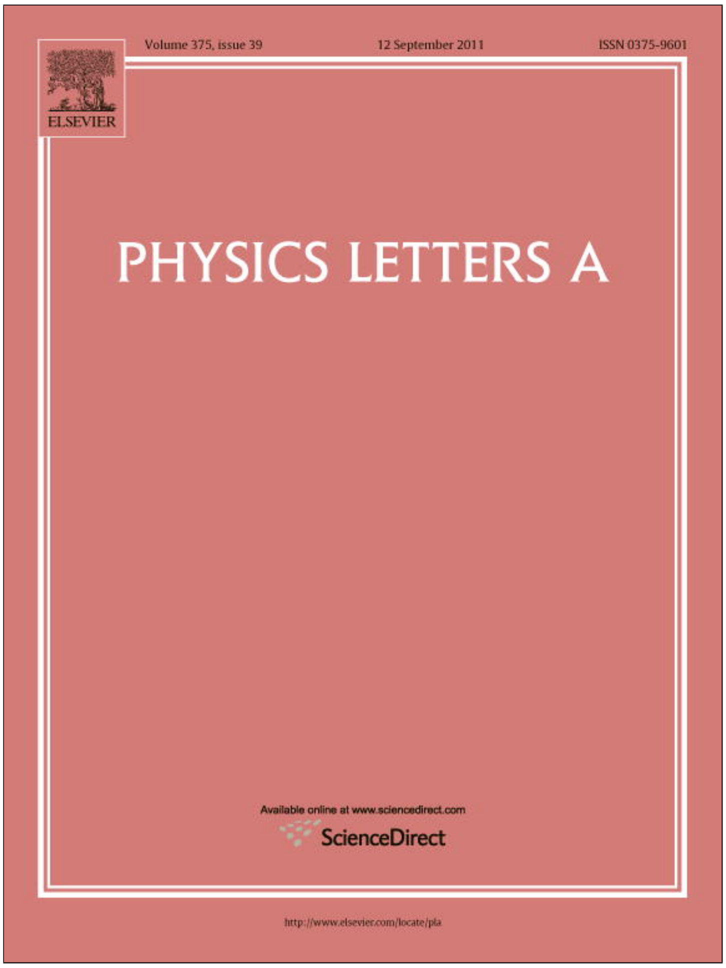
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Central configurations of the collinear three-body problem and singular surfaces in the mass space

Zhifu Xie

Department of Mathematics and Computer Science, Virginia State University, Petersburg, VA 23806, USA

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ABSTRACT

This Letter is to provide a new approach to study the phenomena of degeneracy of the number of the collinear central configurations under geometric equivalence. A direct and simple explicit parametric expression of the singular surface H_3 is constructed in the mass space $(m_1, m_2, m_3) \in (\mathbf{R}^+)^3$. The construction of H_3 is from an inverse respective, that is, by specifying positions for the bodies and then determining the masses that are possible to yield a central configuration. It reveals the relationship between the phenomena of degeneracy and the inverse problem of central configurations. We prove that the number of central configurations is decreased to $3!/2 - 1 = 2$, m_1, m_2, and m_3 are mutually distinct if $m \in H_3$. Moreover, we know not only the number of central configurations but also what the nonequivalent central configurations are.

1. Introduction and main results

In this Letter, we reinvestigate the phenomena of the existence of the singular surface in mass space on which the number of central configurations under geometric equivalence of collinear three-body problem degenerates. The phenomena of degeneracy was studied in [6,18]. This Letter is to provide a new method to study the phenomena of degeneracy of collinear central configurations. We give a direct and simple parametric expression of mass vector for which the number of central configurations decreases. The exact equivalence classes (or the number of central configurations) are known readily from the parametric expression for any given mass vector on the singular surfaces. The construction of the singular surfaces is from an inverse respective, that is, by specifying positions for the bodies and then determining the masses that are possible to yield a central configuration. It reveals the relation between the phenomena of degeneracy and the inverse problem of central configurations. The new method is promising when it is applied to study other collinear n-body problem and it has been applied to study the collinear four-body problem in [14].

The Newtonian n-body problem describes the motion of n point particles with masses $m_k > 0$ and positions $q_k \in \mathbf{R}^3$ by a system of second order differential equations,

$$m_k \ddot{q}_k = \sum_{j=1, j\neq k}^{n} \frac{m_k m_j (q_j - q_k)}{|q_j - q_k|^3}, \quad 1 \leqslant k \leqslant n. \tag{1}$$

When we study homographic solutions of the n-body problem, the problem is reduced to find central configurations for given masses.

Definition 1.1 *(Central Configuration).* The n particles form a *central configuration* (for short CC) if there exists a constant λ such that m_k and q_k satisfy the following system of nonlinear algebraic equations:

$$\lambda(q_k - c) + \sum_{j=1, j\neq k}^{n} \frac{m_j (q_j - q_k)}{|q_j - q_k|^3} = 0, \quad 1 \leqslant k \leqslant n, \tag{2}$$

where c is the center of mass given by $c = C/M$, and $C = m_1 q_1 + \cdots + m_n q_n$, $M = m_1 + \cdots + m_n$.

