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Zhifu Xie

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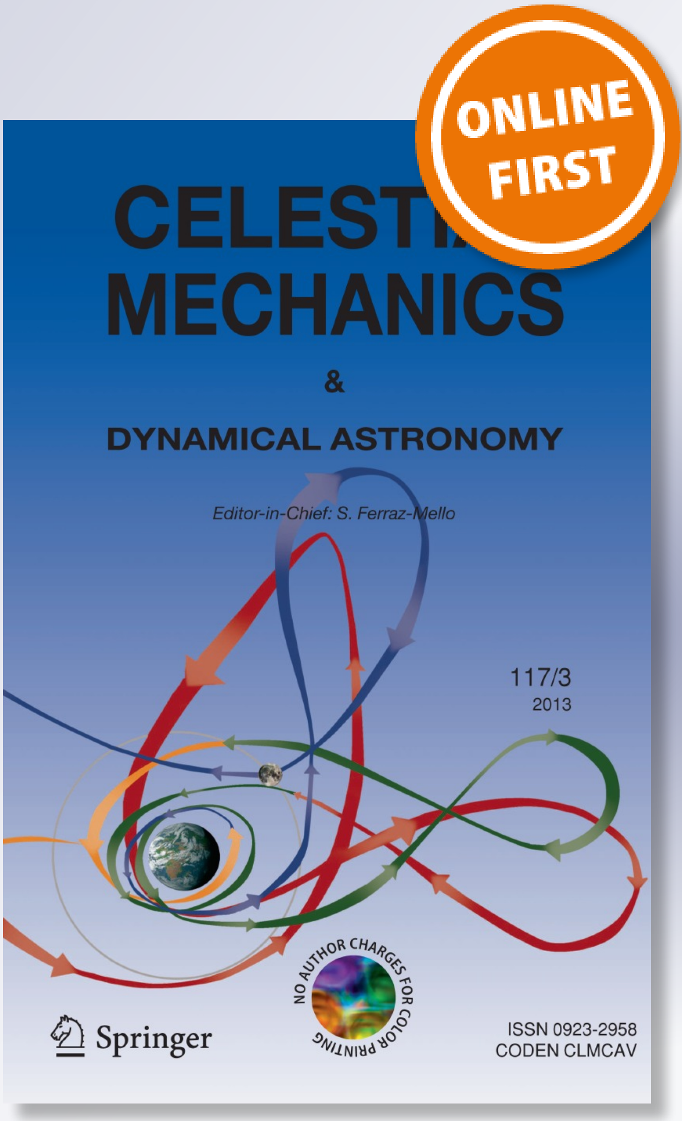

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ORIGINAL ARTICLE

An analytical proof on certain determinants connected with the collinear central configurations in the n-body problem

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Abstract In this paper we give a short analytical proof of the inequalities proved by Albouy–Moeckel through computer algebra, in the cases $n = 5$ and $n = 6$. These inequalities guarantee that, in the n-body problem, the family of mass vectors making a given collinear configuration a central configuration is 2-dimensional. The induction techniques here may be used to prove the inequalities for general n with more subtle estimation but currently the inequalities still remains unproved for $n \geq 7$.

Keywords Skew symmetric matrix · Determinant · Pfaffian · Central configurations

1 Introduction and main results

A configuration $x = (x_1, x_2, \ldots, x_n)^T$ is called a central configuration for a mass vector $m = (m_1, m_2, \ldots, m_n)^T$, if there exists λ such that the following system of algebraic equations holds:

$$\begin{cases} 0 + \frac{m_2(x_2-x_1)}{r_{12}^3} + \frac{m_3(x_3-x_1)}{r_{13}^3} + \cdots + \frac{m_n(x_n-x_1)}{r_{1n}^3} = -\lambda(x_1 - c), \\ \frac{m_1(x_1-x_2)}{r_{12}^3} + 0 + \frac{m_3(x_3-x_2)}{r_{23}^3} + \cdots + \frac{m_n(x_n-x_2)}{r_{2n}^3} = -\lambda(x_2 - c), \\ \frac{m_1(x_1-x_3)}{r_{13}^3} + \frac{m_2(x_2-x_3)}{r_{23}^3} + 0 + \cdots + \frac{m_n(x_n-x_3)}{r_{3n}^3} = -\lambda(x_3 - c) \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ \frac{m_1(x_1-x_n)}{r_{1n}^3} + \frac{m_2(x_2-x_n)}{r_{2n}^3} + \frac{m_3(x_3-x_n)}{r_{3n}^3} + \cdots + 0 = -\lambda(x_n - c). \end{cases}, \tag{1.1}$$

