1. What is the order of equation

$$x^{25} + x^{100}\frac{dy^2}{dx^2} + y^{200} + \frac{dy^{10}}{dx^{10}} = 0$$

A 25 B 100 C 200 D 10

The order of a differential equation is determined by the order of highest derivative.

2. Is $y = 3x + x^2$ a solution to $xy' - y = x^2$?

$y' = 3 + 2x$ $\quad x(3+2x) - (3x+x^2) = (3x+2x^2) - (3x+2x^2) = 0$

$y = 3x + x^2$ is a solution

Is $y = 3x + x^2 + 100$ also a solution?

X

Check by plug it into the equation

or we can solve the equation and compare with general solution

Integrating factor

$y' + p(x)\,y = R(x)$

$\left(e^{\int p(x)dx}\right)' = e^{\int p(x)dx} \quad \left(e^{\int p(x)dx}\right)' = p(x)e^{\int p(x)dx}$

$f(g(x)) = f'(g(x))\,g'(x) \quad f = e^x, \; g = \int p(x)dx$

$e^{\int p(x)dx}y' + p(x)\,e^{\int p(x)dx}y = e^{\int p(x)dx}R(x)$

$e^{\int p(x)dx}y' + \left(e^{\int p(x)dx}\right)' y = e^{\int p(x)dx}R(x)$

$(fg)' = f'g + fg', \quad \left(e^{\int p(x)dx}y\right)' = e^{\int p(x)dx}R(x)$

$$e^{\int p(x)dx}y = \int R(x)\,e^{\int p(x)dx}\,dx$$

Step 1: Find integrating factor $e^{\int p(x)dx}$

Step 2: integrate $\int R(x)\,e^{\int p(x)dx}\,dx$

Tricks: Integration by parts

3、(1) $xy' - y = x^2$

$$y' - \frac{1}{x}y = x \quad \text{I.F.} \quad e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\frac{1}{x}y' - \frac{1}{x^2}y = 1 \Rightarrow \frac{1}{x}y' + \left(\frac{1}{x}\right)'y = 1$$

$$\left(\frac{1}{x}y\right)' = 1 \Rightarrow \frac{1}{x}y' = x + C$$

$$y' = x^2 + Cx$$

Hence, $y = 3x + x^2 + 100$ not a solution

(2) $xy' + 2y = \sin x$, $y(\frac{\pi}{2}) = 1$

$$y' + \frac{2}{x}y = \frac{1}{x}\sin x \qquad e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$x^2y' + 2xy = x\sin x$$

$$x^2y' + (x^2)'y = x\sin x$$

$$(x^2y)' = x\sin x \qquad x^2y = \int x\sin x\,dx$$

$\int f'g\,dx$
$= \int (fg)'dx$
$- \int fg'dx$
$= fg - \int fg'dx$

$$x^2y = \int \underset{g}{x}\,\underset{f'}{\sin x}\,dx \xrightarrow[\text{by parts}]{\text{Integration}} -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C$$

$$y(x) = -\frac{\cos x}{x} + \frac{\sin x}{x^2} + \frac{C}{x^2}$$

$$y\left(\frac{\pi}{2}\right) = \frac{C}{\frac{\pi^2}{4}} + \frac{1}{\frac{\pi^2}{4}} = 1 \Rightarrow$$

$$C = \frac{4}{\pi^2} - 1$$

(3) $xy' + 3y = x\cos x$

$$y' + \frac{3}{x}y = \cos x$$

I.F. $e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$ $\quad x^3y' + 3x^2y = x^3\cos x$

$$x^3y' + (x^3)'y = x^3\cos x$$

$$(x^3y)' = x^3\cos x \Rightarrow x^3y = \int \underset{g}{x^3}\,\underset{f'}{\cos x}\,dx$$

$$\underset{\text{by parts}}{\overset{\text{Integration}}{=\!=}} x^3\sin x - 3\int x^2 \sin x\,dx$$

$$= x^3\sin x + 3x^2\cos x - 6\int x\cos x\,dx$$

$$= x^3\sin x + 3x^2\cos x - 6x\sin x + 6\int \sin x\,dx$$

$$= x^3\sin x + 3x^2\cos x - 6x\sin x - 6\cos x + C$$

$$\Rightarrow \quad y = \sin x + 3\frac{\cos x}{x} - \frac{6}{x^2}\sin x - \frac{6}{x^3}\cos x + \frac{C}{x^3}$$

(4) $y'+y=5\sin 2x$

$e^{\int 1dx}=e^x$ $\quad e^x y' + e^x y = 5e^x\sin 2x$

$$e^x y = 5\int e^x \sin 2x\,dx = 5I$$

$$I = \int \underset{f'}{e^x}\, \underset{g}{\sin 2x}\,dx$$

$$\xrightarrow[\text{by parts}]{\text{Integration}} e^x\sin 2x - 2\int e^x \cos 2x\,dx$$

$$= e^x\sin 2x - 2(e^x\cos 2x + 2\int e^x \sin 2x\,d$$

$$\cancel{+C}$$

$$= e^x\sin 2x - 2e^x\cos 2x - 4I + C$$

$$\Rightarrow 5I = e^x\sin 2x - 2e^x\cos 2x$$

$$\cancel{I = \frac{1}{5}e}$$

$$e^x y = 5I = e^x\sin 2x - 2e^x\cos 2x + C$$

$$y = \sin 2x - 2\cos 2x + C$$

Integration by parts

IBP

polynomials:
Use IBP to lower polynomial order until getting a consta

$\sin x \, e^{2x}$, $e^x \sin 2x$:
Keep using IBP to obtain an equation to the original integral