Research Summary and Future Plans

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My research takes on several directions in the area of celestial mechanics and nonlinear partial differential equations of elliptic type. I have been working on (1a) central configurations of the N-body problem, (1b) collisions and regularization, (1c) stability for Kepler solutions, (1d) index theory for symplectic paths in Hamiltonian system and stability of periodic solutions, and (2a) the asymptotic behavior of large positive solutions for nonlinear elliptic partial differential equations, (2b) steady state solutions and dynamics of singular problems from mathematical biology, such as Turing's diffusion-induced instability for the cross-diffusion systems. I will describe in detail in Section 1 and Section 2. In section 3, I discuss my research plan for the future.

1 Celestial Mechanics

My main research in celestial mechanics centers around the N-body problem, which involves central configurations, collisions and regularization, stability for Kepler solutions, index theory for symplectic paths in Hamiltonian system and stability of periodic solutions.
The $N-$body problem describes a system of $N \geq 2$ point masses interacting according to Newton's law of universal gravitation:

$$m_j\ddot{q}_j = \mathbf{F_j} = \sum_{k=1,k\neq j}^{N} \frac{m_j m_k(q_k - q_j)}{|q_k - q_j|^3} \qquad 1 \leq j \leq N, \tag{1}$$

where $q_j \in \mathbb{R}^d$ $(1 \leq d \leq 3)$ is the position with mass m_j.
The N-body problem is one of the oldest and the most fascinating problems in mathematics. It is an active research area of great potential using the power of mathematics to study challenging celestial mechanics problems. Many important results were obtained during the last three centuries by L.Euler, L.Lagrange, H.Poincaré, G.D.Birkhoff, V.I.Arnold and many others. Despite their best efforts, the problem of solving the N-body problem is far from complete.

1.1 Central Configuration

Simple solutions can be obtained from *central configuration* (C.C.). A central configuration is defined as a configuration of the N bodies such that $\ddot{q}_j = -\lambda q_j, j = 1, \cdots, N$ for some $\lambda > 0$ independent of j, i.e.

$$\sum_{k=1,k\neq j}^{N} \frac{m_k(q_k - q_j)}{|q_k - q_j|^3} = -\lambda q_j, \qquad 1 \leq j \leq N. \tag{2}$$

A great deal of effort has been devoted to understanding the properties of central configurations. The finiteness of the number of distinct C.C. is one of such important problems in $N-$body problem. In 1991, Steve Smale [15] described the finiteness problem as one of the eighteen great problems not solved in the 20th century. Also V.I. Arnold [16], on behalf of the International Mathematical Union, described the finiteness problem as one of great problems for the 21st century. A complete understanding of the nature of central configuration is of fundamental importance to the N-body problem of celestial mechanics. These configurations play an essential role in the global structures of the solutions of the $N-$body problem.
A collinear central configuration is called a *Moulton configuration* after F.R. Moulton [6] who proved that for a fixed mass vector $m = (m_1, \cdots, m_n)$ and a fixed ordering of the bodies along the line, there exists a unique collinear central configuration (up to translation and scaling). Moulton also considered the inverse

problem for the collinear central configuration: Given a configuration, find the mass vectors, if any, for which it is a central configuration. His results depend on whether n is even or odd. Much more recently, Albouy and Moeckel [1] proved that for $n \leq 6$, each configuration determines a one-parameter family of masses (negative masses are allowed) (after normalization of the total mass) and the parameter is the center of mass when n is even and the square of the angular velocity of the corresponding circular periodic orbit when n is odd. For $n \geq 7$, it is still open.
T. Ouyang and I [9] have established an explicit expression for the masses of the central configurations for the collinear four body problem and have found a collection of configurations none of which can form a central configuration for any *positive* masses.

1.2 Regularization of Singularity of Collisions and Periodic Solutions with Collisions

Although the collinear $N-$body problem has a simpler structure than the general case, the dynamical behavior of the collinear $N-$body problem (even for four body) can be very complicated. Thirty years ago, Mather and McGehee [5] gave a proof of the existence of a motion of the collinear four body problem, such that the motion becomes unbounded in finite time by a sequence of binary collisions. If two or more bodies go to same position as $t \rightarrow t^*$, the Newtonian system (1) experiences a singularity of collision. We consider the regularization of singularity of collisions in collinear four-body problem. Let $q(t)$ be a solution of (1) and have a singularity of collision at t_2, if there exists a time rescaling $t = t(\tau)$ such that $q(t(\tau))$ can be extended beyond t_2, the singularity of collision is regularizable.
T. Ouyang and I [10] construct coordinate transforms in new time scale that remove the singularities of simultaneous binary collision (SBC) in collinear four-body problem without any assumption on mass. The regularization is at least of class C^2. Furthermore, the extension in the new time scale can be described as time reverse in a neighborhood of the singular time. Based on the results of regularization of SBC, we construct two types of periodic solution with collisions: (A) periodic solution only involving simultaneous binary collision; (B) periodic solution involving n times single binary collision and one simultaneous binary collision in one period. These periodic obits are bifurcated from a central configuration.

