# Stability Zone of Hamiltonian System and Index Theory for Symplectic Paths

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**Abstract**

In this paper, we apply index theory for symplectic paths introduced by Y. Long to study the stability of a periodic solution $x$ for a Hamiltonian system. We establish a necessary and sufficient condition for stability of the periodic solution $x$ in two dimension. We prove that the solution $x$ is linear stable if and only if its index $ind(x)$ is an odd integer. Furthermore, the necessary and sufficient condition is also successfully expressed in terms of Morse index.

**Key word:** Variational Method, Hamiltonian System, Morse Index, Index for Symplectic Paths, Linear Stability
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## 1 Introduction

The main purpose of this paper is to show that the index theory for symplectic paths is a very flexible tool in the study of the stability. We will establish the relation between the stability of periodic solution for a Hamiltonian system and its index in low dimension.

There are infinitely many ways to define index theories for paths of symplectic matrices. A definition of the index theory for symplectic paths is meaningful if it can be applied to different problem. Historically, in the study of closed geodesics on Riemannian Manifolds, M.Morse in the 1930s successfully developed his index theory. The iteration theory of the Morse index for closed geodesics was developed by R.Bott ([1]) in 1956. Based upon their works, many interesting and deep results on closed geodesics have been obtained via the Morse index theory. In terms of the Morse index of the variational problem with periodic or anti-periodic boundary conditions, Daniel Offin gave necessary and sufficient conditions for stability ([3]) in 2001. Offin also studied hyperbolicity of minimizing geodesics by applying index theory ([4]).

In his book *Index Theory for Symplectic Paths with Applications* ([8]), Yiming Long introduced an index for symplectic paths in symplectic matrix group. He settled down the main geometric features of this index introducing a characteristic class. Also he built the relations with Morse index and Maslov index. This introductive section is very much guided by this book.

Since the fundamental solution of a general linear hamiltonian system with continuous symmetric periodic coefficients is a path in the symplectic matrix group $Sp(2n)$ starting from the identity, to be sufficiently flexible such an index theory needs to be generalized to any symplectic paths. Here the symplectic group is defined by

$$Sp(2n) = \{M \in GL(\mathbb{R}^{2n}) | M^T J M = J\},$$

where $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, $I$ is the identity matrix on $\mathbb{R}^n$, and $M^T$ denotes the transpose of $M$. For $\tau > 0$, we define the set of symplectic matrix paths by

$$\mathcal{P}_\tau(2n) = \{\gamma \in C([0,\tau], Sp(2n)) | \gamma(0) = I\}.$$

Because $Sp(2n)$ is homeomorphic to the product of the unit circle and a simply connected space, a path $\gamma \in \mathcal{P}_\tau(2n)$ rotates naturally in $Sp(2n)$ along this unit circle. The point is to find a way to count this rotation so that the rotation number represents intrinsically the corresponding Morse index of the related Hamiltonian system. For periodic boundary value problems of Hamiltonian systems, because of this consideration, we call a path $\gamma \in \mathcal{P}_\tau(2n)$ *degenerate* if 1 is an eigenvalue of $\gamma(\tau)$, and *non-degenerate* otherwise.

In section 2 the stage is set with a brief review of the properties of the topological space $Sp(2n)$. Because we are going to establish a necessary and sufficient condition for stability in terms of index for symplectic paths in $Sp(2)$, we only introduce the definition of index for symplectic paths in $Sp(2)$ and its properties.

In section 3 an index function $ind(x)$ for a periodic solution $x$ of a Hamiltonian system is introduced. We prove that the periodic solution $x$ of the Hamiltonian system is linear stable if and only if $ind(x)$ is an odd integer. Also we translate such result in terms of classical Morse index.

## 2 Index for Symplectic Paths

In order to study the global structure of the symplectic group $Sp(2n)$ established by I. Gelfand, V. Lidskii, J. Moser, we will use the same notation as in ([8]). Let $U$ be the unit circle in the complex plane $C$. For any $\omega \in U$ and $M \in Sp(2n)$, we define

$$D_\omega(M) = (-1)^{n-1}\omega^{-n}\det(M - \omega I).$$

Then D is a real smooth function on $U \times Sp(2n)$. According to the value of $D_\omega$, we define some subset of $Sp(2n)$. For $\omega \in U$, we define the $\omega-$singular set $Sp(2n)^0_\omega$ of $Sp(2n)$ and its subsets $\mathcal{M}^k_\omega(2n)$ with $0 \le k \le 2n$ by

$$Sp(2n)^0_\omega = \{M \in Sp(2n) | D_\omega(M) = 0\},$$

$$\mathcal{M}^k_\omega(2n) = \{M \in Sp(2n) | \nu_\omega(M) = k\}$$

where $\nu_\omega(M) = dim_C ker_C(M - \omega I)$. We also define the $\omega-$regular sets of $Sp(2n)$ by

$$Sp(2n)^\pm_\omega = \{M \in Sp(2n) | \pm D_\omega(M) < 0\}$$

$$Sp(2n)^*_\omega = Sp(2n)^+_\omega \cup Sp(2n)^-_\omega$$

For $\tau > 0$ and $\omega \in U$, we further define the set of $\omega-$non-degenerate paths by

$$\mathcal{P}^*_{\tau,\omega}(2n) = \{\gamma \in \mathcal{P}_\tau(2n) | \gamma(\tau) \in Sp(2n)^*_\omega\},$$

and the set of $\omega-$degenerate paths by

$$\mathcal{P}^0_{\tau,\omega}(2n) = \mathcal{P}_\tau(2n) \backslash \mathcal{P}^*_{\tau,\omega}(2n)$$

For any symplectic matrix $M \in Sp(2n)$, it can be represented in the form (polar decomposition)

$$M = AU,$$

where $A = \sqrt{MM^T}$ is a symmetric symplectic positive definite matrix, $U$ is a symplectic orthogonal matrix, and they are uniquely determined by $M$. By the polar decomposition, it is easy to prove that the symplectic group $Sp(2n)$ is homeomorphic to the topological product of the unit circle $U$ in the complex plane $C$ and a simply connected topological Space.

