ENGG2430D Tutorial 2

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January 21, 2015

Some Information

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Outline

1. The Binomial Theorem
    - Introduction
    - Formulation
    - Combinatorial Proof
    - Applications
2. Integer Solutions to Indeterminate Equation
3. Monte Carlo Simulation
    - Brief Introduction
    - Examples

The Binomial Theorem

The binomial theorem studies the algebraic expansion of powers of a binomial, i.e.,

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\end{aligned}$$

For a large n, expanding (1) by hand is too tedious. Fortunately, Binomial Theorem gives us the expansion (1) for any nonnegative integer.

The Binomial Theorem

Theorem

For any nonnegative integer n,

$$(x+y)^n = \sum_{k=0}^{n} C_n^k x^k y^{n-k}$$

where

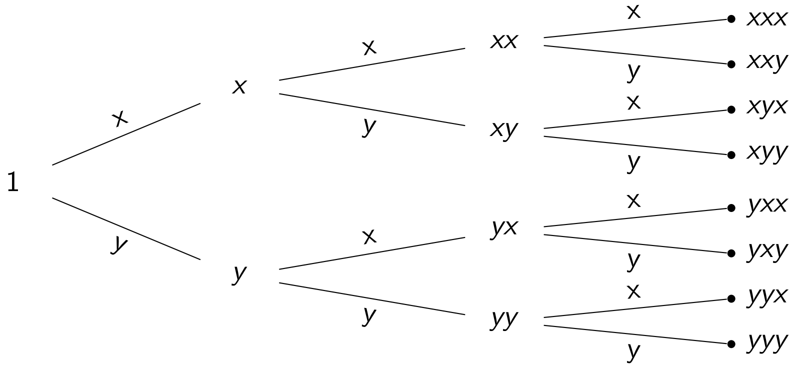
$$C_n^k = \frac{n!}{k!(n-k)!}$$

Two ways to prove:

- **Combinatorial proof**
- Inductive proof

$(x+y)^3 = 1(x+y)(x+y)(x+y)$

How to expand it?
- use sequential model
- analogy to tossing a fair coin 3 times



Remarks: Order does not matter, namely, xxy, xyx, yxx are the same.
Summing up the leaf nodes, we get $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

Combinatorial proof

$$(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{\text{n of those}} = \sum_{k=0}^{n} C_n^k x^k y^{n-k}$$

- The expansion can be expressed as the sum of multiple items of the form: $a_k x^k y^{n-k}$, k is a nonnegative integer.
- The expansion is the same with the total sample space of tossing a fair coin n times, $n \geq 1$.
- Let x, y denote the events "we get a head in one tossing" and "we get a tail in one tossing", respectively. The coefficient a_k in each item equals C_n^k (analogous to the event "in n tossings, we get k heads").

Remarks

Actually, n does not have to be a nonnegative integer; the binomial theorem can be extended to the power of any real number.

Series for e

The Euler's number e is defined as

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Applying binomial theorem:

$$\left(1+\frac{1}{n}\right)^n = 1 + C_n^1\frac{1}{n} + C_n^2\frac{1}{n^2} + \cdots + C_n^n\frac{1}{n^n}$$

Looking into the kth item of the right hand side and take the limit:

$$\lim_{n\to\infty} C_n^k\frac{1}{n^k} = \lim_{n\to\infty}\frac{1}{k!}\cdot\frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k} = \frac{1}{k!}$$

Therefore, e can be written as a series:

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Above is also known as Taylor series of e^x at $x=1$.

Simple Numerical Estimation

Example 1 *Estimate the value of* 1.01^5 *rounding up to 3 decimal places.*
Solution: Using the binomial theorem: let $x = 1, y = 0.01$,

$$\begin{aligned}1.01^5 &= (1+0.01)^5\\ &= 1 + 5(0.01) + 10(0.01)^2 + 10(0.01)^3 + 5(0.01)^4 + (0.01)^5\\ &\approx 1 + 5(0.01) + 10(0.01)^2 = 1.051\end{aligned}$$

Example 2 *Estimate the value of* 1.0309^6 *rounding up to 3 decimal places.*
Solution: Again, calculate this by hand will take us some time. But using the binomial theorem, we can try to expand the following

$$\begin{aligned}(1+x+x^2)^6 &= (1+x(1+x))^6\\ &= 1 + 6x(1+x) + 15x^2(1+x)^2 + 20x^3(1+x)^3 + \cdots\\ &= 1 + 6x + 21x^2 + 50x^3 + \cdots\end{aligned}$$

Let $x = 0.03$, we get $1.0309^6 \approx 1.200$.

