represent an ideal rectangular pulse train in frequency Ω. Like any FS synthesis, it exhibits the Gibbs phenomenon at discontinuities, in this case, at $\Omega = \pm W$ .

- Second, recall that the truncated impulse response is obtained by multiplying the ideal impulse response by a rectangular function:

$$h_{\text{lpf, trunc}}[n] = \Pi\left(\frac{n}{2N_1}\right) \cdot h_{\text{lpf, ideal}}[n]. \tag{60}$$

- By the DTFT multiplication property, the frequency response $H_{\text{lpf,trunc}}\left(e^{j\Omega}\right)$ is a periodic convolution between the ideal rectangular pulse train $H_{\text{lpf,ideal}}\left(e^{j\Omega}\right)$ and the DTFT of $\Pi\left(n/2N_1\right)$, which is $\sin\left(\Omega\left(N_1+\frac{1}{2}\right)\right)/\sin\left(\Omega/2\right)$:

$$H_{\text{lpf,trunc}}\left(e^{j\Omega}\right) = \frac{1}{2\pi}\int_{2\pi} H_{\text{lpf,ideal}}\left(e^{j(\Omega-\theta)}\right)\frac{\sin\left(\theta\left(N_1+\frac{1}{2}\right)\right)}{\sin\left(\theta/2\right)}d\theta. \tag{63}$$

- Recall that a periodic convolution is like an ordinary convolution, except the integration is performed only over one period, in this case, any interval of length 2π.
- In (63), the first function is a periodic rectangular pulse train, while the second function is analogous to a periodic sinc function.
- The convolution between them gives rise to the gradual transition and the ripple and leakage observed in $H_{\text{lpf,trunc}}\left(e^{j\Omega}\right)$.

*Causal Finite Approximation of Ideal Lowpass Filter*

- The symmetric finite lowpass filter is usable on signals that have been recorded and stored. Many applications require a causal system to filter signals in real time.

- We obtain a causal lowpass filter by delaying the symmetric lowpass filter by $N_1$ samples:

$$\begin{aligned} h_{\text{lpf, causal}}[n] &= h_{\text{lpf, trunc}}[n-N_1] \\ &= \begin{cases} \dfrac{W}{\pi}\operatorname{sinc}\left(\dfrac{W}{\pi}(n-N_1)\right) & 0 \le n \le 2N_1 \\ 0 & \text{otherwise} \end{cases}. \end{aligned} \tag{64}$$

- The causal lowpass filter is described by a difference equation

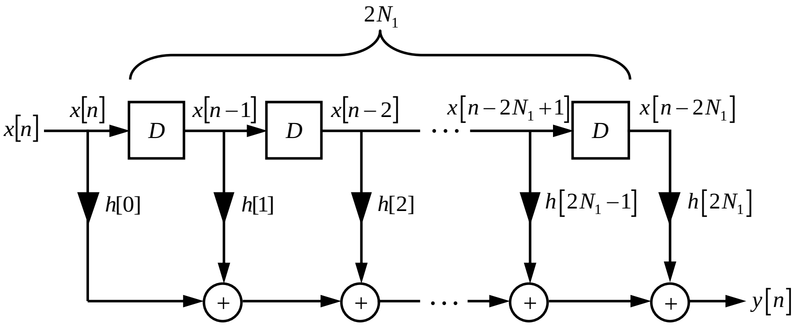
$$y[n] = \frac{W}{\pi}\sum_{k=0}^{2N_1}\operatorname{sinc}\left(\frac{W}{\pi}(k-N_1)\right)x[n-k]. \tag{65}$$

  In (65), the output $y[n]$ is a linear combination of the present and $2N_1$ past inputs.

- The output $y[n]$ in (65) is the same as a convolution of the input $x[n]$ with the impulse response in (64):

$$\begin{aligned} y[n] &= h_{\text{lpf, causal}}[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} h_{\text{lpf, causal}}[k]\,x[n-k] \end{aligned}.$$

- The causal lowpass filter can be realized by the system shown.
  The coefficients $h[0], h[1], h[2], \dots$ correspond to the samples of $h_{\text{lpf, causal}}[n]$.




- Recall the relationship between the impulse responses of the causal and symmetric lowpass filters:

$$h_{\text{lpf, causal}}[n] = h_{\text{lpf, trunc}}[n - N_1]. \tag{64}$$

- Using the DTFT time-shift property, the causal lowpass filter frequency response is the same as the symmetric truncated lowpass filter frequency response, (62) or (63), with a linear phase factor arising from the $N_1$ -sample delay:

$$H_{\text{lpf, causal}}\left(e^{j\Omega}\right) = H_{\text{lpf, trunc}}\left(e^{j\Omega}\right) e^{-jN_1\Omega}. \tag{66}$$

- Alternatively, the frequency response of the causal lowpass filter can be obtained by taking a DTFT of the impulse response (64):

