The Exact Boundary Blow-up Rate of Large Solutions for Semilinear Elliptic Problems

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Abstract

In this paper, we establish the blow up rate of the large positive solutions of the singular boundary value problem $-\triangle u = \lambda u - a(x)u^p, u|_{\partial\Omega} = +\infty$, where Ω is a bounded domain of class C^2 in $\mathbb{R}^N$. The weight function $a(x)$ in front of the nonlinearity can vanish on the boundary of the domain Ω at different rates according to the point x_0 of the boundary. The decay rates of the weight function $a(x)$ may not be approximated by a power function of distance near the boundary $\partial\Omega$. We combine the results in [13] with the localization method introduced in [10] to prove that any large solution $u(x)$ must satisfy

$$\lim_{x\to x_0} \frac{u(x)}{K(b^*_{x_0}(dist(x,\partial\Omega)))^{-\beta}} = 1 \text{ for each } x_0 \in \partial\Omega,$$

where

$$b^*_{x_0}(r) = \int_0^r \int_0^s b_{x_0}(t)dtds, K = [\beta((\beta+1)C_0 - 1)]^{\frac{1}{p-1}},$$

$$\beta = \frac{1}{p-1}, C_0 = \lim_{r\to 0} \frac{(\int_0^r b_{x_0}(t)dt)^2}{b^*_{x_0}(r)b_{x_0}(r)}$$

and $b_{x_0}(r)$ is the boundary normal section of $a(x)$ at $x_0 \in \partial\Omega$, i.e.,

$$b_{x_0}(r) = a(x_0 - r\mathbf{n}_{x_0}), r > 0, r \sim 0,$$

and $\mathbf{n}_{x_0}$ stands for the outward unit normal vector at $x_0 \in \partial\Omega$.

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1 Introduction and Main Results

This paper originated with the sequentially papers ([4], [5], [6], [10], [12], [13]) which contain an exhaustive study of positive solution u to the singular boundary value problem:

$$\begin{cases} -\triangle u &= \lambda(x)u - a(x)u^p & \text{in } \Omega, \\ u &= +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

here Ω is a bounded domain in $\mathbb{R}^N, N \geq 1$, with boundary $\partial\Omega$ of class C^2, $\lambda \in L_\infty(\Omega), p > 1$ and $b \in C(\Omega; \mathbb{R}^+)$, $\mathbb{R}^+ := (0, +\infty)$.The boundary condition in (1.1) is understood as $u(x) \to +\infty$ when $d(x) := dist(x, \partial\Omega) \to 0^+$. The behavior of the potential function of the nonlinear term $a(x)$ approaching to $\partial\Omega$ is closely related to the blow up rate of the solution of (1.1). In the case of $a(x) \geq a_0 > 0$ in $\bar{\Omega}$, many different type of equations are studied in ([1], [2], [7]). In the case of $a(x) \sim C_0 d^\gamma + o(d^\gamma)$ as x goes to $\partial\Omega$, blow up rates and uniqueness are studied in ([4], [6]).

Singular boundary value problem (1.1) arises naturally from a number of different areas and have a long history. Considerable amounts of study have been attracted by such problems. We mention only [4], [5], [9], [12]. Many Other works can be found from the references of these papers. However the blow-up rate of solution near $\partial\Omega$ and uniqueness of solutions for the singular problem (1.1) are the goal of more recent literature. In 1990's, a different type of singular boundary value problem $-\triangle u = a(x)e^u$ with $u|_{\partial\Omega} = +\infty$ is studied in [7](1993) and [1](1994). It is shown that the problem exhibits a unique solution in a smooth domain together with an estimate of the form $u = \log d^{-2} + o(d)$ in [7](where $a(x) \geq a_0 > 0$ as $d \to 0$) and in [1](where $a \equiv 1$). Actually, the more general problem $\triangle u(x) = f(u(x))$ with $u|_{\partial\Omega} = +\infty$, is considered in [8]. In their paper [8], they obtained an asymptotic result for solutions of the above problem under some assumptions on f. Let Ω be a bounded domain in $\mathbb{R}^N, N > 1$, which satisfies a uniform internal sphere condition and a uniform external sphere condition. Let f be a C^1 function which is either defined and positive on $(-\infty, \infty)$ or is defined on a ray $[c, \infty)$ with $f(c) = 0$ and $f(s) > 0$ for $s > c$. They further assume that $f'(s) \geq 0$ for s in the domain of f, and that there exists c_1 such that $f'(s)$ is nondecreasing for $s \geq c_1$. They proved that if

