Super central configurations of the three-body problem under the inverse integer power law

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In this paper, we consider the problem of central configurations of the n-body problem with the general homogenous potential $1/r^{\alpha}$, where α is a positive integer. A configuration $q = (q_1, q_2, \cdots, q_n)$ is called a super central configuration if there exists a positive mass vector $m = (m_1, \cdots, m_n)$ such that q is a central configuration for m with m_i attached to q_i and q is also a central configuration for m', where $m' \neq m$ and m' is a permutation of m. The main result in this paper is the existence and classifications of super central configurations in the rectilinear three-body problem with general homogenous potential. Our results extend the previous work [Xie, Z., J. Math. Phys. **51**, 042902 (2010)] from the case in which $\alpha = 2$ to the case in which α is a positive integer. Descartes' rule of sign is extensively used in the proof of the main theorem. © *2011 American Institute of Physics*. [doi:10.1063/1.3630121]

I. INTRODUCTION

The existence and classifications of super central configurations of the n-body problem are first studied in the paper[8] for the rectilinear three-body problem under Newtonian gravitational law and in the paper[9] for the rectilinear four-body problem. A surprising connection with the golden ratio[10] has been discovered in the super central configurations under general homogenous potential. In this paper, we study the super central configurations of three-body problem under the general inverse integer power law.

Consider the n-body problem in the general law of attraction $\frac{1}{s^{\alpha}}$ with Newton's law of gravitation $(\frac{1}{s^2})$ as a special case:[4]

$$m_k \ddot{q}_k = \sum_{j=1, j\neq k}^{n} m_k m_j \frac{(q_j - q_k)}{|q_j - q_k|^{\alpha+1}} \qquad 1 \leq k \leq n, \tag{1.1}$$

where $m_k > 0$ are the masses of the bodies, $q_k \in \mathbf{R}^d$ (usually with $d = 1$, $d = 2$, or $d = 3$) are their positions, respectively. Let $C = m_1 q_1 + \cdots + m_n q_n$, $M = m_1 + \cdots + m_n$, $c = C/M$ be the first moment, total mass and center of mass of the bodies, respectively.

A central configuration is a special configuration of positions of the n-bodies. The gravitational forces can be exactly balanced among the bodies. A super central configuration is a configuration who is a central configuration for at least two different arrangements of a given mass vector. A configuration q is a *central configuration* (CC for short) for a given mass vector $m = (m_1, m_2, \cdots, m_n) \in (\mathbf{R}^+)^n$ if q is a solution of the following nonlinear algebraic equation system:

$$\sum_{j=1, j\neq k}^{n} m_j \frac{(q_j - q_k)}{|q_j - q_k|^{\alpha+1}} = -\lambda(q_k - c) \qquad 1 \leq k \leq n, \tag{1.2}$$

for a constant λ.

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Because the systems (1.1) and (1.2) are singular when two particles have the same positions, it is natural to assume that the configuration avoids the collision set which is defined by $\triangle = \bigcup\{q = (q_1, q_2, \cdots, q_n) \in (\mathbf{R}^d)^n | q_i = q_j \text{ for some } i \neq j\}$. To avoid singularities we will restrict q to be in $\left\{q = (q_1, q_2, \cdots, q_n) \in (\mathbf{R}^d)^n\right\} \setminus \triangle$.

We introduce the notion of admissible ordered sets of masses before we define super central configurations. Given a configuration $q = (q_1, q_2, \cdots, q_n)$, denote $S(q, \alpha)$ the admissible ordered set of masses for fixed $\alpha > 0$ by

$$S(q, \alpha) = \left\{m = (m_1, \cdots, m_n) | m_i \in \mathbf{R}^+, q \text{ is a CC for } m\right\}. \tag{1.3}$$

For a given $m \in S(q, \alpha)$, let $S_m(q, \alpha)$ be the permutationally admissible set for m, defined by

$$S_m(q, \alpha) = \{m' \in S(q, \alpha) | m' \neq m \text{ and } m' \text{ is a permutation of } m\}. \tag{1.4}$$

Denote the number of elements in $S_m(q, \alpha)$ by ${}^{\#}S_m(q, \alpha)$. Directly from the definition, ${}^{\#}S_m(q, \alpha)$ is finite and has $n! - 1$ as an upper bound.