E-mail address: zxie@vsu.edu.

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Definition 1.2 *(Geometric Equivalence).* Two configurations $q = (q_1, q_2, \dots, q_n)$ and $p = (p_1, p_2, \dots, p_n)$ are *geometric equivalent*, denoted by $q \sim p$, if and only if q and p differ by a rotation followed by a scalar multiplication, or by a permutation of bodies. From now on, the number of central configurations refers to the number of equivalent classes.

The notations in this Letter are similar to those in [6]. For any $n \in \mathbf{N}$ (the set of integers), we denote by $P(n)$ the set of all permutations of $\{1, 2, \dots, n\}$. For any element $\tau \in P(n)$, we use $\tau = (\tau(1), \tau(2), \dots, \tau(n))$ to denote the permutation τ. We also denote a permutation of $(m_1, m_2, \dots, m_n)$ by $m(\tau) = (m_{\tau(1)}, m_{\tau(2)}, \dots, m_{\tau(n)})$ for $\tau \in P(n)$. We define the converse permutation of τ by $\mathrm{conv}(\tau) = (\tau(n), \dots, \tau(1))$ and denote by ${}^{\#}B$ the number of elements in a set B.

A great deal of effort has gone into understanding central configurations. For systematic studies and surveys on CCs we refer to the works of [8,9,16,17] and some other interesting papers [2–4,7,10–21] and reference therein. A configuration $q = (q_1, \dots, q_n)$ is *collinear*, if all the q_is are located on a line. A collinear central configuration is also called a *Moulton configuration* after Moulton [8] who proved that for a fixed mass vector $m = (m_1, \dots, m_n)$ and a fixed ordering of the bodies along the line, there exists a unique collinear central configuration (up to translation and scaling). Without loss generality, we suppose simply that the n bodies are located on the x-axis. Let

$$W(n, m) = \left\{ q = (q_1, q_2, \dots, q_n) \in \mathbf{R}^n \;\middle|\; q_1 < q_2 < \dots < q_n \right\}. \tag{3}$$

Because in $W(n, m)$ we do not allow q_is to change their order, we now allow m_is to change their order. Note that when we say by $q = (q_1, \dots, q_n) \in W(n, m)$ is a collinear CC for $m(\alpha) \equiv (m_{\alpha(1)}, \dots, m_{\alpha(n)})$ with some $\alpha \in P(n)$, we always mean that $m_{\alpha(i)}$ is put on q_i for all $i = 1, 2, \dots, n$. Denote by $L(n, m)$ the set of equivalent classes of n-body collinear central configurations for any given mass vector $m = (m_1, m_2, \dots, m_n) \in (\mathbf{R}^+)^n$.

We need to introduce the six singular surfaces F_i in the half mass spaces $(m_1, m_2, m_3) \in (\mathbf{R}^+)^3$.

Definition 1.3. We denote the six continuous parametric surfaces in the half mass space $(\mathbf{R}^+)^3$ by

$$H_3 = \bigcup_{i=1}^{6} F_i. \tag{4}$$

$$\begin{aligned}
F_1 &= \left\{ (m_1, m_2, m_3) \in \left(\mathbf{R}^+\right)^3 \;\middle|\; m_1 = Mf_1(r), m_2 = Mf_2(r), m_3 = Mf_3(r) \right\}; \\
F_2 &= \left\{ (m_1, m_2, m_3) \in \left(\mathbf{R}^+\right)^3 \;\middle|\; m_1 = Mf_2(r), m_2 = Mf_3(r), m_3 = Mf_1(r) \right\}; \\
F_3 &= \left\{ (m_1, m_2, m_3) \in \left(\mathbf{R}^+\right)^3 \;\middle|\; m_1 = Mf_2(r), m_2 = Mf_1(r), m_3 = Mf_3(r) \right\}; \\
F_4 &= \left\{ (m_1, m_2, m_3) \in \left(\mathbf{R}^+\right)^3 \;\middle|\; m_1 = Mf_1(r), m_2 = Mf_3(r), m_3 = Mf_2(r) \right\}; \\
F_5 &= \left\{ (m_1, m_2, m_3) \in \left(\mathbf{R}^+\right)^3 \;\middle|\; m_1 = Mf_3(r), m_2 = Mf_2(r), m_3 = Mf_1(r) \right\}; \\
F_6 &= \left\{ (m_1, m_2, m_3) \in \left(\mathbf{R}^+\right)^3 \;\middle|\; m_1 = Mf_3(r), m_2 = Mf_1(r), m_3 = Mf_2(r) \right\};
\end{aligned} \tag{5}$$

where $M > 0$, $\underline{r} < r < 1$, and

$$\begin{aligned}
f_1(r) &= \frac{(1 + 2r + r^2 - r^3 + r^4 + 2r^5 + r^6)}{(1 + 2r + r^2 + 2r^3 + r^4)(r^2 + r + 1)}, \qquad f_2(r) = -\frac{(r^6 + 3r^5 + 3r^4 - 3r^3 - 6r^2 - 4r - 1)}{(1 + 2r + r^2 + 2r^3 + r^4)(r^2 + r + 1)}, \\
f_3(r) &= \frac{(r^6 + 4r^5 + 6r^4 + 3r^3 - 3r^2 - 3r - 1)}{(1 + 2r + r^2 + 2r^3 + r^4)(r^2 + r + 1)},
\end{aligned} \tag{6}$$

and $\underline{r}$ is the unique positive root of

$$r^6 + 4r^5 + 6r^4 + 3r^3 - 3r^2 - 3r - 1 = 0. \tag{7}$$

F_i are called singular surfaces.

