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The uniqueness of blow-up solution for radially symmetric semilinear elliptic equation

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Abstract

In this paper we establish a blow up rate of the large positive solutions of the singular boundary value problem $-\Delta u = \lambda u - b(x)u^p$, $u|_{\partial\Omega} = +\infty$ with a ball domain and radially function $b(x)$. All previous results in the literature assumed the decay rate of $b(x)$ to be approximated by a distance function near the boundary $\partial\Omega$. Obtaining the accurate blow up rate of solutions for general $b(x)$ requires more subtle mathematical analysis of the problem.

Keywords: Semilinear elliptic equation; Blow up rate; Large positive solution

1. Introduction and main results

This paper originated with the sequentially papers [9,5,4,6] which contain an exhaustive study of positive solution u to the singular boundary value problem:

$$\begin{cases} -\Delta u = \lambda(x)u - b(x)u^p & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

here Ω is a bounded domain in $\mathbb{R}^N$, $N \geqslant 1$, with boundary $\partial\Omega$ of class C^2, $\lambda \in L_\infty(\Omega)$, $p > 1$ and $b \in C(\Omega; \mathbb{R}^+)$, $\mathbb{R}^+ := (0, +\infty)$. The boundary condition in (1.1) is understood as $u(x) \to +\infty$ when $d(x) := dist(x, \partial\Omega) \to 0^+$. The behavior of the potential function

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of the nonlinear term $b(x)$ approaching to $\partial\Omega$ is closely related to the blow up rate of the solution of (1.1). In the case of $b(x) \geqslant b_0 > 0$ in $\bar{\Omega}$, many different type of equations are studied in [7,1,2]. In the case of $b(x) \sim C_0 d^\gamma + o(d^\gamma)$ as x goes to $\partial\Omega$, blow up rates and uniqueness are studied in [4,6].

Singular boundary value problems such as (1.1) are inherited from the logistical model:

$$\begin{cases} -\Delta u = \lambda(x)u - b(x)u^p & \text{in } \widetilde{\Omega}, \\ u = 0 & \text{on } \partial\widetilde{\Omega}, \end{cases} \tag{1.2}$$

where $\widetilde{\Omega}$ ($\Omega \subset \widetilde{\Omega} \subset \mathbb{R}^N$) is a bounded domain in $\mathbb{R}^N$ and where $b \in C(\overline{\widetilde{\Omega}})$ satisfies $b > 0$ in the proper subdomain $\Omega(\bar{\Omega} \subset \widetilde{\Omega})$ while $b = 0$ in $\widetilde{\Omega} \backslash \bar{\Omega}$.

If $b(x) > 0$ for $x \in \overline{\widetilde{\Omega}}$ with $\lambda(x) = \lambda$ constant in (1.2), then the equation is known as the logistic model and it is well known that (1.2) has a unique positive solution if and only if $\lambda > \lambda_1(\widetilde{\Omega})$, where $\lambda_1(\widetilde{\Omega})$ denotes the first eigenvalue of

$$\begin{cases} -\Delta u = \lambda u & \text{in } \widetilde{\Omega}, \\ u = 0 & \text{on } \partial\widetilde{\Omega}. \end{cases} \tag{1.3}$$

If $b(x)$ in (1.2) is not bounded away from zero in $\widetilde{\Omega}$ and we may assume $b(x) > 0$ in Ω while $b(x) = 0$ on $\widetilde{\Omega} \backslash \bar{\Omega}$, this type of problem (1.2) goes back to the pioneering work [9,4] where they found that (1.2) has a positive solution if and only if $\lambda \in (\lambda_1(\widetilde{\Omega}), \lambda_1(\widetilde{\Omega} \backslash \bar{\Omega}))$. In this case (1.2) has a unique solution u_λ and $\lambda \longmapsto u_\lambda$ is continuous as a map from $(\lambda_1(\widetilde{\Omega}), \lambda_1(\widetilde{\Omega} \backslash \bar{\Omega}))$ to $C^{2+\mu}(\widetilde{\Omega})$. Furthermore, $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to (\lambda_1(\widetilde{\Omega} \backslash \bar{\Omega}))$. The existence of large positive solution of (1.1) is immediately followed from the results in [9,4].

Singular boundary value problem (1.1) arises naturally from a number of different areas and have a long history. Considerable amounts of study have been attracted by such problems. We mention only [9,4,5]. Many other works can be found from the references of these papers. However the blow-up rate of solution near $\partial\Omega$ and uniqueness of solutions for the singular problem (1.1) are the goal of more recent literature. In 1990's, a different type of singular boundary value problem $-\Delta u = b(x)\mathrm{e}^u$ with $u|_{\partial\Omega} = +\infty$ is studied by Lazer and Mckenna [7] and Bandle and Essèn [1]. It is shown that the problem exhibits a unique solution in a smooth domain together with an estimate of the form $u = \log d^{-2} + o(d)$ in [7] (where $b(x) \geqslant b_0 > 0$ as $d \to 0$) and in [1] (where $b \equiv 1$). Actually, the more general problem $-\Delta u(x) = f(u(x))$ with $u|_{\partial\Omega} = +\infty$, is considered in [8]. In their paper [8], they obtained an asymptotic result for solutions of the above problem under some assumptions on f. Let Ω be a bounded domain in $\mathbb{R}^N$, $N > 1$, which satisfies a uniform internal sphere condition and a uniform external sphere condition. Let f be a C^1 function which is either defined and positive on $(-\infty, \infty)$ or is defined on a ray $[a, \infty)$ with $f(a) = 0$ and $f(s) > 0$ for $s > a$. They further assume that $f'(s) \geqslant 0$ for s in the domain of f, and that there exists a_1 such that $f'(s)$ is non-decreasing for $s \geqslant a_1$. They proved that if

$$\lim_{s \to \infty} \frac{f'(s)}{\sqrt{F(s)}} = \infty,$$

where $F'(s) = f(s)$, $F(s) > 0$, then the problem has a unique solution $u(x)$ and moreover,

$$u(x) - Z(d(x)) \to 0 \quad \text{as } d(x) \to 0,$$

where Z is a solution on an interval $(0, b)$, $b > 0$, of the equation $Z''(r) = f(Z(r))$ and such that $Z(r) \to \infty$ as $r \to 0^+$.

