1.

Heat equation: $\alpha^2 u_{xx} = u_t \quad 0 < x < L,\ t > 0,\ \alpha > 0$

Initial condition: $u(x,0) = f(x)$

$u(x,t) = X(x)T(t)$ $\qquad \dfrac{X''(x)}{X(x)} = \dfrac{T'(t)}{\alpha^2 T(t)} = -\lambda = k$

$\alpha^2 X''(x)T(t) = X(x)T'(t)$

$X(x) = c_1 \sin kx + c_2 \cos kx, \qquad T(t) = Ce^{-\alpha^2 k^2 t}$

Boundary Conditions: Case 1: $u(0,t) = u(L,t) = 0$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

$$u(x,0) = f(x) \Rightarrow b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} dx$$

Case 2: $u_x(0,t) = u_x(L,t) = 0$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L} dx, \quad a_0 = \frac{2}{L}\int_0^L f(x)dx$$

Case 3: $u(0,t) = T_1, \quad u(L,t) = T_2$, $\cancel{T_1 \neq T_2}$

$$\phi(x) = T_1 + \frac{x}{L}(T_2 - T_1)$$

$v(x,t) = u(x,t) - \phi(x), \quad v(x,0) = u(x,0) - \phi(x) = f(x) - \phi(x)$

$v(0,t) = v(L,t) = 0$

$v(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}, \quad b_n = \frac{2}{L}\int_0^L [f(x) - \phi(x)] \sin\frac{n\pi x}{L} dx$

$u(x,t) = \phi(x) + v(x,t)$

Wave Eqn:

$$\alpha^2 u_{xx} = u_{tt}, \quad 0<x<L, \ t>0$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

$$u(x,t) = X(x)T(t) \qquad \frac{X''(x)}{X(x)} = \frac{T''(t)}{\alpha^2 T(t)} \Longrightarrow$$

(i) $X_n = \sin\frac{n\pi x}{L}$ (ii) $T_n = \left(A_n \cos\frac{n\pi\alpha t}{L} + B_n \sin\frac{n\alpha\pi t}{L}\right)$

$$u(x,t) = \sum_{n=1}^{\infty} X_n T_n = \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi\alpha}{L}t + B_n \sin\frac{n\pi\alpha t}{L}\right)\sin\frac{n\pi x}{L}$$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \Rightarrow A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty}\frac{n\pi\alpha}{L}B_n \sin\frac{n\pi x}{L} = g(x)$$

$$\Longrightarrow \frac{n\pi\alpha}{L}B_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

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(1) $u_t = u_{xx}, \quad 0<x<\pi, \ u(0,t) = u(\pi,t) = 0$

$$u(x,0) = \sin x + \frac{5}{6}\sin 3x$$

Solution: $u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\, e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}$

$$= \sum_{n=1}^{\infty} b_n \sin(nx)\, e^{-n^2 t}$$

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx) = b_1\sin x + b_2\sin(2x) + b_3\sin(3x) + \cdots$$

$$= \sin x + \frac{5}{6}\sin 3x \Longrightarrow b_1 = 1, \ b_3 = \frac{5}{6}, \ b_2 = b_4 = b_5 = \cdots = 0$$

$$u(x,t) = e^{-t}\sin x + \frac{5}{6}\sin 3x\, e^{-9t}$$

(2) $u_{xx} = u_t$, $0 < x < \pi$, $u_x(0,t) = 0 = u_x(1,t)$,

$u(x,0) = x$

Sol: $u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\pi} e^{-(\frac{n\pi}{\pi})^2 t}$

$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx\, e^{-n^2 t}$

$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{2}{\pi}\left(\frac{x\sin(nx)}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{\sin(nx)}{n}dx\right)$$

$$= \frac{2}{\pi}\left(\frac{\cos(nx)}{n^2}\Big|_0^{\pi}\right) = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$u(x,t) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left[(-1)^n - 1\right]\cos(nx)e^{-n^2 t}$$

(3) $u_{xx} = u_t$, $0 < x < \pi$, $t > 0$, $u(0,t) = 0$, $u(\pi,t) = \pi$

$u(x,0) = \cancel{(\sin x \cos x)}^2 + x$.

sinx cosx

Sol: ~~$\phi(x) = u(0,t) + u(\pi,t) - u(0,$~~

$$\phi(x) = x, \quad \phi(0) = 0, \quad \phi(\pi) = \pi$$

$V(x,t) = u(x,t) - \phi(x) = u(x,t) - x$

$V(\pi,t) = 0, \quad V(0,t) = 0, \quad V(x,0) = \sin^2 x \cos^2 x + x - x$

$V(x,t) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{\pi}x) e^{-(\frac{n\pi}{\pi})^2 t}$

$V(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 t}$

$\cancel{V(x,0) = (\sin x \cos x)^2 = \frac{1}{4}(\sin 2x)^2 = \frac{1}{4} \cdot \frac{1-\cos 4x}{2}}$

$\cancel{\sin^2 x \cos x =}$ $\sin x \cos^2 x = \sin x \cdot \frac{1+\cos 2x}{2}$

$= \frac{1}{2}\sin x + \frac{1}{2}\sin x \cos 2x$

$= \frac{1}{2}\sin x + \frac{1}{2} \cdot \frac{1}{2}[\sin(x-2x) + \sin(x+2x)]$

$= \frac{1}{2}\sin x + \frac{1}{4}[\sin(-x) + \sin 3x]$

$= \frac{1}{4}\sin x + \frac{1}{4}\sin 3x$

$V(x,0) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$

$\Rightarrow b_1 = \frac{1}{4}, \quad b_3 = \frac{1}{4}$

$V(x,t) = \frac{1}{4}\sin x \, e^{-t} + \frac{1}{4}\sin 3x \, e^{-9t}$

$u(x,t) = \frac{1}{4}\sin x \, e^{-t} + \frac{1}{4}\sin 3x \, e^{-9t} + x$

4. $u_{xx} = u_{tt}$, $0 < x < \pi$, $u(0,t) = 0 = u(\pi,t)$

$u(x,0) = \sin 2x - \sin 3x$, $u_t(x,0) = 0$.

Sol: $u(x,t) = \sum_{n=1}^{\infty} (A_n \cos(nt) + B_n \sin(nt)) \sin(nx)$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(nx) = \sin 2x - \sin 3x$$

$$A_2 = 1, \; A_3 = -1, \quad A_1 = A_4 = \cdots = 0$$

$$u_t(x,0) = \sum_{n=1}^{\infty} nB_n \cos(n\cdot 0) \sin(nx)$$

$$= 0 \quad \Rightarrow \quad B_n = 0$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(nt) \sin(nx)$$

$$= \cos(2t)\sin(2x) - \cos(3t)\sin(3x)$$

Comments: Step 1 Determine which case we have

Step 2 Use initial conditions to determine coefficients

Tips: (1) Integration for Fourier Series:
- Integration by parts
- $\sin x, \cos x$ equalities

(2) ~~$\sin x \cos x$~~ matching coefficients

for example: $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$

$f = \sin x + \cos 4x$ ✓ cool $\quad f = \sin \pi x + \sin x$ ✗

$$\sin\alpha \sin\beta = \frac{1}{2}\left(\sin(\alpha - \beta) - \sin(\alpha+\beta)\right)$$

$$\underbrace{\sin 3x} + \underbrace{\sin x}$$

$$\sum_{n=1}^{18} b_n \sin(nx)$$

b_1

$$= b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots$$

$$\cancel{\sin 3x + \sin x}$$