submit H.W. to 301 Amos Eaton

Homepage : zhichaopeng math. github.io

Recitation notes online

HW. Quest 5 (Maxwell one), there is a typo, missing $\frac{1}{t}$ before the integral

---

Euler-Type D.E.

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right), \text{ e.g. } \frac{x+y}{x-y}, \quad \frac{x^2+2xy+y^2}{2y^2}$$

$$\frac{x+y}{x-y} \downarrow \frac{1+\frac{y}{x}}{1-\frac{y}{x}}, \qquad \frac{x^2+2xy+y^2}{2y^2} \downarrow \frac{(\frac{y}{x})^2+2\frac{y}{x}+1}{2(\frac{y}{x})^2}$$

Let $u = \frac{y}{x}$

$$ux = y \xrightarrow[\text{rule}]{\text{Product}} x\,du + u\,dx = dy$$

$$\frac{dy}{dx} = x\frac{du}{dx} + u$$

$$x\frac{du}{dx} + u = F(u), \quad \frac{du}{dx} = \frac{F(u)-u}{x}$$

Seperation of variables to solve this one

1、(1) $\frac{dy}{dx} = \frac{x+y}{x-y}$

Let $u=\frac{y}{x}$, $\frac{dy}{dx} = x\frac{du}{dx} + u = \frac{x+y}{x-y} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}} = \frac{1+u}{1-u}$

$$\frac{du}{dx} = \frac{1}{x}\left(\frac{1+u}{1-u} - u\right) = \frac{1}{x}\cdot\frac{1+u-u+u^2}{1-u} = \frac{1}{x}\cdot\frac{1+u^2}{1-u}$$

$$\frac{(1-u)\,du}{1+u^2} = \frac{dx}{x} \qquad \ln|x| + C = \int\frac{du}{1+u^2} - \int\frac{u}{1+u^2}du$$

$$\int \frac{du}{1+u^2} - \int \frac{u}{1+u^2}\,du = \arctan(\overset{u}{\cancel{1+u^2}}) - \int \frac{u}{1+u^2}\,du$$

$$\xlongequal[dv=2u\,du]{v=u^2} \arctan u - \frac{1}{2}\int \frac{dv}{1+v}$$

$$= \arctan u - \frac{1}{2}\ln(1+v)$$

$$= \arctan u - \frac{1}{2}\ln(1+u^2)$$

$$= \arctan u - \ln\sqrt{1+u^2}$$

$$\parallel$$

$$\ln|x| + C$$

$$e^{-\ln\sqrt{1+u^2} + \arctan u} = |x| \cdot e^{C}$$

$$\Rightarrow \frac{e^{\arctan u}}{\sqrt{1+u^2}} = |x| \cdot e^{C}$$

$$\Rightarrow e^{\arctan\frac{y}{x}} = \sqrt{1+\frac{y^2}{x^2}}\ |x| \cdot \tilde{C}$$

Implicit form

---

(2) $(x^2+3xy+y^2)\,dx - x^2dy = 0$

$$\frac{dy}{dx} = \frac{x^2+3xy+y^2}{x^2} = 1 + 3\frac{y}{x} + \left(\frac{y}{x}\right)^2$$

Let $u = \frac{y}{x}$, $\frac{dy}{dx} = \frac{du}{dx}x + u \Rightarrow x\frac{du}{dx} + u = 1+3u+u^2$

$$\frac{du}{dx} = \frac{1}{x}(1+2u+u^2) = \frac{1}{x}(1+u)^2$$

$$\frac{du}{(1+u^2)} = \frac{dx}{x} \Rightarrow \ln|x| + C = -\frac{1}{1+u} \Rightarrow u = -\frac{1}{\ln|x|+C} - 1$$

$$y = -\frac{x}{\ln|x| + c}$$

Bernolli - Type D.E.

$$y' + p(x)\, y = g(x)\, y^{r}, \quad r \neq 0, 1$$

$r = 0$ or 1, this equation can be solved by integrating factor method $(y^{1-r})'$

Let $z = y^{1-r}$, then $dz = (1-r)\, y^{-r}\, dy$

$$\frac{dy}{dx} = \frac{1}{1-r}\, y^{r} \frac{dz}{dx}$$

$$\frac{1}{1-r}\, y^{r} \frac{dz}{dx} + p(x)\, y = g(x)\, y^{r}$$

Divide y^{r},

$$\frac{1}{1-r} \frac{dz}{dx} + p(x)\, y^{1-r} = g(x)$$

$$\frac{1}{1-r} \frac{dz}{dx} + p(x)\, z = g(x)$$

$$\frac{dz}{dx} + (1-r)\, p(x)\, z = g(x)$$

We use Integrating factor method to solve it

2、(1) $x^2y' + 2xy - y^3 = 0,\ x>0$

$$y' + p(x)\, y = g(x)\, y^r$$

$$y' + \frac{2}{x} y = \frac{1}{x^2} y^3$$

$r=3$, Let $z = y^{1-r} = y^{-2}$,

$$dz = -2y^{-3}\, dy \Rightarrow \frac{dy}{dx} = -\frac{1}{2} y^3 \frac{dz}{dx}$$