Here $m_i > 0$ is the mass of the i-th body, x_i is the position of the i-th body, $r_{ij} = |x_i - x_j|$, $c = \frac{\sum m_i x_i}{M}$ is the center of mass of the bodies, and $M = \sum m_i$ is the total mass. Central

Z. Xie (✉)
Department of Mathematics and Computer Science, Virginia State University,
Petersburg, VA 23806, USA
e-mail: xiezhifu@hotmail.com

configurations lead to the only known cases where the differential equations of the n-body problem are integrable. Any planar central configuration gives rise to a special one-parameter family of periodic solution where each body rotates around the center of mass on its own ellipse. If all the ellipses are circles, the solution is known as a relative equilibrium because it is a fixed point in a rotating coordinate system. Some of the earliest solutions to the three-body problem were of this type. For example, Euler (1767) proved that there exists exactly one central configuration for each ordering of the three masses on a line. Lagrange (1772) discovered the equilateral triangle central configurations. When three bodies with any choice of masses are placed at the vertices of an equilateral triangle, it gives rise to a family of solutions where each body is traveling along a particular Kepler orbit.

If all bodies in a central configuration are in a straight line, the central configuration is called a *Moulton Configuration* after F. R. Moulton. In his celebrated work, Moulton (1910) considered Eq. (1.1) in two ways:

1. Given masses to find the positions. Moulton proved that for a fixed mass vector m and a fixed ordering of the bodies along the line, there exists a unique collinear central configuration (up to translation and scaling).
2. Given the positions to find the masses. Moulton's results depend on whether n is even or odd. Under Moulton's settings, the mass vector of a central configuration is uniquely determined if the corresponding Pfaffian for the given collinear configuration is not zero when n is even. It is also uniquely determined if another consistent Pfaffian is zero when n is odd.

Moulton's results are conditioned on various Pfaffians. Buchanan (1909) claimed that the Pfaffians are nonzero for all even number collinear configurations x. However, Albouy and Moeckel (2000) pointed out that the argument in Buchanan (1909) is incorrect because the zeros of the nth derivatives considered on pages 228 and 229 in Buchanan (1909) may tend to infinity with n. Albouy and Moeckel proved analytically that all corresponding Pfaffians are nonzero for all noncollision configurations when $n \leq 4$. They also described computer assisted proofs for $n = 5, 6$ in the Newtonian case. But the computation involves a large amount of terms. For example, the numerator of the associated Pfaffian in the collinear 6-body problem is a polynomial in five variables with 15158 coefficients and they checked that all the coefficients are positive. They conjectured that these Pfaffians are nonzero for all noncollision configurations for any n.

The main results of central configurations in the collinear n-body problem with $n = 5, 6$ can be restated as follows. These results can be extended to the general collinear n-body problem as in Albouy and Moeckel (2000) if the corresponding Pfaffian can be proved to be nonzero. Our goal is to give an analytical proof of the conjecture and to answer the questions raised by Moulton for the collinear n-body problem with $n \leq 6$. Our approach and the associated Pfaffians are different from Albouy and Moeckel (2000), Moulton (1910). The explicit expressions of masses are given for a central configuration. The masses are not necessarily all positive.

Theorem 1.1 *1.* Albouy and Moeckel (2000) *For any fixed noncollision configuration in the collinear 5-body problem, there exists a unique position c, the center of mass, which only depends on the positions of the bodies. With the appropriate choice of the center of mass, the given configuration determines a two-parameter family of masses making it central.* λ *and* M *can be taken as the two parameters.*

2. For a collinear 5-body central configuration, the configuration is symmetric about the position in the middle if and only if the mass vector is symmetric about the mass in the middle.

Remark 1.2 (1) The center of mass c will be given explicitly as a function of the positions in (2.5). The masses m_i, $1 \leq i \leq 5$ will be given explicitly as a function of the positions and the above two parameters in (2.7).