Remark 1.4. (A) In Definition 1.3, r is the ratio of the distances between $|q_3 - q_2|$ and $|q_2 - q_1|$ in the collinear three-body problem. With the choice of masses $(m_1, m_2, m_3) \in H_3$, $q = (q_1, q_2, q_3)$ is a collinear central configuration for at least two different arrangements of the three masses along a line.

(B) Note that $f_1(r) + f_2(r) + f_3(r) = 1$ for any $r > 0$. If $(m_1, m_2, m_3) \in H_3$, $m_1 + m_2 + m_3 = M$. The six surfaces are corresponding to the six permutations of masses. H_3 can be written as

$$H_3 = \bigcup_{i=1}^{6} \left\{ (m_1, m_2, m_3) \in \left(\mathbf{R}^+\right)^3 \;\middle|\; m_1 = Mf_{\tau_i(1)}(r), m_2 = Mf_{\tau_i(2)}(r), m_3 = Mf_{\tau_i(3)}(r), M > 0, \underline{r} < r < 1 \right\},$$

where τ_i $(i = 1, 2, \dots, 6)$ are the six permutations in $P(3)$.

(C) Because

$$f_2 - f_1 = \frac{r(1 - r)(2r^4 + 7r^3 + 11r^2 + 7r + 2)}{(1 + 2r + r^2 + 2r^3 + r^4)(r^2 + r + 1)} > 0 \quad \text{for } \underline{r} < r < 1,$$

and

$$f_1 - f_3 = \frac{(1 - r)(2r^4 + 7r^3 + 11r^2 + 7r + 2)}{(1 + 2r + r^2 + 2r^3 + r^4)(r^2 + r + 1)} > 0 \quad \text{for } \underline{r} < r < 1,$$

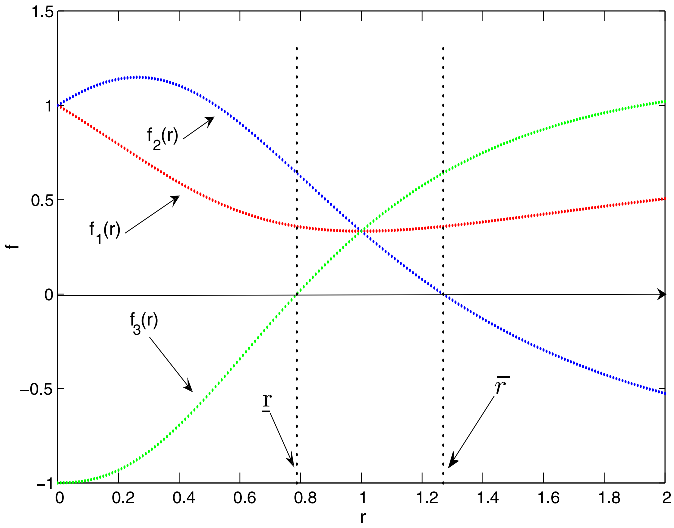

Fig. 1. Graph of $f_1(r)$, $f_2(r)$ and $f_3(r)$.

we have $f_2(r) > f_1(r) > f_3(r) > 0$ for $\underline{r} < r < 1$ (see Fig. 1). Then we have

(1) If $(m_1, m_2, m_3) \in F_1$, $m_2 > m_1 > m_3$;
(2) If $(m_1, m_2, m_3) \in F_2$, $m_1 > m_3 > m_2$;
(3) If $(m_1, m_2, m_3) \in F_3$, $m_1 > m_2 > m_3$;
(4) If $(m_1, m_2, m_3) \in F_4$, $m_3 > m_1 > m_2$;
(5) If $(m_1, m_2, m_3) \in F_5$, $m_2 > m_3 > m_1$;
(6) If $(m_1, m_2, m_3) \in F_6$, $m_3 > m_2 > m_1$.

Theorem 1.5. *For any mass vector $m = (m_1, m_2, m_3) \in (\mathbf{R}^+)^3$, one and only one of the following four cases must apply*:

(i) $^{\#}L(3, m) = 3$, *if m_1, m_2, and m_3 are mutually distinct, and $(m_1, m_2, m_3) \notin H_3$*;
(ii) $^{\#}L(3, m) = 2$, *if m_1, m_2, and m_3 are mutually distinct, and $(m_1, m_2, m_3) \in H_3$*;
(iii) $^{\#}L(3, m) = 2$, *if two of m_1, m_2, and m_3 are equal to each other but not the third*;
(iv) $^{\#}L(3, m) = 1$, *if $m_1 = m_2 = m_3$*.

Historically, apart from Definition 1.2 there exist two other ways to define equivalence relations among collinear central configurations. Because of these different understandings, the number of equivalent classes of CCs for a given $m \in \mathbf{R}^+$ were counted differently in different papers. More details are given in Long and Sun's paper [6].

