Method of variation of parameters

$$y'' + a_1(x)y' + a_2(x)y = g(x)$$

(1) Suppose $y_1(x)$ and $y_2(x)$ are solutions to homogenuous problem.

$$y'' + a_1(x)y' + a_2(x)y = 0.$$

(2) Assume $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ is a particular solution such that

$$y_p'' + a_1(x)y_p' + a_2(x)y_p = g(x).$$

$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

$$y_p'' = u_1''y_1 + 2u_1'y_1' + u_1y_1'' + u_2''y_2 + 2u_2'y_2' + u_2y_2''$$

$$\Rightarrow y_p'' + a_1(x)y_p' + a_2(x)y_p$$

$$= u_1''y_1 + 2u_1'y_1' + u_1y_1'' + u_2''y_2 + 2u_2'y_2' + u_2y_2''$$
$$+ a_1(x)[u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2')$$
$$+ a_2(x)(u_1y_1 + u_2y_2)$$

$$= u_1\overbrace{(y_1'' + a_1(x)y_1' + a_2(x)y_1)}^{0} + u_2\overbrace{(y_2'' + a_1(x)y_2' + a_2(x)y_2)}^{0}$$
$$+ (u_1'y_1 + u_2'y_2)a_1(x) + [u_1''y_1 + u_1'y_1' + u_2''y_2 + u_2'y_2') \rightarrow (u_1'y_1 + u_2'y_2)'$$
$$+ u_1'y_1' + u_2'y_2' = g(x)$$

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 & (1) \\ u_1' y_1' + u_2' y_2' = g(x) & (2) \end{cases}$$

$$(1) \times y_2' - (2) \times y_2 \Rightarrow (y_1 y_2' - y_1' y_2) u_1' = -g(x) y_2$$

$$(1) \times y_1' - (2) \times y_1 \Rightarrow (y_1' y_2 - y_1 y_2') u_2' = -g(x) y_1$$

$$\Rightarrow (y_1 y_2' - y_1' y_2) u_2' = g(x) y_1$$

$\Rightarrow$ Define Wronski determinant as

$$W(x) = y_1 y_2' - y_1' y_2$$

If $W(x) \neq 0$, we say y_1, y_2 are linear independent, which means that above eqn is solvable.

$$\therefore u_1' = -\frac{g(x) y_2}{W(x)}, \quad u_2' = \frac{g(x) y_1}{W(x)}$$

$$\Rightarrow y_p(x) = u_1 y_1 + u_2 y_2 = -y_1 \int \frac{g(x) y_2}{W(x)} dx + y_2 \int \frac{g(x) y_1}{W(x)} dx$$

Step 1: Solve homogenious problem, $y_h = c_1 y_1 + c_2 y_2$

Get two linear independent solution y_1 and y_2

Step 2: Rewrite eqn as $\boxed{y''} + a_1(x) y' + a_2(x) y = g(x)$,

calculate $W(x) = y_1 y_2' - y_1' y_2$,

then obtain $u_1' = -\frac{y_2 g}{W}$, $u_2' = \frac{y_1 g}{W}$, do the integral

and obtain $y_p = u_1 y_1 + u_2 y_2$, final $y = u_1 y_1 + u_2 y_2$

2

(1) $y'' - y' - 6y = 6e^{x}$

Step 1 : $r^2 - r - 6 = 0$, $(r-3)(r+2) = 0$

$r_1 = 3,\ r_2 = -2$, $y_h = c_1 e^{3x} + c_2 e^{-2x}$

Let $y_1 = e^{3x}$, $y_2 = e^{-2x}$

Step 2 : $y_p = u_1 y_1 + u_2 y_2$

$W(x) = y_1 y_2' - y_1' y_2 = -5e^{x}$

$$\Rightarrow u_1' = -\frac{y_2 g}{W} = -\frac{e^{-2x} \cdot 6e^{x}}{-5e^{x}} = \frac{6}{5}e^{-2x}$$

$$u_2' = \frac{y_1 g}{W} = \frac{6e^{x} \cdot e^{3x}}{-5e^{x}} = -\frac{6}{5}e^{3x}$$

$\cancel{y_p = u_1 y_1 + u_2 y_2 = u_1 = \int \frac{6}{5} e^{-2x} dx}$

$\Rightarrow u_1 = -\frac{3}{5}e^{-2x}$, $u_2 = -\frac{2}{5}e^{3x}$

$\Rightarrow y_p = u_1 y_1 + u_2 y_2 = -e^{x}$

$\Rightarrow y = y_h + y_p = c_1 e^{3x} + c_2 e^{-2x} - e^{x}$

Euler - Type

$$x^2y'' + a_1 x y' + a_2 y = 0$$

Let $y = x^r = e^{r\ln x}$, $y' = r x^{r-1}$, $y'' = r(r-1)x^{r-2}$,

$= e^{z}, \; z = \ln x$

$$(r^2 + (a_1 - 1) r + a_2) x^r = (r^2 + (a_1-1) r + a_2) e^{rz} = 0$$

$$z = \ln x$$

$$r_{1,2} = \frac{-(a_1-1) \pm \sqrt{(a_1-1)^2 - 4a_2}}{2}$$

Case 1: r_1, r_2 two different real sol.

