Laplace Transform

1. $Y(s) \equiv \int_0^\infty e^{-st} y(t)\,dt = Ly$

(1) Define $L^{-1}(Y(s))$ s.t. if $L(y) = Y(s)$, then $L^{-1}(Y) = y$

(2) $L(y') = -y(0) + sL(y)$

P.f. $\because L(y') = \int_0^\infty e^{-st} y'(t)\,dt = e^{-st} y(t)\Big|_0^\infty - (-s)\int_0^\infty e^{-st} y(t)\,dt$

$= -y(0) + s\int_0^\infty e^{-st} y(t)\,dt = -y(0) + sL(y)$

Laplace Transform : (i) derivative $\longrightarrow +, \times$

(ii) Inverse Transform: $L(y) \rightarrow y$

Using Laplace Transform to solve differential equation,

Step 1: Rewrite eqn. into an algebra eqn. by applying Laplace Transform

Step 2: Use inverse transform to get back sol.

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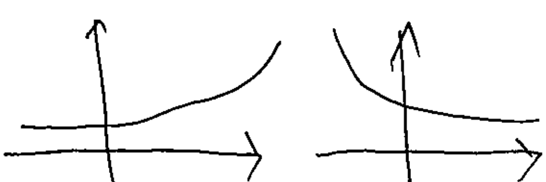
(1) $y(t) = e^{at}$, $Ly = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{(a-s)t}\,dt$

$= \frac{1}{a-s} e^{(a-s)t}\Big|_0^\infty$

$Re(s) > a$ $\quad = -\frac{1}{a-s} = \frac{1}{s-a}$



$\alpha \in \mathbb{R}$: $e^{\alpha t}$ $\alpha > 0$ $\quad e^{\alpha t}$ $\alpha < 0$

Properties: $Y = Ly$, $V = Lv$

check P145

Text book

(1) $L(ay(t) + bv(t)) = aY + bV$

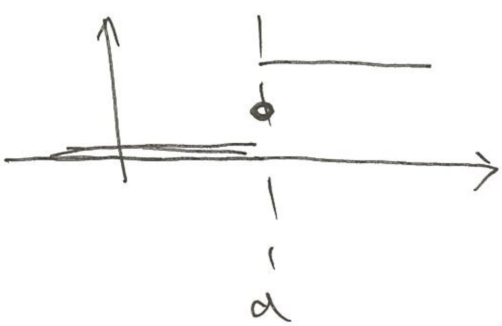
(2) $L^{-1}(Y(s)V(s)) = \int_0^t v(t-r)\,y(r)\,dr$

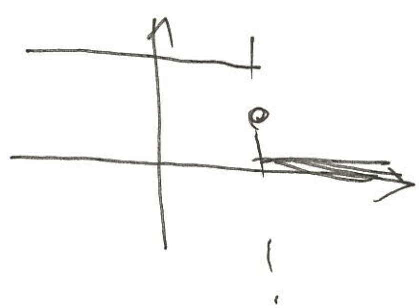
Special case

(3) $L^{-1}\left(\frac{1}{s}Y(s)\right) = \int_0^t y(r)\,dr$, $L(1) = \frac{1}{s}, v=1$

(4) $L^{-1}\left(\frac{1}{s}e^{-as}\right) = H(t-a)$

$$H(t-a) = \begin{cases} 0 & t<a \\ \frac{1}{2} & t=a \\ 1 & t>a \end{cases} \qquad H(a-t) = \begin{cases} 0 & t>a \\ \frac{1}{2} & t=a \\ 1 & t<a \end{cases}$$





$H(a-t) = 1 - H(t-a)$

(5) $L(\delta(t-a)) = e^{-as}$

(6) $L^{-1}(e^{-as}Y(s)) = H(t-a)\,y(t-a)$

(7) $L(e^{at}) = \frac{1}{s-a}$

(8) sin, cos

$y = H(t-1) + H(2-t)$

$Ly = \frac{1}{s}e^{-s} + L(1 - H(t-2)) = \frac{1}{s}e^{-s} + \frac{1}{s} - \frac{1}{s}e^{-2s}$

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① $L(y') = -y(0) + sL(y) = -y(0) + Y$

② $L(y'') = -y'(0) + sL(y') = -y'(0) - sy(0) + Y$

$3y' = -y + e^{-t}, \quad y(0) = 0$

$Ly' = sLy - y(0) = sY$

$L(e^{+at}) = \frac{1}{s-a}$

$3sY = -Y + L(e^{-t}) = -Y + \frac{1}{s+1}$

$Y = \frac{1}{(3s+1)(s+1)} = \frac{\frac{1}{3}}{(s+\frac{1}{3})(s+1)}$

Use Property 9 P1145!

