Regularization of Simultaneous Binary Collisions and Periodic Solutions with Singularity in the Collinear Four-Body Problem

Tiancheng Ouyang and Zhifu Xie
Department of Mathematics, Brigham Young University
Provo, Utah 84602, USA
Email: zhifu@math.byu.edu

Abstract

In this paper we construct new coordinates and time transformation that regularize the singularities of simultaneous binary collisions in collinear four-body problem. The motion in the new coordinates and time scale across simultaneous binary collision at least C^2. Furthermore, we study the behavior of the motion closing, across and after the simultaneous binary collision. Many different types of periodic solutions involving single binary collisions and simultaneous binary collisions are constructed.

Key word: Collinear Four-Body Problem, Single Binary Collision, Simultaneous Binary Collision, Regularization of Singularity, Periodic Solution with Simultaneous Binary collision.
APS classification number 10.000, 60.000.

1 Introduction

We consider the classical collinear four-body problem of celestial mechanics, which is defined by the motion of four mass points on the line under the influence of the Newtonian gravitational force law. Let $x_k \in \mathbb{R}$, $k = 1, 2, 3, 4$, denote the position of k^{th} body on the line with mass $m_k > 0$. Assume, without loss of generality, that $x_1 \leq x_2 \leq x_3 \leq x_4$ and gravitational constant 1. The motion of the four bodies is described by the following set of ordinary differential equations:

$$m_k \frac{d^2 x_k}{dt^2} = \frac{\partial U}{\partial x_k}, \qquad k = 1, 2, 3, 4, \tag{1.1}$$

where U is the potential function,

$$U = \sum_{1 \leq j < i \leq 4} \frac{m_i m_j}{|x_i - x_j|}. \tag{1.2}$$

The total energy

$$H = \sum_{1 \leq i \leq 4} \frac{1}{2} m_i |\dot{x}_i|^2 - \sum_{1 \leq j < i \leq 4} \frac{m_i m_j}{|x_i - x_j|} \tag{1.3}$$

is constant along a solution of (1.1).
We call the space of $x = (x_1, \cdots, x_4) \in \mathbb{R}^4$ the space of positions. Let $\triangle_{ij} := \{x \in \mathbb{R}^4, x_i = x_j\}$ and $\triangle := \bigcup_{1 \leq j < i \leq 4} \triangle_{ij}$. The potential function U, and consequently equation (1.1) are singular on $\triangle$.

Let $x(t) = (x_1(t), \cdots, x_4(t))$ be a solution of equation (1.1) defined on $[t_1, t_2)$, and assume that $x(t) \to L = (L_1, \cdots, L_4)$ as $t \to t_2^-$. We say that $x(t)$ has a singularity of collision at $t = t_2$ if $L \in \triangle$. According to the locations of L in $\triangle$, the singularities of collision are divided into the categories of (I) binary collisions, (II) simultaneous binary collisions, (III) triple collisions and (IV) four-body (total) collisions. In this paper, we study a solution with singularity of *simultaneous binary collision(SBC),* that is, the limit L of the position satisfies $-\infty < L_1 = L_2 < L_3 = L_4 < \infty$. Let us denote the set of L satisfying these restrictions as $\bigwedge$.

For better understanding the behavior of the motion of the particles in a neighborhood of a collision, we make a change of coordinates and of time scale. If, in the new coordinates, the orbits which approach collision can be extended across the collision in a smooth manner with respect to the new time scale, we say that the collision orbits have been regularized. The regularization is of class $C^n, n \geq 0$, or analytic if each collision orbit of the transformed differential equations is C^n or analytic, respectively, as a function of the new time scale in a neighborhood of collision. This type of regularization goes back, in particular, to Sundman ([11]) in his studies of collisions in the three-body problem (see also ([8])).

It is worth to mention that the above concept of regularization just refers to the extension of each individual collision orbit itself across collision. A related question is that of the smoothness of the flow with respect to initial conditions in a neighborhood of a collision orbit. This defines a different type of regularization if the flow also varies smoothly with respect to initial conditions in a neighborhood of a collision orbit. This type of regularization was first studied by Easton ([3]), and later by many other people, see Regina Martinez and Carles Simó ([5]), R. McGehee ([7]) etc. Many other works can be found from the reference of these papers.

It is well known that a single binary collision is regularizable in all senses. Many analytic regularizations are known for a single two-body collision; see the work of Levi-Civita ([13]), Moser ([6]), Belbruno ([1]). There also are many research papers which studied the regularization of simultaneous binary collision with some assumption on masses; see the work of Belbruno ([2]), Punosevac and Wang ([10]), Simo and Lacomba ([12]) etc.

In this paper we construct coordinate transforms in new time scale that remove the singularities of simultaneous binary collision in collinear four-body problem without any assumption on mass. The regularization is at least of class C^2. Base on the results of regularization of SBC in this paper, the behavior of the motion is studied for the motion acrossing collisions. The existence of a family of periodic solutions with simultaneous binary collision is proved in section 4. More periodic solutions involving single binary collision and SBC are constructed in section 5.

We now proceed to the next section and state our results more precisely.

2 Main Results and the Estimation of Collision Rate

Let us now consider equation (1.1) for the collinear four-body problem assuming $x_1 \leq x_2 \leq x_3 \leq x_4$. Without loss of generality, we also put the center of mass at the origin which implies

$$\sum_{k=1}^{4} m_k x_k = 0 \tag{2.1}$$

Related to (2.1), we have

$$\sum_{k=1}^{4} m_k \frac{dx_k}{dt} = 0 \tag{2.2}$$

These help in cutting the dimension of the phase space down by two. Let

$$u_1 = x_2 - x_1, \quad u_2 = x_4 - x_3, \quad u_3 = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \tag{2.3}$$

and

$$K_{31} = \frac{1}{x_3 - x_1} = \frac{\omega_1}{\omega_2 u_1 - \omega_3 u_2 - \omega_4 u_3} \tag{2.4}$$

$$K_{41} = \frac{1}{x_4 - x_1} = \frac{\omega_1}{\omega_2 u_1 + \omega_5 u_2 - \omega_4 u_3} \tag{2.5}$$

$$K_{32} = \frac{1}{x_3 - x_2} = \frac{\omega_1}{-\omega_6 u_1 - \omega_3 u_2 - \omega_4 u_3} \tag{2.6}$$

$$K_{42} = \frac{1}{x_4 - x_2} = \frac{\omega_1}{-\omega_6 u_1 + \omega_5 u_2 - \omega_4 u_3} \tag{2.7}$$

where $\omega_1 = (m_1+m_2)(m_3+m_4), \omega_2 = m_2(m_3+m_4), \omega_3 = m_4(m_1+m_2), \omega_4 = (m_1+m_2)(m_1+m_2+m_3+m_4)$, $\omega_5 = m_3(m_1+m_2)$ and $\omega_6 = m_1(m_3+m_4)$.

Then equation (1.1) reduce to an ordinary differential equation system with six independent variables $\vec{p_1} = (u_1, u_2, u_3, v_1, v_2, v_3)$,

$$\begin{aligned}
\frac{du_1}{dt} &= v_1, & \frac{dv_1}{dt} &= -\frac{m_1+m_2}{u_1^2} + m_3(K_{32}^2 - K_{31}^2) + m_4(K_{42}^2 - K_{41}^2);\\
\frac{du_2}{dt} &= v_2, & \frac{dv_2}{dt} &= -\frac{m_3+m_4}{u_2^2} + m_1(K_{31}^2 - K_{41}^2) + m_2(K_{32}^2 - K_{42}^2);\\
\frac{du_3}{dt} &= v_3, & \frac{dv_3}{dt} &= \frac{m_1m_3}{m_1+m_2}K_{31}^2 + \frac{m_2m_3}{m_1+m_2}K_{32}^2 + \frac{m_1m_4}{m_1+m_2}K_{41}^2 + \frac{m_2m_4}{m_1+m_2}K_{42}^2.
\end{aligned} \tag{2.8}$$

$\vec{u} = (u_1, u_2, u_3) \in \mathbb{R}^2 \times \mathbb{R}^-$ is now the space of positions and $\bigwedge = \{u_1 = u_2 = 0, u_3 \in \mathbb{R}^-\}$ is the singular set for simultaneous binary collisions. $K_{ij}, i = 3,4, j = 1,2$ are bounded on the singular set $\bigwedge$.
It is verified that the total energy (1.3) becomes,

$$\hat{H} = \frac{(\beta_1 v_1^2 + \beta_2 v_2^2 + \beta_3 v_3^2)}{2(m_1+m_2)(m_3+m4)} - (\frac{m_1m_2}{u_1} + \frac{m_3m_4}{u_2} + m_1m_3K_{31} + m_1m_4K_{41} + m_2m_3K_{32} + m_2m_4K_{42}) \tag{2.9}$$

where $\beta_1 = m_1m_2(m_3+m_4), \beta_2 = m_3m_4(m_1+m_2), \beta_3 = (m_1+m_2)^2(m_1+m_2+m_3+m_4)$.
One of the main results of this paper reads as follows.

Theorem 1.1 *Any simultaneous binary collision orbit of collinear four body problem can be extended at least C^1 across $\bigwedge$ with respect to the new time scale, after a change of coordinates and time scale.*

Furthermore, we also prove our extension is at least C^2 in theorem 3.2. The following lemma 1 and its proof are from the work of Belbruno ([2]).

Lemma 1. *Let $\vec{u} = \vec{u}(t)$, $t \in [t_1, t_2)$ denote a simultaneous binary collision orbit encountering $\bigwedge$, where $t = t_2$ corresponds to collision, then*

$$\lim_{t \to t_2} \frac{du_k}{dt} = \lim_{t \to t_2} v_k(t) = \infty, \qquad k = 1, 2 \tag{2.10}$$

$$\lim_{t \to t_2} u_1(t)v_1^2(t) = 2(m_1 + m_2), \quad \lim_{t \to t_2} u_2(t)v_2^2(t) = 2(m_3 + m_4), \tag{2.11}$$

$$\lim_{t \to t_2} u_1(t)v_1(t) = 0, \quad \lim_{t \to t_2} u_2(t)v_2(t) = 0, \tag{2.12}$$

and

$$\lim_{t \to t_2} \frac{u_1(t)}{u_2(t)} = \alpha, \qquad \lim_{t \to t_2} \frac{v_1(t)}{v_2(t)} = \alpha, \tag{2.13}$$

where $\alpha = \left(\frac{m_1+m_2}{m_3+m_4}\right)^{\frac{1}{3}}$.

