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Pyramidal Central Configurations for Spatial 5-Body Problems*

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Abstract: In this paper, classifications are studied for all pyramidal central configurations with a regular trapezoid base. In fact, there are two such central configurations that the base is a regular trapezoid. From the results it is easy to obtain the classfications of pyramidal central configurations with rectangular base.

Key words: N-body problems; pyramidal central configuration; classification
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1 Introduction and Main Results

N-body problems ([1,2]) are related with the motion of masses $m_1, \cdots, m_N$ under the action of Newton's gravitation:

$$m_i \ddot{\bar{q}}_i = \frac{\partial U(\bar{q})}{\partial \bar{q}_i} \quad 1 \leqslant i \leqslant N \tag{1}$$

where $\bar{q}_i \in R^3, \bar{q} = (\bar{q}_1, \cdots, \bar{q}_N)$ and

$$U(\bar{q}) = \sum_{1 \leqslant i < j \leqslant N} \frac{m_i m_j}{|\bar{q}_i - \bar{q}_j|} \tag{2}$$

is the potential engergy of the system.

Define

$$X = \{\bar{q} = (\bar{q}_1, \cdots, \bar{q}_N) \in R^{3N} : \sum_{i=1}^{N} m_i \bar{q}_i = 0\} \tag{3}$$

$$\Delta_{ij} = \{\bar{q} = (\bar{q}_1, \cdots, \bar{q}_N) \in R^{3N} : \bar{q}_i = \bar{q}_j\}$$
$$1 \leqslant i, j \leqslant N \tag{4}$$

$$\Delta = U_{i<j} \Delta_{ij} \tag{5}$$

The set $X \setminus \Delta$ is called the configuration space.

Definition 1([3~6]) A point $\bar{q} = (\bar{q}_1, \cdots, \bar{q}_N) \in X \setminus \Delta$ is a central configuration (c.c.) if there exists a constant λ such that

$$\sum_{j=1, j \neq i}^{N} \frac{m_j m_i}{|\bar{q}_j - \bar{q}_i|^3} (\bar{q}_j - \bar{q}_i) = -\lambda m_i \bar{q}_i$$
$$1 \leqslant i \leqslant N \tag{6}$$

Where

$$\lambda = \frac{U}{I} \tag{7}$$

$$I = \sum_{i=1}^{N} m_i |\bar{q}_i|^2 \tag{8}$$

Definiton 2([2]) A central configuration of N bodies, $N-1$ of which are coplanar, the Nth being off the plane, is called a pyramidal central configuration (p.c.c). Equivalently, we will say that the c.c. has the shape of a pyramid.

Theorem 1 The necessary and sufficient conditions that the configuration with a regular trapezoid base is a pyramidal central configuration of five bodies are

1) The mutual distances are such that $D_{5i} = D_{5j}, 1 \leqslant i, j \leqslant 4$ and

$$h + l - 2g = 0 \tag{9}$$

where $D_{ij} = |\bar{q}_i - \bar{q}_j|$ for $1 \leqslant i, j \leqslant 5$

$$h = \frac{1}{D_{21}^3} = \frac{1}{D_{34}^3} \qquad l = \frac{1}{D_{31}^3} = \frac{1}{D_{24}^3}$$

$$g = \frac{1}{D_{5i}^3} \qquad D_{41} = 2b \qquad D_{23} = 2a$$

and h.l.g can be represented by a, b.

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2) The masses are such that $m_1 = m_4, m_2 = m_3$, m_5 is arbitary and

$$m_2 a(h - g) = m_1 b\left(\frac{1}{(2b)^3} - g\right) \tag{10}$$

$$m_2 a = \left(\frac{1}{(2a)^3} - g\right) = m_1 b(h - g) \tag{11}$$

2 Some general lemmas

Lemma 1 Let $\bar{q} = (\bar{q}_1, \cdots, \bar{q}_N)$ be a p.c.c such that $\bar{q}_N$ is at the top vertex which is off the plane containing $m_1, \cdots, m_{N-1}$, then m_N is equidistant from $m_1, \cdots, m_{N-1}$.