For $n = 1$, Y. Long in ([8]) introduce a geometric representation of $Sp(2)$ in $\mathbb{R}^3$. For any matrix $M \in Sp(2)$, by the polar decomposition, $M$ can be written in the form

$$M = \begin{pmatrix} r & z \\ z & (1+z^2)/r \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{2.1}$$

where $(r, \theta, z) \in \mathbb{R}^+ \times S^1 \times \mathbb{R}$, $\mathbb{R}^+ = \{r \in \mathbb{R} | r > 0\}$, $S^1 = \mathbb{R}/(2\pi\mathbb{R} - \pi)$, and $(r, \theta, z)$ is uniquely determined by $M$. Viewing $(r, \theta, z)$ as the cylindrical coordinates in $\mathbb{R}^3$, we obtain a representation of $Sp(2)$ in $\mathbb{R}^3$. Under this representation, it is easy to see that the two eigenvalues of $M$ are

$$\lambda = \frac{1}{2r}((r^2 + z^2 + 1)\cos\theta \pm \sqrt{(1 + r^2 + z^2)^2 \cos^2\theta - 4r^2}),$$

which are either two reciprocal real numbers or two conjugate complex numbers on the unit circle $U$ in the complex plane $C$. For $\omega = \cos\varphi + \sqrt{-1}\sin\varphi \in U$ and $M$ in the form (2.1), we obtain

$$D_\omega(M) = 2\cos\varphi - \left(r + \frac{1+z^2}{r}\right)\cos\theta.$$

Then we have

$$Sp(2)^\pm_\omega = \{(r, \theta, z) \in \mathbb{R}^+ \times S^1 \times \mathbb{R} | \pm (r^2 + z^2 + 1)\cos\theta > 2r\cos\varphi\},$$

$$Sp(2)^0_\omega = \{(r, \theta, z) \in \mathbb{R}^+ \times S^1 \times \mathbb{R} | \pm (r^2 + z^2 + 1)\cos\theta = 2r\cos\varphi\}.$$

Let $Sp(2)^0_{\omega,\pm} = \{(r, \theta, z) \in Sp(2)^0_\omega | \pm \sin\theta > 0\}$. Then we have $Sp(2)^0_{\pm 1} = Sp(2)^0_{\pm 1,+} \bigcup \{\pm I\} \bigcup Sp(2)^0_{\pm 1,-}$, and $Sp(2)^0_\omega = Sp(2)^0_{\omega,+} \bigcup Sp(2)^0_{\omega,-}$ for $\omega \in U \backslash \mathbb{R}$. Here we are especially interested in the cylindrical coordinate representation of the singular hypersurfaces $Sp(2)^0_\omega$ for $\omega \in U$.

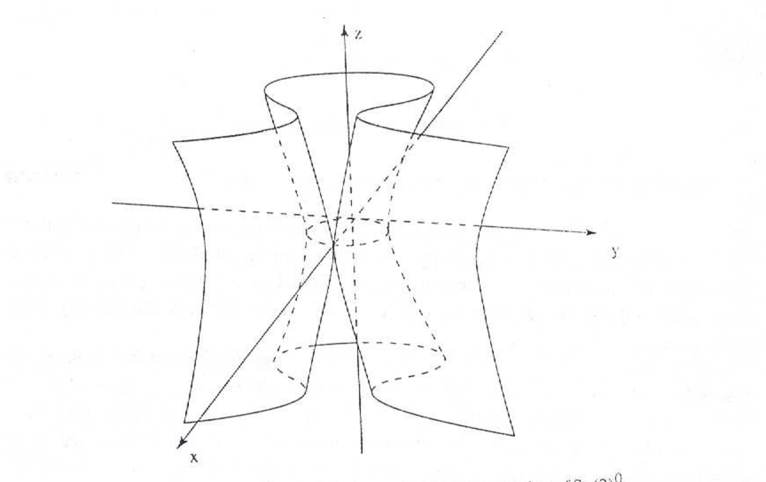



Figure 1: *The* $\mathbb{R}^3$*-cylindrical coordinate representation of* $\mathrm{Sp}(2)_1^0$

In figure 1, we give this $\mathbb{R}^3-$cylindrical coordinate representation of $Sp(2)_1^0$with the Descartes coordinates $(x, y, z) = (r\cos\theta, r\sin\theta, z)$. You also can find this picture in ([8]).

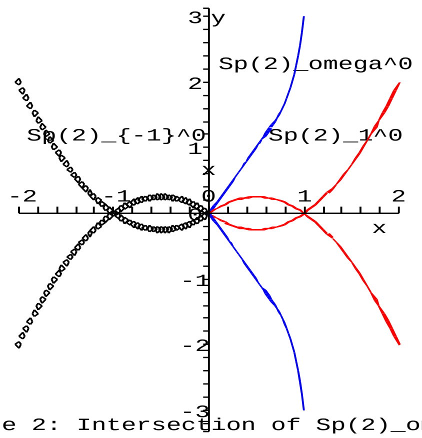



Figure 2: Intersection of Sp(2)_omega^0 with {z=0}

In figure 2, the intersection of the plane $\{z = 0\}$ with $Sp(2)_1^0, Sp(2)_{-1}^0$, and $Sp(2)_\omega^0$ for some $\omega \in U$ are given. It is easy to prove the following propositions.

**Proposition 2.1**. For any $\omega \in U$ the set $Sp(2)_\omega^*$ possesses precisely two path connected components $Sp(2)_\omega^+$ and $Sp(2)_\omega^-$, and it holds that $D(2) = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \in Sp(2)_\omega^+$ and $D(-2) = \begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \in Sp(2)_\omega^-$.

**Proposition 2.2.** Fix an $\omega \in U\backslash\mathbb{R}$. $Sp(2)_1^-, Sp(2)_{-1}^+$ and $Sp(2)_\omega$ are homeomorphic to $\mathbb{R}^3$. $Sp(2)_1^+$ and $Sp(2)_{-1}^-$ are homeomorphic to $\mathbb{R}^3\backslash\{(x, y, z) \in \mathbb{R}^3 | x^2 + z^2 \leq y^2\}$.
Thus for any $\omega \in U$, the set $Sp(2)_\omega^*$ is simply connected in $Sp(2)$, i.e., any closed curve inside $Sp(2)_\omega^+$ or $Sp(2)_\omega^-$ can be continuously contracted inside $Sp(2)$ to a point.

**Proposition 2.3**. When $\omega = 1$, in the singular hypersurface $Sp(2)_1^0$, the identity matrix $I_2$ is the only element which satisfies $\nu_1(I_2) = 2$. The regular part $\mathcal{M}(2)$ of $Sp(2)_1^0$ possesses precisely two path connected components $Sp(2)_{1,+}^0$ and $Sp_{1,-}^0$, both of which are smooth hypersurfaces diffeomorphic to $\mathbb{R}^2\backslash\{0\}$.

Note that from the $\mathbb{R}^3-$cylindrical coordinate represntation introduced above, $Sp(2)_\omega^0$ for $\omega \in U$ is orientable. Now we can define the index for symplectic paths in $Sp(2)$ and index for periodic solutions of Hamiltonian system.
Fix $\tau > 0$, $\omega \in U$, we use the concept of intersection numbers in the algebraic topology to define index for $\omega-$non-degenerate paths in $\mathcal{P}_{\tau,\omega}^*$. Because an orientation of $Sp(2)_\omega^0$ is defined, $Sp(2)_\omega^0$ form a locally finite 2−dimensional singular homological cycle ([6]). For any $\tau > 0$, let $\xi_+(t) = \begin{pmatrix} 1+\frac{t}{\tau} & 0 \\ 0 & \frac{\tau}{t+\tau} \end{pmatrix}$ for all $t \in [0,\tau]$. For any $\tau > 0$ and a path $\beta \in \mathcal{P}_\tau(2)$ denote $\beta^{-1}(t) = \beta(\tau - t)$ for $t \in [0,\tau]$.