Proof. System (2.8) implies

$$\frac{d^2u_1}{dt^2} = -\frac{m_1 + m_2}{u_1^2} + G_1, \qquad \frac{d^2u_2}{dt^2} = -\frac{m_3 + m_4}{u_2^2} + G_2, \tag{2.14}$$

where $G_1 = m_3(K_{32}^2 - K_{31}^2) + m_4(K_{42}^2 - K_{41}^2)$ and $G_2 = m_1(K_{31}^2 - K_{41}^2) + m_2(K_{32}^2 - K_{42}^2)$ are bounded when t is close to t_2. Multiplying (2.14) by $\frac{du_1}{dt}$, $\frac{du_2}{dt}$ respectively, yields

$$\left(\frac{du_1}{dt}\right)^2 = \frac{2(m_1 + m_2)}{u_1} + \tilde{G}_1, \qquad \left(\frac{du_2}{dt}\right)^2 = \frac{2(m_3 + m_4)}{u_2} + \tilde{G}_2$$

where $\tilde{G}_1$ and $\tilde{G}_2$ are also bounded when t is close to t_2. Thus letting $u_k(t) \to 0$ as $t \to t_2, k = 1, 2$, we prove that $\lim_{t\to t_2} \frac{du_1}{dt} = \lim_{t\to t_2} v_1(t) = \infty$. and $\lim_{t\to t_2} \frac{du_2}{dt} = \lim_{t\to t_2} v_2(t) = \infty$. Multiplying by u_1, u_2 respectively, the above equations become

$$u_1 v_1^2 = 2(m_1 + m_2) + u_1\tilde{G}_1, \quad u_2 v_2^2 = 2(m_3 + m_4) + u_2\tilde{G}_2.$$

Then we have

$$\lim_{t \to t_2} u_1(t)v_1^2(t) = 2(m_1 + m_2), \quad \lim_{t \to t_2} u_2(t)v_2^2(t) = 2(m_3 + m_4).$$

Consequently,

$$\lim_{t \to t_2} u_1(t)v_1(t) = 0, \quad \lim_{t \to t_2} u_2(t)v_2(t) = 0.$$

By making us of the fact that both u_1, u_2 tend to 0 monotonically and the results above, we have $\lim_{t\to t_2} \frac{u_1(t)}{u_2(t)} = \alpha$, where $\alpha = \left(\frac{m1+m2}{m_3+m_4}\right)^{1/3}$. We also have

$$\lim_{t \to t_2} \frac{v_1(t)}{v_2(t)} = \lim_{t \to t_2} \frac{\dot{u}_1(t)}{\dot{u}_2(t)} = \lim_{t \to t_2} \frac{u_1(t)}{u_2(t)} = \alpha.$$

3 The proof of Theorem 1

Let $\vec{u} = \vec{u}(t)$ denote a simultaneous binary collision orbit encountering $\bigwedge$ when $t = t_2$, then in a sufficiently small open deleted neighborhood of $t = t_2$, $\vec{u}(t)$ performs no collisions ([2], Belbruno). Therefore we can assume $\vec{p}_1 = (u_1, \cdots, v_3)$ is a solution of (2.8) performing no collisions for $t \in [t_1, t_2)$ and performing a simultaneous binary collision when $t \to t_2^-$. We will construct coordinate transform with a new time scale τ, such that the orbit under the new coordinate can be regularized. Let $\delta > 1$ and $0 < \rho < 1$ be fixed. We only consider solutions of equations (2.8) in $\mathcal{U}_{\delta,\rho}$, where

$$\mathcal{U}_{\delta,\rho} = \{\vec{p}_1 \in (\mathbb{R})^{2+} \times \mathbb{R}^- \times \mathbb{R}^3 : u_1, u_2 < \rho; -\delta < u_3 < -\delta^{-1}\}$$

We are now ready to introduce regularization variables by a Levi-Civita transformation and a time scale. Let $\vec{p}_2 = (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3)$ be the new phase variables with the new time variable as τ.

$$u_1 = \frac{\xi_1^2}{2}, u_2 = \frac{\xi_2^2}{2}, u_3 = -\frac{\xi_3^2}{2}, v_1 = \frac{\eta_1}{\xi_1}, v_2 = \frac{\eta_2}{\xi_2}, v_3 = \frac{\eta_3}{\xi_3} \tag{3.1}$$

and rescale time by

$$dt = (\xi_1^2 + \xi_2^2)d\tau \tag{3.2}$$

One verifies that (2.8) becomes

$$\frac{d\xi_1}{d\tau} = \frac{\xi_1^2 + \xi_2^2}{\xi_1^2}\eta_1, \tag{3.3}$$

$$\frac{d\xi_2}{d\tau} = \frac{\xi_1^2 + \xi_2^2}{\xi_2^2}\eta_2 \tag{3.4}$$

$$\frac{d\xi_3}{d\tau} = -\frac{(\xi_1^2 + \xi_2^2)}{\xi_3^2}\eta_3 \tag{3.5}$$

$$\frac{d\eta_1}{d\tau} = \frac{(\eta_1^2 - 4(m_1 + m_2))}{\xi_1^2}\frac{\xi_1(\xi_1^2 + \xi_2^2)}{\xi_1^2} + \left(m_3(K_{32}^2 - K_{31}^2) + m_4(K_{42}^2 - K_{41}^2)\right)\xi_1(\xi_1^2 + \xi_2^2) \tag{3.6}$$

$$\frac{d\eta_2}{d\tau} = \frac{(\eta_2^2 - 4(m_3 + m_4))}{\xi_2^2}\frac{\xi_2(\xi_1^2 + \xi_2^2)}{\xi_2^2} + \left(m_1(K_{31}^2 - K_{41}^2) + m_2(K_{32}^2 - K_{42}^2)\right)\xi_2(\xi_1^2 + \xi_2^2) \tag{3.7}$$

$$\frac{d\eta_3}{d\tau} = \left(\frac{m_1 m_3}{m_1 + m_2}K_{31}^2 + \frac{m_2 m_3}{m_1 + m_2}K_{32}^2 + \frac{m_1 m_4}{m_1 + m_2}K_{41}^2 + \frac{m_2 m_4}{m_1 + m_2}K_{42}^2\right)\xi_3(\xi_1^2 + \xi_2^2) + \frac{(\xi_1^2 + \xi_2^2)}{\xi_3^3}\eta_3^2 \tag{3.8}$$

where $K_{31}, K_{32}, K_{41}, K_{42}$ are obtained by substituting (3.1) into (2.4)-(2.7), which are bounded and smooth on $\bigwedge$.

Derivations for equations (3.3) to (3.8): For the first equation (3.3) we differentiate $\xi_1^2 = 2u_1$ to obtain

$$\frac{d\xi_1}{d\tau} = \frac{1}{\xi_1}\frac{du_1}{dt}\frac{dt}{d\tau} = \frac{1}{\xi_1}\frac{\eta_1}{\xi_1}(\xi_1^2 + \xi_2^2) = \frac{\xi_1^2 + \xi_2^2}{\xi_1^2}\eta_1.$$

$$\frac{d\eta_1}{d\tau} = \frac{d}{d\tau}(\xi_1 v_1) = \frac{dv_1}{d\tau}\xi_1 + v_1\frac{d\xi_1}{d\tau}$$

$$= \left(-\frac{4(m_1 + m_2)}{\xi_1^4} + m_3(K_{32}^2 - K_{31}^2) + m_4(K_{42}^2 - K_{41}^2)\right)\xi_1(\xi_1^2 + \xi_2^2) + \frac{\eta_1^2(\xi_1^2 + \xi_2^2)}{\xi_1^3}$$

$$= \frac{(\eta_1^2 - 4(m_1 + m_2))}{\xi_1^2}\frac{\xi_1(\xi_1^2 + \xi_2^2)}{\xi_1^2} + \left(m_3(K_{32}^2 - K_{31}^2) + m_4(K_{42}^2 - K_{41}^2)\right)\xi_1(\xi_1^2 + \xi_2^2)$$

$$\frac{d\eta_3}{d\tau} = \xi_3\frac{dv_3}{dt}\frac{dt}{d\tau} + v_3\frac{d\xi_3}{d\tau}$$

$$= \left(\frac{m_1 m_3}{m_1 + m_2}K_{31}^2 + \frac{m_2 m_3}{m_1 + m_2}K_{32}^2 + \frac{m_1 m_4}{m_1 + m_2}K_{41}^2 + \frac{m_2 m_4}{m_1 + m_2}K_{42}^2\right)\xi_3(\xi_1^2 + \xi_2^2) + \frac{(\xi_1^2 + \xi_2^2)}{\xi_3^3}\eta_3^2$$

Other equations can be obtained in a similar way.
The energy (2.9) becomes

$$\begin{aligned}\hat{H} = \tfrac{(\eta_1^2 - 4(m_1+m_2))}{\xi_1^2}\tfrac{m_1 m_2}{2(m_1+m_2)} + \tfrac{\eta_2^2 - 4(m_3+m_4))}{\xi_2^2}\tfrac{m_3 m_4}{2(m_3+m_4)} \\ + \tfrac{\eta_3^2}{\xi_3^2}\tfrac{\beta_3}{2(m_1+m_2)(m_3+m4)} - (m_1 m_3 K_{31} + m_1 m_4 K_{41} + m_2 m_3 K_{32} + m_2 m_4 K_{42})\end{aligned} \tag{3.9}$$

where $\beta_3 = (m_1+m_2)^2(m_1+m_2+m_3+m_4)$.
Remark: We choose $\xi_3 = -\sqrt{-2u_3}$ the negative branch of equation (3.1). The set $\{\xi_1 = \xi_2 = 0, \xi_3 < 0\}$ is the singular set corresponding to $\bigwedge$ the singular set for the simultaneous binary collisions. $K_{31}, K_{32}, K_{41}, K_{42}$ are bounded smooth functions on the singular set.
Let

$$\mathcal{V}_{\delta,\rho} = \{\vec{p}_2 = (\xi_1,\xi_2,\xi_3,\eta_1,\eta_2,\eta_3) : \xi_1^2, \xi_2^2 < 2\rho, -(2\delta)^{1/2} < \xi_3 < -(2\delta^{-1})^{1/2}\}$$

be the correspondence of $\mathcal{U}_{\delta,\rho}$ in phase space $\vec{p}_1$. We will study the solutions of (3.3)-(3.8) in $\mathcal{V}_{\delta,\rho}$.

