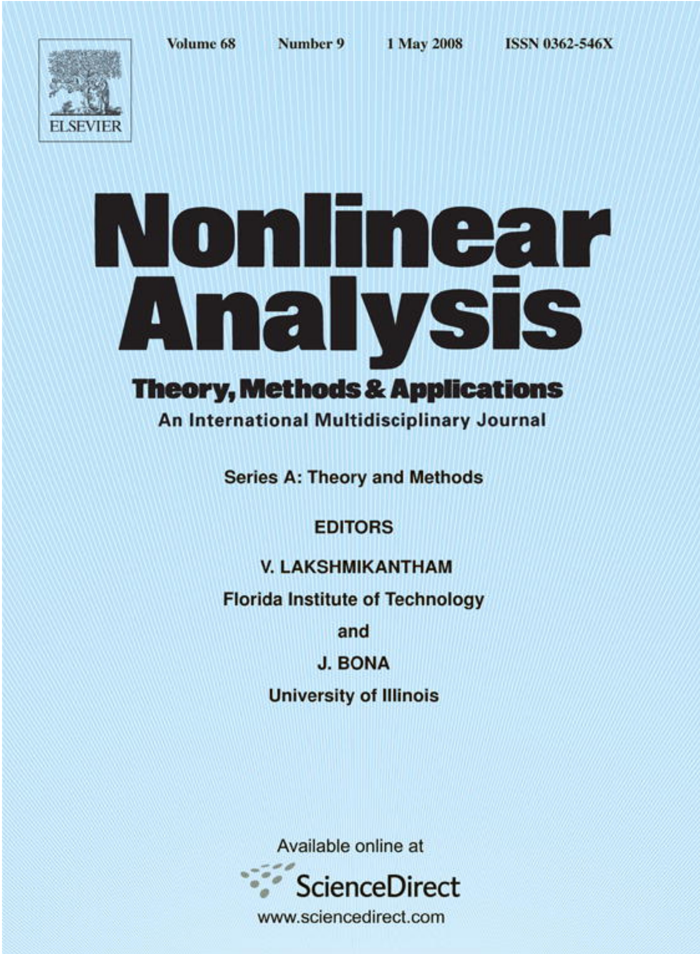
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The exact boundary blow-up rate of large solutions for semilinear elliptic problems

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Abstract

In this paper, we establish the blow-up rate of the large positive solution of the singular boundary value problem $-\triangle u = \lambda u - a(x)u^p$, $u|_{\partial\Omega} = +\infty$, where Ω is a bounded smooth domain in $\mathbb{R}^N$. The weight function $a(x)$ in front of the nonlinearity can vanish on the boundary of the domain Ω at different rates according to the point x_0 of the boundary. The decay rate of the weight function $a(x)$ may not be approximated by a power function of distance near the boundary $\partial\Omega$. We combine the localization method of [J. López-Gómez, The boundary blow-up rate of large solutions, J. Differential Equations 195 (2003) 25–45] with some previous radially symmetric results of [T. Ouyang, Z. Xie, The uniqueness of blow-up solution for radially symmetric semilinear elliptic equation, Nonlinear Anal. 64 (9) (2006) 2129–2142] to prove that any large solution $u(x)$ must satisfy

$$\lim_{x\to x_0} \frac{u(x)}{K(b^*_{x_0}(\mathrm{dist}(x,\partial\Omega)))^{-\beta}} = 1 \quad \text{for each } x_0 \in \partial\Omega,$$

where

$$b^*_{x_0}(r) = \int_0^r \int_0^s b_{x_0}(t)\mathrm{d}t\mathrm{d}s, \qquad K = \left[\beta((\beta+1)C_0 - 1)\right]^{\frac{1}{p-1}},$$

$$\beta = \frac{1}{p-1}, \qquad C_0 = \lim_{r\to 0} \frac{\left(\int_0^r b_{x_0}(t)\mathrm{d}t\right)^2}{b^*_{x_0}(r)b_{x_0}(r)}$$

and $b_{x_0}(r)$ is the boundary normal section of $a(x)$ at $x_0 \in \partial\Omega$, i.e.,

$$b_{x_0}(r) = a(x_0 - r\mathbf{n}_{x_0}), \quad r > 0, r\sim 0,$$

and $\mathbf{n}_{x_0}$ stands for the outward unit normal vector at $x_0 \in \partial\Omega$.

Keywords: Semilinear elliptic equation; Uniqueness; Blow-up rate; Large positive solution

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1. Introduction and main results

This paper continues the studies of the uniqueness of the large solution to the singular boundary value problem

$$\begin{cases} -\Delta u = \lambda(x)u - a(x)u^p & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with boundary $\partial\Omega$ of class C^2, $\lambda \in L_\infty(\Omega)$, $p > 1$ and $a \in C(\Omega; \mathbb{R}^+)$, $\mathbb{R}^+ := (0, +\infty)$. The boundary condition in (1.1) is understood as $u(x) \to +\infty$ when $d(x) := \operatorname{dist}(x, \partial\Omega) \to 0^+$. The behavior of the potential function of the nonlinear term $a(x)$ approaching $\partial\Omega$ is closely related to the blow-up rate of the solution of (1.1). In the case of $a(x) \geq a_0 > 0$ in $\bar{\Omega}$, many different types of equations are studied in [1,2,10]. For the case of $a(x) \sim C_0 d^\gamma + o(d^\gamma)$ as x goes to $\partial\Omega$, blow-up rates and uniqueness are studied in [7,9].