**Definition 2.1** For any $\omega \in U$, $\tau > 0$, and $\gamma \in \mathcal{P}_{\tau,\omega}^*(2)$, we define

$$i_\omega(\gamma) = [Sp(2)_\omega^0 : \gamma * \xi_+^{-1}]. \tag{2.2}$$

For any path $\gamma \in \mathcal{P}_{\tau,\omega}^*(2)$, the two end points of the joint path $\gamma*\xi_+^{-1}$ are not located on $Sp(2)_\omega^0$. Thus the algebraic homological intersection number in (2.2) is well defined. The integer $i_\omega(\gamma)$ is called the index of the symplectic path $\gamma$.

To further explain this definition, fixing an $\omega \in U$, we consider the smooth paths first. Let $\varphi \in C^1([0,\tau], Sp(2))$ such that $\varphi(0) = D(2)$ and $\varphi(\tau) \in Sp(2)_\omega^*$. Then the direction of $\varphi$ at the point $\varphi(t)$ is defined to be the tangent direction $\dot\varphi(t)$ of $\varphi$ at that point. Now we assume the following conditions on $\varphi$.
(1) It holds that

$$\varphi([0,\tau]) \bigcap Sp(2)_\omega^0 \subset \varphi([0,\tau]) \bigcap \mathcal{M}_\omega^1(2) \equiv S(\omega,\varphi).$$

(2) $\varphi$ intersects $\mathcal{M}_\omega^1(2)$ transversally, i.e. at any intersection point $\varphi(t) \in S(\omega,\varphi)$, the tangent vector of $\varphi$ at the point $\varphi(t)$ is not contained in the tangent plane of $\mathcal{M}_\omega^1(2)$ at the same point, i.e. $\dot\varphi(t)\cdot\eta(\omega,\varphi(t)) \neq 0$,, where $\eta(\omega,x)$ is the positively directed unit normal vector of $\mathcal{M}_\omega^1(2)$ at its point $x$.
Denote by $\mathcal{C}_{\tau,reg}^1(2)$ the set of all such $C^1$ curves satisfying (1) and (2), and call them the regular curves in $Sp(2)$. Under these tow conditions, we define the intersection number $\mu(\varphi,\mathcal{M}^1(2),x)$ at $\varphi(t) \in S(\omega,\varphi)$ by

$$\mu(\varphi,\mathcal{M},\varphi(t)) = 1, \text{ if } \dot\varphi(t)\cdot\eta(\omega,\varphi(t)) > 0,$$
$$\mu(\varphi,\mathcal{M},\varphi(t)) = -1, \text{ if } \dot\varphi(t)\cdot\eta(\omega,\varphi(t)) < 0.$$

Then the intersection number $\varphi$ and $Sp(2)_\omega^0$ is defined by

$$[Sp(2)_\omega^0 : \varphi] = \sum_{x\in S(\omega,\varphi)} \mu(\varphi,\mathcal{M},x), \ \forall\varphi \in \mathcal{C}_{\tau,reg}^1(2).$$

Define by $\mathcal{C}_\tau^1(2)$ the set of paths in $C^1([0,\tau], Sp(2))$ started from $D(2)$. Let $\mathcal{C}_{\tau,reg}^1(2, D(2)) = \mathcal{C}_\tau^1(2)\bigcap\mathcal{C}_{\tau,reg}^1(2)$. Now fix a path $\gamma \in \mathcal{P}_\tau(2)$. we consider the $C^0-$ approximations $\varphi \in \mathcal{C}_{\tau,reg}^1(2, D(2))$ of $\gamma * \xi_+^{-1}$ satisfying $\varphi(\tau) = \gamma(\tau)$. It is easy to see that if $\varphi$ is sufficiently $C^0-$close to $\gamma*\xi_+^{-1}$, the intersection umber $[Sp(2)_\omega^0 : \varphi]$ is independent of the particular choices of $\varphi$. Therefore we obtain

$$[Sp(2)_\omega^0 : \gamma * \xi_+^{-1}] = [Sp(2)_\omega^0 : \varphi]$$

for all such smooth $C^0-$approximations of $\gamma * \xi_+^{-1}$. Thus Definition 1 is well defined.

For $\tau > 0$ and $\omega \in U$, given two paths $\gamma_0$ and $\gamma_1 \in \mathcal{P}_\tau(2n)$, if there exists a map $\delta \in C([0,1] \times [0,\tau], Sp(2n))$ such that $\delta(0,\cdot) = \gamma_0(\cdot), \delta(1,\cdot) = \gamma_1(\cdot)$, $\delta(s,0) = I$, and $\nu_\omega(\delta(s,\cdot))$ is constant for $0 \le s \le 1$, then $\gamma_0$ and $\gamma_1$ are $\omega-$homotopic on $[0,\tau]$ along $\delta(\cdot,\tau)$ and we write $\gamma_0 \sim_\omega \gamma_1$.

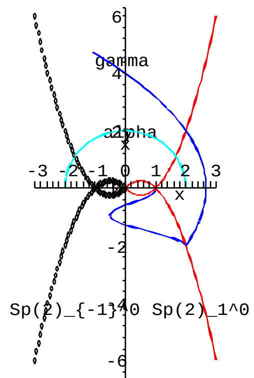



Figure 3: The case of gamma~_1 alpha
The case of gamma not~_-1 alpha

Note that the index function $i_\omega$ is homotopy invariant, i.e., for any $\omega \in U$, $\gamma_0$ and $\gamma_1 \in \mathcal{P}_\tau(2)$, $\gamma_0 \sim_\omega \gamma_1$ implies $i_\omega(\gamma_0) = i_\omega(\gamma_1)$. Conversely it is also true. In figure 3, $\gamma$ is $1-$homotopic to $\alpha_1$ on $[0,\tau]$ which is defined in the proof of theorem 2.1 but $\gamma$ is not $-1-$homotopic to $\alpha_1$ on $[0,\tau]$. In fact, $i_1(\gamma) = 1$ but $i_{-1}(\gamma) = 0$.

