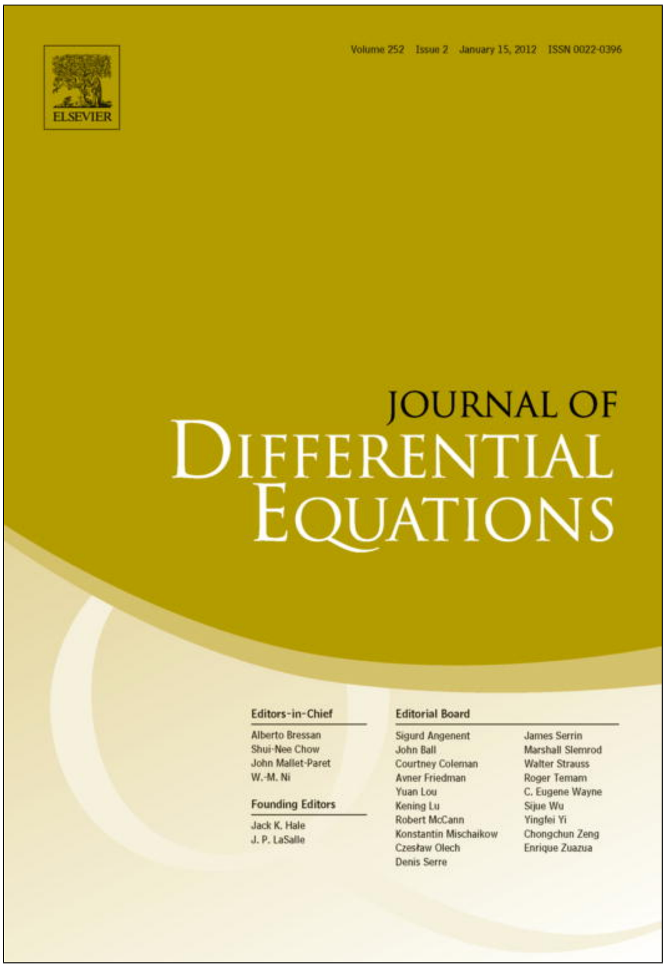
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# Blow-up rate and uniqueness of singular radial solutions for a class of quasi-linear elliptic equations

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ABSTRACT

We establish the uniqueness and the blow-up rate of the large positive solution of the quasi-linear elliptic problem $-\Delta_p u = \lambda u^{p-1} - b(x)h(u)$ in $B_R(x_0)$ with boundary condition $u = +\infty$ on $\partial B_R(x_0)$, where $B_R(x_0)$ is a ball centered at $x_0 \in \mathbb{R}^N$ with radius $R$, $N \geqslant 3$, $2 \leqslant p < \infty$, $\lambda > 0$ are constants and the weight function $b$ is a positive radially symmetrical function. We only require $h(u)$ to be a locally Lipschitz function with $h(u)/u^{p-1}$ increasing on $(0, \infty)$ and $h(u) \sim u^{q-1}$ for large $u$ with $q > p - 1$. Our results extend the previous work [Z. Xie, Uniqueness and blow-up rate of large solutions for elliptic equation $-\Delta u = \lambda u - b(x)h(u)$, J. Differential Equations 247 (2009) 344–363] from case $p = 2$ to case $2 \leqslant p < \infty$.



## 1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geqslant 2)$ be a smooth bounded domain. We consider the uniqueness and the blow-up rate of the large solutions of the quasi-linear elliptic problem with singular boundary value condition as follows:

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} - b(x)h(u) & \text{in } \Omega, \quad \text{(a)} \\ u \geqslant 0 & \text{in } \Omega, \quad \text{(b)} \\ u = \infty & \text{on } \partial\Omega, \quad \text{(c)} \end{cases} \tag{1.1}$$

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where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ with $Du(x) = (\partial_{x_1}u, \partial_{x_2}u, \ldots, \partial_{x_N}u)$, $p \in [2,\infty)$ and $\lambda > 0$ are constants, $b(x) \geqslant 0$ is a weighted function which we will explain later, and the boundary condition (1.1c) is understood as $u(x) \to +\infty$ when $d(x) := dist(x, \partial\Omega) \to 0^+$. The solutions of (1.1a–c) are called *large* (or *blow-up*) *solutions*.

The problem (1.1a–c) appears in the study of non-Newtonian flows, chemotaxis, and biological pattern formation etc. For example, in the study of non-Newtonian flows, the constant $p$ in (1.1a) is a characteristic of medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudo-plastics. If $p = 2$ they are Newtonian fluids (see [8] and the references therein). Especially when $p = 2$, the problem (1.1a–c) becomes as follows:

$$\begin{cases} -\Delta u = \lambda u - b(x)h(u) & \text{in } \Omega, \quad \text{(a)} \\ u \geqslant 0 & \text{in } \Omega, \quad \text{(b)} \\ u = \infty & \text{on } \partial\Omega, \quad \text{(c)} \end{cases} \tag{1.2}$$

and it has been studied extensively. Next let us recollect some related results. In 1916, Bieberbach [3] studied the large solutions for the particular case $-\Delta u = -\exp(u)$ with conditions (1.2b–c) in smooth bounded two-dimensional domains, and showed that there exists a unique solution such that $u(x) - \log(d(x)^{-2})$ is bounded as $x \to \partial\Omega$. Problems of this type also arise in Riemannian geometry. For example, if a Riemannian metric of the form $|ds|^2 = \exp(2u(x))|dx|^2$ has constant Gaussian curvature $-c^2$, then $-\Delta u = -c^2\exp(2u)$. Motivated by a problem in mathematical physics, Rademacher [24] continued the study of the large solutions for the particular case $-\Delta u = -\exp(u)$ on smooth bounded domains in $\mathbb{R}^3$. Bandle and Essén [1] and Lazer and Mckenna [15] extended Bieberbach's and Rademacher's results to general case $-\Delta u = -b(x)\exp(u)$ in bounded domains of $\mathbb{R}^N$ satisfying a uniform external sphere condition, where the function $b(x)$ is continuous and strictly positive on $\overline{\Omega}$, the closure of $\Omega$. It was shown that the problem has a unique solution together with an estimate of the form $u = \log d^{-2} + o(d)$, see [1] for case $b \equiv 1$ and [15] for case $b(x) \geqslant b_0 > 0$ as $d \to 0$.