(2) In Albouy and Moeckel (2000), Moulton (1910), Wintner (1941), it was proved that it is always possible to choose parameters so that all three masses are positive in the collinear three-body central configurations. However, in the collinear 5-body central configurations, it is generally not possible to arrange all masses to be positive. For example, we checked that the configuration $x_1 = -2, x_2 = -1, x_3 = 1, x_4 = 2, x_5 = 4$ does not have all positive masses to make it central. For the sake of simplification, the masses are approximated though the exact expressions are available. By using the explicit solution m in (2.7),

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{bmatrix} = \begin{bmatrix} -4.546744929\,\lambda + 0.07628573854\,M \\ 5.388447451\,\lambda - 0.04113318665\,M \\ -2.138690659\,\lambda + 0.2120383126\,M \\ 3.377152147\,\lambda - 0.09613413336\,M \\ -2.080164010\,\lambda + 0.8489432688\,M \end{bmatrix}.$$

Note that to make $m_1 > 0$, $\frac{M}{\lambda} > 59.60150622$, but to make $m_4 > 0$, $\frac{M}{\lambda} < 35.12958435$. So it is impossible to choose λ and M to make all m_i positive.

(3) The correspondence of symmetry between positions and masses in the collinear 5-body central configuration does not hold in the collinear 4-body central configuration. If $|x_4 - x_3| = |x_2 - x_1|$, m_1, m_2, m_3, m_4 could be all different in a collinear 4-body central configuration Albouy and Moeckel (2000), Ouyang and Xie (2005). But for a collinear 5-body central configuration, $m_1 = m_5$ and $m_2 = m_4$ if and only if $|x_2 - x_1| = |x_5 - x_4|$ and $|x_3 - x_2| = |x_4 - x_3|$. This property was not mentioned in Albouy and Moeckel (2000).

Theorem 1.3 Albouy and Moeckel (2000) *For any fixed noncollision configuration in the collinear 6-body problem, it determines a two-parameter family of masses making it central. λ and c can be taken as the two parameters. Total mass M is independent of the center of mass c and M is linearly dependent on λ.*

Remark 1.4 (1) The relationship of the total mass M, center of mass c, and λ is given by an explicit equation in theorem 2.10.

(2) In the collinear 6-body central configurations, it is also generally not possible to arrange all masses to be positive. For example, we checked that the configuration $x_1 = -3, x_2 = -2, x_3 = -1, x_4 = 1, x_5 = 2, x_6 = 3$ does not have all positive masses to make it central. By using the explicit solution m in (2.13),

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{bmatrix} = \begin{bmatrix} 9.272182018\,\lambda - 5.737904905\,\lambda\,c \\ -0.3164542588\,\lambda + 1.940465451\,\lambda\,c \\ 9.828965816\,\lambda - 5.451909762\,\lambda\,c \\ 9.828965816\,\lambda + 5.451909762\,\lambda\,c \\ -0.3164542588\,\lambda - 1.940465451\,\lambda\,c \\ 9.272182018\,\lambda + 5.737904905\,\lambda\,c \end{bmatrix}.$$

It is easy to see that m_2 and m_5 can not be positive simultaneously for any value of the parameters λ and c.

(3) It is not necessary to have symmetrical masses if the configuration is symmetric. For example, the configuration $x_1 = -5, x_2 = -3, x_3 = -1, x_4 = 1, x_5 = 3, x_6 = 5$ is symmetric about the origin. By formula (2.13), x is a central configuration for $m = (4.547944752, 15.74671018, 11.82552636, 18.65357165, 10.85657254, 13.64383426)^T$ with $c = 0.5$ and $\lambda = 1$.

Our paper is organized as follows. First we briefly describe the associated Pfaffians of the given collinear noncollision configurations. A simple analytical proof is provided for n bodies with $n \leq 6$ in Sect. 2.1. Then we study the collinear 5- and 6-body central configurations and we prove Theorem 1.1 and Theorem 1.3 in Sects. 2.2 and 2.3 respectively.

