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CLASSIFICATION OF PERIODIC ORBITS IN THE PLANAR EQUAL-MASS FOUR-BODY PROBLEM

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Abstract. In the N-body problem, many periodic orbits are found as local Lagrangian action minimizers. In this work, we classify such periodic orbits in the planar equal-mass four-body problem. Specific planar configurations are considered: line, rectangle, diamond, isosceles trapezoid, double isosceles, kite, etc. Periodic orbits are classified into 8 categories and each category corresponds to a pair of specific configurations. Furthermore, it helps discover several new sets of periodic orbits.

1. **Introduction.** "Periodic motions in nature have always been of interest to mankind. All phenomena that have some cyclic behaviors capture our attention because they are a sign for regularity. Therefore, they are indications toward the possibility of understanding the laws of nature. " [4]

"The development of mechanics, the discovery that laws of nature can be written in the language of calculus and that laws of motion can be described in terms of differential equations, opened up the study of periodic solutions of equations of motion. In particular, since Newton and then Poincaré, the main interest has been the understanding of the planetary motion and the solution of the N-body problem. " [4]

Variational method is one of the important tools in the N-body problem. Over the years, it has been applied to construct periodic solutions for the N-body problem, under various types of symmetry constraints or topological constraints. Plenty

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of periodic orbits have been discovered and shown to exist in the three-body or four-body problem as local Lagrangian action minimizers. The trajectories of these orbits are highly symmetric. More importantly, they all have variational properties which are helpful in understanding their stabilities. Is there a way to classify these nontrivial action minimizing periodic solutions? The main challenge is to group these infinitely many periodic orbits into finitely many classes. Actually, Broucke [1] tried to classify periodic orbits in the planar equal-mass four-body problem according to the symmetries of the starting configurations, in which he found several new sets of periodic orbits, such as a remarkable star-shaped choreography [5, 13, 10, 11, 12]. However, this classification is far from complete and the variational properties of his orbits are not clear. In this paper, we intend to classify the planar periodic orbits which are local action minimizers in the equal-mass four-body problem. By introducing a different variational approach, such action minimizing periodic orbits can be characterized as local action minimizers jointing a pair of special configurations. Hence, the classification of such orbits can be done by considering all the possible pairs of special configurations. In such a way, we can claim that these action minimizing periodic orbits can be grouped into finitely many categories. More importantly, the orbits found by our method are all local action minimizers connecting two different configurations, which can be possibly shown to exist via variational arguments.

2. **Variational method.** Inspired by Broucke's work, we intend to classify action minimizing periodic orbits from a variational perspective. Instead of considering only the starting configurations and searching for coefficients of Fourier series, we concentrate on a pair of symmetric configurations as the starting and ending positions respectively. Local action minimizers between two different symmetric configurations could be periodic. By checking all the quadrilateral with certain symmetry, we finally concentrate on 6 different types of shapes: line, rectangle, diamond, isosceles trapezoid, double isosceles and kite. Other special shapes, such as parallelogram, right trapezoid etc., eventually converge to some of the 6 configurations in our numerical our search. After searching local action minimizers connecting any two different configurations among the 6 types, we find that there are only eight categories of nontrivial periodic orbits and all of them have definite variational properties. The main method we use here is a variational method with Structural Prescribed Boundary Conditions (SPBC) (see [5, 6, 13]). Before explaining our main result, we briefly introduce the variational method with SPBC first.

We consider a boundary value problem satisfying an appropriate structural prescribed boundary configurations. A two-step minimizing process is utilized with the proper SPBC to find an appropriate piece of orbit which is proven to be assembled out to a periodic solution (or a quasi-periodic solution). Local action minimizers are obtained in the full space (not in a restricted symmetric space) with the SPBC. Let $Qs = (a_{ij}) \in M^{(N\times d)}$ and $Qe = (b_{ij}) \in M^{(N\times d)}$ are two given matrices of $N \times d$, which represent two configurations of N bodies in d dimensions. Let $\mathcal{P}(Qs, Qe)$ be the set of paths connecting the two given configurations in the functional space $H^1([0, T] \to (\mathbf{R}^d)^N)$,

$$\mathcal{P}(Qs, Qe) = \{\mathbf{q} \in H^1 \mid \mathbf{q}(0) = Qs;\ \mathbf{q}(T) = Qe\}.$$

