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Inverse problem of central configurations and singular curve in the collinear 4-body problem

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Abstract In this paper, we consider the inverse problem of central configurations of n-body problem. For a given $q = (q_1, q_2, \ldots, q_n) \in (\mathbf{R}^d)^n$, let $S(q)$ be the admissible set of masses denoted $S(q) = \{m = (m_1, m_2, \ldots, m_n) | m_i \in \mathbf{R}^+, q \text{ is a central configuration for } m\}$. For a given $m \in S(q)$, let $S_m(q)$ be the permutational admissible set about $m = (m_1, m_2, \ldots, m_n)$ denoted

$$S_m(q) = \{m' | m' \in S(q), m' \neq m \text{ and } m' \text{ is a permutation of } m\}.$$

The main discovery in this paper is the existence of a singular curve $\bar{\Gamma}_{31}$ on which $S_m(q)$ is a nonempty set for some m in the collinear four-body problem. $\bar{\Gamma}_{31}$ is explicitly constructed by a polynomial in two variables. We proved:

(1) If $m \in S(q)$, then either $^{\#}S_m(q) = 0$ or $^{\#}S_m(q) = 1$.
(2) $^{\#}S_m(q) = 1$ only in the following cases:
   (i) If $s = t$, then $S_m(q) = \{(m_4, m_3, m_2, m_1)\}$.
   (ii) If $(s, t) \in \bar{\Gamma}_{31} \setminus \{(\bar{s}, \bar{s})\}$, then either $S_m(q) = \{(m_2, m_4, m_1, m_3)\}$ or $S_m(q) = \{(m_3, m_1, m_4, m_2)\}$.

Keywords Central configuration · Super central configuration · N-body problem · Singular curve · Relative equilibrium · Inverse problem · Descartes' rule

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1 Introduction

The classical n-body problem consists of the study of the dynamics of n point masses interacting according to Newtonian gravity. We consider n particles at $q_i \in \mathbf{R}^d$ (usually with $d = 1, d = 2$, or $d = 3$) with masses $m_i \in \mathbf{R}^+, i = 1, 2, \ldots, n$ and

$$m_i\ddot{q}_i = -\sum_{i\neq j} \frac{m_i m_j}{|q_i - q_j|^3}(q_i - q_j). \tag{1.1}$$

When we study homographic solutions of the n-body problem, the motion at any fixed time must satisfy the following nonlinear algebraic equation system:

$$\lambda(q_i - c) - \sum_{j=1, j\neq i}^{n} \frac{m_j(q_i - q_j)}{|q_i - q_j|^3} = 0, \qquad 1 \leq i \leq n. \tag{1.2}$$

for a constant λ, where $c = (\sum m_i q_i)/M$ is the center of mass and $M = m_1 + m_2 + \cdots + m_n$ is the total mass. By the homogeneity of $U(q)$ of degree (-1), we have $\lambda = U/2I > 0$, where U is the Newtonian potential function

$$U = \sum_{1\leq i<j\leq n} \frac{m_i m_j}{|q_i - q_j|},$$

and I is the moment of inertial of the system, i.e. $I = \frac{1}{2}\sum_{i=1}^{n} m_i|q_i|^2$. Because the potential is singular when two particles have the same position, it is natural to assume that the configuration avoids the collision set which is defined by

$$\triangle = \bigcup\{q = (q_1, q_2, \ldots, q_n) \in (\mathbf{R}^d)^n | q_i = q_j \text{ for some } i \neq j\}. \tag{1.3}$$

Definition 1.1 (*Central Configuration*). A configuration $q = (q_1, q_2, \ldots, q_n) \in (\mathbf{R}^d)^n \backslash \triangle$ is a *central configuration* (CC for short) for $m = (m_1, m_2, \ldots, m_n) \in (\mathbf{R}^+)^n$ if q is a solution of the system (1.2) for some constant $\lambda \in \mathbf{R}$. Two configurations q and $p \in (\mathbf{R}^d)^n \backslash \triangle$ are *equivalent*, denoted by $q \sim p$, if and only if q and p differ by an $SO(3)$ rotation followed by a scalar multiplication. This defines an equivalent relation among elements in CCs for m.

The allowed system of masses which make a configuration central is called admissible. Given a configuration $q = (q_1, q_2, \ldots, q_n) \in (\mathbf{R}^d)^n \backslash \triangle$, denote $S(q)$ the admissible set of masses by

$$S(q) = \left\{m = (m_1, m_2, \ldots, m_n) | m_i \in \mathbf{R}^+, q \text{ is a central configuration for } m\right\}. \tag{1.4}$$

Two mass vectors m and $m' \in S(q)$ are equivalent if and only if $m' = \alpha m$ for some $\alpha \in \mathbf{R}$. This defines an equivalent relation among elements in S as well as CCs for q. We denote by $\tilde{S}(q)$ the set of equivalent classes of $S(q)$.

For a given $m \in S(q)$, let $S_m(q)$ be the permutational admissible set about m, denoted by

$$S_m(q) = \{m' \in S(q) | m' \neq m \text{ and } m' \text{ is a permutation of } m\}. \tag{1.5}$$

Configuration q is called a *Super Central Configuration* if there exists mass m such that $S_m(q)$ is nonempty. The requirements that $m' \neq m$ and m' is a permutation of m in $S_m(q)$ are necessary to exclude some trivial case. For example, if q is a central configuration for $m = (m_1, m_2, m_3, \ldots, m_n)$ with $m_1 = m_2$, then q is also a central configuration $m' = (m_2, m_1, m_3, \ldots, m_n)$ but $m' \notin S_m(q)$. $S_m(q)$ is a finite set and has at most $n! - 1$ elements in $S_m(q)$.

