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A simpler proof of regular polygon solutions of the N-body problem ☆

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Abstract

In this Letter we give simpler proofs of the famous results of Perko–Walter–Elmabsout theorem published in Proc. Amer. Math. Soc. 94 (1985) 301 and Celest. Mech. 41 (1988) 131. © 2000 Elsevier Science B.V. All rights reserved.

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This Letter uses the same notations as paper [5]. For $N \geqslant 2$ the equations of motion of the planar N-body problem can be written in the form

$$\ddot{z}_k = -\sum_{j \neq k} m_j \frac{z_k - z_j}{|z_k - z_j|^3}, \tag{1}$$

where z_k is the complex coordinate of the kth mass m_k in an inertial coordinate system. In Eq. (1) and throughout this paper, unless otherwise restricted, all indices and summations will range from 1 to N.

Let ρ_k denote the N complex kth roots of unity, i.e.,

$$\rho_k = \exp(2\pi i k/N). \tag{2}$$

This equation will also serve to define ρ_k for any number k. The center of mass of N masses, m_k located at the vertices ρ_k of a regular polygon inscribed on the unit circle, is then given by

$$z_0 = \sum_j m_j \rho_j / M,$$

where $M = \sum_j m_j$. The functions describing their rotation about z_0 with angular velocity ω are then given by

$$z_k(t) = (\rho_k - z_0) \exp(i\omega t). \tag{3}$$

There is no loss of generality in assuming that the regular polygon is inscribed in the unit circle since the N-body problem (1) is invariant under the transformation $t \to t/a^{3/2}$, $z \to z/a$ which reduces the functions $a(\rho_k - z_0)\exp(i\omega t)$ to the functions defined in (3).

We will also give simpler proofs of the following results of Perko–Walter–Elmabsout [1,2,4,5].

Theorem 1 (Perko–Walter [5]). *If, for $N \geqslant 2$, the functions $z_k(t)$ given by* (3) *are solutions of the N-body problem* (1), *it follows that $\omega^2 = M\gamma/N$, where*

$$\gamma = \frac{1}{4} \sum_{j \neq N} \csc(\pi j/N). \tag{4}$$

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Corollary 1 (Lagrange [3]). *For $N=3$, the functions $z_k(t)$ given by* (3) *are solutions of the* 3*-body problem* (1) *if and only if $\omega^2 = M/3\sqrt{3}$.*

Theorem 2 (Perko–Walter [5], Elmabsout [2]). *For $N \geqslant 4$ and $m_k > 0$, the functions $z_k(t)$ given by* (3) *with $\omega^2 = M\gamma/N$ and γ given by* (4) *are solutions of the N-body problem* (1) *if and only if $m_1 = m_2 = \cdots = m_N$.*

Proof of Theorem 1. Direct substitution into the differential equation (1) shows that the $z_k(t)$ are a solution of (1) if and only if

$$(\rho_k - z_0)\omega^2 \exp(i\omega t) = \sum_{j\neq k} m_j \frac{\rho_k - \rho_j}{|\rho_k - \rho_j|^3} \exp(i\omega t), \tag{5}$$

or if and only if

$$\sum_{j\neq k} m_j \frac{\rho_k - \rho_j}{|\rho_k - \rho_j|^3} + \omega^2 \left(\frac{1}{M} \sum_j m_j \rho_j - \rho_k \right) = 0, \tag{6}$$

$$\sum_{j\neq k} m_j \frac{\rho_k - \rho_j}{|\rho_k - \rho_j|^3} + \omega^2 \left(\frac{1}{M} \sum_j m_j \rho_j - \frac{1}{M} \sum_j m_j \rho_k \right) = 0, \tag{7}$$

$$\sum_{j\neq k} m_j \left(\frac{1}{|\rho_k - \rho_j|^3} - \frac{\omega^2}{M} \right)(\rho_k - \rho_j) = 0. \tag{8}$$

Multiplying both sides by ρ_{N-k} and noting that $|\rho_k - \rho_j| = |\rho_k||1-\rho_{j-k}| = |1-\rho_{j-k}|$,

$$\sum_{j\neq k} m_j \left(\frac{1}{|\rho_{j-k} - 1|^3} - \frac{\omega^2}{M} \right)(1 - \rho_{j-k}) = 0. \tag{9}$$

Notice that every step from (5) to (9) can be conversed. Now define the $N \times N$ circulant matrix $C = [c_{k,j}]$ as follows:

$$c_{k,j} = 0 \quad \text{for } k = j,$$
$$c_{k,j} = \left(\frac{1}{|\rho_{j-k} - 1|^3} - \frac{\omega^2}{M} \right)(1 - \rho_{j-k}) \quad \text{for } k \neq j. \tag{10}$$

Then (9) holds if and only if

$$C\vec{m} = 0 \tag{11}$$

holds, where $\vec{m} = (m_1, \ldots, m_N)^{\mathrm{T}}$.