$$y = c_1 x^{r_1} + c_2 x^{r_2} = c_1 e^{r_1 z} + c_2 e^{r_2 z}$$

Case 2: $r_1 = r_2$, $\quad y = c_1 x^r + c_2 \ln x \cdot x^r = c_1 e^{rz} + c_2 z e^{rz}$

Case 3: r_1, r_2 two complex, $y = x^{\alpha}(c_1 \sin(\beta \ln x) + c_2 \cos(\beta \ln x))$

$r_{1,2} = \alpha \pm i\beta$

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(2) $xy'' + y' = x$, $y(1) = 1$, $y'(1) = -1$

Step 1: $x^2 y'' + x y' = x^2$,

$$r^2 + (1-1) r = 0 \Rightarrow r_1 = r_2 = 0$$

$$y_h = c_1 + c_2 \ln x, \quad y_1 = 1, \; y_2 = \ln x$$

Step 2: $W(x) = y_1 y_2' - y_1' y_2 = \frac{1}{x}$

Rewrite $\quad y'' + \frac{1}{x} y' = 1$, $g(x) = 1$

3

$$u_1' = -\frac{y_2 g}{W} = -\frac{\ln x}{\frac{1}{x}} = -x\ln x$$

$$u_2' = \frac{y_1 g}{W} = \frac{1}{\frac{1}{x}} = x$$

$$u_1 = \int_1^x -s\ln s\,ds = -\frac{1}{2}s^2 \ln s\Big|_1^x + \frac{1}{2}\int_1^x s^2 \cdot \frac{1}{s}ds$$

$$= -\frac{1}{2}x^2\ln x + \frac{1}{4}x^2 - \frac{1}{4}$$

$$u_2 = \int_1^x s\,ds = \frac{1}{2}x^2 - \frac{1}{2}$$

$$y_p = u_1 y_1 + u_2 y_2 = \frac{1}{4}x^2 - \frac{1}{2}\ln x - \frac{1}{4}$$

step3 : $y = y_h + y_p = C_1 + (C_2 - \frac{1}{2})\ln x + \frac{1}{4}x^2 - \frac{1}{4}$

$$y' = \frac{C_2 - \frac{1}{2}}{x} + \frac{1}{2}x$$

$$y(1) = C_1 = 1, \quad y'(1) = (C_2 - \frac{1}{2}) + \frac{1}{2} = C_2 = -1$$

$$y = 1 - \frac{3}{2}\ln x + \frac{1}{4}x^2 - \frac{1}{4}$$

(3) $x^2y'' + 4xy' + 2y = \frac{1}{x}$, $y(1) = y'(1) = 0$

step1: $r^2 + (4-1)r + 2 = r^2 + 3r + 2 = (r+2)(r+1) = 0$

$r_1 = -1$, $r_2 = -2$, $y_h = C_1x^{-1} + C_2x^{-2}$

$y_1 = x^{-1}$, $y_2 = x^{-2}$

step 2: $y'' + \frac{4}{x}y' + \frac{2}{x^2}y = \frac{1}{x^3}$ $\quad g = \frac{1}{x^3}$

W(x)

$W(x) = y_1 y_2' - y_1' y_2 = -2x^{-4} + x^{-4} = -x^{-4}$

$$u_1' = \frac{y_2 g}{W} = \frac{-x^{-2}\cdot x^{-3}}{-x^{-4}} = x^{-1}$$

$$u_2' = \frac{y_1 g}{W} = \frac{x^{-1}\cdot x^{-3}}{-x^{-4}} = -1$$

$$\Rightarrow \quad u_1 = \ln x, \quad u_2 = -x$$

$$y_p = u_1 y_1 + u_2 y_2 = x^{-1}\ln x - x^{-1}$$

Step3 :

$$y = y_h + y_p = (c_1 - 1)x^{-1} + c_2 x^{-2} + x^{-1}\ln x$$

$$y'(x) = -(c_1-1)x^{-2} - 2c_2 x^{-3} + x^{-2} - x^{-2}\ln x$$

$$y(1) = c_1 + c_2 - 1 = 0$$

$$y'(1) = -(c_1 - 1) - 2c_2 + 1 = 0$$

$$\Rightarrow \quad c_1 + c_2 = 1$$

$$c_1 + 2c_2 = 2$$

$$\Rightarrow \quad c_2 = 1, \quad c_1 = 0$$

$$y = -x^{-1} + x^{-2} + x^{-1}\ln x$$