Recently, the uniqueness of solutions for (1.2a–c) with $h(u) = u^q$ $(q > 1)$ on bounded domains or the whole space $\mathbb{R}^N$ was discussed in many papers (see e.g., [10–26]). The results can be summarized as follows: under the assumption

$$\lim_{d(x,\partial\Omega)\to 0} \frac{b(x)}{d(x,\partial\Omega)^\gamma} = \zeta$$

with $\gamma > 0$ and $\zeta > 0$, an explicit expression for the blow-up rates of solutions of (1.2a–c) was obtained in [10] and [12] as $u = (\alpha(\alpha+1)/\zeta)^{1/(q-1)}d^{-\alpha}(1+o(d))$, $\alpha = (\gamma+2)/(q-1)$. By using the localization method of [17], it was shown in [17,22] that (1.2a–c) with $h(u) = u^q$, $q > 1$ has at most one blow-up solution for the case when $\gamma$ and $\zeta$ vary along $\partial\Omega$. Further improvements of these results can be found in [5,6,18,21,23,25,26] and the references therein.

The radial case of the problem (1.2a–c) on a ball domain $B_R(x_0)$ with $h(u) = u^q$ was studied by López-Gómez [18], and the author obtained the existence and uniqueness of a solution and also established the exact boundary blow-up rate for less restrictions on the weight function $b$, which is a positive non-decreasing function with $b(0) = 0$, $b'(r) \geqslant 0$. The author also extended the results to a general domain by adopting the localization method [17]. Later on, Cano-Casanova and López-Gómez improved the results in [18] for $h(u) = u^q$ to a general function $h(u)$ which satisfies the Keller–Osserman condition [14,20] and $h(u) \sim Hu^q$ ($H > 0$ is a constant and $q > 1$) for sufficiently large $u$ [5,6]. In [21], the authors also considered the problem (1.2a–c) with $h(u) = u^q$ on a ball domain $B_R(x_0)$, but the decay rate of the weight function $b(x)$ was not assumed to be approximated by a distance function near the boundary $\partial\Omega$, i.e., no assumption as $b \sim C_0 d^\gamma + o(d^\gamma)$ or $b(r)$ is a non-decreasing function was needed. They only assumed that $b(x) = b(\|x - x_0\|)$ was a radially symmetric continuous function on a ball $B_R(x_0)$ such that $B(r)/b(r) \in C^1([0,R])$ and $\lim_{r\to R} B(r)/b(r) = 0$, where $B(r) = \int_r^R b(s)\,ds$. Uniqueness and blow-up rates of solution of (1.2a–c) in general domains was also obtained in [22] by combining the localization method with the results in [21]. Also see [25,26] for more results in the direction of general function $h(u)$ in (1.2a–c).

It is often important to know what properties are retained when linear diffusion ($p = 2$) which corresponds to the Laplace operator is replaced by nonlinear diffusion ($p \neq 2$) which corresponds to the degenerate $p$-Laplace operator. We want to point out that it is not always possible to extend the results from Laplace operator to the degenerate $p$-Laplace operator (as many examples have already demonstrated); and even if such an extension is possible, one usually has to overcome many non-trivial technical difficulties since many nice properties inherent to the Laplace operator seem lost or difficult to verify once $p \neq 2$. We refer the readers to [9] for the existence of large positive solutions of the problem (1.1a–c). In this paper, we are interested in the uniqueness and the blow-up rate of solutions to the problem (1.1a–c). Our main theorem extends the results obtained in [25] for the problem (1.2a–c) to the case $2 \leqslant p < \infty$ and can be stated as follows.

**Theorem 1.1.** *Consider the radially symmetric quasi-linear elliptic equation*

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} - b(\|x - x_0\|)h(u) & \text{in } B_R(x_0), \quad \text{(a)} \\ u \geqslant 0 & \text{in } B_R(x_0), \quad \text{(b)} \\ u = \infty & \text{on } \partial B_R(x_0), \quad \text{(c)} \end{cases} \tag{1.3}$$

*where $B_R(x_0)$ is the ball of radius $R$ centered at $x_0 \in \mathbb{R}^N$, $N \geqslant 2$, $2 \leqslant p < \infty$ and $\lambda > 0$. The weight function $b(r)$ satisfies:*

*($\mathcal{A}1$) $b \in C([0, R], (0, \infty))$ satisfies $B(r)/b(r) \in C^1([0, R])$, and*

$$C_0 := \lim_{r \to R} \frac{(B(r))^2}{b^*(r)b(r)} \geqslant 1, \tag{1.4}$$

*where $B(r) = \int_r^R b(s)\,ds$ and $b^*(r) = \int_r^R B(s)\,ds$. Denote $b_0 = b(R) > 0$.*

*The nonlinear function $h(u)$ satisfies:*

*($\mathcal{A}2$) $h(u) \geqslant 0$ is locally Lipschitz continuous on $[0, \infty)$ and $h(u)/u^{p-1}$ is increasing on $(0, \infty)$; and, for some $q > p - 1 > 1$,*

$$H := \lim_{u \to \infty} \frac{h(u)}{u^q} > 0. \tag{1.5}$$

*Then for any solution $u(x)$ of (1.3a–c),*

$$\lim_{d(x) \to 0} \frac{u(x)}{KH^{-2\beta/p}(b^*(\|x - x_0\|))^{-\beta}} = 1, \tag{1.6}$$

*where $d(x) = dist(x, \partial B_R(x_0))$ and $K$ is a constant defined by*

$$K = \left[(p-1)\beta^{p-1}\big((\beta+1)C_0 - 1\big)(b_0 C_0)^{p/2-1}\right]^{2\beta/p}, \quad \beta = \frac{p}{2(q-p+1)}.$$

*Therefore problem (1.3a–c) possesses a unique positive large solution $u(x)$ in $B_R(x_0)$.*

## 2. Some preliminary results

At first let us present some lemmas which will be used in the proof of Theorem 1.1. The following lemmas are mainly from [9] with some notation modifications. Similar results for $p = 2$ can also be found in [7,10,9,17,19].

Consider the problem

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} - b(x)h(u) & \text{in } \Omega, \\ u = \phi \geqslant 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$ with $N \geqslant 2$, $\phi \in C^1(\partial\Omega)$, $h$ satisfies $(\mathcal{A}2)$ and $b \in C(\Omega, \mathbb{R}^+)$.