The singular boundary value problem (1.1) arises naturally from a number of different areas and has a long history. Considerable amounts of study have been inspired by such problems. The blow-up rate of solution near $\partial\Omega$ and uniqueness of solutions for the singular problem (1.1) are the goal of more recent literature (see [7–17] and the references therein). In the 1990's, a different type of singular boundary value problem $-\Delta u = a(x)e^u$ with $u|_{\partial\Omega} = +\infty$ was studied in [10] (1993) and [1] (1994). It is shown that the problem exhibits a unique solution in a smooth domain together with an estimate of the form $u = \log d^{-2} + o(d)$ in [10] (where $a(x) \geq a_0 > 0$ as $d \to 0$) and in [1] (where $a \equiv 1$). In [11], the general problem $\Delta u(x) = f(u(x))$ with $u|_{\partial\Omega} = +\infty$ is considered. An asymptotic result for solutions of the above problem is proved under some assumptions on f. It is assumed that Ω is a bounded domain which satisfies a uniform internal sphere condition and a uniform external sphere condition, and f is a C^1 function which is either defined and positive on $(-\infty, \infty)$ or is defined on a ray $[c, \infty)$ with $f(c) = 0$ and $f(s) > 0$ for $s > c$. It is further assumed that $f'(s) \geq 0$ for s in the domain of f, and that there exists c_1 such that $f'(s)$ is nondecreasing for $s \geq c_1$. The result of [11] is that if

$$\lim_{s\to\infty} \frac{f'(s)}{\sqrt{F(s)}} = \infty$$

where $F(s) = \int_0^s f(t)\mathrm{d}t$, $F(s) > 0$, then the problem has a unique solution $u(x)$ and, moreover,

$$u(x) - Z(d(x)) \to 0 \quad \text{as } d(x) \to 0,$$

where Z is a solution on an interval $(0, b)$, $b > 0$, of the equation $Z''(r) = f(Z(r))$ and such that $Z(r) \to \infty$ as $r \to 0^+$.

More recently, the uniqueness and blow-up rates of $-\Delta u = \lambda(x)u - a(x)u^p$ were treated in [2,7,9,13,17] as well as other work. In [2] (1998) for the case of $-\Delta u = -u^p$, it is shown that the blow-up rate of the solution near the boundary is $u = Ad^{-\alpha}(1 + o(d))$, $\alpha = 2/(p-1)$. Furthermore, the possible presence of a second explosive term in the expansion of u can be obtained when $\alpha > 1 (p < 3)$.

Under the assumption $a(x) = C_0 d^\gamma + o(d^\gamma)$ as $d \to 0^+$ with $\gamma > 0$ and $C_0 > 0$, an explicit expression for the blow-up rates of solutions to $-\Delta u = \lambda(x)u - a(x)u^p$ has been recently proved in [7] (1999) and [9] (2001) as $u = (\alpha(\alpha+1)/C_0)^{1/(p-1)} d^{-\alpha}(1 + o(d))$, $\alpha = (\gamma+2)/(p-1)$. In [9], an explicit expression is obtained for this second term as $u(x) = Ad^{-\alpha}(1 + B(s)d + o(d))$ when $d \to 0^+$ where $B(s) = ((n-1)H(s) - (\alpha+1)C_1)/(\gamma+p+3)$ with $H(s)$ standing for the mean curvature of $\partial\Omega$ at s. All these results are substantially generalized in [5,14,17].

In [5] the blow-up rate of the large positive solution is established for a special case when $a(x)$ decays to 0 on $\partial\Omega$ at a fixed rate along the entire $\partial\Omega$. Assume that $a(x) = a(d(x))$ as $d(x) \to 0$ with $\sqrt{a} \in \mathcal{K}$, where $\mathcal{K}$ is the set of all positive increasing C^1-functions defined on $[0, R]$ such that

$$l_0 := \lim_{r\to 0} \frac{\int_0^r \sqrt{a(s)}\mathrm{d}s}{\sqrt{a(r)}} = 0, \qquad l_1 := \lim_{r\to 0} \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\int_0^r \sqrt{a(s)}\mathrm{d}s}{\sqrt{a(r)}}\right) \in [0, 1].$$

Then problem (1.1) has a unique positive large solution $u(x)$ and, moreover,

$$\lim_{d(x)\to 0} \frac{u(x)}{\xi_0 h(d(x))} = 1, \quad \text{where } \xi_0 = \left(\frac{2 + l_1(p-1)}{p+1}\right)^{\frac{1}{p-1}}$$

and h is defined by the unique solution to the integral equation

$$\int_{h(t)}^{\infty} \frac{\mathrm{d}s}{\sqrt{2\int_0^s u^p \mathrm{d}u}} = \int_0^t \sqrt{a(s)}\mathrm{d}s, \quad \forall t \in (0, R].$$

There was also refined the blow-up rate of $u(x)$ near $\partial\Omega$, by giving the second term in the expansion of $u(x)$ near $\partial\Omega$, in [6] (2003).