Proof Since $\bar{q} = (\bar{q}_1, \cdots, \bar{q}_N)$ forms a c.c then there exists a scalar λ such that

$$\sum_{j=1, j\neq i}^{N} \frac{m_j m_i}{|\bar{q}_j - \bar{q}_i|^3}(\bar{q}_j - \bar{q}_i) = -\lambda m_i \bar{q}_i$$
$$1 \leqslant i \leqslant N \tag{12}$$

Writing $\bar{q}_i = (\bar{x}_i, \bar{y}_i, \bar{z}_i) \in R^3$ in terms of its coordinate, and $D_{ji} = |\bar{q}_j - \bar{q}_i|$ for $1 \leqslant i, j \leqslant N$. Since the masses $m_1, \cdots, m_{N-1}$ lie on a common plane, we may assume then, whithout loss of generality, that this plane is parallel to $G_{\bar{x}\bar{z}}$, hence $\bar{y}_1 = \bar{y}_2 = \cdots = \bar{y}_{N-1}$. Multiplying (12) by $\bar{y}$ which is the unit vector of $\bar{y}$-direction. We obtain

$$\sum_{j=1, j\neq i}^{N} \frac{m_j m_i}{D_{ji}^3}(\bar{q}_j - \bar{q}_i)\bar{y} = -\lambda m_i \bar{q}_i \bar{y},$$
$$1 \leqslant i \leqslant N \tag{13}$$

From (13), for $i = 1, i = 2$, obviously we can get

$$\frac{m_N m_1}{D_{N1}^3}(\bar{y}_N - \bar{y}_1) = -\lambda m_1 \bar{y}_1 \tag{14}$$

$$\frac{m_N m_2}{D_{N2}^3}(\bar{y}_N - \bar{y}_2) = -\lambda m_2 \bar{y}_2 \tag{15}$$

Hence (14), (15) give

$$m_N\left(\frac{1}{D_{N1}^3} - \frac{1}{D_{N2}^3}\right)(\bar{y}_N - \bar{y}_1) = 0 \tag{16}$$

Since $\bar{y}_N - \bar{y}_1 \neq 0$ otherwise $m_1, \cdots, m_N$ are coplanar which is contradict to definition 2 then

$$D_{N1} = D_{N2}$$

Similarily, we readily obtain

$$D_{Ni} = D_{Nj} \qquad 1 \leqslant i, j \leqslant N - 1$$

i.e. m_N is equidistant from $m_1, \cdots, m_{N-1}$.

Remark The masses $m_1, \cdots, m_{N-1}$ are concyclic.

In fact, they lie on the intersection of a plane with a sphere, for they are coplanar by assumption, and they belong to a sphere centered at m_N by Lemma 1.

Lemma 2 If $\bar{q} = (\bar{q}_1, \cdots, \bar{q}_N)$ is a p.c.c then λ mg, where $m = m_1 + \cdots + m_N$ is the total masse and $= \frac{1}{D_{Ni}^3}, 1 \leqslant i \leqslant N - 1$.

Proof Denote by Oxyz, the coordinate system ob tained from $G\bar{x}\bar{y}\bar{z}$ by parallel translation to a new or gin $O \in P$ containing $m_1, \cdots, m_{N-1}$. Let $q_1, \cdots, q_N$ b the position vectors of $m_1, \cdots, m_N$ in $Oxyz$. Obviously

$$\overline{OG} = \frac{1}{m}\sum_{j=1}^{N} m_j q_j \tag{17}$$

Since $\bar{q} = (\bar{q}_1, \cdots, \bar{q}_N)$ is a c.c., there exists a λ suc that

$$\sum_{j=1, j\neq i}^{N} \frac{m_j m_i}{|\bar{q}_j - \bar{q}_i|^3}(\bar{q}_j - \bar{q}_i) = -\lambda m_i \bar{q}_i$$
$$1 \leqslant i \leqslant N \tag{18}$$