More recently, the uniqueness and blow up rates of $-\Delta u = \lambda(x)u - b(x)u^p$ are treated in [2,4,6] etc. In [2] the case of $-\Delta u = -u^p$, it is shown that the blow up rate of solution near boundary is $u = Ad^{-\alpha}(1 + o(d))$. Furthermore, the possible presence of a second explosive term in the expansion of u can be obtained when $\alpha > 1 (p < 3)$. Under the assumption $b(x) = C_0 d^\gamma + o(d^\gamma)$ as $d \to 0^+$ with $\gamma > 0$ and $C_0 > 0$, an explicit expression for the blow-up rates of $-\Delta u = \lambda(x)u - b(x)u^p$ has been recently proved in [4,6] as $u = (\alpha(\alpha + 1)/C_0)^{1/(p-1)} d^{-\alpha}(1 + o(d))$, $\alpha = (\gamma + 2)/(p - 1)$. García Melián [6] also gives an explicit expression for this second term as $u(x) = Ad^{-\alpha}(1 + B(s)d + o(d))$ when $d \to 0^+$ where $B(s) = ((n-1)H(s) - (\alpha + 1)C_1)/(\gamma + p + 3)$ with $H(s)$ standing for the mean curvature of $\partial\Omega$ at s. Notice that all the blow up rates are obtained by assuming $b(x) \sim C_0 d^\gamma$ near boundary. The blow up rates are determined by the asymptotic properties of the potential function of the nonlinear term $b(x)$ as x approaching boundary.

In this paper, we assume the potential function $b(x) \in C(\Omega)$, where Ω is a ball $B_R(x_0)$ satisfying $b(x) = b(\|x - x_0\|)$ and $b(x) > 0 \in \Omega$. Then $b(r)$ is a real function defined on $[0, R]$. We also assume $B(r)/b(r) \in C^1([0, R])$ and $\lim_{r \to R} B(r)/b(r) = 0$, where $B(r) = \int_r^R b(s)\,\mathrm{d}s$. Then a very accurate blow up rate of solution is established without any further assumption on the decay rate of $b(x)$ near the boundary $\partial\Omega$. Furthermore, the uniqueness of singular boundary value problem (1.2) is proved (Theorem 1 in this paper). The idea and the formula in Theorem 1 will be used for more general domain $\Omega \subset \mathbb{R}^n$ and for more general potential function $b(x)$ in our next paper.

Theorem 1. *Consider the radially symmetric semilinear elliptic equation*:

$$\begin{cases} -\Delta u = \lambda u - b(\|x - x_0\|)u^p & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega = B_R(x_0)$ *is the ball of radius* R *centered at* x_0. $\lambda \in \mathbb{R}$, $b \in C([0, R]; [0, \infty))$ *satisfying* $b > 0$ *in* $[0, R)$, $B(r)/b(r) \in C^1([0, R])$, $\lim_{r \to R} B(r)/b(r) = 0$ *where* $B(r) = \int_r^R b(s)\,\mathrm{d}s$. *Then the problem exists a unique solution* u *satisfying*

$$\lim_{d(x) \to 0} \frac{u(x)}{K(b^*(\|x - x_0\|))^{-\beta}} = 1,$$

where $d(x) = dist(x, \partial B_R(x_0))$ *and*

$$b^*(r) = \int_r^R \int_s^R b(t)\,\mathrm{d}t\,\mathrm{d}s.$$

K is a constant defined by

$$K = [\beta((\beta + 1)C_0 - 1)]^{1/(p-1)}, \quad \beta := \frac{1}{p - 1}$$

and C_0 is defined in Lemma 2.6. *With a little more effort, our arguments can be carried out when constant λ is replaced by function $\lambda(x) \in L^\infty$.*

Remark. (1) The assumption $B(r)/b(r) \in C^1([0, R])$, $\lim_{r\to R} B(r)/b(r) = 0$ is satisfied by all analytical functions. In fact, we prove that Theorem 1 holds under a weaker condition on the potential function $b(x)$ in Theorem 1. More details are given in Lemma 2.6 and its remarks.

(2) If $b(r) = b_0 > 0$ is a constant, then $b^*(r) = \frac{1}{2}b_0(R - r)^2$, $C_0 = 2$, $K = (\beta(2\beta + 1))^{1/(p-1)} = ((p + 1)/(p - 1)^2)^{1/(p-1)}$, $\beta = 1/(p - 1)$. Therefore, $u \sim K(b^*)^{-\beta} = ((p + 1)/(p - 1)^2)^{1/(p-1)}(\frac{1}{2}b_0(R - r)^2)^{-\beta}$ as x goes to boundary which is same as [1].