For an $\omega-$degenerate path, i.e., fixed $\tau > 0$, and $\omega \in U$, $\forall \gamma \in \mathcal{P}^0_{\tau,\omega}(2)$, there are two methods to define the index function $i_\omega(\gamma)$. One is perturbation method and another is minimizing method. More details can be found at Chapter 5.1 in ([8]). So for any path $\gamma \in \mathcal{P}_\tau(2)$, an index function $i_\omega(\gamma)$ is well defined for all $\omega \in U$. Here we give two important theorems which establish a relation between index function for a syplectic path in $Sp(2)$ and the ending point of the path.

**Theorem 2.1** For $\tau > 0$, $\forall \gamma \in \mathcal{P}^*_{\tau,1}(2)$, we have (1) $i_1(\gamma)$ is an even integer if and only if $\gamma(\tau) \in Sp(2)^+_1$. Furthermore, if $\lambda_1, \lambda_2$ are two eigenvalues of $\gamma(\tau)$, then $0 < \lambda_1 < 1 < \lambda_2$ and $\lambda_1 = \frac{1}{\lambda_2}$. (2) $i_1(\gamma)$ is an odd integer if and only if $\gamma(\tau) \in Sp(2)^-_1$.
**Proof.** For $\tau > 0$, $\gamma \in \mathcal{P}^*_{\tau,1}(2)$, assume $i_1(\gamma) = k$, we define a *zigzag standard path* $\alpha_k$ in $\mathcal{P}^*_{\tau,1}(2)$ such that $\alpha_k \sim_1 \gamma$ as follows. Set

$$\phi_{\tau,\theta}(t) = \begin{pmatrix} \cos(\theta\frac{t}{\tau}) & -\sin(\theta\frac{t}{\tau}) \\ \sin(\theta\frac{t}{\tau}) & \cos(\theta\frac{t}{\tau}) \end{pmatrix}, \forall t \in [0,\tau], \theta \in \mathbb{R},$$

$$\alpha_0(t) = \xi_+(t), \forall t \in [0,\tau],$$

For $0 \le t \le \tau$ we define

$$\alpha_k(t) = [D(2)\phi_{\tau,k\pi}] * \xi_+(t), \text{ if } k \in \mathbb{Z}\backslash\{0\}.$$

Then $\alpha_k(\tau) = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ if $k$ is even and $\alpha_k(\tau) = \begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ if $k$ is odd. $\alpha_k \in \mathcal{P}^*_{\tau,1}(2)$ and $i_1(\alpha_k) = k$.

Claim: Both $\gamma(\tau)$ and $\alpha_k(\tau)$ are in the same 1-regular subset $Sp(2)_1^+$ or in $Sp(2)_1^-$.
Because $i_1(\gamma) = k = i_1(\alpha_k)$, $\gamma \sim_1 \alpha_k$. There exists a map $\delta \in C([0,1] \times [0,\tau], Sp(2))$ such that $\delta(0,\cdot) = \gamma(\cdot)$, $\delta(1,\cdot) = \alpha_k(\cdot)$, $\delta(s,0) = I$, and $\nu_1(\delta(s,\cdot)) = \nu_1(\delta(s,\tau)) = dim_C ker_C(\delta(s,\tau) - I) = 0$ for $0 \leq s \leq 1$. By the continuity of $\delta(\cdot,\tau)$, both $\gamma(\tau)$ and $\alpha_k(\tau)$ must be in the same 1-regular subset $Sp(2)_1^+$ or in $Sp(2)_1^-$.
Then if $i_1(\gamma) = k$ is even, $\alpha_k(\tau) \in Sp(2)_1^+$. We complete the proof of the first part of (1). As for the second part of (1), it is directed from the properties of eigenvalues of sypmlectic matrix. The proof for (2) is similar as the proof for (1). ♮

**Theorem 2.2** For $\tau > 0$, $\forall \gamma \in \mathcal{P}^*_{\tau,-1}(2)$, we have (1) $i_{-1}(\gamma)$ is an even integer if and only if $\gamma(\tau) \in Sp(2)^+_{-1}$. (2) $i_{-1}(\gamma)$ is an odd integer if and only if $\gamma(\tau) \in Sp(2)^-_{-1}$.Furthermore, if $\lambda_1, \lambda_2$ are two eigenvalues of $\gamma(\tau)$, then $0 > \lambda_1 > -1 > \lambda_2$ and $\lambda_1 = \frac{1}{\lambda_2}$.
We will omit the proof of theorem 2.2 because it is similar to the proof of theorem 2.1. From theorem 2.1 and 2.2, we obtain

**Corollary 2.1** For $\tau > 0$, $\forall \gamma \in \mathcal{P}^*_{\tau,\pm 1}(2)$, $i_1(\gamma)$ is an odd integer and $i_{-1}(\gamma)$ is an even integer if and only if all eigenvalues of $\gamma(\tau)$ are on the unit circle $U$.

# 3 Index for periodic solutions of Hamiltonian system and its stability zone

In this section, first we define the index for linear Hamiltonian system. Then we establish the stability zone according to the properties of index for the linear Hamiltonian system. Theorem 2.1 and 2.2 will play important roles.

**Definition 3.1** For $\tau > 0$, let $B \in C(\mathbb{R}/(\tau\mathbb{Z}), \mathcal{L}_s(\mathbb{R}^2))$ be a periodic continuous map into the set of symmetric square real matrix. Classically, the fundamental solution $\gamma_B$ of the linear Hamiltonian system

$$\dot{x}(t) = JB(t)x(t) \tag{3.1}$$

satisfies $\gamma_B \in \mathcal{P}_\tau(2)$, and is called the associated symplectic path of $B$. For $\omega \in U$, we define the index function of $B$ via that of the $\gamma_B$:

$$i_\omega(B) = i_\omega(\gamma_B), \text{ for all } \omega \in U \tag{3.2}$$

As usual, eigenvalues of $\gamma_B(\tau)$ are called Floquet multipliers of the system (3.1) (or $B$).

Let $H \in C^1(\mathbb{R}/(\tau\mathbb{Z}) \times \mathbb{R}^2, \mathbb{R})$. Suppose $x$ is a $\tau-$periodic solution of the Hamiltonian system

$$\dot{x}(t) = JH'(t, x(t)), \tag{3.3}$$

such that $H$ is $C^2$ along the orbit $x(\mathbb{R})$ of x. The associated symplectic path of $x$ is defined to be the fundamental solution $\gamma_x \equiv \gamma_B$ of the linearized Hamiltonian system with $B(t) = H''(t, x(t))$ for all $t$. For $\omega \in U$, we define the index function of $x$ via that of its associated symplectic path $\gamma_x : i_\omega(x) = i_\omega(\gamma_x)$.
As usual, eigenvalues of $\gamma_x(\tau)$ are called Floquet multipliers of the solution $x$ of the Hamiltonian system. By Floquet theorem, the periodic solution is linear stable if all the Floquet multipliers

are on unit circle and $\gamma_x(\tau)$ is diagonalizable.