1.3 Linear Instability of Kepler Orbits in the Rhombus Four-Body Problem

T. Ouyang and I [11] study the instability of Kepler orbits for the rhombus four body problem as an example. If a central configuration is obtained for given four proper masses with each body at the vertex of a rhombus in the plane, there exists a family of periodic solutions for which each body is at the vertex of a rhombus and travels along an elliptic Kepler orbit. The difficulty of studying the stability of the periodic solution lies in the high dimension. Instead of studying the 8 degrees of freedom Hamilton system for the planar four-body problem, we reduce the degree of freedom by means of some symmetries to derive a two degrees of freedom system. The new system with two degrees then can be used to determine the linear instability of the periodic solutions. After making a clever change of coordinates, a two dimensional ordinary differential equation system is obtained, which governs the linear instability of the periodic solutions. The system is surprisingly simple and depends only on the length of the sides of the rhombus and the eccentricity e of the Kepler orbit. We have proved that the family of Kepler periodic solutions is linear unstable.

1.4 Index Theory for Symplectic Paths and Stability for Periodic Solutions

The variational structure of Hamiltonian systems was observed more than one hundred years ago. Morse-type index, Maslov index and Conley index have been applied to the variational study of Hamiltonian systems and closed geodesics on Riemannian manifolds. We apply index theory for symplectic matrix paths introduced by Y. Long [4] to study the stability of a periodic solution x for a nonlinear Hamiltonian system, particularly for N-body problem. In [12], Ouyang and I first study the topological properties of the symplectic group $Sp(2n) = \{M \in GL(\mathbb{R}^{2n}) | M^T JM = J\}$. An index function is defined for each symplectic path $\gamma(t)$ starting from identity. Because the fundamental solution of the linearized Hamiltonian system along the periodic solution x is a symplectic path (called the associated symplectic path of the periodic solution), an index $ind(x)$ is defined for x by the index of its associated symplectic path. Then the periodic solution x for a two dimensional Hamiltonian system is linear stable if and only if its index $ind(x)$ is an odd integer. The index can also be expressed in terms of Morse index. For higher dimension, the conditions for stability and instability are more complicated. We establish some relations between the stability and index theory.

2 Nonlinear Partial Differential Equations and Applications

2.1 Blow-up rate of large positive solutions

We consider a semilinear elliptical partial differential equation with singular boundary value:

$$-\triangle u = \lambda u - b(x)u^p, x \in \Omega; \qquad u|_{\partial\Omega} = +\infty, x \in \partial\Omega \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^N, N \geq 1$, with boundary $\partial\Omega$ of class C^2, $\triangle$ is the Laplacian operator in $\mathbb{R}^N$, $\lambda \in L_\infty(\Omega), p > 1$ and $b \in C(\Omega; \mathbb{R}^+)$. The boundary condition in (3) is understood as $u(x) \to +\infty$ when $d(x) := dist(x, \partial\Omega) \to 0^+$.
Equation (3) have their roots from many mathematical and physical fields, e.g., the well-known scalar curvature equation in the study of Riemannian geometry, the scalar field equation for the standing wave of nonlinear Schrödinger and Klein-Görden equations, the Matukuma equation describing the dynamics of globular cluster of stars, the evolution of a single species obeying a generalized logistic growth law, etc. Singular boundary value problem (3) has a long history. Considerable amounts of study have been attracted by such problems. However the blow-up rate of solution near $\partial\Omega$ and uniqueness of solutions for the singular problem (3) are the goal of more recent literature.
In [13] Ouyang and I consider the singular boundary value problem (3) with a ball domain and radial function $b(x)$. In this case, we establish the blow up rate of the large positive solutions of equation (3) in [13]. All previous results in the literature assumed the decay rate of $b(x)$ to be approximated by a distance function near the boundary $\partial\Omega$. We assume the potential function $b(x) \in C(\Omega)$, where Ω is a ball $B_R(x_0)$ satisfying $b(x) = b(\|x - x_0\|)$ and $b(x) > 0 \in \Omega$. Then $b(r)$ is a real function defined on $[0, R]$. We also assume $\frac{B(r)}{b(r)} \in C^1([0, R])$ and $\lim_{r \to R} \frac{B(r)}{b(r)} = 0$, where $B(r) = \int_r^R b(s)ds$. Then a very accurate blow up rate of solution is established without any further assumption on the decay rate of $b(x)$ near the boundary $\partial\Omega$. Furthermore, the existence and uniqueness of the large positive solution to the singular boundary value problem (3) have also been proved.