$L^{-1}\left(\frac{cs+d}{(s+a)(s+b)}\right)$

$= \frac{1}{b-a}\left[(bc-d)e^{-bt} - (ac-d)e^{-at}\right]$

or $Y = \frac{1}{2}\left[\frac{1}{s+\frac{1}{3}} - \frac{1}{s+1}\right]$

$y = L^{-1}(Y) = \frac{1}{2}\left(L^{-1}\left(\frac{1}{s+\frac{1}{3}}\right) - L^{-1}\left(\frac{1}{s+1}\right)\right]$

$= \frac{1}{2}e^{-\frac{1}{3}t} - \frac{1}{2}e^{-t}$

$y'' + 3y' = 3t - 1, \quad y(0) = 1, \quad y'(0) = 0$

(1) Solve holomogenuous problem:

$y_h'' + 3y_h' = 0, \quad r^2 + 3r = 0, \quad y_h = c_1 + c_2e^{-3t}$

$y_h(0) = 1, \quad y_h'(0) = 0 \quad \Rightarrow y_h = 1$

yp(0) should be 0

(2) Find y_p s.t. $y_p'' + 3y_p' = 3t - 1$, $y_p(0) = 1$, $y_p'(0) = 0$

(i) $L(y_p') = sY_p$, $L(y_p'') = s^2 Y_p$

$L(t^n) = \frac{n!}{s^{n+1}}, n \geq 0$

$$(s^2+3s)Y_p = L(3t-1)$$

$$(s^2+3s)Y_p = \frac{3}{s^2} - \frac{1}{s}$$

$$\Rightarrow Y_p = \frac{3}{s^3(s+3)} - \frac{1}{s^2(s+3)}$$

$$L^{-1}\left(\frac{1}{s^2(s+3)}\right) = L^{-1}\left(\frac{1}{s} \cdot \frac{1}{3}\left(\frac{1}{s} - \frac{1}{s+3}\right)\right)$$

$$= L^{-1}\left(\frac{1}{3} \cdot \frac{1}{s^2}\right) - L^{-1}\left(\frac{1}{3} \cdot \frac{1}{s} \cdot \frac{1}{s+3}\right)$$

$$= \frac{1}{3} L^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{9} L^{-1}\left(\frac{1}{s} - \frac{1}{s+3}\right)$$

$$= \frac{1}{3}t - \frac{1}{9} + \frac{1}{9}e^{-3t}$$

$$L^{-1}\left(\frac{1}{s^3(s+3)}\right) = L^{-1}\left(\frac{1}{s} \cdot \frac{1}{s^2(s+3)}\right)$$

$$= \int_0^t \left(\frac{1}{3}r - \frac{1}{9} + \frac{1}{9}e^{-3r}\right) dr$$

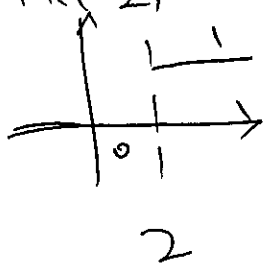
$$= \frac{1}{6}t^2 - \frac{1}{9}t - \frac{1}{27}e^{-3t} + \frac{1}{27}$$

$$y_p = 3L^{-1}\left(\frac{1}{s^3(s+3)}\right) - L^{-1}\left(\frac{1}{s^2(s+3)}\right) = \frac{1}{2}t^2 - \frac{2}{3}t - \frac{2}{9}e^{-3t} + \frac{2}{9}$$

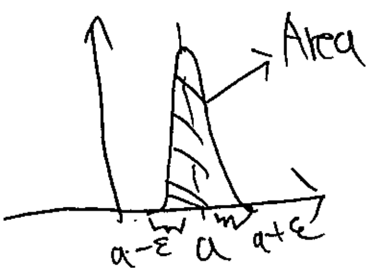
$$y = y_h + y_p$$

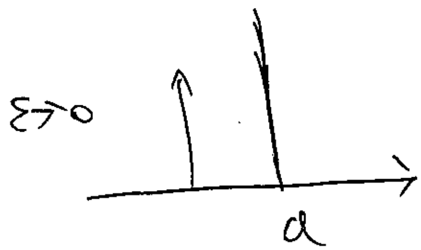
$$y' - 4y = 2H(t-2) - \delta(t-1), \quad y(0) = 0$$

H(t-2)



δ(t-a)





$$L^{-1}(e^{-as}Y(s)) = H(t-a)\, y(t-a)$$

$$L(\delta(t-a)) = e^{-as}$$

$$Ly' = sY - y(0) = sY$$

$$(s-4)Y = \frac{2}{s}e^{-2s} - e^{-s}$$

$$Y = \frac{2}{s(s-4)}e^{-2s} - \frac{1}{s-4}e^{-s}$$

$$L^{-1}\left(\frac{2}{s(s-4)}\right) = L^{-1}\left(\frac{1}{2}\cdot\left(\frac{1}{s-4} - \frac{1}{s}\right)\right)$$

$$= \frac{1}{2}(e^{4t} - 1)$$

$$L^{-1}\left(\frac{1}{s-4}\right) = e^{4t}$$

$$L^{-1}Y = y = \frac{1}{2}(e^{4(t-2)} - 1)H(t-2)$$

$$- e^{4(t-1)}H(t-1)$$

$$\cancel{y'' + y = \delta(t-3)}$$