In [13], a localization method is developed to establish the blow-up rate of the large positive solutions of singular boundary problem (1.1) with the weight function $a(x)$ that vanishes on the boundary $\partial\Omega$ at different rates according to the point of the boundary and further assuming that $a(x)$ can be approximated by a power of a distance function, i.e. $\lim_{x\to x_\infty} a(x)\beta^{-1}(x_\infty)[\mathrm{dist}(x, \partial\Omega)]^{-\gamma(x_\infty)} = 1$ for $x_\infty \in \partial\Omega$.

In [14], a rather explicit boundary blow-up rate is established for the large solution of (1.1) in a radially symmetric domain with a radial weight function a. Assume that $a \in C[0, \infty)$ satisfies $a(0) = 0$, $a(t) \geq a(s) > 0$ if $t \geq s > 0$, and

$$\lim_{t\to 0} \frac{A(t)A''(t)}{[A'(t)]^2} = I_0 \in (0, \infty),$$

where

$$A(t) := \int_t^{\infty} \left(\int_0^s f^{\frac{1}{p+1}}\right)^{-\frac{p+1}{p-1}} \mathrm{d}s, \quad t > 0.$$

Then problem (1.1) has a unique positive large solution $u(x)$ and, moreover,

$$\lim_{t\to 0} \frac{u(x)}{A(d(x))} = I_0^{-\frac{p}{p-1}} \left(\frac{p+1}{p-1}\right)^{\frac{p+1}{p-1}}.$$

In [17], the present authors established the blow-up rate of the positive large solution to (1.1) with a ball domain and a radial weight function $a(x)$ which is more general without assuming the decay rate of $a(x)$ to be approximated by a power of the distance function near the boundary $\partial\Omega$. Suppose $a(x) = a(d(x))$ as $d(x) \to 0$ such that $a(r)$ is a positive continuous function defined on $[0, R]$ with $a(0) = 0$ and

$$\lim_{r\to 0} \frac{\int_0^r \int_0^t a(s)\mathrm{d}s\mathrm{d}t}{\int_0^r a(s)\mathrm{d}s} = \lim_{r\to 0} \frac{\int_0^r a(s)\mathrm{d}s}{a(r)} = 0, \qquad C_0 := \lim_{r\to 0} \frac{\left(\int_0^r a(s)\mathrm{d}s\right)^2}{a(r)\int_0^r \int_0^t a(s)\mathrm{d}s\mathrm{d}t} \geq 1.$$

Note that under these conditions it is not necessary that $a(r)$ is a nondecreasing function near the boundary. Here is one example: $a(r) = r(2 + 3r\sin r^{-1})$; then $a'(r) = 2 + 6r\sin\left(r^{-1}\right) - 3\cos\left(r^{-1}\right)$. As $r \to 0^+$, $a'(r)$ oscillates between -1 and 1. On the other hand,

$$0 \leq \frac{\int_0^r s\left(2 + 3s\sin\frac{1}{s}\right)\mathrm{d}s}{r\left(2 + 3r\sin\frac{1}{r}\right)} \leq \frac{\int_0^r s(2+3s)\mathrm{d}s}{r\left(2 + 3r\sin\frac{1}{r}\right)} \leq \frac{r(r(2+3r))}{r\left(2 + 3r\sin\frac{1}{r}\right)} \to 0, \quad \text{as } r \to 0^+.$$

Similarly we can check that $a(r)$ satisfies the conditions above and $C_0 = \frac{3}{2}$. Thus theorem in [17] works for this oscillating weight function $a(r)$ which is not previously covered in [4,5,14], etc. The main result in [17] is restated here as Theorem 2.5 in Section 2.

In this paper we will produce sharper results in a general domain by combining the localization method of [13] with the result of [17]. Now we turn to stating the main results of this paper more precisely.