The motivation to study the set $S_m(q)$ emanates from the example of the equilateral triangle configuration (Lagrange 1772) in the planar three-body problem. If q is the equilateral triangle configuration and $m = (m_1, m_2, m_3)$, then q is also a central configuration for each permutation of m. Therefore, for three distinct masses, the set $S_m(q)$, which has five elements, consists of all the permutations of (m_1, m_2, m_3). Xie (2010) provides a nontrivial example of super central configuration in the collinear three-body problem. The set of super central configurations in the collinear three-body problem is classified.

However, there are no other obvious examples such that $S_m(q)$ is nonempty. In fact,

$$S_m(q) \text{ is empty for any } m \in S(q) \text{ if } q$$
$$\text{is a planar configuration in the 4-body problem.}$$

This is an immediate consequence from the following result.

Lemma 1.2 (Macmillan and Bartky 1932, p. 872) *Associated with each admissible quadrilateral there is one and only one set of mass ratios, with the single exception of three equal masses at the vertices of an equilateral triangle and a fourth arbitrary mass at the center of gravity of the other three.*

The possible nonempty $S_m(q)$ may exist in the collinear central configurations of four-body problem. Because central configuration is invariant up to translation and scaling, we can choose the coordinate system so that all the four bodies are on the x-axis with positions $q_1 = -s-1, q_2 = -1, q_3 = 1, q_4 = t+1$ where $s, t > 0$. This is a general form of collinear 4-body configuration. The main discovery in this paper is the existence of a singular curve $\bar{\Gamma}_{31}$ (see definition 2.1 and Fig. 1) on which $S_m(q)$ is a nonempty set for some m. $\bar{\Gamma}_{31}$ is explicitly constructed by a polynomial in two variables (s, t) in the first quadrant of st-plane. Our main conclusion is as follows and detailed classifications of $S_m(q)$ for the collinear four-body problem are given in theorem 2.2.

Theorem 1.3 *Let $q = (-s-1, -1, 1, t+1)$ with $s > 0, t > 0$. If $(s, t) \in \bar{\Gamma}_{31}$, then there exist some $m \in S(q)$ such that $S_m(q)$ is a nonempty set.*

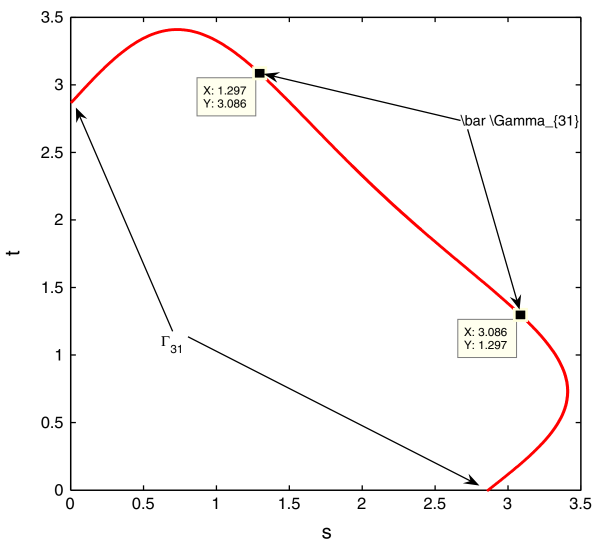


Fig. 1 Singular curve Γ_{31} and $\bar{\Gamma}_{31}$ in the st-plane

The mass m for which $S_m(q)$ is a nonempty set may decrease the number of collinear central configurations of four-body problem under geometrical equivalent classes (see Long and Sun 2004, 2002a, Ouyang and Xie 2009). The question on the number of central configurations for a given mass vector $m = (m_1, m_2, \ldots, m_n)$ is still a challenging problem for 21st century mathematicians (see Smale 1998). The finiteness was proved for $n = 4$ by Hampton and Moeckel (2006), and it is still open for general n. In fact, an exact count is known only for the equal masses case (Albouy 1995, 1996). Some partial results of central configurations are given in Albouy et al. (2008), Bernat et al. (2009), Long (2003), Long and Sun (2002b), Perez-Chavela and Santoprete (2007), Schmidt (1988), Shi and Xie (2010), Xia (2004) for the four-body problem with some equal masses, in Llibre and Mello (2008), Lee and Santoprete (2009), Williams (1938) for the five-body problem, and in Arribas et al. (2007), Woodlin and Xie (2009) for general homogenous or quasi-homogeneous potentials. Existence of different types of central configurations can be found in Alvarez et al. (2008), Corbera and Llibre (2008), Hampton and Santoprete (2007), Lei and Santoprete (2006), Ouyang et al. (2004), Roberts (1999), Zhang (2001) and the references therein. For the importance and additional properties of central configurations and related topics, we refer to the works of Moeckel (1990), Saari (1980), and the books Meyer and Hall (1992), Meyer et al. (2009).

It is natural to consider the inverse problem: given a configuration, find mass vectors, if any, for which it is a central configuration. Moulton (1910), Wintner (1952) considered the inverse problem for collinear n-body problem. His results depend on whether n is even or odd. When n is even, if the center of mass is fixed at the origin (and another assumption on the configuration), there is a one parameter family of mass vectors for which the configuration is central. When n is odd, the inverse problem could not always be solved if the center of mass is fixed at $c = 0$. Albouy and Moeckel (2000) proved that a given configuration determines a two-parameter family of masses making it central where masses are allowed to be negative. Ouyang and Xie (2005) found the region for collinear 4-body configurations for which there exists a positive mass vector making it central. Long and Sun (2004, 2002a) and Xie (2010) studied the collinear 3-body problem and some general properties in the n-body problem.

Our paper is organized as follows. In Sect. 2, we describe the singular curve $\bar{\Gamma}_{31}$ and the main theorem 2.2. The solutions of the collinear four-body central configurations and some properties are provided in Sect. 3. The proof for the main theorem 2.2 is conducted in Sect. 4. Section 5 carries out the proof for some propositions which are used in the proof of the main theorem.