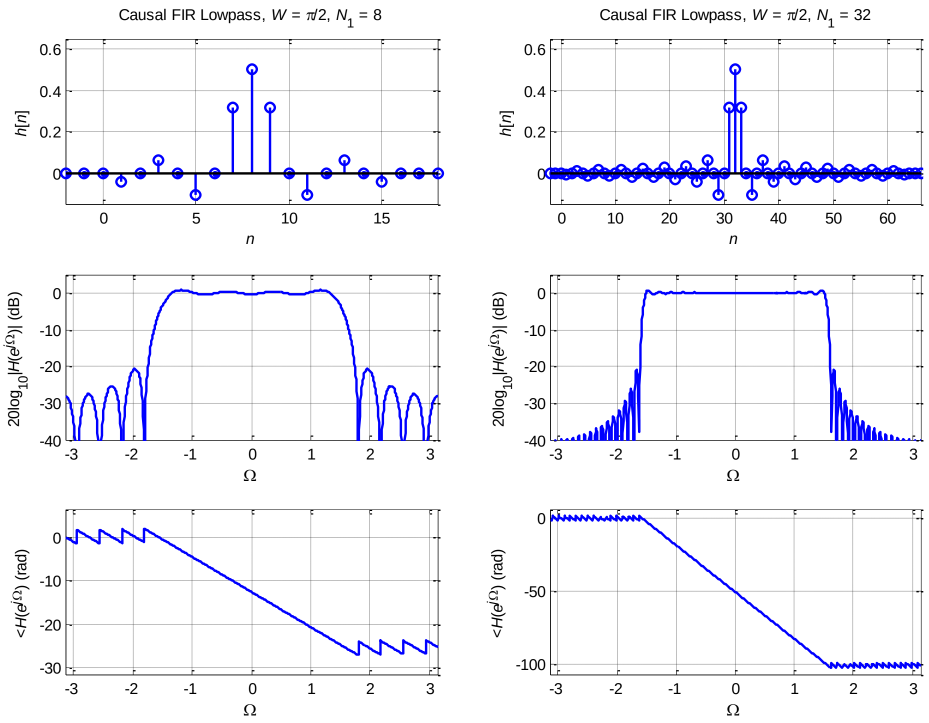
$$\begin{aligned} H_{\text{lpf,causal}}\left(e^{j\Omega}\right) &= \sum_{n=-\infty}^{\infty} h_{\text{lpf, causal}}[n]e^{-jn\Omega} \\ &= \frac{W}{\pi}\sum_{n=0}^{2N_1}\operatorname{sinc}\left(\frac{W}{\pi}(n-N_1)\right)e^{-jn\Omega} \end{aligned} . \tag{67}$$

- Finally, the frequency response $H_{\text{lpf,causal}}\left(e^{j\Omega}\right)$ could instead be obtained by evaluating the DTFT of the difference equation (65) using the DTFT time-shift property and solving for $H_{\text{lpf,causal}}\left(e^{j\Omega}\right) = Y\left(e^{j\Omega}\right)/X\left(e^{j\Omega}\right)$.

*On the following slide*

- The impulse response $h_{\text{lpf, causal}}[n]$ and frequency response $H_{\text{lpf, causal}}\left(e^{j\Omega}\right)$ are shown for a cutoff frequency $W = \pi/2$ for $N_1 = 8$ and $N_1 = 32$.

- The MATLAB **`unwrap`** command was used on the phase $\angle H_{\text{lpf, causal}}\left(e^{j\Omega}\right)$ to avoid 2π phase jumps. This highlights the linearity of the phase in the passband.




*Observations*

- As $N_1$ increases:
  - The nominal cutoff frequency $\Omega = \pm W$ does not change.
  - The transition at $\Omega = \pm W$ becomes more abrupt.
  - The passband ripple and stopband leakage do not diminish in magnitude. They are confined to a narrower range near $\Omega = \pm W$ .
  - The passband group delay

    $$-\frac{d\angle H_{\text{lpf, causal}}\left(e^{j\Omega}\right)}{d\Omega} = N_1$$

    increases.
- The filter induces minimal amplitude distortion within the passband.
- Its phase is linear with integer slope, so it induces no phase distortion.

## Overview of Fourier Representations

- The four Fourier representations we have studied are summarized in the figure below.

<table>
<tr><th rowspan="2"></th><th colspan="2">Continuous Time</th><th colspan="2">Discrete Time</th></tr>
<tr><th>Time Domain</th><th>Frequency Domain</th><th>Time Domain</th><th>Frequency Domain</th></tr>
<tr><td>Fourier Series</td><td>$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$<br>Continuous<br>Periodic</td><td>$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt$<br>Discrete<br>Aperiodic<br>duality</td><td>$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n}$<br>Discrete<br>Periodic<br>duality</td><td>$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\Omega_0 n}$<br>Discrete<br>Periodic</td></tr>
<tr><td>Fourier Transform</td><td>$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$<br>Continuous<br>Aperiodic<br>duality</td><td>$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$<br>Continuous<br>Aperiodic</td><td>$x[n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\Omega}\right)e^{j\Omega n}d\Omega$<br>Discrete<br>Aperiodic</td><td>$X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$<br>Continuous<br>Periodic</td></tr>
</table>

*Organizing Principles*

- The key attributes – discrete vs. continuous and periodic vs. aperiodic – follow a simple pattern:

discrete in one domain ↔ periodic in the other domain

continuous in one domain ↔ aperiodic in the other domain

- "One domain" can denote either time or frequency.
  "The other domain" can denote either frequency or time.

