

I'll post a sample Maple code online, and the due date for Maple will be Oct 16th.

The method of underdetermined coefficient

$$y'' + a_1 y' + a_2 y = g(x)$$

Step 1: solve $y_h'' + a_1 y_h' + a_2 y_h = g(x)$, using the characteristic equation

Step 2: Based on RHS "g" and characteristic roots, we make assumptions for y_p . Then, we substitute y_p into the original equation, compare the coefficients and get y_p .

Final general sol.: $y = y_h + y_p$

1. $g(x) = (b_0 + b_1 x + \dots + b_n x^n) e^{\lambda x} = p_n(x) e^{\lambda x}$

characteristic eqn: $r^2 + a_1 r + a_2 = 0$, roots r_1, r_2

$\frac{d}{dx}(q_n e^{\lambda x}) = p_n \frac{d}{dx}(e^{\lambda x})$

if q_n is a n -th order polynomial,

$$(q_n e^{\lambda x})' = q_n' e^{\lambda x} + \lambda q_n e^{\lambda x} = (\lambda q_n + q_n') e^{\lambda x}$$

(1) $\lambda \neq r_1, \lambda \neq r_2$, $y_p = (A_0 + \dots + A_n x^n) e^{\lambda x}$

(2) $\lambda \neq r_1, \lambda = r_2$ or $\lambda = r_1, \lambda \neq r_2$

$$y_p = (A_0 + \dots + A_n x^n) x e^{\lambda x}$$

(3) $\lambda = r_1 = r_2$, $y_p = (A_0 + \dots + A_n x^n) x^2 e^{\lambda x}$

$$g(x) = e^{\alpha x} (c_1 \sin \beta x + c_2 \cos \beta x) \quad (\text{both } \alpha x + \dots + \beta n x)$$

$$e^{(\alpha + i\beta)x}$$

$$\lambda = \alpha + i\beta \quad \leftarrow \text{"}e^{\lambda x} \cdot \text{polynomial"}\text{"}$$

$$= e^{\alpha x} (\cos \beta x + i \sin \beta x)$$

$$(1) \lambda \neq r_1, r_2 \rightarrow \text{"characteristic roots"}$$

$$y_p = e^{\alpha x} [(A_0 + A_1 x + \dots + A_n x^n) \sin \beta x + (B_0 + B_1 x + \dots + B_n x^n) \cos \beta x]$$

$$(2) \lambda = \alpha + i\beta, \quad \lambda = r_1, \text{ or } \lambda = r_2$$

$\lambda \neq r_2 \qquad \qquad \lambda \neq r_1$

$$y_p = [(A_0 + A_1 x + \dots + A_n x^n) \sin \beta x + (B_0 + B_1 x + \dots + B_n x^n) \cos \beta x] \bar{x} e^{\alpha x}$$

$$1. y'' - y' - 6y = 6x e^x$$

$$\text{step 1: } r^2 - r - 6 = 0, (r-3)(r+2) = 0$$

$$\Rightarrow r_1 = 3, r_2 = -2, y_h = c_1 e^{3x} + c_2 e^{-2x}$$

$$\text{step 2: Assume } y_p = (A_0 + A_1 x) e^{\lambda x}, \quad (A_0 + A_1 x) e^x$$

$$\lambda = 1: y_p' = A_1 e^x + \cancel{(A_0 + A_1 x) e^x} + \cancel{A_1 x e^x}$$

$$r_1 = 3: = (A_0 + A_1 x) e^x + A_1 x e^x$$

$$r_2 = -2:$$

$$y_p'' = A_1 e^x + (A_0 + A_1 x) e^x + A_1 x e^x$$

$$= (A_0 + 2A_1) e^x + A_1 x e^x$$

$$y_p'' - y_p' - 6y_p = 6x e^x$$

$$(A_0 + 2A_1 x) e^x + A_1 x e^x$$

$$- (A_0 + A_1) e^x - A_1 x e^x$$

$$- 6A_0 e^x - 6A_1 x e^x$$

$$= (A_1 - 6A_0) e^x - 6A_1 x e^x = 6x e^x + 0 \cdot e^x$$

Compare coefficients : $\begin{cases} A_1 - 6A_0 = 0 \\ -6A_1 = 6 \end{cases} \Rightarrow \begin{cases} A_0 = -\frac{1}{6} \\ A_1 = -1 \end{cases}$

$$y_p = (-\frac{1}{6} - x) e^x$$

step 3: $y = -(\frac{1}{6} + x) e^x + c_1 e^{3x} + c_2 e^{-2x}$

2. $y'' - 5y' - 6y = 10x \sin(3x)$

step 1: $r^2 - 5r - 6 = 0 \Rightarrow (r-6)(r+1) = 0$

$$r_1 = 6, r_2 = -1, y_h = c_1 e^{6x} + c_2 e^{-x}$$

sin(3x) step 2: $y_p = (A_0 + A_1 x) \sin 3x + (B_0 + B_1 x) \cos 3x$

$$y_p' = (A_1 - B_0 - 3B_1 x) \sin 3x + (B_1 + 3A_0 + 3A_1 x) \cos 3x$$

$$y_p'' = (-6B_1 - 9A_0 - 9A_1 x) \sin 3x + (6A_1 - 9B_0 - 9B_1 x) \cos 3x$$

$e^{0 \cdot x} \sin(3x)$
 \downarrow
 $\lambda = 0 + 3i$

$$y_p'' - 5y_p' - 6y_p = 10x \sin(3x)$$

$$\begin{aligned} & \left(\underbrace{-6B_1 - 9A_0 - 9A_1 x}_{y_p''} \underbrace{- 5A_1 + 15B_0 + 15B_1 x}_{-5y_p'} \right) \sin 3x \\ & \quad \underbrace{- 6A_0 - 6A_1 x}_{-6y_p} \end{aligned}$$