2 Inverse problem for collinear central configurations

There will be no loss of generality in selecting the notation so that the position vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbf{R}^n$ with $x_1 < x_2 < \cdots < x_n$ for the collinear n-body problem. With this choice of notation the system of Eq. (1.1) becomes

$$Am = -\lambda(x - cL), \tag{2.1}$$

where $A = A(x_1, x_2, \ldots, x_n) = (a_{ij})$, $a_{ij} = \frac{1}{r_{ij}^2}$ and $a_{ji} = -a_{ij}$ if $i < j$ and $a_{ii} = 0$, $L = (1, 1, \ldots, 1)^T$. λ and c are parameters. Matrix A is called the associated matrix of the configuration x.

Matrix A is skew-symmetric and its determinant can be written as a square of a polynomial in the matrix entries a_{ij}. This polynomial is called the Pfaffian of the matrix. The term Pfaffian was introduced by Cayley (1852) who named them after Johann Friedrich Pfaff. Explicitly for a skew-symmetric matrix A,

$$\det(A) = (\mathrm{Pf}(A))^2 \tag{2.2}$$

which was first proved by Thomas Muir in 1882. When n is odd, $\det(A) = 0$ since $\det(A) = \det(A^T) = (-1)^n \det(A) = -\det(A)$. When n is even, $\det(A)$ is not in general zero as remarked by Moulton (1910) and it was conjectured that $\det(A)$ is not zero by Albouy and Moeckel (2000).

Definition 2.1 By convention, the Pfaffian of the 0×0 matrix is 1. When $n = 2k$ is even, The Pfaffian of the skew-symmetric $2k \times 2k$ matrix A with $k > 0$ can be computed recursively as

$$\mathrm{Pf}(A) = \sum_{i=1}^{n-1} (-1)^{i+1} a_{in} \mathrm{Pf}(A_{\hat{i}\hat{n}}), \tag{2.3}$$

where $A_{\hat{i}\hat{n}}$ denotes the matrix A with both the n-th and i-th rows and columns removed.

Proposition 2.2 *(1) For any skew-symmetric matrix A and any vector v, $v^T Av = 0$.*
(2) If A is an invertible skew-symmetric matrix, then A^{-1} is also a skew-symmetric matrix.

Proof (1) Because A is a skew-symmetric matrix, $A^T = -A$. $(v^T Av)^T = v^T Av$ because it is a real number. In addition, $(v^T Av)^T = (v^T A^T (v^T)^T) = v^T(-A)v = -v^T Av$, so $v^T Av = 0$.

(2) Because $(AA^{-1})^T = I$ and $(A^{-1})^T A^T = I$, $(A^{-1})^T = (A^T)^{-1} = (-A)^{-1} = -A^{-1}$. So A^{-1} is also a skew-symmetric matrix.

□

If $\mathrm{Pf}(A) \neq 0$, A^{-1} exists and we can get A^{-1} by the following.

Proposition 2.3 *Let A be an $n \times n$ skew-symmetric matrix. For each $i < j$, let D_{ij} the Pfaffian of the matrix obtained from A by removing the rows and columns indexed by i, j. Let B be the skew-symmetric matrix with (i, j) entry equal to $(-1)^{i+j} D_{ij}$ whenever $i < j$. Then $AB = BA = \mathrm{Pf}(A) I_n$. In particular, if $\mathrm{Pf}(A)$ is nonzero, then $A^{-1} = \frac{1}{\mathrm{Pf}(A)} B$.*

2.1 Pfaffians in the collinear n-body problem with even $n \leq 6$

When $n = 2$, $\mathrm{Pf}(A) = a_{12} > 0$, $\det(A) = a_{12}^2$ and $A^{-1} = -A/\det(A)$.

When $n = 4, 6$, we have the following theorems.

Theorem 2.4 *Let $x = (x_1, x_2, x_3, x_4)^T$ with $x_1 < x_2 < x_3 < x_4$ and A be the associate matrix of x.*

(1) $a_{12}a_{34} - a_{13}a_{24} > 0$ and $a_{14}a_{23} - a_{13}a_{24} > 0$.
(2) $\mathrm{Pf}(A) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$ is always positive and the inverse of A is

$$A^{-1} = \frac{1}{\mathrm{Pf}(A)} \begin{bmatrix} 0 & -a_{34} & a_{24} & -a_{23} \\ a_{34} & 0 & -a_{14} & a_{13} \\ -a_{24} & a_{14} & 0 & -a_{12} \\ a_{23} & -a_{13} & a_{12} & 0 \end{bmatrix}.$$

(3) When fixed x_2, x_3, x_4, $\mathrm{Pf}(A)$ is decreasing as x_1 decreases. When fixed x_1, x_2, x_3, $\mathrm{Pf}(A)$ is decreasing as x_4 increases.