**Lemma 2.1.** *If $\underline{u}, \bar{u} \in C^1(\Omega)$ are both positive in $\Omega$ such that*

$$\begin{cases} -\Delta_p \underline{u} \leqslant \lambda \underline{u}^{p-1} - b(x)h(\underline{u}) & \text{in } \Omega, \\ -\Delta_p \bar{u} \geqslant \lambda \bar{u}^{p-1} - b(x)h(\bar{u}) & \text{in } \Omega \end{cases} \tag{2.2}$$

*hold in the sense of distribution, and $\underline{u} \leqslant \phi \leqslant \bar{u}$ on $\partial\Omega$, then $\underline{u}(x) \leqslant \bar{u}(x)$ on $\overline{\Omega}$.*

**Remark.** We refer the readers to [13] and [16] for maximum and comparison principles for elliptic equations involving $p$-Laplacian. This lemma can be proved similarly to the proof of Lemma 1.1 in [19] (see also [9] and [4]), that goes back to Benguria, Brezis and Lieb [2]. For convenience, next we give a proof.

**Proof of Lemma 2.1.** By the hypotheses in (2.2), we have

$$\Delta_p \underline{u} + \lambda \underline{u}^{p-1} - b(x)h(\underline{u}) \geqslant 0 \quad \text{in } \Omega \tag{2.3}$$

and

$$\Delta_p \bar{u} + \lambda \bar{u}^{p-1} - b(x)h(\bar{u}) \leqslant 0 \quad \text{in } \Omega. \tag{2.4}$$

Let $w_1$ and $w_2$ be nonnegative $C^1$ functions on $\Omega$ and both of them vanish near $\partial\Omega$. Multiplying (2.3) by $w_1$ and (2.4) by $w_2$, and applying integration by parts, and substraction, we easily obtain

$$\begin{aligned} &-\int_\Omega \big[|\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla w_1 - |\nabla \bar{u}|^{p-2}\nabla \bar{u}\nabla w_2\big]\,dx + \int_{\partial\Omega} \left[|\nabla \underline{u}|^{p-2}\frac{\partial \underline{u}}{\partial \nu} w_1 - |\nabla \bar{u}|^{p-2}\frac{\partial \bar{u}}{\partial \nu} w_2\right] dS \\ &\quad \geqslant \int_\Omega \big[b(x)\big(h(\underline{u})w_1 - h(\bar{u})w_2\big)\big]\,dx - \lambda \int_\Omega \big[\underline{u}^{p-1}w_1 - \bar{u}^{p-1}w_2\big]\,dx. \end{aligned} \tag{2.5}$$

Because $w_1$ and $w_2$ vanish near $\partial\Omega$,

$$\begin{aligned} &-\int_\Omega \big[|\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla w_1 - |\nabla \bar{u}|^{p-2}\nabla \bar{u}\nabla w_2\big]\,dx \\ &\quad \geqslant \int_\Omega \big[b(x)\big(h(\underline{u})w_1 - h(\bar{u})w_2\big)\big]\,dx - \lambda \int_\Omega \big[\underline{u}^{p-1}w_1 - \bar{u}^{p-1}w_2\big]\,dx. \end{aligned} \tag{2.6}$$

Let $\epsilon_2 > \epsilon_1 > 0$ and denote

$$v_1 = \left[\frac{(\underline{u}+\epsilon_1)^{2p-2} - (\bar{u}+\epsilon_2)^{2p-2}}{(\underline{u}+\epsilon_1)^{p-1}}\right]^+, \tag{2.7}$$

$$v_2 = \left[\frac{(\underline{u}+\epsilon_1)^{2p-2} - (\bar{u}+\epsilon_2)^{2p-2}}{(\bar{u}+\epsilon_2)^{p-1}}\right]^+. \tag{2.8}$$

Since $v_i$ can be approximated arbitrarily closely by $C^1$ functions vanishing near $\partial\Omega$, we see that (2.6) holds when $w_i$ is replaced by $v_i$. Denote

$$\Omega_+(\epsilon_1,\epsilon_2) = \{x \in \Omega \mid \underline{u}+\epsilon_1 > \bar{u}+\epsilon_2\}.$$

We note that the integrands in (2.6) (with $w_i = v_i$) vanish outside this set. Then the left-hand side of (2.6) is

$$\begin{aligned}
&- \int\limits_{\Omega_+(\epsilon_1,\epsilon_2)} \left[|\nabla\underline{u}|^{p-2}\nabla\underline{u}\nabla v_1 - |\nabla\bar{u}|^{p-2}\nabla\bar{u}\nabla v_2\right] dx \\
&= -(p-1) \int\limits_{\Omega_+(\epsilon_1,\epsilon_2)} \left[|\nabla\underline{u}|^p(\underline{u}+\epsilon_1)^{p-2}\left(1+\left(\frac{\bar{u}+\epsilon_2}{\underline{u}+\epsilon_1}\right)^{2p-2}\right)\right] dx \\
&\quad + 2(p-1) \int\limits_{\Omega_+(\epsilon_1,\epsilon_2)} \left[\nabla\bar{u}\nabla\underline{u}\left((\bar{u}+\epsilon_2)^{p-2}\left(\frac{\bar{u}+\epsilon_2}{\underline{u}+\epsilon_1}\right)^{p-1}|\nabla\underline{u}|^{p-2}\right)\right] \\
&\quad - (p-1) \int\limits_{\Omega_+(\epsilon_1,\epsilon_2)} \left[|\nabla\bar{u}|^p(\bar{u}+\epsilon_2)^{p-2}\left(1+\left(\frac{\underline{u}+\epsilon_1}{\bar{u}+\epsilon_2}\right)^{2p-2}\right)\right] dx \\
&\quad + 2(p-1) \int\limits_{\Omega_+(\epsilon_1,\epsilon_2)} \left[\nabla\bar{u}\nabla\underline{u}\left((\underline{u}+\epsilon_1)^{p-2}\left(\frac{\underline{u}+\epsilon_1}{\bar{u}+\epsilon_2}\right)^{p-1}|\nabla\bar{u}|^{p-2}\right)\right] \\
&= -(p-1) \int\limits_{\Omega_+(\epsilon_1,\epsilon_2)} (\underline{u}+\epsilon_1)^{p-2}\left(\frac{\bar{u}+\epsilon_2}{\underline{u}+\epsilon_1}\right)^{2p-2}|\nabla\underline{u}|^{p-2}\left|\nabla\underline{u} - \nabla\bar{u}\left(\frac{\underline{u}+\epsilon_1}{\bar{u}+\epsilon_2}\right)\right|^2 dx \\
&\quad - (p-1) \int\limits_{\Omega_+(\epsilon_1,\epsilon_2)} (\bar{u}+\epsilon_2)^{p-2}\left(\frac{\underline{u}+\epsilon_1}{\bar{u}+\epsilon_2}\right)^{2p-2}|\nabla\bar{u}|^{p-2}\left|\nabla\bar{u} - \nabla\underline{u}\left(\frac{\bar{u}+\epsilon_2}{\underline{u}+\epsilon_1}\right)\right|^2 dx \\
&\quad - (p-1) \int\limits_{\Omega_+(\epsilon_1,\epsilon_2)} (\underline{u}+\epsilon_1)^{p-2}\left(|\nabla\underline{u}|^{p-2} - |\nabla\bar{u}|^{p-2}\left(\frac{\underline{u}+\epsilon_1}{\bar{u}+\epsilon_2}\right)^{p-2}\right) \\
&\quad \times \left(|\nabla\underline{u}|^2 - |\nabla\bar{u}|^2\left(\frac{\bar{u}+\epsilon_2}{\underline{u}+\epsilon_1}\right)^{2p-4}\right) dx.
\end{aligned}$$