For a fixed boundary value problem, it is known that a corresponding minimizer of the Lagrangian action functional $\mathcal{A}(\mathbf{q}) = \int_0^T L(\mathbf{q}, \dot{\mathbf{q}})dt$ over the path space $\mathcal{P}(Qs, Qe)$ is a collision free solution in the interior $(0, T)$ of the N-body problem. Let

$$\mathcal{A}(Qs, Qe) = \inf_{\mathbf{q} \in \mathcal{P}(Qs,Qe)} \mathcal{A}(\mathbf{q}) = \inf_{\mathbf{q} \in \mathcal{P}(Qs,Qe)} \int_0^T L(\mathbf{q}, \dot{\mathbf{q}})dt.$$

Is there a pair of configurations (Qs, Qe) such that a corresponding minimizing path coincides with part of a periodic orbit? The answer is yes and the method to find such a pair (Qs, Qe) is to do a second minimizing process under the appropriate structural prescribed boundary conditions, which can be described by a set of relation $\{(Qs, Qe)|G(Qs, Qe) = 0\}$. The local minimizer over a proper SPBC

$$\operatorname*{localmin}_{\{G(Qs,Qe)=0\}} \mathcal{A}(Qs, Qe) = \operatorname*{localmin}_{\{G(Qs,Qe)=0\}} \inf_{\mathbf{q} \in \mathcal{P}(Qs,Qe)} \mathcal{A}(\mathbf{q})$$

can generate a periodic solution (or a quasi-periodic solution).

This method works for general N-body problem with/without any constraints on masses or symmetries. Many periodic orbits in planar four-body problem have been found by the variation method with SPBC and the structures of these periodic orbits are determined by the geometric configurations in the SPBC $\{(Qs, Qe)|G(Qs, Qe) = 0\}$. In this paper, the SPBC is restricted in the simple form of $\{(Qs, Qe)|Qs \in \mathbf{A}, Qe \in \mathbf{B}\}$, where $\mathbf{A}$ and $\mathbf{B}$ are two proper linear subspaces of $(\mathbf{R}^2)^4$. $G(Qs, Qe) = 0$ is chosen to be $\mathbf{A} \cap \mathbf{B} = \{0\}$. Usually, Qs, Qe are configurations with symmetries. In the next section, we consider all the possible pairs of special configurations (Qs, Qe) and check if there exists an orbit connecting them.

3. **Classification of periodic orbits.** The configuration of Qs or Qe could be a line, parallelogram, rectangle, square, diamond, trapezoid, double isosceles, kite, etc. We search all the possible pairs of these configurations and try to find if a local minimizing path connecting them is nontrivial and periodic. Our numerical result suggests that the following six configurations generate most of the periodic solutions in the equal-mass four-body problem. They are line, rectangle, diamond, isosceles trapezoid, double isosceles and kite. We list them in Fig. 1. We then classify action minimizing periodic orbits by different pairs of configurations. In general, it should be 15 different pairs. However, nontrivial periodic orbits are only found in 8 pairs. The other 7 pairs either lead to solutions of the 8 pairs or non-periodic orbits. The 8 categories of periodic orbits are presented here. From now on, we use $Q(0)$ to denote Qs at $t = 0$ and $Q(1)$ to denote Qe at $t = 1$.

(I): From a symmetric line to a rectangle. In this category, the two boundary configurations $Q(0)$ and $Q(1)$ are defined in Eqn (1).

$$Q(0) = \begin{bmatrix} -a & 0 \\ -b & 0 \\ b & 0 \\ a & 0 \end{bmatrix}, \quad Q(1) = \begin{bmatrix} -c & d \\ -c & -d \\ c & d \\ c & -d \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{1}$$

There are five variables in $Q(0)$ and $Q(1)$: a, b, c, d and θ. We set a, b, c and d as parameters and treat θ as a constant. For special choices of θ, local minimizing paths $\mathcal{P}_{l,\theta}$ can be generated as follows

$$\mathcal{A}(\mathcal{P}_{l,\theta}) = \operatorname*{localmin}_{\{a,b,c,d\}} \inf_{\{q(0)=Q(0),\, q(1)=Q(1)\}} \int_0^1 \mathcal{A}.$$