Recall that $\vec{p}_1 = (u_1,u_2,u_3,v_1,v_2,v_3)$ is a solution in $\mathcal{U}_{\delta,\rho}$ defined in $[t_1,t_2)$ and assume that $\vec{p}_1 \to \bigwedge$ as $t \to t_2^-$. Let

$$\tau(t) = \tau_0 + \int_{t_1}^{t} \frac{1}{2(u_1(s)+u_2(s))} ds \tag{3.10}$$

Theorem 3.1 Let $\vec{p}_1(t)$, $t \in [t_1,t_2)$ be a solution of equation (2.8) in $\mathcal{U}_{\delta,\rho}$ with simultaneous binary collision at $t = t_2$, in other words, $\vec{p}_1(t) \to \bigwedge$ as $t \to t_2^-$. Let $\tau(t)$ be defined by (3.10) and $\vec{p}_2(\tau)$, $\tau \in [\tau_1,\tau_2)$ be the functions obtained from $\vec{p}_1(t)$ through (3.1). Then
(1) $\vec{p}_2(\tau)$ is a solution of equations (3.3)-(3.8) in $[\tau_1,\tau_2)$;
(2) $\tau_2 := \tau(t_2) < \infty$, and $\vec{p}_2(\tau_2) := \lim_{\tau\to\tau_2} \vec{p}_2(\tau)$ is well defined; and
(3) the solution $\vec{p}_2$ of (3.3)-(3.8) can be at least C^1 smoothly extended through τ_2.
Proof. (1) This follows from the derivations of equations. We caution that (3.1) allows different ways to convert $\vec{p}_1(t)$ to $\vec{p}_2(\tau)$ because $\xi_k, k = 1,2,3$ can have different signs. This is a well known characteristic of Levi-Civita variables. For definiteness, let us choose the positive sign $\xi_k = \sqrt{2u_k}, k = 1,2$ and negative sign $\xi_3 = -\sqrt{-2u_3}$.
(2) It is well know that when a collision singularity occurs at t_2,

$$u_1(t) + u_2(t) \sim (t - t_2)^{2/3}.$$

Then it follows that

$$\tau_2 = \tau_0 + \int_{t_1}^{t_2} \frac{1}{2(u_1(t)+u_2(t))} dt < \infty.$$

It is easy to show that $v_3(t)$ and $u_k(t) \to$ a definite limit as $t \to t_2^-$, which we denote by $v_3(t_2), u_k(t_2), k = 1,2,3$. Now for $\vec{p}_2(\tau_2)$: we let $\xi_k(\tau_2) = \sqrt{2u_k(t_2)}, k = 1,2$ and $\xi_3(\tau_2) = -\sqrt{-2u_3(t_2)}, \eta_3(\tau_2) = \xi_3 v_3(t_2)$. By the assumption, $\xi_1(\tau_2) = 0, \xi_2(\tau_2) = 0$.
From above, we have

$$\lim_{\tau\to\tau_2} \eta_1^2(\tau) = \lim_{t\to t_2} 2u_1v_1^2 = 4(m_1+m_2),$$

and

$$\lim_{\tau\to\tau_2} \eta_2^2(\tau) = \lim_{t\to t_2} 2u_2v_2^2 = 4(m_3+m_4),$$

from which it follows that $\eta_1(\tau_2) = -2\sqrt{m_1+m_2}$, $\eta_2(\tau_2) = -2\sqrt{m_3+m_4}$. They are negative because we have chosen positive sigh for $\xi_k(\tau_2), k = 1,2$. Therefore $\vec{p}_2(\tau_2) := \lim_{\tau\to\tau_2} \vec{p}_2(\tau)$ is well defined.
Before we prove (3), we need the following lemma 2.

Lemma 2. *Let $\vec{p}_1(t)$ be a solution of (2.8). $\vec{p}_2(\tau)$ is obtained from $\vec{p}_1(t)$ as above. Then*

$$\lim_{\tau\to\tau_2} \frac{\xi_1^2(\tau)}{\xi_2^2(\tau)} = \alpha \tag{3.11}$$

$$\lim_{\tau\to\tau_2} \frac{\xi_1^2(\tau)+\xi_2^2(\tau)}{\xi_2^2(\tau)} = 1+\alpha \tag{3.12}$$

$$\lim_{\tau\to\tau_2}\frac{\xi_1^2(\tau)+\xi_2^2(\tau)}{\xi_1^2(\tau)}=1+\frac{1}{\alpha} \tag{3.13}$$

$$\lim_{\tau\to\tau_2}\frac{\frac{\eta_1^2(\tau)}{m_1+m_2}}{\frac{\eta_2^2(\tau)}{m_3+m_4}}=1 \tag{3.14}$$

Proof. By directional computation and lemma 1, we can check that

$$\lim_{\tau\to\tau_2}\frac{\xi_1^2(\tau)}{\xi_2^2(\tau)}=\lim_{t\to t_2}\frac{u_1}{u_2}=\alpha,$$

$$\lim_{\tau\to\tau_2}\frac{\xi_1^2(\tau)+\xi_2^2(\tau)}{\xi_2^2(\tau)}=\lim_{\tau\to\tau_2}(1+\frac{\xi_1^2(\tau)}{\xi_2^2(\tau)})=1+\alpha$$

$$\lim_{\tau\to\tau_2}\frac{\xi_1^2(\tau)+\xi_2^2(\tau)}{\xi_1^2(\tau)}=\lim_{\tau\to\tau_2}(1+\frac{\xi_2^2(\tau)}{\xi_1^2(\tau)})=1+\frac{1}{\alpha}$$

$$\lim_{\tau\to\tau_2}\frac{\frac{\eta_1^2(\tau)}{m_1+m_2}}{\frac{\eta_2^2(\tau)}{m_3+m_4}}=\lim_{t\to t_2}\frac{2u_1v_1^2(t)(m_3+m_4)}{2u_2v_2^2(t)(m_1+m_2)}=1. \qquad \sharp$$

The proof of (3). From (3.9), we have

$$\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}\frac{m_1m_2}{2(m_1+m_2)}+\frac{\eta_2^2-4(m_3+m_4))}{\xi_2^2}\frac{m_3m_4}{2(m_3+m_4)}=\hat{H}-\Big(\frac{(\beta_3\eta_3^2)}{2\xi_3^2(m_1+m_2)(m_3+m4)} \\ -(m_1m_3K_{31}+m_1m_4K_{41}+m_2m_3K_{32}+m_2m_4K_{42})\Big). \tag{3.15}$$

Because $\hat{H}$ is a constant along any solution $\vec{p}_2(\tau)$, the right side in (3.15) is bounded in $[\tau_1,\tau_2)$ and the limit of the right side in (3.15) is finite defined by L as $\tau\to\tau_2$, i.e

$$\lim_{\tau\to\tau_2}\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}\frac{m_1m_2}{2(m_1+m_2)}+\frac{(\eta_2^2-4(m_3+m_4))}{\xi_2^2}\frac{m_3m_4}{2(m_3+m_4)}=L.$$

In addition,

$$\lim_{\tau\to\tau_2}\frac{\xi_1^2(\tau)}{\xi_2^2(\tau)}=\alpha,\quad \lim_{\tau\to\tau_2}\xi_1(\tau)=0,\quad \lim_{\tau\to\tau_2}\xi_2(\tau)=0,\quad \lim_{\tau\to\tau_2}\frac{\frac{\eta_1^2(\tau)}{m_1+m_2}}{\frac{\eta_2^2(\tau)}{m_3+m_4}}=1,$$

and

$$\lim_{\tau\to\tau_2}\frac{\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}\frac{m_1m_2}{2(m_1+m_2)}}{\frac{\eta_2^2-4(m_3+m_4))}{\xi_2^2}\frac{m_3m_4}{2(m_3+m_4)}}=\frac{m_1m_2(m_3+m_4)}{\alpha m_3m_4(m_1+m_2)}\lim_{t\to t_2}\frac{u_1v_1^2-2(m_1+m_2)}{u_2v_2^2-2(m_3+m_4)}$$

$$=\frac{m_1m_2(m_3+m_4)}{m_3m_4(m_1+m_2)}\lim_{t\to t_2}\frac{v_1^2-\frac{2(m_1+m_2)}{u_1}}{v_2^2-\frac{2(m_3+m_4)}{u_2}}=\frac{m_1m_2(m_3+m_4)}{m_3m_4(m_1+m_2)}\lim_{t\to t_2}\frac{2v_1\frac{dv_1}{dt}+\frac{2(m_1+m_2)}{u_1^2}}{2v_2\frac{dv_2}{dt}+\frac{2(m_3+m_4)}{u_2^2}}$$

$$=\frac{m_1m_2(m_3+m_4)}{m_3m_4(m_1+m_2)}\lim_{t\to t_2}\frac{\frac{2(m_1+m_2)}{u_1^2}(-v_1+1)}{\frac{2(m_3+m_4)}{u_2^2}(-v_2+1)}=\frac{m_1m_2}{\alpha^2m_3m_4}\lim_{t\to t_2}\frac{(-v_1+1)}{(-v_2+1)}$$

$$=\frac{m_1m_2}{\alpha m_3m_4}$$

Then $\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}$ and $\frac{(\eta_2^2-4(m_3+m_4))}{\xi_2^2}$ are well defined when $\tau \to \tau_2$ by making use of (3.15) and $\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}\frac{\xi_1(\xi_1^2+\xi_2^2)}{\xi_1^2}$ and $\frac{(\eta_2^2-4(m_3+m_4))}{\xi_2^2}\frac{\xi_1(\xi_1^2+\xi_2^2)}{\xi_2^2}$ in (3.6)-(3.7) go to zero as $\tau \to \tau_2$. In fact, we can prove this by direct computation as follows:

$$\lim_{\tau\to\tau_2}\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}\frac{\xi_1(\xi_1^2+\xi_2^2)}{\xi_1^2} = \lim_{\tau\to\tau_2}\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1}\lim_{\tau\to\tau_2}\frac{(\xi_1^2+\xi_2^2)}{\xi_1^2}$$

$$= (1+\frac{1}{\alpha})\lim_{t\to t_2}\frac{u_1v_1^2-2(m_1+m_2)}{\sqrt{2u_1}} = (1+\frac{1}{\alpha})\lim_{t\to t_2}\frac{u_1\tilde{G}_1}{\sqrt{2u_1}} = 0.$$