$$\lim_{s\to\infty} \frac{f'(s)}{\sqrt{F(s)}} = \infty$$

where $F'(s) = f(s)$, $F(s) > 0$, then the problem has a unique solution $u(x)$ and moreover,

$$u(x) - Z(d(x)) \to 0 \text{ as } d(x) \to 0,$$

where Z is a solution on an interval $(0, b)$, $b > 0$, of the equation $Z''(r) = f(Z(r))$ and such that $Z(r) \to \infty$ as $r \to 0^+$.

More recently, the uniqueness and blow up rates of $-\triangle u = \lambda(x)u - b(x)u^p$ are treated in [2],[4],[6],[10],[13] etc. In [2](1998) the case of $-\triangle u = -u^p$, it is shown that the blow up rate of solution near boundary is $u = Ad^{-\alpha}(1 + o(d))$. Furthermore, the possible presence of a second explosive term in the expansion of u can be obtained when $\alpha > 1 (p < 3)$.

Under the assumption $a(x) = C_0 d^\gamma + o(d^\gamma)$ as $d \to 0^+$ with $\gamma > 0$ and $C_0 > 0$, an explicit expression for the blow-up rates of $-\triangle u = \lambda(x)u - a(x)u^p$ has been recently proved in [4](1999) and [6](2001) as $u = (\frac{\alpha(\alpha+1)}{C_0})^{\frac{1}{p-1}} d^{-\alpha}(1 + o(d))$, $\alpha = \frac{\gamma+2}{p-1}$. In [6] they also give an explicit expression for this second term as $u(x) = Ad^{-\alpha}(1 + B(s)d + o(d))$ when $d \to 0^+$ where $B(s) = \frac{(n-1)H(s)-(\alpha+1)C_1}{\gamma+p+3}$ with $H(s)$ standing for the mean curvature of $\partial\Omega$ at s.

In [10], Juliá López-Gómez (2003) ascertain the blow -up rate of the large positive solutions of singular boundary problem (1.1) with the weight function $a(x)$ that vanishes on the boundary $\partial\Omega$ at different rates according to the point of the boundary and further assuming that $a(x)$ can be approximated by a distance function, i.e. $\lim_{x\to x_\infty} \frac{a(x)}{\beta(x_\infty)[dist(x,\partial\Omega)]^{\gamma(x_\infty)}} = 1$

for $x_\infty \in \partial\Omega$.

In [13], T. Ouyang and Z. Xie (2006) establish the blow up rate of the large positive solution to (1.1) with a ball domain and a radially weight function $a(x)$ which is more general without assuming the decay rate of $a(x)$ to be approximated by a distance function near the boundary $\partial\Omega$.

In this paper we will produce sharper results by combining the results (which is restated here as Theorem 2.5 in section 2) in [13] and the localization method introduced in [10]. Now we turn to state the main results more precisely.

Consider the singular boundary value problem:

$$\begin{cases} -\triangle u &= \lambda u - a(x)u^p \qquad \text{in } \Omega, \\ u &= \infty \qquad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\lambda \in \mathbb{R}$, Ω is a bounded smooth domain in $\mathbb{R}^N$, and the weight function $a(x) > 0$ in Ω. We also assume that the following conditions on $a(x)$ are satisfied.