Definition 1.1: [Super central configuration]. A configuration q is called a *super central configuration* (SCC for short) for some $\alpha > 0$ if there exists a positive mass vector m such that $S_m(q, \alpha)$ is not empty.

The main results of the current paper are the existence and classifications of super central configuration in the rectilinear three-body problem under general inverse integer power law. Descartes' rule and intermediate value theorem will play a major role in the proof. Problems on central configurations have been studied in Refs. 2, 3, 7, and 10 under quasi-homogeneous or homogeneous potential. Because central configurations are invariant up to translation and scaling, we can choose the coordinate system so that all the three bodies are on the x-axis with positions $q_1 = 0$, $q_2 = 1$, and $q_3 = 1 + r$, where $r > 0$. This is a general form of rectilinear three-body configuration. Fix $r > 0$. Let $\alpha > 0$ be a positive integer and $q = (0, 1, 1 + r)$. Then $f_1(r, \alpha)$, $f_2(r, \alpha)$, $f_3(r, \alpha)$, and $g(r, \alpha)$ are polynomial functions which are defined as follows:

$$f_1(r, \alpha) = (1 + r)^{\alpha} - r^{\alpha+1} + (1 + r)^{\alpha} r^{\alpha+2},$$

$$f_2(r, \alpha) = (1 + r)^{\alpha+2} - (1 + r)^{\alpha+1} r^{\alpha+1} - r^{\alpha+2},$$

$$f_3(r, \alpha) = -r^{\alpha} - (1 + r)^{\alpha+1} + (1 + r)^{\alpha+2} r^{\alpha},$$

$$g(r, \alpha) = ((1 + r)^{\alpha} r^{\alpha} - r^{\alpha} + (1 + r)^{\alpha}) \left(1 + r + r^2\right).$$

For each fixed $\alpha > 0$, we also define $\Gamma(\alpha)$ by

$$\Gamma(\alpha) = \{r | f_2(r, \alpha) > 0, f_3(r, \alpha) > 0, r > 0\}. \tag{1.5}$$

Lemma 1.2: For each fixed positive integer α, (1) $f_1(r, \alpha) > 0$ and $g(r, \alpha) > 0$ for any positive r; (2) $f_2(r, \alpha) = 0$ has exactly one positive root $\bar{r}$ which is greater than one; (3) $f_3(r, \alpha) = 0$ has exactly one positive root $\underline{r}$ which is less than one; (4) $f_2(r, \alpha) > f_1(r, \alpha) > f_3(r, \alpha) > 0$ when $\underline{r} < r < 1$; $f_1(r, \alpha) = f_2(r, \alpha) = f_3(r, \alpha) > 0$ when $r = 1$; and $f_3(r, \alpha) > f_1(r, \alpha) > f_2(r, \alpha) > 0$ when $1 < r < \bar{r}$.

Our main results state as follows.

Theorem 1.3:

(1) *For any $m = (m_1, m_2, m_3) \in S(q, \alpha)$, $S_m(q, \alpha)$ has at most one element, i.e., either ${}^{\#}S_m(q, \alpha) = 0$ or ${}^{\#}S_m(q, \alpha) = 1$.*

(2) *$q = (0, 1, 1 + r)$ is a super central configuration if and only if $r \in \Gamma(\alpha)\backslash\{1\}$. If $r \notin \Gamma(\alpha)$ or $r = 1$, $S_m(q, \alpha)$ is empty for any $m \in S(q, \alpha)$.*

(3) *If $r \in \Gamma(\alpha)\backslash\{1\}$, then ${}^{\#}S_m(q, \alpha) = 1$ only in the following two cases:*

