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REGULARIZATION OF SIMULTANEOUS BINARY COLLISIONS AND SOLUTIONS WITH SINGULARITY IN THE COLLINEAR FOUR-BODY PROBLEM

Tiancheng Ouyang

Department of Mathematics
Brigham Young University
Provo, Utah 84602, USA

Zhifu Xie

Department of Mathematics and Computer Science
Virginia State University
Petersburg, Virginia 23806, USA

Dedicated to Professor Peter Bates on the occasion of his sixtieth birthday

Abstract. The purpose of this paper is to analyze the asymptotic properties of collision orbits of Newtonian N-body problems. We construct new coordinates and time transformation that regularize the singularities of simultaneous binary collisions in the collinear four-body problem. The motion in the new coordinates and time scale across simultaneous binary collisions at least C^2. The explicit formulae are given in detail for the transformations and the extension of solutions. Furthermore, we study the behaviors of the motion approaching, across and after the simultaneous binary collision. Numerical simulations have been conducted for the special case in which the bodies are distributed symmetrically about the center of mass.

1. **Introduction.** In this paper we analyze the collision orbits of Newtonian N-body problems. It is well known that binary collisions are regularizable but general triple collisions are nonregularizable ([8]). Triple collisions can be regularizable in some special, highly symmetrical case. To study the behavior of the motion of the particles in a neighborhood of a collision, we usually make a change of coordinates and of time scale. If, in the new coordinates, the orbits which approach collision can be extended across the collision in a smooth manner with respect to the new time scale, we say that the collision orbits have been regularized. The regularization is of class $C^n, n \geq 0$, or analytic if each collision orbit of the transformed differential equations is C^n or analytic, respectively, as a function of the new time scale in a neighborhood of collision. The regularization here is on the extension of each individual collision orbit across collision. We will refer to this as regularization with respect to time. This type of regularization goes back, in particular, to Sundman ([12]) in his studies of collisions in the three-body problem (see also ([9])).

In 1906 Sundman ([12]) showed that any solution of the three body problem that ends in a single binary collision as $t \to 0$ can be written as a convergent power series

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in a new time variable $s = t^{1/3}$. Thus the solution can be analytically continued in the complex t plane to an ejection solution in a time reverse manner.

Belbruno ([2]) proved that the simultaneous binary collision (SBC) can be C^1 regularized with respect to time in collinear four-body problem, provided the masses of the four bodies are suitably restricted, i.e., in the following two cases: (1) $\{m_i > 0 | m_1 = m_4, m_2 = m_3\}$; (2) $\{m_i > 0 | m_1 = m_3, m_2 = m_4\}$.

Saari ([11]) studied the nature of simultaneous binary collision orbits (collisions occur at $t = 0$). After the change of the independent variable $s = t^{1/3}$, the collision points will be "blown up". He showed that for any number of simultaneous binary collisions the singularity is removable and the solution of each of the particles can be expanded in a power series of $t^{1/3}$ which converges in a neighborhood of $t = 0$.

ElBialy ([4]) studied simultaneous binary collision singularities of many binaries in the planar N-body problem. He introduced a generalized Levi-Civita transformation and composed with it by a new transformation (he called the projective transformation) near a SBC singularity. He showed that the collection of collision and ejection orbits together with the singularity form a real analytic submanifold. Each collision orbit corresponds to a unique ejection orbit. He also studied the asymptotic behavior of SBC orbits near collision. In particular, it was proved that the energy of each collision pair converges as fast as $t^{4/3}$ on a collision orbit.

It is worthwhile mentioning that the above concept of regularization refers to the extension of each individual collision orbit itself across collision. Another analytical question concerning regularization is that of the smoothness of the flow with respect to initial conditions in a neighborhood of a collision orbit. This defines a different type of regularization if the flow also varies smoothly with respect to initial conditions in a neighborhood of a collision orbit. We will refer to this as regularization with respect to initial data. This type of regularization was first studied by Easton ([3]), and later by many other people, see Martinez and Simo ([5]), McGehee ([8]) etc. Many other works can be found from the reference of these papers.

The main goal of this paper is to give a detailed proof of the regularization of the simultaneous binary collisions with respect to time in the case of the collinear four-body problem. We construct new coordinate transforms in new time scale of the Levi-Civita-like time, which is not the physical time scale (like in ([11], [4]). The equations of motion are rewritten in the new set of variables in the new time scale. Then the singularity of simultaneous binary collisions can be removed in the new equations for any masses. The total energy plays an essential role in the proof. Furthermore, we will analyze the asymptotic properties of the SBC orbits (before and approaching collision), the natural extensions of the SBC orbits (after collision) as well as their applications in constructing orbits involving single binary collisions and simultaneous binary collisions. All transformation and formulae are given explicitly with detailed computation.

In order to make the precise main statements of the paper, we recall some definitions. Let $x_k \in \mathbb{R}$, $k = 1, 2, 3, 4$, denote the position of k-th body on the line with mass $m_k > 0$. Assume, without loss of generality, that $x_1 \leq x_2 \leq x_3 \leq x_4$ and gravitational constant one. The motion of the four bodies under the influence of the Newtonian gravitational force law is described by the following set of ordinary differential equations:

$$m_k \frac{d^2 x_k}{dt^2} = \frac{\partial U}{\partial x_k}, \qquad k = 1, 2, 3, 4, \tag{1}$$

where U is the potential function,

$$U = \sum_{1 \le j < i \le 4} \frac{m_i m_j}{|x_i - x_j|}. \tag{2}$$

The total energy is defined by

$$H = \sum_{1 \le i \le 4} \frac{1}{2} m_i |\dot{x}_i|^2 - \sum_{1 \le j < i \le 4} \frac{m_i m_j}{|x_i - x_j|}. \tag{3}$$

We call the space of $x = (x_1, \cdots, x_4) \in \mathbb{R}^4$ the space of positions. Let $\triangle_{ij} := \{x \in \mathbb{R}^4, x_i = x_j\}$. This set corresponds to where the ith and the jth particles collide. Let $\triangle := \bigcup_{1 \le j < i \le 4} \triangle_{ij}$. The potential function U, and consequently equation (1) are singular on $\triangle$.

Let $x(t) = (x_1(t), \cdots, x_4(t))$ be a solution of equation (1) defined on $[t_1, t_2)$, and assume that $x(t) \to L = (L_1, \cdots, L_4)$ as $t \to t_2^-$. We say that $x(t)$ has a singularity of collision at $t = t_2$ if $L \in \triangle$. According to the locations of L in $\triangle$, the singularities of collision are divided into the categories of (I) binary collisions, (II) simultaneous binary collisions, (III) triple collisions and (IV) four-body (total) collisions. In this paper, we study a solution with singularity of *simultaneous binary collision(SBC)*, that is, the limit L of the position satisfies $-\infty < L_1 = L_2 < L_3 = L_4 < \infty$, i.e., x_1, x_2 form a collision pair and x_3, x_4 form a collision pair. Let us denote the set of L satisfying these restrictions as $\bigwedge$.

The object of this paper is to prove the following theorems.

Theorem 1.1. *Any simultaneous binary collision orbit of the collinear four body problem can be extended at least C^2 across $\bigwedge$ with respect to the new time scale, after a change of coordinates and time scale.*

Theorem 1.2. *Near a simultaneous binary collision singularity, the following are true:*

(1) Simultaneous binary collisions must occur in such a way that the two pairs of particles each collide symmetrically with respect to the origin.

(2) The velocities of the particles in each collision pair are inversely proportional to their masses and are in opposite directions, which is independent of their initial velocities and positions.

(3) When the orbits approach simultaneous binary collision, the ratio of the distances in each collision pair is determined by their mass ratio, i.e.

$$\lim_{t \to t_2^-} \frac{x_2 - x_1}{x_4 - x_3} = \left(\frac{m_1 + m_2}{m_3 + m_4} \right)^{1/3}.$$

Theorem 1.3. *Assume that $x^0 = (-s-1, -1, 1, s+1)$ with masses $(m, 1, 1, m)$ is the initial position. If s, m fall into the region of $\frac{s^2(s+2)^2}{16(1+s)} < m$ in the first quadrant of sm-plane, then the orbit obtained by releasing the four bodies with zero velocity at x^0 has at least one simultaneous binary collision before a single binary collision between m_2 and m_3.*

Theorem 1.4. *Assume that $x^0 = (-s_0 - 1, -1, 1, s_0 + 1)$ with mass $(m, 1, 1, m)$ form a central configuration.*

(1) There exists a sequence $\{\tilde{s}_n\}$ such that $0 < \tilde{s}_1 < \tilde{s}_2 < \cdots < \tilde{s}_n < \cdots < s_0$. If the motion starts with initial position $y^0 = (-s_0 + s_n - 1, -1, 1, s_0 - s_n + 1)$ and

zero initial velocity, where $\tilde{s}_{n-1} < s_n < \tilde{s}_n < s_0$, *then the motion has exactly* n *discrete moments where four bodies undergo simultaneous binary collision before a single binary collision between* m_2 *and* m_3 *in* $[0, T]$, *where* T *corresponds to the first time of single binary collision.*

(2) If four particles are released from $y^0 = (-s_0 + \tilde{s}_n - 1, -1, 1, s_0 - \tilde{s}_n + 1)$ *with zero initial velocity,* $n = 1, 2, \cdots$, *then the motion has a total collision after* n *times simultaneous binary collisions.*

(3) There exists a sequence $\{s_n^*\}$ *such that* $0 < s_1^* < s_2^* < \cdots < s_n^* < \infty$. *If the motion starts with initial position* $y^0 = (-s_0 - s_n - 1, -1, 1, s_0 + s_n + 1)$ *and zero initial velocity, where* $s_{n-1}^* < s_n < s_n^*$, *then the motion has exactly* n *discrete moments where masses* m_2, m_3 *undergo single binary collision before a simultaneous binary collision in* $[0, T]$, *where* T *corresponds to the first time of simultaneous binary collision.*

(4)If four particles are released from $y^0 = (-s_0 - s_n^* - 1, -1, 1, s_0 + s_n^* + 1)$ *with zero initial velocity,* $n = 1, 2, \cdots$, *then the motion has a total collision after* n *times single binary collisions between* m_2 *and* m_3.

The rest of the paper is organized as follows. In section 2, we give our first transformation to reduce the dimension and prove a very useful lemma. In section 3, we introduce the new coordinates and new time variables and we prove Theorem 1.1. In section 4, Theorem 1.2 and Theorem 1.3 are proved. In particular, we describe the asymptotic behavior of the SBC orbits and its extensions. In section 5, we further classify the solutions with SBC and single binary collisions for the special case in which the bodies are distributed symmetrically about the center of mass.