Definition 1.6 *(Mass Equivalence).* For a given $m \in (\mathbf{R}^+)^n$, let $q = (q_1, q_2, \ldots, q_n)$ and $p = (p_1, p_2, \ldots, p_n) \in W(n, m)$ be collinear CCs for $m(\alpha)$ and $m(\beta)$ with α and $\beta \in P(n)$ respectively. Then q and p are *mass equivalent*, denoted by $q \sim_M p$, if and only if either $m(\alpha) = m(\beta)$ and $q = ap$ for some $a > 0$, or $m(\alpha) = m(\mathrm{conv}(\beta))$ and $q = ap$ for some $a < 0$. For any given $m \in (\mathbf{R}^+)^n$, we denote by $L_M(n, m)$ the set of all mass equivalent classes of n-body collinear CCs for $m(\tau)$ with all $\tau \in P(n)$.

Clearly we have $^{\#}L(n, m) \leqslant {}^{\#}L_M(n, m)$ for all $m \in (\mathbf{R}^+)^n$. Directly from Theorem 1.5, we have

Corollary 1.7. *For any mass vector $m = (m_1, m_2, m_3) \in (\mathbf{R}^+)^3$,*

(1) $^{\#}L_M(3, m) = {}^{\#}L(3, m)$ *if and only if $(m_1, m_2, m_3) \notin H_3$*;
(2) $^{\#}L_M(3, m) = {}^{\#}L(3, m) + 1$ *if and only if $(m_1, m_2, m_3) \in H_3$*.

Remark 1.8. (A) We consider the plane P defined by $m_1 + m_2 + m_3 = 1$ in $(m_1, m_2, m_3) \in \mathbf{R}^3$ which intersects the positive mass quadrant $(m_1, m_2, m_3) \in (\mathbf{R}^+)^3$ and produces an equilateral triangle shown in Fig. 2. The vertices of the equilateral triangle are given by $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. The three bisection lines of the three interior angles of this triangle are given by $m_i = m_j$ for $i \neq j$. For $i = 1, 2, \ldots, 6$, the intersection curves G_i of the plane P with the six singular surfaces F_i are shown in Fig. 2 and G_i is given by the parametric form:

$$G_i = \left\{(m_1, m_2, m_3) \in \left(\mathbf{R}^+\right)^3 \;\middle|\; m_1 = f_{\tau_i(1)}(r), m_2 = f_{\tau_i(2)}(r), m_3 = f_{\tau_i(3)}(r), \underline{r} < r < 1\right\}.$$

All the intersection curves of the plane $m_1 + m_2 + m_3 = M$ with the six singular surfaces F_i are similar to the curves G_i in the plane P shown in Fig. 2. H_3 has six connected components F_i. The six singular surfaces F_i are connected at the line $(\frac{t}{3}, \frac{t}{3}, \frac{t}{3})$ with $t > 0$.

(B) If $(m_1, m_2, m_3) \in H_3$, then m_1, m_2 and m_3 are mutually distinct and positive.

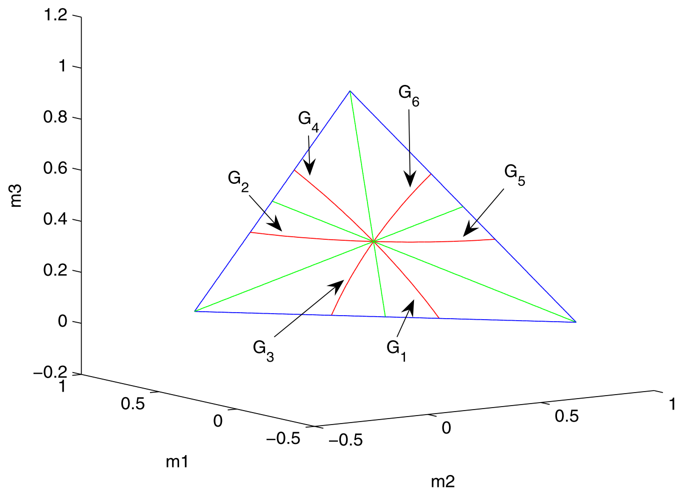

Fig. 2. Six singular curves G_i in the plane $m_1 + m_2 + m_3 = 1$. $m_2 > m_1 > m_3$ on G_1; $m_1 > m_3 > m_2$ on G_2; $m_1 > m_2 > m_3$ on G_3; $m_3 > m_1 > m_2$ on G_4; $m_2 > m_3 > m_1$ on G_5; $m_3 > m_2 > m_1$ on G_6.