Consider the singular boundary value problem

$$\begin{cases} -\Delta u = \lambda u - a(x)u^p & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\lambda \in \mathbb{R}$, Ω is a bounded smooth domain in $\mathbb{R}^N$, and the weight function $a(x) > 0$ in Ω. We also assume that the following conditions on $a(x)$ are satisfied. For each $x \in \partial\Omega$, we define the boundary normal sections $b_x(r)$ as

$$b_x(r) = a(x - r\mathbf{n}_x), \quad r \geq 0, r \sim 0. \tag{1.3}$$

For any $x_0 \in \partial\Omega$, suppose there exists $\tau > 0$, such that $a(x) \in C^1(\bar{B}_\tau(x_0) \bigcap \bar{\Omega})$ and

$$b_{x_0}(r) \in C^1(0, \tau), \qquad b'_{x_0}(r) > 0 \quad \text{for each } t \in (0, \tau) \tag{1.4}$$

and

$$\lim_{x \in \partial\Omega, x \to x_0, r \to 0^+} \frac{b_x(r)}{b_{x_0}(r)} = 1. \tag{1.5}$$

Furthermore, let $B_{x_0}(r) = \int_0^r b_{x_0}(s)\mathrm{d}s$, $b^*_{x_0}(r) = \int_0^r B_{x_0}(s)\mathrm{d}s$. We assume that $\lim_{r\to 0} \frac{B_{x_0}(r)}{b_{x_0}(r)} = 0$ and

$$C_0 = \lim_{r \to 0} \frac{(B_{x_0}(r))^2}{b^*_{x_0}(r) b_{x_0}(r)} \geq 1.$$

Our main result is

Theorem 1.1. *For each $x_0 \in \partial\Omega$, any positive solution u of (1.2) satisfies*

$$\lim_{r \to 0} \frac{u(x_0 - r\mathbf{n}_{x_0})}{K(b^*_{x_0}(r))^{-\beta}} = 1, \tag{1.6}$$

where K is a constant defined by

$$K = [\beta((\beta + 1)C_0 - 1)]^{\frac{1}{p-1}}, \qquad \beta := \frac{1}{p-1}.$$

Moreover, if the condition (1.5) is uniformly satisfied on $\partial\Omega$, then for any positive solution u of (1.2),

$$\lim_{r \to 0} \frac{u(x_0 - r\mathbf{n}_{x_0})}{K(b^*_{x_0}(r))^{-\beta}} = 1, \quad \text{uniformly for } x_0 \in \partial\Omega. \tag{1.7}$$

Therefore, the problem (1.2) possesses a unique positive solution $u(x)$ in Ω.

Remark. (1) For any $x_0 \in \partial\Omega$, if there exist $\beta(=\beta(x_0)) > 0, \gamma(=\gamma(x_0)) \geq 0$ such that $\lim_{x \to x_0} a(x)\beta^{-1}(\mathrm{dist}(x, \partial\Omega))^{-\gamma} = 1$, then $b_{x_0}(r) = \beta_{x_0} r^{\gamma_{x_0}}$ in $(0, \tau)$. Under these assumptions, the theorem of López-Gómez ([13], 2003) is an immediate consequence of Theorem 1.1.

(2) Theorem 1.1 can be applied to other functions $a(x)$ which may not be approximated by a power function of distance. Given $x_0 \in \partial\Omega$, for example, let $b_{x_0}(r) = \exp(-r^{-2})$, $B_{x_0}(r) = \int_0^r b_{x_0}(s)\mathrm{d}s$, $b^*_{x_0}(r) = \int_0^r \int_0^s b_{x_0}(t)\mathrm{d}t\mathrm{d}s$; then $C_0 = 1$, $K = [\beta((\beta + 1) - 1)]^\beta$, $\beta := 1/(p - 1)$. Therefore, the unique solution $u \sim K(b^*_{x_0})^{-\beta}$ when x approaches x_0 goes to ∞ near the boundary faster than any power function.

2. Some preliminary results

In this section we collect some important comparison results which will be used in the proof of Theorem 1.1. As an immediate consequence from the papers [3,7,13,9,17], Theorems 2.1–2.6 were proved and Theorems 2.2–2.4 are borrowed from [13].

Consider the problem

$$\begin{cases} -\Delta u = \lambda(x)u - b(x)u^p & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain with smooth boundary, $\phi \in C(\partial\Omega)$, $p > 1$ and $b \in C(\Omega, \mathbb{R}^+)$.

Theorem 2.1 ([7], Lemma 2.1). *Let $\underline{u}, \overline{u} \in C^2(\bar{\Omega})$ both be positive in $\bar{\Omega}$ such that*

$$-\triangle \underline{u} \le \lambda(x)\underline{u} - b(x)\underline{u}^p \quad \text{in } \Omega,$$
$$-\triangle \overline{u} \ge \lambda(x)\overline{u} - b(x)\overline{u}^p \quad \text{in } \Omega.$$

If $\underline{u} \le \overline{u}$ on $\partial\Omega$, then $\underline{u}(x) \le \overline{u}(x)$ on $\bar{\Omega}$.

Definition. If $\underline{u}$ (resp. $\overline{u}$) satisfies the conditions in Theorem 2.1 and $\underline{u} \le \phi$ on $\partial\Omega$ (resp. $\overline{u} \ge \phi$), then $\underline{u}$ (resp. $\overline{u}$) is called a subsolution (resp. supersolution) of (2.1).