$$-\frac{1}{2} y^3 \frac{dz}{dx} + \frac{2}{x} y = \frac{1}{x^2} y^3$$

$$-\frac{1}{2} \frac{dz}{dx} + \frac{2}{x} y^{-2} = \frac{1}{x^2}$$

$$\frac{dz}{dx} - \frac{4}{x} z = -\frac{2}{x^2}$$

$$e^{\int -\frac{4}{x}dx} = \frac{1}{x^4} \qquad \frac{1}{x^4} z' - \frac{4}{x^5} z = -\frac{2}{x^6}$$

$$\left(\frac{1}{x^4} z\right)' = -\frac{2}{x^6}$$

$$\frac{1}{x^4} z = -\int \frac{2}{x^6} dx = \frac{2}{5} \cdot \frac{1}{x^5} + C$$

$$z = y^{-2} = \frac{2}{5} \cdot \frac{1}{x} + Cx^4 \Rightarrow y = \pm\sqrt{\frac{5x}{2+5Cx^4}}$$

(2) $y' = 5y + e^{-2x}y^{-2}, \quad y(0) = 2$

Sol: $y' - 5y = e^{-2x}y^{-2}, \quad r = -2$

$$z = y^{1-(-2)} = y^3, \quad dz = 3y^2 dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{3}y^{-2}\frac{dz}{dx}$$

$$\frac{1}{3}y^{-2}\frac{dz}{dx} - 5y = e^{-2x}y^{-2}$$

$$\frac{1}{3}z' - 5y^3 = e^{-2x}$$

$$z' - 15z = 3e^{-2x}$$

$$(e^{-15x}z)' = e^{-15x}z' - 15e^{-15x}z = 3e^{-17x}$$

$$e^{-15x}z = 3\int e^{-17x}dx = -\frac{3}{17}e^{-17x} +$$

$$z = ce^{15x} - \frac{3}{17}e^{-2x}$$

$x=0 \Rightarrow y(0)=2, \quad z(0) = y^3(0) = 8 = C - \frac{3}{17}$

$$\Rightarrow c = \frac{139}{17} \quad \Rightarrow y(x) = z^{\frac{1}{3}} = \left(\frac{139 - 3e^{-2x}}{17}\right)^{\frac{1}{3}}$$

(3) $y' + \frac{y}{x} - \sqrt{y} = 0, \quad y(1) = 0$

In Berndli Type, r may not be integer, r could be any real number

$$y' + \frac{1}{x}y = \sqrt{y} = y^{\frac{1}{2}} \quad r = \frac{1}{2}$$

$$\text{Let } z = y^{1-r} = y^{\frac{1}{2}}$$

$$dz = \frac{1}{2} y^{-\frac{1}{2}} dy$$

$$\frac{dy}{dx} = 2y^{\frac{1}{2}} \frac{dz}{dx}$$

$$2y^{\frac{1}{2}} \frac{dz}{dx} + \frac{1}{x}y = y^{\frac{1}{2}}$$

$$\frac{dz}{dx} + \frac{1}{2x} z = \frac{1}{2}$$

$$e^{\int \frac{1}{2x} dx} = \sqrt{x} \qquad \sqrt{x} z' + \frac{1}{2} \cdot \frac{1}{\sqrt{x}} z = \frac{1}{2}\sqrt{x}$$

$$(\sqrt{x} z)' = \frac{1}{2}\sqrt{x}$$

$$\sqrt{x} z = \frac{1}{2} \int x^{\frac{1}{2}} dx = \frac{1}{3} x^{\frac{3}{2}} + C$$

$$\Rightarrow z = \frac{1}{3} x + C x^{-\frac{1}{2}}$$

$$y(1) = 0 \Rightarrow z(1) = \sqrt{y(1)} = 0 = \frac{1}{3} + C \Rightarrow C = -\frac{1}{3}$$

$$y = \left(\frac{1}{3}x - \frac{1}{3\sqrt{x}}\right)^2$$

(1) Figure out how to determine z and u

(2) Rewrite $\frac{dy}{dx} = F(\frac{dz}{dx}, x, y)$ based on the relation between dz and dy

(3) Simplify eqn., use integrating factor / Sep. of Var.

$$y' + y = 5\sin \overset{3x}{\cancel{2x}}$$

$$(e^x y)' = e^x y' + e^x y = 5\sin 2x\, e^x$$

$$e^x y = 5\int e^x \sin 2x\,dx = 5I$$

$$I = \int e^x \sin 2x\,dx$$

Integration by parts

$$= e^x \sin 2x - 2\int e^x \cos 2x\,dx$$

$$= e^x \sin 2x - 2\left(e^x \cos 2x + 2\int e^x \sin 2x\,dx\right)$$

$$= e^x \sin 2x - 2e^x \cos 2x - 4I$$

$$5I = e^x (\sin 2x - 2\cos 2x)$$ integral

$$\Rightarrow y = \sin 2x - 2\cos 2x$$

I.B.P.: $e^x \sin 2x$, $e^x \cos 2x$, $e^{2x} \sin x$, keep doing I.B.P. to obtain