2. **Estimate of collision rate.**

Let us now consider equation (1) for the collinear four-body problem assuming $x_1 \le x_2 \le x_3 \le x_4$. Without loss of generality, we also put the center of mass at the origin which implies

$$\sum_{k=1}^{4} m_k x_k = 0. \tag{4}$$

Related to (4), we have

$$\sum_{k=1}^{4} m_k \frac{dx_k}{dt} = 0. \tag{5}$$

These help in cutting the dimension of the phase space down by two. Let

$$u_1 = x_2 - x_1, \quad u_2 = x_4 - x_3, \quad u_3 = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \tag{6}$$

and

$$K_{31} = \frac{1}{x_3 - x_1} = \frac{\omega_1}{\omega_2 u_1 - \omega_3 u_2 - \omega_4 u_3}, \tag{7}$$

$$K_{41} = \frac{1}{x_4 - x_1} = \frac{\omega_1}{\omega_2 u_1 + \omega_5 u_2 - \omega_4 u_3}, \tag{8}$$

$$K_{32} = \frac{1}{x_3 - x_2} = \frac{\omega_1}{-\omega_6 u_1 - \omega_3 u_2 - \omega_4 u_3}, \tag{9}$$

$$K_{42} = \frac{1}{x_4 - x_2} = \frac{\omega_1}{-\omega_6 u_1 + \omega_5 u_2 - \omega_4 u_3}, \tag{10}$$

where $\omega_1 = (m_1 + m_2)(m_3 + m_4), \omega_2 = m_2(m_3 + m_4), \omega_3 = m_4(m_1 + m_2), \omega_4 = (m_1 + m_2)(m_1 + m_2 + m_3 + m_4)$, $\omega_5 = m_3(m_1 + m_2)$ and $\omega_6 = m_1(m_3 + m_4)$.

Then equations (1) reduce to an ordinary differential equation system with six independent variables $\vec{p}_1 = (u_1, u_2, u_3, v_1, v_2, v_3)$,

$$\begin{aligned}
\frac{du_1}{dt} &= v_1, \quad \frac{dv_1}{dt} = -\frac{m_1 + m_2}{u_1^2} + m_3(K_{32}^2 - K_{31}^2) + m_4(K_{42}^2 - K_{41}^2);\\
\frac{du_2}{dt} &= v_2, \quad \frac{dv_2}{dt} = -\frac{m_3 + m_4}{u_2^2} + m_1(K_{31}^2 - K_{41}^2) + m_2(K_{32}^2 - K_{42}^2);\\
\frac{du_3}{dt} &= v_3, \quad \frac{dv_3}{dt} = \frac{m_1 m_3}{m_1 + m_2}K_{31}^2 + \frac{m_2 m_3}{m_1 + m_2}K_{32}^2 + \frac{m_1 m_4}{m_1 + m_2}K_{41}^2 + \frac{m_2 m_4}{m_1 + m_2}K_{42}^2.
\end{aligned} \tag{11}$$

$\vec{u} = (u_1, u_2, u_3) \in \mathbb{R}^2 \times \mathbb{R}^-$ is now the space of positions and $\bigwedge = \{u_1 = u_2 = 0, u_3 \in \mathbb{R}^-\}$ is the singular set for simultaneous binary collisions; $K_{ij}, i = 3, 4, j = 1, 2$ are bounded on the singular set $\bigwedge$. It can be verified that the total energy (3) becomes

$$\begin{aligned}
\hat{H} &= \frac{(\beta_1 v_1^2 + \beta_2 v_2^2 + \beta_3 v_3^2)}{2(m_1 + m_2)(m_3 + m4)}\\
&\quad - (\frac{m_1 m_2}{u_1} + \frac{m_3 m_4}{u_2} + m_1 m_3 K_{31} + m_1 m_4 K_{41} + m_2 m_3 K_{32} + m_2 m_4 K_{42})
\end{aligned} \tag{12}$$

where $\beta_1 = m_1 m_2(m_3 + m_4)$, $\beta_2 = m_3 m_4(m_1 + m_2)$, $\beta_3 = (m_1 + m_2)^2(m_1 + m_2 + m_3 + m_4)$.

Lemma 2.1. *Let $\vec{u} = \vec{u}(t)$, $t \in [t_1, t_2)$ denote a simultaneous binary collision orbit encountering $\bigwedge$, where $t = t_2$ corresponds to collision. Then*

$$\lim_{t\to t_2} \frac{du_k}{dt} = \lim_{t\to t_2} v_k(t) = \infty, \qquad k = 1, 2, \tag{13}$$

$$\lim_{t\to t_2} u_1(t)v_1^2(t) = 2(m_1 + m_2), \ \lim_{t\to t_2} u_2(t)v_2^2(t) = 2(m_3 + m_4), \tag{14}$$

$$\lim_{t\to t_2} u_1(t)v_1(t) = 0, \ \lim_{t\to t_2} u_2(t)v_2(t) = 0, \tag{15}$$

and

$$\lim_{t\to t_2} \frac{u_1(t)}{u_2(t)} = \alpha, \qquad \lim_{t\to t_2} \frac{v_1(t)}{v_2(t)} = \alpha, \tag{16}$$

where $\alpha = \left(\frac{m_1+m_2}{m_3+m_4}\right)^{\frac{1}{3}}$.

Proof of Lemma 2.1. The system (11) implies that

$$\frac{d^2u_1}{dt^2} = -\frac{m_1 + m_2}{u_1^2} + G_1, \qquad \frac{d^2u_2}{dt^2} = -\frac{m_3 + m_4}{u_2^2} + G_2, \tag{17}$$

where $G_1 = m_3(K_{32}^2 - K_{31}^2) + m_4(K_{42}^2 - K_{41}^2)$ and $G_2 = m_1(K_{31}^2 - K_{41}^2) + m_2(K_{32}^2 - K_{42}^2)$ are bounded when t is close to t_2. Multiplying (17) by $\frac{du_1}{dt}$, $\frac{du_2}{dt}$ respectively, we obtain

$$\left(\frac{du_1}{dt}\right)^2 = \frac{2(m_1 + m_2)}{u_1} + \tilde{G}_1, \qquad \left(\frac{du_2}{dt}\right)^2 = \frac{2(m_3 + m_4)}{u_2} + \tilde{G}_2,$$

where $\tilde{G}_1$ and $\tilde{G}_2$ are also bounded when t is close to t_2. Thus letting $u_k(t) \to 0$ as $t \to t_2, k = 1, 2$, we prove that $\lim_{t\to t_2} \frac{du_1}{dt} = \lim_{t\to t_2} v_1(t) = \infty$. and $\lim_{t\to t_2} \frac{du_2}{dt} = \lim_{t\to t_2} v_2(t) = \infty$. Multiplying by u_1, u_2 respectively, the above equations become

$$u_1 v_1^2 = 2(m_1 + m_2) + u_1\tilde{G}_1, \quad u_2 v_2^2 = 2(m_3 + m_4) + u_2\tilde{G}_2.$$

Then we have

$$\lim_{t\to t_2} u_1(t)v_1^2(t) = 2(m_1 + m_2), \quad \lim_{t\to t_2} u_2(t)v_2^2(t) = 2(m_3 + m_4).$$

Consequently,

$$\lim_{t\to t_2} u_1(t)v_1(t) = 0, \quad \lim_{t\to t_2} u_2(t)v_2(t) = 0.$$

By making use of the fact that both u_1, u_2 tend to 0 monotonically and the results above, we have $\lim_{t\to t_2} \frac{u_1(t)}{u_2(t)} = \alpha$, where $\alpha = \left(\frac{m1+m2}{m_3+m_4}\right)^{1/3}$. We also have

$$\lim_{t\to t_2} \frac{v_1(t)}{v_2(t)} = \lim_{t\to t_2} \frac{\dot{u}_1(t)}{\dot{u}_2(t)} = \lim_{t\to t_2} \frac{u_1(t)}{u_2(t)} = \alpha.$$

□

Remark 1. Lemma 2.1 and its proof are motivated by the work of Belbruno ([2]).

3. **The proof of Theorem 1.1.** Let $\vec{u} = \vec{u}(t)$ denote a simultaneous binary collision orbit encountering $\bigwedge$ when $t = t_2$, then in a sufficiently small open deleted neighborhood of $t = t_2$, $\vec{u}(t)$ performs no collisions (Belbruno [2]). Therefore we can assume $\vec{p}_1 = (u_1, \cdots, v_3)$ is a solution of (11) performing no collisions for $t \in [t_1, t_2)$ and performing a simultaneous binary collision when $t \to t_2^-$. We will construct coordinate transform with a new time scale τ, such that the orbit under the new coordinate can be regularized. Let $\delta > 1$ and $0 < \rho < 1$ be fixed. We only consider solutions of equations (11) in $\mathcal{U}_{\delta,\rho}$, where

$$\mathcal{U}_{\delta,\rho} = \{\vec{p}_1 \in (\mathbb{R})^{2+} \times \mathbb{R}^- \times \mathbb{R}^3 : u_1, u_2 < \rho; -\delta < u_3 < -\delta^{-1}\}.$$

We are now ready to introduce regularization variables by a Levi-Civita transformation and a time scale. Let $\vec{p}_2 = (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3)$ be the new phase variables with the new time variable as τ.

$$u_1 = \frac{\xi_1^2}{2}, u_2 = \frac{\xi_2^2}{2}, u_3 = -\frac{\xi_3^2}{2}, v_1 = \frac{\eta_1}{\xi_1}, v_2 = \frac{\eta_2}{\xi_2}, v_3 = \frac{\eta_3}{\xi_3}, \tag{18}$$

and rescale time by

$$dt = (\xi_1^2 + \xi_2^2)d\tau. \tag{19}$$

One verifies that (11) becomes

$$\frac{d\xi_1}{d\tau} = \frac{\xi_1^2 + \xi_2^2}{\xi_1^2}\eta_1, \tag{20}$$

$$\frac{d\xi_2}{d\tau} = \frac{\xi_1^2 + \xi_2^2}{\xi_2^2}\eta_2, \tag{21}$$

$$\frac{d\xi_3}{d\tau} = -\frac{(\xi_1^2 + \xi_2^2)}{\xi_3^2}\eta_3, \tag{22}$$

$$\begin{aligned}\frac{d\eta_1}{d\tau} =& \frac{(\eta_1^2 - 4(m_1 + m_2))}{\xi_1^2}\frac{\xi_1(\xi_1^2 + \xi_2^2)}{\xi_1^2} \\ &+ \left(m_3(K_{32}^2 - K_{31}^2) + m_4(K_{42}^2 - K_{41}^2)\right)\xi_1(\xi_1^2 + \xi_2^2),\end{aligned} \tag{23}$$

$$
\begin{aligned}
\frac{d\eta_2}{d\tau} =&\frac{(\eta_2^2-4(m_3+m_4))}{\xi_2^2}\frac{\xi_2(\xi_1^2+\xi_2^2)}{\xi_2^2} \\
&+\left(m_1(K_{31}^2-K_{41}^2)+m_2(K_{32}^2-K_{42}^2)\right)\xi_2(\xi_1^2+\xi_2^2),
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\frac{d\eta_3}{d\tau} =&\Big(\frac{m_1m_3}{m_1+m_2}K_{31}^2+\frac{m_2m_3}{m_1+m_2}K_{32}^2+\frac{m_1m_4}{m_1+m_2}K_{41}^2 \\
&+\frac{m_2m_4}{m_1+m_2}K_{42}^2\Big)\xi_3(\xi_1^2+\xi_2^2)+\frac{(\xi_1^2+\xi_2^2)}{\xi_3^3}\eta_3^2,
\end{aligned}
\tag{25}
$$

where $K_{31}, K_{32}, K_{41}, K_{42}$ are obtained by substituting (18) into (7)-(10), which are bounded and smooth on $\bigwedge$.