Theorem 2.2. *Suppose $\phi \in C(\partial\Omega)$ and (2.1) possesses a nonnegative solution. Let u be any nonnegative solution of (2.1). Then $u(x) > 0$ for each $x \in \Omega$ and $\partial_{\mathbf{n}} u(x) < 0$ for any $x \in \partial\Omega$ such that $u(x) = 0$; $\mathbf{n}$ stands for the outward unit normal to Ω. Moreover, the positive solution is unique and if we denote it by Ψ and $\underline{u}$ (resp. $\overline{u}$) is a nonnegative subsolution (resp. supersolution) of (2.1) then $\underline{u} \le \Psi$ (resp. $\Psi \le \overline{u}$).*

Theorem 2.3. *If $\underline{u}, \overline{u} \in C^2(\Omega)$ are both positive in Ω such that*

$$-\triangle \underline{u} \le \lambda(x)\underline{u} - b(x)\underline{u}^p \quad \text{in } \Omega,$$
$$-\triangle \overline{u} \ge \lambda(x)\overline{u} - b(x)\overline{u}^p \quad \text{in } \Omega,$$
$$\lim_{\mathrm{dist}(x,\partial\Omega)\to 0} \underline{u}(x) = \infty, \qquad \lim_{\mathrm{dist}(x,\partial\Omega)\to 0} \overline{u}(x) = \infty,$$

and $\underline{u}(x) \le \overline{u}(x)$ in Ω, then there exists at least one solution $u \in C^2(\Omega)$ of (1.1) satisfying $\underline{u}(x) \le u \le \overline{u}(x)$ in Ω.

Theorem 2.4. *Suppose (1.1) possesses a nonnegative solution, say Ψ; then the problem (2.1) possesses a unique nonnegative solution for each $\phi \in C(\partial\Omega, R^+)$ denoted by u_ϕ and $u_\phi \le \Psi$ in Ω. Furthermore,*

$$\Psi_L := \limsup_{\inf_{\partial\Omega} \phi \to \infty} u_\phi$$

provides us with the minimal positive solution of (1.1).

Theorem 2.5 ([17], Theorem 1). *Consider the radially symmetric semilinear elliptic equation*

$$\begin{cases} -\triangle u = \lambda u - b(r)u^p & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $\Omega = B_R(x_0)$ is the ball of radius R centered at x_0 and $r = R - \|x - x_0\| = \mathrm{dist}(x, \partial B_R(x_0))$. $\lambda \in \mathbb{R}$, $b \in C([0, R]; [0, \infty))$ satisfying $b > 0$ in $(0, R]$. Let $B(r) = \int_0^r b(s)\mathrm{d}s$ and $b^*(r) = \int_0^r B(s)\mathrm{d}s$. Suppose

$$\lim_{r\to 0} \frac{B(r)}{b(r)} = 0 \tag{2.3}$$

$$C_0 = \lim_{r\to 0} \frac{(B(r))^2}{b^*(r)b(r)}. \tag{2.4}$$

Then there exists a unique solution u satisfying

$$\lim_{d(x)\to 0} \frac{u(x)}{K(b^*(r))^{-\beta}} = 1 \tag{2.5}$$

where $d(x) = \mathrm{dist}(x, \partial B_R(x_0))$ and K and β are constants defined by

$$K = [\beta((\beta+1)C_0 - 1)]^{\frac{1}{p-1}}, \qquad \beta := \frac{1}{p-1}. \tag{2.6}$$

As an immediate consequence from Theorem 2.5, combining a translation together with a reflection about

$$r_0 := \frac{R_1 + R_2}{2},$$

it readily follows that the corresponding result can be proved in each of the annuli

$$A_{R_1,R_2}(x_0) := \{x \in \mathbb{R}^N : 0 < R_1 < \|x - x_0\| < R_2\}.$$

Theorem 2.6. *Consider the problem*

$$\begin{cases} -\Delta u = \lambda u - b(r)u^p & \text{in } A_{R_1,R_2}(x_0), \\ u = \infty & \text{on } \partial A_{R_1,R_2}(x_0), \end{cases} \tag{2.7}$$

where $\lambda \in \mathbb{R}$, $r = \mathrm{dist}(x, \partial A_{R_1,R_2}(x_0))$, *and* $R = \frac{R_2 - R_1}{2}$; $b \in C([0, R]; [0, \infty))$ *satisfying* $b > 0$ *in* $(0, R]$. *Let* $B(r) = \int_0^r b(s)\mathrm{d}s$ *and* $b^*(r) = \int_0^r B(s)\mathrm{d}s$. *Suppose* (2.3) *and* (2.4) *are satisfied. Then there exists a unique solution* u *satisfying*

$$\lim_{x \to \partial A_{R_1,R_2}(x_0)} \frac{u(x)}{K(b^*(r))^{-\beta}} = 1$$

where K *and* β *are defined in* (2.6).

3. Proof of Theorem 1.1

Let u be a large positive solution of (1.2). We first construct a large supersolution locally for each $x_0 \in \partial\Omega$.