Integer Solutions to Indeterminate Equation

Problem 1

Suppose we have a equation as follows

$$x_1 + x_2 + x_3 + \cdots + x_n = m$$

where x_i, n, m are positive integers, and $\forall i \in \{1, 2, 3, 4, \cdots, n\}$. Then, how many solutions are there? (hint: use combinations.)

Solution

Consider m to be m "1"s summing up together, i.e.,

$$m = \underbrace{1 + 1 + 1 + \cdots + 1}_{\text{m ones}}$$

Every solution of $x_1, x_2, \cdots, x_n$ can be matched to an event of selecting $n - 1$ "+" between those "1"s.

Integer Solutions to Indeterminate Equation

Because of the one-to-one matching, the number of solutions is equal to the frequency of the event "picking $n-1$ from $m-1$", that is C_{m-1}^{n-1}

Problem 2

What if x_i are nonnegative, how many solutions are there? (hint: convert back into the previous problem)

Solution

Since x_i can be zero, so for each item we can add 1 to x_i. In this way, we can make it positive again. Note that whenever we add 1 in x_i, we should also add 1 to the right hand side. Let $x_i' = x_i + 1$ which is a positive integer, so the problem is equivalent to

$$x_1' + x_2' + x_3' + \cdots + x_n' = m + n$$

. So the number of solution is C_{m+n-1}^{n-1}

Monte Carlo Simulation

Definition

Monte Carlo simulation is a broad class of computational algorithms that rely on repeated random sampling to obtain numerical results.

Remarks:

- Monte Carlo simulation is useful when it is difficult or impossible to obtain a closed-form expression, or infeasible to apply a deterministic algorithm.
- Monte Carlo simulation is based on *Law of Large Numbers*:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n}{n} = E(X)$$

where $X_1, X_2, \cdots$ are a sequence of independent identically random variables. Namely, we repeatedly run a experiment for a sufficiently large number of times, the result will finally converge to its expectation (probability), e.g., coin toss.

Monte Carlo Simulation

Example: extimate π

Consider a circle inscribed in a unit square. Given that the circle and the square have a ratio of areas that is $\pi/4$, how can we approximate the value of π?

Solution: Using a Monte Carlo simulation:

- Draw a square and inscribe a circle within it.
- *Uniformly* scatter some objects of uniform size (grains of rice or sand) over the square.
- Count the number of objects inside the circle and the total number of objects.
- The ratio of the two counts is an estimate of the ratio of the two areas, which is $\pi/4$. Multiply the result by 4 to estimate π

Monte Carlo Simulation

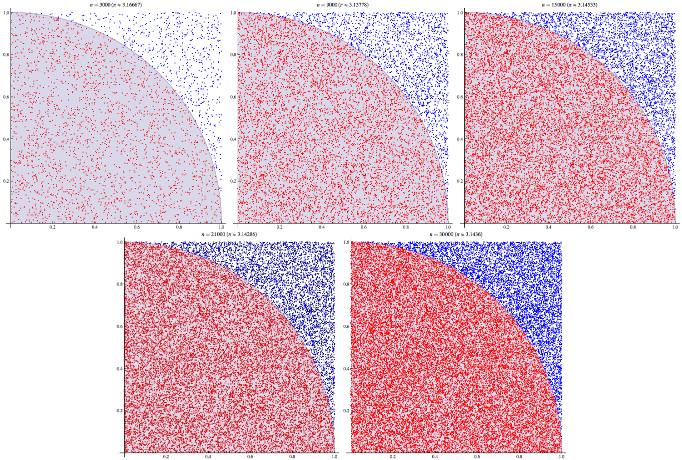


Figure 1: Estimate π using Monte Carlo simulation.

Thank you!