Derivations for equations (20) to (25): For the first equation (20) we differentiate $\xi_1^2 = 2u_1$ to obtain

$$
\frac{d\xi_1}{d\tau} = \frac{1}{\xi_1}\frac{du_1}{dt}\frac{dt}{d\tau} = \frac{1}{\xi_1}\frac{\eta_1}{\xi_1}(\xi_1^2+\xi_2^2) = \frac{\xi_1^2+\xi_2^2}{\xi_1^2}\eta_1,
$$

$$
\frac{d\eta_1}{d\tau} = \frac{d}{d\tau}(\xi_1 v_1) = \frac{dv_1}{d\tau}\xi_1 + v_1\frac{d\xi_1}{d\tau}
$$

$$
= \left(-\frac{4(m_1+m_2)}{\xi_1^4}+m_3(K_{32}^2-K_{31}^2)+m_4(K_{42}^2-K_{41}^2)\right)\xi_1(\xi_1^2+\xi_2^2)+\frac{\eta_1^2(\xi_1^2+\xi_2^2)}{\xi_1^3}
$$

$$
= \frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}\frac{\xi_1(\xi_1^2+\xi_2^2)}{\xi_1^2}+\left(m_3(K_{32}^2-K_{31}^2)+m_4(K_{42}^2-K_{41}^2)\right)\xi_1(\xi_1^2+\xi_2^2),
$$

$$
\begin{aligned}
\frac{d\eta_3}{d\tau} &= \xi_3\frac{dv_3}{dt}\frac{dt}{d\tau}+v_3\frac{d\xi_3}{d\tau} \\
&= \Big(\frac{m_1m_3}{m_1+m_2}K_{31}^2+\frac{m_2m_3}{m_1+m_2}K_{32}^2+\frac{m_1m_4}{m_1+m_2}K_{41}^2 \\
&\quad+\frac{m_2m_4}{m_1+m_2}K_{42}^2\Big)\xi_3(\xi_1^2+\xi_2^2)+\frac{(\xi_1^2+\xi_2^2)}{\xi_3^3}\eta_3^2.
\end{aligned}
$$

Other equations can be obtained in a similar way.

The energy function (12) becomes

$$
\begin{aligned}
\hat{H} =&\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}\frac{m_1m_2}{2(m_1+m_2)}+\frac{\eta_2^2-4(m_3+m_4))}{\xi_2^2}\frac{m_3m_4}{2(m_3+m_4)} \\
&+\frac{\eta_3^2}{\xi_3^2}\frac{\beta_3}{2(m_1+m_2)(m_3+m4)} \\
&-(m_1m_3K_{31}+m_1m_4K_{41}+m_2m_3K_{32}+m_2m_4K_{42}),
\end{aligned}
\tag{26}
$$

where $\beta_3 = (m_1+m_2)^2(m_1+m_2+m_3+m_4)$.

Remark 2. We choose $\xi_3 = -\sqrt{-2u_3}$ which is the negative branch of equation (18). The set $\{\xi_1 = \xi_2 = 0, \xi_3 < 0\}$ is the singular set corresponding to $\bigwedge$ the singular set for the simultaneous binary collisions. $K_{31}, K_{32}, K_{41}, K_{42}$ are bounded smooth functions on the singular set.

Let

$$
\mathcal{V}_{\delta,\rho} = \{\vec{p}_2 = (\xi_1,\xi_2,\xi_3,\eta_1,\eta_2,\eta_3) : \xi_1^2, \xi_2^2 < 2\rho, -(2\delta)^{1/2} < \xi_3 < -(2\delta^{-1})^{1/2}\}
$$

be the correspondence of $\mathcal{U}_{\delta,\rho}$ in phase space $\vec{p}_1$. We will study the solutions of (20)-(25) in $\mathcal{V}_{\delta,\rho}$.

Recall that $\vec{p}_1 = (u_1, u_2, u_3, v_1, v_2, v_3)$ is a solution in $\mathcal{U}_{\delta,\rho}$ defined in $[t_1, t_2)$ and assume that $\vec{p}_1 \to \bigwedge$ as $t \to t_2^-$. Let

$$\tau(t) = \tau_0 + \int_{t_1}^{t} \frac{1}{2(u_1(s) + u_2(s))} ds. \tag{27}$$

Theorem 3.1. *Let $\vec{p}_1(t)$, $t \in [t_1, t_2)$ be a solution of equation (11) in $\mathcal{U}_{\delta,\rho}$ with simultaneous binary collision at $t = t_2$, in other words, $\vec{p}_1(t) \to \bigwedge$ as $t \to t_2^-$. Let $\tau(t)$ be defined by (27) and $\vec{p}_2(\tau)$, $\tau \in [\tau_1, \tau_2)$ be the functions obtained from $\vec{p}_1(t)$ through (18). Then*
(1) $\vec{p}_2(\tau)$ is a solution of equations (20)-(25) in $[\tau_1, \tau_2)$;
(2) $\tau_2 := \tau(t_2) < \infty$, and $\vec{p}_2(\tau_2) := \lim_{\tau\to\tau_2} \vec{p}_2(\tau)$ is well defined; and
(3) the solution $\vec{p}_2$ of (20)-(25) can be at least C^1 smoothly extended through τ_2.

Proof. (1) This follows from the derivations of equations. We caution that (18) allows different ways to convert $\vec{p}_1(t)$ to $\vec{p}_2(\tau)$ because $\xi_k, k = 1, 2, 3$ can have different signs. This is a well known characteristic of Levi-Civita variables. For definiteness, let us choose the positive sign $\xi_k = \sqrt{2u_k}, k = 1, 2$ and negative sign $\xi_3 = -\sqrt{-2u_3}$.
(2) It is well know that when a collision singularity occurs at t_2,

$$u_1(t) + u_2(t) \sim (t - t_2)^{2/3}.$$

Then it follows that

$$\tau_2 = \tau_0 + \int_{t_1}^{t_2} \frac{1}{2(u_1(t) + u_2(t))} dt < \infty.$$

It is easy to show that $v_3(t)$ and $u_k(t) \to$ a definite limit as $t \to t_2^-$, which we denote by $v_3(t_2), u_k(t_2), k = 1, 2, 3$. Now for $\vec{p}_2(\tau_2)$: we let $\xi_k(\tau_2) = \sqrt{2u_k(t_2)}, k = 1, 2$ and $\xi_3(\tau_2) = -\sqrt{-2u_3(t_2)}, \eta_3(\tau_2) = \xi_3 v_3(t_2)$. By the assumption, $\xi_1(\tau_2) = 0, \xi_2(\tau_2) = 0$. From above, we have

$$\lim_{\tau\to\tau_2} \eta_1^2(\tau) = \lim_{t\to t_2} 2u_1 v_1^2 = 4(m_1 + m_2),$$

and

$$\lim_{\tau\to\tau_2} \eta_2^2(\tau) = \lim_{t\to t_2} 2u_2 v_2^2 = 4(m_3 + m_4),$$

from which it follows that $\eta_1(\tau_2) = -2\sqrt{m_1 + m_2}$, $\eta_2(\tau_2) = -2\sqrt{m_3 + m_4}$. They are negative because we have chosen positive sigh for $\xi_k(\tau_2), k = 1, 2$. Therefore $\vec{p}_2(\tau_2) := \lim_{\tau\to\tau_2} \vec{p}_2(\tau)$ is well defined.
Before we prove (3), we need the following Lemma 3.2.

Lemma 3.2. *Let $\vec{p}_1(t)$ be a solution of (2.8). $\vec{p}_2(\tau)$ is obtained from $\vec{p}_1(t)$ as above. Then*

$$\lim_{\tau\to\tau_2} \frac{\xi_1^2(\tau)}{\xi_2^2(\tau)} = \alpha, \tag{28}$$

$$\lim_{\tau\to\tau_2} \frac{\xi_1^2(\tau) + \xi_2^2(\tau)}{\xi_2^2(\tau)} = 1 + \alpha, \tag{29}$$

$$\lim_{\tau\to\tau_2} \frac{\xi_1^2(\tau) + \xi_2^2(\tau)}{\xi_1^2(\tau)} = 1 + \frac{1}{\alpha}, \tag{30}$$

$$\lim_{\tau\to\tau_2}\frac{\frac{\eta_1^2(\tau)}{m_1+m_2}}{\frac{\eta_2^2(\tau)}{m_3+m_4}}=1. \tag{31}$$

Proof of Lemma 3.2. By directional computation and Lemma 2.1, we can check that

$$\lim_{\tau\to\tau_2}\frac{\xi_1^2(\tau)}{\xi_2^2(\tau)}=\lim_{t\to t_2}\frac{u_1}{u_2}=\alpha,$$

$$\lim_{\tau\to\tau_2}\frac{\xi_1^2(\tau)+\xi_2^2(\tau)}{\xi_2^2(\tau)}=\lim_{\tau\to\tau_2}(1+\frac{\xi_1^2(\tau)}{\xi_2^2(\tau)})=1+\alpha,$$

$$\lim_{\tau\to\tau_2}\frac{\xi_1^2(\tau)+\xi_2^2(\tau)}{\xi_1^2(\tau)}=\lim_{\tau\to\tau_2}(1+\frac{\xi_2^2(\tau)}{\xi_1^2(\tau)})=1+\frac{1}{\alpha},$$

$$\lim_{\tau\to\tau_2}\frac{\frac{\eta_1^2(\tau)}{m_1+m_2}}{\frac{\eta_2^2(\tau)}{m_3+m_4}}=\lim_{t\to t_2}\frac{2u_1v_1^2(t)(m_3+m_4)}{2u_2v_2^2(t)(m_1+m_2)}=1.$$