2 Main theorem and the singular curve $\bar{\Gamma}_{31}$

We first denote the polynomials $f_i(s,t)$, $i = 1, 2, \ldots, 5$ by:

$$
\begin{aligned}
f_1(s,t) &= (s+t+2)^2\,(s+2)^2\,s^2\left(-4\,t^2-16\,t-16+t^5+5\,t^4+8\,t^3\right),\\
f_2(s,t) &= (s+t+2)^2\,(s+2)^2\,s^2\left(16\,t+16+t^4+4\,t^3+4\,t^2\right),\\
f_3(s,t) &= 4\;(t+2)^2\,s^2(16+48\,s+16\,t+5\,s^2t^2+4\,t^2+56\,s^2+36\,s^2t\\
&\quad +8\,st^2+40\,st+s^3t^2+14\,s^3t+2\,s^4t+9\,s^4+32\,s^3+s^5\\
&\quad -s^2t^3-2\,st^4-6\,st^3-t^5-5\,t^4-8\,t^3),\\
f_4(s,t) &= 4\;(t+2)^2\,s^2\left(s^4+8\,s^3+24\,s^2+2\,s^3t+12\,s^2t+24\,st+32\,s\right.\\
&\quad \left.+16\,t+16+s^2t^2+2\,st^3+4\,st^2+t^4+4\,t^3+4\,t^2\right),
\end{aligned}
$$

$$\begin{aligned} f_5(s,t) &= 16\,(s+2)^4 + 16\,(s+4)\,(s+2)^3\,t + 4\,\left(s^2+12\,s+24\right)(s+2)^2\,t^2 \\ &\quad + 4\,(s+2)\left(s^3+2\,s^2+12\,s+16\right)t^3 + \left(16+16\,s+4\,s^3+s^4+4\,s^2\right)t^4, \\ h(s,t) &= 16\,(t-2)\left(t^2+2\,t+4\right)(t+2)^2 + (16\,(t-2)\left(t^2+2\,t+4\right)(t+2)^2)s \\ &\quad + \left(4\,(t-2)\left(t^2+2\,t+4\right)(t+2)^2\right)s^2 + \left(64+64\,t+16\,t^2\right. \\ &\quad \left. + 64\,t^3+28\,t^4+4\,t^5\right)s^3 + \left(64+64\,t+16\,t^2+28\,t^3+10\,t^4+t^5\right)s^4 \\ &\quad + \left(16+16\,t+4\,t^2+4\,t^3+t^4\right)s^5. \end{aligned}$$

Note that $f_5(s,t) = f_5(t,s)$. For the sake of convenience, we let

$$f_{ii}(s,t) = f_i(t,s), \qquad i = 1,2,3,4. \tag{2.1}$$

By convention f_i and f_{ii} mean $f_i(s,t)$ and $f_{ii}(s,t)$. To study $S_m(q)$ in the 4-body collinear case, we need to introduce an algebraic curve $\bar{\Gamma}_{31}$ in the first quadrant of st-plane.

Definition 2.1 Let

$$\begin{aligned} g_{31}(s,t) = -(&f_{33}\,f_2{}^2 + f_{33}\,f_{44}\,f_4 - f_{44}\,f_2\,f_1 - f_{44}{}^2 f_1 \\ &+ f_{44}\,f_2\,f_{33} + f_{44}{}^2 f_3 - f_{11}\,f_2{}^2 - f_{11}\,f_{44}\,f_4 \\ &+ f_{22}\,f_3\,f_2 + f_{22}\,f_4\,f_1 - f_{22}\,f_4\,f_{33} - f_{22}\,f_2\,f_1), \end{aligned} \tag{2.2}$$

and

$$\Gamma_{31} = \{(s,t) | g_{31}(s,t) = 0, s > 0, t > 0\}, \tag{2.3}$$

$$\bar{\Gamma}_{31} = \{(s,t) \in \Gamma_{31} | s_0 < s, t < s_1\}, \tag{2.4}$$

where s_0, s_1 are constant and are determined later in proposition 2.5. Numerically $s_0 = 1.297093169$, $s_1 = 3.086044724$ (see Fig. 1). Let $\bar{s}$ be the unique positive root of the polynomial:

$$s^7 + 7\,s^6 + 19\,s^5 - 84\,s^3 - 152\,s^2 - 112\,s - 32 = 0 \tag{2.5}$$

and numerically $\bar{s} = 2.162120398$. We call g_{31} and $\bar{\Gamma}_{31}$ the *singular polynomial* and *singular curve*.

To understand the singular curve $\bar{\Gamma}_{31}$, note that by Descartes' rule of signs (see proposition 5.1 below), the equation $g_{31}(s,t) = 0$ defines an implicit function of s for $0 < s < \bar{s}$, and an implicit function of t for $0 < t < \bar{s}$. Further details on properties of the singular curve $\bar{\Gamma}_{31}$ are given in proposition 2.5 below. Our main theorem 1.3 is restated as follows.

Theorem 2.2 *Let* $q = (-s-1, -1, 1, t+1)$ *with* $s > 0, t > 0$ *and* $m = (m_1, m_2, m_3, m_4)$ *with* $m_i > 0 (i = 1,2,3,4)$. *Suppose* G *is the open region given by* $h(s,t) > 0$ *in the first quadrant* (*see* Fig. 2).

