第二类拉格朗日方程

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本讲主要内容

1、第二类拉格朗日方程的推导

2、第二类拉格朗日方程的应用

3、拉格朗日方程的初积分

1、第二类拉格朗日方程的推导

设由n个质点组成的系统受m个理想完整约束作用，系统具有N=3n-m 个自由度。

设$q_1, q_2, \ldots, q_N$为系统的一组广义坐标，则每个质点的位置：

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1,\ q_2, \cdots,\ q_N,\ t) \quad (i = 1, 2, \cdots,\ n)$$

上式两端进行等时变分运算得到：

$$\delta \boldsymbol{r}_i = \frac{\partial \boldsymbol{r}_i}{\partial q_1}\delta q_1 + \ldots + \frac{\partial \boldsymbol{r}_i}{\partial q_N}\delta q_N + \frac{\partial \boldsymbol{r}_i}{\partial t}\delta t = \sum_{k=1}^{N}\frac{\partial \boldsymbol{r}_i}{\partial q_k}\delta q_k$$

主动力在任意虚位移上所作的虚功之和为：

$$\sum_{i=1}^{n}\boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i = \sum_{k=1}^{N} Q_k \cdot \delta q_k$$

将以上两式代入动力学普遍方程：

$$\sum_{i=1}^{n}(\boldsymbol{F}_i - m_i\ddot{\boldsymbol{r}}_i)\cdot\delta\boldsymbol{r}_i = 0 \Longrightarrow \sum_{k=1}^{N}(Q_k - \sum_{i=1}^{n} m_i\ddot{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_k})\delta q_k = 0$$

对于完整约束系统，广义坐标相互独立，因此 δq_k是任意的，上式成立的话，恒有：

$$Q_k - \sum_{i=1}^{n} m_i\ddot{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_k} = 0 \quad (k = 1, 2, \ldots, N)$$

$$Q_k - \sum_{i=1}^{n} m_i \ddot{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_k} = 0 \quad (k = 1, 2, \ldots, N)$$

广义惯性力

上式不便于直接应用，为此可作如下变换：

(1) $$\frac{\partial \boldsymbol{r}_i}{\partial q_k} = \frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_k}$$

证明：

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1,\ q_2 \cdots,\ q_N,\ t) \qquad (i = 1,2,\cdots,\ n)$$

$$\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \dot{\boldsymbol{r}}_i = \frac{\partial \boldsymbol{r}_i}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial \boldsymbol{r}_i}{\partial q_k}\dot{q}_k + \cdots + \frac{\partial \boldsymbol{r}_i}{\partial t}$$

注意 $\frac{\partial \boldsymbol{r}_i}{\partial q_k}$ 和 $\frac{\partial \boldsymbol{r}_i}{\partial t}$ 只是广义坐标和时间的函数（不含有广义速度项），并且上式只在第k项含有 $\dot{q}_k$

$$\Longrightarrow \frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_k} = \frac{\partial \boldsymbol{r}_i}{\partial q_k}$$

(2) $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_k}\right)=\frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_k}$$

证明：这实际是一个交换求导次序的问题

$$\boldsymbol{r}_i=\boldsymbol{r}_i(q_1\,,\ q_2\cdots,\ q_N\,,\ t)\qquad (i=1,2,\cdots,\ n)$$

$$\dot{\boldsymbol{r}}_i=\frac{\partial \boldsymbol{r}_i}{\partial q_1}\dot{q}_1+\cdots+\frac{\partial \boldsymbol{r}_i}{\partial q_N}\dot{q}_N+\frac{\partial \boldsymbol{r}_i}{\partial t}=\sum_{j=1}^{N}\frac{\partial \boldsymbol{r}_i}{\partial q_j}\dot{q}_j+\frac{\partial \boldsymbol{r}_i}{\partial t}$$

$$\frac{\partial \boldsymbol{r}_i}{\partial q_k}=\frac{\partial \boldsymbol{r}_i}{\partial q_k}(q_1,\ q_2,\cdots,\ q_N,\ t)$$