Let the above equality be

$$- \int\limits_{\Omega_+(\epsilon_1,\epsilon_2)} \left[|\nabla\underline{u}|^{p-2}\nabla\underline{u}\nabla v_1 - |\nabla\bar{u}|^{p-2}\nabla\bar{u}\nabla v_2\right] dx = J_1 + J_2 + J_3.$$

Then the inequality (2.6) becomes

$$J_1 + J_2 + J_3 \geqslant \int\limits_{\Omega_+(\epsilon_1,\epsilon_2)} \left[ b(x)\left( \frac{h(\underline{u})}{(\underline{u}+\epsilon_1)^{p-1}} - \frac{h(\bar{u})}{(\bar{u}+\epsilon_2)^{p-1}} \right)\left( (\underline{u}+\epsilon_1)^{2p-2} - (\bar{u}+\epsilon_2)^{2p-2} \right) \right] dx$$
$$- \lambda \int\limits_{\Omega_+(\epsilon_1,\epsilon_2)} \left[ \underline{u}^{p-1} v_1 - \bar{u}^{p-1} v_2 \right] dx. \tag{2.9}$$

Note that $J_1$ and $J_2$ are non-positive. As $\epsilon_2 > \epsilon_1 \to 0$, the second term on the right-hand side of (2.9) converges to 0 and the first term on the right-hand side of (2.9) converges to

$$\int\limits_{\Omega_+(0,0)} \left[ b(x)\left( \frac{h(\underline{u})}{(\underline{u})^{p-1}} - \frac{h(\bar{u})}{(\bar{u})^{p-1}} \right)\left( (\underline{u})^{2p-2} - (\bar{u})^{2p-2} \right) \right] dx$$

which is positive unless $\Omega_+(0,0)$ is empty. The proof of Lemma 2.1 is complete. □

Next we present the definition of sub-solution and super-solution as follows.

**Definition 1.** If $\underline{u}$ (resp. $\bar{u}$) satisfies the conditions in Lemma 2.1 and $\underline{u} \leqslant \phi$ (resp. $\bar{u} \geqslant \phi$) on $\partial\Omega$, then $\underline{u}$ (resp. $\bar{u}$) is called a sub-solution (resp. super-solution) of (2.1).

**Lemma 2.2.** *If $\underline{u}, \bar{u} \in C^1(\Omega)$ are both positive in $\Omega$ such that*

$$-\Delta_p \underline{u} \leqslant \lambda \underline{u}^{p-1} - b(x)h(\underline{u}) \quad \text{in } \Omega,$$
$$-\Delta_p \bar{u} \geqslant \lambda \bar{u}^{p-1} - b(x)h(\bar{u}) \quad \text{in } \Omega,$$
$$\lim_{d(x)\to 0} \underline{u}(x) = +\infty, \qquad \lim_{d(x)\to 0} \bar{u}(x) = +\infty,$$

*hold in the sense of distribution, and $\underline{u}(x) \leqslant \bar{u}(x)$ in $\Omega$, then there exists at least one solution $u \in C^1(\Omega)$ to (2.1) satisfying $\underline{u}(x) \leqslant u \leqslant \bar{u}(x)$ in $\Omega$.*

We note that $\underline{u} = 0$ is a sub-solution of (2.1), while $\bar{u} = L$ is a super-solution of (2.1) if $L$ is large enough, so (2.1) has at least a solution $u_\phi$. Thus the proof of Lemma 2.2 follows exactly the same arguments as in the proof of Theorem 3.2 in López-Gómez [18] and it is omitted for brevity: consider (2.1) in domains $\Omega_n := \{x \in \Omega \mid d(x, \partial\Omega) > \frac{1}{n}\}$ with $u = (\underline{u} + \bar{u})/2$ on $\partial\Omega_n$ and we make $n \to \infty$ through a diagonal process. The limit of the diagonal sequence provides us with a solution satisfying all the required conditions.

## 3. Proof of Theorem 1.1

Next we consider the corresponding singular problem in one dimension

$$\begin{cases} -\left[ \left( |\psi'|^{p-2}\psi' \right)' + \dfrac{N-1}{r}|\psi'|^{p-2}\psi' \right] = \lambda\psi^{p-1} - b(r)h(\psi) \quad \text{in } (0,R), \\ \lim\limits_{r\to R} \psi(r) = \infty, \\ \psi'(0) = 0. \end{cases} \tag{3.1}$$

We first show that for each $\epsilon > 0$, problem (3.1) has a positive large solution $\psi_\epsilon$ such that

$$1-\epsilon \leqslant \liminf_{r\to R} \frac{\psi_\epsilon(r)}{KH^{-2\beta/p}(b^*(r))^{-\beta}} \leqslant \limsup_{r\to R} \frac{\psi_\epsilon(r)}{KH^{-2\beta/p}(b^*(r))^{-\beta}} \leqslant 1+\epsilon. \tag{3.2}$$