(1) ${}^{\#}\tilde{S}(q) = 0$ *if* $(s,t) \notin G$; ${}^{\#}\tilde{S}(q) \geq 2$ *if* $(s,t) \in G$.
(2) *If* $m \in S(q)$, *then either* ${}^{\#}S_m(q) = 0$ *or* ${}^{\#}S_m(q) = 1$.
(3) ${}^{\#}S_m(q) = 1$ *only in the following three cases.*

Fig. 2 Implicit function $h(s,t) = 0$ in st-plane

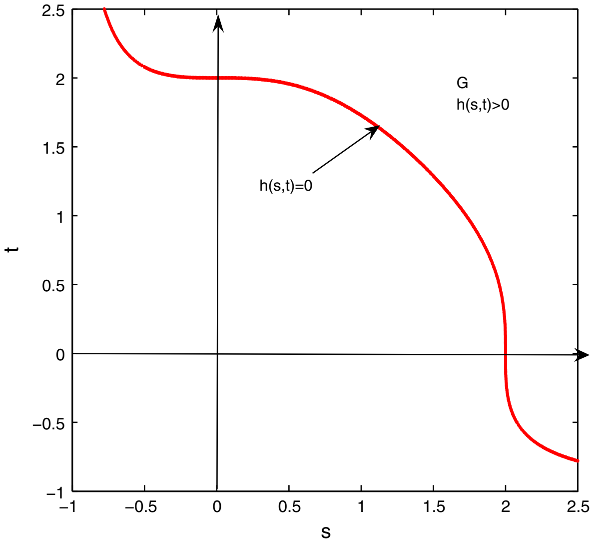
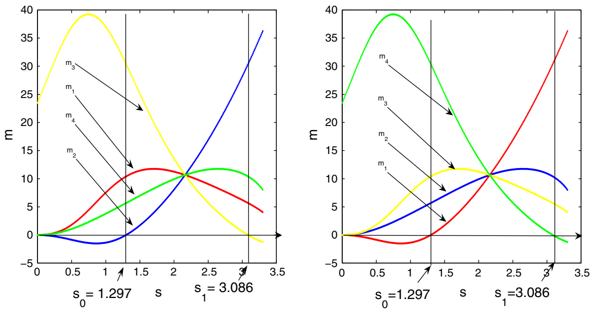


Fig. 3 $\lambda = 1$. *Left*: the graph of m given by (2.6) along $\bar{\Gamma}_{31}$. m is positive and $m_3 > m_1 > m_4 > m_2$ for $s_0 < s < \bar{s}$ and $m_2 > m_4 > m_1 > m_3$ for $\bar{s} < s < s_1$. *Right*: the graph of m given by (2.7) along $\bar{\Gamma}_{31}$. m is positive and $m_4 > m_3 > m_2 > m_1$ for $s_0 < s < \bar{s}$ and $m_1 > m_2 > m_3 > m_4$ for $\bar{s} < s < s_1$. At $\bar{s}$, $m_1 = m_2 = m_3 = m_4$



(*i*) *When* $s = t$, $S_m(q) = \{(m_4, m_3, m_2, m_1)\}$.

(*ii*) *When* $(s,t) \in \bar{\Gamma}_{31} \setminus \{(\bar{s}, \bar{s})\}$, *then* $S_m(q) = \{(m_2, m_4, m_1, m_3)\}$ *if* $m = (m_1, m_2, m_3, m_4) \in S(q)$ *given by* (*see the left of* Fig. 3)

$$
\begin{aligned}
m_1 &= \frac{\lambda\,\left(f_1\,f_{44}\,f_4 - f_2\,f_{44}\,f_1 + {f_2}^2 f_{33} + f_2\,f_{44}\,f_3\right)}{\left({f_2}^2 + f_{44}\,f_4\right)\,f_5},\\
m_2 &= \frac{\lambda\,\left(f_3\,{f_2}^2 + f_4\,f_2\,f_1 + f_1\,f_{44}\,f_4 - f_4\,f_2\,f_{33}\right)}{\left({f_2}^2 + f_{44}\,f_4\right)\,f_5},\\
m_3 &= \frac{\lambda\,\left({f_2}^2 f_{33} + f_{33}\,f_{44}\,f_4 - f_2\,f_{44}\,f_1 - {f_{44}}^2 f_1 + f_{44}\,f_2\,f_{33} + {f_{44}}^2 f_3\right)}{\left({f_2}^2 + f_{44}\,f_4\right)\,f_5},\\
m_4 &= -\frac{\lambda\,\left(-f_{11}\,{f_2}^2 - f_{11}\,f_{44}\,f_4 - f_{22}\,f_2\,f_1 - f_{22}\,f_{44}\,f_1 + f_{22}\,f_2\,f_{33} + f_{22}\,f_{44}\,f_3\right)}{\left({f_2}^2 + f_{44}\,f_4\right)\,f_5}.
\end{aligned}
\tag{2.6}
$$

(*iii*) *When* $(s,t) \in \bar{\Gamma}_{31} \setminus \{(\bar{s},\bar{s})\}$, *then* $S_m(q) = \{(m_3, m_1, m_4, m_2)\}$ *if* $m = (m_1, m_2, m_3, m_4) \in S(q)$ *given by* (*see the right of* Fig. 3)

$$\begin{aligned}
m_1 &= \frac{\lambda\left(f_1 f_4{}^2 + f_2 f_3 f_{22} + f_2 f_4 f_3 - f_2 f_4 f_{11}\right)}{\left(f_2 f_{22} + f_4{}^2\right) f_5}, \\
m_2 &= \frac{\lambda\left(f_2 f_3 f_{22} + f_4 f_1 f_{22} - f_4 f_3 f_{22} + f_4{}^2 f_{11}\right)}{\left(f_2 f_{22} + f_4{}^2\right) f_5}, \\
m_3 &= \frac{\lambda\left(f_{33} f_2 f_{22} + f_{33} f_4{}^2 - f_{44} f_1 f_{22} + f_{44} f_3 f_{22} + f_{44} f_4 f_3 - f_{44} f_4 f_{11}\right)}{\left(f_2 f_{22} + f_4{}^2\right) f_5}, \\
m_4 &= \frac{\lambda\left(f_{11} f_2 f_{22} + f_4{}^2 f_{11} + f_1 f_{22}{}^2 - f_3 f_{22}{}^2 - f_4 f_3 f_{22} + f_{22} f_4 f_{11}\right)}{\left(f_2 f_{22} + f_4{}^2\right) f_5}.
\end{aligned} \tag{2.7}$$