(C) The exact number $^{\#}L(3, m)$ in collinear three body problem was studied by Long and Sun in [6] for any given masses. They constructed a union $\tilde{H}_3$ of singular surfaces in the mass space which changes the number of collinear central configurations. $\tilde{H}_3 = \tilde{F}_1 \cup \tilde{F}_2 \cup \tilde{F}_3$, where $\tilde{F}_i$ are the surfaces in $\mathbf{R}^+ \times \mathbf{R}^2$ *implicitly* defined by $\tilde{f}_i(x, y, z) = 0$ $(i = 1, 2, 3)$, $x = m_1$, $y = m_2 - m_1$, $z = m_3 - m_2$, and

$$\tilde{f}_1(x, y, z) = z^6 + (4y + 2x)z^5 + (6y + 5x)yz^4 + (3y + 4x)y^2z^3 - (3y + 4x)y^3z^2 - (3y + 5x)y^4z - (y + 2x)y^5$$

for $yz > 0$. $\tilde{f}_2(x, y, z)$ and $\tilde{f}_3(x, y, z)$ are also homogeneous polynomials of degree 6 in three variables, which are defined in similar forms. The relation between the singular surfaces $\tilde{H}_3$ in [6] and H_3 in this Letter could be stated as follows. For any $(m_1, m_2, m_3) \in F_3 \cup F_6$, $\tilde{f}_1(m_1, m_2 - m_1, m_3 - m_2) = 0$ by direct computation. Similarly for any $(m_1, m_2, m_3) \in F_1 \cup F_4$, $\tilde{f}_2(m_1, m_2 - m_1, m_3 - m_2) = 0$. For any $(m_1, m_2, m_3) \in F_2 \cup F_5$, $\tilde{f}_3(m_1, m_2 - m_1, m_3 - m_2) = 0$.

(D) Furthermore, for any given $m = (m_1, m_2, m_3) \in H_3$, we know not only $^{\#}L(3, m) = 2$ but also what the nonequivalent central configurations are. For example: if $m = (m_1, m_2, m_3) \in F_1$, $L(3, m)$ consists two nonequivalent central configurations corresponding to the mass orders (m_1, m_2, m_3) and (m_2, m_1, m_3). Moreover, the central configuration for mass order (m_1, m_2, m_3) is equivalent to the central configuration for mass order (m_1, m_3, m_2). More details are given in the proof of next section.

(E) After we reinvestigated the number of the collinear central configurations of three-body problem, we follow the approaches developed in current Letter to study the collinear four-body problem. A singular surface H_4 has been constructed in [15]. We proved that if $(m_1, m_2, m_3, m_4) \in H_4$, the number of collinear central configurations $^{\#}L(4, m) = 4!/2 - 1 = 11$ and m_1, m_2, m_3 and m_4 are mutually distinct. Moreover, if $(m_1, m_2, m_3, m_4) \notin H_4$, the number of collinear central configurations $^{\#}L(4, m) = ^{\#} L_M(4, m)$.

2. The proof of Theorem 1.5

Because central configuration is invariant up to translation and scaling, we can choose the coordinate system so that all the three bodies are on the x-axis in $\mathbf{R}^3$. Let $q_1 = 0$, $q_2 = 1$, $q_3 = 1 + r$ where $r > 0$. Then the set $W(n, m)$ of (3) can be modified to

$$W(3, m) = \{q = (0, 1, 1 + r) \in \mathbf{R}^3 \mid r > 0\}. \tag{8}$$

Because we fix q_is with their order $q_1 < q_2 < q_3$, we now allow m_is to change their order. When we say by $q = (q_1, q_2, q_3)$ is a collinear CC for $m(\alpha) \equiv (m_{\alpha(1)}, m_{\alpha(2)}, m_{\alpha(3)})$ with some $\alpha \in P(3)$, we always mean that $m_{\alpha(i)}$ is attached to q_i for all $i = 1, 2, 3$.

We first prove the equivalence of collinear three-body central configurations in our settings.

Proposition 2.1 *(Equivalence for collinear CC in 3-body problem). For a given $m \in (\mathbf{R}^+)^3$, let $q = (q_1, q_2, q_3) = (0, 1, 1 + r_1)$ and $p = (p_1, p_2, p_3) = (0, 1, 1 + r_2) \in W(3, m)$ be two collinear CCs for $m(\alpha)$ and $m(\beta)$ with α and $\beta \in P(3)$ respectively. Then q and p are equivalent, if and only if $r_2 = r_1$ or $r_2 = 1/r_1$.*

Proof. By Definition 1.2, q and p are equivalent if and only if q and p differ by a rotation followed by a scalar multiplication. So q and p are equivalent if and only if $r_2 = r_1$ or $r_2 = 1/r_1$ where q can be obtained from p by a rotation of an angle π and followed by a scaler multiplication r_1. □

Proof of Theorem 1.5. Fix $m = (m_1, m_2, m_3) \in (\mathbf{R}^+)^3$. To get all the 3-body collinear CCs, we use (2) and choose $q_1 = 0$, $q_2 = 1$, and $q_3 = 1 + r$ with $r > 0$. Substitute $c = (m_2 + m_3(1 + r))/M$ into (2), where M is the total mass which is chosen as a parameter. Then (2) is equivalent to

$$m_2\left(-1 + \frac{\lambda}{M}\right) + m_3\left(-(1 + r)^{-2} + \frac{\lambda(1 + r)}{M}\right) = 0, \tag{9}$$

$$m_1+\frac{m_2\lambda}{M}+m_3\left(-r^{-2}+\frac{\lambda(1+r)}{M}\right)=\lambda, \tag{10}$$

$$\frac{m_1}{(1+r)^2}+m_2\left(r^{-2}+\frac{\lambda}{M}\right)+\frac{m_3\lambda(1+r)}{M}=\lambda(1+r). \tag{11}$$