**Theorem 3.1** Suppose $x$ is a $\tau-$periodic solution of the Hamiltonian system (3.3) and $\gamma_x$ is the associated symplectic path of $x$. If $\gamma_x \in \mathcal{P}_{\tau,1}(2)^*$ and $i_1(\gamma_x)$ is an even integer, then the periodic solution $x$ is linear unstable.
**Proof.** Because the index $i_1(\gamma_x)$ of the periodic solution is an even integer, $\gamma_x(\tau) \in Sp(2)_1^+$. So the Floquet multipliers are all real and one is bigger than 1. Therefore the periodic solution is linear unstable.

**Theorem 3.2** Suppose $x$ is a $\tau-$periodic solution of the Hamiltonian system (3.3) and $\gamma_x$ is the associated symplectic path of $x$. If $\gamma_x \in \mathcal{P}_{\tau,-1}(2)^*$ and $i_{-1}(\gamma_x)$ is an odd integer, then the periodic solution $x$ is linear unstable.

**Corollary 3.1** Suppose $x$ is a $\tau-$periodic solution of the Hamiltonian system (3.3) and $\gamma_x$ is the associated symplectic path of $x$. If $\gamma_x \in \mathcal{P}_{\tau,\pm1}(2)^*$ and the periodic solution $x$ is linear stable, then $i_1(\gamma_x)$ is an odd integer and $i_{-1}(\gamma_x)$ is an even integer.

Our final result shows that these two theorems are in fact sufficient for stability as well. For higher-dimensional problems, that is when $n > 1$, there is much less that can be stated for linear systems with periodic coefficients in these simple terms. But for $n = 2$, we have some similar results for instability. Also it can be applied to study the instability for isosceles three body problem ([5]). Suppose $x$ is a $\tau-$periodic solution of the Hamiltonian system (3.3) and $\gamma_x$ is the associated symplectic path of $x$. If $\gamma_x \in \mathcal{P}_{\tau,\pm1}(2)^*$, we denote the index of the periodic solution $x$ by $ind(x) = i_1(\gamma_x) + i_{-1}(\gamma_x)$.

**Theorem 3.3** Suppose $x$ is a $\tau-$periodic solution of the Hamiltonian system (3.3) and $\gamma_x$ is the associated symplectic path of $x$. If $\gamma_x \in \mathcal{P}_{\tau,\pm1}(2)^*$ and $ind(x)$ is odd if and only if the periodic solution $x$ of the Hamiltonian system (3.3) is linear stable. Or equivalently, $ind(x)$ is even if and only if the periodic solution $x$ of the Hamiltonian system (3.3) is linear unstable.
**Proof.** $Ind(x)$ is even if and only if $i_1(\gamma_x)$ and $i_{-1}(\gamma_x)$ are both even or odd. If $i_1(\gamma_x)$ and $i_{-1}(\gamma_x)$ are both even, by theorem (3.1) $i_1(\gamma_x)$ even implies that the periodic solution $x$ is linear unstable. If $i_1(\gamma_x)$ and $i_{-1}(\gamma_x)$ are both odd, by theorem (3.2) $i_{-1}(\gamma_x)$ odd implies that the periodic solution $x$ is linear unstable.
Conversely, assuming $x$ is linear unstable, if $i_1(\gamma_x)$ is an even integer, then $\gamma_x(\tau) \in Sp(2)_1^+$ by theorem 2.1. But if $i_{-1}(\gamma_x)$ is an odd integer, then $\gamma_x(\tau) \in Sp(2)_{-1}^-$ by theorem 2.2. Because $Sp(2)_1^+ \bigcap Sp(2)_{-1}^-$ is empty, there are no such symplectic paths. So $i_{-1}(\gamma_x)$ must be also an even integer. Therefore $ind(x)$ is even. Similar discussion can be apply to other case ($i_1(\gamma_x)$ is an odd integer ). $\natural$

In chapter 7.3 ([8]), the relationship between the index function defined above and the Morse index or the index functions of R.Bott defined in ([1]) are studied. We restate these results in our needs. Let's consider the Hamiltonian equations with periodic coefficients

$$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial x} \tag{3.4}$$

where $H = H(t,x,y) = H(t+\tau,x,y)$, and $(x,y) \in \mathbb{R}^2$. We assume that $H$ satisfies (The Legendre convexity condition) $\frac{\partial^2 H}{\partial y^2} > 0$. Under this assumption, the equations (3.4) are equivalent to Euler-Lagrange equation

$$\frac{d}{dt}L_p(t,x,\dot{x}) = L_x(t,x,\dot{x}). \tag{3.5}$$

where the Lagrangian is defined classically via the Legendre transform

$$L(t,x,p) = \max_y [yp - H(t,x,y)] = Yp - H(x,Y,t),$$

where $p = \frac{\partial H(x,Y,t)}{\partial y}$. Let

$$F(x) = \int_0^\tau L(t,x,\dot{x})dt, \forall x \in \Psi \tag{3.6}$$

where $\Psi$ is the space of curves $\Psi = \{x \in C^1([0,\tau],\mathbb{R})\}$. Boundary conditions will be introduced by restricting $\Psi$ to the set of curves $\Psi_R = \{x \in \Psi; x \text{ satisfies boundary condition } R\}$. For example, $R$ is the periodic boundary value condition

$$x(\tau) = x(0), \qquad \dot{x}(\tau) = \dot{x}(0). \tag{3.7}$$

or boundary $\omega \in U$ value condition

$$x(\tau) = \omega x(0), \qquad \dot{x}(\tau) = \omega\dot{x}(0). \tag{3.8}$$

A critical point $x$ of $F$ in $\Psi_R$ with $R$ being the periodic boundary value condition, i.e., an extreme loop of $F$, corresponds to a solution of the periodic boundary value problem of the Lagrangian system (3.5) with periodic boundary condition (3.7). Fix such an extremal loop $x$ and define

$$\begin{cases} P(t) & = L_{p,p}(t,x(t),\dot{x}(t)), \\ Q(t) & = L_{x,p}(t,x(t),\dot{x}(t)), \\ G(t) & = L_{x,x}(t,x(t),\dot{x}(t)) \end{cases} \tag{3.9}$$

where $L_x$ and $L_p$ denote the corresponding gradients. Note that in this case, $P, Q, G$ are periodic. The Hessian of $F$ at critical point $x$ is given by

$$\langle D^2F(x)y, z\rangle = \int_0^1 ((P\dot{y} + Qy)\cdot\dot{z} + Q^T\dot{y}\cdot z + Gy\cdot z)dt$$

for all $y$ and $z$ in tangent space of $\Psi_R$. Therefore the linearized system of (3.5), (3.7) at $x$ is given by the **Sturm system**:

$$-(P\dot{y} + Qy) + Q^T\dot{y} + Gy = 0 \tag{3.10}$$

Denote the Morse index of $F$ at $x$ by $m^-(x)$, i.e., the total multiplicities of all the negative eigenvalues of $D^2F(x)$.