Remark 2.3 (i), (ii) and (iii) are proved in Cases 1, 3, and 9, respectively, in Sect. 4. A numerical example here is given to show a relation between (ii) and (iii).
For $(s,t) = (1.8, 2.542561504) \in \bar{\Gamma}_{31} \setminus \{(\bar{s},\bar{s})\}$, Eq. 2.6 gives

$$m = (11.69283646\lambda, 5.205495232\lambda, 17.74345114\lambda, 8.831033413\lambda)$$

and $S_m(q) = \{(5.205495232\lambda, 8.831033413\lambda, 11.69283646\lambda, 17.74345114\lambda)\}$.
For the same (s,t), Eq. 2.7 gives

$$m = (5.205495232\lambda, 8.831033413\lambda, 11.69283646\lambda, 17.74345114\lambda)$$

and $S_m(q) = \{(11.69283646\lambda, 5.205495232\lambda, 17.74345114\lambda, 8.831033413\lambda)\}$.
So (ii) and (iii) are corresponding to $m' \in S_m(q)$ and $m \in S_{m'}(q)$. This property is in proposition 2.5 and its proof is given in Sect. 5.

Remark 2.4 If $(s,t) \notin \bar{\Gamma}_{31}$, then ${}^{\#}S_m(q) = 0$ for any $m \in S(q)$. If $(s,t) \in \bar{\Gamma}_{31} \setminus \{(\bar{s},\bar{s})\}$ but m is not given by either (2.6) or (2.7), then ${}^{\#}S_m(q) = 0$.

Proposition 2.5 (*i*) $g_{31}(s,t) = g_{31}(t,s)$ *holds for all* $(s,t) \in (\mathbf{R}^+)^2$. *Then the points on* $g_{31}(s,t) = 0$ *are symmetric about* $s = t$.

(*ii*) *The unique intersection point of* $g_{31}(s,t) = 0$ *and* $s = t$ *is* $(\bar{s},\bar{s})$ *given by the unique positive root of the polynomial:*

$$s^7 + 7s^6 + 19s^5 - 84s^3 - 152s^2 - 112s - 32 = 0$$

and numerically $\bar{s} = 2.162120398$.

(*iii*) *For* $0 < s < \bar{s}$, t *is a continuous function of* s *implicitly defined by* $g_{31}(s,t) = 0$. *For* $0 < t < \bar{s}$, s *is a continuous function of* t *implicitly defined by* $g_{31}(s,t) = 0$. *Furthermore the curve* $\Gamma_{31} = \{(s,t) | g_{31}(s,t) = 0\}$ *is in the region* $\{(s,t) | 0 < s < 4, 0 < t < 4\}$ (*see Fig.* 1).

(*iv*) *If* m *is given by* (2.6), *then* $m_i > 0 (i = 1,2,3,4)$ *along the curve*

$$\bar{\Gamma}_{31} = \{(s,t) \in \Gamma_{31} | s_0 < s, t < s_1, \},$$

where (s_0,t_0) *is the intersection point of* $m_2 = 0$ *and* $g_{31} = 0$ *and* (s_1,t_1) *is the intersection point of* $m_3 = 0$ *and* $g_{31} = 0$. *Numerically* $s_0 = 1.297093169$, $s_1 = 3.086044724$, $t_0 = s_1$, *and* $t_1 = s_0$. *Furthermore, for* $s_0 < s < \bar{s}$, $m_3 > m_1 > m_4 > m_2$. *For* $\bar{s} < s < s_1$, $m_2 > m_4 > m_1 > m_3$ (*see the left of Fig.* 3).

(*v*) *If* m *is given by* (2.7) *then* $m_i > 0 (i = 1,2,3,4)$ *along the curve* $\bar{\Gamma}_{31}$ *and* m *is a permutation of those in Eq.* 2.6. *Furthermore, for* $s_0 < s < \bar{s}$, $m_4 > m_3 > m_2 > m_1$. *For* $\bar{s} < s < s_1$, $m_1 > m_2 > m_3 > m_4$ (*see the right of Fig.* 3).

3 Solutions of the collinear 4-body central configurations and some lemmas

Up to translation and scaling, the general collinear four-body configuration can be chosen as $q_1 = -s-1, q_2 = -1, q_3 = 1$, and $q_4 = t+1$ with $s > 0, t > 0$. Ouyang and Xie (2005) (pp. 151 and 152) found explicitly the unique solution of masses to the central configuration Eq. 1.2 by standard row reduction:

$$
\begin{aligned}
m_1 &= \left(\left(-\frac{-\lambda-\lambda c}{t^2}+\frac{-\lambda-\lambda c}{(t+2)^2}\right)t^{-2}-\frac{-\lambda(-t-1)-\lambda c}{4t^2}\right) \\
&\quad\times\left(\left(-\frac{1}{s^2t^2}+\frac{1}{(s+2)^2(t+2)^2}\right)t^{-2}-\frac{1}{4t^2(s+t+2)^2}\right)^{-1}; \qquad (3.1)\\
m_2 &= \left(\left(-\frac{-\lambda(s+1)-\lambda c}{t^2}+\frac{-\lambda-\lambda c}{(s+t+2)^2}\right)t^{-2}-\frac{-\lambda(-t-1)-\lambda c}{t^2(s+2)^2}\right) \\
&\quad\times\left(\left(\frac{1}{s^2t^2}+\frac{1}{4(s+t+2)^2}\right)t^{-2}-\frac{1}{(t+2)^2t^2(s+2)^2}\right)^{-1}; \qquad (3.2)\\
m_3 &= \left(\left(-\frac{-\lambda(t+1)+\lambda c}{s^2}+\frac{-\lambda+\lambda c}{(s+t+2)^2}\right)s^{-2}-\frac{-\lambda(-s-1)+\lambda c}{s^2(t+2)^2}\right) \\
&\quad\times\left(\left(\frac{1}{s^2t^2}+\frac{1}{4(s+t+2)^2}\right)s^{-2}-\frac{1}{s^2(t+2)^2(s+2)^2}\right)^{-1}; \qquad (3.3)\\
m_4 &= \left(\left(-\frac{-\lambda+\lambda c}{s^2}+\frac{-\lambda+\lambda c}{(s+2)^2}\right)s^{-2}-\frac{-\lambda(-s-1)+\lambda c}{4s^2}\right) \\
&\quad\times\left(\left(-\frac{1}{s^2t^2}+\frac{1}{(s+2)^2(t+2)^2}\right)s^{-2}-\frac{1}{4s^2(s+t+2)^2}\right)^{-1}. \qquad (3.4)
\end{aligned}
$$