Solving λ, M from Eqs. (9) and (10), we obtain:

$$M=\frac{(m_3(1+r)+m_2)((m_1+m_2)r^4+2(m_1+m_2)r^3+(m_1+m_2)r^2-m_3(2r-1))}{(m_2+2m_2r+m_2r^2+m_3)r^2},$$

$$\lambda=\frac{(m_1+m_2)r^4+2(m_1+m_2)r^3+(m_1+m_2)r^2-m_3(2r-1)}{(1+2r+r^2)r^2}.$$

Substituting M and λ into (11) and multiplying $(1+2r+r^2)r^2$ on both sides and rearranging terms, we have the following equation which is similar to an equation in [5,6,18]:

$$-(m_3+m_2)-(3m_3+2m_2)r-(3m_3+m_2)r^2+(m_2+3m_1)r^3+(2m_2+3m_1)r^4+(m_1+m_2)r^5=0. \tag{12}$$

By Descartes' Rule of signs for polynomials, Eq. (12) has one and only one positive solution $r>0$ for any choice of the mass vector $(m_1,m_2,m_3)\in(\mathbf{R}^+)^3$. For each permutation of (m_1,m_2,m_3), there is a unique solution $(r,M,\lambda)\in(\mathbf{R}^+)^3$ and this r produces one collinear central configuration $q=(q_1,q_2,q_3)=(0,1,1+r)$. It is well known that two central configurations $q(\alpha)$ and $q(\beta)$ are equivalent for permutations $\alpha=(1,2,3)$ and $\beta=(3,2,1)$. In fact $r(\alpha)=1/r(\beta)$. So we have $^{\#}L(3,m)\leqslant 3=\frac{3!}{2}$. The possible three nonequivalent central configurations are corresponding to the three permutations $\alpha_1=(1,2,3)$, $\alpha_2=(1,3,2)$, and $\alpha_3=(2,1,3)$, and the corresponding solution are defined by (r_1,M,λ_1), (r_2,M,λ_2), and (r_3,M,λ_3) respectively.

By the method of Gaussian elimination, we can find solutions of Eqs. (9), (10) and (11) with parameters (r_i,M,λ_i).

$$m_{\alpha_i(1)}=\frac{(1+r_i)^2(M-\lambda_i{r_i}^3)}{{r_i}^4+2{r_i}^3+{r_i}^2+2r_i+1},\qquad m_{\alpha_i(2)}=\frac{{r_i}^2(-M+\lambda_i(1+r_i)^3)}{{r_i}^4+2{r_i}^3+{r_i}^2+2r_i+1},\qquad m_{\alpha_i(3)}=\frac{{r_i}^2(1+r_i)^2(M-\lambda_i)}{{r_i}^4+2{r_i}^3+{r_i}^2+2r_i+1}. \tag{13}$$

Remarkably, the center of mass only depends on r_i

$$c=\frac{{r_i}^3(3+3r_i+{r_i}^2)}{{r_i}^4+2{r_i}^3+{r_i}^2+2r_i+1}, \tag{14}$$

and total mass is $M=m_{\alpha_i(1)}+m_{\alpha_i(2)}+m_{\alpha_i(3)}$. Albouy and Moeckel had the above solutions in [1] and they also proved that the mass set depends on two parameters to make the fixed configuration central for collinear n-body problem with $3\leqslant n\leqslant 6$. When $\max(1,r_i^3)<M/\lambda_i<(1+r_i)^3$, all three masses $m_{\alpha_i(1)}, m_{\alpha_i(2)}, m_{\alpha_i(3)}$ are positive.

Case 1: Three equal masses: $m_1=m_2=m_3$.

The unique solution to Eqs. (9), (10) and (11) for three equal masses is $(r,M,\lambda)=(1,3m_1,\frac{5}{4}m_1)$. So $^{\#}L(3,m)=1$.

Case 2: Two equal masses: Without loss of generality, assume $m_1=m_2\neq m_3$.

By the existence and uniqueness of the solutions to Eqs. (9), (10) and (11) for any given mass (m_1,m_2,m_3), r_1 must be equal to r_3 because $m(\alpha_1)=m(\alpha_3)=(m_2,m_2,m_3)$. So $q(\alpha_1)\sim q(\alpha_3)$ and $^{\#}L(3,m)\leqslant 2$.

It is easy to check that $r_2=1$, $M=2m_1+m_3$, and $\lambda_2=m_3+\frac{m_1}{4}$ for $m(\alpha_2)$. If $r_1=r_2$, we must have $\lambda_1=\lambda_2$ from $m_1=m_{\alpha_1(1)}=m_{\alpha_2(1)}$ in Eq. (13). Thus $m_2=m_{\alpha_1(2)}=\frac{{r_1}^2(-M+\lambda_1(1+r_1)^3)}{{r_1}^4+2{r_1}^3+{r_1}^2+2r_1+1}=\frac{{r_2}^2(-M+\lambda_2(1+r_2)^3)}{{r_2}^4+2{r_2}^3+{r_2}^2+2r_2+1}=m_{\alpha_2(2)}=m_3$ which contradicts to the assumption $m_2\neq m_3$. This contradiction proves that $q(\alpha_1)$ is not equivalent to $q(\alpha_2)$ and $^{\#}L(3,m)=2$.

