ENGG2430D Tutorial 6

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Outline

Homework 2

Problem 1: Independence

Assuming A, B, C are **independent**, with $P(A) = 0.1$, $P(B) = 0.2$, $P(C) = 0.3$, compute:

1. $P(A \cap B)$.
   Solution: $P(A)P(B) = 0.02$
2. $P(A \cup B)$.
   Solution: $P(A) + P(B) - P(A)P(B) = 0.28$
3. $P(A \cup B \cup C)$.
   Solution: $P(A) + P(B) + P(C) - P(A)P(B) - P(B)P(C) - P(A)P(C) + P(A)P(B)P(C) = 0.496$
4. $P((B \cup C)^c \cap A)$.
   Solution: $P(A) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C) = 0.056$

Problem 2: Bayes' Formula

Solve the following questions.

1. Suppose that $P(E) = 0.3$, $P(F) = 0.6$, and $P(E \cap F) = 0.1$. Compute $P(E|F)$.
   Solution: $P(E|F) = P(E \cap F)/P(F) = 1/6$
2. Suppose that $P(E) = 0.3$, $P(F) = 0.6$, and that E and F are independent events. Compute $P(E|F)$.
   Solution:
   $P(E|F) = P(E \cap F)/P(F) = (P(E)P(F))/P(F) = P(E) = 0.3$

Problem 3: Independence & Bayes' Formula

Assume A and B are independent events with $P(A) = 0.1$ and $P(B) = 0.4$. Let C be the event that **at least one** of A or B occurs, and let D be the event that **exactly one** of A or B occurs.

1. Find $P(C)$.
   Solution: The event C is just the union of A and B, so
   $P(C) = P(A \cup B) = P(A) + P(B) - P(A)P(B) = 0.46$
2. Find $P(D)$.
   Solution: We can see from the Venn diagram that D consists of the union of A and B minus the overlap. Thus,
   $P(D) = P(A \cup B) - P(A \cap B) = 0.42$

Problem 3: Independence & Bayes' Formula (cont'd)

3. Find $P(A|D)$ and $P(D|A)$.
   Solution:
   $P(A|D) = P(A \cap D)/P(D) = (P(A) - P(A \cap B))/P(D) = 1/7.$
   $P(D|A) = P(A \cap D)/P(A) = 0.6$
3. Determine whether A and D are independent.
   Solution: Since $P(A|D) \neq P(A)$, A and D are not independent.

Problem 4: De Morgan's Law

Prove one of the De Morgan' laws: let S_k, $1 \leq k \leq n$, be n sets, then $(\cap_{k=1}^{n} S_k)^c = \cup_{k=1}^{n} S_k^c$.

Solution:

If $x \in (\cap_{k=1}^{n} S_k)^c$, meaning $x \notin \cap_{k=1}^{n} S_k$, there exits $1 \leq K \leq n$, such that $x \notin S_K$, i.e. $x \in S_K^c$. Then we can get $x \in \cup_{k=1}^{n} S_k^c$.
$\Rightarrow (\cap_{k=1}^{n} S_k)^c \subseteq \cup_{k=1}^{n} S_k^c$.

If $x \in \cup_{k=1}^{n} S_k^c$, meaning there exists $1 \geq K \leq n$, such that $x \notin S_K$, then we can get $x \notin \cap_{k=1}^{n} S_k$, i.e. $x \in (\cap_{k=1}^{n} S_k)^c$. $\Rightarrow \cup_{k=1}^{n} S_k^c \subseteq (\cap_{k=1}^{n} S_k)^c$

Then we can obtain $(\cap_{k=1}^{n} S_k)^c = \cup_{k=1}^{n} S_k^c$

Problem 5: Bayes' Formula

Finalphobia is a rare disease in which the victim has the delusion that he or she is being subjected to an intense mathematical examination.

- A person selected uniformly at random has finalphobia with probability $1/200$.
- A person with finalphobia has shaky hands with probability $19/20$.
- A person without finalphobia has shaky hands with probability $1/20$.

What is the probablility that a randomly-selected person has finalphobia, given that he or she has shaky hands?