Taking the scalar multiple of upper equation with $\bar{y}$ which is a unit vector in $\bar{y}$-direction for $i = 1, \cdots, N$ an using $\bar{q}_i = q_i - \overline{OG}$. We get

$$\sum_{j=1, j\neq i}^{N} \frac{m_j m_i}{|q_j - q_i|^3}(q_j - q_i) = -\lambda m_i(q_i - \overline{OG}) \tag{19}$$

that is

$$\sum_{j=1, j\neq i}^{N} \frac{m_j m_i}{|\bar{q}_j - \bar{q}_i|^3}(\bar{q}_j - \bar{q}_i) = -\lambda m_i\left(\frac{1}{m}\sum_{j=1}^{N} m_j q_i - \frac{1}{m}\sum_{j=1}^{N} m_j q_j\right) \tag{20}$$

or

$$\sum_{j=1, j\neq i}^{N} \frac{m_j m_i}{|q_j - q_i|^3}(q_j - q_i) = \frac{\lambda}{m} m_i \sum_{j=1, j\neq i}^{N} m_j(q_j - q_i) \tag{21}$$

or

$$\sum_{j=1, j\neq i}^{N} m_j m_i\left(\frac{1}{D_{ji}^3} - \frac{\lambda}{m}\right)(q_j - q_i) = 0 \tag{22}$$

then

$$\sum_{j=1, j\neq i}^{N} m_j m_i\left(\frac{1}{D_{ji}^3} - \frac{\lambda}{m}\right)(q_j - q_i)\bar{y} = 0 \tag{23}$$

But $\bar{y}$ is perpendicular to the plane P containing the vectors $q_1, \cdots, q_{N-1}$ then

$$m_N m_i\left(\frac{1}{D_{ji}^3} - \frac{\lambda}{m}\right) q_N \bar{y} = 0 \tag{24}$$

Hence

$$\lambda = \frac{m}{D_{Ni}^3} \qquad 1 \leqslant i \leqslant N - 1 \tag{25}$$

Note: The equation (18) holds if and only if the equation (22) holds.

3 Necessary Conditions of Pyramidal Central Configuration for Five-bodies

Assume m_1, m_2, m_3, m_4 be coplanar and counterclockwise and $\boldsymbol{P}_1\boldsymbol{P}_4$ parallel to $\boldsymbol{P}_2\boldsymbol{P}_3$. Since $m_1, \cdots, m_4$ are concyclic, then $D_{21} = D_{34}$, $D_{42} = D_{31}$. Denote by $Oxyz$, the coordinate system obtained from $G\bar{x}\bar{y}\bar{z}$ by parallel translation to a new origin $O \in P$ containing $m_1, \cdots, m_4$, at the middle point between $\boldsymbol{P}_1$ and $\boldsymbol{P}_4$ (see Figure 1). Let $\boldsymbol{q}_1, \cdots, \boldsymbol{q}_5$ be the position vectors of $m_1, \cdots, m_5$ in $Oxyz$. Then

$$q_1 = (0,0,b), \boldsymbol{q}_2 = (c,0,a),$$
$$q_3 = (c,0,-a), \boldsymbol{q}_4 = (0,0,-b)$$

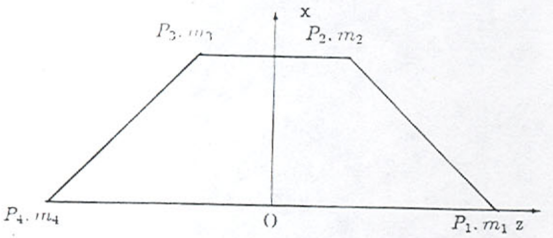

Figure 1 The regular trapezoid base

where c is distance between the line $\boldsymbol{P}_1\boldsymbol{P}_4$ and the line $\boldsymbol{P}_2\boldsymbol{P}_3$. Obviously $c>0$, otherwise if $c=0$, then m_1, m_2, m_3, m_4 are colinear. But m_5 is equidistant from $m_1, \cdots, m_4$, then $\boldsymbol{q}_1 = \boldsymbol{q}_2$, $\boldsymbol{q}_3 = \boldsymbol{q}_4$. This is contradict to the definition of configuration space.

