1、 f integrable $[-L, L]$, period of $2L$

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$, $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$,

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$

(1) $f(x) = f(-x)$, even

$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$, $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$

(2) $f(x) = -f(-x)$, odd

$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$, $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$

2、(1) $f(x)$ $0 < x < L$, cosine series

$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$, $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$

(2) sin series

$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$, $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$

3、Period $x \in [0, T]$, T period

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi x}{T} + b_n \sin \frac{2n\pi x}{T}\right)$$

$$a_n = \frac{2}{T} \int_0^T f(x) \cos \frac{2n\pi x}{T} dx, \quad b_n = \frac{2}{T} \int_0^T f(x) \sin \frac{2n\pi x}{L} dx$$

4、Complex: $[-L, L]$, $2L$ period

$$f(x) \sim \sum_{n=-\infty}^{+\infty} c_n e^{i\frac{n\pi x}{L}}, \quad c_n = \frac{1}{2\pi} \int_{-L}^{L} f(x) e^{-i\frac{n\pi x}{L}} d$$

(1) $f(x) = 1 - x^2$, $-1 < x < 1$

$f(x) = f(-x)$, even, $a_0 = 2\int_0^1 (1-x^2) dx = \frac{4}{3}$

$$a_n = 2 \int_0^1 (1-x^2) \cos(n\pi x) dx$$

$$= \frac{2\sin(n\pi x)}{n\pi}\Big|_0^1 - 2\int_0^1 x^2 \cos(n\pi x) dx$$

$$= -2x^2 \frac{\sin(n\pi x)}{n\pi}\Big|_0^1 + \frac{4}{n\pi} \int_0^1 x \sin(n\pi x) dx$$

$$= -\frac{4}{n^2\pi^2} x \cos(n\pi x)\Big|_0^1 + \frac{4}{n^2\pi^2} \int_0^1 \cos(n\pi x) dx$$

$$= -\frac{4}{(n\pi)^2} \cos n\pi + \frac{4}{n^3\pi^3} \sin(n\pi x)\Big|_0^1 = \frac{4}{n^2\pi^2}(-1)^{n+1}$$

$$f(x) \sim \frac{2}{3} + \sum \frac{4}{n^2\pi^2} (-1)^{n+1} \cos(n\pi x)$$

$\cos^2 x - \sin^2 x = \cos 2x$

$\sin^2 x + \cos^2 x = 1$

$\sin x \cos x = \frac{1}{2}\sin 2x$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-$

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+$

$f(x) = (\sin x)^3 \cos x \quad -\pi < x < \pi$

$f(-x) = [\sin(-x)]^3 \cos(-x)$

$= -\cos x \, (\sin x)^3 = -f(x)$,

odd

$b_n = \frac{2}{\pi}\int_0^{\pi} (\sin x)^3 \cos x \cdot \sin(\frac{n\pi x}{\pi})\,dx$

$= \frac{2}{\pi} \cdot \int_0^{\pi} \sin^2 x \quad \sin x \cos x \quad \sin(nx)\,dx$

$= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2}(1-\cos 2x)\,\frac{1}{2}\sin 2x \, \sin(nx)\,dx$

$= \frac{1}{2\pi}\left[\int_0^{\pi} \sin 2x \cdot \sin nx\,dx - \frac{1}{2}\int_0^{\pi} \sin 4x \cdot \sin(nx)\,dx\right]$

$= \frac{1}{2\pi}\left[\frac{1}{2}\int_0^{\pi}[\cos(n-2)x - \cos(n+2)x]\,dx\right.$

$\left. - \frac{1}{4}\int_0^{\pi}\frac{1}{2}[\cos(n-4)x - \cos(n+4)x]\,dx\right]$

$= \begin{cases} \frac{1}{4} & n=2 \\ -\frac{1}{8} & n=4 \\ 0 & \text{otherwise} \end{cases}$

$\int_0^{\pi} \cos kx\,dx = \frac{1}{k}\sin kx \Big|_0^{\pi} = 0, \quad k \neq 0$

$\int_0^{\pi} 1\,dx = \pi$

Method 2: $f(x) = \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x$

$\downarrow$

Directly using equalities for sin, cos.

$f(x) = x(L-x)$, $0 < x < L$,

Get cosine series:

$$a_0 = \frac{2}{L}\int_0^L (xL - x^2)\,dx = \frac{L^2}{3}$$

$$a_n = \frac{2}{L}\int_0^L (xL - x^2)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\left.\frac{(xL-x^2)\sin(\frac{n\pi}{L}x)}{\frac{n\pi}{L}}\right|_0^L$$

$$- \frac{2}{L}\cdot\frac{1}{\frac{n\pi}{L}}\int_0^L (L-2x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{-2}{n\pi}(L-2x)\left.\frac{-\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right|_0^L - \frac{2}{\frac{n\pi}{L}}\int_0^L \cos\frac{n\pi x}{L}\,dx$$

$$= -\frac{2L^2}{n^2\pi^2}\left[\cos(n\pi) + 1\right] = -\frac{2L^2}{n^2\pi^2}\left[(-1)^n + 1\right]$$