Solution: Let F be the event that the randomly-selected person has finalphobia, and let S be the event that he or she has shaky hands. The probability that a person has finalphobia, given that he or she has shaky hands is:

Problem 5: Bayes' Formula(cont'd)

$$
\begin{aligned}
P(F|S) &= \frac{P(F \cap S)}{P(S)} \\
&= \frac{P(S|F)P(F)}{P(S)} \\
&= \frac{P(S|F)P(F)}{P(F^c)P(S|F^c) + P(F)P(S|F)} \\
&= \frac{\frac{19}{20}\frac{1}{200}}{\frac{199}{200}\frac{1}{20} + \frac{1}{200}\frac{19}{20}} \\
&= \frac{19}{218}
\end{aligned}
$$

Homework 3

Problem 1

Please show that two events, A and B, are not independent if their relationship is characterized by either one of the diagrams in Fig. 1.

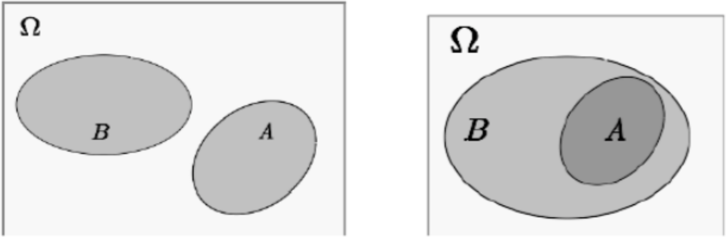

Figure 1: two relationships between A and B

Problem 1 (cont'd)

Solution: When the relationship is characterized by the diagram on the left, A and B are disjoint, so $P(AB) = 0$, but $P(A)$ and $P(B)$ are positive, then $P(A)P(B) \neq P(AB)$. When the relationship is characterized by the diagram on the right, $A \subset B$, meaning $A \cap B = A$, so $P(AB) = P(A)$, with the fact that $P(B) < 1$, then $P(AB) < P(A)P(B)$. We can have that A and B are not independent for both cases.

Problem 2

Given a random variable X, whose distribution is given by

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, ...$$

where λ is a specified constant. Suppose we know that $\lambda = E(X) = Var(X)$. Now we define a new random variable $Y = X^2$, please compute the expectation of Y, i.e., $E[Y]$.

Solution. For any random variable X, we can have

$$Var[X] = E[X^2] - (E[X])^2.$$

Then $E[Y] = E[X^2] = Var[X] + (E[X])^2 = \lambda + \lambda^2$.

P.S: X is actually the Poisson random variable.

Problem 3

Consider a coin. When you flip it, it will show a head with probability $p = \frac{1}{3}$ and a tail with probability $q = 1 - p = \frac{2}{3}$. Suppose the results of different flips are independent. Calculate the PMF, expectation and variance of the random variable X in the following cases.

(a) (Bernoulli) X is the number of heads when you flip the coin once.
Solution: The PMF function is

$$p_X(x) = \begin{cases} p = \frac{1}{3}, & X = 1; \\ q = \frac{2}{3}, & X = 0. \end{cases}$$

The expectation is

$$E[X] = p = \frac{1}{3}.$$

The variance is

$$\mathrm{var}(X) = pq = \frac{2}{9}.$$

Problem 3 (cont'd)

(b) (Binomial) X is the number of heads when you flip the coin 4 times.
Solution: The PMF function is

$$p_X(x) = \binom{4}{x} p^x q^{4-x} = \binom{4}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{4-x}, x = 0, 1, 2, 3, 4.$$

More specifically, the PMF is shown in the following table.

x	0	1	2	3	4
$p_X(x)$	$\frac{16}{81}$	$\frac{32}{81}$	$\frac{24}{81}$	$\frac{8}{81}$	$\frac{1}{81}$

The expectation is

$$E[X] = 4p = \frac{4}{3}.$$

The variance is

$$\mathrm{var}(X) = 4pq = \frac{8}{9}.$$

Problem 3 (cont'd)

(c) (Geometric) X is the number of flips until you see a head the first time.
Solution: The PMF function is

$$p_X(x) = q^{x-1}p = \frac{2^{x-1}}{3^x}, x = 1, 2, 3, \cdots$$

The expectation is

$$E[X] = \frac{1}{p} = 3.$$

The variance is

$$\mathrm{var}(X) = \frac{q}{p^2} = 6.$$

Problem 4

let us consider throwing a fair die twice. Throwing the die once will equally likely show all 6 results (1, 2, 3, 4, 5, or 6 dots). Suppose the results of the first throw and the second throw are independent. And define the following random variables.

$$X_1 = \{\text{The number of dots in the first throw}\}$$
$$X_2 = \{\text{The number of dots in the second throw}\}$$
$$Z = X_1 - X_2$$