Therefore, for each $x_0 \in \mathbb{R}^N$, the function

$$u_\epsilon(x) := \psi_\epsilon(r); \qquad r := \|x - x_0\|$$

provides us with a radially symmetric positive large solution of (1.3a–c) with the assumptions in Theorem 1.1 and the solution satisfies

$$1-\epsilon \leqslant \liminf_{d(x)\to 0} \frac{u_\epsilon(x)}{KH^{-2\beta/p}(b^*(\|x-x_0\|))^{-\beta}} \leqslant \limsup_{d(x)\to 0} \frac{u_\epsilon(x)}{KH^{-2\beta/p}(b^*(\|x-x_0\|))^{-\beta}} \leqslant 1+\epsilon. \tag{3.3}$$

In order to prove (3.2), we construct a super-solution and a sub-solution of (3.1). For each $\epsilon > 0$ we claim that

$$\overline{\psi}_\epsilon(r) = A + B_+\left(\frac{r}{R}\right)^2 \left(b^*(r)\right)^{-\beta} \tag{3.4}$$

provides us a super-solution, where $A > 0$ and $B_+ > 0$ have to be determined later. Then

$$\overline{\psi}'_\epsilon(r) = 2B_+\frac{r}{R^2}\left(b^*(r)\right)^{-\beta} - \beta B_+\left(\frac{r}{R}\right)^2\left(b^*(r)\right)^{-\beta-1}\left(b^*(r)\right)',$$

$$\begin{aligned}\overline{\psi}''_\epsilon(r) &= 2B_+\frac{1}{R^2}\left(b^*(r)\right)^{-\beta} - 4\beta B_+\frac{r}{R^2}\left(b^*(r)\right)^{-\beta-1}\left(b^*(r)\right)' \\ &\quad + \beta(\beta+1)B_+\left(\frac{r}{R}\right)^2\left(b^*(r)\right)^{-\beta-2}\left[\left(b^*(r)\right)'\right]^2 - \beta B_+\left(\frac{r}{R}\right)^2\left(b^*(r)\right)^{-\beta-1}\left(b^*(r)\right)''.\end{aligned}$$

We have $\overline{\psi}_\epsilon(r) \to \infty$ as $r \to R$ because $b^*(r) \to 0$ as $r \to R$ and $\beta > 0$. Also $\overline{\psi}'_\epsilon(r) \geqslant 0$ for $r \in [0, R]$ and $\overline{\psi}'_\epsilon(r) \to 0$ as $r \to 0$. Then $\overline{\psi}_\epsilon(r)$ is a super-solution if and only if

$$-\left[\left(\left|\overline{\psi}'_\epsilon(r)\right|^{p-2}\overline{\psi}'_\epsilon(r)\right)' + \frac{N-1}{r}\left|\overline{\psi}'_\epsilon(r)\right|^{p-2}\overline{\psi}'_\epsilon(r)\right] \geqslant \lambda\overline{\psi}^{p-1}_\epsilon(r) - b(r)h\left(\overline{\psi}_\epsilon(r)\right), \tag{3.5}$$

i.e.

$$-(p-1)\left(\overline{\psi}'_\epsilon(r)\right)^{p-2}\overline{\psi}''_\epsilon(r) - \frac{N-1}{r}\left(\overline{\psi}'_\epsilon(r)\right)^{p-1} \geqslant \lambda\overline{\psi}^{p-1}_\epsilon(r) - b(r)h\left(\overline{\psi}_\epsilon(r)\right). \tag{3.6}$$

By the assumption ($\mathcal{A}2$) on $h$, it is easy to see that for the same $\epsilon > 0$,

$$(1-\epsilon)H\overline{\psi}^q_\epsilon(r) \leqslant h\left(\overline{\psi}_\epsilon(r)\right) \leqslant (1+\epsilon)H\overline{\psi}^q_\epsilon(r) \tag{3.7}$$

for all $r \in [0, R)$ by choosing $A$ sufficiently large, say $A \geqslant A_0$. The inequality (3.6) holds if

$$-(p-1)\left(\overline{\psi}'_\epsilon(r)\right)^{p-2}\overline{\psi}''_\epsilon(r) - \frac{N-1}{r}\left(\overline{\psi}'_\epsilon(r)\right)^{p-1} \geqslant \lambda\overline{\psi}^{p-1}_\epsilon(r) - b(r)(1-\epsilon)H\overline{\psi}^q_\epsilon(r). \tag{3.8}$$

Note that $(b^*(r))' = -B(r)$, $(b^*(r))'' = b(r)$. The inequality (3.8) is equivalent to

$$
\begin{aligned}
&-(p-1)\left[2B_+\frac{r}{R^2}\left(b^*(r)\right)^{-\beta}+\beta B_+\left(\frac{r}{R}\right)^2\left(b^*(r)\right)^{-\beta-1}B(r)\right]^{p-2}\cdot\left[2B_+\frac{1}{R^2}\left(b^*(r)\right)^{-\beta}\right.\\
&\quad+4\beta B_+\frac{r}{R^2}\left(b^*(r)\right)^{-\beta-1}B(r)+\beta(\beta+1)B_+\left(\frac{r}{R}\right)^2\left(b^*(r)\right)^{-\beta-2}\left(B(r)\right)^2\\
&\quad\left.-\beta B_+\left(\frac{r}{R}\right)^2\left(b^*(r)\right)^{-\beta-1}b(r)\right]-\frac{N-1}{r}\left[2B_+\frac{r}{R^2}\left(b^*(r)\right)^{-\beta}+\beta B_+\left(\frac{r}{R}\right)^2\left(b^*(r)\right)^{-\beta-1}B(r)\right]^{p-1}\\
&\geqslant\lambda\left[A+B_+\left(\frac{r}{R}\right)^2\left(b^*(r)\right)^{-\beta}\right]^{p-1}-b(r)(1-\epsilon)H\left[A+B_+\left(\frac{r}{R}\right)^2\left(b^*(r)\right)^{-\beta}\right]^q.
\end{aligned}
$$