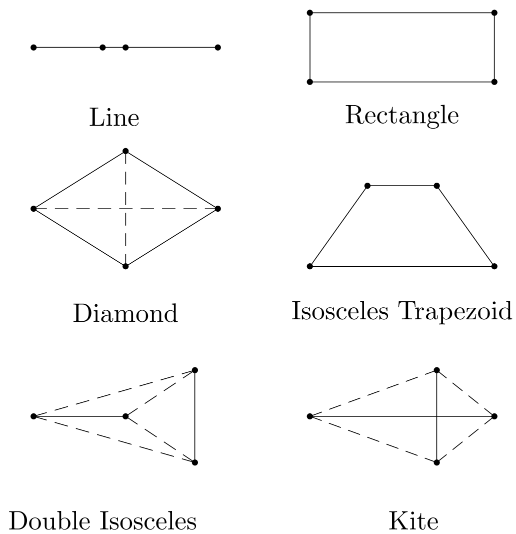


FIGURE 1. 6 special configurations

In this category, two types of periodic orbits are found. One can be called as a retrograde orbit, and the other can be called as a prograde orbit. Actually, the retrograde orbit with $\theta = \pi/n$ has outward corners; while the corners of the prograde orbit with $\theta = \pi/n$ are inward (as in Fig. 2). From our numerical search, the action of a retrograde orbit is always smaller than that of a prograde orbit with the same rotation angle θ. Eight pictures of motion of category I are presented here in Fig. 2. Part of them are referred to as doubledouble orbits in [9]. Note that the orbits in this category is symmetric. At any time, the four masses form a parallelogram. Most of the orbits in this category are shown to exist by Ferrario and Terracini [7] and Chen [2] independently.

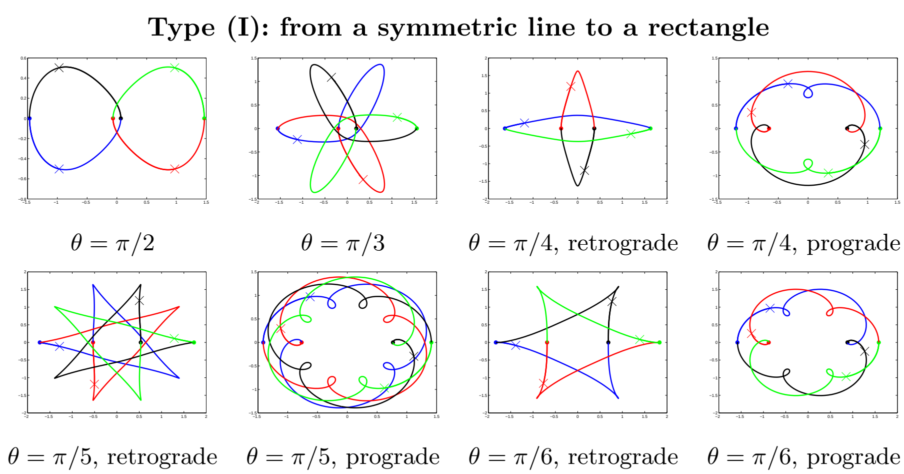


FIGURE 2. 8 orbits of Type (I)

(II): From a symmetric line to a diamond. $Q(0)$ and $Q(1)$ are defined as follows.

$$Q(0) = \begin{bmatrix} -a & 0 \\ -b & 0 \\ b & 0 \\ a & 0 \end{bmatrix}, \quad Q(1) = \begin{bmatrix} -c & 0 \\ 0 & d \\ 0 & -d \\ c & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{2}$$

The first six pictures of these types of motion are listed in Fig. 3. Besides that, there is actually a special orbit jointing a line and a square. $Q(0)$ still follows the definition in Eqn. (1) and $Q(1)$ is set to be

$$Q(1) = \begin{bmatrix} -c & d \\ -d & -c \\ d & c \\ c & -d \end{bmatrix}.$$

The orbit is the last one in Fig. 2. Part of the orbits in this category are shown to exist by Chen [3].