(3) If $b(r) = b_0(R - r)^\gamma$, $(\gamma \geqslant 0)$ is a distance function as r goes to R, then $b^*(r) = (1/(\gamma + 1)(\gamma + 2))b_0(R - r)^{\gamma+2}$, $C_0 = (\gamma + 2)/(\gamma + 1)$, $K = [\beta((\beta + 1)(\gamma + 2)/(\gamma + 1) - 1)]^{1/(p-1)}$. Therefore,

$$
\begin{aligned}
u \sim K(b^*)^{-\beta} &= \left[\beta\left((\beta+1)\frac{\gamma+2}{\gamma+1} - 1\right)\right]^{1/(p-1)} \\
&\quad \times \left(\frac{1}{(\gamma+1)(\gamma+2)} b_0(R - r)^{\gamma+2}\right)^{-1/(p-1)} \\
&= \left(\frac{\alpha(\alpha+1)}{b_0}\right)^{1/(p-1)} (R - r)^{-\alpha},
\end{aligned}
$$

$$
\alpha = \frac{\gamma+2}{p-1}
$$

as x goes to boundary which is same as [4,6].

(4) We give a special example which illustrate that our theorem can be applied more complicate potential functions than [2,4,6] etc.'s theorems can. Let $b(r) = \exp(-1/(R-r)^2)$, $B(r) = \int_r^R b(s)\,\mathrm{d}s$, $b^*(r) = \int_r^R \int_t^R b(t)\,\mathrm{d}t\,\mathrm{d}s$, then $\lim_{r\to R} B(r)/b(r) = \lim_{r\to R} 2(R-r)^3 = 0$ and $(B(r)/b(r))' = -1 + 2B(r)/b(r)(R - r)^3$ is continuous and approaches zero as r goes to R. $C_0 = 1$, $K = [\beta((\beta+1) - 1)]^{1/(p-1)}$, $\beta := 1/(p - 1)$. Therefore, there is a unique solution $u \sim K(b^*)^{-\beta}$ which goes to ∞ near the boundary faster than any power function.

2. Some preliminary results

In this section we collect some important comparison results that we are going to use in the proof of Theorem 1. As an immediate consequence from the papers [3,4,6], Theorems 2.1–2.4 were proved. Lemmas 2.5, 2.6 are new and are proved in this section.

Consider the problem

$$
\begin{cases}
-\Delta u = \lambda(x)u - b(x)u^p & \text{in } \Omega, \\
u = \phi & \text{on } \partial\Omega,
\end{cases}
\tag{2.1}
$$

where Ω is a bounded domain with smooth boundary, $\phi \in C(\partial\Omega)$, $p > 1$ and $b \in C(\Omega, \mathbb{R}^+)$.

Theorem 2.1 (*Du and Huang [4, Lemma 2.1]*). *Let $\underline{u}, \overline{u} \in C^2(\bar{\Omega})$ be both positive in $\bar{\Omega}$ such that*

$$-\Delta \underline{u} \leqslant \lambda(x)\underline{u} - b(x)\underline{u}^p \quad \text{in } \Omega,$$

$$-\Delta \overline{u} \geqslant \lambda(x)\overline{u} - b(x)\overline{u}^p \quad \text{in } \Omega.$$

If $\underline{u} \leqslant \phi \leqslant \overline{u}$ on $\partial\Omega$, then $\underline{u}(x) \leqslant \overline{u}(x)$ on $\bar{\Omega}$.

Definition. If $\underline{u}$ (resp. $\overline{u}$) satisfy the conditions in Theorem 2.1 and $\underline{u} \leqslant \phi$ on $\partial\Omega$ (resp. $\overline{u} \geqslant \phi$), then $\underline{u}$ (resp. $\overline{u}$) is called subsolution (resp. supsolution) of (2.1).

Theorem 2.2. *Suppose $\phi \in C(\partial\Omega)$ and (2.1) possesses a non-negative solution. Let u be any non-negative solution of (2.1). Then $u(x) > 0$ for each $x \in \Omega$ and $\partial_\nu u(x) < 0$ for any $x \in \partial\Omega$ such that $u(x) = 0$; ν stands for the outward unit normal to Ω. Moreover, the positive solution is unique and if we denote it by Ψ and $\underline{u}$ (resp. $\overline{u}$) is a non-negative subsolution (resp. supersolution) of (2.1) then $\underline{u} \leqslant \Psi$ (resp. $\Psi \leqslant \overline{u}$)*

Theorem 2.3. *If $\underline{u}, \overline{u} \in C^2(\Omega)$ are both positive in Ω such that*

$$-\Delta \underline{u} \leqslant \lambda(x)\underline{u} - b(x)\underline{u}^p \quad \text{in } \Omega,$$

$$-\Delta \overline{u} \geqslant \lambda(x)\overline{u} - b(x)\overline{u}^p \quad \text{in } \Omega,$$

$$\lim_{dist(x,\partial\Omega)\to 0} \underline{u}(x) = \infty, \qquad \lim_{dist(x,\partial\Omega)\to 0} \overline{u}(x) = \infty,$$

and $\underline{u}(x) \leqslant \overline{u}(x)$ in Ω, then there exists at least one solution $u \in C^2(\Omega)$ of (1.1) satisfying $\underline{u}(x) \leqslant u \leqslant \overline{u}(x)$ in Ω.

Theorem 2.4. *Suppose (1.1) possesses a non-negative solution, say Ψ, then the problem (2.1) possesses a unique non-negative solution for each $\phi \in C(\partial\Omega, R^+)$ denoted by u_ϕ and $u_\phi \leqslant \Psi$ in Ω. Furthermore,*

$$\Psi_L := \limsup_{\inf_{\partial\Omega} \phi \to \infty} u_\phi$$

provides us with the minimal positive solution of (1.1).

Now we are going to find the asymptotic properties of the potential function of the nonlinear term $b(x)$ as x approaching boundary. For convenience of state and proof, we first prove the following lemma.

Lemma 2.5. *Let $f(r) : [0, R] \mapsto [0, \infty)$ be continuous function such that $f(r) > 0$ for $r \in (0, R]$. Define*

$$F(r) = \int_0^r f(s)\,\mathrm{d}s, \quad f^*(r) := \int_0^r F(s)\,\mathrm{d}s.$$

Then the following two statements are equivalent.