For each $x \in \partial\Omega$, we define the boundary normal sections $b_x(r)$ as

$$b_x(r) = a(x - r\mathbf{n}_x), r \geq 0, r \sim 0. \tag{1.3}$$

For any $x_0 \in \partial\Omega$, suppose there exists $\tau > 0$, such that $a(x) \in C^1(\bar{B}_\tau(x_0) \bigcap \bar{\Omega}$ and

$$b_{x_0}(r) \in C^1(0,\tau), \qquad b'_{x_0}(r) > 0 \text{ for each } t \in (0,\tau) \tag{1.4}$$

and

$$\lim_{x\in\partial\Omega, x\to x_0, r\to 0} \frac{b_x(r)}{b_{x_0}(r)} = 1. \tag{1.5}$$

Further, let $B_{x_0}(r) = \int_0^r b_{x_0}(s)ds, b^*_{x_0}(r) = \int_0^r B_{x_0}(s)ds$. We assume that $\frac{B_{x_0}(r)}{b_{x_0}(r)} \in C^1([0,\tau])$, $\lim_{r\to 0} \frac{B_{x_0}(r)}{b_{x_0}(r)} = 0$.

In this paper, we show the following results.

Theorem 1.1 *For each $x_0 \in \partial\Omega$, any large positive solution u of (1.2) satisfies*

$$\lim_{r\to 0} \frac{u(x_0 - r\mathbf{n}_{x_0})}{K(b^*_{x_0}(r))^{-\beta}} = 1, \tag{1.6}$$

where K is a constant defined by

$$K = [\beta((\beta+1)C_0 - 1)]^{\frac{1}{p-1}}, \beta := \frac{1}{p-1},$$

and C_0 is defined by

$$C_0 = \lim_{r\to 0} \frac{(B_{x_0}(r))^2}{b^*_{x_0}(r) b_{x_0}(r)}.$$

Moreover, if the conditions (1.3), (1.4), (1.5) are uniformly satisfied on $\partial\Omega$, then for any large positive solution u of (1.2),

$$\lim_{r\to 0} \frac{u(x_0 - r\mathbf{n}_{x_0})}{K(b^*_{x_0}(r))^{-\beta}} = 1, \quad \textit{uniformly in } x_0 \in \partial\Omega. \tag{1.7}$$

Therefore, the problem (1.4) possesses a unique large positive solution $u(x)$ in Ω.

Remark:

(1) For any $x_0 \in \partial\Omega$, if there exist $\beta(=\beta(x_0)) > 0, \gamma(=\gamma(x_0)) \geq 0$ such that $\lim_{x\to x_0} \frac{a(x)}{\beta(dist(x,\partial\Omega))^\gamma} = 1$, then $b_{x_0}(r) = \beta_{x_0}(r)^{\gamma_{x_0}}$ in $(0,\tau)$. Under these assumptions, López-Gómez ([10], 2003) showed the same results as in theorem 1.1 by a different argument.

(2) If the domain Ω is a ball and the weight function $a(x)$ is a radially function on the ball, T. Ouyang and Z. Xie ([13], 2006) showed the same results as in theorem 1.1.

(3) The assumption $\frac{B_{x_0}(r)}{b_{x_0}(r)} \in C^1([0,\tau])$, $\lim_{r\to 0} \frac{B_{x_0}(r)}{b_{x_0}(r)} = 0$ is satisfied by all analytical functions. It also implies that C_0 is finite and greater than or equal to 1.

(4) The theorem 1.1 can be applied more complicate potential functions $a(x)$ which may be not approximated by a power function of distance. Given $x_0 \in \partial\Omega$, for example, let $b_{x_0}(r) = \exp(-\frac{1}{r^2})$, $B_{x_0}(r) = \int_0^r b_{x_0}(s)ds$, $b^*_{x_0}(r) = \int_0^r \int_0^s b_{x_0}(t)dtds$, then $C_0 = 1$, $K = [\beta((\beta+1)-1)]^{\frac{1}{p-1}}$, $\beta := \frac{1}{p-1}$. Therefore, the unique solution $u \sim K(b^*_{x_0})^{-\beta}$ when x approaches x_0, which goes to ∞ near the boundary faster than any power function.

2 Some Preliminary Results

In this section we collect some important comparison results that we are going to use in the proof of Theorem 1.1. As an immediate consequence from the papers [3], [4], [6], [13], theorem 2.1 to 2.6 were proved.

Consider the problem

$$\begin{cases} -\triangle u &= \lambda(x)u - b(x)u^p \qquad \text{in } \Omega, \\ u &= \phi \qquad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain with smooth boundary, $\phi \in C(\partial\Omega), p > 1$ and $b \in C(\Omega, \mathbb{R}^+)$.