(i) *$S_m(q, \alpha) = \{(m_2, m_3, m_1)\}$ and m is given by*

$$m_1 = \frac{M f_1(r, \alpha)}{g(r, \alpha)}, \quad m_2 = \frac{M f_2(r, \alpha)}{g(r, \alpha)}, \quad m_3 = \frac{M f_3(r, \alpha)}{g(r, \alpha)}. \tag{1.6}$$

(ii) *$S_m(q, \alpha) = \{(m_3, m_1, m_2)\}$ and m is given by*

$$m_1 = \frac{M f_2(r, \alpha)}{g(r, \alpha)}, \quad m_2 = \frac{M f_3(r, \alpha)}{g(r, \alpha)}, \quad m_3 = \frac{M f_1(r, \alpha)}{g(r, \alpha)}. \tag{1.7}$$

Remark 1.4: The properties of $f_i(r, \alpha)$ are shown in Sec. II. The masses m_1, m_2, m_3 given by (1.6) and (1.7) are mutually distinct. More precisely, if masses m_1, m_2, m_3 is given by (1.6), then $m_2 > m_1 > m_3$ for $\underline{r} < r < 1$, $m_1 = m_2 = m_3$ for $r = 1$, and $m_3 > m_1 > m_2$ for $1 < r < \bar{r}$. If masses m_1, m_2, m_3 is given by (1.7), then $m_1 > m_3 > m_2$ for $\underline{r} < r < 1$, $m_1 = m_2 = m_3$ for $r = 1$, and $m_2 > m_3 > m_1$ for $1 < r < \bar{r}$. The main tools are Descartes' rule of sign and intermediate value theorem. Since Descartes' rule of sign works for the polynomials, we are able to extend the classifications of super central configurations from the case in which $\alpha = 2$ to the case in which α is a positive integer. A new method has been applied to the case of general homogenous potential for non-polynomials and a similar result has been obtained in Ref. 10.

II. THE PROOF OF LEMMA 1.2

Because $(1 + r)^\alpha > r^\alpha > r^{\alpha+1}$ for $0 < r < 1$ and $(1 + r)^\alpha r^{\alpha+2} > r^{\alpha+1}$ for $r \geq 1$, $f_1(r, \alpha) > 0$ and $g(r, \alpha) > 0$ for any positive r.

Because α is a positive integer, we can use binomial formula to expand $f_2(r, \alpha)$ as follows:

$$f_2(r, \alpha) = 1 + \binom{\alpha+2}{1} r + \binom{\alpha+2}{2} r^2 + \cdots + \left(\binom{\alpha+2}{\alpha+1} - 1\right) r^{\alpha+1}$$

$$- \binom{\alpha+1}{1} r^{\alpha+2} - \binom{\alpha+1}{2} r^{\alpha+3} \cdots - \binom{\alpha+1}{\alpha+1} r^{2\alpha+2}.$$

Since $f_2(r, \alpha)$ has only one change of sign, it has exactly one positive root by Descartes' rule. We also note that $f_2(1, \alpha) = 2^{\alpha+1} - 1 > 0$ and $\lim_{r\to\infty} f_2(r, \alpha) = -\infty$. By intermediate value theorem, the unique positive root $\bar{r}$ of $f_2(r, \alpha) = 0$ is greater than 1. By similar arguments, we can prove the property for $f_3(r, \alpha)$.