We further simplify them to get:

$$
\begin{array}{ll}
m_1 = \frac{\lambda f_1 - f_2 u}{f_5}, & m_2 = \frac{\lambda f_3 + f_4 u}{f_5}, \\
m_3 = \frac{\lambda f_{33} - f_{44} u}{f_5}, & m_4 = \frac{\lambda f_{11} + f_{22} u}{f_5},
\end{array} \qquad (3.5)
$$

where c is the center of mass, $u = \lambda c$ and $f_i, f_{jj} (i = 1, 2, \ldots, 5; j = 1, 2, 3, 4)$ are polynomials of (s, t) defined in (2.1). So the solution m is uniquely determined by s, t, λ, u. On the other hand, Moulton (1910) proved that for a fixed mass vector $m = (m_1, m_2, \ldots, m_n)$ and a fixed ordering of the bodies along the line, there exists a unique collinear central configuration (up to translation and scaling). So the parameters s, t, λ and u are also uniquely determined by m.

By direct computation, we have

Lemma 3.1 *For any $s > 0, t > 0$,*

$$
\begin{aligned}
&(1) \qquad f_2 > f_4 > 0,\ f_{22} > f_{44} > 0, &(3.6)\\
&(2) \qquad f_2 - f_4 + f_{44} - f_{22} = 0, &(3.7)\\
&(3) \qquad f_1 + f_{11} + f_3 + f_{33} > 0, &(3.8)\\
&(4) \qquad f_2 = f_{22},\ f_4 = f_{44} \text{ if only if } s = t. &(3.9)
\end{aligned}
$$

The notations and definitions in this paper are similar to those in Long and Sun (2004), Ouyang and Xie (2005) and Xie (2010). For any natural number $n \in \mathbf{N}$, we denote by $P(n)$ the set of all permutations of $\{1, 2, \ldots, n\}$. For any element $\tau \in P(n)$, we use $\tau = (\tau(1), \tau(2), \ldots, \tau(n))$ to denote the permutation τ. We also denote a permutation of

$(m_1, m_2, \ldots, m_n)$ by $m(\tau) = (m_{\tau(1)}, m_{\tau(2)}, \ldots, m_{\tau(n)})$ for $\tau \in P(n)$. We define the converse permutation of τ by $con(\tau) = (\tau(n), \ldots, \tau(1))$ and define the converse of $q = (q_1, q_2, q_3, q_4)$ by $con(q) = -(q_4, q_3, q_2, q_1)$.

In order to study the set $S_m(q)$ for given $q = (-s-1, -1, 1, t+1)$ and $m = (m_1, m_2, m_3, m_4) \in S(q)$, we have to discuss the corresponding twenty-four permutations of the given mass m:

$$\begin{aligned}
\tau_1 &= (1,2,3,4), & \tau_2 &= (2,1,4,3), & \tau_3 &= (2,4,1,3);\\
\tau_4 &= (3,4,2,1), & \tau_5 &= (2,3,4,1), & \tau_6 &= (4,3,1,2);\\
\tau_7 &= (4,1,2,3), & \tau_8 &= (1,3,4,2), & \tau_9 &= (1,4,2,3);\\
\tau_{10} &= (2,3,1,4), & \tau_{11} &= (1,3,2,4), & \tau_{12} &= (3,1,2,4);
\end{aligned}$$

and the 12 converse permutations $con(\tau_i), i = 1, 2, \ldots, 12$. If q is a central configuration for $m(\tau)$ with $\tau \in P(4)$ (which always means that $m_{\tau(i)}$ is put on q_i for all $i = 1, 2, 3, 4$), by Eq. 3.5 we have

$$\begin{aligned}
m_{\tau(1)} &= \tfrac{\lambda(\tau) f_1 - f_2 u(\tau)}{f_5}, & m_{\tau(2)} &= \tfrac{\lambda(\tau) f_3 + f_4 u(\tau)}{f_5},\\
m_{\tau(3)} &= \tfrac{\lambda(\tau) f_{33} - f_{44} u(\tau)}{f_5}, & m_{\tau(4)} &= \tfrac{\lambda(\tau) f_{11} + f_{22} u(\tau)}{f_5}.
\end{aligned} \tag{3.10}$$

If $\tau = \tau_1 = (1, 2, 3, 4)$, λ, u will be used as in Eq. 3.5 instead of $\lambda(\tau_1)$ or $u(\tau_1)$. We first prove the following lemma.

Lemma 3.2 *Fix $q = (-s-1, -1, 1, t+1)$ with $(s, t) \in G$. Suppose $m = (m_1, m_2, m_3, m_4) \in S(q)$ and $\tau \in P(4)$.*

(i) If $m(\tau) \in S(q)$, then $\lambda(\tau) = \lambda$.
(ii) If $m(\tau) \in S(q)$ and $m_{\tau(i)} = m_i$ for some i, then $u(\tau) = u$. Therefore $m_{\tau(i)} = m_i$ for all $i = 1, 2, 3, 4$, i.e. $m(\tau) = m$.