The critical point $x$ gives rise to a periodic solution $(x,y)$ of Hamiltonian system (3.4) with periodic boundary condition, where $y(t) = \frac{\partial L(t,x,\dot{x})}{\partial p}$. The associate symplectic path $\gamma_x$ is the fundamental solution of the linearized Hamiltonian system along the periodic solution.

**Lemma 3.1** (Theorem 7.3.1 of ([8])). Under the above conditions, the Morse index $m^-(x)$ is equal to $i_1(\gamma_x)$.

Similarly, the Morse index of the Lagrangian system (3.5) with boundary value condition (3.8) is defined by $\Lambda(\omega)$ for $\omega \in U$, i.e. the total multiplicities of all the negative eigenvalues of $D^2F$ at solution of (3.5), (3.8). In particular, $\Lambda(1)$ is the Morse index of periodic solution and $\Lambda(-1)$ is the Morse index of anti-periodic solution. Y. Long proved the following lemma in ([9]) of 1999.

**Lemma 3.2** (Theorem 7.3.4 of ([8])). Under the above conditions, the Morse index $\Lambda(\omega)$ is equal to $i_\omega(\gamma_x)$.
Then we can restate the necessary and sufficient condition for stability of a periodic solution $x$ of a Hamiltonian system in terms of the Morse index.
**Theorem 3.4** If either $\Lambda(1)$ is even, or $\Lambda(-1)$ is odd, then the system (3.4) is linear unstable.
**Theorem 3.5** $\Lambda(1)+\Lambda(-1)$ is odd if and only if the system (3.4) is linear stable.
This theorem coincides with corollary 4.1, theorem 4.1 of ([3]).

For higher dimensional problems, that is when $n>1$, there is much less that can be stated for linear systems with periodic coefficients in these simple terms. But for $n=2$, some instability and stability results can be determined by parity of index. Those results can be applied to study the stability of N-body problem such as isosceles three body problem (described by Daniel Offin ([5]) and figure eight solution ([2],[7]).

# 4 Stability and $Sp(4)$

For any $M\in Sp(2n)$, it has a unique polar decomposition $M=AU$. We denote by

$$U(M)=\begin{pmatrix} u_1(M) & -u_2(M)\\ u_2(M) & u_1(M)\end{pmatrix}$$

the orthogonal and symplectic part of its unique polar decomposition. Then $u(M)=u_1(M)+\sqrt{-1}u_2(M)\in U(n,\mathbb{C})$. So in such a way, for every path $\gamma\in\mathcal{P}_\tau(2n)$ we can uniquely associate to it a path,

$$u_\gamma(t)=u(\gamma(t)),\qquad \forall t\in[0,\tau]$$

in the unitary group $U(n,\mathbb{C})$. For any $\gamma\in C([0,\tau],Sp(2n))$, let $\triangle:[0,\tau]\to\mathbb{R}$ be any continuous real function satisfying

$$\det u_\gamma(t)=\exp(\sqrt{-1}\triangle(t)),\qquad \forall t\in[0,\tau]. \tag{4.1}$$

We define the rotation number of $\gamma$ by

$$\triangle_\tau(\gamma)=\triangle(\tau)-\triangle(0).$$

Then $\triangle_\tau(\gamma)$ depends only on $\gamma$ but not on the choice of the function $\triangle$ satisfying (4.1).
For any $\omega\in U$ and $\gamma\in\mathcal{P}^*_{\tau,\omega}(2n)$, by Theorem 2.4.1 in ([8]) we can connect $\gamma(\tau)$ to $M_n^+$ or $M_n^-$ by a path $\beta:[0,\tau]\to Sp(2n)^*_\omega$. Under these conditions, define

$$k\equiv\triangle_\tau(\beta*\gamma)/\pi. \tag{4.2}$$

Then $k$ is an integer; is independent of the choice of the path $\beta$, and

$$\begin{cases} k & \text{is odd,}\quad \text{if } \beta(\tau)=M_n^-,\\ k & \text{is even,}\quad \text{if } \beta(\tau)=M_n^+,\end{cases} \tag{4.3}$$

We denote by $\mathcal{P}^k_{\tau,\omega}(2n)$ the set of all such paths in $\mathcal{P}^*_{\tau,\omega}(2n)$, that possess the property (4.2).
**Definition 4.1** For any $\omega\in U$ and $\tau>0$, we define the index of a symplectic path $\gamma$ by

$$i_\omega(\gamma)=k,\qquad \text{if } \gamma\in\mathcal{P}^k_{\tau,\omega}(2n). \tag{4.4}$$

**Lemma 4.1** For $M \in Sp(4)$, its characteristic polynomial has the form

$$f(\lambda) = \lambda^4 - 4A\lambda^3 + B\lambda^2 - 4A\lambda + 1 \tag{4.5}$$

where $A = \frac{\mathrm{tr}(M)}{4}, B = \frac{1}{2}((\mathrm{tr}(M))^2 - \mathrm{tr}(M^2))$. Then

$1^o$ (4.5) possesses one pair of conjugate double roots if and only if $B = 4A^2 + 2$.

$2^o$ $+1$ is one root of (4.5) if and only if $B = 8A - 2$.

$3^o$ $-1$ is one root of (4.5) if and only if $B = -8A - 2$.

$4^o$ $i = \sqrt{-1}$ (or $-i = -\sqrt{-1}$) is one root of (4.5) if and only if $B = 2$.

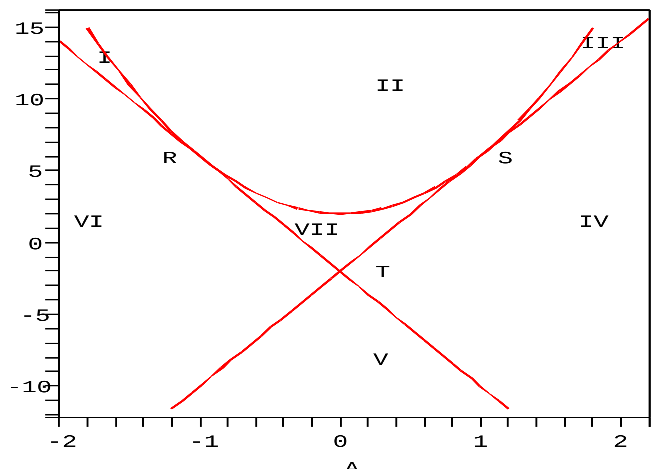
**Proof.** By direct computation. $\natural$

The Figure 4 in $AB$-plane is obtained as follows.