If $\bar{\boldsymbol{q}} = (\bar{\boldsymbol{q}}_1, \cdots, \bar{\boldsymbol{q}}_5)$ is a p.c.c., we have had

1) $D_{5i} = D_{5j}, 1 \leqslant i, j \leqslant 4$

2) $\lambda = mg = \dfrac{m}{D_{5i}^3}, 1 \leqslant i \leqslant 4$

By definition of p.c.c. we get

$$\sum_{j=1, j\neq i}^{5} \frac{m_j}{D_{ji}^3}(\bar{\boldsymbol{q}}_j - \bar{\boldsymbol{q}}_i) = -\lambda \bar{\boldsymbol{q}}_i \qquad 1 \leqslant i \leqslant 5 \qquad (26)$$

that is

$$\sum_{j=1, j\neq i}^{5} \frac{m_j}{D_{ji}^3}(\boldsymbol{q}_j - \boldsymbol{q}_i) = -\lambda(\boldsymbol{q}_i - \overline{\boldsymbol{OG}}) \qquad (27)$$

or

$$\sum_{j=1, j\neq i}^{5} m_j\left(\frac{1}{D_{ji}^3} - \frac{\lambda}{m}\right)(\boldsymbol{q}_j - \boldsymbol{q}_i) = 0 \qquad (28)$$

using $\dfrac{1}{D_{5i}^3} = \dfrac{\lambda}{m}, 1 \leqslant i \leqslant 4$

$$\sum_{j=1, j\neq i}^{4} m_j\left(\frac{1}{D_{ji}^3} - \frac{\lambda}{m}\right)(\boldsymbol{q}_j - \boldsymbol{q}_i) = 0 \qquad (29)$$

Taking the scalar multiple with $\boldsymbol{x}$ which is a unit vector in $\boldsymbol{x}$-direction we get for $i=1$

$$m_2\left(\frac{1}{D_{21}^3} - \frac{\lambda}{m}\right)c + m_3\left(\frac{1}{D_{31}^3} - \frac{\lambda}{m}\right)c = 0 \qquad (30)$$

for $i=2$

$$m_1\left(\frac{1}{D_{12}^3} - \frac{\lambda}{m}\right)(-c) + m_4\left(\frac{1}{D_{42}^3} - \frac{\lambda}{m}\right)(-c) = 0 \qquad (31)$$

for $i=3$

$$m_1\left(\frac{1}{D_{13}^3} - \frac{\lambda}{m}\right)(-c) + m_4\left(\frac{1}{D_{43}^3} - \frac{\lambda}{m}\right)(-c) = 0 \qquad (32)$$

for $i=4$

$$m_2\left(\frac{1}{D_{24}^3} - \frac{\lambda}{m}\right)c + m_3\left(\frac{1}{D_{43}^3} - \frac{\lambda}{m}\right)c = 0 \qquad (33)$$

Using $c>0$, and

$$\frac{1}{D_{21}^3} = \frac{1}{D_{34}^3} = h; \frac{1}{D_{24}^3} = \frac{1}{D_{31}^3} = l; g = \frac{\lambda}{m} = \frac{1}{D_{5i}^3} \qquad (34)$$