*Example: the CTFS*

- Signal $x(t)$ is a periodic function of a continuous-valued time variable *t*.
- The time-domain description is *periodic*, so the frequency-domain description must be *discrete*.
- The time-domain description is *continuous*, so the frequency-domain description must be *aperiodic*.
- In fact, the CTFS coefficients $a_k$ are an aperiodic function of a discrete variable *k*.

*Duality in the CTFT*

- The time-domain description $x(t)$ is continuous and aperiodic.
  The frequency-domain description $X(j\omega)$ is continuous and aperiodic.

- The CTFT (analysis) and inverse CTFT (synthesis) are of very similar mathematical forms.

- Every CTFT pair $x(t)\overset{F}{\leftrightarrow}X(j\omega)$ corresponds to a pair $y(t)\overset{F}{\leftrightarrow}Y(j\omega)$ in which the functional forms of $x(t)$ and $X(j\omega)$ are similar to the functional forms of $Y(j\omega)$ and $y(t)$.
  For example:

$$1\overset{F}{\leftrightarrow}2\pi\delta(\omega) \quad \text{and} \quad \delta(t)\overset{F}{\leftrightarrow}1$$

$$\Pi\left(\frac{t}{2T_1}\right)\overset{F}{\leftrightarrow}2T_1\text{sinc}\left(\frac{\omega T_1}{\pi}\right) \quad \text{and} \quad \left(\frac{W}{\pi}\right)\text{sinc}\left(\frac{Wt}{\pi}\right)\overset{F}{\leftrightarrow}\Pi\left(\frac{\omega}{2W}\right)$$

- Every property of the CTFT corresponds to a property for the inverse CTFT.
  For example, the convolution property and multiplication properties:

$$x(t)*y(t)\overset{F}{\leftrightarrow}X(j\omega)Y(j\omega) \quad \text{and} \quad x(t)y(t)\overset{F}{\leftrightarrow}\frac{1}{2\pi}X(j\omega)*Y(j\omega)$$

*Duality in the DTFS*

- The time-domain description $x[n]$ is discrete and periodic.

  The frequency-domain description $a_k$ is discrete and periodic.

- The DTFS synthesis and analysis equations are of very similar mathematical forms.
- Every DTFS pair $x[n] \overset{FS}{\longleftrightarrow} a_k$ corresponds to a pair $y[n] \overset{FS}{\longleftrightarrow} b_k$ in which the functional forms of $x[n]$ and $a_k$ are similar to the functional forms of $b_k$ and $y[n]$.
- Every property of the DTFS synthesis corresponds to a property of the DTFS analysis.

*Duality between the CTFS and the DTFT*

*CTFS*

- Time-domain description is continuous and periodic.
  Frequency-domain description is discrete and aperiodic.

*DTFT*

- Time-domain description is discrete and aperiodic.
  Frequency-domain description is continuous and periodic.

- CTFS synthesis equation is of a mathematical form similar to DTFT analysis equation.
  CTFS analysis equation has a mathematical form similar to DTFT synthesis equation.

- Every CTFS pair $x(t) \overset{FS}{\leftrightarrow} a_k$ corresponds to a DTFT pair $y[n] \overset{F}{\leftrightarrow} Y\left(e^{j\Omega}\right)$ in which the functional forms of $x(t)$ and $Y\left(e^{j\Omega}\right)$ are similar, while the functional forms of $a_k$ and $y[n]$ are similar.

- For example, CTFS pair describing a rectangular pulse train:

$$\sum_{l=-\infty}^{\infty} \Pi\left(\frac{t-lT_0}{2T_1}\right) \overset{FS}{\longleftrightarrow} \frac{\omega_0 T_1}{\pi}\operatorname{sinc}\left(\frac{\omega_0 T_1}{\pi}k\right)$$

and DTFT pair describing an ideal lowpass filter:

$$\frac{W}{\pi}\operatorname{sinc}\left(\frac{W}{\pi}n\right) \overset{F}{\longleftrightarrow} \sum_{l=-\infty}^{\infty} \Pi\left(\frac{\Omega - l2\pi}{2W}\right).$$

- Every property of CTFS synthesis equation corresponds to a property of DTFT analysis equation. Every property of CTFS analysis equation corresponds to a property of DTFT synthesis equation.