$$y'' + a_1 y' + a_2 y = 0$$

Assume, $y = e^{rx}$, $y' = re^{rx}$, $y'' = r^2 e^{rx}$

$$\Rightarrow (r^2 + a_1 r + a_2) e^{rx} = 0$$

characteristic equation: $r^2 + a_1 r + a_2 = 0$

$$\Delta = a_1^2 - 4a_2, \quad r_{1,2} = \frac{-a_1 \pm \sqrt{\Delta}}{2}$$

(1) $\Delta > 0$, r_1, r_2 two different real roots

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

(2) $\Delta < 0$, r_1, r_2 two different complex roots

$$r_{1,2} = -\frac{a_1 \pm \sqrt{\Delta}}{2} = -\frac{a_1 \pm i\sqrt{|\Delta|}}{2} = \alpha \pm i\beta$$

$$e^{(\alpha + i\beta)x} = e^{\alpha x}(\cos\beta x \pm i \sin\beta x)$$

$$y = \tilde{C}_1 e^{r_1 x} + \tilde{C}_2 e^{r_2 x} \Rightarrow y = e^{\alpha x}(C_1 \sin\beta x + C_2 \cos\beta x)$$

(3) $\Delta = 0$, $r_1 = r_2 = -\frac{a_1}{2}$

two repeated roots

$$y = C_1 e^{r_1 x} + C_2 x e^{r_1 x} \rightarrow \text{Related to Jordan Form}$$

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1. (1) $y'' + y' - 2y = 0$, $y(0) = 1$, $y'(0) = -1$

$$\Rightarrow r^2 + r - 2 = 0 \Rightarrow (r+2)(r-1) = 0 \Rightarrow r_1 = -2, r_2 = 1$$

$$y = C_1 e^{-2x} + C_2 e^{x}$$

$$y(0) = C_1 + C_2 = 1, \quad y'(0) = -2C_1 e^{-2\cdot 0} + C_2 e^{0} = -1$$

$$\Rightarrow C_1 = \frac{2}{3}, C_2 = \frac{1}{3} \Rightarrow y = \frac{2}{3}e^{-2x} + \frac{1}{3}e^{x}$$

(2) $y'' + 2y' + 2y = 0$

$r^2 + 2r + 2 = 0 \Rightarrow \Delta = 4 - 4\cdot 2 = -4$

$r_{1,2} = \dfrac{-2 \pm \sqrt{-4}}{2} = \overset{\alpha}{-1} \pm \overset{\beta i}{i}$

$y = c_1 e^{-x}\sin x + c_2 e^{-x}\cos x$

(3) $9y'' - 12y' + 4y = 0,\ y(0) = 2,\ y'(0) = -1$

$9r^2 - 12r + 4 = 0 \Rightarrow (3r-2)^2 = 0 \Rightarrow r_1 = r_2 = \frac{2}{3}$

$y = c_1 e^{\frac{2}{3}x} + c_2 x e^{\frac{2}{3}x}$

$y(0) = c_1 e^{\frac{2}{3}\cdot 0} + 0\cdot c_2 e^{\frac{2}{3}\cdot 0} = c_1 = 2$

$y'(0) = \frac{2}{3}c_1 e^{\frac{2}{3}\cdot 0} + c_2 e^{\frac{2}{3}\cdot 0} + \frac{2}{3}c_2\cdot 0\cdot e^{\frac{2}{3}\cdot 0}$ *

$= \frac{2}{3}c_1 + c_2 = -1$

$\Rightarrow c_1 = 2,\ c_2 = -\frac{7}{3},\ y = 2e^{\frac{2}{3}x} - \frac{7}{3}xe^{\frac{2}{3}x}$

(4) $y'' + 8y' - 9y = 0$

$r^2 + 8r - 9 = 0 \Rightarrow (r+9)(r-1) = 0$

$\Rightarrow y = c_1 e^{-9x} + c_2 e^{x}$

Step 1: Write out and solve characteristic equation

Step 2: Determine which cases we are considering and obtain General solution

(Step 3: Solve I.V.P.)