Proof The proof could be done by direct computation as in Ouyang and Xie (2005). We give a simple proof here. Let $t_i = x_{i+1} - x_i$ for $i = 1, 2, 3$.

(1) $a_{12}a_{34} - a_{13}a_{24} = (t_1 t_3)^{-2} - ((t_1 + t_2)(t_2 + t_3))^{-2} > 0$.

$$\begin{aligned} a_{14}a_{23} - a_{13}a_{24} &= ((t_1 + t_2 + t_3)(t_2))^{-2} - ((t_1 + t_2)(t_2 + t_3))^{-2} \\ &= \frac{t_1\, t_3\, \left(2\, t_1\, t_2 + t_1\, t_3 + 2\, t_2{}^2 + 2\, t_2\, t_3\right)}{(t_1 + t_2 + t_3)^2\, t_2{}^2\, (t_1 + t_2)^2\, (t_2 + t_3)^2} > 0 \end{aligned}$$

(2) This is an immediate consequence of (1) and Proposition 2.3.
(3) Since

$$\begin{aligned} \mathrm{Pf}(A) &= t_1^{-2} t_3^{-2} - (t_1 + t_2)^{-2} (t_2 + t_3)^{-2} + (t_1 + t_2 + t_3)^{-2} t_2^{-2}, \\ \frac{d}{dt_1} \mathrm{Pf}(A) &= -2(t_1^{-3} t_3^{-2} - (t_1 + t_2)^{-3} (t_2 + t_3)^{-2} + (t_1 + t_2 + t_3)^{-3} t_2^{-2}), \end{aligned}$$

and

$$\frac{d}{dt_3} \mathrm{Pf}(A) = -2(t_1^{-2} t_3^{-3} - (t_1 + t_2)^{-2} (t_2 + t_3)^{-3} + (t_1 + t_2 + t_3)^{-3} t_2^{-2}).$$

These expressions are similar to those in (1) but only with different exponents. By similar arguments, we have $\frac{d}{dt_1}\mathrm{Pf}(A) < 0$ and $\frac{d}{dt_3}\mathrm{Pf}(A) < 0$ for positive t_i. This completes the proof. □

Theorem 2.5 *Let $x = (x_1, x_2, \dots, x_6)$ with $x_1 < x_2 < \cdots < x_6$. Let A be the associate matrix of x. Then* $\mathrm{Pf}(A) > 0$.

Proof Let $D = \mathrm{diag}\left(1, \frac{1}{a_{12}}, \frac{1}{a_{13}}, \frac{1}{a_{14}}, \frac{1}{a_{15}}, \frac{1}{a_{16}}\right)$. Then $DAD = \tilde{A} = (\tilde{a}_{ij})$ is skew-symmetric such that the first row of $\tilde{A}$ equals $(0, 1, 1, \dots, 1)$, and for $1 < i < j \leq 6$,

$$\tilde{a}_{ij} = \frac{a_{ij}}{a_{1i}a_{1j}} = (x_1 - x_i)^2(x_1 - x_j)^2(x_i - x_1 + x_1 - x_j)^{-2} = (y_i - y_j)^{-2},$$

where

$$y_i = (x_1 - x_i)^{-1} \qquad \text{for } i = 2, \dots, 6.$$

Note that $y_2 < y_3 < \cdots < y_6 < 0$ since $x_1 < x_2 < \cdots < x_6$ and $\det(\tilde{A}) = \det(DAD) = \det(D^2)\det(A)$. Denote by $E(i, j) = \tilde{A}(i, j)$ the submatrix of $\tilde{A}$ obtained from $\tilde{A}$ by deleting its ith and jth rows and columns. The expansion of the Pfaffian is

$$\mathrm{Pf}(\tilde{A}) = \mathrm{Pf}(E(1,2)) - \mathrm{Pf}(E(1,3)) + \mathrm{Pf}(E(1,4)) - \mathrm{Pf}(E(1,5)) + \mathrm{Pf}(E(1,6)), \tag{2.4}$$

where $E(1,2) = E(y_3, y_4, y_5, y_6)$, $E(1,3) = E(y_2, y_4, y_5, y_6)$, $E(1,5) = E(y_2, y_3, y_4, y_6)$, $E(1,6) = E(y_2, y_3, y_4, y_5)$. By the property (3) in Theorem 2.4, we see that

$$\mathrm{Pf}(E(1,2)) - \mathrm{Pf}(E(1,3)) > 0 \quad \text{and} \quad -\mathrm{Pf}(E(1,5)) + \mathrm{Pf}(E(1,6)) > 0.$$