According to lemma 2, it is clear that the functions on the right-hand side of (3.3)-(3.8) have a well-defined finite limit as $\tau \to \tau_2$ along $\vec{p}_2(\tau)$ given in the above. Moreover, $(\xi_1(\tau), \xi_2(\tau), \xi_3(\tau))$ intersects the simultaneous binary collision set $\bigwedge = \{\xi_1 = 0, \xi_2 = 0, \xi_3 < 0\}$ transversally, because letting $\tau \to \tau_2$ in (3.3), (3.4) implies

$$\lim_{\tau\to\tau_2}\frac{d\xi_1}{d\tau} = \lim_{\tau\to\tau_2}\frac{\xi_1^2+\xi_2^2}{\xi_1^2}\eta_1 = 2(1+\frac{1}{\alpha})\sqrt{m_1+m_2} > 0,$$

$$\lim_{\tau\to\tau_2}\frac{d\xi_2}{d\tau} = \lim_{\tau\to\tau_2}\frac{\xi_1^2+\xi_2^2}{\xi_2^2}\eta_2 = 2(1+\alpha)\sqrt{m_3+m_4} > 0.$$

Thus, $(\xi_1(\tau), \xi_2(\tau), \xi_3(\tau))$ can be extended across $\bigwedge$. The solution $\vec{p}_2(\tau) = (\xi_1(\tau), \xi_2(\tau), \xi_3(\tau), \eta_1(\tau), \eta_2(\tau), \eta_3(\tau))$ can be extended for $\tau > \tau_2$ by solving the differential equations (3.3)-(3.8) with initial condition $\vec{p}_(\tau) = \vec{p}_2(\tau_2)$ when $\tau = \tau_2$. The vector field given by (3.3)-(3.8) is clearly continuous at $\tau = \tau_2$ and therefore, the components $\xi_1(\tau), \xi_2(\tau), \xi_3(\tau), \eta_1(\tau), \eta_2(\tau), \eta_3(\tau)$ are continuously differentiable functions of τ when $\tau = \tau_2$. So the singularity of simultaneous binary collision in equation (2.8) is removed by transferring to equation (3.3)-(3.8). This concludes the proof of theorem 3.1. which yields the proof of theorem 1.1. $\sharp$
Furthermore, we even can prove that the regularization is C^2 in theorem 3.2.

Theorem 3.2 *The equations (3.3)-(3.8) give rise to a C^2 extension of $\vec{p}_2(\tau)$ with respect to τ at $\vec{p}_2(\tau_2)$ the simultaneous binary collision.*
Proof. Let $F(\tau) = \frac{\xi_1(\tau)}{\xi_2(\tau)}$. Then $F(\tau_2) = \alpha^{(1/2)}$. From equations (3.3)-(3.8) and lemma 2, we have at $\tau = \tau_2$,

$$\frac{d\xi_1}{d\tau} = (1+\alpha^{-1})(-2\sqrt{m_1+m2}), \frac{d\xi_2}{d\tau} = (1+\alpha)(-2\sqrt{m_3+m_4}), \frac{d\eta_1}{d\tau} = 0, \frac{d\eta_2}{d\tau} = 0.$$

$$\lim_{\tau\to\tau_2}\frac{dF}{d\tau} = \lim_{\tau\to\tau_2}\frac{\frac{d\xi_1}{d\tau}\xi_2 - \xi_1\frac{d\xi_2}{d\tau}}{\xi_2^2} = \lim_{\tau\to\tau_2}(1+F^{-2})\frac{\eta_1 - F^3\eta_2}{\xi_2}$$

$$= (1+\alpha^{-2})\lim_{\tau\to\tau_2}\frac{\frac{d\eta_1}{d\tau} - F^3\frac{d\eta_2}{d\tau} - 3F^2\eta_2\frac{dF}{d\tau}}{d\xi_2} = -3\alpha^{-1/2}\lim_{\tau\to\tau_2}\frac{dF}{d\tau}$$

So $\lim_{\tau\to\tau_2}\frac{dF}{d\tau} = 0$ and

$$\lim_{\tau\to\tau_2}\frac{d^2\xi_1}{d\tau^2} = \lim_{\tau\to\tau_2}\frac{d((1+F^{-2})\eta_1)}{d\tau} = \lim_{\tau\to\tau_2}(1+F^{-2})\frac{d\eta_1}{d\tau} + \lim_{\tau\to\tau_2}(1-2F^{-3}\frac{dF}{d\tau})\eta_1 = -2\sqrt{m1+m2}.$$

Similarly, we can prove the limits of the second derivative of $\xi_1(\tau), \xi_2(\tau), \xi_3(\tau), \eta_1(\tau), \eta_2(\tau), \eta_3(\tau)$ exist at $\tau = \tau_2$. Therefore the second derivatives are continuously differentiable functions of τ when $\tau = \tau_2$. The extension of the simultaneous collision orbit is C^2. $\sharp$

4 The Behavior of the Solutions Close to SBC Singularity and Periodic Solutions with SBC

Let us recall our notation. x_1, x_2, x_3, x_4 are the positions of collinear four body problem with the center of mass at origin, i.e. (2.1) holds. $u_1 = x_2 - x_1$ is the difference of the first two bodies and $u_2 = x_4 - x_3$ is the difference of the last two bodies. $u_3 = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$ is the center of mass of the first two bodies. v_i are the derivatives corresponding to u_i, $i = 1, 2, 3$. Furthermore, we have a new coordinates and time scale given by (3.1) and (3.2). $\bigwedge = \{x_1 = x_2, x_3 = x_4, x_2 \neq x_3\} = \{u_1 = u_2 = 0, u_3 < 0\} = \{\xi_1 = \xi_2 = 0, \xi_3 \neq 0\}$ are the sets of simultaneous binary collision in the respective coordinates. In this section we always assume that $x = (x_1, x_2, x_3, x_4)$ be a simultaneous binary collision solution which is defined in $t \in [t_1, t_2)$ encountering with singular set $\bigwedge$ at $t = t_2$. $\vec{p}_1(t) = (u_1(t), u_2(t), u_3(t), v_1(t), v_2(t), v_3(t))$ is obtained by transformation (2.3) and $\vec{p}_2(\tau) = (\xi_1(\tau), \xi_2(\tau), \xi_3(\tau), \eta_1(\tau), \eta_2(\tau), \eta_3(\tau))$ is obtained by transformation (3.1) and new time scale (3.2). By theorem 3.1 and theorem 3.2, $\vec{p}_2(\tau)$ is a C^2 solution of equation (3.3)-(3.8) without singularity at $\tau = \tau_2$. Furthermore, there exit $\tau_4 > \tau_2$, such that the behavior of the extension of $\vec{p}_2(\tau)$ can be described by time reverse in (τ_2, τ_4) as follows.

Theorem 4.1. *Suppose that $\vec{p}_2(\tau) = (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3)$ is defined as a simultaneous binary collision solution of (3.3)-(3.8) in (τ_1, τ_2) and is extended to (τ_1, τ_3) in theorem 3.1, where $\tau_1 < \tau_2 < \tau_3$. Let $\vec{p}_3(\tau) = (\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3)$ denote as follows,*

$$\tilde{\xi}_1(\tau) = \begin{cases} \xi_1(\tau) & \tau_1 < \tau \leq \tau_2, \\ -\xi_1(2\tau_2 - \tau) & \tau_2 < \tau < \tau_4. \end{cases} \tag{4.1}$$

$$\tilde{\xi}_2(\tau) = \begin{cases} \xi_2(\tau) & \tau_1 < \tau \leq \tau_2, \\ -\xi_2(2\tau_2 - \tau) & \tau_2 < \tau < \tau_4. \end{cases} \tag{4.2}$$

$$\tilde{\xi}_3(\tau) = \begin{cases} \xi_3(\tau) & \tau_1 < \tau \leq \tau_2, \\ \xi_3(2\tau_2 - \tau) & \tau_2 < \tau < \tau_4. \end{cases} \tag{4.3}$$

$$\tilde{\eta}_1(\tau) = \begin{cases} \eta_1(\tau) & \tau_1 < \tau \leq \tau_2, \\ \eta_1(2\tau_2 - \tau) & \tau_2 < \tau < \tau_4. \end{cases} \tag{4.4}$$

$$\tilde{\eta}_2(\tau) = \begin{cases} \eta_2(\tau) & \tau_1 < \tau \leq \tau_2, \\ \eta_2(2\tau_2 - \tau) & \tau_2 < \tau < \tau_4. \end{cases} \tag{4.5}$$

$$\tilde{\eta}_3(\tau) = \begin{cases} \eta_3(\tau) & \tau_1 < \tau \leq \tau_2, \\ -\eta_3(2\tau_2 - \tau) & \tau_2 < \tau < \tau_4. \end{cases} \tag{4.6}$$

where $\tau_4 = \min\{2\tau_2 - \tau_1, \tau_3\}$. Then $\vec{p}_3(\tau) = \vec{p}_2(\tau)$ for $\tau \in (\tau_1, \tau_4)$.