2、(1) $y'' + 6y' + 9y = 2e^{-3t}$

$r^2 + 6r + 9 = 0 \Rightarrow r_1 = r_2 = -3$

$\Rightarrow y_p = A_0 t^2 e^{-3t}$

$y_h = C_1 e^{-3t} + C_2 t e^{-3t}$

Determine A_0: $y_p' = 2A_0 t e^{-3t} - 3A_0 t^2 e^{-3t}$

$y_p'' = 2A_0 e^{-3t} - 12 A_0 t e^{-3t} + 9A_0 t^2 e^{-3t}$

$y_p'' + 6y_p' + 9y_p = 2A_0 e^{-3t} - 12 A_0 t e^{-3t} + \cancel{9A_0 t^2 e^{-3t}}\ 9A_0 t^2 e^{-3t}$
$+ 12 A_0 t e^{-3t} - 18 A_0 t^2 e^{-3t}$
$+ 9A_0 t^2 e^{-3t}$

$= 2A_0 e^{-3t}$

$= 2e^{-3t} \Rightarrow A_0 = 1$

$y = C_1 e^{-3t} + C_2 t e^{-3t} + t^2 e^{-3t}$

(2) $y'' + y' - 2y = 2e^{-x} x$

$r^2 + r - 2 = 0 \quad (r+2)(r-1) = 0$

$\Rightarrow r_1 = -2,\ r_2 = 1 \Rightarrow y_h(x) = C_1 e^{-2x} + C_2 e^{x}$

$y_p = (A_0 + A_1 x) e^{-x}$

consider $y'' + a_1 y' + a_2 y = g(x)$

General solution $y_h(x)$ to homogenuous problem $y'' + a_1 y' + a_2 y = 0$

Particular solution $y_p(x)$ s.t. $y_p'' + a_1 y_p' + a_2 y_p = g(x)$

then $y(x) = y_h(x) + y_p(x)$.

Particular $g(x) = \underbrace{(b_0 + b_1 x + \cdots + b_n x^n)}_{n\text{-th order polynomial}} e^{\lambda x}$

$y'' + a_1 y' + a_2 y = 0$

Characteristic Eqn. $r^2 + a_1 r + a_2 = 0$

solution r_1, r_2

Case 1: $\lambda \neq r_1$, $\lambda \neq r_2$, $y_p = (\underbrace{A_0 + A_1 x + \cdots + A_n x^n}_{n\text{-th order polynomial}}) e^{\lambda x}$

Plug y_p into the original eqn., then we can obtain equations for coefficients $A_0, A_1, \cdots, A_n$

Case 2 (i) $\lambda = r_1 \neq r_2$ or $\lambda = r_1 \neq r_2$

$$y_p = (A_0 + A_1 x + \cdots + A_n x^n) x e^{\lambda x}$$

(ii) $\lambda = r_1 = r_2$, $y_p = (A_0 + \cdots + A_n x) x^2 e^{\lambda x}$

$$\cancel{y_p'' = -(A_1 - A_0)e^{-x} - A_1}$$

$$y_p' = (A_1 - A_0)e^{-x} - A_1 x e^{-x}$$

$$y_p'' = -(A_1 - A_0)e^{-x} - A_1 e^{-x} + A_1 x e^{-x}$$

$$= (-2A_1 + A_0)e^{-x} + A_1 x e^{-x}$$

$$\Rightarrow y_p'' + y_p' - 2y_p = (-2A_1 + A_0)e^{-x} + A_1 x e^{-x}$$

$$+ (A_1 - A_0)e^{-x} - A_1 x e^{-x}$$

$$- 2A_0 e^{-x} - 2A_1 x e^{-x}$$

$$= 2xe^{-x}$$

$$\Rightarrow \quad -A_1 - 2A_0 = 0, \quad -2A_1 = 2$$

$$\Rightarrow A_1 = -1, \ A_0 = \frac{1}{2}, \ y_p = -xe^{-x} + \frac{1}{2}e^{-x}$$