Multiplying both sides of this inequality by $\frac{(b^*(r))^{(\beta+1)(p-1)+1}}{(B(r))^p}$ and taking into consideration that $\beta=\frac{p}{2(q-p+1)}$, we have

$$
\begin{aligned}
&-(p-1)\left[2B_+\frac{r}{R^2}\frac{b^*(r)}{B(r)}+\beta B_+\left(\frac{r}{R}\right)^2\right]^{p-2}\cdot\left[2B_+\frac{1}{R^2}\left(\frac{b^*(r)}{B(r)}\right)^2+4\beta B_+\frac{r}{R^2}\frac{b^*(r)}{B(r)}\right.\\
&\quad\left.+\beta(\beta+1)B_+\left(\frac{r}{R}\right)^2-\beta B_+\left(\frac{r}{R}\right)^2\frac{b^*(r)b(r)}{B^2(r)}\right]\\
&\quad-\frac{N-1}{r}\frac{b^*(r)}{B(r)}\left[2B_+\frac{r}{R^2}\frac{b^*(r)}{B(r)}+\beta B_+\left(\frac{r}{R}\right)^2\right]^{p-1}\\
&\geqslant\lambda\left(\frac{b^*(r)}{B(r)}\right)^p\left[A\left(b^*(r)\right)^\beta+B_+\left(\frac{r}{R}\right)^2\right]^{p-1}\\
&\quad-b(r)\left(b^*(r)\right)^{-\beta p}\frac{(b^*(r))^{(\beta+1)(p-1)+1}}{(B(r))^p}(1-\epsilon)H\left[A\left(b^*(r)\right)^\beta+B_+\left(\frac{r}{R}\right)^2\right]^q\\
&=\lambda\left(\frac{b^*(r)}{B(r)}\right)^p\left[A\left(b^*(r)\right)^\beta+B_+\left(\frac{r}{R}\right)^2\right]^{p-1}\\
&\quad-b^{1-p/2}(r)\left(\frac{b^*(r)b(r)}{B^2(r)}\right)^{p/2}(1-\epsilon)H\left[A\left(b^*(r)\right)^\beta+B_+\left(\frac{r}{R}\right)^2\right]^q.
\end{aligned}
$$

Since when $r\to R$, $b^*(r)\to 0$, $\frac{b^*(r)}{B(r)}\to 0$, $\frac{(B(r))^2}{b^*(r)b(r)}\to C_0\geqslant 1$ by the assumption $(\mathcal{A}1)$, then as $r\to R$,

$$
-(p-1)(\beta B_+)^{p-2}\left[\beta(\beta+1)B_+-\beta B_+\frac{1}{C_0}\right]\geqslant -b_0^{1-p/2}C_0^{-p/2}(1-\epsilon)HB_+^p,
$$

which is

$$
B_+\geqslant\left[(p-1)\beta^{p-1}\left((\beta+1)C_0-1\right)(b_0C_0)^{p/2-1}\right]^{\frac{1}{q-p+1}}\left[(1-\epsilon)H\right]^{\frac{-1}{q-p+1}}.
$$

Let $B_+=(1+\epsilon)[(1-\epsilon)H]^{-2\beta/p}[(p-1)\beta^{p-1}((\beta+1)C_0-1)(b_0C_0)^{p/2-1}]^{2\beta/p}=(1+\epsilon)(1-\epsilon)^{-2\beta/p}H^{-2\beta/p}K$. Therefore, by making the choice $B_+$, the inequality (3.8) is satisfied in a left neighborhood of $r=R$, say $(R-\delta,R]$, for some $\delta=\delta(\epsilon)>0$. Finally, by choosing $A$ sufficiently large (larger than $A_0$) it is clear that the inequality is satisfied in the whole interval $[0,R]$ since $q>p-1$ and $b^*(r)$ is bounded away from zero in $[0,R-\delta]$. Then $\overline{\psi}_\epsilon$ is our required super-solution of problem (3.1).

Next, we construct a sub-solution with the same blow-up rate as the above super-solution. Due to the assumption $(\mathcal{A}2)$ on $h$, for $u \geqslant A_0$ large,

$$(1-\epsilon)Hu^q \leqslant h(u) \leqslant (1+\epsilon)Hu^q.$$

For given $A_0 > 0$ and $0 < R_0 < R$, we consider the auxiliary problem

$$\begin{cases} -\left[\left(|\psi'|^{p-2}\psi'\right)' + \dfrac{N-1}{r}|\psi'|^{p-2}\psi'\right] = \lambda\psi^{p-1} - b(r)h(\psi) \quad \text{in } (0, R_0), \\ \psi(R_0) = A_0, \\ \psi'(0) = 0. \end{cases} \tag{3.9}$$

By the assumptions on $b$ and $h$, we have

$$\min_{r\in[0,R_0]} b(r) > 0, \qquad h(0) = 0, \quad \text{and} \quad h(u)/u \to \infty \quad \text{as } u \to \infty.$$

Then it is easy to know that

$$\underline{\psi}_{A_0} := 0, \qquad \overline{\psi}_{A_0} := A_0$$

provides us with an ordered sub-super-solution pair of (3.9). Thus (3.9) possesses a solution $\psi_{A_0}$ such that $\psi_{A_0}(r) \in [0, A_0]$ for all $r \in [0, R_0]$.

For each $\epsilon > 0$ sufficiently small, we claim that there exists $0 < C < A_0$ for which the function

$$\underline{\psi}_\epsilon(r) = \begin{cases} \psi_{A_0}(r), & r \in [0, R_0], \\ \max\{A_0, C + B_-(\frac{r}{R})^2(b^*(r))^{-\beta}\}, & r \in (R_0, R], \end{cases}$$

provides a sub-solution, where $R_0$ and $C$ are to be determined later and

$$B_- = (1-\epsilon)(1+\epsilon)^{-2\beta/p}H^{-2\beta/p}K.$$

In fact, denoting $f_C(r) = C + B_-(\frac{r}{R})^2(b^*(r))^{-\beta}$ we have

$$\begin{aligned} f_C'(r) &= 2B_-\frac{r}{R^2}\left(b^*(r)\right)^{-\beta} - \beta B_-\left(\frac{r}{R}\right)^2\left(b^*(r)\right)^{-\beta-1}\left(b^*(r)\right)' \\ &= 2B_-\frac{r}{R^2}\left(b^*(r)\right)^{-\beta} + \beta B_-\left(\frac{r}{R}\right)^2\left(b^*(r)\right)^{-\beta-1}B(r) \end{aligned}$$

which is strictly bigger than zero in $(0, R)$. It follows that $f_C(r)$ is increasing and

$$\lim_{r\to R} f_C(r) = +\infty, \qquad \lim_{r\to 0} f_C(r) = C < A_0.$$

By the continuity of $f_C(r)$ and the intermediate-value theorem, there exists a unique $Z = Z(C) \in (0, R)$ such that

$$C + B_-\left(\frac{r}{R}\right)^2\left(b^*(r)\right)^{-\beta} < A_0 \quad \text{when } r \in \left[0, Z(C)\right),$$

$$C + B_-\left(\frac{r}{R}\right)^2\left(b^*(r)\right)^{-\beta} \geqslant A_0 \quad \text{when } r \in \left[Z(C), R\right].$$