Proof of Lemma 3.2 (i) For given $q = (-s-1, -1, 1, t+1)$ and $\tau \in P(4)$,

$$M = m_1 + m_2 + m_3 + m_4 = m_{\tau(1)} + m_{\tau(2)} + m_{\tau(3)} + m_{\tau(4)}.$$

From Eqs. 3.5 and 3.10,

$$\begin{aligned}
M &= \frac{\lambda(f_1 + f_3 + f_{11} + f_{33}) - u(f_2 - f_4 + f_{44} - f_{22})}{f_5}\\
&= \frac{\lambda(\tau)(f_1 + f_3 + f_{11} + f_{33}) - u(\tau)(f_2 - f_4 + f_{44} - f_{22})}{f_5}.
\end{aligned}$$

By Lemma 3.1, $f_1 + f_3 + f_{11} + f_{33} > 0$ and $f_2 - f_4 + f_{44} - f_{22} = 0$, we have $\lambda(\tau) = \lambda$. From now on, λ will be used for $m(\tau)$ instead of $\lambda(\tau)$.

(ii) If $m_{\tau(i)} = m_i$ for some i, and if $i = 1$ we have

$$m_{\tau(1)} = \frac{\lambda f_1 - f_2 u(\tau)}{f_5} = m_1 = \frac{\lambda f_1 - f_2 u}{f_5}$$

which implies $u(\tau) = u$ because $f_2 > 0$. Then $m_{\tau(j)} = m_j$ for all $j = 1, 2, 3, 4$ because Eq. 3.5 is same as Eq. 3.10 for same s, t, λ, u. Similarly, $m_{\tau(i)} = m_i$ is true for $i = 2, 3, 4$.

Thus the proof of Lemma 3.2 is complete. □

Remark 3.3 Suppose $m = (m_1, m_2, m_3, m_4)$ and $m \in S(q)$. If $m(\tau_8) \in S(q)$, we must have $m = m(\tau_8)$ from (ii) of Lemma 3.2 because $m_1 = m_{\tau_8(1)}$. So $m(\tau_8) \notin S_m(q)$. Similarly, $m(\tau_i) \notin S_m(q),\ i = 8, 9, 10, 11, 12$, and $m(con(\tau_i)) \notin S_m(q), i = 4, 5, \ldots, 12$ for any q. To find all q such that $S_m(q)$ is not empty, we only need to find conditions for q such that $m(\tau_i) \in S_m(q),\ i = 2, 3, 4, 5, 6, 7$ or $m(con(\tau_j)) \in S_m(q),\ j = 1, 2, 3$. So we only need to study nine cases.

4 Proof of the main Theorem 2.2

Proof (1) Here we first shall borrow Lemma 4.1 from Ouyang and Xie (2005). □

Lemma 4.1 (Ouyang and Xie 2005 Theorem 2). *Let $q = (q_1, q_2, q_3, q_4) = (-s-1, -1, 1, t+1)$ be a configuration in the collinear four-body problem. Then there exists an unbounded region G (see Fig. 2) in the first quadrant ($s > 0, t > 0$) bounded away from the origin by the implicit function $h(s,t) = 0$, such that for any $(s,t) \in G$, there exists a positive mass vector $m = (m_1, m_2, m_3, m_4)$ making q as a central configuration. Conversely for any (s,t) in $E = (\mathbf{R}^+)^2 \setminus G$, there is no positive mass $m = (m_1, m_2, m_3, m_4)$ making $q = (-s-1, -1, 1, t+1)$ central, where $(\mathbf{R}^+)^2$ is the first quadrant in the plane.*

If q is in the region $G, {}^{\#}\tilde{S}(q) \geq 1$ and if q is not in the region $G, {}^{\#}\tilde{S}(q) = 0$ by lemma 4.1. Given any configuration $q \in G$, there exists at least one u such that m_1, m_2, m_3, m_4 are all positive. By continuity, $m = (m_1, m_2, m_3, m_4)$ is positive in a neighborhood of u from Eq. 3.5. Then the set $S(q)$ has $u = \lambda c$ as a natural parameter and ${}^{\#}\tilde{S}(q) \geq 2$.

(2) We prove it by contradiction. Suppose $m(\alpha) \in S_m(q)$ and $m(\beta) \in S_m(q)$ where $\alpha \in P(4)$ and $\beta \in P(4)$. By Lemma 3.2 we have

$$\lambda = \lambda(\alpha) = \lambda(\beta).$$

Without loss of generality, we can assume $u(\alpha) < u(\beta)$. So u, $u(\alpha)$ and $u(\beta)$ give us three distinct solutions m, $m(\alpha)$ and $m(\beta)$ through Eq. 3.10 for same s, t, λ. Note that $m_{\alpha(i)},\ i = 1, 2, 3, 4$ are linear functions of $u(\alpha)$. Because $f_2 > 0$, $f_4 > 0$, $f_{22} > 0$, and $f_{44} > 0$ for all $s > 0, t > 0$, $m_{\alpha(1)}, m_{\alpha(3)}$ are decreasing with respect to $u(\alpha)$ and $m_{\alpha(2)}, m_{\alpha(4)}$ are increasing with respect to $u(\alpha)$ (see Fig. 4). Then $\max\{m_{\alpha(i)}, i = 1, 2, 3, 4\}$ must be either $m_{\alpha(1)}$ or $m_{\alpha(3)}$ i.e.

$$\max\{m_{\alpha(i)}, i = 1, 2, 3, 4\} = \max\{m_{\alpha(i)}, i = 1, 3\}.$$

If not, i.e $\max\{m_{\alpha(i)}, i = 1, 2, 3, 4\}$ is either $m_{\alpha(2)}$ or $m_{\alpha(4)}$, then

$$\max\{m_{\beta(i)}, i = 2, 4\} > \max\{m_{\alpha(i)}, i = 2, 4\} = \max\{m_{\alpha(i)}, i = 1, \ldots, 4\}$$

because $u(\beta) > u(\alpha)$. Then

$$\begin{aligned}\max\{m_{\beta(i)}, i = 1, 2, 3, 4\} &\geq \max\{m_{\beta(i)}, i = 2, 4\}\\ &> \max\{m_{\alpha(i)}, i = 1, 2, 3, 4\}\end{aligned}$$

which contradicts $\{m_{\beta(i)}, i = 1, 2, 3, 4\} = \{m_{\alpha(i)}, i = 1, 2, 3, 4\}$. The contradiction confirms that

$$\max\{m_{\alpha(i)}, i = 1, 2, 3, 4\} = \max\{m_{\alpha(1)}, m_{\alpha(3)}\}.$$