For a sufficiently small $\epsilon > 0$, thanks to (1.5), there exist $\rho = \rho(\epsilon) \in (0, \tau)$ and $\mu = \mu(\epsilon) > 0$ such that

$$1 - \epsilon < \frac{b_x(r)}{b_{x_0}(r)} = \frac{a(x - r\mathbf{n}_x)}{a(x_0 - r\mathbf{n}_{x_0})} < 1 + \epsilon, \tag{3.1}$$

for any $x \in \partial\Omega \bigcap \bar{B}_\rho(x_0)$, $r \in [0, \mu]$. Let

$$\mathcal{B} = \left\{x - r\mathbf{n}_x | (x, r) \in \left[\partial\Omega \bigcap \bar{B}_\rho(x_0)\right] \times [0, \mu]\right\}. \tag{3.2}$$

Because $\partial\Omega$ is smooth, ρ, μ can be shortened, if necessary, so that for each $y \in \mathcal{B}$ there exists a unique $(\pi(y), r(y)) \in [\partial\Omega \bigcap \bar{B}_\rho(x_0)] \times [0, \mu]$ and $y = \pi(y) - r(y)\mathbf{n}_{\pi(y)}$ and $r(y) = |y - \pi(y)| = \mathrm{dist}(y, \partial\Omega)$. From now on, we assume that ρ, μ satisfy these requirements. Furthermore, there exists $R_0 \in (0, \min\{\frac{\rho}{2}, \frac{\mu}{2}\})$ such that

$$B_{R_0}(x_0 - R_0\mathbf{n}_{x_0}) \subset \Omega \bigcap \mathrm{Int}\,\mathcal{B} \quad \text{and} \quad \bar{B}_{R_0}(x_0 - R_0\mathbf{n}_{x_0}) \bigcap \partial\Omega = \{x_0\}. \tag{3.3}$$

Then for any $\delta \in [0, \delta_0]$ (δ_0 is chosen to be very small), the family of the small ball is in $\Omega \bigcap \mathrm{Int}\,\mathcal{B}$ and

$$B_{R_0-\delta}(x_0 - R_0\mathbf{n}_{x_0}) \subset \bar{B}_{R_0}(x_0 - R_0\mathbf{n}_{x_0}).$$

By (1.4) and (1.5), for each $\delta \in [0, \delta_0]$ and $y \in \bar{B}_{R_0-\delta}(x_0 - R_0)$, we have

$$\begin{aligned} a(y) &= a(\pi(y) - r(y)\mathbf{n}_{\pi(y)}) > (1 - \epsilon)a(x_0 - r(y)\mathbf{n}_{x_0}) \\ &= (1 - \epsilon)b_{x_0}(r(y)) = (1 - \epsilon)b_{x_0}(\mathrm{dist}(y, \partial\Omega)) \\ &\geq (1 - \epsilon)b_{x_0}(\mathrm{dist}(y, \partial B_{R_0-\delta}(x_0 - R_0\mathbf{n}_{x_0}))). \end{aligned} \tag{3.4}$$

Therefore, for all $y \in \bar{B}_{R_0-\delta}(x_0 - R_0\mathbf{n}_{x_0})$,

$$a(y) \geq (1 - \epsilon)b_{x_0}(r_\delta), \tag{3.5}$$

where $r_\delta = (R_0 - \delta) - \|y - x_0 + R_0\mathbf{n}_{x_0}\| = \mathrm{dist}(y, \partial B_{R_0-\delta}(x_0 - R_0\mathbf{n}_{x_0}))$.

Thanks to (3.5), for any $\delta \in [0, \delta_0]$, the restriction

$$\underline{u}_\delta := u|_{B_{R_0-\delta}}(x_0 - R_0\mathbf{n}_{x_0}) \tag{3.6}$$

is a positive smooth subsolution of

$$\begin{cases} -\Delta u = \lambda u - (1 - \epsilon)b_{x_0}(r)u^p & \text{in } B_{R_0-\delta}(x_0 - R_0\mathbf{n}_{x_0}), \\ u = \infty & \text{on } \partial B_{R_0-\delta}(x_0 - R_0\mathbf{n}_{x_0}). \end{cases} \tag{3.7}$$

Thanks to Theorem 2.5, for each $\delta \in [0, \delta_0]$, there exists a unique solution Φ_δ of (3.7) satisfying

$$\lim_{x \to \partial B_{R_0-\delta}(x_0 - R_0 \mathbf{n}_{x_0})} \frac{\Phi_\delta(x)}{K((1-\epsilon) b^*_{x_0}(r_\delta))^{-\beta}} = 1 \tag{3.8}$$

where $r_\delta = \operatorname{dist}(x, \partial B_{R_0-\delta}(x_0 - R_0 \mathbf{n}_{x_0}))$ and

$$b^*_{x_0}(r_\delta) = \int_0^{r_\delta} \int_s^{r_\delta} b_{x_0}(t) \mathrm{d}t \mathrm{d}s.$$