Theorem 2.1. [Y. Du and Q. Huang [4], Lemma 2.1] Let $\underline{u}, \overline{u} \in C^2(\bar{\Omega})$ be both positive in $\bar{\Omega}$ such that

$$-\triangle \underline{u} \leq \lambda(x)\underline{u} - b(x)\underline{u}^p \qquad \text{in } \Omega,$$
$$-\triangle \overline{u} \geq \lambda(x)\overline{u} - b(x)\overline{u}^p \qquad \text{in } \Omega.$$

If $\underline{u} \leq \phi \leq \overline{u}$ on $\partial\Omega$, then $\underline{u}(x) \leq \overline{u}(x)$ on $\bar{\Omega}$.

Definition: If $\underline{u}$(resp. $\overline{u}$) satisfy the conditions in Theorem 2.1 and $\underline{u} \leq \phi$ on $\partial\Omega$ (resp. $\overline{u} \geq \phi$, then $\underline{u}$ (resp.$\overline{u}$) is called subsolution (resp. supsolution) of (2.1).

Theorem 2.2. Suppose $\phi \in C(\partial\Omega)$ and (2.1) possesses a non-negative solution. Let u be any non-negative solution of (2.1). Then $u(x) > 0$ for each $x \in \Omega$ and$\partial_\nu u(x) < 0$ for any $x \in \partial\Omega$ such that $u(x) = 0$; ν stands for the outward unit normal to Ω. Moreover, the positive solution is unique and if we denote it by Ψ and $\underline{u}$ (resp. $\overline{u}$) is a non-negative subsolution (resp. supersolution) of (2.1) then $\underline{u} \leq \Psi$ (resp. $\Psi \leq \overline{u}$)

Theorem 2.3. If $\underline{u}, \overline{u} \in C^2(\Omega)$ are both positive in Ω such that

$$-\triangle \underline{u} \leq \lambda(x)\underline{u} - b(x)\underline{u}^p \qquad \text{in } \Omega,$$

$$-\triangle \overline{u} \geq \lambda(x)\overline{u} - b(x)\overline{u}^p \qquad \text{in } \Omega,$$

$$\lim_{dist(x,\partial\Omega)\to 0} \underline{u}(x) = \infty, \quad \lim_{dist(x,\partial\Omega)\to 0} \overline{u}(x) = \infty,$$

and $\underline{u}(x) \leq \overline{u}(x)$ in Ω, then there exists at least one solution $u \in C^2(\Omega)$ of (1.1) satisfying $\underline{u}(x) \leq u \leq \overline{u}(x)$ in Ω.

Theorem 2.4. Suppose (1.1) possesses a non-negative solution, say Ψ, then the problem (2.1) possesses a unique non-negative solution for each $\phi \in C(\partial\Omega, R^+)$ denoted by u_ϕ and $u_\phi \leq \Psi$ in Ω. Furthermore,

$$\Psi_L := \limsup_{\inf_{\partial\Omega} \phi \to \infty} u_\phi$$

provides us with the minimal positive solution of (1.1).

Theorem 2.5 [T. Ouyang and Z. Xie [13], Theorem 1] Consider the radially symmetric semilinear elliptic equation:

$$\begin{cases} -\triangle u &= \lambda u - b(r)u^p \qquad \text{in } \Omega, \\ u &= \infty \qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $\Omega = B_R(x_0)$ is the ball of radius R centered at x_0 and $r = R - \|x - x_0\| = dist(x, \partial B_R(x_0))$. $\lambda \in \mathbb{R}$, $b \in C([0,R];[0,\infty))$ satisfying $b > 0$ in $(0,R]$, $\frac{B(r)}{b(r)} \in C^1([0,R])$, $\lim_{r\to 0} \frac{B(r)}{b(r)} = 0$ where $B(r) = \int_0^r b(s)ds$. Then the problem exists a unique solution u satisfying