Type (II): from a symmetric line to a diamond

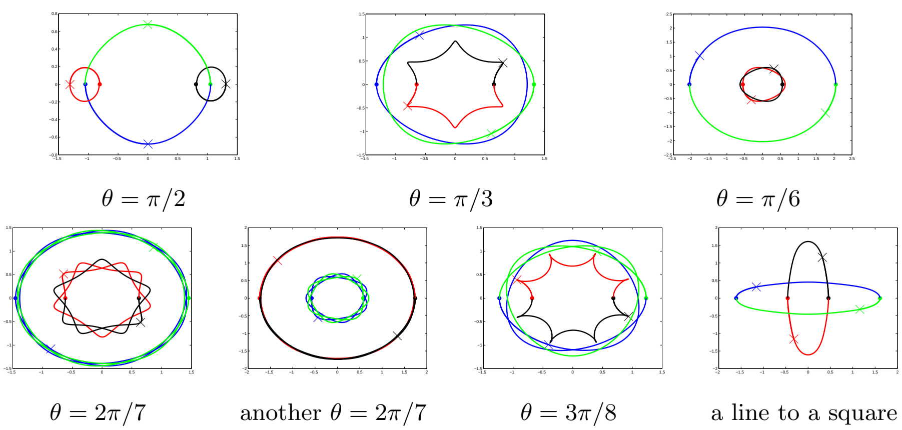


FIGURE 3. 7 orbits of type (II)

(III): From a line to an isosceles trapezoid. In Category III, $Q(0)$ and $Q(1)$ are defined in Eqn. (3).

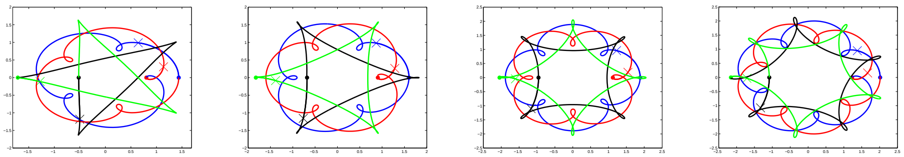
$$Q(0) = \begin{bmatrix} a & 0 \\ b & 0 \\ -c & 0 \\ -a-b+c & 0 \end{bmatrix}, Q(1) = \begin{bmatrix} d & e \\ d & -e \\ -d & -f \\ -d & f \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{3}$$

Four pictures of motion are listed in Fig. 4. In these pictures, it is interesting to notice that one pair runs on one curve and the other pair runs on another, which can be seen as a combination of two orbits.

(IV): From a line to a double isosceles. We define $Q(0)$ and $Q(1)$ as follows.

$$Q(0) = \begin{bmatrix} 0 & a \\ 0 & b \\ 0 & -c \\ 0 & -a-b+c \end{bmatrix}, Q(1) = \begin{bmatrix} 0 & d \\ 0 & e \\ -f & -(d+e)/2 \\ f & -(d+e)/2 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{4}$$

Type (III): from a line to an isosceles trapezoid

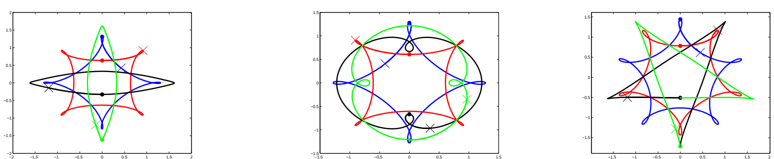


$\theta = \pi/5$ $\theta = \pi/6$ $\theta = \pi/8$ $\theta = \pi/9$

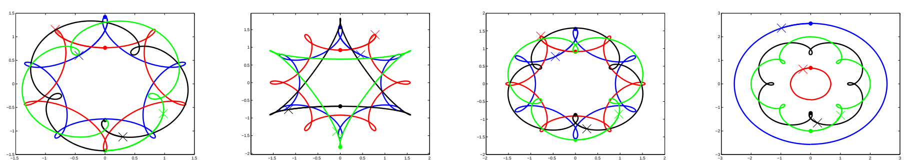
FIGURE 4. 4 orbits of Type (III)

Similar to category I, we find two sets of orbits: retrograde orbit and prograde orbit in this category. In a retrograde orbit with $\theta = \pi/n$, it looks like a combination of two convex orbits: one has $2n$ corners and the other has n corners. In a prograde orbit with $\theta = \pi/n$, it looks like a combination of a convex orbit with $2n$ corners and a concave orbit with n corners (as in Fig. 5).