□

The proof of (3). From (26), we have

$$\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}\frac{m_1m_2}{2(m_1+m_2)}+\frac{\eta_2^2-4(m_3+m_4))}{\xi_2^2}\frac{m_3m_4}{2(m_3+m_4)}=\hat{H}- \left(\frac{(\beta_3\eta_3^2)}{2\xi_3^2(m_1+m_2)(m_3+m4)}-(m_1m_3K_{31}+m_1m_4K_{41}+m_2m_3K_{32}+m_2m_4K_{42})\right). \tag{32}$$

Because $\hat{H}$ is a constant along any solution $\vec{p}_2(\tau)$, the right side in (32) is bounded in $[\tau_1,\tau_2)$ and the limit of the right side in (32) is finite defined by L as $\tau\to\tau_2$, *i.e.*

$$\lim_{\tau\to\tau_2}\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}\frac{m_1m_2}{2(m_1+m_2)}+\frac{(\eta_2^2-4(m_3+m_4))}{\xi_2^2}\frac{m_3m_4}{2(m_3+m_4)}=L.$$

In addition,

$$\lim_{\tau\to\tau_2}\frac{\xi_1^2(\tau)}{\xi_2^2(\tau)}=\alpha,\quad \lim_{\tau\to\tau_2}\xi_1(\tau)=0,\quad \lim_{\tau\to\tau_2}\xi_2(\tau)=0,\quad \lim_{\tau\to\tau_2}\frac{\frac{\eta_1^2(\tau)}{m_1+m_2}}{\frac{\eta_2^2(\tau)}{m_3+m_4}}=1,$$

and

$$\begin{aligned}
&\lim_{\tau\to\tau_2}\frac{\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}\frac{m_1m_2}{2(m_1+m_2)}}{\frac{\eta_2^2-4(m_3+m_4))}{\xi_2^2}\frac{m_3m_4}{2(m_3+m_4)}}\\
&=\frac{m_1m_2(m_3+m_4)}{\alpha m_3m_4(m_1+m_2)}\lim_{t\to t_2}\frac{u_1v_1^2-2(m_1+m_2)}{u_2v_2^2-2(m_3+m_4)}\\
&=\frac{m_1m_2(m_3+m_4)}{m_3m_4(m_1+m_2)}\lim_{t\to t_2}\frac{v_1^2-\frac{2(m_1+m_2)}{u_1}}{v_2^2-\frac{2(m_3+m_4)}{u_2}}\\
&=\frac{m_1m_2(m_3+m_4)}{m_3m_4(m_1+m_2)}\lim_{t\to t_2}\frac{2v_1\frac{dv_1}{dt}+\frac{2(m_1+m_2)}{u_1^2}}{2v_2\frac{dv_2}{dt}+\frac{2(m_3+m_4)}{u_2^2}}\\
&=\frac{m_1m_2(m_3+m_4)}{m_3m_4(m_1+m_2)}\lim_{t\to t_2}\frac{\frac{2(m_1+m_2)}{u_1^2}(-v_1+1)}{\frac{2(m_3+m_4)}{u_2^2}(-v_2+1)}\\
&=\frac{m_1m_2}{\alpha^2m_3m_4}\lim_{t\to t_2}\frac{(-v_1+1)}{(-v_2+1)}\\
&=\frac{m_1m_2}{\alpha m_3m_4}.
\end{aligned}$$

Then $\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}$ and $\frac{(\eta_2^2-4(m_3+m_4))}{\xi_2^2}$ are well defined when $\tau\to\tau_2$ by making use of (32) and $\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}\frac{\xi_1(\xi_1^2+\xi_2^2)}{\xi_1^2}$ and $\frac{(\eta_2^2-4(m_3+m_4))}{\xi_2^2}\frac{\xi_1(\xi_1^2+\xi_2^2)}{\xi_2^2}$ in (23)-(24) go to zero as $\tau\to\tau_2$. In fact, we can prove this by direct computation as follows:

$$\lim_{\tau\to\tau_2}\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}\frac{\xi_1(\xi_1^2+\xi_2^2)}{\xi_1^2}=\lim_{\tau\to\tau_2}\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1}\lim_{\tau\to\tau_2}\frac{(\xi_1^2+\xi_2^2)}{\xi_1^2}$$

$$=(1+\frac{1}{\alpha})\lim_{t\to t_2}\frac{u_1v_1^2-2(m_1+m_2)}{\sqrt{2u_1}}=(1+\frac{1}{\alpha})\lim_{t\to t_2}\frac{u_1\tilde{G}_1}{\sqrt{2u_1}}=0.$$

According to Lemma 3.2, it is clear that the functions on the right-hand side of (20)-(25) have a well-defined finite limit as $\tau\to\tau_2$ along $\vec{p}_2(\tau)$ given in the above. Moreover, $(\xi_1(\tau),\xi_2(\tau),\xi_3(\tau))$ intersects the simultaneous binary collision set $\bigwedge=\{\xi_1=0,\xi_2=0,\xi_3<0\}$ transversally, because letting $\tau\to\tau_2$ in (20), (21) implies

$$\lim_{\tau\to\tau_2}\frac{d\xi_1}{d\tau}=\lim_{\tau\to\tau_2}\frac{\xi_1^2+\xi_2^2}{\xi_1^2}\eta_1=-2(1+\frac{1}{\alpha})\sqrt{m_1+m_2}<0,$$

$$\lim_{\tau\to\tau_2}\frac{d\xi_2}{d\tau}=\lim_{\tau\to\tau_2}\frac{\xi_1^2+\xi_2^2}{\xi_2^2}\eta_2=-2(1+\alpha)\sqrt{m_3+m_4}<0.$$

Thus, $(\xi_1(\tau),\xi_2(\tau),\xi_3(\tau))$ can be extended across $\bigwedge$. The solution $\vec{p}_2(\tau)=(\xi_1(\tau),\xi_2(\tau),\xi_3(\tau),\eta_1(\tau),\eta_2(\tau),\eta_3(\tau))$ can be extended for $\tau>\tau_2$ by solving the differential equations (20)-(25) with initial condition $\vec{p}(\tau)=\vec{p}_2(\tau_2)$ when $\tau=\tau_2$. The vector field given by (20)-(25) is clearly continuous at $\tau=\tau_2$ and therefore, the components $\xi_1(\tau),\xi_2(\tau),\xi_3(\tau),\eta_1(\tau),\eta_2(\tau),\eta_3(\tau)$ are continuously differentiable functions of τ when $\tau=\tau_2$. So the singularity of simultaneous binary collision in equation (11) is removed by transferring to equation (20)-(25). This concludes the proof of Theorem 3.1. □

Furthermore, we even can prove that the regularization is C^2 in Theorem 3.3.

Theorem 3.3. *The equations (20)-(25) give rise to a C^2 extension of $\vec{p}_2(\tau)$ with respect to τ at the simultaneous binary collision $\vec{p}_2(\tau_2)$.*

Proof. Let $F(\tau) = \frac{\xi_1(\tau)}{\xi_2(\tau)}$. Then $F(\tau_2) = \alpha^{(1/2)}$. From equations (20)-(25) and lemma 3.2, we have at $\tau = \tau_2$,

$$\frac{d\xi_1}{d\tau} = (1+\alpha^{-1})(-2\sqrt{m_1+m2}), \frac{d\xi_2}{d\tau} = (1+\alpha)(-2\sqrt{m_3+m_4}), \frac{d\eta_1}{d\tau} = 0, \frac{d\eta_2}{d\tau} = 0.$$

$$\lim_{\tau\to\tau_2}\frac{dF}{d\tau} = \lim_{\tau\to\tau_2}\frac{\frac{d\xi_1}{d\tau}\xi_2 - \xi_1\frac{d\xi_2}{d\tau}}{\xi_2^2} = \lim_{\tau\to\tau_2}(1+F^{-2})\frac{\eta_1 - F^3\eta_2}{\xi_2}$$

$$= (1+\alpha^{-2})\lim_{\tau\to\tau_2}\frac{\frac{d\eta_1}{d\tau} - F^3\frac{d\eta_2}{d\tau} - 3F^2\eta_2\frac{dF}{d\tau}}{d\xi_2} = -3\alpha^{-1/2}\lim_{\tau\to\tau_2}\frac{dF}{d\tau}.$$

So $\lim_{\tau\to\tau_2}\frac{dF}{d\tau} = 0$ and

$$\lim_{\tau\to\tau_2}\frac{d^2\xi_1}{d\tau^2} = \lim_{\tau\to\tau_2}\frac{d((1+F^{-2})\eta_1)}{d\tau}$$

$$= \lim_{\tau\to\tau_2}(1+F^{-2})\frac{d\eta_1}{d\tau} + \lim_{\tau\to\tau_2}(1-2F^{-3}\frac{dF}{d\tau})\eta_1 = -2\sqrt{m1+m2}.$$

Similarly, we can prove the limits of the second derivative of $\xi_1(\tau), \xi_2(\tau), \xi_3(\tau), \eta_1(\tau), \eta_2(\tau), \eta_3(\tau)$ exist at $\tau = \tau_2$. Therefore the second derivatives are continuously differentiable functions of τ when $\tau = \tau_2$. The extension of the simultaneous collision orbit is C^2. □

4. **The behavior of the solutions approaching SBC singularity and solutions with SBC.** Let us recall our notation. x_1, x_2, x_3, x_4 are the positions of the masses of the collinear four body problem with the center of mass at origin, i.e. (4) holds. $u_1 = x_2 - x_1$ is the difference of the first two bodies and $u_2 = x_4 - x_3$ is the difference of the last two bodies. $u_3 = \frac{m_1x_1+m_2x_2}{m_1+m_2}$ is the center of mass of the first two bodies. v_i are the derivatives corresponding to u_i, $i = 1, 2, 3$. Furthermore, we have new coordinates and time scale given by (18) and (19). $\bigwedge = \{x_1 = x_2, x_3 = x_4, x_2 \neq x_3\} = \{u_1 = u_2 = 0, u_3 < 0\} = \{\xi_1 = \xi_2 = 0, \xi_3 \neq 0\}$ are the sets of simultaneous binary collisions in the respective coordinates. In this section we always assume that $x = (x_1, x_2, x_3, x_4)$ be a simultaneous binary collision solution which is defined in $t \in [t_1, t_2)$ encountering with singular set $\bigwedge$ at $t = t_2$. $\vec{p}_1(t) = (u_1(t), u_2(t), u_3(t), v_1(t), v_2(t), v_3(t))$ is obtained by transformation (6) and $\vec{p}_2(\tau) = (\xi_1(\tau), \xi_2(\tau), \xi_3(\tau), \eta_1(\tau), \eta_2(\tau), \eta_3(\tau))$ is obtained by transformation (18) and new time scale (19). By Theorem 3.1 and Theorem 3.3, there exist $\tau_3 > \tau_2$ such that $\vec{p}_2(\tau)$ ($\tau \in [\tau_1, \tau_3]$) is a C^2 solution of equation (20)-(25) without singularity at $\tau = \tau_2$.