Proof. We only need to verify the extension $\vec{p}_3(\tau)$ of $\vec{p}_2(\tau)$ also satisfies the differential equations (3.3)-(3.8) in (τ_1, τ_4) and the smoothness $\vec{p}_3(\tau)$ at $\tau = \tau_2$.

For $\tau_1 < \tau \leq \tau_2$, $\vec{p}_3(\tau) = \vec{p}_2(\tau)$ then $\vec{p}_3(\tau)$ obviously satisfies the differential equations (3.3)-(3.8). For $\tau_2 < \tau < \tau_4$,

$$\frac{d\tilde{\xi}_1}{d\tau} = \frac{d(-\xi_1(2\tau_2 - \tau))}{d\tau} = \frac{d\xi_1}{d\tau}\bigg|_{2\tau_2 - \tau}$$

$$= \frac{(\xi_1(2\tau_2 - \tau))^2 + (\xi_2^2(2\tau_2 - \tau))^2}{(\xi_1(2\tau_2 - \tau))^2}\eta_1(2\tau_2 - \tau) = \frac{\tilde{\xi}_1^2 + \tilde{\xi}_2^2}{\tilde{\xi}_1^2}\tilde{\eta}_1.$$

Because $\xi_1(\tau_2) = 0$, the time reverse extension $\tilde{\xi}_1$ of ξ_1 is continuously differentiable at τ_2. ξ_1 across the singular set because the derivative of ξ_1 at τ_2 is not zero. In fact, it is negative. The figure 1 illustrates the extension for ξ_1.

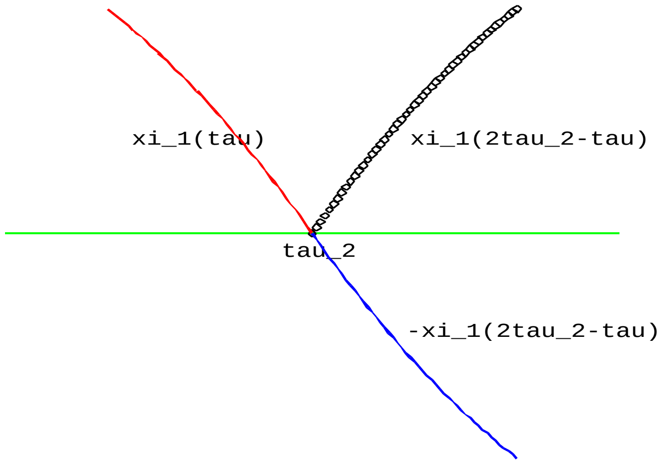


Figure 1: The extension of xi_1 to tilde{xi_1}

For $\tau_2 < \tau < \tau_4$, because $\frac{d\eta_1}{d\tau}|_{\tau_2} = 0$ and

$$\frac{d\tilde{\eta}_1}{d\tau} = \frac{d(\eta_1(2\tau_2 - \tau))}{d\tau} = -\frac{d\eta_1}{d\tau}|_{2\tau_2-\tau}$$

$$= \frac{(\tilde{\eta}_1^2 - 4(m_1+m_2))}{\tilde{\xi}_1^2}\frac{\tilde{\xi}_1(\tilde{\xi}_1^2+\tilde{\xi}_2^2)}{\tilde{\xi}_1^2} + \left(m_3(K_{32}^2 - K_{31}^2) + m_4(K_{42}^2 - K_{41}^2)\right)\tilde{\xi}_1(\tilde{\xi}_1^2+\tilde{\xi}_2^2),$$

where K_{ij} only involve square terms ξ_i^2, the time reverse extension $\tilde{\eta}_1$ of η_1 also satisfies the differential equation (3.6). The figure 2 is an example of $\tilde{\eta}_1$.

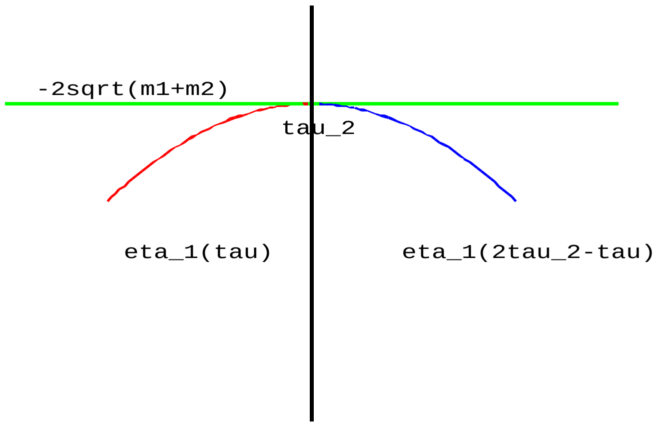


Figure 2: The extension of eta_1 to tilde{eta_1}

The proof for extension $\tilde{\xi}_2, \tilde{\eta}_3$ and $\tilde{\xi}_3, \tilde{\eta}_2$ are similar to the proof for extension $\tilde{\xi}_1$ and $\tilde{\eta}_1$ respectively. So $\vec{p}_3(\tau)$ is also a solution of differential equation (3.3)-(3.8) and it is same to $\vec{p}_2(\tau)$ when $\tau \leq \tau_2$. By the uniqueness theorem of ordinary differential equations, $\vec{p}_3(\tau)$ must equal to $\vec{p}_2(\tau)$ for $\tau \in (\tau_1, \tau_4)$. $\sharp$

Now we are going to describe the behavior of the simultaneous binary collision solution when it is closing to and at the singular set $\bigwedge$ in the original coordinate.

Theorem 4.2 *Let $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ be the extended simultaneous binary collision solution of (1.1) by converting the C^2 solution $\vec{p}_3(\tau)$ into original system. Then the solution $x(t)$ has the following properties.*

(a) x is defined in $t \in [t_1, t_4]$, where $t_4 = t(\tau_4)$ and $t_1 < t_2 < t_4 \leq 2t_2 - t_1$. x encounters with singular set $\bigwedge$ when $t = t_2$.

(b) Let $C_1 = \frac{m_1x_1+m_2x_2}{m_1+m_2}$ be the center of mass of m_1, m_2 and $C_2 = \frac{m_3x_3+m_4x_4}{m_3+m_4}$ be the center of mass of m_3, m_4. Then

$$\lim_{t\to t_2} \frac{C_1}{C_2} = -\frac{m_3+m_4}{m_1+m_2},$$

where $C_1 < 0$ and $C_2 > 0$ are both finite.

(c) The ratio of velocity $\frac{dx_i}{dt}$ and $\frac{dx_{i+1}}{dt}$ approaches a finite number as $t \to t_2$, where $i = 1, 3$, more precisely, $\lim_{t\to t_2} \frac{dx_i}{dt} / \frac{dx_{i+1}}{dt} = -\frac{m_{i+1}}{m_i}$. The negative sign implies that the velocities of collision pairs are in opposite direction as $t \to t_2$, which is independent of the initial positions and the initial velocities.

(d) The ratio of the distance $u_1 = x_2 - x_1$ and the distance $u_2 = x_4 - x_3$ is determined by their mass ratio, more precisely, $\lim_{t\to t_2} \frac{u_1}{u_2} = \left(\frac{m_1+m_2}{m_3+m_4}\right)^{\frac{1}{3}}$. The ratio of their velocities is also determined by their mass ratio, i.e. $\lim_{t\to t_2} \frac{du_1/dt}{du_2/dt} = \left(\frac{m_1+m_2}{m_3+m_4}\right)^{\frac{1}{3}}$.

(e) In the original time scale, the velocities are unbounded, i.e., $\lim_{t\to t_2} \frac{dx_i}{dt} = \infty$. But in the new time scale, the velocities are bounded and $\lim_{\tau\to\tau_2} \frac{dx_i}{d\tau} = 0$, where $i = 1, \cdots, 4$.

Proof. (a) is directly from the theorem 3.1 and theorem 4.1. By making using of the center of mass at origin, it is easy to prove (b). (c) is directly from the equation (1.1) and L'Hopital's rule, in fact,

$$\lim_{t\to t_2} \frac{\frac{dx_1}{dt}}{\frac{dx_2}{dt}} = \lim_{t\to t_2} \frac{\frac{d^2x_1}{dt^2}}{\frac{d^2x_2}{dt^2}} = \lim_{t\to t_2} \frac{\sum_{j\neq 1} \frac{m_j(x_j-x_1)}{|x_j-x_1|^3}}{\sum_{j\neq 2} \frac{m_j(x_j-x_2)}{|x_j-x_2|^3}} = -\frac{m_2}{m_1},$$

and $\lim_{t\to t_2} \frac{\frac{dx_3}{dt}}{\frac{dx_4}{dt}} = -\frac{m_4}{m_3}$. (d) is directly from lemma 1. Now we turn to prove (e).
In the new coordinates and new time scale, as the solution $\vec{p}_2(\tau)$ approaches the singular set $\bigwedge$, we already have, from the proof of theorem 3.1 and theorem 4.1,

$$\xi_1(\tau_2) = 0, \xi_2(\tau_2) = 0, \xi_3(\tau_2) < 0, \eta_1(\tau_2) = -2\sqrt{m_1+m_2}, \eta_2(\tau_2) = -2\sqrt{m_3+m_4}$$

and $\eta_3(\tau_2)$ is finite. So in the new time scale, it slows down the motion to a finite speed (η_i are related to the velocity of the particles). Recall that $\frac{du_1}{dt} \to \infty$ as t goes to t_2, but $u_1 = \frac{\xi_1^2}{2}$ implies

$$\frac{du_1}{d\tau} = \xi_1 \frac{d\xi_1}{d\tau} = 0 \text{ at } \tau = \tau_2.$$

From which we have,

$$\frac{d(x_2 - x_1)}{d\tau} = \frac{dx_2}{d\tau} - \frac{dx_1}{d\tau} = 0 \text{ at } \tau = \tau_2.$$

From 2 above, we have

$$\lim_{\tau\to\tau_2}\frac{\frac{dx_2}{d\tau}}{\frac{dx_1}{d\tau}}=\lim_{t\to t_2}\frac{\frac{dx_2}{dt}\frac{dt}{d\tau}}{\frac{dx_1}{dt}\frac{dt}{d\tau}}=\lim_{t\to t_2}\frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}}=-\frac{m_2}{m_1}$$

Therefore,

$$\frac{dx_2}{d\tau}=0 \text{ and } \frac{dx_1}{d\tau}=0 \text{ at } \tau=\tau_2.$$

$$\eta_3(\tau)=\xi_3(\tau)v_3(\tau)=\xi_3(\tau)\frac{m_1\frac{dx_1}{d\tau}+m_2\frac{dx_2}{d\tau}}{m_1+m_2}=0 \text{ at } \tau=\tau_2.$$

We complete the proof of theorem 4.2. ♯

Using the above properties, we construct a family of periodic solutions of the collinear four body problem with simultaneous binary collisions.