We notice that Eq. (9), or equivalently (10), is equivalent to Eq. (8) of [5]. According to (9) and (11),

$$\sum_k \sum_{j\neq k} m_j \left(\frac{1}{|\rho_{j-k} - 1|^3} - \frac{\omega^2}{M} \right)(1 - \rho_{j-k}) = 0$$
$$\Leftrightarrow \left(\sum_k m_k \right) \sum_{j\neq N} \left(\frac{1}{|\rho_j - 1|^3} - \frac{\omega^2}{M} \right)(1 - \rho_j) = 0$$
$$\Leftrightarrow \sum_{j\neq N} \frac{1-\rho_j}{|\rho_j - 1|^3} = \frac{\omega^2}{M} \sum_{j\neq N} (1 - \rho_j)$$
$$\Leftrightarrow \frac{\omega^2 N}{M} = \sum_{j\neq N} \frac{1-\rho_j}{|\rho_j - 1|^3} = \frac{1}{4} \sum_{j\neq N} \csc(\pi j/N),$$

i.e.,

$$\gamma = \frac{1}{4} \sum_{j\neq N} \csc(\pi j/N). \qquad \square$$

Proof of Corollary 1.

(1) The proof of the necessary conditions: For $N=3$, formula (4) of Theorem 1 gives $\omega^2 = M/3\sqrt{3}$.

(2) The proof of the sufficient conditions: The proof hangs on showing that certain eigenvalues of the circulant matrix C are zero. This is accomplished by using the general formulas (4) for the eigenvalues λ_k and the eigenvectors $\vec{v}_k$ of a circulant matrix $C = [c_{k,j}]$:

$$\lambda_k = \sum_j c_{1,j} \rho_{k-1}^{j-1}, \tag{12}$$

$$\vec{v}_k = \left(\rho_{k-1}, \rho_{k-1}^2, \ldots, \rho_{k-1}^N \right)^{\mathrm{T}}. \tag{13}$$

According to the relations between the eigenvalues λ_k and the eigenvectors $\vec{v}_k$, one has

$$C\vec{v}_k = \lambda_k \vec{v}_k, \quad 1 \leqslant k \leqslant N.$$

Then in order to find the solution of (11), it is enough to find the positive real eigenvectors with zero eigenvalue.

We notice that $c_{k,j} = 0$ for $j = k$ and

$$c_{k,j} = \left(\frac{1}{|\rho_{j-k} - 1|^3} - \frac{\omega^2}{M} \right)(1 - \rho_{j-k}) = 0$$

for $j \neq k$, since

$$\frac{1}{|\rho_{j-k} - 1|^3} = \frac{1}{3\sqrt{3}} \quad \text{and} \quad \frac{\omega^2}{M} = \frac{1}{3\sqrt{3}}.$$

So C is a zero matrix, then any $\vec{m}$ is a solution of (11). □

Proof of Theorem 2.

(1) The proof of the necessary conditions of Theorem 2: Since

$$\frac{1}{|\rho_1-1|^3} \neq \frac{1}{|\rho_2-1|^3},$$

then $|c_{1,2}|+|c_{1,3}| \neq 0$. So C is not a zero matrix. According to the relations between the eigenvalues and eigenvectors,

$$C\vec{v}_k = \lambda_k \vec{v}_k, \quad 1 \leqslant k \leqslant N.$$

In order to find the positive real solution of (11), it is enough to find the zero eigenvalue λ_k with positive real eigenvector $\vec{v}_k$.

(i) N is an odd number. Only $\vec{v}_1 = (1,1,\ldots,1)^{\mathrm{T}}$ is the positive real eigenvector. At the same time

$$\lambda_1 = \sum_j c_{1,j}\rho_{1-1}^{j-1} = \sum_j c_{1,j} = 0$$

(see the proof of Theorem 1). That is, $C\vec{v}_1 = 0$ and $\vec{m} = (m_1, m_1, \ldots, m_1)^{\mathrm{T}}$, $m_1 > 0$ is the unique solution of (11).

(ii) N is an even number. Only $\vec{v}_1 = (1,1,\ldots,1)^{\mathrm{T}}$ and $\vec{v}_{N/2+1} = (-1,1,\ldots,-1,1)^{\mathrm{T}}$ are real eigenvector, but only $\vec{v}_1$ is a positive real eigenvector. The corresponding eigenvalue is

$$\lambda_1 = \sum_j c_{1,j}\rho_{1-1}^{j-1} = \sum_j c_{1,j} = 0$$

(see the proof of Theorem 1). That is, $C\vec{v}_1 = 0$ and $\vec{m} = (m_1, m_1, \ldots, m_1)^{\mathrm{T}}$, $m_1 > 0$ is the unique solution of (11).

(2) The proof of sufficient conditions of Theorem 2: Assume $m_1 = m_2 = \cdots = m_N > 0$,

$$\frac{\omega^2}{M} = \frac{1}{4N}\sum_{j\neq N}\csc(\pi j/N),$$

then $\vec{m}$ is a solution of (11) or (9):

$$\begin{aligned}
&\sum_{j\neq k} m_j\left(\frac{1}{|\rho_{j-k}-1|^3} - \frac{\omega^2}{M}\right)(1-\rho_{j-k}) \\
&\quad = m_1\sum_{j\neq k}\left(\frac{1}{|\rho_{j-k}-1|^3} - \frac{\omega^2}{M}\right)(1-\rho_{j-k}) \\
&\quad = m_1\sum_{j\neq N}\left(\frac{1}{|\rho_j-1|^3} - \frac{\omega^2}{M}\right)(1-\rho_j) \\
&\quad = m_1\left(\sum_{j\neq N}\frac{1-\rho_j}{|\rho_j-1|^3} - \frac{\omega^2}{M}\sum_{j\neq N}(1-\rho_j)\right) \\
&\quad = m_1\left(\sum_{j\neq N}\frac{1-\rho_j}{|\rho_j-1|^3} - \frac{\omega^2}{M}N\right) = 0.
\end{aligned}$$

Thus the proof is complete. □

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