To prove the order of $f_i(r, \alpha)$ in Lemma 1.2, we will compute $f_2(r, \alpha) - f_1(r, \alpha)$ and $f_1(r, \alpha) - f_3(r, \alpha)$. By direct computation, we have

$$\begin{aligned}
&f_2(r, \alpha) - f_1(r, \alpha) = \\
&2r\,(1 + r)^\alpha + r^2\,(1 + r)^\alpha + r^{\alpha+1} - (1 + r)^{\alpha+1}\, r^{\alpha+1} - r^{\alpha+2} - (1 + r)^\alpha\, r^{\alpha+2} \\
&= 2\binom{\alpha}{0} r + \left[2\binom{\alpha}{1} + \binom{\alpha}{0}\right] r^2 + \left[2\binom{\alpha}{2} + \binom{\alpha}{1}\right] r^3 + \cdots
\end{aligned}$$

$$
\begin{aligned}
&+\left[2\binom{\alpha}{\alpha-1}+\binom{\alpha}{\alpha-2}\right]r^{\alpha}+\left[2\binom{\alpha}{\alpha}+\binom{\alpha}{\alpha-1}\right]r^{\alpha+1}\\
&-\left[\binom{\alpha+1}{1}+\binom{\alpha}{0}\right]r^{\alpha+2}-\left[\binom{\alpha+1}{2}+\binom{\alpha}{1}\right]r^{\alpha+3}-\cdots\\
&-\left[\binom{\alpha+1}{\alpha+1}+\binom{\alpha}{\alpha}\right]r^{2\alpha+2}.
\end{aligned}
$$

By the Descartes' rule, $f_2(r,\alpha)-f_1(r,\alpha)=0$ has exactly one positive root. In addition, by the directly computation, $f_2(1,\alpha)-f_1(1,\alpha)=0$. We also note that $f_2(r,\alpha)-f_1(r,\alpha)$ is positive when $r>0$ is sufficiently small since the term with lowest degree $2\binom{\alpha}{0}r$ is positive. So $f_2(r,\alpha)>f_1(r,\alpha)>0$ when $\underline{r}<r<1$; $f_1(r,\alpha)=f_2(r,\alpha)>0$ when $r=1$; and $f_1(r,\alpha)>f_2(r,\alpha)>0$ when $1<r<\bar{r}$.

By similar argument on $f_1(r,\alpha)-f_3(r,\alpha)$, we can prove that $f_1(r,\alpha)>f_3(r,\alpha)>0$ when $\underline{r}<r<1$; $f_1(r,\alpha)=f_3(r,\alpha)>0$ when $r=1$; and $f_3(r,\alpha)>f_1(r,\alpha)>0$ when $1<r<\bar{r}$. This completes the proof of Lemma 1.2.

III. THE PROOF OF THEOREM 1.3

Denote the six permutations in $P(3)$ by

$$
\tau_1=(1,2,3),\qquad \tau_2=(3,1,2),\qquad \tau_3=(2,3,1),
$$

$$
\tau_4=(1,3,2),\qquad \tau_5=(2,1,3),\qquad \tau_6=(3,2,1).
$$

Fix $\alpha>0$ and $r>0$. Let $q=(0,1,1+r)$ be a central configuration for some appropriately chosen positive mass vector $m=(m_1,m_2,m_3)$ where m_i is attached to q_i. So $S(q,\alpha)$ is not an empty set. Conversely, r is uniquely determined by the mass vector (see Refs. 1, 5, 6, and 7). Since $S_m(q,\alpha)$ is a subset of $\{m(\tau)|\tau\in P(3)\}$, we only need to check whether the other five permutations of mass m are also in $S(q,\alpha)$. Because $q=(0,1,1+r)$ is fixed, when we say $m(\tau)\equiv(m_{\tau(1)},m_{\tau(2)},m_{\tau(3)})\in S(q,\alpha)$ for some $\tau\in P(3)$, we always mean that $m_{\tau(i)}$ is attached to q_i for all $i=1,2,3$.