K and β are given by

$$K = [\beta((\beta+1)C_0 - 1)]^{\frac{1}{p-1}}, \qquad \beta := \frac{1}{p-1}$$

and C_0 is defined as

$$C_0 = \lim_{r_\delta \to R_0} \frac{(B_{x_0}(r_\delta))^2}{b^*_{x_0}(r_\delta) b_{x_0}(r_\delta)}.$$

For each $\delta \in [0, \delta_0]$, Theorem 2.4 guarantees

$$\underline{u}_\delta = u|_{B_{R_0-\delta}}(x_0 - R_0 \mathbf{n}_{x_0}) \le \Phi_\delta$$

for x in $B_{R_0-\delta}(x_0 - R_0 \mathbf{n}_{x_0})$. Thus

$$\limsup_{x \to \partial B_{R_0-\delta}(x_0 - R_0 \mathbf{n}_{x_0})} \frac{u_\delta(x)}{K((1-\epsilon) b^*_{x_0}(r_\delta))^{-\beta}} \le 1, \tag{3.9}$$

and passing to the limit as $\delta \to 0$ gives

$$\limsup_{r \to 0} \frac{u(x_0 - r \mathbf{n}_{x_0})}{K((1-\epsilon) b^*_{x_0}(r))^{-\beta}} \le 1. \tag{3.10}$$

In particular, (3.10) is valid for any sufficiently small $\epsilon > 0$; then

$$\limsup_{r \to 0} \frac{u(x_0 - r \mathbf{n}_{x_0})}{K(b^*_{x_0}(r))^{-\beta}} \le 1. \tag{3.11}$$

To prove (1.6), we will build up a large subsolution having adequate growth at $x_0 \in \partial \Omega$ so that we can show

$$1 \le \limsup_{r \to 0} \frac{u(x_0 - r \mathbf{n}_{x_0})}{K(b^*_{x_0}(r))^{-\beta}}. \tag{3.12}$$

Since Ω has a smooth boundary, for any $x_0 \in \partial \Omega$, there exist $R_2 > R_1 > 0$ and $\delta_0 > 0$ such that

$$\Omega \subset \bigcap_{\delta \in [0, \delta_0]} A_{R_1-\delta, R_2-\delta}(x_0 + R_1 \mathbf{n}_{x_0})$$

and

$$\partial \Omega \bigcap \bar{A}_{R_1, R_2}(x_0 + R_1 \mathbf{n}_{x_0}) = \{x_0\}.$$

Moreover, R_2 can be taken arbitrarily large. We suppose R_2 has been chosen to satisfy

$$\Omega \subset A_{R_1, \frac{R_2}{3}}(x_0 + R_1 \mathbf{n}_{x_0})$$

and R_1 can be taken arbitrarily small.

Fix a sufficiently small $\epsilon > 0$ and $x_0 \in \partial \Omega$. Thanks to (3.1), we have

$$1 - \epsilon < \frac{b_x(r)}{b_{x_0}(r)} = \frac{a(x - r \mathbf{n}_x)}{a(x_0 - r \mathbf{n}_{x_0})} < 1 + \epsilon, \tag{3.13}$$

for any $x \in \partial\Omega \bigcap \bar{B}_\rho(x_0)$, $r \in [0, \mu]$. Pick up a small $2\eta < \min\{\rho, \mu\}$; for each $y \in B_{2\eta}(x_0) \bigcap \bar{\Omega}$,

$$\begin{aligned} a(y) &= a(\pi(y) - r(y)\mathbf{n}_{\pi(y)}) \le (1+\epsilon)b_{x_0}(r(y)) \\ &= (1+\epsilon)b_{x_0}(\operatorname{dist}(y, \partial\Omega)) \\ &\le (1+\epsilon)b_{x_0}(\operatorname{dist}(y, \partial B_{R_1}(x_0 + R_1\mathbf{n}_{x_0}))) \\ &= (1+\epsilon)b_{x_0}(\operatorname{dist}(y, \partial A_{R_1,R_2}(x_0 + R_1\mathbf{n}_{x_0}))). \end{aligned} \tag{3.14}$$

Thus we can construct a radially symmetric function

$$\hat{a} : A_{R_1,R_2}(x_0 + R_1\mathbf{n}_{x_0}) \mapsto [0, \infty),$$

such that

$$\hat{a} \ge a \quad \text{in } \Omega, \tag{3.15}$$

by extending the function

$$\hat{a}(y) = \hat{a}(r) = (1+\epsilon)b_{x_0}(r), \tag{3.16}$$

where

$$r = \operatorname{dist}(y, \partial A_{R_1,R_2}(x_0 + R_1\mathbf{n}_{x_0})) \quad \text{and} \quad y \in B_{2\eta}(x_0) \bigcap \bar{\Omega}.$$

Further, for $y \in \bar{\Omega}$, $\hat{a}$ also satisfies

$$\hat{a}(r_\delta) = \hat{a}(\operatorname{dist}(y, \partial A_{R_1-\delta,R_2-\delta}(x_0 + R_1\mathbf{n}_{x_0}))) \ge a(y)$$

for a sufficiently small δ_0 and any $0 \le \delta \le \delta_0$.