S$_1$: *There exists a differentiable function*

$$g \in C^1([0, R])$$

such that $g(0) = 0$, $g'(0) \geqslant 0$ *and* $\lim_{r\to 0}(F(r)/g(r)f(r)) = a_0 > 0$.

S$_2$:

$$(1)\quad \frac{F(r)}{f(r)} \to 0 \quad \textit{as } r \to 0,$$

$$(2)\quad \frac{f^*(r)}{F(r)} \to 0 \quad \textit{as } r \to 0,$$

$$(3)\quad \lim_{r\to 0}\frac{(F(r))^2}{f^*(r)f(r)} = C_0 = 1 + a_0 g'(0) \geqslant 1.$$

Proof. Assuming S$_1$, we are going to show S$_2$.

(1) $F(r)/f(r) = (F(r)/g(r)f(r))g(r) \to 0$.

(2) Since $f(r)$ is positive in $(0, R]$, $F(r)$ is positive and non-decreasing in $(0, R]$. Then

$$0 \leqslant f^*(r) = \int_0^r F(s)\,\mathrm{d}s \leqslant rF(r),$$

$$0 \leqslant \frac{f^*(r)}{F(r)} \leqslant r.$$

Therefore $\lim_{r\to 0} f^*(r)/F(r) = 0$. Or we directly get by L'Hospital rule

$$\lim_{r\to 0}\frac{f^*(r)}{F(r)} = \lim_{r\to R}\frac{F(r)}{f(r)} = 0.$$

(3)

$$\begin{aligned}\lim_{r\to 0}\frac{(F(r))^2}{f^*(r)f(r)} &= \lim_{r\to 0}\frac{g(r)F(r)}{f^*(r)}\frac{F(r)}{g(r)f(r)}\\ &= \lim_{r\to 0}\frac{g(r)F(r)}{f^*(r)}\lim_{r\to 0}\frac{F(r)}{g(r)f(r)}\\ &= \lim_{r\to 0}\frac{g(r)f(r)+g'(r)F(r)}{F(r)}a_0 = \left(\frac{1}{a_0}+g'(0)\right)a_0\\ &= 1 + a_0 g'(0) = C_0 \geqslant 1.\end{aligned}$$

Now, assuming S$_2$, we turn to show S$_1$.

By (2), we can define

$$g(r) = \begin{cases}\dfrac{f^*(r)}{F(r)}, & 0 < r \leqslant R,\\ 0, & r = 0.\end{cases}$$

Then g is differentiable since $F(r)$ and $f^*(r)$ are differentiable. Then $g'(0)=\lim_{r\to 0} g'(r)=\lim_{r\to 0} F^2(r) - f^*(r)f(r)/F^2(r) = 1 - 1/C_0 \geqslant 0$ and

$$\lim_{r\to 0} \frac{F(r)}{g(r)f(r)} = C_0 \geqslant 1 > 0. \qquad \square$$

Remark. The results in Lemma 2.5 hold for all polynomial functions and analytical functions. Furthermore, if $f(r) = ar^\alpha(1 + o(r))$ near zero, where $\alpha \geqslant 0$, a is a constant, then we may choose $g(r) = r$. So (1)–(3) hold and $C_0 = (\alpha + 2)/(\alpha + 1)$ which depends only on the least power α. From the proof in Lemma 2.5, (2) holds for any non-negative continuous functions. Generally, if $f(r)$ is increasing in a neighborhood of 0 which implies $0 \leqslant F(r) \leqslant rf(r)$, then $\lim_{r\to 0} F(r)/f(r) = 0$ and $0 \leqslant F(r)/rf(r) \leqslant 1$ in the neighborhood of 0.

Moreover, if $F(r)/f(r) \in C^1([0, R])$ and $\lim_{r\to 0} F(r)/f(r) = 0$ then we can choose $g(r) = F(r)/f(r)$. We have $g(0) = 0$, $g'(0) \geqslant 0$ and $\lim_{r\to 0} F(r)/g(r)f(r) = 1 > 0$, so S_2 hold. But the question is that, under which conditions on $f(r)$, $g(r) = F(r)/f(r)$ satisfies the above requirements. We know that if $f(r)$ is increasing in a neighborhood of 0 then (1) and (2) in Lemma 1 hold. Is it also sufficient to (3)? We leave it as an open problem.

For example, if $f(r) = \exp(-1/r^2)$ then

$$\lim_{r\to 0} \frac{\int_0^r \exp(-1/s^2)\,\mathrm{d}s}{\exp(-1/r^2)} = \lim_{r\to 0} \frac{\exp(-1/r^2)}{2r^{-3}\exp(-1/r^2)} = 0.$$

Let $g(r) = \int_0^r \exp(-1/s^2)\,\mathrm{d}s/\exp(-1/r^2)$. It is easy to check that $g(r)$ satisfies S_1 in Lemma 2.5. So S_2 hold for $f(r) = \exp(-1/r^2)$.

As an immediate consequence of Lemma 2.5, we have:

Lemma 2.6. *Let $b(r) : [0, R] \mapsto [0, \infty)$ be continuous function such that $b(r) > 0$ for $r \in [0, R)$. Define*

$$B(r) = \int_r^R b(s)\,\mathrm{d}s, \quad b^*(r) := \int_r^R B(s)\,\mathrm{d}s.$$

If $g(r) = B(r)/b(r)$ is differentiable in $[0, R]$ and $\lim_{r\to R} g(r) = 0$, $\lim_{r\to R} g'(r) \leqslant 0$, then we have

(1) $\dfrac{B(r)}{b(r)} \to 0$ *as* $r \to R$,

(2) $\dfrac{b^*(r)}{B(r)} \to 0$ *as* $r \to R$,

(3) $\lim_{r\to R} \dfrac{(B(r))^2}{b^*(r)b(r)} = C_0 \geqslant 1.$

In the proof of Theorem 1, we only need the limits of (1)–(3) in Lemma 2.6 and the continuity of potential function $b(r)$.