Then (1.2) is equivalent to

$$
\begin{cases}
m_2+\frac{m_3}{(1+r)^{\alpha}}-\lambda\, c=0,\\
-m_1+\frac{m_3}{r^{\alpha}}+\lambda\,(1-c)=0,\\
-\frac{m_1}{(1+r)^{\alpha}}-\frac{m_2}{r^{\alpha}}+\lambda\,(1+r-c)=0,
\end{cases}
\tag{3.1}
$$

where $c=\frac{m_2+m_3(1+r)}{M}$ is the center of mass. M would be regarded as a parameter. Given any $r>0$, we want to know whether there exist positive masses m_1,m_2,m_3 which is a solution for (3.1) with appropriate choices of λ, M. Substituting c into above equation, (3.1) becomes a linear equation of the masses m_1,m_2,m_3. By row reduction, we have

$$
\begin{cases}
m_1=-\frac{(1+r)^{\alpha}\left(-M+r^{\alpha+1}\lambda\right)}{(1+r)^{\alpha}r^{\alpha}-r^{\alpha}+(1+r)^{\alpha}},\\
m_2=\frac{r^{\alpha}\left((1+r)^{\alpha+1}\lambda-M\right)}{(1+r)^{\alpha}r^{\alpha}-r^{\alpha}+(1+r)^{\alpha}},\\
m_3=-\frac{(\lambda-M)(1+r)^{\alpha}r^{\alpha}}{(1+r)^{\alpha}r^{\alpha}-r^{\alpha}+(1+r)^{\alpha}}.
\end{cases}
\tag{3.2}
$$

By direct computation and simplification, we have the total mass M:

$$
M=m_1+m_2+m_3,
$$

and the center of mass c:

$$
c=\frac{m_2+m_3(1+r)}{M}=\frac{r^{\alpha}\left((1+r)^{\alpha+1}-1\right)}{(1+r)^{\alpha}\,r^{\alpha}-r^{\alpha}+(1+r)^{\alpha}},
\tag{3.3}
$$

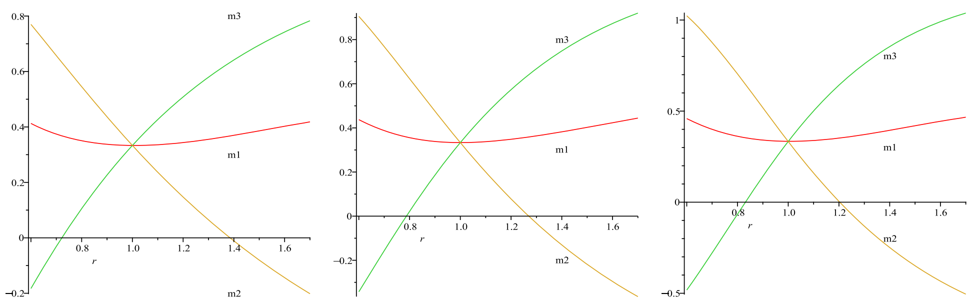

FIG. 1. (Color online) The figures of masses are corresponding to Eqs. (1.6) with $\alpha = 1, 2, 3$ from left to right.

which is independent of the choices of masses. The independence of masses will play an essential role in the classification of super central configurations. The solutions (3.2) were given in the paper.[7]

Proof of Theorem 1.3: Through the following series of claims we will prove the property (1) in Theorem 1.3 that ${}^{\#}S_m(q,\alpha)$ is 0 or 1. Similar methods as in Refs. 8 and 10 are applied here.

Claim 1: If $r = 1$, i.e., $q = (0, 1, 2)$, then $S_m(q,\alpha)$ is an empty set for any $m \in S(q,\alpha)$.

In fact, if $r = 1$, then $m_1 = \frac{(M-\lambda)2^\alpha}{2^{1+\alpha}-1}$, $m_2 = \frac{2^{1+\alpha}\lambda - M}{2^{1+\alpha}-1}$, $m_3 = \frac{(M-\lambda)2^\alpha}{2^{1+\alpha}-1}$ from Eq. (3.2). So m_1 must be equal to m_3. Therefore (m_1, m_3, m_2) is not in $S_m(q,\alpha)$, for otherwise $m_1 = m_2$ from above equations for m_1 and m_2. Similarly, $(m_2, m_1, m_3) \notin S_m(q,\alpha)$. This proves the claim.