We have

for $i=1$ $\quad m_2(h-g) + m_3(l-g) = 0 \qquad (35)$

for $i=2$ $\quad m_1(h-g) + m_4(l-g) = 0 \qquad (36)$

for $i=3$ $\quad m_1(l-g) + m_4(h-g) = 0 \qquad (37)$

for $i=4$ $\quad m_2(l-g) + m_3(h-g) = 0 \qquad (38)$

Summing all the equation (35~38), obtain

$$(m_1 + m_2 + m_3 + m_4)(h + l - 2g) = 0$$

that is $h + l - 2g = 0 \qquad (39)$

By (35~37), we obtain $m_2(h-l) + m_3(l-h) = 0$

That is $\quad (m_2 - m_3)(h - l) = 0$

Since h is not equal to l, hence

$$m_2 = m_3 \qquad (40)$$

Similarily, (36), (37) which gives

$$m_1 = m_4 \qquad (41)$$

Taking the scalar multiple with z which is a unit vector in z-direction for (29), we obtain

for $i=1$

$$m_2\left(\frac{1}{D_{21}^3} - \frac{\lambda}{m}\right)(a-b) + m_3\left(\frac{1}{D_{31}^3} - \frac{\lambda}{m}\right)(-a-b) + m_4\left(\frac{1}{D_{41}^3} - \frac{\lambda}{m}\right)(-2b) = 0 \qquad (42)$$

for $i=2$

$$m_1\left(\frac{1}{D_{21}^3} - \frac{\lambda}{m}\right)(b-a) + m_3\left(\frac{1}{D_{32}^3} - \frac{\lambda}{m}\right)(-2a) +$$

$$m_4\left(\frac{1}{D_{42}^3}-\frac{\lambda}{m}\right)(-b-a)=0 \tag{43}$$

for $i=3$

$$m_1\left(\frac{1}{D_{31}^3}-\frac{\lambda}{m}\right)(b+a)+m_2\left(\frac{1}{D_{32}^3}-\frac{\lambda}{m}\right)(2a)+$$

$$m_4\left(\frac{1}{D_{43}^3}-\frac{\lambda}{m}\right)(-b+a)=0 \tag{44}$$

for $i=4$

$$m_1\left(\frac{1}{D_{41}^3}-\frac{\lambda}{m}\right)(2b)+m_2\left(\frac{1}{D_{42}^3}-\frac{\lambda}{m}\right)(a+b)+$$

$$m_3\left(\frac{1}{D_{43}^3}-\frac{\lambda}{m}\right)(b-a)=0 \tag{45}$$

that is

$$m_2(h-g)(a-b)+m_2(l-g)(-a-b)+$$

$$m_1\left(\frac{1}{(2b)^3}-g\right)(-2b)=0 \tag{46}$$

$$m_1(h-g)(b-a)+m_2\left(\frac{1}{(2a)^3}-g\right)(-2a)+$$

$$m_1(l-g)(-b-a)=0 \tag{47}$$

$$m_1(l-g)(b+a)+m_2\left(\frac{1}{(2a)^3}-g\right)(2a)+$$

$$m_1(h-g)(-b+a)=0 \tag{48}$$

$$m_1\left(\frac{1}{(2b)^3}-g\right)(2b)+m_2(l-g)(a+b)+$$

$$m_2(h-g)(b-a)=0 \tag{49}$$

Since $h+l-2g=0$, then

$$m_2a(l-g)+m_1b\left(\frac{1}{(2b)^3}-g\right)=0 \tag{50}$$

$$m_2a\left(\frac{1}{(2a)^3}-g\right)+m_1b(l-g)=0 \tag{51}$$

$$m_1a(l-g)+m_2a\left(\frac{1}{(2a)^3}-g\right)=0 \tag{52}$$

$$m_1b\left(\frac{1}{(2b)^3}-g\right)+m_2a(l-g)=0 \tag{53}$$

That is

$$m_2a(h-g)=m_1b\left(\frac{1}{(2b)^3}-g\right) \tag{54}$$

$$m_2a\left(\frac{1}{(2a)^3}-g\right)=m_1b(h-g) \tag{55}$$

Thus we have proved:

Theorem 2 Given $m_1,\cdots,m_5$ such that m_5 is at the top vertex of the pyramid and $m_1,\cdots,m_4$ are at the vertices of the base, then

1) mutual distances satisfy:

$$D_{5i}=D_{5j} \qquad 1\leqslant i,j\leqslant 4 \tag{56}$$

$$h+l-2g=0 \tag{57}$$

2) the masses satisfy:

m_5 is arbitrary, $m_1=m_4$, $m_2=m_3$ and

$$m_2a(h-g)=m_1b\left(\frac{1}{(2b)^3}-g\right) \tag{58}$$

$$m_2a\left(\frac{1}{(2a)^3}-g\right)=m_1b(h-g) \tag{59}$$

where $\frac{1}{D_{5i}^3}=g$, $\frac{1}{D_{21}^3}=\frac{1}{D_{34}^3}=h$, $\frac{1}{D_{31}^3}=\frac{1}{D_{42}^3}=l$, $D_{41}=2b$, $D_{23}=2a$.

4 Sufficient Conditions of Pyramidal Central Configuration for Five-bodies

Theorem 3 If the mutual distances among $m_1,\cdots,m_4$ at the vertices of a regular trapezoid and m_5 at the top vertex are given by Theorem 2 (1) and the masses $m_1,\cdots,m_5$ are given by Theroem 2(2), then $\bar{q}=(\bar{q}_1,\cdots,\bar{q}_5)$ forms a pyramidal central configuration.

Proof Consider a coordinate system $G\bar{x}\bar{y}\bar{z}$ with origin at the center of the masses $m_1,\cdots,m_5$ such that $G\bar{x}\bar{z}$ is parallel to P containing $m_1,\cdots,m_4$. Let $\bar{q}_i$ be the position vector of m_i, $1\leqslant i\leqslant 5$, in the coordinate system $G\bar{x}\bar{y}\bar{z}$. And let $Oxyz$ be the parallet translate of $G\bar{x}\bar{y}\bar{z}$ with origin O at the middle point of $\boldsymbol{P}_1\boldsymbol{P}_4$.

By lemma 2, we can choose

$$\lambda=\frac{m}{D_{5i}^3},1\leqslant i\leqslant 4 \tag{60}$$

Since(the proof can be seen in the proof lemma 2)

$$\sum_{j=1,j\neq i}^{5}\frac{m_jm_i}{D_{ji}^3}(\bar{q}_j-\bar{q})=-\lambda m_i\bar{q}_i \tag{61}$$

holds if and only if

$$\sum_{j=1,j\neq i}^{4}m_j\left(\frac{1}{D_{ji}^3}-\frac{\lambda}{m}\right)(q_j-q_i)=0 \tag{62}$$

Thus we only need prove that

$$\sum_{j=1,j\neq i}^{4}m_j\left(\frac{1}{D_{ji}^3}-\frac{\lambda}{m}\right)(q_j-q_i)=0 \tag{63}$$

Since $\boldsymbol{q}_1=(0,0,b)$, $\boldsymbol{q}_2=(c,0,a)$, $\boldsymbol{q}_3=(c,0,-a)$, $\boldsymbol{q}_4=(0,0,-b)$

Obviously

$$\sum_{j=1,j\neq i}^{4}m_j\left(\frac{1}{D_{ji}^3}-\frac{\lambda}{m}\right)(q_j-q_i)y=0 \tag{64}$$

and seeing (35～38) we can get

$$\sum_{j=1,j\neq i}^{4}m_j\left(\frac{1}{D_{ji}^3}-\frac{\lambda}{m}\right)(q_j-q_i)x=0 \tag{65}$$

and seeing from (42) to (53) we can get

$$\sum_{j=1,j\neq i}^{4}m_j\left(\frac{1}{D_{ji}^3}-\frac{\lambda}{m}\right)(q_j-q_i)z=0 \tag{66}$$