Type (IV): from a line to a double isosceles



$\theta = \pi/4$, retrograde $\theta = \pi/4$, prograde $\theta = \pi/5$, retrograde



$\theta = \pi/5$, prograde $\theta = \pi/6$, retrograde $\theta = \pi/6$, prograde New shape of $\theta = \pi/8$

FIGURE 5. 7 orbits of type (IV)

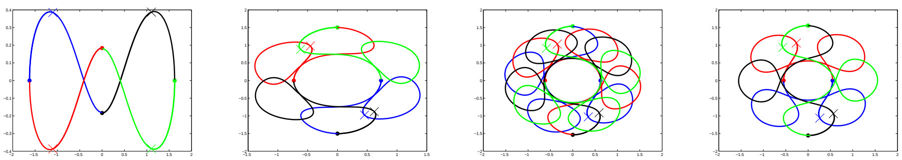
Furthermore, a different type of periodic orbit is found when $\theta = \pi/8$ (last picture in Fig. 5). In this orbit, two bodies run on its own circle-like curve and the other two run on a concave orbit with eight corners. Pictures of motion of this category are given in Fig. 5.

(V): From a diamond to a rectangle. In this category, $Q(0)$ and $Q(1)$ are defined in Eqn. (5). There exist periodic orbits when $\theta = \pi/n \, (n \geq 6)$. We only present four pictures in Fig. 6 here. More pictures can be found in Ouyang and Xie's work [6].

$$Q(0) = \begin{bmatrix} -a & 0 \\ 0 & b \\ 0 & -b \\ a & 0 \end{bmatrix}, Q(1) = \begin{bmatrix} -c & d \\ -c & -d \\ c & d \\ c & -d \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{5}$$

(VI): From a diamond to an isosceles trapezoid. $Q(0)$ and $Q(1)$ are defined in Eqn. (6). In this category, we only find one periodic orbit as Fig. 7. Its shape is like a figure-eight. Actually in our search, the trapezoid $Q(1)$ easily converges to a

Type (V): from a diamond to a rectangle



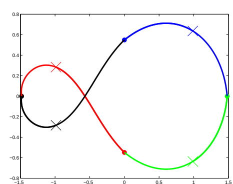
$\theta = 0$ $\theta = \pi/6$ $\theta = \pi/7$ $\theta = \pi/8$

FIGURE 6. 4 orbits of type (V)

rectangle, which becomes an orbit in Category V.

$$Q(0) = \begin{bmatrix} -a & 0 \\ 0 & b \\ 0 & -b \\ a & 0 \end{bmatrix}, Q(1) = \begin{bmatrix} -c & -e \\ c & d \\ -c & e \\ c & -d \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{6}$$

Type (VI): from a diamond to an isosceles trapezoid



$\theta = 0$

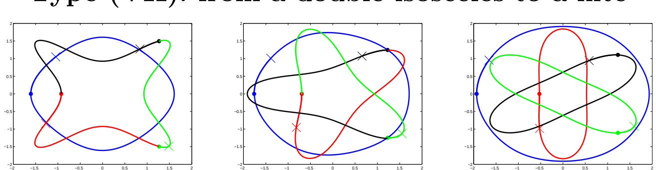
FIGURE 7. 1 orbit of type (VI)

(VII): From a double isosceles to a kite. We define $Q(0)$ and $Q(1)$ in Eqn. (7).

$$Q(0) = \begin{bmatrix} -2a + c & 0 \\ -c & 0 \\ a & b \\ a & -b \end{bmatrix}, Q(1) = \begin{bmatrix} -d & 0 \\ e & -f \\ e & f \\ d - 2e & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{7}$$

Three pictures of motion are presented in Fig. 8. Actually, for every $\theta = -\pi/n\, (n \geq 4)$, there exists such an action minimizing periodic orbit.