Because of Claim 1 and the uniqueness of central configuration for a given order of masses m, $S_m(q,\alpha)$ is an empty set if m_1, m_2, m_3 are not mutually distinct.

We assume $r \neq 1$ and m_1, m_2, m_3 are mutually distinct from now on. Suppose $m = (m_1, m_2, m_3) \in S(q,\alpha)$.

Claim 2: $m(\tau_4)$, $m(\tau_5)$, and $m(\tau_6)$ are not in $S_m(q,\alpha)$.

Note that the center of mass c is fixed for a given r by (3.3). The center of mass is $(m_2 + m_3(1 + r))/M$ for $m(\tau_1)$ and the center of mass is $(m_3 + m_2(1 + r))/M$ for $m(\tau_4)$. If $m(\tau_4) \in S_m(q,\alpha)$, we have $m_2 + m_3(1 + r) = m_3 + m_2(1 + r)$ which implies $m_2 = m_3$. This contradiction proves that $m(\tau_4)$ is not in $S_m(q,\alpha)$. Similar arguments prove that $m(\tau_5)$, and $m(\tau_6)$ are not in $S_m(q,\alpha)$.

Claim 3: $m(\tau_2)$ and $m(\tau_3)$ cannot be in $S_m(q,\alpha)$ simultaneously.

If so, the center of mass for $m = m(\tau_1)$, the center of mass for $m(\tau_2)$, and the center of mass for $m(\tau_3)$ should be same, i.e.,

$$m_2 + m_3(1 + r) = m_1 + m_2(1 + r) = m_3 + m_1(1 + r),$$

which implies $m_1 = m_2 = m_3$. This contradiction proves the claim.

The three claims prove that either ${}^{\#}S_m(q,\alpha) = 0$ or ${}^{\#}S_m(q,\alpha) = 1$ must hold for any $m \in S(q,\alpha)$ and any $q = (0, 1, 1 + r)$. When $S_m(q,\alpha) = 1$, either $m(\tau_2) \in S_m(q,\alpha)$ or $m(\tau_3) \in S_m(q,\alpha)$.

Case 1: $S_m(q,\alpha) = \{(m_2, m_3, m_1)\}$.

Because $m(\tau_3) = (m_2, m_3, m_1)$ is a permutation of m, $M = m_1 + m_2 + m_3 = m_{\tau_3(1)} + m_{\tau_3(2)} + m_{\tau_3(3)}$ is a constant. If $m(\tau_3)$ is in $S_m(q,\alpha)$, then $m(\tau_3)$ should be given by Eq. (3.2) with different λ, say $\lambda(\tau_3)$, i.e.,

$$\begin{aligned} m_{\tau_3(1)} &= (1+r)^2\left(M - \lambda(\tau_3)r^3\right)/d(r), \\ m_{\tau_3(2)} &= r^2\left(-M + \lambda(\tau_3)\,(1+r)^3\right)/d(r), \qquad (3.4)\\ m_{\tau_3(3)} &= r^2\,(1+r)^2\,(M - \lambda(\tau_3))/d(r). \end{aligned}$$

Inverse law ($\alpha = 1$)	Inverse square law ($\alpha = 2$)	Inverse cubic law ($\alpha = 3$)
$(\underline{r}, \bar{r}) = (0.7221, 1.3849)$	$(\underline{r}, \bar{r}) = (0.7875, 1.2698)$	$(\underline{r}, \bar{r}) = (0.8311, 1.2033)$
$m_2 > m_1 > m_3 > 0$	$m_1 = m_2 = m_3 > 0$	$m_3 > m_1 > m_2 > 0$
for $\underline{r} < r < 1$	for $r = 1$	for $1 < r < \bar{r}$

$S_m(q, \alpha) = \{(m_2, m_3, m_1)\}$. $\underline{r}$ is determined by $f_3(r, \alpha) = 0$.
$\bar{r}$ is determined by $f_2(r, \alpha) = 0$. We observed that $\underline{r}$ is increasing and $\bar{r}$ is decreasing as α increases.

