~~[illegible]~~

I'll post a sample Maple code online, and the due date for Maple will be Oct 16th.

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The method of underdetermined coefficient

$$y'' + a_1 y' + a_2 y = g(x)$$

step 1: Solve $y_h'' + a_1 y_h' + a_2 y_h = g(x)$, using the characteristic equation

Step 2: Based on RHS "g" and characteristic roots, we make assumptions for $y_p$. Then, we substitute $y_p$ into the original equation, compare the coefficients and get $y_p$.

Final general sol.: $y = y_h + y_p$

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1、 $g(x) = (b_0 + b_1 x + \cdots + b_n x^n) e^{\lambda x} = p_n(x) e^{\lambda x}$

charcteristic eqn: $r^2 + a_1 r + a_2 = 0$, roots $r_1, r_2$

(1) $\lambda \neq r_1, \lambda \neq r_2$, $y_p = (A_0 + \cdots + A_n x^n) e^{\lambda x}$

(2) $\lambda \neq r_1, \lambda = r_2$ or $\lambda = r_1, \lambda \neq r_2$

$$y_p = (A_0 + \cdots + A_n x^n)\, x\, e^{\lambda x}$$

(3) $\lambda = r_1 = r_2$, $y_p = (X_0 + \cdots + A_n X^n) x^2 e^{\lambda x}$

$\boxed{q_n e^{\lambda x} = \text{poly} \cdot e^{\lambda x}}$

if $q_n$ is a n-th order polynomial,

$(q_n e^{\lambda x})'$

$= q_n' e^{\lambda x} + \lambda q_n e^{\lambda x}$

$= (\lambda q_n + q_n') e^{\lambda x}$

$$y_p'' - y_p' - 6y_p \qquad = \qquad 6x e^x$$

$$(A_0+2A_1)e^x + \quad A_1 x e^x$$

$$-(A_0+A_1)e^x - \quad A_1 x e^x$$

$$-\quad 6A_0 e^x \quad - 6A_1 x e^x$$

$$= (A_1 - 6A_0)e^x - 6A_1 x e^x = 6xe^x + 0\cdot e^x$$

compare coefficients : $\begin{cases} A_1 - 6A_0 = 0 \\ -6A_1 = 6 \end{cases} \Rightarrow \begin{cases} A_0 = -\frac{1}{6} \\ A_1 = -1 \end{cases}$

$$y_p = \left(-\tfrac{1}{6} - x\right)e^x$$

Step 3: $y = -\left(\frac{1}{6}+x\right)e^x + c_1 e^{3x} + c_2 e^{-2x}$

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2、 $y'' - 5y' - 6y = 10x \sin(3x)$

step 1: $r^2 - 5r - 6 = 0 \Rightarrow (r-6)(r+1) = 0$

$r_1 = 6, r_2 = -1$ , $y_h = c_1 e^{6x} + c_2 e^{-x}$

$\sin(3x)$

$= e^{0\cdot x}\sin(3x)$

$\lambda = 0 + 3i$

step 2: $y_p = (A_0 + A_1 x)\sin 3x + (B_0 + B_1 x)\cos 3x$

$$y_p' = (A_1 - 3B_0 - 3B_1 x)\sin 3x + (B_1 + 3A_0 + 3A_1 x)\cos 3x$$

$$y_p'' = (-6B_1 - 9A_0 - 9A_1 x)\sin 3x + (6A_1 - 9B_0 - 9B_1 x)\cos 3x$$

$$g(x) = e^{\alpha x}(c_1 \sin\beta x + c_2 \cos\beta x)(b_0 + b_1 x + \cdots + b_n x^n)$$

$e^{(\alpha + i\beta)x}$

$= e^{\alpha x}(\cos\beta x + i\sin\beta x)$

$$\lambda = \alpha + i\beta \longleftarrow \text{"}e^{\lambda x} \cdot \text{polynomial"}$$

(1) $\lambda \neq r_1, r_2$ → "charateristic roots"

$$y_p = e^{\alpha x}\left[(A_0 + A_1 x + \cdots + A_n x^n)\sin\beta x + (B_0 + B_1 x + \cdots + B_n x^n)\cos\beta x\right]$$

(2) $\lambda = \alpha + i\beta$, $\underset{\lambda \neq r_2}{\lambda = r_1}$, or $\underset{\lambda \neq r_1}{\lambda = r_2}$

$$y_p = \left[(A_0 + A_1 x + \cdots + A_n x^n)\sin\beta x + (B_0 + B_1 x + \cdots + B_n x^n)\right] \boxed{x}\, e^{\alpha x}$$