Type (VII): from a double isosceles to a kite



$\theta = -\pi/4$ $\theta = -\pi/5$ $\theta = -\pi/6$

FIGURE 8. 3 orbits of type (VII)

(VIII): From a double isosceles to an isosceles trapezoid. In the last category, it is surprising that four sets of periodic orbits are found. By defining $Q(0)$ and $Q(1)$

in Eqn. (8) and minimizing the action $\mathcal{A}$ over 6 parameters: a, b,c, d, e and f, we find the first set of orbits.

$$\text{(VIII) a:} \quad Q(0) = \begin{bmatrix} -2a+c & 0 \\ -c & 0 \\ a & b \\ a & -b \end{bmatrix}, \; Q(1) = \begin{bmatrix} -d & -f \\ -e & f \\ e & f \\ d & -f \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{8}$$

Eight pictures are presented in (VIII) a of Fig. 9. In (VIII) a, there are two types of orbits: retrograde orbit and prograde orbit. When $\theta = \pi/2n\,(n \geq 5)$, the retrograde orbit has outward corners; while the prograde orbit has inward corners.

A different setting of $Q(1)$ in Eqn. (9) creates another new set of periodic solutions. This set of orbits also contains two types: retrograde orbit and prograde orbit. In a retrograde orbit with $\theta = \pi/n\,(n \geq 2)$, one pair runs on a convex orbit and another pair runs on two different circle-shaped curves. In a prograde orbit with $\theta = \pi/n\,(n \geq 4)$, one pair runs on a concave orbit and the other two are on two circle-shaped curves. Six pictures of motion are presented in (VIII) b of Fig. 9.

$$\text{(VIII) b:} \quad Q(0) = \begin{bmatrix} -2a+c & 0 \\ -c & 0 \\ a & b \\ a & -b \end{bmatrix}, \; Q(1) = \begin{bmatrix} -d & -e \\ -d & e \\ d & f \\ d & -f \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{9}$$

A third setting of $Q(0)$ and $Q(1)$ is defined in Eqn. (10). (VIII) c of Fig. 9 shows six orbits for different values of θ. Similarly, this set has two types of orbits: retrograde orbit and prograde orbit. A retrograde orbit with $\theta = \pi/n\,(n \geq 3)$ is a combination of two convex orbits: one has $2n$ corners and the other has n corners. While a prograde orbit with $\theta = \pi/n\,(n \geq 4)$ is a combination of one convex orbit with $2n$ corners and one concave orbit with n corners.

$$\text{(VIII) c:} \quad Q(0) = \begin{bmatrix} -c & 0 \\ -2a+c & 0 \\ a & b \\ a & -b \end{bmatrix}, \; Q(1) = \begin{bmatrix} -d & -e \\ -d & e \\ d & f \\ d & -f \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{10}$$

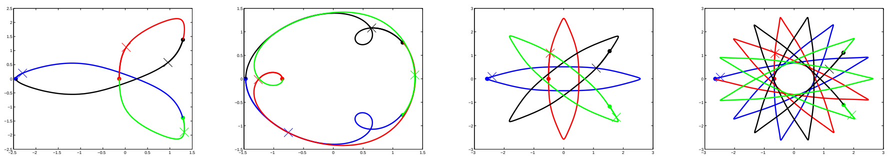
Only one orbit is found at $\theta = 0$ for the setting in Eqn. (11), which is a choreography. Picture of motion is show in (VIII) d of Fig. 10.

$$\text{(VIII) d:} \quad Q(0) = \begin{bmatrix} -2a+c & 0 \\ -c & 0 \\ a & b \\ a & -b \end{bmatrix}, \; Q(1) = \begin{bmatrix} -d & e \\ -d & -f \\ d & -e \\ d & f \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{11}$$

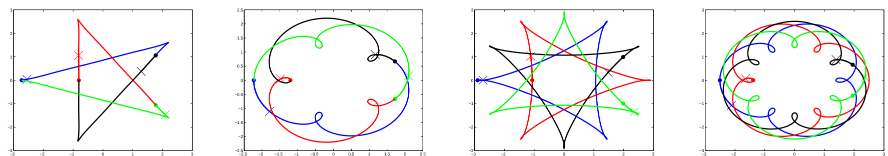
4. **Conclusion.** In conclusion, we present a way to classify periodic orbits which can be characterized as local action minimizers in the planar equal-mass four-body problem. The classification is realized according to pairs of special configurations each orbit passing by. Each pair of special configurations corresponds to one category. Basically, a special configuration could be a line, parallelogram, rectangle, square, diamond, trapezoid, double isosceles, kite, etc. By checking all possible pairs of configurations, the periodic orbits are mainly found between the following 6 configurations: line, rectangle, diamond, isosceles trapezoid, double isosceles, kite. In general, there should be 15 different pairs of configurations, which corresponds to 15 different categories. However, only eight categories are located. Each category may contain several sets of periodic orbits. As a byproduct, two sets of new orbits