Now we are going to describe the behavior of the simultaneous binary collision solution when it approaches the singular set $\bigwedge$ in the original coordinate.

Theorem 4.1. *Let $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ be the extended simultaneous binary collision solution of equation (1) by converting the C^2 solution $\vec{p}_3(\tau)$ into original system. Then the solution $x(t)$ has the following properties.*

(a) x is defined in $t \in [t_1, t_3]$, where $t_3 = t(\tau_3)$ and $t_1 < t_2 < t_3$. x encounters with singular set $\bigwedge$ when $t = t_2$.

(b) Let $C_1 = \frac{m_1x_1+m_2x_2}{m_1+m_2}$ be the center of mass of m_1, m_2 and $C_2 = \frac{m_3x_3+m_4x_4}{m_3+m_4}$ be the center of mass of m_3, m_4. Then

$$\lim_{t\to t_2} \frac{C_1}{C_2} = -\frac{m_3+m_4}{m_1+m_2},$$

where $C_1 < 0$ and $C_2 > 0$ are both finite.

(c) The ratio of velocities $\frac{dx_i}{dt}$ and $\frac{dx_{i+1}}{dt}$ approaches a finite number as $t \to t_2$, where $i = 1, 3$, more precisely, $\lim_{t\to t_2} \frac{dx_i}{dt} / \frac{dx_{i+1}}{dt} = -\frac{m_{i+1}}{m_i}$. The negative sign implies that the velocities of collision pairs are in opposite direction as $t \to t_2$, which is independent of the initial positions and the initial velocities.

(d) The ratio of the distance $u_1 = x_2 - x_1$ and the distance $u_2 = x_4 - x_3$ is determined by their mass ratio, more precisely, $\lim_{t\to t_2} \frac{u_1}{u_2} = \left(\frac{m_1+m_2}{m_3+m_4}\right)^{\frac{1}{3}}$. The ratio of their velocities is also determined by their mass ratio, i.e. $\lim_{t\to t_2} \frac{du_1/dt}{du_2/dt} = \left(\frac{m_1+m_2}{m_3+m_4}\right)^{\frac{1}{3}}$.

(e) In the original time scale, the velocities are unbounded, i.e., $\lim_{t\to t_2} \frac{dx_i}{dt} = \pm\infty$. But in the new time scale, the velocities are bounded and $\lim_{\tau\to\tau_2} \frac{dx_i}{d\tau} = 0$, where $i = 1, \cdots, 4$.

Proof. (a) is directly from Theorem 3.1. By making use of the center of mass at origin, it is easy to prove (b). (c) is directly from the equation (1) and L'Hopital's rule, in fact,

$$\lim_{t\to t_2} \frac{\frac{dx_1}{dt}}{\frac{dx_2}{dt}} = \lim_{t\to t_2} \frac{\frac{d^2x_1}{dt^2}}{\frac{d^2x_2}{dt^2}} = \lim_{t\to t_2} \frac{\sum_{j\neq 1} \frac{m_j(x_j-x_1)}{|x_j-x_1|^3}}{\sum_{j\neq 2} \frac{m_j(x_j-x_2)}{|x_j-x_2|^3}} = -\frac{m_2}{m_1},$$

and $\lim_{t\to t_2} \frac{\frac{dx_3}{dt}}{\frac{dx_4}{dt}} = -\frac{m_4}{m_3}$. (d) is directly from Lemma 2.1. Now we turn to prove (e). In the new coordinates and new time scale, as the solution $\vec{p}_2(\tau)$ approaches the singular set $\bigwedge$, we already have, from the proof of Theorem 3.1,

$$\xi_1(\tau_2) = 0, \xi_2(\tau_2) = 0, \xi_3(\tau_2) < 0, \eta_1(\tau_2) = -2\sqrt{m_1+m_2}, \eta_2(\tau_2) = -2\sqrt{m_3+m_4},$$

and $\eta_3(\tau_2)$ is finite. So in the new time scale, it slows down the motion to a finite speed (η_i are related to the velocity of the particles). Recall that $\frac{du_1}{dt} \to \infty$ as t goes to t_2, but $u_1 = \frac{\xi_1^2}{2}$ implies

$$\frac{du_1}{d\tau} = \xi_1 \frac{d\xi_1}{d\tau} = 0 \text{ at } \tau = \tau_2.$$

From which we have,

$$\frac{d(x_2-x_1)}{d\tau} = \frac{dx_2}{d\tau} - \frac{dx_1}{d\tau} = 0 \text{ at } \tau = \tau_2.$$

From c above, we have

$$\lim_{\tau\to\tau_2}\frac{\frac{dx_2}{d\tau}}{\frac{dx_1}{d\tau}}=\lim_{t\to t_2}\frac{\frac{dx_2}{dt}\frac{dt}{d\tau}}{\frac{dx_1}{dt}\frac{dt}{d\tau}}=\lim_{t\to t_2}\frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}}=-\frac{m_2}{m_1}$$

Therefore,

$$\frac{dx_2}{d\tau}=0 \text{ and } \frac{dx_1}{d\tau}=0 \text{ at } \tau=\tau_2.$$

We complete the proof of Theorem 4.1. Then Theorem 1.2 is proved. □

Using the above properties, we construct a family of solutions of the collinear four body problem with simultaneous binary collisions before a single binary collision between m_2 and m_3. Let $x^0=(x_1^0,x_2^0,x_3^0,x_4^0)$ denote the initial positions of the masses of the collinear four body problem with $-\infty<x_1^0<x_2^0<0<x_3^0<x_4^0<\infty$. We assume that x^0 possesses symmetries on positions and masses, i.e. $x_1^0=-x_4^0, x_2^0=-x_3^0$ and $m_1=m_4, m_2=m_3$. Without loss of generality, let $s=x_2^0-x_1^0=x_4^0-x_3^0>0$, $x_2^0=-1, x_3^0=1$, and $m_1=m_4=m, m_2=m_3=1$. Theorem 1.3 read as follows.

Theorem 4.2. *If s,m fall into the region of $\frac{s^2(s+2)^2}{16(1+s)}<m$ in the first quadrant of sm-plane (see Figure 1), then the orbit by releasing the four bodies with zero velocity at x^0 has at least one simultaneous binary collision before a single binary collision between m_2 and m_3.*

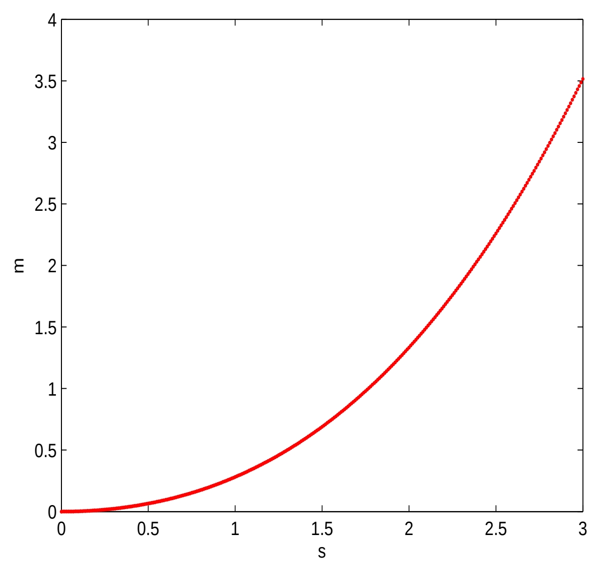

FIGURE 1. The region of simultaneous collision in sm-plane:

$$\frac{s^2(s^2+4s+4)}{16(1+s)}<m.$$

Proof. By Newton's law, the accelerations of the four particles are respectively,

$$a_1 = m_2 s^{-2} + m_3(s+2)^{-2} + m_4(2s+2)^{-2},$$
$$a_2 = -m_1 s^{-2} + \frac{m_3}{4} + m_4(s+2)^{-2},$$
$$a_3 = -m_1(s+2)^{-2} - \frac{m_2}{4} + m_4 s^{-2},$$
$$a_4 = -m_1(2s+2)^{-2} - m_2(s+2)^{-2} - m_3 s^{-2}.$$

Note that $m_1 = m_4 = m, m_3 = m_2 = 1$ then no matter the choice of s and m, $a_1 > 0$ and $a_4 < 0$. It is easy to prove that a_2 keeps the negative sign and a_3 keeps the positive sign before simultaneous binary collisions if they are released with zero velocities and if $a_2 < 0$ and $a_3 > 0$ at the initial time. Therefore, if s, m can be chosen such that the acceleration $a_2 < 0$ but $a_3 > 0$, then x_0 leads to a simultaneous binary collision solution because of the symmetry of positions and masses. Therefore it is extended to a solution with simultaneous binary collision before single binary collision.