3. Proof of Theorem 1

To prove Theorem 1, firstly consider the corresponding singular problem in one dimension

$$\begin{cases} -\psi'' - \dfrac{N-1}{r}\psi' = \lambda\psi - b(r)\psi^p \quad \text{in } (0, R), \\ \lim_{r\to R}\psi(r) = \infty, \\ \psi'(0) = 0. \end{cases} \tag{3.1}$$

We claim that for each $\varepsilon > 0$, problem (3.1) possesses a positive large solution ψ_ε such that

$$1-\varepsilon \leqslant \liminf_{r\to R} \frac{\psi_\varepsilon(r)}{K(b^*(r))^{-\beta}} \leqslant \limsup_{r\to R} \frac{\psi_\varepsilon(r)}{K(b^*(r))^{-\beta}} \leqslant 1+\varepsilon, \tag{3.2}$$

where we have denoted

$$\beta := \frac{1}{p-1}, \quad b^*(r) := \int_r^R \int_s^R b(t)\,\mathrm{d}t\,\mathrm{d}s,$$
$$K = [\beta((\beta+1)C_0 - 1)]^{1/(p-1)}, \tag{3.3}$$

and C_0 is in Lemma 2.6.

Therefore, for each $x_0 \in \mathbb{R}^N$, the function

$$u_\varepsilon(x) := \psi_\varepsilon(r); \quad r := \|x - x_0\|$$

provides us a radially symmetric positive large solution of (1.4) and the solution satisfies

$$1-\varepsilon \leqslant \liminf_{d(x)\to 0} \frac{u_\varepsilon(x)}{K(b^*(\|x-x_0\|))^{-\beta}} \leqslant \limsup_{d(x)\to 0} \frac{u_\varepsilon(x)}{K(b^*(\|x-x_0\|))^{-\beta}} \leqslant 1+\varepsilon. \tag{3.4}$$

To prove the claim, first of all, we construct a supersolution of (3.1) for each $\varepsilon > 0$. Let

$$\bar{\psi}_\varepsilon(r) = A + B_+\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta},$$

where $\beta = 1/(p-1)$ and $A > 0$, $B_+ > 0$ have to be determined later. Then

$$\bar{\psi}'_\varepsilon(r) = 2B_+\frac{r}{R^2}(b^*(r))^{-\beta} - \beta B_+\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta-1}(b^*(r))'.$$

$$\begin{aligned}\bar{\psi}''_\varepsilon(r) &= 2B_+\frac{1}{R^2}(b^*(r))^{-\beta} - 4\beta B_+\frac{r}{R^2}(b^*(r))^{-\beta-1}(b^*(r))' \\ &\quad + \beta(\beta+1)B_+\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta-2}[(b^*(r))']^2 \\ &\quad - \beta B_+\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta-1}(b^*(r))''.\end{aligned}$$

$\bar{\psi}_\varepsilon(r) \to \infty$ as $r \to R$ because $b^*(r) \to 0$ as $r \to R$. Also $\bar{\psi}'_\varepsilon(r) \to 0$ as $r \to 0$. Then $\bar{\psi}_\varepsilon(r)$ is a supersolution if, and only if,

$$-\bar{\psi}''_\varepsilon(r) - \frac{N-1}{r}\bar{\psi}'_\varepsilon(r) \geqslant \lambda\bar{\psi}_\varepsilon(r) - b(r)\bar{\psi}^p_\varepsilon(r).$$

That is,

$$
\begin{aligned}
&-2N\frac{B_+}{R^2}(b^*(r))^{-\beta}+[N+3]\beta B_+\frac{r}{R^2}(b^*(r))^{-\beta-1}(b^*(r))'\\
&\quad-\beta(\beta+1)B_+\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta-2}[(b^*(r))']^2+\beta B_+\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta-1}(b^*(r))''\\
&\quad\geqslant\lambda(b^*(r))^{-\beta}\left[A(b^*(r))^{\beta}+B_+\left(\frac{r}{R}\right)^2\right]\\
&\qquad-b(r)(b^*(r))^{-p\beta}\left[A(b^*(r))^{\beta}+B_+\left(\frac{r}{R}\right)^2\right]^p.
\end{aligned}
$$

Multiplying both sides of this inequality by $(b^*(r))^{p\beta}/b(r)$ and taking into consideration that $p\beta=\beta+1$.

$$
\left\{
\begin{aligned}
&-2N\frac{B_+}{R^2}\frac{b^*(r)}{b(r)}+[N+3]\beta B_+\frac{r}{R^2}\frac{(b^*(r))'}{b(r)}\\
&\quad-\beta(\beta+1)B_+\left(\frac{r}{R}\right)^2\frac{[(b^*(r))']^2}{b^*(r)b(r)}+\beta B_+\left(\frac{r}{R}\right)^2\frac{(b^*(r))''}{b(r)}\\
&\qquad\geqslant\lambda\frac{b^*(r)}{b(r)}\left[A(b^*(r))^{\beta}+B_+\left(\frac{r}{R}\right)^2\right]-\left[A(b^*(r))^{\beta}+B_+\left(\frac{r}{R}\right)^2\right]^p.
\end{aligned}
\right.
\tag{3.5}
$$

