09/06/2019

Integrating Factor

$$y' + p(t)\, y = R(t)$$

$(f(g(y)))' = f'(g(y))\, g'(y)$

$$\underline{\left(e^{\int p(t)\,dt}\right)'} = \left(\int p(t)\,dt\right)' e^{\int p(t)\,dt}$$

$$= \underline{p(t)\, e^{\int p(t)\,dt}}$$

$$e^{\int p(t)\,dt}\, y' + p(t) e^{\int p(t)\,dt}\, y = R(t)\, e^{\int p(t)\,dt}$$

$(fg)' = f'g + fg'$

$$e^{\int p(t)\,dt}\, y' + \left(e^{\int p(t)\,dt}\right)' y = R(t)\, e^{\int p(t)\,dt}$$

$$\left(e^{\int p(t)\,dt}\, y\right)' = R(t)\, e^{\int p(t)\,dt}$$

Step 1: figure out integrating factor

$$e^{\int p(t)\,dt}$$

Step 2: Integrate the R.H.S.

(1) Integration by parts,

$\int t^9 \sin t\,dt \quad \int \sin(t)\, e^t\,dt \quad \int t^9 e^t\,dt$

(2) Change of variable

If IVP, $y(0) = \pi$. Plug $t = 0$ into the general solution

$y' + p(x)y = R(x)$

$p(x) = \frac{1}{2}$

$e^{\int p(x)dx}$

$(e^{\int p(x)dx} y)'$

1、 $2y' = -y + e^{-x} - 2, \quad y(0) = 1$

$$y' = -\frac{1}{2}y + e^{-x} - 2$$

$$y' + \frac{1}{2}y = \frac{1}{2}e^{-x} - \cancel{2}\,1$$

$$e^{\frac{1}{2}x}y' + \frac{1}{2}e^{\frac{1}{2}x}y = e^{\frac{1}{2}x}\left(\frac{1}{2}e^{-x} - 1\right)$$

$$(e^{\frac{1}{2}x}y)' = e^{\frac{1}{2}x}\left(\frac{1}{2}e^{-x} - 1\right)$$

$$e^{\frac{1}{2}x}y = \int \underline{\frac{1}{2}e^{-\frac{1}{2}x}}\,dx - \int e^{\frac{1}{2}x}dx$$

$$e^{\frac{1}{2}x}y = -e^{-\frac{1}{2}x} - 2e^{\frac{1}{2}x} + c$$

$$y = -e^{-x} - 2 + ce^{-\frac{1}{2}x}$$

$$y(0) = -1 - 2 + c = 1 \Rightarrow c = 4$$

$$y(t) = -e^{-t} - 2 + 4e^{-\frac{1}{2}t}$$

Seperation of variables

$$y' = F(x)\,G(y)$$

$$y' = \frac{dy}{dx} = F(x)\,G(y)$$

$$\frac{dy}{G(y)} = F(x)\,dx$$

$$\int \frac{dy}{G(y)} = \int F(x)\,dx$$

(1) $y' = \frac{x^2}{y}$ $\quad \frac{dy}{dx} = \frac{x^2}{y}$

$$y\,dy = x^2\,dx \Rightarrow \frac{1}{2}y^2 = \frac{1}{3}x^3 + C$$

$$y = \pm\sqrt{\frac{2}{3}x^3 + C}$$

(2) $y' + y^2 \sin x = 0$

$$\frac{dy}{dx} + y^2 \sin x = 0 \quad \frac{dy}{y^2} = -\sin x\,dx$$

$$\int \frac{dy}{y^2} = \int -\sin x\,dx \Rightarrow -\frac{1}{y} = \cos x + C$$

$$y = -\frac{1}{\cos x + C}$$

(3) $y' = (1-2x)y^2$; $y(0) = -\frac{1}{6}$

$$\frac{dy}{y^2} = (1-2x)\,dx$$

Integration: $-\frac{1}{y} = x - x^2 + C$

$$y = -\frac{1}{x - x^2 + C}$$

$$y(0) = -\frac{1}{C} = -\frac{1}{6} \Rightarrow C = 6$$

$$y = -\frac{1}{-x^2+x+6} = \frac{1}{x^2-x-6}$$

(4) $$\frac{dy}{dx} = \frac{x^2}{1+y^2}$$

$$(1+y^2)\,dy = x^2dx \Rightarrow y + \frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$

Implicit expression

(5)