In order that $a_2 < 0$ and $a_3 > 0$, we only need make $a_2 < 0$ by choosing proper s, m because $a_2 = -a_3$. The numerator of a_2 is

$$na_2 = -16ms - 16m + s^4 + 4s^3 + 4s^2,$$

and the denominator of a_2 is

$$da_2 = 4s^2(s+2)^2.$$

So when s, m fall into the region of $\frac{s^2(s^2+4s+4)}{16(1+s)} < m$, it has $a_2 < 0$ and leads to simultaneous binary collisions (see Figure 2) before single binary collisions. □

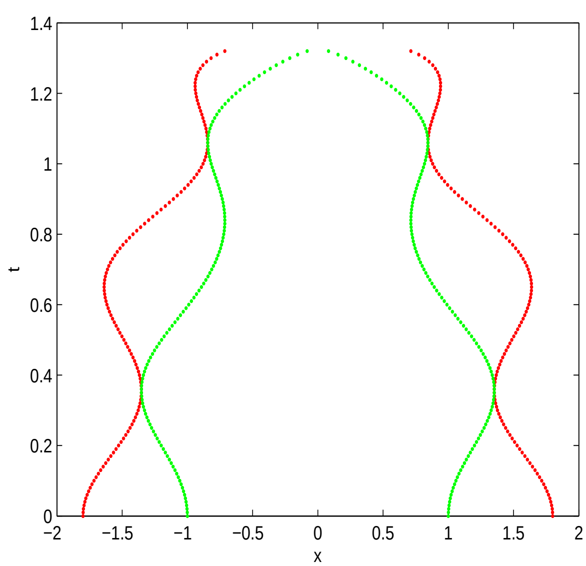

FIGURE 2. Numerical simulations of two simultaneous binary collisions before a single binary collision. $s = 0.8, m = 1$, i.e. $m_1 = m_2 = m_3 = m_4 = 1$ $x_1^0 = -1.8$, $x_2^0 = -1$, $x_3^0 = 1$, $x_4^0 = 1.8$

Remark 3. The conditions that s, m fall into the region of $\frac{s^2(s+2)^2}{16(1+s)} < m$ in the first quadrant of sm-plane is sufficient but not necessary to give a solution with simultaneous binary collisions before single binary collisions. In fact, when $s = 2, m = 1$, they give an example for which a solution has one simultaneous binary collision before a single binary collision (see Figure 3) but s, m do not satisfy the condition in Theorem 4.1. The acceleration of m_2 is positive instead of negative as in Theorem 4.1 at the beginning of the motion.

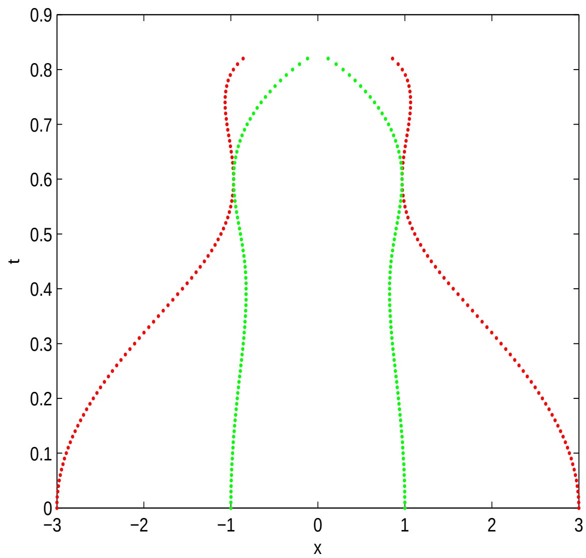

FIGURE 3. Numerical simulations of one simultaneous binary collision before a single binary collision. $s = 2, m = 1$, i.e. $m_1 = m_2 = m_3 = m_4 = 1$ $x_1^0 = -3$, $x_2^0 = -1$, $x_3^0 = 1$, $x_4^0 = 3$.

5. **Classifications of solutions with single binary collisions and simultaneous binary collisions.** The behavior of the motion for the pair closing the single binary collision can be described as time reverse plus a higher order term in a very short time neighborhood. At the moment of single binary collision, the velocities of the particles involving the collision approach infinity. By changing the time scale, the velocities of the particles remain bounded as slow motion. The motion can be extended to cross the collision point. Any solution of the collinear four body problem must involve collisions if four bodies are released from zero velocities. In this section we study the number of simultaneous binary collisions before a single binary collision occurs and we also study the number of single binary collisions before a simultaneous binary collision occurs in the collinear four body problem. The central configuration of collinear four body problem plays an important role in our construction.

Let $x^0 = (x_1^0, x_2^0, x_3^0, x_4^0)$ denote the initial positions of collinear four body problem with $-\infty < x_1^0 < x_2^0 < 0 < x_3^0 < x_4^0 < \infty$. We assume that x^0 possesses symmetries on positions and masses, i.e. $x_1^0 = -x_4^0, x_2^0 = -x_3^0$ and $m_1 = m_4, m_2 = m_3$. Without loss of generality, let $s_0 = x_2^0 - x_1^0 = x_4^0 - x_3^0 > 0$, $x_2^0 = -1, x_3^0 = 1$, and $m_1 = m_4 = m, m_2 = m_3 = 1$.

$x^0 = (-s_0 - 1, -1, 1, s_0 + 1)$ with mass $(m, 1, 1, m)$ forms a central configuration if and only if

$$m = \frac{(s_0 + 1)^2 \left({s_0}^5 + 5\,{s_0}^4 + 8\,{s_0}^3 - 4\,{s_0}^2 - 16\,s_0 - 16\right)}{17\,{s_0}^4 + 68\,{s_0}^3 + 100\,{s_0}^2 + 64\,s_0 + 16}. \tag{33}$$

For $s_0 > 1.396812289$, there is a positive $m > 0$ such that $x^0 = (-s_0-1, -1, 1, s_0+1)$ with mass $(m, 1, 1, m)$ forms a central configuration. The result is a special case of Theorem 1 in ([10]). In this section, all the motion are obtained by releasing the four bodies at initial position with zero velocities. Theorem 1.4 is proved by the following two theorems.

Theorem 5.1. *Assume that* $x^0 = (-s_0 - 1, -1, 1, s_0 + 1)$ *with mass* $(m, 1, 1, m)$ *forms a central configuration. Then* $y^0 = (-s_0 + s - 1, -1, 1, s_0 - s + 1)$ *with mass* $(m, 1, 1, m)$ *leads to a solution involving simultaneous binary collisions before a single binary collision occurs if the four bodies are released from zero velocities, where* $0 < s < s_0$.

Proof. It is well known that there is a unique central configuration with the fixed order of four given masses in collinear four body problem. Because $x^0 = (-s_0 - 1, -1, 1, s_0 + 1)$ with mass $(m, 1, 1, m)$ form a central configuration (The formula of central configuration for s_0 and m is given as above), then $y^0 = (-s_0 + s - 1, -1, 1, s_0 - s + 1)$ with mass $(m, 1, 1, m)$ can not lead to a total collision with $0 < s < s_0$. By symmetry, y^0 only can lead to either a single binary collision first between m_2 and m_3 or a simultaneous binary collision first.

Claim:For $0 < s < s_0$, y^0 can not first have a single binary collision between m_2 and m_3.

At x^0, the accelerations of the four particles are respectively,

$$\begin{aligned}
ax_1 &= m_2 s_0^{-2} + m_3(s_0 + 2)^{-2} + m_4(2s_0 + 2)^{-2}, \\
ax_2 &= -m_1 s_0^{-2} + \frac{m_3}{4} + m_4(s_0 + 2)^{-2}, \\
ax_3 &= -m_1(s_0 + 2)^{-2} - \frac{m_2}{4} + m_4 s_0^{-2}, \\
ax_4 &= -m_1(2s_0 + 2)^{-2} - m_2(s_0 + 2)^{-2} - m_3 s_0^{-2}.
\end{aligned}$$

They lead to a total collision. When the initial condition changes to $y^0 = (-s_0 + s - 1, -1, 1, s_0 - s + 1)$, the accelerations of the four particles are respectively,

$$\begin{aligned}
ay_1 &= m_2(-s_0 + s)^{-2} + m_3(s - s_0 - 2)^{-2} + m_4(2(s_0 - s) + 2)^{-2}, \\
ay_2 &= -m_1(-s_0 + s)^{-2} + \frac{m_3}{4} + m_4((s - s_0 - 2)^{-2}, \\
ay_3 &= -m_1(s - s_0 - 2)^{-2} - \frac{m_2}{4} + m_4(-s_0 + s)^{-2}, \\
ay_4 &= -m_1(2(s_0 - s) + 2)^{-2} - m_2(s_0 + 2)^{-2} - m_3 s_0^{-2}.
\end{aligned}$$

By direct computation, it is easy to see $0 < ax_1 < ay_1$ but $ax_2 > ay_2$ and symmetrically for other two bodies. This implies that m_1 and m_2 shall collide before m_2 and m_3 collides by comparing this motion with the motion having total collision. So the motion with initial position y^0 and zero initial velocities can not have a single binary collision first between m_2 and m_3. By symmetry, it must lead to a simultaneous binary collision. Figure 3 illustrates a case that there has exact one SBC before single binary collisions when the four bodies with equal unit mass are released from $y^0 = (-s_0 + s - 1, -1, 1, s_0 - s + 1)$ where $s_0 = 2.1622$ and $s = 0.1622$. Figure 3 also shows that m_2 and m_3 move inside first then turn back to a simultaneous binary collision. Figure 4 gives two examples that have three SBCs and twelve SBCs before single binary collision. $\square$

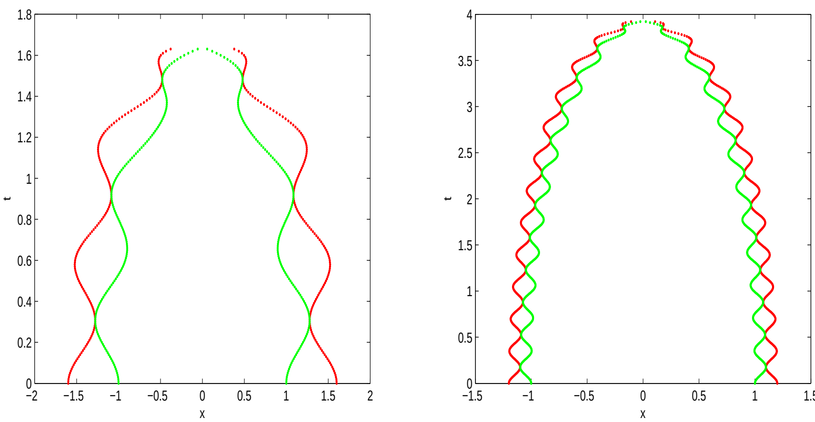

FIGURE 4. Numerical simulations of simultaneous binary collisions before a single binary collision, where $m_1 = m_2 = m_3 = m_4 = 1$. Left (three SBCs): $x_1^0 = -1.6$, $x_2^0 = -1$, $x_3^0 = 1$, $x_4^0 = 1.6$. Right (twelve SBCs): $x_1^0 = -1.2$, $x_2^0 = -1$, $x_3^0 = 1$, $x_4^0 = 1.2$.