Since when $r\to R$, $b^*(r)/b(r)\to 0, (b^*(r))'/b(r)\to 0$, $[(b^*(r))']^2/b^*(r)b(r)\to C_0\geqslant 1$ and $(b^*(r))''/b(r)\to 1$ by Lemma 2.6, then the inequality (3.5) becomes into

$$-\beta(\beta+1)B_+C_0+\beta B_+\geqslant -B_+^p$$

as $r\to R$ which is

$$B_+\geqslant[\beta((\beta+1)C_0-1)]^{1/(p-1)}.$$

Let $B_+=(1+\varepsilon)[\beta((\beta+1)C_0-1)]^{1/(p-1)}=(1+\varepsilon)K$. Therefore, by making the choice B_+, the inequality (3.5) is satisfied in a left neighborhood of $r=R$, say $(R-\delta, R]$, for some $\delta=\delta(\varepsilon)>0$. Finally, by choosing A sufficiently large it is clear that the inequality is satisfied in the whole interval $[0, R]$ since $p>1$ and $b^*(r)$ is bounded away from zero in $[0, R-\delta]$. Then $\bar{\psi}_\varepsilon$ is our required supersolution of problem (3.1).

Second of all, we will construct a subsolution with the same blow-up rate as supersolution above has. For doing this we shall distinguish two different cases according to the sign of the parameter λ. First, we assume $\lambda\geqslant 0$. For each $\varepsilon>0$ sufficiently small, we claim that there exists $C<0$ for which the function

$$\underline{\psi}_\varepsilon(r)=\max\left\{0, C+B_-\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta}\right\}$$

provides us a subsolution, where $\beta=1/(p-1)$ and

$$B_-=(1-\varepsilon)[\beta((\beta+1)C_0-1)]^{1/(p-1)}=(1-\varepsilon)K.$$

In fact, denoting $f_C(r) = C + B_-(r/R)^2(b^*(r))^{-\beta}$ we have

$$\begin{aligned} f_C'(r) &= 2B_- \frac{r}{R^2}(b^*(r))^{-\beta} - \beta B_- \left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta-1}(b^*(r))' \\ &= 2B_- \frac{r}{R^2}\,(b^*(r))^{-\beta} + \beta B_- \left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta-1} \int_r^R b(s)\,\mathrm{d}s \end{aligned}$$

which is strictly bigger than zero in $(0, R)$. It follows that $f_C(r)$ is increasing and

$$\lim_{r\to R} f_C(r) = +\infty, \quad \lim_{r\to 0} f_C(r) = C < 0.$$

By the continuity of $f_C(r)$ and the intermediate-value theorem, there exists a unique $Z = Z(C) \in (0, R)$ such that

$$C + B_-\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta} < 0 \quad \text{when } r \in [0, Z(C)),$$

$$C + B_-\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta} \geqslant 0 \quad \text{when } r \in [Z(C), R].$$

Moreover, $Z(C)$ is decreasing and

$$\lim_{C\to -\infty} Z(C) = R, \quad \lim_{C\to 0} Z(C) = 0.$$

From the definition of $\underline{\psi}_\varepsilon(r)$ and $Z(C)$, $\underline{\psi}_\varepsilon(r) \equiv 0$ in $[0, Z(C)]$, and then the inequality $-\underline{\psi}_\varepsilon'' - ((N-1)/r)\underline{\psi}_\varepsilon' \leqslant \lambda \underline{\psi}_\varepsilon - b(r)\underline{\psi}_\varepsilon^p$ holds in $[0, Z(C)]$. So $\underline{\psi}_\varepsilon(r)$ is a subsolution if the following inequality are satisfied in $[Z(C), R]$

$$\left\{ \begin{aligned} &-2N \frac{B_-}{R^2}\frac{b^*(r)}{b(r)} + [N+3]\beta B_- \frac{r}{R^2}\frac{(b^*(r))'}{b(r)} \\ &\quad -\beta(\beta+1)B_-\left(\frac{r}{R}\right)^2 \frac{[(b^*(r))']^2}{b^*(r)b(r)} + \beta B_-\left(\frac{r}{R}\right)^2 \frac{(b^*(r))''}{b(r)} \\ &\quad\quad \leqslant \lambda \frac{b^*(r)}{b(r)}\left[C(b^*(r))^\beta + B_-\left(\frac{r}{R}\right)^2\right] - \left[C(b^*(r))^\beta + B_-\left(\frac{r}{R}\right)^2\right]^p. \end{aligned} \right. \tag{3.6}$$

In order to make the proof of the inequality (3.6) clear, let us rewrite (3.6) as $-I_1 + I_2 - I_3 + I_4 \leqslant J_1 - J_2$. Since $\lambda > 0$, for each $r \in [Z(C), R)$, $-I_1 + I_2 \leqslant 0 \leqslant J_1$, i.e.

$$\begin{aligned} &-2N\,\frac{B_-}{R^2}\,\frac{b^*(r)}{b(r)} + [N+3]\beta B_- \frac{r}{R^2}\,\frac{(b^*(r))'}{b(r)} \\ &\quad = -\left(2N \frac{B_-}{R^2}\frac{b^*(r)}{b(r)} + [N+3]\beta B_- \frac{r}{R^2}\frac{\int_r^R b(s)\,\mathrm{d}s}{b(r)}\right) \\ &\quad \leqslant 0 \leqslant \lambda \frac{b^*(r)}{b(r)}\left[C(b^*(r))^\beta + B_-\left(\frac{r}{R}\right)^2\right]. \end{aligned}$$

Then inequality (3.6) holds if $-I_3 + I_4 \leqslant -J_2$, i.e.

$$-\beta(\beta+1)B_-\left(\frac{r}{R}\right)^2\frac{[(b^*(r))']^2}{b^*(r)b(r)} + \beta B_-\left(\frac{r}{R}\right)^2 \leqslant -\left[C(b^*(r))^\beta + B_-\left(\frac{r}{R}\right)^2\right]^p$$

for each $r \in [Z(C), R]$. At $r = R$, it becomes into

$$-\beta(\beta+1)B_-C_0 + \beta B_- \leqslant -B_-^p.$$

That is

$$\mathrm{B}_- \leqslant [\beta((\beta+1)C_0 - 1)]^{1/(p-1)}.$$

By making the choice $B_- = (1-\varepsilon)[\beta((\beta+1)C_0-1)]^{1/(p-1)}$ and using the continuity, it is easy to see that a constant $\delta = \delta(\varepsilon) > 0$ exists for which the inequality is satisfied in $[R-\delta, R)$, then we choose C such that $Z(C) = R - \delta(\varepsilon)$. For this choice of C, it readily follows that $\underline{\psi}_\varepsilon$ provides us a subsolution of the problem.