$y'+p(x)y=R(x),$ $\qquad xy'+y=1+y+y^2$

$P(x)=\frac{1}{x}$ $\qquad y'+\frac{1}{x}y=(1+y+y^2)\frac{1}{x}$

Can not use integrating factor method

$$xy'+y=1+y+y^2$$

$$xy'=1+y^2$$

$$\frac{dy}{dx}=\frac{1+y^2}{x}$$

$(\arctan(y))'$ $\qquad \dfrac{dy}{1+y^2}=\dfrac{dx}{x}$

$=\frac{1}{1+y^2}$ $\qquad \arctan y=\ln x+C$

$$y=\tan(\ln x+c)$$

(6) $(4+e^{x})\,y' + e^{x}y^{2} = 0, \quad y(0) = 1$

Sol: $(4+e^{x})\dfrac{dy}{dx} + e^{x}y^{2} = 0$

$$\frac{dy}{y^2} = \frac{-e^{x}dx}{4+e^{x}}$$

$$-\frac{1}{y} = -\int \frac{e^{x}}{4+e^{x}}\,dx$$

$(e^{x})' = e^{x}$ $\qquad \dfrac{u = e^{x}}{du = e^{x}dx}$ $\qquad -\displaystyle\int \frac{e^{x}dx}{4+e^{x}} = -\int \frac{du}{4+u}$

$$= -\ln(4+u) + C$$

$$= +\ln(4+e^{x}) + C$$

$$y = +\frac{1}{\ln(4+e^{x}) + C}$$

$$y(0) = \frac{1}{\ln 5 + C} = 1 \qquad C = 1 - \ln 5$$

$$y(x) = \frac{1}{\ln(4+e^{x}) + 1 - \ln 5}$$

(7) $y' = \sqrt{1-y^2}$, $y(0)=0$

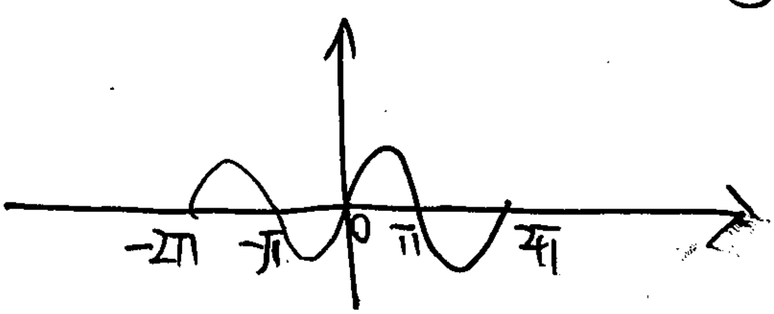
sol: $\frac{dy}{dx} = \sqrt{1-y^2}$ $\quad \frac{dy}{\sqrt{1-y^2}} = dx$

$$\int \frac{dy}{\sqrt{1-y^2}} \quad \underset{dy = \cos\theta d\theta}{\overset{y=\sin\theta}{=\!=}} \quad \int \frac{\cos\theta \, d\theta}{\cos\theta} = \theta$$

$1 = \sin^2\theta + \cos^2\theta$ $\qquad = \arcsin y$

$$\arcsin y = x + C$$

$$y = \sin(x + c)$$



$y(0) = \sin C = 0$

$C = k\pi, k \in \mathbb{N}$

$$y = \sin(x + k\pi)$$

(8) $y' = 1+\cos y$, $y(0) = \frac{\pi}{2}$

$$\frac{dy}{dx} = 1+\cos y$$

$$\frac{dy}{1+\cos y} = \int dx \cancel{0} = C+x$$

$$\int \frac{dy}{1+\cos y} = \int \frac{dy}{\sin^2\frac{y}{2}+\cos^2\frac{y}{2}+\cos^2\frac{y}{2}-\sin^2\frac{y}{2}}$$

$$= \int \frac{dy}{2\cos^2\frac{y}{2}}$$

$$= \int \frac{1}{2}\frac{dy}{\cos^2\frac{y}{2}}$$

$$= \tan\left(\frac{y}{2}\right)$$

$\tan\left(\frac{y}{2}\right) = C+x \Rightarrow y = 2\arctan(x+c)$

$y(0) = \frac{\pi}{2} = 2\arctan C$ $\quad C = \tan\frac{\pi}{4}$

$\sin^2 y + \cos^2 y = 1$

$\sin(2y) = 2\sin y\cos y$

$\cos(2y) = \cos^2 y - \sin^2 y$

$\sin^2 y = 1-\cos^2 y$ $\quad 2\cos^2 y - 1$

$\cos^2 y = 1-\sin^2 y$ $\quad 1-2\sin^2 y$

$(\sin y)' = \cos y$

$(\cos y)' = -\sin y$

$(\tan y)' = \frac{1}{\cos^2 y}$