(a) Find the PMF of X_1, X_2, and Z, i.e., $p_{X_1}(x_1)$, $p_{X_2}(x_2)$ and $p_Z(z)$.
Solution: The PMF function is

$$p_{X_1}(x_1) = \frac{1}{6}, x_1 = 1, 2, 3, 4, 5, 6.$$

$$p_{X_2}(x_2) = \frac{1}{6}, x_2 = 1, 2, 3, 4, 5, 6.$$

Problem 4 (cont'd)

z	-5	-4	-3	-2	-1	0	1	2	3	4	5
$p_Z(z)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(b) Find the expectation and variance of X_1, X_2 and Z.
Solution:

$$E[X_1] = E[X_2] = \frac{7}{2}, E[Z] = 0.$$

$$\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = \frac{35}{12}, \mathrm{Var}(Z) = \frac{35}{6}.$$

Problem 4 (cont'd)

(c) Find the joint PMF of X_1 and Z, i.e., $p_{X_1,Z}(x_1, z)$.

$p_{X_1,Z}(x_1,z)$	$z=-5$	-4	-3	-2	-1	0	1	2	3	4	5
$x_1=1$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	0	0	0	0	0
$x_1=2$	0	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	0	0	0	0
$x_1=3$	0	0	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	0	0	0
$x_1=4$	0	0	0	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	0	0
$x_1=5$	0	0	0	0	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	0
$x_1=6$	0	0	0	0	0	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$

(d) Determine whether X_1 and Z are independent or not.
Solution: Consider $X_1 = 1, Z = 1$. Since

$$p_{X_1}(1) = \frac{1}{6}, p_Z(1) = \frac{5}{36}, p_{X_1,Z}(1,1) = 0,$$

we get that

$$p_{X_1,Z}(1,1) \neq p_{X_1}(1) p_Z(1).$$

Therefore, X_1 and Z are not independent.

Problem 5

Given two **independent** random variables X_1 and X_2, we know that they follow the geometric distribution with parameters p_1 and p_2 respectively. Now we define another random variable

$$X = \min\{X_1, X_2\}.$$

(a) Given a positive integer k, please compute the probabilities that X_1 and X_2 are no larger than k respectively, i.e., $P(X_1 \leq k)$ and $P(X_2 \leq k)$.

Solution: We know that $P(X_i = n) = (1 - p_i)^{n-1} p_i$. Then

$$P(X_i \leq k) = \sum_{n=1}^{k} P(X_i = n) = \sum_{n=1}^{k} (1 - p_i)^{n-1} p_i = 1 - (1 - p_i)^k$$

we can have $P(X_1 \leq k) = 1 - (1 - p_1)^k$ and $P(X_2 \leq k) = 1 - (1 - p_2)^k$.

Problem 5 (cont'd)

(b) Please find the probability that X is no larger than k, i.e., $P(X \leq k)$.
Hint: $P(X_i \leq k) + P(X_i > k) = 1$,
$P(X > k) = P(X_1 > k, and X_2 > k)$ and use the fact that X_1 and X_2 are independent.
Solution: By the result in (a), we can have
$P(X_i > k) = 1 - P(X_i \leq k) = (1 - p_i)^k$. The next computation is as follows,

$$
\begin{aligned}
P(X > k) &= P(X_1 > k, and X_2 > k) \quad \text{By the definition of } X. \\
&= P(X_1 > k)P(X_2 > k) \quad \text{By the independence of } X_1 \text{ and } X_2. \\
&= (1 - p_1)^k (1 - p_2)^k \\
&= [(1 - p_1)(1 - p_2)]^k .
\end{aligned}
$$

And

$$P(X \leq k) = 1 - P(X > k) = 1 - [(1 - p_1)(1 - p_2)]^k .$$

Problem 5 (cont'd)

(c) Next, compute $P(X = k)$. Hopefully you will find that X is a geometric random variable.

$$\begin{aligned} P(X = k) &= P(X \leq k) - P(X \leq k-1) \\ &= [(1-p_1)(1-p_2)]^{k-1} - [(1-p_1)(1-p_2)]^{k} \\ &= [(1-p_1)(1-p_2)]^{k-1} [1-(1-p_1)(1-p_2)] \end{aligned}$$

By letting $p = 1 - (1-p_1)(1-p_2)$, we can have $P(X = k) = (1-p)^{k-1}p$. It is easy to see that X is a geometric random variable with parameter p.