In the case of $\lambda < 0$, let $\mu = ((N-1)/R)\sqrt{-\lambda} > 0$ and $\underline{\psi}_\varepsilon$ be a subsolution as above of

$$\begin{cases} -\psi'' - \dfrac{N-1}{r}\psi' = \mu\psi - b(r)\psi^p \quad \text{in } (0, R), \\ \lim_{r\to R}\psi(r) = \infty, \\ \psi'(0) = 0. \end{cases} \tag{3.7}$$

Then

$$\underline{\tilde{\psi}}_\varepsilon(r) = \exp\left(\sqrt{-\lambda}(r-R)\right)\cdot\underline{\psi}_\varepsilon(r)$$

provides us a subsolution of (3.1) for $\lambda < 0$. In fact,

$$\underline{\tilde{\psi}}'_\varepsilon(r) = \sqrt{-\lambda}\exp\left(\sqrt{-\lambda}(r-R)\right)\cdot\underline{\psi}_\varepsilon(r) + \exp\left(\sqrt{-\lambda}(r-R)\right)\cdot\underline{\psi}'_\varepsilon(r),$$

$$\begin{aligned}\underline{\tilde{\psi}}''_\varepsilon(r) = &-\lambda\exp\left(\sqrt{-\lambda}(r-R)\right)\cdot\underline{\psi}_\varepsilon(r) + 2\sqrt{-\lambda}\exp\left(\sqrt{-\lambda}(r-R)\right)\cdot\underline{\psi}'_\varepsilon(r) \\ &+ \exp\left(\sqrt{-\lambda}(r-R)\right)\cdot\underline{\psi}''_\varepsilon(r).\end{aligned}$$

Similarly $\underline{\tilde{\psi}}_\varepsilon(r) \to \infty$ as $r \to R$. $\underline{\tilde{\psi}}'_\varepsilon(r) \to 0$ as $r \to 0$. Therefore $\underline{\tilde{\psi}}'_\varepsilon(r)$ is a subsolution if the following inequality is satisfied

$$\begin{aligned}&\lambda\exp\left(\sqrt{-\lambda}(r-R)\right)\cdot\underline{\psi}_\varepsilon(r) \\ &\quad - 2\sqrt{-\lambda}\exp\left(\sqrt{-\lambda}(r-R)\right)\cdot\underline{\psi}'_\varepsilon(r) - \exp\left(\sqrt{-\lambda}(r-R)\right)\cdot\underline{\psi}''_\varepsilon(r) \\ &\quad - \frac{N-1}{r}\exp\left(\sqrt{-\lambda}(r-R)\right)\cdot\underline{\psi}'_\varepsilon(r) - \frac{N-1}{r}\sqrt{-\lambda}\exp\left(\sqrt{-\lambda}(r-R)\right)\cdot\underline{\psi}_\varepsilon(r) \\ &\qquad \leqslant \lambda\exp\left(\sqrt{-\lambda}(r-R)\right)\cdot\underline{\psi}_\varepsilon(r) - b(r)\left(\exp\left(\sqrt{-\lambda}(r-R)\right)\right)\cdot\underline{\psi}_\varepsilon(r))^p.\end{aligned}$$

Simplifying above inequality by both sides multiplying $\exp(-\sqrt{-\lambda}(r-R))$, one gets

$$\begin{aligned}&-\underline{\psi}_\varepsilon''(r)-\frac{N-1}{r}\underline{\psi}_\varepsilon'(r)-2\sqrt{-\lambda}\underline{\psi}_\varepsilon'(r)\\&\leqslant\frac{N-1}{r}\sqrt{-\lambda}\underline{\psi}_\varepsilon(r)-b(r)\Big(\exp\Big(\sqrt{-\lambda}(r-R)\Big)\Big)^{p-1}\cdot\underline{\psi}_\varepsilon(r)^p.\end{aligned}$$

By the fact of $\underline{\psi}_\varepsilon'\geqslant 0$, $\exp\big(\sqrt{-\lambda}(r-R)\big)<1$ and $\underline{\psi}_\varepsilon(r)$ is a subsolution of (3.7), above inequality holds. So we have constructed a subsolution for any λ.