Let $x^0=(x_1^0,x_2^0,x_3^0,x_4^0)$ denote the initial positions of collinear four body problem with $-\infty<x_1^0<x_2^0<0<x_3^0<x_4^0<\infty$. We assume that x^0 possesses symmetries on positions and masses, i.e. $x_1^0=-x_4^0, x_2^0=-x_3^0$ and $m_1=m_4, m_2=m_3$. Without loss of generality, let $s=x_2^0-x_1^0=x_4^0-x_3^0>0$, $x_2^0=-1, x_3^0=1$, and $m_1=m_4=m, m_2=m_3=1$

Theorem 4.3 *If s,m fall into the region of $\frac{s^2(s+2)^2}{16(1+s)}<m$ in the first quadrant of sm-plane (see Figure 3), then the orbit by releasing the four bodies with zero velocity at x^0 is a periodic orbit with simultaneous binary collisions .*

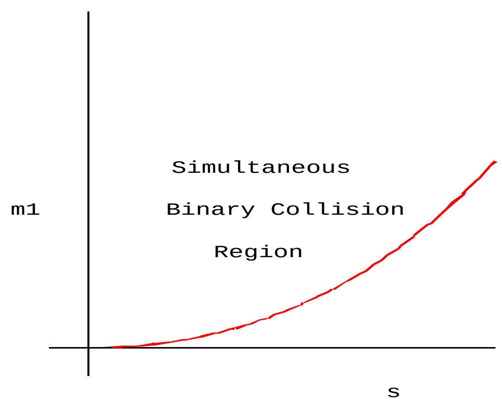


Figure 3

Before we prove theorem 4.3, we prove the following lemma. Let $x=(x_1,x_2,x_3,x_4)$ denote the positions of our four particles on the line with positive mass (m_1,m_2,m_3,m_4). We assume, without loss of generality, that $x_1\le x_2\le x_3\le x_4$ and the center of mass is at origin.

Lemma 3 (Periodic Solution with Simultaneous Binary Collision) *Let $x(t)=(x_1(t),x_2(t),x_3(t),x_4(t))$ be a smooth solution of (1.1) in the interval $[t_0,t_2)$ with initial condition $x(0)=x^0=(x_1^0,x_2^0,x_3^0,x_4^0)$ and $\frac{dx}{dt}(0)=0$, where $t_2>t_0=0$. If the solution $x(t)$ has a simultaneous binary collision at $t_2=T>0$, then the solution $x(t)$ can be extended to a periodic solution with period $2T$ as follows, in the sense of regularization given by theorem 3.1,*

$$\tilde{x}(t)=\begin{cases} x(t-2nT) & 2nT\le t\le(2n+1)T,\\ x((2n+2)T-t) & (2n+1)T\le t\le(2n+2)T.\end{cases}\tag{4.7}$$

Proof. Because the motion is governed by Newton's differential equation (1.1), it encounters a singularity at $t_2 = T$ which the velocities of the bodies approaching collision go to infinity. So the equation can not give information of the motion in a neighborhood of t_2. But the singularity at $t_2 = T$ causing by simultaneous binary collision can be removed in the sense of theorem 3.1. Then the orbit can be obtained in the following steps.

Step 1: The four particles are released at $t_0 = 0$ with initial positions $x^0 = x(t_0) = (x_1^0, x_2^0, x_3^0, x_4^0)$ and zero initial velocity. During time interval $(t_0, t_1]$, $t_1 < t_2$, the motion of the four particles are described by Newton's differential equation (1.1) (see Figure 4).There is no any collision in the time interval $(t_0, t_1]$ and $x(t_1) = (x_1^1, x_2^1, x_3^1, x_4^1)$ close to simultaneous binary collision, i.e. $0 < x_2^1 - x_1^1 < \rho, -\delta < \frac{m_1 x_1^1 + m_2 x_2^1}{m_1 + m_2} < -\delta^{-1}$.

Step 2: Because $x(t_1)$ falls into $\mathcal{U}_{\rho,\delta}$ and leads to a simultaneous binary collision solution, the motion of x can be described by the differential equations (3.3)-(3.8) in the new coordinates (3.1) and the new time scale (3.2). During the time (t_1, t_2), x approaches a simultaneous binary collision and encountering the singular set $\bigwedge$ at $t = t_2 = T$.

Step 3: By theorem 4.1, the motion can be extended as a time reverse, i.e. $\tilde{x}(t) = x(2T - t)$ for $t \in (t_2, t_3), t_3 = 2t_2 - t_1$

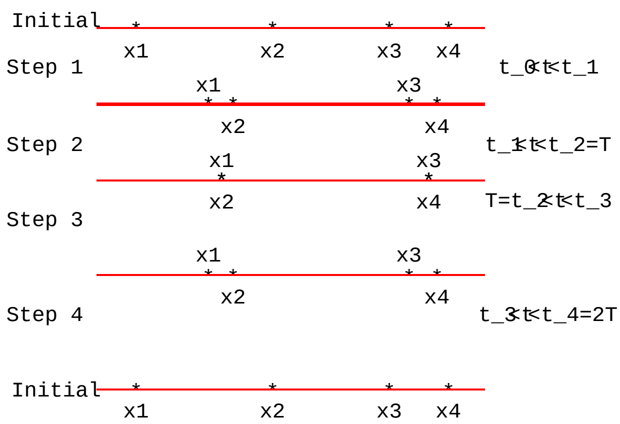


Figure 4

Step 4: The position $x(t_3)$ of the particles at t_3 is equal to the position $x(t_1)$ but their velocity is just opposite by step 3. Then in the following time interval $(t_3, t_4], t_4 = 2T$, the particles go back to the initial position at t_4 and they have zero velocity at t_4. The motion in $(t_3, t_4]$ is described by equations (1.1).

Then the orbit completes one period in $[0, 2T]$ and it repeats step 1, step 2, step 3, step 4. So the solution is extended to a periodic solution with simultaneous collision at $t = (2n + 1)T$, where n is an integer.

Note that the time scale is τ and new coordinates are ξ_i and η_i, $i = 1, 2, 3$ in step 2 and step 3. But we can change back to x and t by (3.1) and (3.2). $\sharp$

The proof of theorem 4.3. We only need check whether the conditions in theorem 4.3 lead to a simultaneous binary collision without any other collisions. By Newton's law, the accelerations of the four particles are respectively,

$$a_1 = m_2 s^{-2} + m_3(s+2)^{-2} + m_4(2s+2)^{-2},$$

$$a_2 = -m_1 s^{-2} + \frac{m_3}{4} + m_4(s+2)^{-2},$$

$$a_3 = -m_1(s+2)^{-2} - \frac{m_2}{4} + m_4 s^{-2},$$

$$a_4 = -m_1(2s+2)^{-2} - m_2(s+2)^{-2} - m_3 s^{-2}.$$

Note that $m_1 = m_4 = m, m_3 = m_2 = 1$ then no matter the choice of s and m, $a_1 > 0$ and $a_4 < 0$. If s, m can be chosen such that the acceleration $a_2 < 0$ but $a_3 > 0$, then x_0 leads to a simultaneous binary solution if it is released with zero velocity because of the symmetry of positions and masses. Therefore it is extended to a periodic solution with singularity.
In order that $a_2 < 0$ and $a_3 > 0$, we only need make $a_2 < 0$ by choosing proper s, m_1 because $a_2 = -a_3$. The numerator of a_2 is

$$na_2 = -16m_1 s - 16m_1 + s^4 + 4s^3 + 4s^2,$$

and the denominator of a_2 is

$$da_2 = 4s^2(s+2)^2.$$

So when s, m_1 fall into the region of $\frac{s^2(s^2+4s+4)}{16(1+s)} < m$, it has $a_2 < 0$ and leads to a simultaneous binary collision (see Figure 4). $\sharp$

5 Periodic Solutions with single Binary Collision and Simultaneous Binary Collision

The behavior of the motion for the pair closing the single binary collision can be described as time reverse plus a higher order term in a very short time neighborhood. At the moment of single binary collision, the velocities of the particles involving the collision approach to infinity. BY changing the time scale, the velocities of the particles remain bounded as slow motion. The motion can be extended to cross the collision point. Any periodic solution of collinear four body problem involves collisions. In this section many periodic solutions are constructed with both a sequence of single single binary collisions and a sequence of simultaneous binary collisions in the collinear four body problem. The central configuration of collinear four body problem plays an important role in our construction. It separates periodic solutions into two categories (a) not involving single binary collisions and (b) involving single binary collisions.
Let $x^0 = (x_1^0, x_2^0, x_3^0, x_4^0)$ denote the initial positions of collinear four body problem with $-\infty < x_1^0 < x_2^0 < 0 < x_3^0 < x_4^0 < \infty$. We assume that x^0 possesses symmetries on positions and masses, i.e. $x_1^0 = -x_4^0, x_2^0 = -x_3^0$ and $m_1 = m_4, m_2 = m_3$. Without loss of generality, let $s_0 = x_2^0 - x_1^0 = x_4^0 - x_3^0 > 0$, $x_2^0 = -1, x_3^0 = 1$, and $m_1 = m_4 = m, m_2 = m_3 = 1$.
$x^0 = (-s_0 - 1, -1, 1, s_0 + 1)$ with mass $(m, 1, 1, m)$ forms a central configuration if and only if

$$m = \frac{(s_0+1)^2 \left({s_0}^5 + 5\,{s_0}^4 + 8\,{s_0}^3 - 4\,{s_0}^2 - 16\,s_0 - 16\right)}{17\,{s_0}^4 + 68\,{s_0}^3 + 100\,{s_0}^2 + 64\,s_0 + 16}. \tag{5.1}$$

For $s_0 > 1.396812289$, there is a positive $m > 0$ such that $x^0 = (-s_0 - 1, -1, 1, s_0 + 1)$ with mass $(m, 1, 1, m)$ forms a central configuration. The result is a special case of theorem 1 in ([9]). In this section, all the motion are obtained by releasing the four bodies at initial position with zero velocity.