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1、 $y'' - y' - 6y = 6x e^x$

step 1: $r^2 - r - 6 = 0$, $(r-3)(r+2) = 0$

$\Rightarrow r_1 = 3, r_2 = -2$, $y_h = c_1 e^{3x} + c_2 e^{-2x}$

step 2: Assume $y_p = (A_0 + A_1 x)e^{\lambda x}$, $(A_0 + A_1 x)e^x$

$\lambda = 1$

$r_1 = 3$

$r_2 = -2$

$$y_p' = A_1 e^x + \cancel{(A_0 + A_1 e^x) + A_1 e^x}$$

$$= (A_0 + A_1)e^x + A_1 x e^x$$

$$y_p'' = A_1 e^x + (A_0 + A_1 x)e^x + A_1 e^x$$

$$= (A_0 + 2A_1)e^x + A_1 x e^x$$

$$y_p'' - 5y_p' - 6y_p = 10x\sin(3x)$$

$$\big(\overbrace{-6B_1 - 9A_0 - 9A_1x}^{y_p''} \; \overbrace{-5A_1 + 15B_0 + 15B_1x}^{-5y_p'} \; \overbrace{-6A_0 - 6A_1x}^{-6y_p}\big)\sin 3x$$

$$+\big(\overbrace{6A_1 - 9B_0 - 9B_1x}^{y_p''} \; \overbrace{-5B_1 - 15A_0 - 15A_1x}^{-5y_p'} \; + 6B_0 + 6B_1x\big)\cos 3x = 10x\sin(3x)$$

$$(-6B_1 + 15B_0 - 15A_0 - 5A_1)\sin 3x$$
$$+(-15A_1 + 15B_1)\,x\sin 3x$$
$$+(6A_1 - 15B_0 - 5B_1 - 15A_0)\cos 3x$$
$$+(-15B_1 - 15A_1)\,x\cos 3x = 10x\sin(3x)$$

$$\begin{cases} \cancel{-6}\ -6B_1 + 15B_0 - 15A_0 - 5A_1 = 0 & \cdots \sin 3x \\ -15A_1 + 15B_1 = 10 & \cdots x\sin 3x \\ 6A_1 - 15B_0 - 5B_1 - 15A_0 = 0 & \cdots \cos 3x \\ -15B_1 - 15A_1 = 0 & \cdots x\cos 3x \end{cases}$$

$$\Rightarrow \quad A_1 = -\frac{1}{3},\ B_1 = \frac{1}{3},\ A_0 = -\frac{2}{15},\ B_0 = -\frac{1}{9}$$

$$y_p = \left(-\frac{2}{15} - \frac{1}{3}x\right)\sin(3x) + \left(-\frac{1}{9} + \frac{1}{3}x\right)\cos(3x)$$

step 3: $y = y_h + y_p = \underline{\quad \cdots \quad}$

$$y'' + a_1 y' + a_2 y = g(x) = g_1(x) + g_2(x)$$

$g_1 = e^x$ ; $g_2 = x^3 e^{2x}$ ; $g = e^x + x^3 e^{-2x} + x^5 \sin(3x)$

(1) Solve Homegenuous problem,

$$y_h'' + a_1 y_h' + a_2 y_h = 0$$

(2) Find $y_{p_1}$, s.t. $y_{p_1}'' + a_1 y_{p_1}' + a_2 y_{p_1} = g_1(x)$

(Based on $r_1, r_2, g_1(x)$.

(3) Find $y_{p_2}$, s.t. $y_{p_2}'' + a_1 y_{p_2}' + a_2 y_{p_2} = g_2(x)$

Final Solution $y = y_h + y_{p_1} + y_{p_2}$

(1) $y'' + \frac{1}{3} y' - \frac{2}{3} y = e^{-2x} - \frac{1}{3} e^{-x}$,

$y(0) = \frac{3}{8}$, $y'(0) = -\frac{5}{8}$

step 1: $r^2 + \frac{1}{3} r - \frac{2}{3} = 0$, $(r+1)(r - \frac{2}{3}) = 0$

$r_1 = -1$, $r_2 = \frac{2}{3}$, $y_h = C_1 e^{-x} + C_2 e^{\frac{2}{3}x}$

Step 2: Assume $y_{p_1} = A_0 e^{-2x}$

$e^{-2x}$
$\lambda = -2$

$$y_{p_1}'' + \frac{1}{3} y_{p_1}' - \frac{2}{3} y_{p_1} = (4A_0 - \frac{2}{3}A_0 - \frac{2}{3}A_0) e^{-2x}$$
$$= e^{-2x}$$

$$\Rightarrow A_0 = \frac{3}{8}$$

step 2: $\cancel{y_{p_2}'' + \frac{1}{3}}$ $y_{p_2} = B_0 \cdot x e^{-x}$

$\lambda = -1 = r_1$,
$r_1 e^{-x}$

$$y_{p_2}'' + \frac{1}{3} y_{p_2}' - \frac{2}{3} y_p = (\quad - B_0 + \frac{1}{3} B_0) e^{-x}$$
$$= -\frac{1}{3} e^{-x}$$

$B_0 = -\frac{1}{?}$

$$y = C_1 e^{-x} + C_2 e^{\frac{2}{3}x} + \frac{3}{8}e^{-2x} - \frac{1}{2}xe^{-2x}$$

$$y(0) = C_1 + C_2 + \frac{3}{8} = \cancel{0}\,\frac{3}{8}$$

$$y'(0) = -C_1 + \frac{2}{3}C_2 - \frac{3}{4} - \frac{1}{2} = -\frac{5}{8} \Rightarrow C_1 = C_2 = 0$$

$$y = \frac{3}{8}e^{-2x} - \frac{1}{2}x\,e^{-x}$$