Moreover, $Z(C)$ is decreasing and

$$\lim_{C\to-\infty} Z(C)=R, \qquad \lim_{C\to A_0} Z(C)=0.$$

Let $R_0=Z(C)$. From the definition of $\underline{\psi}_\epsilon(r)$ and $R_0$, $\underline{\psi}_\epsilon(r)\equiv\psi_{A_0}(r)$ in $[0,Z(C)]$, and then the inequality $-(p-1)(\underline{\psi}'_\epsilon(r))^{p-2}\underline{\psi}''_\epsilon(r)-\frac{N-1}{r}(\underline{\psi}'_\epsilon(r))^{p-1}\leqslant\lambda\underline{\psi}^{p-1}_\epsilon(r)-b(r)h(\underline{\psi}_\epsilon(r))$ holds in $[0,Z(C)]$. So $\underline{\psi}_\epsilon(r)$ is a sub-solution of (3.1) if the following inequality is satisfied in $[Z(C),R]$

$$-(p-1)\big(\underline{\psi}'_\epsilon(r)\big)^{p-2}\underline{\psi}''_\epsilon(r)-\frac{N-1}{r}\big(\underline{\psi}'_\epsilon(r)\big)^{p-1}\leqslant\lambda\underline{\psi}^{p-1}_\epsilon(r)-b(r)h\big(\underline{\psi}_\epsilon(r)\big). \tag{3.10}$$

For each $r\in[Z(C),R]$,

$$-\frac{N-1}{r}\big(\underline{\psi}'_\epsilon(r)\big)^{p-1}\leqslant 0\leqslant\lambda\underline{\psi}^{p-1}_\epsilon(r).$$

By using the fact $h(\underline{\psi}_\epsilon(r))\leqslant(1+\epsilon)H\underline{\psi}_\epsilon(r)^p$ in $[Z(C),R]$, (3.10) holds if

$$-(p-1)\big(\underline{\psi}'_\epsilon(r)\big)^{p-2}\underline{\psi}''_\epsilon(r)\leqslant-b(r)(1+\epsilon)H\underline{\psi}^q_\epsilon(r),\quad\text{for each } r\in\big[Z(C),R\big],$$

i.e.

$$\begin{aligned}&-(p-1)\left[2B_-\frac{r}{R^2}\frac{b^*(r)}{B(r)}+\beta B_-\left(\frac{r}{R}\right)^2\right]^{p-2}\cdot\left[2B_-\frac{1}{R^2}\left(\frac{b^*(r)}{B(r)}\right)^2+4\beta B_-\frac{r}{R^2}\frac{b^*(r)}{B(r)}\right.\\&\qquad\left.+\beta(\beta+1)B_-\left(\frac{r}{R}\right)^2-\beta B_-\left(\frac{r}{R}\right)^2\frac{b^*(r)b(r)}{B^2(r)}\right]\\&\quad\leqslant-b^{1-p/2}(r)\left(\frac{b^*(r)b(r)}{B^2(r)}\right)^{p/2}(1+\epsilon)H\left[C\big(b^*(r)\big)^\beta+B_-\left(\frac{r}{R}\right)^2\right]^q.\end{aligned}$$

Taking $r\to R$, it becomes

$$-(p-1)(\beta B_-)^{p-2}\left[\beta(\beta+1)B_--\beta B_-\frac{1}{C_0}\right]\leqslant-b_0^{1-p/2}C_0^{-p/2}(1+\epsilon)HB_-^p,$$

which is

$$B_-\leqslant\big[(p-1)\beta^{p-1}\big((\beta+1)C_0-1\big)(b_0C_0)^{p/2-1}\big]^{\frac{1}{q-p+1}}\big[(1+\epsilon)H\big]^{\frac{-1}{q-p+1}}.$$

Let $B_-=(1-\epsilon)[(1+\epsilon)H]^{-2\beta/p}[(p-1)\beta^{p-1}((\beta+1)C_0-1)(b_0C_0)^{p/2-1}]^{2\beta/p}=(1-\epsilon)(1+\epsilon)^{-2\beta/p}H^{-2\beta/p}K$. It is easy to see that a constant $\delta=\delta(\epsilon)>0$ exists for which the inequality (3.10) is satisfied in $[R-\delta,R)$, then we choose $C$ such that $Z(C)=R-\delta(\epsilon)$ (therefore $R_0=R-\delta(\epsilon)$). For this choice of $C$, it readily follows that $\underline{\psi}_\epsilon$ is a sub-solution to the problem.

So we have constructed a sub-solution and a super-solution with the same blow-up rate of problem (3.1). Because $\overline{\psi}_\epsilon(r)\geqslant\underline{\psi}_\epsilon(r)$ in $[0,R)$ and $\lim_{r\to R}\overline{\psi}_\epsilon(r)=\lim_{r\to R}\underline{\psi}_\epsilon(r)=\infty$, then by Lemma 2.2 there exists a solution $\Psi_\epsilon(r)$ of (3.1) such that

$$1-\epsilon\leqslant\liminf_{r\to R}\frac{\Psi_\epsilon(r)}{KH^{-2\beta/p}(b^*(r))^{-\beta}}\leqslant\limsup_{r\to R}\frac{\Psi_\epsilon(r)}{KH^{-2\beta/p}(b^*(r))^{-\beta}}\leqslant1+\epsilon.$$

**Proof of uniqueness.** The proof of uniqueness basically follows the proofs in [10,12,25]. Let $u$ be an arbitrary solution of (1.3a–c) with assumptions on nonlinear function $h(u)$ and weight function $b$ as in Theorem 1.1. We first show that

$$\lim_{d(x)\to 0} \frac{u(x)}{KH^{-2\beta/p}(b^*(\|x-x_0\|))^{-\beta}} = 1.$$

Consequently, for any pair of solutions $u, v$ of (1.3a–c)

$$\lim_{d(x)\to 0} \frac{u(x)}{v(x)} = 1.$$

In doing so, for any $\epsilon > 0$, there exists a radially symmetric positive large solution $u_\epsilon$ of (1.3a–c) satisfying (3.3). Choose $0 < \delta < \frac{R}{3}$ small, fix $0 < \tau < \frac{\delta}{4}$ and introduce the region

$$Q_\tau := \left\{ x \;\middle|\; \tau < d\big(x, \partial B_R(x_0)\big) < \frac{\delta}{2} \right\}.$$

