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System of linear equations

(A) Eigen vector and eigen values

(1) A is a matrix, if λ is a solution to $\det(\lambda I - A) = 0$, then λ is an eigen value.

(2) If vector v satisfies $Av = \lambda v$, then v is a vector corresponding to eigen value λ. $\Leftrightarrow (\lambda I - A)v = 0$

1. $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$\det A = a_{11}a_{22} - a_{12}a_{21}$ $\quad \det A = (a_{11}a_{22}a_{33} + a_{13}a_{32}a_{21} + a_{12}a_{23}a_{31}) - (a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31} + a_{11}a_{32}a_{23})$

2. $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\quad \lambda I - A = \begin{pmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{pmatrix}$

$\det(\lambda I - A) = (\lambda-1)^2 - 4 = (\lambda-1-2)(\lambda-1+2)$

$= (\lambda-3)(\lambda+1) = 0$

$\lambda_1 = 3, \lambda_2 = -1$

3. $A = \begin{pmatrix} -3 & -2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda+3 & 2 & 0 \\ -1 & \lambda+1 & 0 \\ 0 & 0 & \lambda+2 \end{pmatrix}$$

$$= (\lambda+3)(\lambda+1)(\lambda+2) - (-1)\times 2(\lambda+2)$$

$$= (\lambda+2)(\lambda^2+4\lambda+3+2)$$

$$= (\lambda+2)(\lambda^2+4\lambda+5)$$

$$= (\lambda+2)((\lambda+2)^2+1)$$

$$= (\lambda+2)(\lambda+2+i)(\lambda+2-i)$$

$\lambda_1 = -2$, $\lambda_2 = -2-i$, $\lambda_3 = -2+i$

$$X' = Ax$$

Case 1: if $\lambda_1, \cdots, \lambda_n$ are distinguished eigen
$K_1, \cdots, K_n$ are corresponding eigen value

then $X = c_1 K_1 e^{\lambda_1 t} + \cdots + c_n K_n e^{\lambda_n t}$

is a solution.

Intuition : (i) $x' = \lambda x$, $x = ce^{\lambda t}$ is a solution (ii) $AK = \lambda K$,

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Case 2: λ is a repeated eigen value for 2 time

Case 1° A is 2×2,

(1) $A k_1 = \lambda k_1 \rightarrow$ eigen-vector

(2) $(A-\lambda I) k_2 = k_1$

then $c_1 k_1 e^{\lambda t} + c_2 (k_2 + k_1 t) e^{\lambda t}$ is a solution

2° A is 3×3, λ is repeated 3 times.

(1) $A k_1 = \lambda k_1$

(2) $(A-\lambda I) k_2 = k_1$

(3) $(A-\lambda I) k_3 = k_2$

$$y = c_1 k_1 e^{\lambda t} + c_2 (k_2 + k_1 t) e^{\lambda t} + c_3 \left(k_3 + k_2 t + k_1 \frac{t^2}{2}\right) e^{\lambda t}$$

Step 1: Calculate eigen-value
Step 2: Obtain ~~eigen vector k, or~~ k_1, k_2, k_3
Step 3: Get solution

Tip: $\lambda = \alpha + \beta i$

$$e^{\lambda t} = e^{(\alpha+\beta i)t}$$

$$= e^{\alpha t}(\cos\beta t \pm i \sin\beta t)$$

$$X' = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} X$$

Step 1: ~~Calculate ei~~

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & -1 \\ -6 & \lambda-3 \end{vmatrix} = (\lambda-3)(\lambda-2)-6$$

$$= \lambda^2 - 5\lambda = 0$$

$$\Rightarrow \lambda_1 = 5, \quad \lambda_2 = 0$$

Step 2: (i)

$$(5I-A)u = \begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

$$\begin{cases} 3u_1 - u_2 = 0 \\ -6u_1 + 2u_2 = 0 \end{cases} \qquad u_2 = 3u_1$$

Let $u_1 = 1$, $u_2 = 3$, $\quad u = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

(ii)

$$(0I-A)v = \begin{pmatrix} 0-2 & -1 \\ -6 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow -2v_1 = v_2, \quad v = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Solution:

$$X = c_1\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{0\cdot t}$$

$$= \begin{pmatrix} c_1 e^{5t} + c_2 \\ c_1 \cdot 3e^{5t} - 2c_2 \end{pmatrix}.$$

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(2) $X' = \begin{pmatrix} 0 & -9 \\ 1 & 0 \end{pmatrix} X$

step 1:

$\det(\lambda I - A) = \begin{pmatrix} \lambda & 9 \\ -1 & \lambda \end{pmatrix} = \lambda^2 + 9 = 0$

$\Rightarrow \lambda_1 = 3i, \quad \lambda_2 = -3i$

step 2: (1)

$(3i - A)u = \begin{pmatrix} 3i & 9 \\ -1 & 3i \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$

$u_1 = 3i u_2 \Rightarrow u_0 = \begin{pmatrix} 3i \\ 1 \end{pmatrix}$

$u_2 = 1$

(2) $(-3i - A)v = \begin{pmatrix} -3i & 9 \\ -1 & -3i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad v_1 = -3i v_2$

$v_2 = 1, \quad V = \begin{pmatrix} -3i \\ 1 \end{pmatrix}$

step 3: $y = c_1 e^{3it} u + c_2 e^{-3i} V$

$= (\cos(3t) + i\sin(3t))\begin{pmatrix} 3i \\ 1 \end{pmatrix} + c_2(\cos(3t) - i\sin(3t))$

$= \begin{pmatrix} (3\cos(3t) i + 3i^2 \sin(3t)) c_1 + (-3i\cos(3t) + 3i^2 \sin(3t)) c_2 \\ (\cos(3t) + i\sin(3t)) c_1 + (\cos(3t) - i\sin(3t)) c_2 \end{pmatrix}$

$$= \begin{pmatrix} 3c_1 \cos 3t\, i - 3\sin 3t\, c_1 - 3c_2 \cos 3t\, i - 3\sin 3t\, c_2 \\ c_1 \cos 3t + \sin 3t\, c_1 i + c_2 \cos 3t - c_2 i \sin 3t \end{pmatrix}$$

$$= \begin{pmatrix} -3(c_1+c_2)\sin 3t \\ (c_1+c_2)\cos 3t \end{pmatrix} + \begin{pmatrix} 3(c_1-c_2)\cos 3t \\ (c_1-c_2)\sin 3t \end{pmatrix} i$$

General real solution

$$\Rightarrow y = \hat{c}_1 \begin{pmatrix} -3\sin 3t \\ \cos 3t \end{pmatrix} + \hat{c}_2 \begin{pmatrix} 3\cos 3t \\ \sin 3t \end{pmatrix}$$

(3) $$x' = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} x$$

Step 1: $$\lambda I - A = \begin{pmatrix} \lambda+2 & -1 \\ 0 & \lambda+2 \end{pmatrix} = (\lambda+2)^2 = 0$$

$$\lambda_1 = \lambda_2 = -2$$

Step 2: (i) $$(\lambda I - A)u = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

$$\Rightarrow u_2 = 0, \quad \text{Let } u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(ii) $$(\lambda I - A)v = u \quad \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow v_2 = -1 \quad v = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Step 3: $$x = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + c_2\left(\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t\right) e^{-2t} = \begin{pmatrix} c_1 e^{-2t} + c_2 t e^{-2t} \\ -c_2 e^{-2t} \end{pmatrix}$$