Type (VIII): from a double isosceles to an isosceles trapezoid

(VIII) a

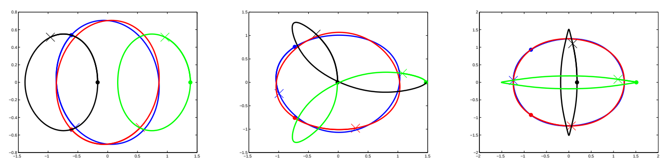


$\theta = \pi/6$, retrograde $\quad \theta = \pi/6$, prograde $\quad \theta = \pi/8$, retrograde $\quad \theta = \pi/9$

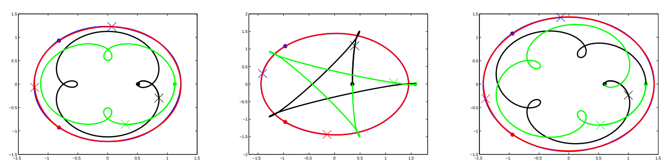


$\theta = \pi/10$, retrograde $\quad \theta = \pi/10$, prograde $\quad \theta = \pi/12$, retrograde $\quad \theta = \pi/12$, prograde

(VIII) b

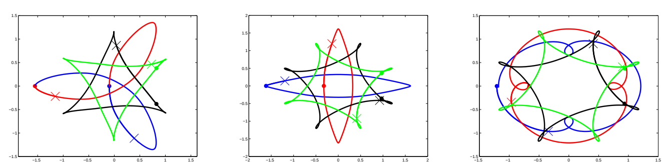


$\theta = \pi/2 \quad \theta = \pi/3 \quad \theta = \pi/4$, retrograde

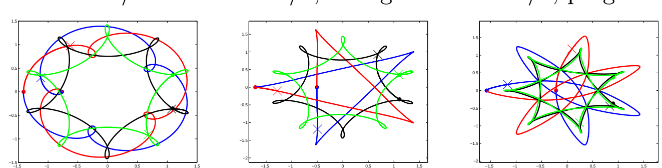


$\theta = \pi/4$, prograde $\quad \theta = \pi/5$, retrograde $\quad \theta = \pi/5$, prograde

(VIII) c



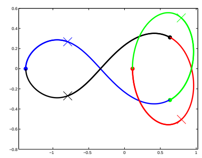
$\theta = \pi/3 \quad \theta = \pi/4$, retrograde $\quad \theta = \pi/4$, prograde



$\theta = \pi/5$, retrograde $\quad \theta = \pi/5$, prograde $\quad \theta = 2\pi/7$

FIGURE 9. orbits of type (VIII) a, b, c

Type (VIII): from a double isosceles to an isosceles trapezoid (continued)
(VIII) d



$\theta = 0$

FIGURE 10. orbit of type (VIII) d

((VIII) b and (VIII) c in Fig. 9) are found and also many new periodic shapes are found. Following our method, the infinitely many action-minimizing periodic orbits can be grouped into only finitely many classes. Mathematically, the existence proofs of these orbits can be done by our variational method. The two main difficulties are the coercivity of the action functional and the exclusion of possible boundary collision. Let $\{(Qs, Qe)|Qs = Q(0) \in \mathbf{A}, Qe = Q(1) \in \mathbf{B}\}$. In fact, when the linear subsets A and B satisfies $A \cap B = \{0\}$, the coercivity of the action functional has been proved in [14]. The exclusion of boundary collision at $Q(0)$ or $Q(1)$ will be done case by case for the 6 configurations by introducing local deformation analysis.

Different from the standard variational approach in [9, 1], we do not directly concentrate on the search of periodic orbits. It is known that, in a period of any nontrivial symmetric periodic orbit, it has two or more special configurations. Reversely, if given two special configurations, should there exist a periodic orbit connecting them? Actually, after choosing the shapes of two different special configurations and freeing some of their magnitudes, there exist one or more local action minimizers. When these local action minimizers are periodic? Our method helps find such minimizers and identify whether they are periodic or not. It will also be interesting to see the application of this method to periodic orbits in the planar five-body problem or the unequal mass case in the four-body problem.

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