Theorem 5.2. *Let $x^0 = (-s_0 - 1, -1, 1, s_0 + 1)$ with mass $(m, 1, 1, m)$ form a central configuration, where $s_0 > 0$ is implicitly defined in equation (33).*
(1) Then there exists a sequence $0 < s_1^* < s_2^* < \cdots$ such that the motion has exact n times single binary collision between m_2 and m_3 before a simultaneous binary collision in $[0, T]$, where T corresponds to the first time of simultaneous binary collision, if the motion starts with initial position $y^0 = (-s_0 - s_n - 1, -1, 1, s_0 + s_n + 1)$ and zero initial velocity, where $s_{n-1}^* < s_n < s_n^*$.
(2)If four particles are released from $y^0 = (-s_0 - s_n^* - 1, -1, 1, s_0 + s_n^* + 1)$ with zero initial velocity, $n = 1, 2, \cdots$, then the motion has a total collision after n times single binary collision between m_2 and m_3.

Proof. The proof can be done by induction base on the proof of the following claims. Figure 5 illustrates an example that m_2 and m_3 collide 4 times before a simultaneous binary collision.

Claim 1: For small $s > 0$, the motion with initial position y^0 and zero velocity at $t = 0$ has exact one single binary collision between m_2 and m_3 before a simultaneous binary collision in $[0, T]$.

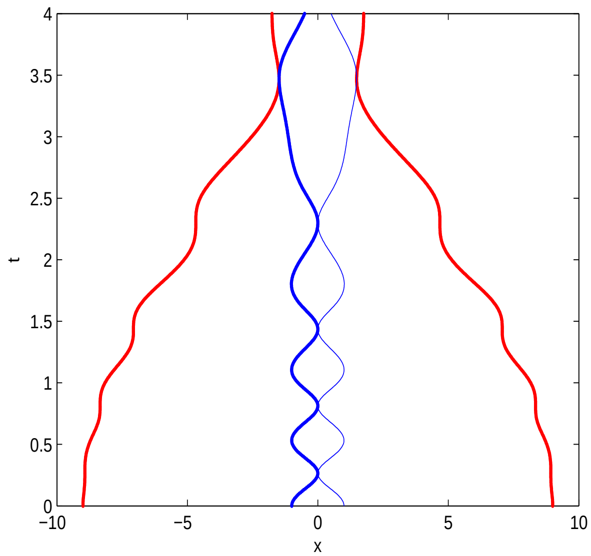

FIGURE 5. Simultaneous binary collision occurs after four single binary collisions. $m_1 = m_2 = m_3 = m_4 = 1$ $x_1^0 = -9$, $x_2^0 = -1$, $x_3^0 = 1$, $x_4^0 = 9$.

Proof of Claim 1. By similar argument as in the proof of Theorem 5.1, the accelerations of m_1 and m_2 at time $t = 0$ satisfy $0 < ay_1 < ax_1$ but $ax_2 < ay_2$ and symmetrically for other two bodies. This implies that m_2 and m_3 shall collide at origin before m_1 and m_2 collides by comparing the motion with total collision. So the motion with initial position $y^0 = (-s_0 - s - 1, -1, 1, s_0 + s + 1)$ has a single binary collision between m_2 and m_3 first at time $0 < t_1 < T$. It can be regularized and then the motion will continue and keep symmetry. m_1 continue to right and m_2 bounds back.

If m_1 and m_2 don't collide after t_1, then there exists a time t_2 with $t_1 < t_2$ such that m_2 turns back to right, that is, the velocity of m_2 at t_2 is zero. Let $y(t) = (y_1(t), y_2(t), y_3(t), y_4(t))$ be the solution with initial position y^0 and zero initial velocity. Comparing the orbit of $y(t)$ in $[0, t_1)$ with the orbit of $y(t')$ in $(t_1, t_2]$, we shall have $y_2(0) = y_2(t_2) < 0$ if there is no force on m_2 from m_1 and m_4. But when the position $y_2(t)$ of m_2 in $[0, t_1)$ is equal to the position $y_2(t')$ of m_2 in $[t_1, t_2)$, $\ddot{y}_2(t) > \ddot{y}_2(t') > 0$ because m_1 and m_4 are closer to m_2 at t' than at t. Therefore we have $y_2(t_2) < y_2(0) < 0$. Similarly, $0 < y_3(0) < y_3(t_2)$.

If s is small enough, m_1 can go over position $y_2(0)$ at time t_1, i.e. $-1 < y_1(t_1) < 0$. So m_1 and m_2 must collide, say at time T, after the single binary collision between m_2 and m_3 by continuity argument and by comparing with total collision. Then the collision must be a simultaneous binary collision by symmetry. The orbit has exact one single binary collision (at t_1) and one simultaneous binary collision (at T) in $[0, T]$. $\square$

Claim 2: There exists a $\tilde{s} > 0$, such that the motion with initial position $y^0 = (-s_0 - \tilde{s} - 1, -1, 1, s_0 + \tilde{s} + 1)$ has at least two single binary collisions between m_2

and m_3 before m_1 and m_4 are involved in any collisions.

Proof of Claim 2. Consider an auxiliary system $z = (-r, z_2, z_3, r)$ with mass $(m, 1, 1, m)$ by fixing $z_1 = -r, z_4 = r$ under Newton's law. So $z_1(t) = -r, z_2(t) = r$ and z_2, z_3 are determined by the following equations,

$$\ddot{z}_2(t) = -\frac{m}{(z_2+r)^2} + \frac{1}{(z_2-z_3)^2} + \frac{m}{(z_2-r)^2}, \tag{34}$$

$$\ddot{z}_3(t) = -\frac{m}{(z_3+r)^2} - \frac{1}{(z_3-z_2)^2} + \frac{m}{(z_3-r)^2}, \tag{35}$$

with initial position $z(0) = (-r, -1, 1, r)$ and zero initial velocity. By the symmetry of differential equations and initial conditions, $z_3(t) = -z_2(t)$.
In fact, equation (34) is a Hamiltonian system with

$$H = \frac{1}{2}|\dot{z}_2|^2 + \left(-\frac{m}{(z_2+r)} + \frac{1}{4z_2} + \frac{m}{(z_2-r)}\right).$$

H is a constant along solution $z_2(t)$. At $t = 0$, $z_2(0) = -1, \dot{z}_2(0) = 0$, then $H \equiv C = \left(-\frac{m}{(-1+r)} - \frac{1}{4} + \frac{m}{(-1-r)}\right) = -(\frac{2mr}{r^2-1} + \frac{1}{4}) < 0$. Assume that $z_2(t)$ travels from -1 to 0 in $[0, t_1]$. We have

$$dt = \frac{dz_2}{\sqrt{2\left(C - \left(-\frac{m}{(z_2+r)} + \frac{1}{4z_2} + \frac{m}{(z_2-r)}\right)\right)}}.$$

So

$$\begin{aligned} t_1(m,r) &= \int_{-1}^{0} \frac{dz_2}{\sqrt{2\left(C - \left(-\frac{m}{(z_2+r)} + \frac{1}{4z_2} + \frac{m}{(z_2-r)}\right)\right)}} \\ &= \int_{-1}^{0} \frac{dz_2}{\sqrt{2\left(C - \frac{2mr}{(z_2^2-r^2)} - \frac{1}{4z_2}\right)}} \\ &\geq \int_{-1}^{0} \frac{dz_2}{\sqrt{2\left(C - \frac{2mr}{((-1)^2-r^2)} - \frac{1}{4z_2}\right)}} \\ &= \int_{-1}^{0} \frac{dz_2}{\sqrt{2\left(-\frac{1}{4} - \frac{1}{4z_2}\right)}} = \frac{\sqrt{2}\pi}{2}. \end{aligned}$$

From equation (34), the acceleration of m_2 can be always positive if r is large, in fact,

$$\begin{aligned} \ddot{z}_2(t) &= -\frac{m}{(z_2+r)^2} + \frac{1}{(z_2-z_3)^2} + \frac{m}{(z_2-r)^2} \\ &= \frac{2mrz_2}{(z_2^2-r^2)^2} + \frac{1}{4z_2^2} \geq \frac{-2mr}{(1-r^2)^2} + \frac{1}{4z_2^2} > 0 \end{aligned}$$

if $m < \frac{(1-r^2)^2}{8r}$ and $z_2 \in [-1, 0]$. Then we have

$$t_1(m,r) \leq \int_{-1}^{0} \frac{dz_2}{\sqrt{2\left(C - \frac{2mrz_2}{(1-r^2)} - \frac{1}{4z_2}\right)}} < \infty. \tag{36}$$

The above integral is the time that the z_2 moves from -1 to 0 with the smaller acceleration $\frac{-2mr}{(1-r^2)^2} + \frac{1}{4z_2^2} > 0$. For any finite time T' with $3t_1(m,r) < T' < \infty$, there exists a large r such that m_2 and m_3 collide at origin at least two times in the finite time interval $[0, T']$.

Consider another auxiliary system $w = (w_1, -r, r, w_4)$ with mass $(m, 1, 1, m)$ by fixing $w_2 = -r, w_3 = r$ under Newton's law. So w_1, w_4 are determined by the following equations,

$$\ddot{w}_1(t) = \frac{1}{(w_1+r)^2} + \frac{1}{(w_1-r)^2} + \frac{m}{(w_1-w_4)^2} \tag{37}$$

$$\ddot{w}_4(t) = -\frac{1}{(w_4-r)^2} - \frac{1}{(w_4+r)^2} - \frac{m}{(w_4-w_1)^2} \tag{38}$$

with initial positions $(-\tilde{s}, -r, r, \tilde{s})$ and zero initial velocity. Within the motion of w_1 from $\tilde{s}$ to $-r$, we have

$$\ddot{w}_1(t) \leq \frac{k}{(w_1+r)^2}$$

because $(w_1+r)^2 < (w_1-r)^2 < 4w_1^2$, where $k = \max\{1, m\}$. By the similar argument as $t_1(m,r)$, the time t_2 for the motion of m_1 from $-\tilde{s}$ to $-r$ has

$$t_2(m, \tilde{s}) \geq \int_{-\tilde{s}}^{-r} \frac{dw_1}{\sqrt{2(-C_2 - \frac{k}{w_1+r})}} = \frac{k\pi}{(2C_2)^{3/2}} = \frac{\pi}{2\sqrt{2k}}(\tilde{s}-r)^{3/2}, \tag{39}$$

where $C_2 = \frac{k}{\tilde{s}-r}$. After comparing (36) with (39), we know that if $\tilde{s}$ is large enough, w_1 can not across $-r$ and w_4 can not across r in the finite time $[0, T']$.