$$\lim_{d(x)\to 0} \frac{u(x)}{K(b^*(r))^{-\beta}} = 1$$

where $d(x) = dist(x, \partial B_R(x_0))$ and

$$b^*(r) = \int_0^r \int_0^s b(t)dtds.$$

K and β are constant defined by

$$K = [\beta((\beta+1)C_0 - 1)]^{\frac{1}{p-1}}, \beta := \frac{1}{p-1}$$

and C_0 is defined as

$$C_0 = \lim_{r\to 0} \frac{(B(r))^2}{b^*(r)b(r)}.$$

As an immediate consequence from theorem 2.5, combining a translation together with a reflection about

$$r_0 := \frac{R_1 + R_2}{2},$$

it readily follows the corresponding result in each of the annuli

$$A_{R_1,R_2}(x_0) := \{x \in \mathbb{R}^N : 0 < R_1 < \|x - x_0\| < R_2\}.$$

Theorem 2.6. Consider the problem

$$\begin{cases} -\triangle u &= \lambda u - b(r)u^p \qquad \text{in } A_{R_1,R_2}(x_0), \\ u &= \infty \qquad\qquad\qquad \text{on } \partial A_{R_1,R_2}(x_0), \end{cases} \tag{2.3}$$

where $\lambda \in \mathbb{R}$, $r = dist(x, \partial A_{R_1,R_2}(x_0))$, and $R = \frac{R_2 - R_1}{2}$. $b \in C([0, R]; [0, \infty))$ satisfying $b > 0$ in $(0, R]$, $\frac{B(r)}{b(r)} \in C^1([0, R])$, $\lim_{r \to 0} \frac{B(r)}{b(r)} = 0$ where $B(r) = \int_0^r b(s)ds$. Then the problem exists a unique solution u satisfying

$$\lim_{x \to \partial A_{R_1,R_2}(x_0)} \frac{u(x)}{K(b^*(r))^{-\beta}} = 1$$

where $b^*(r), K$ and β are defined as in theorem 2.5.

3 Proof of Theorem 1.1

Let u be a large positive solution of (1.2). We first construct a large supsolution locally for each $x_0 \in \partial\Omega$.

Fix a sufficiently small $\epsilon > 0$, thanks to (1.5), there exist $\rho = \rho(\epsilon) \in (0, \tau)$ and $\mu = \mu(\epsilon) > 0$ such that

$$1 - \epsilon < \frac{b_x(r)}{b_{x_0}(r)} = \frac{a(x - r\mathbf{n}_x)}{a(x_0 - r\mathbf{n}_{x_0})} < 1 + \epsilon, \tag{3.1}$$

for any $x \in \partial\Omega \bigcap \bar{B}_\rho(x_0), r \in [0, \mu]$. Let

$$\mathcal{B} = \{x - r\mathbf{n}_x | (x, r) \in [\partial\Omega \bigcap \bar{B}_\rho(x_0)] \times [0, \mu]\}. \tag{3.2}$$

Because $\partial\Omega$ is smooth, ρ, μ can be shortened, if necessary, so that for each $y \in \mathcal{B}$ there exists a unique $(\pi(y), r(y)) \in [\partial\Omega \bigcap \bar{B}_\rho(x_0)] \times [0, \mu]$ and $y = \pi(y) - r(y)\mathbf{n}_{\pi(y)}$ and $r(y) = |y - \pi(y)| = dist(y, \partial\Omega)$. From now on, we assume that ρ, μ satisfy these requirements. Furthermore, there exists $R_0 \in (0, \min\{\frac{\rho}{2}, \frac{\mu}{2}\})$, such that

$$B_{R_0}(x_0 - R_0\mathbf{n}_{x_0}) \subset \Omega \bigcap \text{ Int } \mathcal{B} \qquad \text{and} \qquad \bar{B}_{R_0}(x_0 - R_0\mathbf{n}_{x_0}) \bigcap \partial\Omega = \{x_0\}. \tag{3.3}$$

Then for any $\delta \in [0, \delta_0]$ (δ_0 is chosen to be very small), the family of small ball is in $\Omega \bigcap$ Int $\mathcal{B}$ and

$$B_{R_0 - \delta}(x_0 - R_0\mathbf{n}_{x_0}) \subset \bar{B}_{R_0}(x_0 - R_0\mathbf{n}_{x_0}).$$