Case 3: m_1, m_2, and m_3 are mutually distinct.

First we note that $r_i\neq 1$ for $i=1,2,3$, otherwise $m_{\alpha_i(1)}=m_{\alpha_i(3)}=\frac{4M}{7}-\frac{4\lambda_i}{7}$ by (13).

Second $r_1\neq r_2$. In fact, if $r_1=r_2=r$, then their corresponding centers of mass must be same by Eq. (14). We have

$$c=\frac{m_{\alpha_1(2)}+m_{\alpha_1(3)}(1+r_1)}{M}=\frac{m_{\alpha_2(2)}+m_{\alpha_2(3)}(1+r_2)}{M},$$

or

$$c=\frac{m_2+m_3(1+r)}{M}=\frac{m_3+m_2(1+r)}{M},$$

which implies $m_2=m_3$. This contradiction shows that $r_1\neq r_2$. Similarly, we must have $r_1\neq r_3$ and $r_2\neq 1/r_3$.

Step 1: *Non-equivalency*, $1<{}^{\#}L(3,m)\leqslant 3$.

$^{\#}L(3,m)$ must be strictly bigger than 1 for three mutually distinct masses. Otherwise, $q(\alpha_1)\sim q(\alpha_2)\sim q(\alpha_3)$ implies that at least one of the following four cases is true. (1) $r_1=r_2=r_3$; (2) $r_1=r_2=1/r_3$; (3) $r_1=1/r_2=r_3$; (4) $1/r_1=r_2=r_3$.

Because $r_1\neq r_2$ and $r_1\neq r_3$ for three distinct masses, the first three cases are impossible. For the fourth case, the centers of mass by Eq. (14) are same for $m(con(\alpha_1))$, $m(\alpha_2)$, and $m(\alpha_3)$, i.e.

$$c=\frac{m_{\alpha_1(2)}+m_{\alpha_1(1)}(1+1/r_1)}{M}=\frac{m_{\alpha_2(2)}+m_{\alpha_2(3)}(1+r_2)}{M}=\frac{m_{\alpha_3(2)}+m_{\alpha_3(3)}(1+r_3)}{M},$$

or

$$c=\frac{m_2+m_1(1+r_3)}{M}=\frac{m_3+m_2(1+r_3)}{M}=\frac{m_1+m_3(1+r_3)}{M},$$

which implies

$$r_3 = \frac{m_2 - m_1}{m_3 - m_2} = \frac{m_3 - m_2}{m_1 - m_3}.$$

Since $r_3 > 0$, we have $m_1 < m_2 < m_3$ or $m_3 < m_2 < m_1$ from first equation and we have $m_1 < m_3 < m_2$ or $m_2 < m_3 < m_1$ from second equation. But we cannot have positive m_1, m_2, m_3 satisfying the inequalities simultaneously. So it is impossible to have $r_1 = 1/r_2 = 1/r_3$. This proves that $1 <^{\#} L(3, m)$.

Step 2: *Equivalency*, ${}^{\#}L(3, m) = 2$.

In this step, we give the necessary and sufficient conditions of three distinct masses for which ${}^{\#}L(3, m) = 2$. We only need check the conditions for the following three cases.

(1) $q(\alpha_1) \sim q(\alpha_2)$;

Since $r_1 \neq r_2$, we are going to find the conditions such that $r_2 = 1/r_1$. From Eq. (13) for permutation α_2 with $r_2 = 1/r_1$, we have

$$m_{\alpha_2(1)} = \frac{(r_1+1)^2(Mr_1{}^3 - \lambda_2)}{r_1(r_1{}^4 + 2r_1{}^3 + r_1{}^2 + 2r_1 + 1)}, \qquad m_{\alpha_2(2)} = \frac{\lambda_2(r_1+1)^3 - Mr_1{}^3}{r_1(r_1{}^4 + 2r_1{}^3 + r_1{}^2 + 2r_1 + 1)},$$
$$m_{\alpha_2(3)} = \frac{(r_1+1)^2(-\lambda_2 + M)}{r_1{}^4 + 2r_1{}^3 + r_1{}^2 + 2r_1 + 1}. \tag{15}$$

Using Eqs. (13) for α_1 and (15) for α_2, we solve λ_1 and λ_2 by setting $m_{\alpha_2(1)} = m_1$ and $m_{\alpha_2(2)} = m_3$.

$$\lambda_1 = -\frac{M(-1 - 3r_1 - 4r_1{}^2 + 2r_1{}^4 + r_1{}^5)}{r_1{}^2(r_1{}^4 + 3r_1{}^3 + 4r_1{}^2 + 3r_1 + 1)}, \qquad \lambda_2 = \frac{r_1 M(4{-}1 - 2r_1 + r_1{}^3 + 3r_1{}^4 + r_1{}^5)}{(r_1{}^4 + 3r_1{}^3 + 4r_1{}^2 + 3r_1 + 1)}.$$