2.2 Turing type instability in reaction-diffusion systems

A pure diffusion process usually leads to a smoothering effect so that the system tends to a constant equilibrium state. Turing [2] suggested that, under certain conditions, chemicals can react and diffuse in such a way as to produce non-constant equilibrium solutions, which represent spatial patterns of chemical or morphogen concentration. He considered a reaction-diffusion system

$$\begin{cases} u_t = D_u \Delta u + f(u, v), & t > 0, \\ v_t = D_v \Delta v + g(u, v), & t > 0, \end{cases} \tag{4}$$

and its corresponding kinetic equation

$$(5) \qquad \begin{cases} u' = f(u,v), & t > 0, \\ v' = g(u,v), & t > 0. \end{cases}$$

He said that if, in the absence of the diffusion (considering (5)), u and v tend to a linearly stable uniform steady state, then, with the presence of diffusion and under certain conditions, the uniform steady state can become unstable, and spatial inhomogeneous patterns can evolve through bifurcations. In another word, a constant equilibrium can be asymptotically stable with respect to (5), but it is unstable with respect to (4). Therefore this constant equilibrium solution becomes unstable because of the diffusion, which is called *diffusion driven instability.*
J. Shi and I [14] further explore Turing's diffusion-induced instability for the cross-diffusion systems. Our main results following Turing's idea can be summarized as follows: assume that in the absence of diffusion and cross-diffusion, there is a spatial homogeneous steady state; in the presence of diffusion but not cross-diffusion, this steady state remains stable hence it does not belong to the classical Turing instability scheme, but it could become unstable when cross-diffusion also comes to play a role in the system; thus it is a *cross-diffusion induced instability.* On the other hand, if Turing instability does occur, *i.e.* a spatial uniform steady state is stable with respect to the diffusion-free system, and it is unstable when diffusion (but not cross-diffusion) presents; this steady state could become stable with the inclusion of cross-diffusion influence, which represents a *cross-diffusion induced stability.* We apply our general analysis to a reaction-diffusion system modeling vegetation patterns and desertification by J. von Hardenberg, E. Meron, etc in [18] and [7].

3 Plans for Future Research

In the near future, I plan to further investigate the N-body problem, in particular, on the central configurations, the linear stability for periodic solutions of the N-body problem, index theory of symplectic paths for Hamiltonian system, and also variational problems. I also plan to continue with study on nonlinear elliptic equations.
1. *Classification of Central Configurations* We are interested in the problem of the finiteness of the number of central configurations. We want to find all possible central configurations for 4-body problem by studying the classification of central configuration including collinear and planar cases as follows. We will firstly consider the central configuration for four bodies in plane assuming that two masses are equal and that the positions of the four bodies are convex. By setting up a special symmetry, we will show there is only one symmetric central configuration. Then change one of the masses continuously to any finite positive value. As that mass changes continuously, the corresponding positions also change so that the total four bodies still form a central configuration. If there are no bifurcations from the case in which two masses are equal, then the number of central configuration is the same as the number of central configuration with two equal masses. If there are bifurcations, the number of central configurations is determined by the bifurcations and the number of central configurations with two equal masses.
2. *Dynamics of the Collinear Four-body Problem* Relate to prior work, we will find explicitly initial conditions which lead to periodic solutions, bounded solutions, or unbounded solutions in infinite time or unbounded solutions in finite time (like Mather and McGehee's solution). One difficulty is that there is no explicit formula for such conditions. We will further study the dynamics of the collinear four-body problem. We will lead more discussion of orbits involving single binary and simultaneous binary collisions. This would be interesting if it turns up any surprises in the dynamics of the problem. We also want to generalize the results to planar four body problem or the general collinear $N-$body problem.
3. *Stability and Instability on Reduced Hamiltonian System* The approach of reducing the degrees of freedom can be generalized for Hamiltonian systems with some constraints under some symmetries. Let

Hamiltonian system H_R be obtained from Hamiltonian system H by the reduction of degrees of freedom. If a solution x in H could be converted to a solution x_R in H_R, it will be of interest to determine how the properties of the solution x in original system H affect the properties of the solution x_R in the system H_R. For example, does the stability of x_R in H_R imply the stability of x in H? I will collaborate with Guangyu Zhao on this question.

4. *Index Theory and Stability* I will collaborate with Daniel Offin to study the relation between stability and Maslov index. I look forward to using existing theoretical techniques and developing new ones to advance our understanding of dynamical behavior in celestial mechanics, particularly to use index theory or variational methods to explain the stability or instability of certain periodic solutions.

5. *Blow-up rate of large positive solution* We are working on the existence, uniqueness, and blow-up rate of the large solution to the semilinear elliptic equation (3) in a general domain and for a general function $H(u)$ instead of $H(u) = u^p$. We apply the result to study the asymptotic behavior of the positive solutions of logistic population model.

6. *Turing reaction-diffusion system with advection* J. Shi and I in [14] established the affection of the cross-diffusion without advection. We are also interested in find the effect of the advection (simulating the effect of hill slopes) on the pattern. J. von Hardenberg, E. Merton, etc. have observed that the model predicts the existence of wide precipitation ranges where different stable states coexist, e.g. a bare soil state and a pot pattern, a spot pattern and a strip pattern, and so on. We expect to give a mathematical explanation by using analysis and abstract bifurcation theory.

I have undertaken a few of the initial research directions I am interested in. These are starting points for exploration, and many interesting and unexpected research challenges are sure to arise during the process. I hope to extend my research even further in the future. I have benefited considerably from continual collaboration with my advisors, and I will continue to collaborate with researchers in these fields.

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