Together with the fact that $\mathrm{Pf}(E(1,4)) > 0$, we see that $\mathrm{Pf}(DAD) > 0$. Hence, $\det(DAD) = \mathrm{Pf}(DAD)^2 > 0$ and thus $\det(A) > 0$. □

2.2 Inverse problem for collinear 5-body central configurations

Now, let us consider the inverse problem of the collinear 5-body central configurations. Theorem 1.1 can be proved by the following theorems. To find solution for algebraic equation (2.1) with $n = 5$, we solve for m_1, m_2, m_3, m_4 from first 4 equations by transposing the terms containing m_5 and substitute into the last equation.

Theorem 2.6
1. *The center of mass c for any 5-body collinear central configuration $x = (x_1, x_2, \dots, x_5)^T$ only depends on the configuration x and it is independent of the parameter λ and the corresponding mass m. More precisely it is given by*

$$c = \frac{x_5 + v^T B^{-1} y}{1 + v^T B^{-1} L}, \tag{2.5}$$

   where B is the associated matrix of the configuration $y = (x_1, x_2, x_3, x_4)^T$, $v = (a_{15}, a_{25}, a_{35}, a_{45})^T$, and $L = (1, 1, 1, 1)^T$. The denominator $1 + v^T B^{-1} L$ of c in (2.5) is always positive for any configuration x.
2. *With the appropriate choice of the center of mass c given by (2.5), m can be given by a function with two parameters λ and m_5*

$$m = \begin{bmatrix} -\lambda B^{-1}(y - cL) - m_5 B^{-1} v \\ m_5 \end{bmatrix}. \tag{2.6}$$

3. *With the appropriate choice of the center of mass c given by (2.5), m can be given by a function with two parameters λ and M*

$$m = \begin{bmatrix} -\lambda B^{-1}\left(y - cL + \frac{(L^T B^{-1} y)v}{1+v^T B^{-1} L}\right) - \frac{M B^{-1} v}{1+v^T B^{-1} L} \\ \frac{\lambda L^T B^{-1} y + M}{1+v^T B^{-1} L} \end{bmatrix}, \tag{2.7}$$

where M is the total mass.

Proof 1. The first four equations in (2.1) are equivalent to

$$B(m_1, m_2, m_3, m_4)^T = -\lambda(y - cL) - m_5 v.$$

Because B is invertible, there is a unique solution

$$(m_1, m_2, m_3, m_4)^T = -\lambda B^{-1}(y - cL) - m_5 B^{-1} v. \tag{2.8}$$

Substituting the solution into the last (5th) equation in (2.1), we have

$$-\lambda v^T B^{-1}(y - cL) - m_5 v^T B^{-1} v = \lambda(x_5 - c).$$

Since B^{-1} is skew-symmetric, $v^T B^{-1} v$ is zero. Solving for c, we get Eq. (2.5). Now we prove the denominator of the center of mass c is positive.Let B_i be the associate matrix of the configuration that is obtained by deleting the x_i from the configuration $(x_1, x_2, \ldots, x_5)$. In particular, $\mathrm{Pf}(B_5) = \mathrm{Pf}(B)$.

$$\begin{aligned} 1 + v^T B^{-1} L &= \frac{1}{\mathrm{Pf}(B)} \left(\mathrm{Pf}(B) + (a_{15}, a_{25}, a_{35}, a_{45}) \begin{bmatrix} 0 & -a_{34} & a_{24} & -a_{23} \\ a_{34} & 0 & -a_{14} & a_{13} \\ -a_{24} & a_{14} & 0 & -a_{12} \\ a_{23} & -a_{13} & a_{12} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right) \\ &= \frac{1}{\mathrm{Pf}(B)} \left(\mathrm{Pf}(B) + (\mathrm{Pf}(B1), -\mathrm{Pf}(B_2), \mathrm{Pf}(B_3), -\mathrm{Pf}(B_4))(1,1,1,1)^L \right) \\ &= \frac{1}{\mathrm{Pf}(B)} (\mathrm{Pf}(B_1) - \mathrm{Pf}(B_2) + \mathrm{Pf}(B_3) - \mathrm{Pf}(B_4) + \mathrm{Pf}(B_5)). \end{aligned}$$