Finally, since $1-\varepsilon\leqslant\lim_{r\to R}\underline{\psi}_\varepsilon(r)/K(b^*(r))^{-\beta}\leqslant\lim_{r\to R}\bar{\psi}_\varepsilon(r)/K(b^*(r))^{-\beta}\leqslant 1+\varepsilon$, then by Theorem 2.3, there exists a solution ψ_ε of (3.1) such that

$$1-\varepsilon\leqslant\liminf_{r\to R}\frac{\psi_\varepsilon(r)}{K(b^*(r))^{-\beta}}\leqslant\limsup_{r\to R}\frac{\psi_\varepsilon(r)}{K(b^*(r))^{-\beta}}\leqslant 1+\varepsilon.$$

Proof of uniqueness. The proof of uniqueness basically follows the proof in [4,6]. Let u be an arbitrary solution of (1.4). We firstly show that

$$\lim_{d(x)\to 0}\frac{u(x)}{K(b^*(\|x-x_0\|))^{-\beta}}=1.$$

Consequently, for any pair of solution u, v of (1.4)

$$\lim_{d(x)\to 0}\frac{u(x)}{v(x)}=1.$$

In doing so, for any $\varepsilon>0$, there exists a radially symmetric positive large solution u_ε of (1.4) satisfying (3.4). Choose $0<\delta<R/3$ small, fix $0<\tau<\delta/4$ and introduce the region

$$Q_\tau:=\left\{x|\tau<d(x,\partial B_R(x_0))<\frac{\delta}{2}\right\}.$$

Let $M\geqslant\max_{\|x-x_0\|\leqslant(R-\delta/4)}u(x)$ large. Thus for every $\tau\in(0,\delta/4)$,

$$\bar{V}_\varepsilon(x)=u_\varepsilon\left(x+\tau\frac{(x-x_0)}{\|x-x_0\|}\right)+M=\psi_\varepsilon(\|x-x_0\|+\tau)+M$$

is a supersolution to

$$\begin{cases}-\Delta v=\lambda v-bv^p & \text{in } Q_\tau,\\ v=u & \text{on } \partial Q_\tau\end{cases}\tag{3.8}$$

with u an arbitrary fixed solution to (1.4) since $\bar{V}_\varepsilon(x)\geqslant u$ for $x\in\partial Q_\tau$, $\tau\in(0,\delta/4)$. Note that $\bar{V}_\varepsilon(x)\to\infty$ as $x\to\partial B_{R-\tau}(x_0)$. $\bar{V}_\varepsilon(x)\geqslant M\geqslant u$ as $x\to\partial B_{R-\delta/2}(x_0)$. In addition, the auxiliary problem (3.8) has $v=u$ as its unique solution. Since 0 is a subsolution, we conclude $u(x)\leqslant\bar{V}_\varepsilon(x)=u_\varepsilon(\|x-x_0\|+\tau)+M$ for every $x\in Q_\tau$, $0<\tau<\delta/4$. Letting $\tau\to 0^+$, we arrive at $u(x)\leqslant u_\varepsilon(x)+M$ for every $x\in A_{\delta/2,R}(x_0)$ and we obtain

$$\limsup_{d(x)\to 0}\frac{u(x)}{K(b^*(\|x-x_0\|))^{-\beta}}\leqslant\limsup_{d(x)\to 0}\frac{u_\varepsilon(x)}{K(b^*(\|x-x_0\|))^{-\beta}}\leqslant 1+\varepsilon.$$

Our next objective is finding a subsolution with the same blow up rate as in the supersolution above. For any $\varepsilon > 0$, there exists a radially symmetric positive large solution u_ε of (1.4) satisfying (3.4). Choose $0 < \delta < R/3$ small, fix $0 < \tau < \delta/4$ and introduce the annuli region

$$A_{R-\delta,R+\tau} = \{x : R - \delta < \|x - x_0\| < R + \tau\}.$$

Let $M_1 \geqslant \max_{R-(\delta+\delta/4)\leqslant \|x-x_0\|\leqslant R-\delta} u(x)$ large. Thus for every $\tau \in (0, \delta/4)$,

$$\underline{V}_\varepsilon(x) = \max\left\{u_\varepsilon\left(x - \tau\frac{(x - x_0)}{\|x - x_0\|}\right) - M_1, 0\right\} = \max\{\psi_\varepsilon(\|x - x_0\| - \tau) - M_1, 0\}$$

is a subsolution to

$$\begin{cases} -\Delta v = \lambda v - bu^p & \text{in } A_{R-\delta,R+\tau}, \\ v = u & \text{on } \partial A_{R-\delta,R+\tau} \end{cases} \tag{3.9}$$

for all $\tau \in (0, \delta/4)$. It readily gets

$$1 - \varepsilon \leqslant \liminf_{d(x)\to 0} \frac{u_\varepsilon(x)}{K(b^*(\|x - x_0\|))^{-\beta}} \leqslant \liminf_{d(x)\to 0} \frac{u}{K(b^*(\|x - x_0\|))^{-\beta}}.$$

As a result, we obtain

$$1 - \varepsilon \leqslant \lim_{d(x)\to 0} \frac{u}{K(b^*(\|x - x_0\|))^{-\beta}} \leqslant 1 + \varepsilon.$$

Letting $\varepsilon \to 0^+$, we have

$$\lim_{d(x)\to 0} \frac{u}{K(b^*(\|x - x_0\|))^{-\beta}} = 1.$$

Now let u and v be large positive solutions to (1.4). By virtue of above, u and v satisfy $\lim_{d(x)\to 0} u/v = 1$. Thus, for every $\varepsilon > 0$, we can find $\delta > 0$ (as small as we please) such that

$$(1 - \varepsilon)v(x) \leqslant u(x) \leqslant (1 + \varepsilon)v(x)$$

when $0 < d(x) \leqslant \delta$. On the other hand $\underline{w} = (1 - \varepsilon)v(x)$ and $\bar{w} = (1 + \varepsilon)v(x)$ are sub and supersolutions to

$$\begin{cases} -\Delta w = \lambda w - bw^p & \text{in } B_{R-\delta}(x_0), \\ w = u & \text{on } \partial B_{R-\delta}(x_0). \end{cases} \tag{3.10}$$

The unique solution to this problem is $w = u$. Then by Theorem 2.1

$$(1 - \varepsilon)v(x) \leqslant u(x) \leqslant (1 + \varepsilon)v(x)$$

holds in $B_{R-\delta}(x_0)$, therefore it is true in $B_R(x_0)$. Letting $\varepsilon \to 0$ we arrive at $u = v$.

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