$m_{\alpha_2(3)} = m_2$ if λ_1 and λ_2 are as above. Substituting λ_1 into (13) for $m(\alpha_1)$ and replace r_1 by r, we have

$$m_1 = \frac{M(1 + 2r + r^2 - r^3 + r^4 + 2r^5 + r^6)}{(1 + 2r + r^2 + 2r^3 + r^4)(r^2 + r + 1)}, \qquad m_2 = -\frac{M(r^6 + 3r^5 + 3r^4 - 3r^3 - 6r^2 - 4r - 1)}{(1 + 2r + r^2 + 2r^3 + r^4)(r^2 + r + 1)},$$
$$m_3 = \frac{M(r^6 + 4r^5 + 6r^4 + 3r^3 - 3r^2 - 3r - 1)}{(1 + 2r + r^2 + 2r^3 + r^4)(r^2 + r + 1)}. \tag{16}$$

Note that $m_1 > 0$ for all $r > 0$. $m_2 > 0$ for $0 < r < \bar{r}$ where $\bar{r}$ is the unique positive root of

$$g_1(r) = -\bigl(r^6 + 3r^5 + 3r^4 - 3r^3 - 6r^2 - 4r - 1\bigr).$$

$m_3 > 0$ for $\underline{r} < r < \infty$ where $\underline{r}$ is the unique positive root of

$$g_2(r) = r^6 + 4r^5 + 6r^4 + 3r^3 - 3r^2 - 3r - 1.$$

In order to $m_i > 0, i = 1, 2, 3$, r must satisfy $\underline{r} < r < \bar{r}$ (see Fig. 1 for $M = 1$). Because $r^6 g_1(1/r) = g_2(r)$, $\bar{r} = 1/\underline{r}$ and numerically $\underline{r} = 0.7875161542$ and $\bar{r} = 1.269815222$. By direct computation $m_1(r) = m_1(1/r)$, $m_2(r) = m_3(1/r)$ and $m_3(r) = m_2(1/r)$.

When $\underline{r} < r < 1$,

$$m_2 - m_1 = \frac{Mr(1-r)(2r^4 + 7r^3 + 11r^2 + 7r + 2)}{(1 + 2r + r^2 + 2r^3 + r^4)(r^2 + r + 1)} > 0, \qquad m_1 - m_3 = \frac{M(1-r)(2r^4 + 7r^3 + 11r^2 + 7r + 2)}{(1 + 2r + r^2 + 2r^3 + r^4)(r^2 + r + 1)} > 0.$$

So $m_2 > m_1 > m_3$ for $\underline{r} < r < 1$. At $r = 1$, $m_1 = m_2 = m_3 = \frac{M}{3}$.

So $q(\alpha_1) \sim q(\alpha_2)$ if and only if $(m_1, m_2, m_3) \in F_1$ or $(m_1, m_2, m_3) \in F_4$, where

$$\begin{aligned} F_1 &= \bigl\{(x_1, x_2, x_3) \bigm| x_i = Mf_1(r), x_2 = Mf_2(r), x_3 = Mf_3(r), M > 0, \underline{r} < r < 1\bigr\}; \\ F_4 &= \bigl\{(x_1, x_2, x_3) \bigm| x_1 = Mf_1(r), x_2 = Mf_2(r), x_3 = Mf_3(r), M > 0, 1 < r < \bar{r}\bigr\} \\ &= \bigl\{(x_1, x_2, x_3) \bigm| x_1 = Mf_1(r), x_2 = Mf_3(r), x_3 = Mf_2(r), M > 0, \underline{r} < r < 1\bigr\}; \end{aligned} \tag{17}$$

and

$$f_1(r) = \frac{(1 + 2r + r^2 - r^3 + r^4 + 2r^5 + r^6)}{(1 + 2r + r^2 + 2r^3 + r^4)(r^2 + r + 1)}, \qquad f_2(r) = -\frac{(r^6 + 3r^5 + 3r^4 - 3r^3 - 6r^2 - 4r - 1)}{(1 + 2r + r^2 + 2r^3 + r^4)(r^2 + r + 1)},$$
$$f_3(r) = \frac{(r^6 + 4r^5 + 6r^4 + 3r^3 - 3r^2 - 3r - 1)}{(1 + 2r + r^2 + 2r^3 + r^4)(r^2 + r + 1)}. \tag{18}$$

(2) $q(\alpha_1) \sim q(\alpha_3)$;

Since $r_1 \neq r_3$, we are going to find the conditions such that $r_3 = 1/r_1$. By same arguments as in (1), we have $q(\alpha_1) \sim q(\alpha_3)$ if and only if $(m_1, m_2, m_3) \in F_2$ or $(m_1, m_2, m_3) \in F_5$.

(3) $q(\alpha_2) \sim q(\alpha_3)$;

Since $r_2 \neq 1/r_3$, we are going to find the conditions such that $r_2 = r_3$. By same arguments as in (1), we have $q(\alpha_2) \sim q(\alpha_3)$ if and only if $(m_1, m_2, m_3) \in F_3$ or $(m_1, m_2, m_3) \in F_6$. □

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