Now for the large $\tilde{s}$, the motion $y(t) = (y_1(t), y_2(t), y_3(t), y_4(t))$ with initial position $y^0 = (-s_0 - \tilde{s} - 1, -1, 1, s_0 + \tilde{s} + 1)$ and zero initial velocity has the following properties. (1) In the finite time T', y_1 can not across $-r$ and y_4 can not across r by comparing with the auxiliary system $w(t)$ because $0 < \ddot{y}_1(t) < \ddot{w}_1(t)$. and $0 > \ddot{y}_4(t) > \ddot{w}_4(t)$. (2) In the finite time T', y_2 and y_3 should collide at least two times by comparing with the auxiliary system $z(t)$. □

Claim 3: If the motion with initial position $y^1 = (-s_0 - s_1 - 1, -1, 1, s_0 + s_1 + 1)$ and zero initial velocity has n times single binary collision between m_2 and m_3 in [0,T], where T corresponds to the first time of simultaneous binary collision, then $y^2 = (-s_0 - s_2 - 1, -1, 1, s_0 + s_2 + 1)$ leads to at least n times single binary collision before simultaneous binary collision for $0 < s_1 < s_2$ in $[0, T]$.

Proof of Claim 1. At time $t = 0$, $\ddot{y}_1^1 > \ddot{y}_1^2 > 0$ but $0 < \ddot{y}_2^1 < \ddot{y}_2^2$, this implies that y_1^2 goes slower than y_1^1 but y_2^2 goes faster than y_1^2. Therefore, y^2 takes shorter time to have the first single binary collision between m_2 and m_3. By similar argument, we can prove that y^2 also take shorter time to have the second binary collision between m_2 and m_3. Then in $[0, T]$, y^2 has at leat n times single binary collision before simultaneous binary collision. □

Now assume that $s_n^* = \sup\{s > 0 : y^0 = (-s_0 - s - 1, -1, 1, s_0 + s + 1)$ *leads to a solution involving exact n times single binary collisions between m_2 and m_3 before a simultaneous binary collision.* $\}$

Claim 1 proves the existence of s_1^*. Claim 2 and Claim 3 prove s_1^* is finite and unique. The existence of s_n^* can be proved by induction and claim 3. The second part of Theorem 5.2 is an immediate result of continuation arguments. □

Remark 4. The orbits we discussed above are not complete. Some of them encounter simultaneous binary collisions before a single binary collision and some of them encounter single binary collisions before a simultaneous binary collisions. We don't give the dynamics of the orbits when the type of collisions changes. One of referees suggest that we may give a more complete discussion of the orbits. We have done some numerical simulations which show that the dynamics of the orbits could be very complicated. This type of numerical analysis can be found in the paper of Sweatman [14]. In his investigation, the energy is fixed at -1 and the initial condition of those orbits is characterized by initial position and initial velocity.

Here we give a selection of typical and special symmetrical orbits (see Figure 6 to Figure 10). The initial condition of these orbits is characterized by initial position for different mass and the energy of the system is not fixed. The various four-body orbits in our study meet the following assumptions. Four bodies have mass $(m, 1, 1, m)$, have initial position at $(-s-1, -1, 1, s+1)$ and have zero initial velocities. The ordinates are the x-coordinates of the bodies and t-coordinates of the time. Upon an individual graph each curve represents one of the four bodies and the initial value of (s, m) appears in the right above corner of the graph.

To illustrate the processes we present the study of Figure 6. This system starts with initial position at $(-16, -1, 1, 16)$ with mass $(5, 1, 1, 5)$. The central pair of bodies collide 8 times before they each collide with one of the outer bodies (simultaneous binary collision). This collision is followed by a central single binary collision, a simultaneous binary collision, then another two this pattern collisions and a last central collision before the system divides into two separating binaries near time 8.

The majority of orbits may be divided into two categories: scattering orbits where the system finishes as distinct subsystems of binaries and single bodies; and bound orbits where the four bodies remain together for all time. Scattering orbits are further subdivided into central-scattering and simultaneous-scattering orbits. In central-scattering orbits the system is divided into two subsystems with central single binary collisions and two outer single bodies escaping to infinity. In simultaneous-scattering orbits the system is divided into two subsystems with two binaries themselves separating to infinity. (It is not possible for the system to separate into four single bodies as they start with zero velocities).

Figures 6, 7 and 8 are all examples of scattering orbits. They show the four possible combinations for final state of the system in the two sub-categories and for the first collisions. Figures 9 and 10 are bounded orbits in a very long time and we are unable to prove they are bounded for ever.

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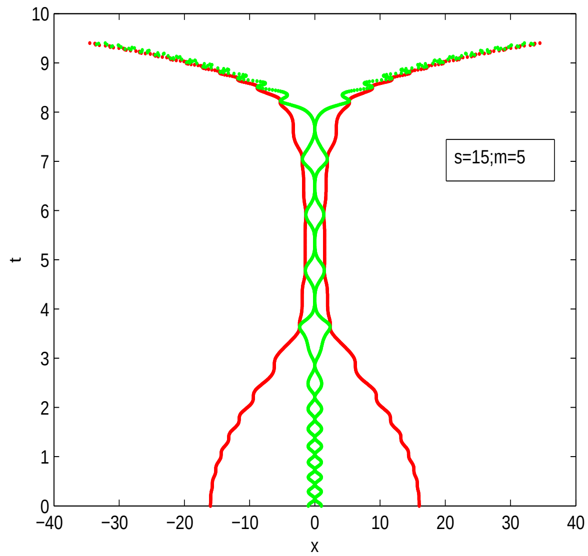

FIGURE 6. Simultaneous-scattering orbit

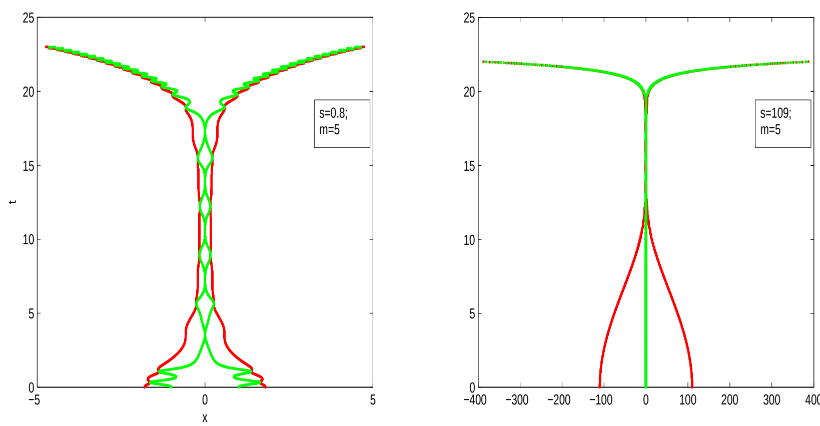

FIGURE 7. Simultaneous-scattering orbits: Left one starts from simultaneous binary collision; Right one starts from single binary collision.

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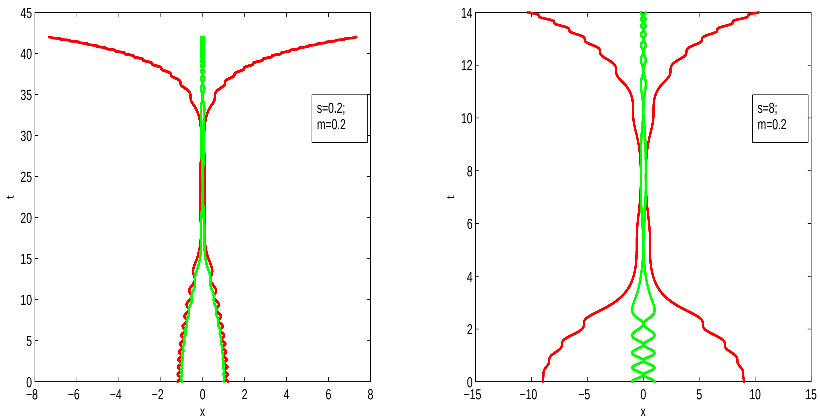

FIGURE 8. Central-scattering orbits: Left one starts from simultaneous binary collision; Right one starts from single binary collision.

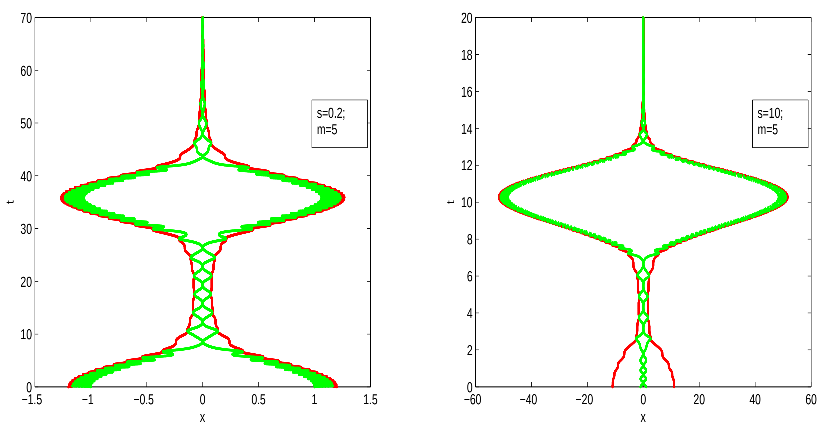

FIGURE 9. Bounded orbits (four bodies all together): Left one starts from simultaneous binary collision; Right one starts from single binary collision.

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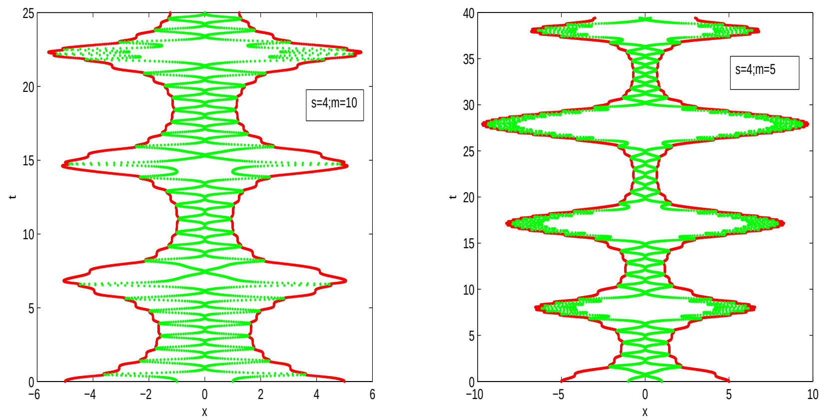

FIGURE 10. Bounded orbits: Left one starts from simultaneous binary collision; Right one starts from single binary collision.

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E-mail address: `ouyang@math.byu.edu`
E-mail address: `zxie@vsu.edu`