By Theorem 2.4 (3), $\mathrm{Pf}(B_1) > \mathrm{Pf}(B_2) > 0$ and $\mathrm{Pf}(B_5) > \mathrm{Pf}(B_4) > 0$. So the denominator is positive. The center of mass c is uniquely determined by the configuration x.

2. With the choice of the center of mass as above, m_5 can be taken as a free variable and m is given by (2.6).
3. Let $M = m_1 + m_2 + m_3 + m_4 + m_5$ be the total mass. By using Eq. (2.8) and the fact $L^T B^{-1} L = 0$, we obtain

$$\begin{aligned} M &= L^T (m_1, m_2, m_3, m_4)^L + m_5 = -\lambda L^T B^{-1} y - m_5 L^T B^{-1} v + m_5. \\ m_5 &= \frac{M + \lambda L^T B^{-1} y}{1 - L^T B^{-1} v} = \frac{M + \lambda L^T B^{-1} y}{1 + v^T B^{-1} L}. \end{aligned}$$

Substituting m_5 into Eq. (2.8), we get (2.7).

□

Remark 2.7 If m is given by (2.7), then it is easy to check that $m_1 x_1 + m_2 x_2 + \cdots + m_5 x_5 = Mc$ and $m_1 + m_2 + \cdots + m_5 = M$.

Theorem 2.8 *Let $x = (x_1, x_2, \ldots, x_5)$ with $x_1 < x_2 < \cdots < x_5$ be a central configuration for positive masses $m = (m_1, m_2, \ldots, m_5)$. $m_1 = m_5$ and $m_2 = m_4$ if and only if $|x_2 - x_1| = |x_5 - x_4|$ and $|x_3 - x_2| = |x_4 - x_3|$. In this symmetric case, the center of mass is at $c = x_3$.*

Proof We first prove the necessary condition, i.e., assuming that the configuration is symmetric. Without loss of generality, the configuration with symmetry can be chosen as $x_1 = -s - 1, x_2 = -1, x_3 = 0, x_4 = 1, x_5 = s + 1$ with $s > 0$ after translation and rescaling due to the invariance of central configurations. It is easy to check that the center of mass c is at $x_3 = 0$. The solution (2.7) is

$$m_i = \frac{\lambda f_i(s) + M g_i(s)}{h(s)}, \quad i = 1, 2, \ldots, 5, \tag{2.9}$$

where

$$\begin{aligned}
h(s) &= 256 + 1024\,s + 1664\,s^2 + 2432\,s^3 + 3216\,s^4 + 2720\,s^5 + 1368\,s^6 + 392\,s^7 + 49\,s^8,\\
f_1(s) &= -4\,s^2\,(2+s)^2\left(7\,s^7 + 49\,s^6 + 133\,s^5 + 175\,s^4 + 112\,s^3 + 52\,s^2 + 48\,s + 16\right),\\
g_1(s) &= 4s^2\,(2+s)^2\left(7\,s^4 + 28\,s^3 + 52\,s^2 + 48\,s + 16\right),\\
f_2(s) &= 4s^2\,(2+s)^2\left(8\,s^7 + 56s^6 + 152\,s^5 + 209\,s^4 + 164\,s^3 + 100\,s^2 + 64s + 16\right)\\
g_2(s) &= -4s^2\,(2+s)^2\left(s^4 + 4\,s^3 + 4\,s^2 + 16\,s + 16\right),\\
f_3(s) &= -8s^3\,(2+s)^3\left(s^5 + 5\,s^4 + 9\,s^3 + 16\,s^2 + 20\,s + 8\right),\\
g_3(s) &= \left(s^8 + 8\,s^7 + 24\,s^6 + 160\,s^5 + 656\,s^4 + 1408\,s^3 + 1664\,s^2 + 1024\,s + 256\right),
\end{aligned}$$

and $f_5(s) = f_1(s), f_4(s) = f_2(s), g_5(s) = g_1(s), g_4(s) = g_2(s)$. Therefore, $m_5 = m_1, m_4 = m_2$. Now we prove the sufficient condition, i.e., assuming that $m_5 = m_1$ and $m_4 = m_2$. By the results of Moulton (1910), there is a unique central configuration for a given order of masses up to translation, scaling, rotation.We only need to prove that there is a solution λ, M, s of the Eq. (2.9) for any given positive masses m_1, m_2, m_3.