$$\begin{aligned} & + \left(\underbrace{6A_1 - 9B_0 - 9B_1 x}_{y_p''} \underbrace{- 5B_1 - 15A_0 - 15A_1 x}_{-5y_p'} \right) \cos 3x \\ & \quad + 6B_0 + 6B_1 x \end{aligned} = 10x \sin(3x)$$

$$(-6B_1 + 15B_0 - 15A_0 - 5A_1) \sin 3x$$

$$+ (-15A_1 + 15B_1) x \sin 3x$$

$$+ (6A_1 - 15B_0 - 5B_1 - 15A_0) \cos 3x$$

$$+ (-15B_1 - 15A_1) x \cos 3x = 10x \sin(3x)$$

$$\left\{ \begin{aligned} -6B_1 + 15B_0 - 15A_0 - 5A_1 &= 0 \quad \dots \sin 3x \\ -15A_1 + 15B_1 &= 10 \quad \dots x \sin 3x \\ 6A_1 - 15B_0 - 5B_1 - 15A_0 &= 0 \quad \dots \cos 3x \\ -15B_1 - 15A_1 &= 0 \quad \dots x \cos 3x \end{aligned} \right.$$

$$\Rightarrow A_1 = -\frac{1}{3}, B_1 = \frac{1}{3}, A_0 = -\frac{2}{15}, B_0 = -\frac{1}{9}$$

$$y_p = \left(-\frac{2}{15} - \frac{1}{3}x\right) \sin(3x) + \left(-\frac{1}{9} + \frac{1}{3}x\right) \cos(3x)$$

$$\text{step 3: } y = y_h + y_p = \underline{\hspace{10em}}$$

$$y'' + a_1 y' + a_2 y = g(x) = g_1(x) + g_2(x)$$

$$g_1 = e^x \quad (1) \text{ Solve Homogeneous problem,}$$

$$g_2 = x^3 e^{2x} \quad y_h'' + a_1 y_h' + a_2 y_h = 0$$

$$g = e^x + x^3 e^{2x}$$

$$(2) \text{ Find } y_{p1}, \text{ s.t. } y_{p1}'' + a_1 y_{p1}' + a_2 y_{p1} = g_1(x)$$

$$(3) \text{ (Based on } r_1, r_2, g_1(x)).$$

$$(3) \text{ Find } y_{p2}, \text{ s.t. } y_{p2}'' + a_1 y_{p2}' + a_2 y_{p2} = g_2(x)$$

$$\text{Final Solution } y = y_h + y_{p1} + y_{p2}$$

$$(1) y'' + \frac{1}{3} y' - \frac{2}{3} y = e^{-2x} - \frac{1}{3} e^{-x},$$

$$y(0) = \frac{3}{8}, \quad y'(0) = -\frac{5}{8}$$

$$\text{Step 1: } r^2 + \frac{1}{3} r - \frac{2}{3} = 0, \quad (r+1)(r-\frac{2}{3}) = 0$$

$$r_1 = -1, r_2 = \frac{2}{3}, \quad y_h = C_1 e^{-x} + C_2 e^{\frac{2}{3}x}$$

$$\text{Step 2: Assume } y_{p1} = A_0 e^{-2x}$$

$$\lambda = -2: \quad y_{p1}'' + \frac{1}{3} y_{p1}' - \frac{2}{3} y_{p1} = (4A_0 - \frac{2}{3}A_0 - \frac{2}{3}A_0) e^{-2x} = e^{-2x}$$

$$\Rightarrow A_0 = \frac{3}{8}$$

$$\text{Step 2:}$$

$$\cancel{y_{p2}'' + \frac{1}{3} y_{p2}'} = B_0 \cdot x e^{-x}$$

$$y_{p2}'' + \frac{1}{3} y_{p2}' - \frac{2}{3} y_{p2} = (0 - B_0 + \frac{1}{3} B_0) e^{-x} = -\frac{1}{3} e^{-x}$$

$$B_0 = -\frac{1}{1}$$

$$y = C_1 e^{-x} + C_2 e^{\frac{2}{3}x} + \frac{3}{8} e^{-2x} - \frac{1}{2} x e^{-2x}$$

$$y(0) = C_1 + C_2 + \frac{3}{8} = \frac{3}{8}$$

$$y'(0) = -C_1 + \frac{2}{3} C_2 - \frac{3}{4} - \frac{1}{2} = -\frac{5}{8} \Rightarrow C_1 = C_2 = 0$$

$$y = \frac{3}{8} e^{-2x} - \frac{1}{2} x e^{-2x}$$