Theorem 5.1 *Assume that $x^0 = (-s_0 - 1, -1, 1, s_0 + 1)$ with mass $(m, 1, 1, m)$ forms a central configuration. Then $y^0 = (-s_0 + s - 1, -1, 1, s_0 - s + 1)$ with mass $(m, 1, 1, m)$ leads to a periodic solution only involving simultaneous binary collision if the four bodies are released with zero velocity, where $0 < s < s_0$.*

Proof. It is well known that there is a unique central configuration with the fixed order of four given masses in collinear four body problem. Because $x^0 = (-s_0 - 1, -1, 1, s_0 + 1)$ with mass $(m, 1, 1, m)$ form a central configuration (The formula of central configuration for s_0 and m is given as above), then $y^0 = (-s_0 + s - 1, -1, 1, s_0 - s + 1)$ with mass $(m, 1, 1, m)$ can not lead to a total collision with $0 < s < s_0$. By symmetry, y^0 only can lead to either a single binary collision first between m_2 and m_3 or a simultaneous binary collision first. By lemma 3, if y^0 leads to a simultaneous binary collision first, then $y^0 = (-s_0 + s - 1, -1, 1, s_0 - s + 1)$ leads to a periodic solution only involving simultaneous binary collision.
Claim:For $0 < s < s_0$, y^0 can not lead to a single binary collision between m_2 and m_3.
At x^0, the accelerations of the four particles are respectively,

$$ax_1 = m_2 s_0^{-2} + m_3(s_0 + 2)^{-2} + m_4(2s_0 + 2)^{-2},$$

$$ax_2 = -m_1 s_0^{-2} + \frac{m_3}{4} + m_4(s_0 + 2)^{-2},$$

$$ax_3 = -m_1(s_0 + 2)^{-2} - \frac{m_2}{4} + m_4 s_0^{-2},$$

$$ax_4 = -m_1(2s_0 + 2)^{-2} - m_2(s_0 + 2)^{-2} - m_3 s_0^{-2}.$$

They lead to a total collision. When the initial condition changes to $y^0 = (-s_0+s-1, -1, 1, s_0-s+1)$, the accelerations of the four particles are respectively,

$$ay_1 = m_2(-s_0 + s)^{-2} + m_3(s - s_0 - 2)^{-2} + m_4(2(s_0 - s) + 2)^{-2},$$

$$ay_2 = -m_1(-s_0 + s)^{-2} + \frac{m_3}{4} + m_4((s - s_0 - 2)^{-2},$$

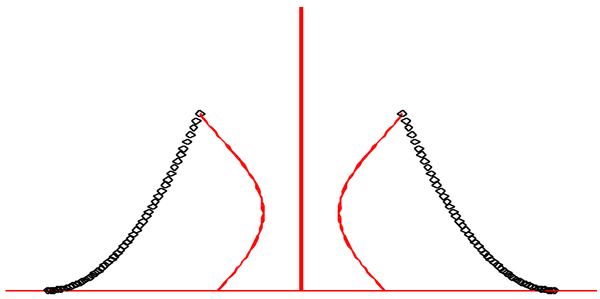
$$ay_3 = -m_1(s - s_0 - 2)^{-2} - \frac{m_2}{4} + m_4(-s_0 + s)^{-2},$$

$$ay_4 = -m_1(2(s_0 - s) + 2)^{-2} - m_2(s_0 + 2)^{-2} - m_3 s_0^{-2}.$$

By direct computation, it is easy to see $0 < ax_1 < ay_1$ but $ax_2 > ay_2$ and symmetrically for other two bodies. This implies that m_1 and m_2 shall collide before m_2 and m_3 collides by comparing this motion with the motion having total collision. So the motion with initial position y^0 and zero initial velocity can not have a single binary collision first between m_2 and m_3. By symmetry, it must lead to a simultaneous binary collision. Therefore, by lemma 3, it leads to a periodic solution only involving simultaneous binary collision. Figure 5 illustrates a case that m_2 and m_3 move inside first then turn back to a simultaneous binary collision.

Figure 5: Simultaneous Binary Collision

without single binary collision



Lemma 4 *Let $x^0 = (-s_0 - 1, -1, 1, s_0 + 1)$ with mass $(m, 1, 1, m)$ form a central configuration, where $s_0 > 0$. Then there exist a unique $s_1^* > 0$, such that, for any $0 < s < s_1^*$, $y^0 = (-s_0 - s - 1, -1, 1, s_0 + s + 1)$ leads to a periodic solution involving exact one single binary collision between m_2 and m_3 before a simultaneous binary collision in $[0, T]$, where T corresponds to the first time of simultaneous binary collision after releasing the four bodies from y^0 at time $t = 0$ with zero velocity.*

Proof. Claim 1: For small $s > 0$, the motion with initial position y^0 and zero velocity at $t = 0$ has exact one single binary collision between m_2 and m_3 before a simultaneous binary collision in $[0, T]$.

Proof of the claim 1. By similar argument as in the proof of theorem 5.1, the accelerations of m_1 and m_2 at time $t = 0$ satisfy $0 < ay_1 < ax_1$ but $ax_2 < ay_2$ and symmetrically for other two bodies. This implies that m_2 and m_3 shall collide at origin before m_1 and m_2 collides by comparing the motion with total collision. So the motion with initial position $y^0 = (-s_0 - s - 1, -1, 1, s_0 + s + 1)$ has a single binary collision between m_2 and m_3 first at time $0 < t_1 < T$. It can be regularized and then the motion will continue and keep symmetry. m_1 continue to right and m_2 bounds back.

If m_1 and m_2 don't collide after t_1, then there exists a time t_2 with $t_1 < t_2$ such that m_2 turns back to right, that is, the velocity of m_2 at t_2 is zero. Let $y(t) = (y_1(t), y_2(t), y_3(t), y_4(t))$ be the solution with initial position y^0 and zero initial velocity. Comparing the orbit of $y(t)$ in $[0, t_1)$ with the orbit of $y(t')$ in $(t_1, t_2]$, we shall have $y_2(0) = y_2(t_2) < 0$ if there is no force on m_2 from m_1 and m_4. But when the position $y_2(t)$ of m_2 in $[0, t_1)$ is equal to the position $y_2(t')$ of m_2 in $[t_1, t_2)$, $\ddot{y}_2(t) > \ddot{y}_2(t') > 0$ because m_1 and m_4 are closer to m_2 at t' than at t. Therefore we have $y_2(t_2) < y_2(0) < 0$. Similarly, $0 < y_3(0) < y_3(t_2)$.

If s is small enough, m_1 can go over position $y_2(0)$ at time t_1, i.e. $-1 < y_1(t_1) < 0$. So m_1 and m_2 must collide, say at time T, after the single binary collision between m_2 and m_3 by continuity argument and by comparing with total collision. Then the collision must be a simultaneous binary collision by symmetry. The orbit can be extended to a periodic solution with exact one single binary collision (at t_1) and one simultaneous binary collision (at T) in $[0, T]$.

Claim 2: There exist a $\tilde{s} > 0$, such that the motion with initial position $y^0 = (-s_0 - \tilde{s} - 1, -1, 1, s_0 + \tilde{s} + 1)$ has at least two single binary collision between m_2 and m_3 before m_1 and m_4 are involved in any collisions.

Proof of claim 2. Consider an auxiliary system $z = (-r, z_2, z_3, r)$ with mass $(m, 1, 1, m)$ by fixing $z_1 = -r, z_4 = r$ under Newton's law. So $z_1(t) = -r, z_2(t) = r$ and z_2, z_3 are determined by the following equations,

$$\ddot{z}_2(t) = -\frac{m}{(z_2+r)^2} + \frac{1}{(z_2-z_3)^2} + \frac{m}{(z_2-r)^2} \tag{5.2}$$

$$\ddot{z}_3(t) = -\frac{m}{(z_3+r)^2} - \frac{1}{(z_3-z_2)^2} + \frac{m}{(z_3-r)^2} \tag{5.3}$$

with initial position $z(0) = (-r, -1, 1, r)$ and zero initial velocity. By the symmetry of differential equations and initial conditions, $z_3(t) = -z_2(t)$.
In fact, equation (5.2) is a Hamiltonian system with

$$H = \frac{1}{2}|\dot{z}_2|^2 + \left(-\frac{m}{(z_2+r)} + \frac{1}{4z_2} + \frac{m}{(z_2-r)}\right).$$

H is a constant along solution $z_2(t)$. At $t=0$, $z_2(0) = -1$, $\dot{z}_2(0) = 0$, then $H \equiv C = \left(-\frac{m}{(-1+r)} - \frac{1}{4} + \frac{m}{(-1-r)}\right) = -(\frac{2mr}{r^2-1} + \frac{1}{4}) < 0$. Assume that $z_2(t)$ travels from -1 to 0 in $[0, t_1]$. We have

$$dt = \frac{dz_2}{\sqrt{2\left(C - \left(-\frac{m}{(z_2+r)} + \frac{1}{4z_2} + \frac{m}{(z_2-r)}\right)\right)}}.$$

So

$$\begin{aligned} t_1(m,r) &= \int_{-1}^{0} \frac{dz_2}{\sqrt{2\left(C - \left(-\frac{m}{(z_2+r)} + \frac{1}{4z_2} + \frac{m}{(z_2-r)}\right)\right)}} \\ &= \int_{-1}^{0} \frac{dz_2}{\sqrt{2\left(C - \frac{2mr}{(z_2^2-r^2)} - \frac{1}{4z_2}\right)}} \\ &\geq \int_{-1}^{0} \frac{dz_2}{\sqrt{2\left(C - \frac{2mr}{((-1)^2-r^2)} - \frac{1}{4z_2}\right)}} \\ &= \int_{-1}^{0} \frac{dz_2}{\sqrt{2\left(-\frac{1}{4} - \frac{1}{4z_2}\right)}} = \frac{\sqrt{2}\pi}{2} \end{aligned}$$