对时间 t 求微分

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_k}\right)=\sum_{j=1}^{N}\frac{\partial}{\partial q_j}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_k}\right)\dot{q}_j+\frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_k}\right)=\sum_{j=1}^{N}\frac{\partial^2 \boldsymbol{r}_i}{\partial q_k\partial q_j}\dot{q}_j+\frac{\partial^2 \boldsymbol{r}_i}{\partial q_k\partial t}$$

而 $$\frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_k}=\frac{\partial}{\partial q_k}\left(\sum_{j=1}^{N}\frac{\partial \boldsymbol{r}_i}{\partial q_j}\dot{q}_j+\frac{\partial \boldsymbol{r}_i}{\partial t}\right)=\sum_{j=1}^{N}\frac{\partial^2 \boldsymbol{r}_i}{\partial q_k\partial q_j}\dot{q}_j+\frac{\partial^2 \boldsymbol{r}_i}{\partial q_k\partial t}$$

若函数 $\boldsymbol{r}_i=\boldsymbol{r}_i(q_1,\ q_2,\cdots,\ q_N,\ t)$ 的一阶和二阶偏导数连续

$$\Rightarrow\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_k}\right)=\frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_k}$$

将 $\dfrac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_k} = \dfrac{\partial \boldsymbol{r}_i}{\partial q_k}$ 和 $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \boldsymbol{r}_i}{\partial q_k}\right) = \dfrac{\partial \dot{\boldsymbol{r}}_i}{\partial q_k}$ 代入动力学普遍方程的广义惯性力项中：

$$\sum_{i=1}^{n} m_i \ddot{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_k} = \sum_{i=1}^{n} m_i \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\boldsymbol{r}} \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_k}) - \sum_{i=1}^{n} m_i \dot{\boldsymbol{r}}_i \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial \boldsymbol{r}_i}{\partial q_k})$$

$$= \sum_{i=1}^{n} m_i \frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{\boldsymbol{r}} \cdot \frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_k}\right] - \sum_{i=1}^{n} m_i \dot{\boldsymbol{r}}_i \cdot \frac{\partial \dot{\boldsymbol{r}}}{\partial q_k}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{n} \left[m_i \dot{\boldsymbol{r}}_i \cdot \frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_k}\right] - \frac{\partial}{\partial q_k} \sum_{i=1}^{n} (\frac{1}{2} m_i \dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i)$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}_k} \sum_{i=1}^{n} (\frac{1}{2} m_i v_i^2)\right] - \frac{\partial}{\partial q_k} \sum_{i=1}^{n} (\frac{1}{2} m_i v_i^2)$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial T}{\partial \dot{q}_k}) - \frac{\partial T}{\partial q_k}$$

记 $T = \sum_{i=1}^{n} (\frac{1}{2} m_i v_i^2)$

将前述结果代入动力学普遍方程: $\sum\limits_{k=1}^{N}(Q_k - \sum\limits_{i=1}^{n} m_i \ddot{\boldsymbol{r}}_i \cdot \dfrac{\partial \boldsymbol{r}_i}{\partial q_k})\delta q_k = 0$

得到 $$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} - Q_k = 0 \quad (k = 1,2,\cdots, \quad N)}$$ —第二类拉格朗日方程

二阶常微分方程组，方程式的数目等于质点系的自由度数。

特别地，如果作用在质点系上的主动力都是有势力（保守力），则:

$$Q_k = -\frac{\partial V}{\partial q_k} \qquad (k = 1,2,\cdots, \quad N)$$

拉格朗日方程可以写成 $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = 0 \quad (k = 1,2,\cdots, \quad N)$$

引入拉格朗日函数（又称为动势） $L = T - V$

则拉格朗日方程又可以写成 $$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \quad (k = 1,2,\cdots, \quad N)}$$

拉格朗日方程是解决完整约束系统动力学问题的普遍方程。它形式简洁、便于计算，广泛应用于求解复杂质点系的动力学问题。

—保守系统的拉格朗日方程