By (1.4) and (1.5), for each $\delta \in [0, \delta_0]$ and $y \in \bar{B}_{R_0 - \delta}(x_0 - R_0)$, we have

$$\begin{aligned} a(y) = a(\pi(y) - r(y)\mathbf{n}_{\pi(y)}) &> (1 - \epsilon)a(x_0 - r(y)\mathbf{n}_{x_0}) \\ &= (1 - \epsilon)b_{x_0}(r(y)) = (1 - \epsilon)b_{x_0}(dist(y, \partial\Omega)) \\ &\geq (1 - \epsilon)b_{x_0}(dist(y, \partial B_{R_0 - \delta}(x_0 - R_0\mathbf{n}_{x_0})). \end{aligned} \tag{3.4}$$

Therefore, for all $y \in \bar{B}_{R_0-\delta}(x_0 - R_0\mathbf{n}_{x_0})$,

$$a(y) \geq (1-\epsilon)b_{x_0}(r_\delta), \tag{3.5}$$

where $r_\delta = (R_0-\delta) - \|y - x_0 + R_0\mathbf{n}_{x_0}\| = dist(y, \partial B_{R_0-\delta}(x_0 - R_0\mathbf{n}_{x_0}))$.

Thanks to (3.5), for any $\delta \in [0, \delta_0]$, the restriction

$$\underline{u}_\delta := u|_{B_{R_0-\delta}}(x_0 - R_0\mathbf{n}_{x_0}) \tag{3.6}$$

is a positive smooth subsolution of

$$\begin{cases} -\triangle u & = \lambda u - (1-\epsilon)b_{x_0}(r)u^p \qquad \text{in } B_{R_0-\delta}(x_0 - R_0\mathbf{n}_{x_0}), \\ u & = \infty \qquad\qquad\qquad\qquad \text{on } \partial B_{R_0-\delta}(x_0 - R_0\mathbf{n}_{x_0}), \end{cases} \tag{3.7}$$

According to theorem 2.5, for each $\delta \in [0, \delta_0]$, there exists a unique solution Φ_δ of (3.7) satisfying

$$\lim_{x \to \partial B_{R_0-\delta}(x_0 - R_0\mathbf{n}_{x_0})} \frac{\Phi_\delta(x)}{K((1-\epsilon)b^*_{x_0}(r_\delta))^{-\beta}} = 1 \tag{3.8}$$

where $r_\delta = dist(x, \partial B_{R_0-\delta}(x_0 - R_0\mathbf{n}_{x_0}))$ and

$$b^*_{x_0}(r_\delta) = \int_0^{r_\delta} \int_s^{r_\delta} b_{x_0}(t)dtds.$$

K and β are constant defined by

$$K = [\beta((\beta+1)C_0 - 1)]^{\frac{1}{p-1}}, \beta := \frac{1}{p-1}$$

and C_0 is defined as

$$C_0 = \lim_{r_\delta \to R_0} \frac{(B_{x_0}(r_\delta))^2}{b^*_{x_0}(r_\delta)b_{x_0}(r_\delta)}.$$

For each $\delta \in [0, \delta_0]$, theorem 2.4 guarantee

$$\underline{u}_\delta = u|_{B_{R_0-\delta}}(x_0 - R_0\mathbf{n}_{x_0}) \leq \Phi_\delta$$

for x in $B_{R_0-\delta}(x_0 - R_0\mathbf{n}_{x_0})$. Thus

$$\limsup_{x \to \partial B_{R_0-\delta}(x_0 - R_0\mathbf{n}_{x_0})} \frac{u_\delta(x)}{K((1-\mu)b^*_{x_0}(r_\delta))^{-\beta}} \leq 1, \tag{3.9}$$

passing to the limit as $\delta \to 0$ gives

$$\limsup_{r \to 0} \frac{u(x_0 - r\mathbf{n}_{x_0})}{K((1-\epsilon)b^*_{x_0}(r))^{-\beta}} \leq 1. \tag{3.10}$$

In particular, (3.10) is valid for any sufficiently small $\epsilon > 0$, then

$$\limsup_{r \to 0} \frac{u(x_0 - r\mathbf{n}_{x_0})}{K(b^*_{x_0}(r))^{-\beta}} \leq 1. \tag{3.11}$$