$\langle 1 \rangle$ Let $R$ be the tangent point of the line $B = -8A - 2$ to the curve $B = 4A^2 + 2$. Let $S$ be the tangent point of the line $B = 8A - 2$ to the curve $B = 4A^2 + 2$. Denote by $T$ the intersection point of the line $B = -8A - 2$ and the line $B = 8A - 2$. It is easy to check that $R = (-1, 6), S = (1, 6), T = (0, -2)$.

$\langle 2 \rangle$ We denote by $I$ the open region bounded by the curve $B = 4A^2 + 2$ from left to $R$ and the line $B = -8A - 2$ from left to $R$. Other open regions are illustrated as in Figure 4.

Figure4:AB-Plane




$\langle 3 \rangle$ Note that for any $M \in Sp(4)$, there corresponds a unique point $(A, B)$ in the $AB-$plane, where $A, B$ are the parameters in the characteristic polynomial of M in (4.5). If $(A, B)$ is in region $I$, we may say the symplectic matrix $M$ is in region $I$ and similarly for other regions.

Recall for any $\omega \in U, M \in Sp(2n)$, the $\omega-$nullity $\nu_\omega(M) = \dim_C \ker_C(M - \omega I)$ and $\sigma(M)$ is the set of eigenvalues of $M$. We define the *hyperbolic index* $\alpha(M)$ of $M$ b the mod 2 number of the total algebraic multiplicity of negative eigenvalues of $M$ which are strictly less than $-1$, and the *elliptic height* $e(M)$ of $M$ by the total algebraic multiplicity of all eigenvalues of $M$ on $U$. An $M \in Sp(2n)$is

$$\begin{array}{cl} \text{truly hyperbolic ,} & \text{if } e(M) = 0, \\ \text{hyperbolic ,} & \text{if } 1 \in \sigma(M) \text{ and } e(M) = 2, \\ \text{elliptic ,} & \text{if } e(M) = 2n, \\ \text{strongly elliptic ,} & \text{if } \sigma(M) \subset U \backslash \{1\}. \end{array}$$

We denote by $Sp^{th}(2n), Sp^h(2n), Sp^e(2n)$, and $Sp^{se}(2n)$ the subsets of all truly hyperbolic, hyperbolic, elliptic, strongly elliptic matrices in $Sp(2n)$ respectively.
We need the following
**Lemma 4.2** The truly hyperbolic matrices $Sp^{th}(4)$ in $Sp(4)$ consist of those symplectic matrix in the regions $I, II, III, V$ and the curve $B = 4A^2 + 2$ from left to $R$ and then from $S$ to right. Further more, $Sp^{th}(4)_1$ is the region $V$ and $Sp^th(4)_0 = Sp^th(4)\backslash Sp^{th}(t)_1$ is the region $I, II, III$ and the curve $B = 4A^2 + 2$ from left to $R$ and then from $S$ to right.

**Lemma 4.3** The singular set $Sp(4)_1^0$ is the line $B = 8A - 2$. The singular set $Sp(4)_{-1}^0$ is the line $B = -8A - 2$. The singular set $Sp(4)_i^0$ is the line $B = 2$.
**Lemma 4.4** The regular set $Sp(4)_1^+$ consist of those symplectic matrix in the regions $I, II, III, VI, VII$ and those curves left to the line $B = 8A - 2$. Or equivalently, $Sp(4)_1^+$ is the left half plane to the line $B = 8A - 2$. $Sp(4)_1^-$ is the right half plane to the line $B = 8A - 2$.
**Proof.** $Sp(4)_1^+ = \{M|D_1(M) =$

**Theorem 4.1** For any $\gamma \in \mathcal{P}_{\tau,1}^*(4)$, the index $i_1(\gamma)$ is odd if and only if $\gamma(\tau) \in Sp(4)_1^-$. Equivalently, the index $i_1(\gamma)$ is even if and only if $\gamma(\tau) \in Sp(4)_1^+$.
**Theorem 4.2** Suppose $x$ is a $\tau-$periodic solution of the Hamiltonian system and $\gamma_x$ is the associated symplectic path of $x$. If $\gamma_x \in \mathcal{P}_{\tau,1}^*(4)$ and $i_1(\gamma_x)$ is an odd integer, then the periodic solution $x$ is linear unstable.
**Theorem 4.3** Suppose $x$ is a $\tau-$periodic solution of the Hamiltonian system and $\gamma_x$ is the associated symplectic path of $x$. If $\gamma_x \in \mathcal{P}_{\tau,-1}^*(4)$ and $i_{-1}(\gamma_x)$ is an odd integer, then the periodic solution $x$ is linear unstable.
**Theorem 4.4** Suppose $x$ is a $\tau-$periodic solution of the Hamiltonian system and $\gamma_x$ is the associated symplectic path of $x$. If $\gamma_x \in \mathcal{P}_{\tau,1}^*(4) \bigcap \mathcal{P}_{\tau,i}^*(4) \bigcap \mathcal{P}_{\tau,-1}^*(4)$ and $i_1(\gamma_x)$ is an even integer, $i_{-1}(\gamma_x)$ is an even integer and $i_i(\gamma_x)$ is an odd integer, then the periodic solution $x$ is linear stable.

# 5 Application in Isosceles Three Body Problem

**5.1 The Statement of Isosceles Three Body Problem**
The $N$-body problem configuration $q = (q_1, \cdots, q_N)$ describes spatial positions of $N$ masses $(m_1, \cdots, m_N)$. Interaction between the masses is determined by the Newtonian potential function $V = -\sum_{1\le k<j\le N} \frac{m_k m_j}{|q_k - q_j|}$ on the set of noncollision configurations(where $q_i \neq q_j, i \neq j$). The Hamiltonian for the N-body problem is the sum of kinetic plus potential

$$H(q_1, \cdots, q_N, p_1, \cdots, p_N) = \sum_{1\le i\le N} \frac{1}{2m_i}|p_i|^2 - \sum_{1\le j<i\le N} \frac{m_i m_j}{|x_i - x_j|} \tag{5.1}$$

We study the Hamiltonian *N-body equations of motion*

$$\dot{q}_i = \frac{1}{m_i} p_i, \qquad \dot{p}_i = \frac{\partial U(\mathbf{q})}{\partial q_i}, \qquad i = 1, 2, \cdots, N, \tag{5.2}$$

where the function $U(q) = -V(q)$ is called the force function.
The isosceles three body problem can be described as the special motions of the three body problem whose triangular configurations always describe an isosceles triangle (Wintner (1941)). It is known that this can only occur if two of the masses are the same, and the third mass lies

on the symmetry axis described by the binary pair. the symmetry axis can be fixed or rotating. We will consider below the case where the symmetry axis is fixed.