By setting $m_1 = m_{\tau_3(3)}$ and $m_2 = m_{\tau_3(1)}$ for the corresponding equations in (3.2), we get two linear equations in $\lambda = \lambda(\tau_1)$ and $\lambda(\tau_3)$, which solve as λ and $\lambda(\tau_3)$,

$$\lambda = \frac{M\big((1+r)^{-a}+r^{-a}+r^{-a+1}-r\big)}{1+r+r^2},$$
$$\lambda(\tau_3) = \frac{M\big(r(1+r)^{-a}-r^{-a}+1+r\big)}{1+r+r^2}. \tag{3.5}$$

By direct computation, we have $m_3 = m_{\tau_3(2)}$ if λ and $\lambda(\tau_3)$ are taken as in (3.5). So $m(\tau_3)$ is a permutation of m for the above $\lambda(\tau_3)$. Substituting λ into (3.2), we have

$$m_1 = Mf_1(r, \alpha)/g(r, \alpha),\, m_2 = Mf_2(r, \alpha)/g(r, \alpha),\, m_3 = Mf_3(r, \alpha)/g(r, \alpha),$$

which are the same as (1.6). By Lemma 1.2, $m_2 > m_1 > m_3 > 0$ for $\underline{r} < r < 1$, $m_1 = m_2 = m_3 > 0$ for $r = 1$, and $m_3 > m_1 > m_2 > 0$ for $1 < r < \bar{r}$. This completes the proof of case 1.

Case 2: $S_m(q, \alpha) = \{(m_3, m_1, m_2)\}$.

The proof for $m(\tau_2) = (m_3, m_1, m_2) \in S_m(q)$ is very similar to the proof for $m(\tau_3) \in S_m(q)$ and thus the proof is omitted.

We end our paper by giving a numerical comparison table of super central configurations among inverse law ($\alpha = 1$), inverse square law ($\alpha = 2$), and inverse cubic law ($\alpha = 3$). The corresponding masses of super central configurations under different law is shown in Fig. 1. □

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[1] Albouy, A. and Moeckel, R., "The inverse problem for collinear central configuration," Celest. Mech. Dyn. Astron. **77**, 77–91 (2000).
[2] Arribas, M., Elipe, A., Kalvouridis, T., and Palacios, M., "Homographic solutions in the planar $n + 1$ body problem with quasi-homogeneous potentials," Celest. Mech. Dyn. Astron. **99**, 1–12 (2007).
[3] Diacu, F., Perez-Chavela, E., and Santoprete, M., "Central configurations and total collisions for quasihomogeneous n-body problems," Nonlinear Anal. **65**, 1425–1439 (2006).
[4] Pollard, H., *Celestial Mechanics* (Mathematical Association of America (MAA), Oberlin, OH, 1976).
[5] Long, Y. and Sun, S., "Collinear central configurations and singular surfaces in the mass space," Arch. Ration. Mech. Anal. **173**, 151–167 (2004).
[6] Moulton, F. R., "The straight line solutions of the problem of N-bodies," Ann. Math. **2**(12), 1–17 (1910); Moulton, F. R., *Periodic Orbits* (Carnegie, Pittsburgh, 1920), pp. 28–298.
[7] Woodlin, M. and Xie, Z., "Collinear central configurations in the n-body problem with general homogeneous potential," J. Math. Phys. **50**, 102901 (2009).
[8] Xie, Z., "Super Central configurations of the n-body problem," J. Math. Phys. **51**, 042902 (2010).
[9] Xie, Z., "Inverse problem of central configurations and singular curve in the collinear 4-body problem," Celest. Mech. Dyn. Astron. **107**, 353–376 (2010).
[10] Xie, Z., "The golden ratio and super central configurations of the n-body problem," J. Differ. Equations **251**, 58–72 (2011).