To prove (1.6), we will build up a large subsolution having the adequate growth at $x_0 \in \partial\Omega$ so that we can show

$$1 \leq \limsup_{r\to 0} \frac{u(x_0 - r\mathbf{n}_{x_0})}{K(b^*_{x_0}(r))^{-\beta}}. \tag{3.12}$$

Since Ω has smooth boundary, for any $x_0 \in \partial\Omega$, there exist $R_2 > R_1 > 0$ and $\delta_0 > 0$ such that

$$\Omega \subset \bigcap_{\delta\in[0,\delta_0]} A_{R_1-\delta,R_2-\delta}(x_0 + R_1\mathbf{n}_{x_0})$$

and

$$\partial\Omega \bigcap \bar{A}_{R_1,R_2}(x_0 + R_1\mathbf{n}_{x_0}) = \{x_0\}.$$

Moreover, R_2 can be taken arbitrarily large. We suppose R_2 has been chosen to satisfy

$$\Omega \subset A_{R_1,\frac{R_2}{3}}(x_0 + R_1\mathbf{n}_{x_0}).$$

and R_1 can be taken arbitrarily small.

Fix a sufficiently small $\epsilon > 0$ and $x_0 \in \partial\Omega$. Thanks to (3.1),we have

$$1 - \epsilon < \frac{b_x(r)}{b_{x_0}(r)} = \frac{a(x - r\mathbf{n}_x)}{a(x_0 - r\mathbf{n}_{x_0})} < 1 + \epsilon, \tag{3.13}$$

for any $x \in \partial\Omega \bigcap \bar{B}_\rho(x_0), r \in [0,\mu]$. Pick up a small $2\eta < \min\{\rho,\mu\}$, for each $y \in B_{2\eta}(x_0) \bigcap \bar{\Omega}$,

$$\begin{aligned} a(y) &= a(\pi(y) - r(y)\mathbf{n}_{x_{\pi(y)}}) \leq (1+\epsilon)b_{x_0}(r(y)) \\ &= (1+\epsilon)b_{x_0}(dist(y,\partial\Omega)) \\ &\leq (1+\epsilon)b_{x_0}(dist(y,\partial B_{R_1}(x_0 + R_1\mathbf{n}_{x_0}))) \\ &= (1+\epsilon)b_{x_0}(dist(y,\partial A_{R_1,R_2}(x_0 + R_1\mathbf{n}_{x_0}))) \end{aligned} \tag{3.14}$$

Thus we can construct a radially symmetric function

$$\hat{a} : A_{R_1,R_2}(x_0 + R_1\mathbf{n}_{x_0}) \mapsto [0,\infty),$$

such that

$$\hat{a} \geq a \text{ in } \Omega, \tag{3.15}$$

by extending the function

$$\hat{a}(y) = \hat{a}(r) = (1+\epsilon)b_{x_0}(r), \tag{3.16}$$

where

$$r = dist(y,\partial A_{R_1,R_2}(x_0 + R_1\mathbf{n}_{x_0})) \qquad \text{and} \qquad y \in B_{2\eta}(x_0) \bigcap \bar{\Omega}.$$

Further, for $y \in \bar{\Omega}$, $\hat{a}$ also satisfies

$$\hat{a}(r_\delta) = \hat{a}(dist(y,\partial A_{R_1-\delta,R_2-\delta}(x_0 + R_1\mathbf{n}_{x_0}))) \geq a(y)$$

for a sufficiently small δ_0 and any $0 \leq \delta \leq \delta_0$.
For each sufficiently small $\delta > 0$, $\delta \in (0,\delta_0]$, consider the auxiliary problem

$$\begin{cases} -\triangle u &= \lambda u - \hat{a}(r)u^p \qquad \text{in } A_{R_1-\delta,R_2-\delta}(x_0 + R_1\mathbf{n}_{x_0}), \\ u &= \infty \qquad \text{on } \partial A_{R_1-\delta,R_2-\delta}((x_0 + R_1\mathbf{n}_{x_0}), \end{cases} \tag{3.17}$$