Let $m_i = \lambda^* f_i + M^* g_i, i = 1, 2, 3$. Solving for λ^* and M^* from first two equations, we obtain $\lambda^* = -\frac{g_1 m_2 - g_2 m_1}{-g_1 f_2 + g_2 f_1}$ and $M^* = \frac{-m_1 f_2 + f_1 m_2}{-g_1 f_2 + g_2 f_1}$. Note that the denominator

$$\begin{aligned}
-g_1 f_2 + g_2 f_1 = -16\,s^5\,(2+s)^4\left(s^2 + 3\,s + 3\right)\Big(&49\,s^8 + 392\,s^7 + 1368\,s^6 + 2720\,s^5\\
&+ 3216\,s^4 + 2432\,s^3 + 1664\,s^2 + 1024\,s + 256\Big)
\end{aligned}$$

is nonzero for any positive s. So λ^* and M^* are well defined. Substituting into the third equation, the resulting equation is equivalent to

$$\begin{aligned}
&-(16\,m_1 + 16\,m_2) - (64\,m_1 + 48\,m_2)\,s - (52\,m_2 + 100\,m_1)\,s^2\\
&\quad + (-68\,m_1 + 48\,m_3 - 16\,m_2)\,s^3 + (-17\,m_1 + 96\,m_3 + 17\,m_2)\,s^4\\
&\quad + (19\,m_2 + 76\,m_3)\,s^5 + (7\,m_2 + 28\,m_3)\,s^6 + (4\,m_3 + m_2)\,s^7 = 0
\end{aligned} \tag{2.10}$$

For any given positive masses m_1, m_2, m_3, it is impossible that $(-68\,m_1 + 48\,m_3 - 16\,m_2) > 0$ and $(-17\,m_1 + 96\,m_3 + 17\,m_2) < 0$ simultaneously. Then the number of sign change of the coefficients is exact once in the polynomial of s in the equation (2.10). By Descartes' rule, there exists a unique positive root s_0. Now let $\lambda = \lambda^* h(s_0),\ M = M^* h(s_0)$ and $s = s_0$. Then λ, M, s satisfy the Eq. (2.9) for the given positive masses m_1, m_2, m_3.

Remark 2.9 If the masses are allowed to be negative, the existence of positive s is not guaranteed. For example, if $m_1 = 1, m_2 = -2, m_3 = 2$, then Eq. (2.10) becomes

$$16 + 32\,s + 4\,s^2 + 60\,s^3 + 141\,s^4 + 114\,s^5 + 42\,s^6 + 6\,s^7 = 0$$

which has no positive root by the Descartes' rule.

2.3 Inverse problem for the collinear 6-body central configurations

Theorem 1.3 can be restated as follows.

Theorem 2.10 *For any fixed noncollision configuration x in the collinear 6-body problem, λ and c can be taken as the two parameters for m,*

$$m = \lambda(-A^{-1}x + cA^{-1}L) \tag{2.11}$$

to make x central. Total mass M is independent of the center of mass c and M is linearly dependent on λ, i.e.

$$M = -\lambda L^T A^{-1} x. \tag{2.12}$$

Proof For any fixed noncollision configuration x in the collinear 6-body problem, $\det(A) = (\mathrm{Pf}(A))^2 > 0$ by Theorem 2.5. To make x be a central configuration for mass m,

$$m = -\lambda A^{-1}x + \lambda c A^{-1} L \tag{2.13}$$

by Eq. (2.1). Since A^{-1} is skew symmetric, $L^T A^{-1} L = 0$. So

$$M = m_1 + m_2 + \cdots + m_6 = L^T m = -\lambda L^T A^{-1} x + \lambda c L^T A^{-1} L = -\lambda L^T A^{-1} x.$$

$\square$

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