Let $M \geqslant \max_{\|x-x_0\| \leqslant (R-\frac{\delta}{4})} u(x)$ be large. Thus for every $\tau \in (0, \frac{\delta}{4})$,

$$\overline{V}_\epsilon(x) = u_\epsilon\left(x + \tau \frac{(x-x_0)}{\|x-x_0\|}\right) + M = u_\epsilon\big(\|x-x_0\| + \tau\big) + M$$

is a super-solution to

$$\begin{cases} -\Delta_p v = \lambda v^{p-1} - bh(v) & \text{in } Q_\tau, \\ v = u & \text{on } \partial Q_\tau \end{cases} \tag{3.11}$$

with $u$ an arbitrary fixed solution to (1.3a–c) since $\overline{V}_\epsilon(x) \geqslant u$ for $x \in \partial Q_\tau, \tau \in (0, \frac{\delta}{4})$. Note that $\overline{V}_\epsilon(x) \to \infty$ as $x \to \partial B_{R-\tau}(x_0)$. $\overline{V}_\epsilon(x) \geqslant M \geqslant u$ as $x \to \partial B_{R-\frac{\delta}{2}}(x_0)$. In addition, the auxiliary problem (3.11) has $v = u$ as its unique solution. Since 0 is a sub-solution ($h(0) = 0$ by the assumption $h(u)/u$ is increasing), we conclude $u(x) \leqslant \overline{V}_\epsilon(x) = u_\epsilon(\|x-x_0\| + \tau) + M$ for every $x \in Q_\tau, 0 < \tau < \frac{\delta}{4}$. Letting $\tau \to 0^+$, we arrive at $u(x) \leqslant u_\epsilon(x) + M$ for every $x \in A_{\frac{\delta}{2},R}(x_0)$ and we obtain

$$\limsup_{d(x)\to 0} \frac{u(x)}{KH^{-2\beta/p}(b^*(\|x-x_0\|))^{-\beta}} \leqslant \limsup_{d(x)\to 0} \frac{u_\epsilon(x)}{KH^{-2\beta/p}(b^*(\|x-x_0\|))^{-\beta}} \leqslant 1 + \epsilon.$$

We now prove

$$1 - \epsilon \leqslant \liminf_{d(x)\to 0} \frac{u_\epsilon(x)}{KH^{-2\beta/p}(b^*(\|x-x_0\|))^{-\beta}} \leqslant \liminf_{d(x)\to 0} \frac{u}{KH^{-2\beta/p}(b^*(\|x-x_0\|))^{-\beta}}.$$

For the same $\epsilon > 0$ and the radially symmetric positive large solution $u_\epsilon$ of (1.3a–c) satisfying (3.3), we choose $0 < \delta < \frac{R}{3}$ small, fix $0 < \tau < \frac{\delta}{4}$ and introduce the annuli region

$$A_{R-\delta, R-\tau} = \big\{x\colon R - \delta < \|x-x_0\| < R - \tau\big\}.$$

Let $M_1 \geqslant \max_{\|x-x_0\| \leqslant R-\frac{3\delta}{4}} u(x)$ be large. Thus for every $\tau \in (0, \frac{\delta}{4})$,

$$\tilde{V}_\epsilon(x) = \max\left\{u\left(x + \tau\frac{(x-x_0)}{\|x-x_0\|}\right) - M_1, 0\right\} = \max\bigl\{u\bigl(\|x-x_0\| - \tau\bigr) - M_1, 0\bigr\}$$

is a sub-solution to

$$\begin{cases} -\Delta_p v = \lambda v^{p-1} - bh(v) & \text{in } A_{R-\delta, R-\tau}, \\ v = u_\epsilon & \text{on } \partial A_{R-\delta, R-\tau} \end{cases} \tag{3.12}$$

for all $\tau \in (0, \frac{\delta}{4})$. By the same argument as above, we obtain

$$1 - \epsilon \leqslant \liminf_{d(x)\to 0} \frac{u_\epsilon}{KH^{-2\beta/p}(b^*(\|x-x_0\|))^{-\beta}} \leqslant \liminf_{d(x)\to 0} \frac{u}{KH^{-2\beta/p}(b^*(\|x-x_0\|))^{-\beta}}.$$

Then

$$1 - \epsilon \leqslant \liminf_{d(x)\to 0} \frac{u}{KH^{-2\beta/p}(b^*(\|x-x_0\|))^{-\beta}} \leqslant \limsup_{d(x)\to 0} \frac{u}{KH^{-2\beta/p}(b^*(\|x-x_0\|))^{-\beta}} \leqslant 1 + \epsilon.$$

Letting $\epsilon \to 0^+$, we have

$$\lim_{d(x)\to 0} \frac{u}{KH^{-2\beta/p}(b^*(\|x-x_0\|))^{-\beta}} = 1.$$

Now let $u$ and $v$ be large positive solutions to (1.3a–c). By virtue of above, $u$ and $v$ satisfy $\lim_{d(x)\to 0} \frac{u}{v} = 1$. Thus, for every $\epsilon > 0$, we can find $\delta > 0$ (as small as we are please) such that

$$(1-\epsilon)v(x) \leqslant u(x) \leqslant (1+\epsilon)v(x)$$

when $0 < d(x) \leqslant \delta$. On the other hand $\underline{w} = (1-\epsilon)v(x)$ and $\overline{w} = (1+\epsilon)v(x)$ are a sub-solution and a super-solution respectively to

$$\begin{cases} -\Delta_p w = \lambda w^{p-1} - bh(w) & \text{in } B_{R-\delta}(x_0), \\ w = u & \text{on } \partial B_{R-\delta}(x_0), \end{cases} \tag{3.13}$$

where we use the property $\frac{h((1-\epsilon)v)}{1-\epsilon} \leqslant h(v)$ and $\frac{h((1+\epsilon)v)}{1+\epsilon} \geqslant h(v)$. The unique solution to this problem is $w = u$. Then

$$(1-\epsilon)v(x) \leqslant u(x) \leqslant (1+\epsilon)v(x)$$

holds in $B_{R-\delta}(x_0)$, therefore it is true in $B_R(x_0)$. Letting $\epsilon \to 0$ we arrive at $u = v$.

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