where $r = dist(x, \partial A_{R_1-\delta,R_2-\delta}(x_0 + R_1\mathbf{n}_{x_0})$.
From theorem (2.6), there exists a unique large positive solution $\Phi_{\epsilon,\delta}$ such that

$$\lim_{x\to\partial A_{R_1-\delta,R_2-\delta}(x_0+R_1\mathbf{n}_{x_0})} \frac{\Phi_{\epsilon,\delta}(x)}{K((1+\epsilon)b^*_{x_0}(r))^{-\beta}} = 1$$

where $b^*_{x_0}(r)$ and K are defined as before. Moreover, by construction, the restriction $\Phi_{\epsilon,\delta}|_\Omega$ provides us with a subsolution of (1.2). Thus, for each $\delta \in (0, \delta_0]$, we have

$$\Phi_{\epsilon,\delta}(x) \le u(x)$$

for each $x \in A_{R_1-\delta,R_2-\delta}(x_0 + R_1\mathbf{n}_{x_0}) \bigcap \Omega$, then

$$\lim_{r\to 0} \frac{u(x_0 - r\mathbf{n}_{x_0})}{K((1+\epsilon)b^*_{x_0}(r))^{-\beta}} = 1. \tag{3.18}$$

By passing $\mu \to 0$, it readily gives

$$\liminf_{x\to x_0, x\in A_{R_1,R_2}(x_0+R_1\mathbf{n}_{x_0})\bigcap\Omega} \frac{u(x)}{K(b^*_{x_0}(r))^{-\beta}} \ge 1. \tag{3.19}$$

So (3.12) and (3.19) conclude the proof of (1.6).

If the conditions (1.3), (1.4) and (3.5) are satisfied uniformly on $\partial\Omega$. To prove that (1.7) is satisfied uniformly on $\partial\Omega$, we may check the proof above whether it is true uniformly. It is clear that ρ, τ can be chosen small enough so that (3.1) is true for each $x, x_0 \in \partial\Omega$ sufficiently close. R_0 and R_1 can be chosen to be the same for each $x_0 \in \partial\Omega$. Also note that the limit of $b^*_{x_0}(r)$ is independent of the choice of δ in (3.9) and (3.18). Therefore

$$\lim_{r\to 0} \frac{u(x_0 - r\mathbf{n}_{x_0})}{K(b^*_{x_0}(r))^{-\beta}} = 1, \quad \text{uniformly in } x_0 \in \partial\Omega.$$

Proof of uniqueness. The proof of uniqueness basically follows the proof in [4], [6] and [10]. For any pair of solution u, v of (1.2),

$$\lim_{dist(y,\partial\Omega)\to 0} \frac{u(y)}{v(y)} = \lim_{r(y)\to 0} \frac{u(\pi(y) - r(y)\mathbf{n}_{\pi(y)})}{v(\pi(y) - r(y)\mathbf{n}_{\pi(y)}))} = 1, \quad \text{uniformly on } \partial\Omega.$$

Thus, for every $\epsilon > 0$, we can find $\delta > 0$(as small as we please) such that

$$(1-\epsilon)v(x) \le u(x) \le (1+\epsilon)v(x), \qquad x \in \Omega\backslash\Omega_\delta,$$

where, for each small enough $\delta > 0$, we have denoted

$$\Omega_\delta := \{x \in \Omega : dist(x, \partial\Omega) > \delta\}.$$

On the other hand $\underline{w} = (1-\epsilon)v(x)$ and $\bar{w} = (1+\epsilon)v(x)$ are sub and super solutions to

$$\begin{cases} -\triangle w &= \lambda w - a w^p \qquad \text{in } \Omega_\delta, \\ w &= u \qquad\qquad\quad \text{on } \partial\Omega_\delta. \end{cases} \tag{3.20}$$

The unique solution to this problem is $w = u$. Then by theorem 2.1

$$(1-\epsilon)v(x) \le u(x) \le (1+\epsilon)v(x)$$

holds in Ω_δ, therefore it is true in Ω. Letting $\epsilon \to 0$ we arrive at $u = v$.
Finally, thanks to the uniqueness, it follows from the abstract existence theory of [11] that, the problem (1.2) possesses a unique positive solution. This concludes the proof of the theorem. $\sharp$

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