1、 Eigen ~~Vec~~ Values

(1) A is a matrix, if λ is ~~an eigen value~~ a solution to $\det(\lambda I - A) = 0$, then λ is an eigenvalue.

(2) If $\vec{v} \neq 0$, satisfies $A\vec{v} = \lambda\vec{v}$ (or $(A-\lambda I)\vec{v} = 0$), then $\vec{v}$ is an eigenvector corresponds to λ.

(i) $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\det A = a_{11}a_{22} - a_{12}a_{21}$

(ii) $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ $A = (a_{11}a_{22}a_{33} + a_{13}a_{32}a_{21} + a_{12}a_{23}a_{31}) - (a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31} + a_{11}a_{32}a_{23})$

$A = \begin{pmatrix} -3 & -2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ $\det(\lambda I - A) = \begin{vmatrix} \lambda+3 & 2 & 0 \\ -1 & \lambda+1 & 0 \\ 0 & 0 & \lambda+2 \end{vmatrix}$

$= (\lambda+2)(\lambda^2 + 4\lambda + 3 + 2)$

$= (\lambda+2)[(\lambda+2)^2 + 1]$

$\lambda_1 = -2, \quad \lambda_2 = -2-i, \quad \lambda_3 = -2+i$

$x' = Ax$

Case 1: If $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$ are eigenvalues of A, $\vec{k_1}, \cdots, \vec{k_n}$ are corresponding eigenvectors,

then $\vec{x} = c_1 e^{\lambda_1 t} \vec{k_1} + \cdots + c_n \vec{k_n} e^{\lambda_n t}$

Case 2: λ is repeated

~~Case 2a: $\lambda = \lambda_1 = \lambda_2$ is repeated for 2~~

1° we can find linearly independent eigenvectors $\vec{k_1}, \cdots, \vec{k_m}$ corresponding to the repeated eigenvalue $\lambda = \lambda_1 = \cdots = \lambda_m$, then

$$\vec{x} = c_1 \vec{k_1} e^{\lambda t} + c_2 \vec{k_2} e^{\lambda t} + \cdots + c_m \vec{k_m} e^{\lambda t}$$

is a solution.

2° $A \in \mathbb{R}^{2\times 2}$, and $\lambda_1 = \lambda_2$, but matrix is degenerate (can not find two linearly independent eigenvectors).

(1) $(A - \lambda I)\vec{k_1} = \vec{k_2}$

(2) $(A - \lambda I)\vec{k_2} = \vec{k_1}$

then $c_1 k_1 e^{\lambda t} + c_2 (k_2 + k_1 t) e^{\lambda t}$ is a solution

$$X' = \begin{pmatrix} 0 & -9 \\ 1 & 0 \end{pmatrix} X$$

Step 1: $\det(\lambda I - A) = \begin{vmatrix} \lambda & 9 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 9 = 0$

$\Rightarrow \lambda_1 = 3i, \ \lambda_2 = -3i$

Step 2: (1) $(3i - A)\vec{k_1} = \begin{pmatrix} 3i & 9 \\ -1 & 3i \end{pmatrix} \begin{pmatrix} k_{11} \\ k_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{cases} 3i k_{11} + 9 k_{12} = 0 \\ -k_{11} + 3i k_{12} = 0 \end{cases} \Rightarrow k_{11} = 3i k_{12} \quad \vec{k_1} = \begin{pmatrix} 3i \\ 1 \end{pmatrix}$

(2) $(-3i - A)\vec{k_2} = \begin{pmatrix} -3i & 9 \\ -1 & -3i \end{pmatrix} \begin{pmatrix} k_{21} \\ k_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$k_{21} = -3i k_{22} \quad \vec{k_2} = \begin{pmatrix} -3i \\ 1 \end{pmatrix}$

Step 3: $y = c_1 e^{3it} \vec{k_1} + c_2 e^{-3it} \vec{k_2}$

$= c_1 (\cos(3t) + i\sin(3t)) \begin{pmatrix} 3i \\ 1 \end{pmatrix} + c_2 (\cos(3t) - i\sin(3t)) \begin{pmatrix} -3i \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -3(c_1 + c_2)\sin 3t \\ (c_1 + c_2)\cos 3t \end{pmatrix} + i \begin{pmatrix} 3(c_1 - c_2)\cos 3t \\ (c_1 - c_2)\sin 3t \end{pmatrix}$

$$y = \hat{c}_1 \begin{pmatrix} -3\sin 3t \\ \cos 3t \end{pmatrix} + \hat{c}_2 \begin{pmatrix} 3\cos 3t \\ \sin 3t \end{pmatrix}$$

$$X' = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

step 1: $\det(\lambda I - A) = \begin{vmatrix} \lambda+2 & -1 \\ 0 & \lambda+2 \end{vmatrix} = (\lambda+2)^2 = 0$

$\lambda_1 = \lambda_2 = -2$

step 2: (1) $(\lambda I - A)\vec{k_1} = 0 \quad \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} k_{11} \\ k_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow k_{12} = 0$

One can not find 2 linearly independent eigenvectors

$\vec{k_1} = \begin{pmatrix} k_{11} \\ 0 \end{pmatrix} \Rightarrow \vec{k_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(2) $(\lambda I - A)\vec{k_2} = \vec{k_1} \Rightarrow \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} k_{21} \\ k_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$k_{22} = -1, \quad k_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

step 3: $X = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + c_2 \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t \right) e^{-2t} = \begin{pmatrix} c_1 e^{-2t} + c_2 t e^{-2t} \\ -c_2 e^{-2t} \end{pmatrix}$

$X(0) = \begin{pmatrix} c_1 \\ -c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow X = \begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix}$

$$X' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} X$$

Step 1: $\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} = \lambda^3 - 3\lambda - 2$

$= \lambda^2 - \lambda - 2\lambda - 2$

$= \lambda(\lambda^2 - 1) - 2(\lambda + 1)$

$= \lambda(\lambda+1)(\lambda-1) - 2(\lambda+1)$

$= (\lambda+1)(\lambda^2 - \lambda - 2)$

$= (\lambda+1)^2(\lambda - 2)$

$\lambda_1 = \lambda_2 = -1, \quad \lambda_3 = 2$

step 2: (1) $(\lambda_1 I - A)\vec{k_1} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} k_{11} \\ k_{12} \\ k_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow k_{11} = -(k_{12} + k_{13}), \quad \vec{k} = \begin{pmatrix} -k_{12} - k_{13} \\ k_{12} \\ k_{13} \end{pmatrix}$

two linearly independent eigenvector

$\vec{k_1} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{k_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

(2) $(\lambda_3 I - A)\overrightarrow{k_3} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \overrightarrow{k_3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow k_{31} = k_{32} = k_{33}$

$\Rightarrow \overrightarrow{k_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

step3: $X = C_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + C_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}$

$= \begin{pmatrix} -C_1 e^{-t} - C_2 e^{-t} + C_3 e^{2t} \\ C_1 e^{-t} + C_3 e^{2t} \\ C_2 e^{-t} + C_3 e^{2t} \end{pmatrix}$

Step 1: Get eigenvalues

Step 2: Calculate eigenvectors,

and if there are not enough eigenvectors,

calculate $(\lambda I - A)\overrightarrow{k_2} = \overrightarrow{k_1}$

Step 3: $\overrightarrow{X} =$ ...