Show $(Im(E), Im(H))^T$ is a fixed point of Π

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Suppose $(E, H)^T$ is the solutions of the non-dimensionalized equation

$$
\begin{aligned}
i\omega E &= \nabla \times H - J, && (1)\\
i\omega H &= -\nabla \times E, && (2)
\end{aligned}
$$

We will obtain

$$
\begin{aligned}
-\omega Im(E) &= \nabla \times Re(H) - Re(J), && (3)\\
\omega Re(E) &= \nabla \times Im(H) - Im(J), && (4)\\
-\omega Im(H) &= -\nabla \times Re(E), && (5)\\
\omega Re(H) &= -\nabla \times Im(E). && (6)
\end{aligned}
$$

Utilizing this relation, one can verify that

$$
\begin{aligned}
\tilde{E}(t) &= Im(E)\cos(\omega t) + Re(E)\sin(\omega t), && (7)\\
\tilde{H}(t) &= Im(H)\cos(\omega t) + Re(H)\sin(\omega t). && (8)
\end{aligned}
$$

is a solution of

$$
\begin{aligned}
\partial_t \tilde{E} &= \nabla \times \tilde{H} - \sin(\omega t)Re(J) - \cos(\omega t)Im(J), && (9)\\
\partial_t \tilde{H} &= -\nabla \times \tilde{E}. && (10)
\end{aligned}
$$

Substitute $(\tilde{E}_0, \tilde{H}_0)^T = (Im(E), Im(H))^T$ and $(\tilde{E}(t), \tilde{H}(t))^T$ into the definition of filtering operator Π, we have

$$
\begin{aligned}
\Pi \begin{pmatrix} Im(E) \\ Im(H) \end{pmatrix} &= \Pi\left(\tilde{E}_0 \tilde{H}_0\right) = \frac{2}{T}\int_0^T (\cos(\omega t) - \frac{1}{4}) \begin{pmatrix} \tilde{E}(t) \\ \tilde{H}(t) \end{pmatrix} dt \\
&= \frac{2}{T}\int_0^T (\cos(\omega t) - \frac{1}{4}) \begin{pmatrix} Im(E)\cos(\omega t) + Re(E)\sin(\omega t) \\ Im(H)\cos(\omega t) + Re(H)\sin(\omega t) \end{pmatrix} dt \\
&= \begin{pmatrix} Im(E) \\ Im(H) \end{pmatrix}. && (11)
\end{aligned}
$$