For each sufficiently small $\delta > 0$, $\delta \in (0, \delta_0]$, consider the auxiliary problem

$$\begin{cases} -\Delta u = \lambda u - \hat{a}(r)u^p & \text{in } A_{R_1-\delta,R_2-\delta}(x_0 + R_1\mathbf{n}_{x_0}), \\ u = \infty & \text{on } \partial A_{R_1-\delta,R_2-\delta}(x_0 + R_1\mathbf{n}_{x_0}), \end{cases} \tag{3.17}$$

where $r = \operatorname{dist}(x, \partial A_{R_1-\delta,R_2-\delta}(x_0 + R_1\mathbf{n}_{x_0}))$.

From Theorem 2.6, there exists a unique large positive solution $\Phi_{\epsilon,\delta}$ such that

$$\lim_{x \to \partial A_{R_1-\delta,R_2-\delta}(x_0+R_1\mathbf{n}_{x_0})} \frac{\Phi_{\epsilon,\delta}(x)}{K((1+\epsilon)b^*_{x_0}(r))^{-\beta}} = 1$$

where $b^*_{x_0}(r)$ and K are defined as before. Moreover, by construction, the restriction $\Phi_{\epsilon,\delta}|_\Omega$ provides us with a subsolution of (1.2). Thus, for each $\delta \in (0, \delta_0]$, we have

$$\Phi_{\epsilon,\delta}(x) \le u(x)$$

for each $x \in A_{R_1-\delta,R_2-\delta}(x_0 + R_1\mathbf{n}_{x_0}) \bigcap \Omega$; then

$$\lim_{r \to 0} \frac{u(x_0 - r\mathbf{n}_{x_0})}{K((1+\epsilon)b^*_{x_0}(r))^{-\beta}} \ge 1. \tag{3.18}$$

By passing to $\mu \to 0$, this readily gives

$$\liminf_{x \to x_0, x \in A_{R_1,R_2}(x_0+R_1\mathbf{n}_{x_0}) \bigcap \Omega} \frac{u(x)}{K(b^*_{x_0}(r))^{-\beta}} \ge 1. \tag{3.19}$$

So (3.12) and (3.19) conclude the proof of (1.6).

If the conditions (1.3), (1.4) and (3.5) are satisfied uniformly on $\partial\Omega$, to prove that (1.7) is satisfied uniformly on $\partial\Omega$, we may check the proof above, whether it is true uniformly. It is clear that ρ, τ can be chosen small enough so that (3.1) is true for each $x, x_0 \in \partial\Omega$ sufficiently close to each other. R_0 and R_1 can be chosen to be the same for each

$x_0 \in \partial\Omega$. Also note that the limit of $b^*_{x_0}(r)$ is independent of the choice of δ in (3.9) and (3.18). Therefore

$$\lim_{r\to 0}\frac{u(x_0 - r\mathbf{n}_{x_0})}{K(b^*_{x_0}(r))^{-\beta}} = 1, \quad \text{uniformly in } x_0 \in \partial\Omega.$$

Proof of the uniqueness. The proof of the uniqueness basically follows the proof in [7,9,13]. For any pair of solutions u, v of (1.2),

$$\lim_{\operatorname{dist}(y,\partial\Omega)\to 0}\frac{u(y)}{v(y)} = \lim_{r(y)\to 0}\frac{u(\pi(y) - r(y)\mathbf{n}_{\pi(y)})}{v(\pi(y) - r(y)\mathbf{n}_{\pi(y)})} = 1, \quad \text{uniformly on } \partial\Omega.$$

Thus, for every $\epsilon > 0$, we can find $\delta > 0$ (as small as we please) such that

$$(1-\epsilon)v(x) \le u(x) \le (1+\epsilon)v(x), \quad x \in \Omega \setminus \Omega_\delta,$$

where, for each small enough $\delta > 0$, we have defined

$$\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}.$$

On the other hand $\underline{w} = (1-\epsilon)v(x)$ and $\bar{w} = (1+\epsilon)v(x)$ are a subsolution and a supersolution to

$$\begin{cases} -\triangle w = \lambda w - a w^p & \text{in } \Omega_\delta, \\ w = u & \text{on } \partial\Omega_\delta. \end{cases} \tag{3.20}$$

The unique solution to this problem is $w = u$. Then by Theorem 2.1

$$(1-\epsilon)v(x) \le u(x) \le (1+\epsilon)v(x)$$

holds in Ω_δ; therefore it is true in Ω. Letting $\epsilon \to 0$ we arrive at $u = v$.

Finally, thanks to the uniqueness, it follows from the abstract existence theory of [12] that the problem (1.2) possesses a unique positive solution. This concludes the proof of the theorem. ♯

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