From (5.1), the acceleration of m_2 can be always positive if r is large, in fact,

$$\begin{aligned} \ddot{z}_2(t) &= -\frac{m}{(z_2+r)^2} + \frac{1}{(z_2-z_3)^2} + \frac{m}{(z_2-r)^2} \\ &= \frac{2mrz_2}{(z_2^2-r^2)^2} + \frac{1}{4z_2^2} \geq \frac{-2mr}{(1-r^2)^2} + \frac{1}{4z_2^2} > 0 \end{aligned}$$

if $m < \frac{(1-r^2)^2}{8r}$ and $z_2 \in [-1, 0]$. Then we have

$$t_1(m,r) \leq \int_{-1}^{0} \frac{dz_2}{\sqrt{2\left(C - \frac{2mrz_2}{(1-r^2)} - \frac{1}{4z_2}\right)}} < \infty. \tag{5.4}$$

The above integral is the time that the z_2 moves from -1 to 0 with the smaller acceleration $\frac{-2mr}{(1-r^2)^2} + \frac{1}{4z_2^2} > 0$. For any finite time T' with $3t_1(m,r) < T' < \infty$, there exists a large r such that m_2 and m_3 collide at origin at least two times in the finite time interval $[0, T']$.

Consider another auxiliary system $w = (w_1, -r, r, w_4)$ with mass $(m, 1, 1, m)$ by fixing $w_2 = -r, w_3 = r$ under Newton's law. So w_1, w_4 are determined by the following equations,

$$\ddot{w}_1(t) = \frac{1}{(w_1 + r)^2} + \frac{1}{(w_1 - r)^2} + \frac{m}{(w_1 - w_4)^2} \tag{5.5}$$

$$\ddot{w}_4(t) = -\frac{1}{(w_4 - r)^2} - \frac{1}{(w_4 + r)^2} - \frac{m}{(w_4 - w_1)^2} \tag{5.6}$$

with initial positions $(-\tilde{s}, -r, r, \tilde{s})$ and zero initial velocity. Within the motion of w_1 from $\tilde{s}$ to $-r$, we have

$$\ddot{w}_1(t) \leq \frac{k}{(w_1 + r)^2}$$

because $(w_1 + r)^2 < (w_1 - r)^2 < 4w_1^2$, where $k = \max\{1, m\}$. By the similar argument as $t_1(m, r)$, the time t_2 for the motion of m_1 from $-\tilde{s}$ to $-r$ has

$$t_2(m, \tilde{s}) \geq \int_{-\tilde{s}}^{-r} \frac{dw_1}{\sqrt{2(-C_2 - \frac{k}{w_1 + r})}} = \frac{k\pi}{(2C_2)^{3/2}} = \frac{\pi}{2\sqrt{2k}}(\tilde{s} - r)^{3/2}, \tag{5.7}$$

where $C_2 = \frac{k}{\tilde{s} - r}$. After comparing (5.4) with (5.7), we know that if $\tilde{s}$ is large enough, w_1 can not across $-r$ and w_4 can not across r in the finite time $[0, T']$.
Now for the large $\tilde{s}$, the motion $y(t) = (y_1(t), y_2(t), y_3(t), y_4(t))$ with initial position $y^0 = (-s_0 - \tilde{s} - 1, -1, 1, s_0 + \tilde{s} + 1)$ and zero initial velocity has the following properties. (1) In the finite time T', y_1 can not across $-r$ and y_4 can not across r by comparing with the auxiliary system $w(t)$ because $0 < \ddot{y}_1(t) < \ddot{w}_1(t)$. and $0 > \ddot{y}_4(t) > \ddot{w}_4(t)$. (2) In the finite time T', y_2 and y_3 should collide at least two times by comparing with the auxiliary system $z(t)$.

Claim 3: If the motion with initial position $y^1 = (-s_0 - s_1 - 1, -1, 1, s_0 + s_1 + 1)$ and zero initial velocity has n times single binary collision between m_2 and m_3 in [0,T], where T corresponds to the first time of simultaneous binary collision, then $y^2 = (-s_0 - s_2 - 1, -1, 1, s_0 + s_2 + 1)$ leads to at least n times single binary collision before simultaneous binary collision for $0 < s_1 < s_2$ in $[0, T]$.
Proof of Claim 3. At time $t = 0$, $\ddot{y}_1^1 > \ddot{y}_1^2 > 0$ but $0 < \ddot{y}_2^1 < \ddot{y}_2^2$, this implies that y_1^2 goes slower than y_1^1 but y_2^2 goes faster than y_1^2. Therefore, y^2 takes shorter time to have the first single binary collision between m_2 and m_3. By similar argument, we can prove that y^2 also take shorter time to have the second binary collision between m_2 and m_3. Then in $[0, T]$, y^2 has at leat n times single binary collision before simultaneous binary collision.

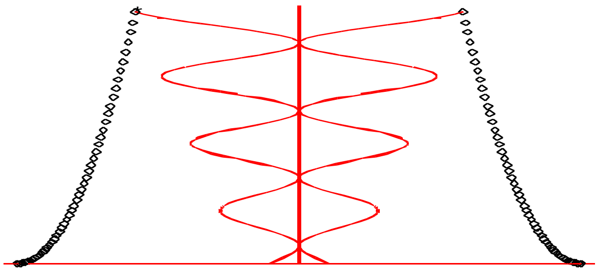
Now assume that $s_1^* = \sup\{s > 0 : y^0 = (-s_0 - s - 1, -1, 1, s_0 + s + 1)$ *leads to a periodic solution involving only one single binary collision between m_2 and m_3 and one simultaneous binary collision in one period.* $\}$

The claim 1 proves the existence of s_1^*. The claim 2 and claim 3 prove s_1^* is finite and unique. This completes the proof of lemma 4.♯

Theorem 5.2 *Let $x^0 = (-s_0 - 1, -1, 1, s_0 + 1)$ with mass $(m, 1, 1, m)$ form a central configuration, where $s_0 > 0$ is implicitly defined in (5.1).*
(1) Then there exists a sequence $0 < s_1^* < s_2^* < \cdots$ such that the motion has exact n times single binary collision between m_2 and m_3 before a simultaneous binary collision in $[0, T]$, where

T corresponds to the first time of simultaneous binary collision, if the motion starts with initial position $y^0 = (-s_0 - s_n - 1, -1, 1, s_0 + s_n + 1)$ and zero initial velocity, where $s^*_{n-1} < s_n < s^*_n$. (2)If four particles are released from $y^0 = (-s_0 - s^*_n - 1, -1, 1, s_0 + s^*_n + 1)$ with zero initial velocity, $n = 1, 2, \cdots$, then the motion ends at a total collision after n times single binary collision between m_2 and m_3.

Figure 6: Simultaneous Binary Collision
with 4 single binary collisions



Proof. The proof can be done by induction base on the proof of lemma 4. Figure 6 illustrates an example that m_2 and m_3 collide 4 times before a simultaneous binary collision.

Remark: Let $x^0 = (-s_0 - 1, -1, 1, s_0 + 1)$ with mass $(m, 1, 1, m)$ form a central configuration, where $s_0 > 0$. Assume $y^0 = (-s_0 - s_n - 1, -1, 1, s_0 + s_n + 1)$ denote the initial position of the collinear four bodies, where $s^*_{n-1} < s_n < s^*_n$. Let $y(t) = (y_1(t), y_2(t), y_3(t), y_4(t))$ be the periodic solutions involving single binary collisions and simultaneous binary collisions with $y(0) = y^0$ and $\frac{dy}{dt}(0) = 0$. Then there exist a time sequence $0 = t_1 < t_2 < \cdots < t_n < T$ such that $y_2(t_{i+1}) \le y_2(t_i) < 0, 0 < y_3(t_i) \le y_3(t_{i+1})$, and $\frac{dy_j}{dt}(t_i) = 0$, where $j = 2, 3, i = 1, \cdots, n-1$. Figure 6 illustrates an example that m_3 goes further each time.

References

[1] E. A. Belbruno, The two-body problem under the inverse square central force and equivalent geodesic flows, Celestial mech. 15 (1977), 467-476

[2] E. A. Belbruno, On simultanenous double collision in the collinear four-body problem, J. Differential Equations 52 (1984), 415-431

[3] R. Easton, Regularization of vector fields by surgery, J. Differential Equations 10 (1971), 92-99

[4] Kenneth Meyer and Quidong Wang, The Collinear Three-Body Problem with Negative Energy, Journal of Differential Equations 119 (1995), P284-309

[5] Regina Martinez and Carles SimÓ, Simultaneous binary collisions in the planar four-body problem, Nonlinearity 12(1999) 903-930

[6] J. Moser, Regularization of Kepler's Problem and the Averaging Method on a Manifold, Comm. Pure Appl. Math. 23 (1970), 609-636

[7] R. McGehee, Triple collision in the collinear three-body problem, Inv. Math. 27(1974) 191-227

[8] R.McGehee, Double collisions for a classical particle system with nongravitational interactions, Comment. Math. Helvetici 56 (1981), P524-557

[9] Tiancheng Ouyang, Zhifu Xie, Collinear Central Configuration in Four-Body Problem, Celestial Mechanics and Dynamical Astronomy, Vol.93 (2005),No.1 p147-166

[10] Predrag Punoševac and Qiudong Wang, Regularization of Simultaneous Binary Collisions in Some Gravitational Systems, preprint

[11] C.L. Siegel and J.K. Moser, Lecture on celestial mechanics, Springer-Verlag, (1971)

[12] C. Simo and E. Lacomba, Regularization of simultaneous binary collisions in the n-body problem, J. Differential equations 98 (1992), 241-259

[13] E. Stiefel and G. Scheifele, Linear and Regular Celestial Mechanics, Springer-Verlag, New York/Heidelberg/Berlin, 1971