Configurations of the isosceles problem require simple constraints: we shall assume that $m_1 = m_2 = m$ and that if $e_1, e_2, e_3$ denote the standard orthogonal unit vectors of $\mathbb{R}^3$,

$$\sum m_i q_i = 0, \qquad \langle q_1 - q_2, e_3 \rangle = 0, \qquad \langle q_1 + q_2, e_i \rangle = 0, i = 1, 2 \tag{5.3}$$

We consider the collision-free configuration manifold $M_{iso}$,

$$M_{iso} = \{q = (q_1, q_2, q_3) | q_i \neq q_j, q \text{ satisfies (5.3)}\} \tag{5.4}$$

We restrict the potential $V$ to the manifold $M_{iso}$. Counting dimensions we see that $M_{iso}$ is three dimensional. When all three masses lie in the horizontal plane, the third mass must be at the origin and the three masses aree collinear. The set of collinear configurations is two dimensional, and will be denoted by $\mathcal{S}$.
We will use cylindrical coordinates $(r, \theta, z)$ on the manifold $M_{iso}$, where $z$ denotes the vertical height of the mass $m_3$ above the horizontal plane, and $(r, \theta)$ denotes the horizontal position of mass $m_1$, relative to the axis of symmetry. In these coordinates with corresponding momenta $(p_r, p_\theta, p_z)$, the Hamiltonian is

$$H = \frac{p_r^2}{4m} + \frac{p_\theta^2}{4mr^2} + \frac{p_z^2}{2m_3(\frac{m_3}{2m} + 1)} - \frac{m^2}{2r} - \frac{2mm_3}{\sqrt{r^2 + z^2(\frac{m_3}{2m} + 1)^2}} \tag{5.5}$$

Because $\theta$ is ignorant variable in $H$, $p_\theta$ must be a constant along any flow of the Hamiltonian vector field. Setting $p_\theta = c$ and substituting in equation (5.5) gives $H_c = H(r, 0, z, p_r, c, p_z)$ and the reduced Hamiltonian vector field is

$$\dot{r} = \frac{\partial H_c}{\partial p_r}, \qquad \dot{z} = \frac{\partial H_c}{\partial p_z}, \qquad \dot{p}_r = \frac{\partial H_c}{\partial r}, \qquad \dot{p}_z = \frac{\partial H_c}{\partial z}. \tag{5.6}$$

**5.2 Variational Method**

Professor Ouyang, could you please write this part? How do you set up the problem and what results can you get? Thank you.

**5.3 Stability of periodic solutions of isosceles three problem**

From 5.2, we get the initial positions and velocities which lead a periodic solution of isosceles three body problem. Here are two examples:

$$M_1 = \begin{bmatrix} 1.0240, & 0, & 0, & 0.0008, & 1.0486, & -0.2182, & 1 \\ -1.0240, & 0, & 0, & -0.0008, & -1.0486, & -0.2182, & 1 \\ 0, & 0, & 0, & 0.0000, & 0.0000, & 0.4364, & 1 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0.744965690370485, & 0, & 0, & 0, & 0.733358921776412, & -0.673423272793242, & 1 \\ -0.744965690370485, & 0, & 0, & 0, & -0.733358921776412, & -0.673423272793242, & 1 \\ 0, & 0, & 0, & 0, & 0, & 1.34684654558649, & 1 \end{bmatrix},$$

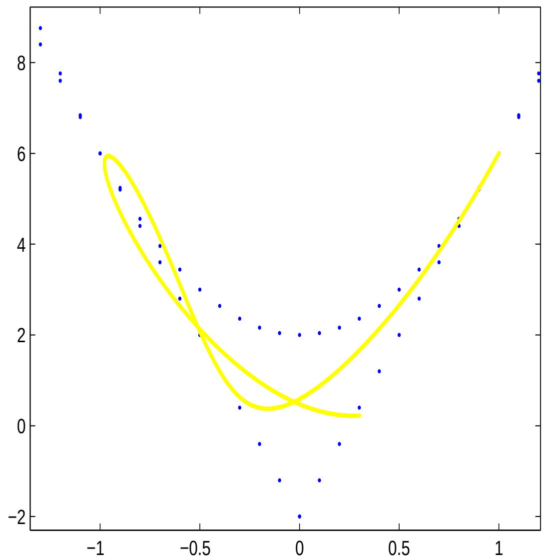
where $M_i (i = 1, 2)$ is a $3 \times 7$ matrix. Each row of $M_i$ represents the initial position, the initial velocity and mass for one body.

Converting the initial data to polar coordinates, $M_1$ becomes $PM_1(r, z, p_r, p_z) = [1.0240, 0, 0.0016, 0.6546]$ with constant $p_\theta = 2.1475$ and $M_2$ becomes $PM_2(r, z, p_r, p_z) = [0.7450, 0, 0, 2.0203]$ with $p_\theta = 1.0927$.

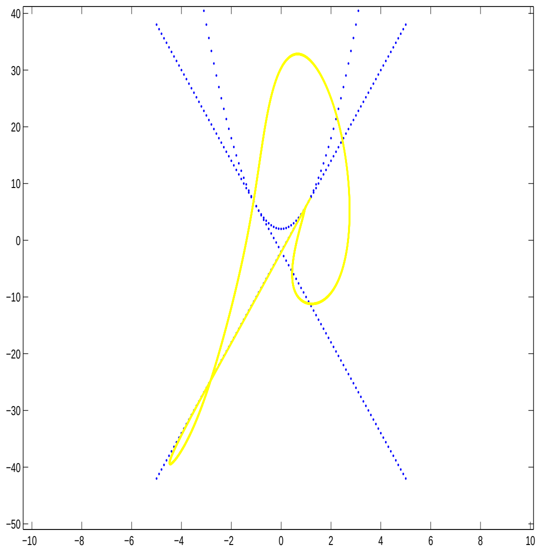
**Method 1:**
Let $m = m_3 = 1$. The solution of ordinary differential equation (5.6) with initial point $PM_i(r, z, p_r, p_z), i = 1, 2$ is a periodic solution, called $x_i(t)$ with $x_i(0) = PM_i$. Then the fundamental matrix path $\gamma_i(t)$ of the linearization for the periodic solution is a symplectic path in $Sp(4)$ with $\gamma_i(0) = I$. Because $\gamma_i(T)$ has two eigenvalues 1, (4.5) implies $B = 8A - 2$ and

$$f_{\gamma_i(T)}(\lambda) = (\lambda^2 - 1)(\lambda^2 + (2 - 4A)\lambda + 1).$$

So the solution $x_i(t)$ is linear stable if and only if $(2-4A)^2-4 = 16A(A-1) < 0$,i.e. $0 < A < 1$, where $A = \frac{tr(\gamma_i(T))}{4}$.
For $x_1$, $0 < A_1 = 0.3021 < 1$, then it is linear stable.




For $x_2$,$1 < A_2 = 1.1708$, then it is linear unstable.




**Method 2:** Application of Index theory.

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