ENGG2430D Tutorial 6

Zhibo Yang

Department of Information Engineering
The Chinese University of Hong Kong

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Outline

1. Expectation of the Sum of Random Variables
    - Linearity of Expecation
    - Proof
    - Example
2. Variance of the Sum of Random Variables
    - Definition of Covariance
    - Independence and Covariance
    - Properties of Covariance
    - Variance of Sum of Random Variables
    - Example

Linearity of Expecation

We learned the following proposition from the class:

Proposition

Let X_1, X_2, $\cdots$, X_n be random variables, we have

$$E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$$

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Actually, we can be more ambitious:

Linearity of Expectation

Let $X_1, X_2, \cdots, X_n$ be random variables, we have

$$E[\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n] = \alpha_1 E[X_1] + \alpha_2 E[X_2] + \cdots + \alpha_n E[X_n]$$

Note that $X_1, X_2, \cdots, X_n$ do not have to be independent!

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\begin{aligned}
E[\alpha X_1 + \beta X_2] &= \sum_{x_1}\sum_{x_2}(\alpha x_1 + \beta x_2) Pr(X_1 = x_1, X_2 = x_2) \\
&= \alpha \sum_{x_1} x_1 \sum_{x_2} Pr(X_1 = x_1, X_2 = x_2) + \\
&\quad \beta \sum_{x_2} x_2 \sum_{x_1} Pr(X_1 = x_1, X_2 = x_2) \\
&= \alpha \sum_{x_1} x_1 Pr(X_1 = x_1) + \beta \sum_{x_2} x_2 Pr(X_2 = x_2) \\
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The proof is similar for $n > 2$.

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Recalling that such a random variable X represents the number of successes in n trials when each trial has probability p of being a success, we have

$$X = X_1 + X_2 + \cdots + X_n$$

where X_i are Bernoulli variables, i.e.,

$$X_i = \begin{cases} 1, & \text{if the } i\text{th trial is a success} \\ 0, & \text{if the } i\text{th trial is a failure} \end{cases}$$

whose expectation is p. Thus,

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n] = np$$

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Remarks:

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$$\begin{aligned} Cov(X, Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E[XY - YE[X] - XE[Y] + E[X]E[Y]] \\ &= E[XY] - E[Y]E[X] - E[X]E[Y] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

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3. Covariance characterizes the degree of the dependence between two random variables.

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Thus we have $Cov(X, Y) = P\{X = 1, Y = 1\} - P\{X = 1\}P\{Y = 1\}$.

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1. $Cov(X, Y) = 0$ is equivalent to X and Y are independent.
2. $Cov(X, Y) > 0$ indicates that the outcome that $X = 1$ makes it more likely that $Y = 1$.

$$\begin{aligned} Cov(X, Y) > 0 &\Leftrightarrow P\{X = 1, Y = 1\} > P\{X = 1\}P\{Y = 1\} \\ &\Leftrightarrow \frac{P\{X = 1, Y = 1\}}{P\{Y = 1\}} > P\{X = 1\} \\ &\Leftrightarrow P\{Y = 1 \mid X = 1\} > P\{X = 1\} \end{aligned}$$

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3. If $Cov(X, Y) < 0$, then it's the other way around.

Some Properties of Covariance

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Properties

1. $Cov(X, X) = Var(X)$,
2. $Cov(X, Y) = Cov(Y, X)$
3. $Cov(cX, Y) = cCov(X, Y)$
4. $Cov(X, Y + Z) = Cov(X, Y) + Cov(X, Z)$

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Proof of 4:

$$\begin{aligned} Cov(X, Y + Z) &= E[X(Y + Z)] - E[X]E[Y + Z] \\ &= E[XY] + E[XZ] - E[X]E[Y] - E[X]E[Z] \\ &= (E[XY] - E[X]E[Y]) + (E[XZ] - E[X]E[Z]) \\ &= Cov(X, Y) + Cov(X, Z) \end{aligned}$$

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Corollary of 4:

$$Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} Cov(X_i, Y_j)$$

Variance of Sum of Random Variables

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Var\left(\sum_{i=1}^{n} X_i\right) &= Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} Cov(X_i, X_j) \\
&= \sum_{i=1}^{n} Cov(X_i, X_i) + \sum_{i=1}^{n}\sum_{j\neq i} Cov(X_i, X_j) \\
&= \sum_{i=1}^{n} Var(X_i) + 2\sum_{i=1}^{n}\sum_{j<i} Cov(X_i, X_j)
\end{aligned}
$$

Variance of Sum of Random Variables (cont'd)

If $X_i, i = 1, \ldots, n$ are independent with each other, then we have $Cov(X_i, X_j) = 0$ where $i \neq j$, thus

$$Var\Big(\sum_{i=1}^{n} X_i\Big) = \sum_{i=1}^{n} Var(X_i)$$

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Example Calculate the variance of a binomial random variable having parameters n and p without using the its PMF.

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Example Calculate the variance of a binomial random variable having parameters n and p without using the its PMF.

Setting are the same as before, i.e., $X = X_1 + X_2 + \cdots + X_n$ and

$$X_i = \begin{cases} 1, & \text{if the } i\text{th trial is a success} \\ 0, & \text{if the } i\text{th trial is a failure} \end{cases}$$

whose variance is $Var(X_i) = E[X^2] - E[X]^2 = E[X] - E[X]^2 = p - p^2$. thus,

$$Var(X) = \sum_{i=1}^{n} Var(X_i) = np(1-p)$$