Show $(\mathrm{Im}(E), \mathrm{Im}(H))^T$ is a fixed point of Π

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Suppose $(E, H)^T$ is the solutions of the non-dimensionalized equation

$$
\begin{aligned}
i\omega E &= \nabla \times H - J, && (1)\\
i\omega H &= -\nabla \times E. && (2)
\end{aligned}
$$

Decompose E and H as $E = \mathrm{Re}(E) + i\mathrm{Im}(E)$ and $H = Re(H) + i\mathrm{Im}(H)$, we will obtain

$$
\begin{aligned}
-\omega \mathrm{Im}(E) &= \nabla \times \mathrm{Re}(H) - Re(J), && (3)\\
\omega \mathrm{Re}(E) &= \nabla \times \mathrm{Im}(H) - \mathrm{Im}(J), && (4)\\
-\omega \mathrm{Im}(H) &= -\nabla \times \mathrm{Re}(E), && (5)\\
\omega \mathrm{Re}(H) &= -\nabla \times \mathrm{Im}(E). && (6)
\end{aligned}
$$

Utilizing this relation, one can verify that

$$
\begin{aligned}
\tilde{E}(t) &= \mathrm{Im}(E)\cos(\omega t) + \mathrm{Re}(E)\sin(\omega t), && (7)\\
\tilde{H}(t) &= \mathrm{Im}(H)\cos(\omega t) + \mathrm{Re}(H)\sin(\omega t). && (8)
\end{aligned}
$$

is a solution of

$$
\begin{aligned}
\partial_t \tilde{E} &= \nabla \times \tilde{H} - \sin(\omega t)\mathrm{Re}(J) - \cos(\omega t)\mathrm{Im}(J), && (9)\\
\partial_t \tilde{H} &= -\nabla \times \tilde{E}. && (10)
\end{aligned}
$$

Substitute $(\tilde{E}_0, \tilde{H}_0)^T = (\mathrm{Im}(E), \mathrm{Im}(H))^T$ and $(\tilde{E}(t), \tilde{H}(t))^T$ into the definition of filtering operator Π, we have

$$
\begin{aligned}
\Pi \begin{pmatrix} \mathrm{Im}(E) \\ \mathrm{Im}(H) \end{pmatrix} = \Pi \left(\tilde{E}_0 \tilde{H}_0\right) &= \frac{2}{T}\int_0^T (\cos(\omega t) - \frac{1}{4}) \begin{pmatrix} \tilde{E}(t) \\ \tilde{H}(t) \end{pmatrix} dt \\
&= \frac{2}{T}\int_0^T (\cos(\omega t) - \frac{1}{4}) \begin{pmatrix} \mathrm{Im}(E)\cos(\omega t) + \mathrm{Re}(E)\sin(\omega t) \\ \mathrm{Im}(H)\cos(\omega t) + \mathrm{Re}(H)\sin(\omega t) \end{pmatrix} dt \\
&= \begin{pmatrix} \mathrm{Im}(E) \\ \mathrm{Im}(H) \end{pmatrix}. && (11)
\end{aligned}
$$