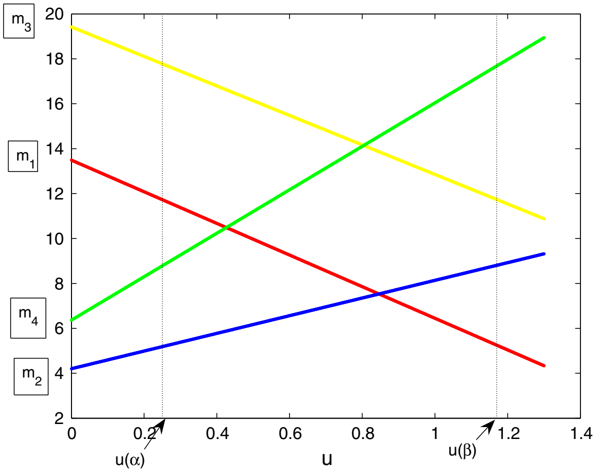


Fig. 4 The graph m_i of (3.5) for fixed $(s,t) = (1.8, 2.542561504)$ and $\lambda = 1$. $m(\beta)$ is a permutation of $m(\alpha)$

Similar arguments imply

$$\max\{m_{\beta(i)}, i = 1, 2, 3, 4\} = \max\{m_{\beta(2)}, m_{\beta(4)}\}.$$

Because $u, u(\alpha), u(\beta)$ are distinct, then one of the following cases must apply: (a) $u < u(\alpha)$, (b) $u(\alpha) < u < u(\beta)$, (c) $u(\beta) < u$. However, none of the three cases is true.

In fact, suppose (a) $u < u(\alpha)$ is true. Then

$$\begin{aligned}\max\{m_i, i = 1, 2, 3, 4\} &\geq \max\{m_1, m_3\}\\ &> \max\{m_{\alpha(1)}, m_{\alpha(3)}\} = \max\{m_{\alpha(i)}, i = 1, 2, 3, 4\},\end{aligned}$$

which contradicts $\{m_i, i = 1, 2, 3, 4\} = \{m_{\alpha(i)}, i = 1, 2, 3, 4\}$.

Suppose (b) $u(\alpha) < u < u(\beta)$ is true.

There are two cases. If $\max\{m_i, i = 1, 2, 3, 4\} = \max\{m_1, m_3\}$

$$\begin{aligned}\max\{m_{\alpha(1)}, m_{\alpha(3)}\} &= \max\{m_{\alpha(i)}, i = 1, 2, 3, 4\}\\ &> \max\{m_i, i = 1, 2, 3, 4\},\end{aligned}$$

and if $\max\{m_i, i = 1, 2, 3, 4\} = \max\{m_2, m_4\}$,

$$\max\{m_{\beta(2)}, m_{\beta(4)}\} = \max\{m_{\beta(i)}, i = 1, \ldots, 4\} > \max\{m_i, i = 1, \ldots, 4\}.$$

Both of them contradict

$$\{m_i, i = 1, \ldots, 4\} = \{m_{\alpha(i)}, i = 1, \ldots, 4\} = \{m_{\beta(i)}, i = 1, \ldots, 4\}.$$

Suppose (c) $u(\beta) < u$, then

$$\begin{aligned}\max\{m_{\beta(i)}, i = 1, 2, 3, 4\} &= \max\{m_{\beta(2)}, m_{\beta(4)}\}\\ &< \max\{m_2, m_4\} \leq \max\{m_i, i = 1, 2, 3, 4\},\end{aligned}$$

which contradicts $\{m_i, i = 1, 2, 3, 4\} = \{m_{\beta(i)}, i = 1, 2, 3, 4\}$.

The contradictions prove that $S_m(q)$ does not have more than two elements, i.e. ${}^{\#}S_m(q) = 0$ or ${}^{\#}S_m(q) = 1$.

(3) By Remark 3.3, we prove part (3) in Theorem 2.2 by checking nine cases one by one. We suppose $m=(m_1,m_2,m_3,m_4)$, $q=(-s-1,-1,1,t+1)$, $m\in S(q)$ and $m(\tau)\neq m$ for each case.

Case 1 $con(\tau_1)=(4,3,2,1)$: If $s=t$, $m(con(\tau_1))\in S_m(q)$.

For a given $q=(-s-1,-1,1,t+1)$, if $m=(m_1,m_2,m_3,m_4)\in S(q)$, then $m(con(\tau_1))\in S(con(q))$ with $u(con(\tau_1))=-u$. By the uniqueness of central configuration for a given order of mass, $m(con(\tau_1))\in S_m(q)$ if and only if $con(q)=q$ if and only if $s=t$. In order to have $m(con(\tau_1))\neq m$, u must be nonzero.

Case 2 $\tau_2=(2,1,4,3)$: $m(\tau_2)\notin S_m(q)$ for any q.

If $m(\tau_2)\in S_m(q)$, $m_1=m_{\tau_2(2)}$ and $m_2=m_{\tau_2(1)}$. From Eqs. 3.5 and 3.10 we have

$$\lambda f_1-f_2u=\lambda f_3+f_4u(\tau_2),\qquad \lambda f_3+f_4u=\lambda f_1-f_2u(\tau_2). \tag{4.1}$$

Adding the two equations and rearranging the terms, we have $(-f_2+f_4)u=(-f_2+f_4)u(\tau_2)$ which implies that $u(\tau_2)=u$ because $f_2>f_4>0$ for any q. Then $m(\tau_2)=m$, i.e. $m(\tau_2)$ is not in $S_m(q)$ by definition.

Case 3 $\tau_3=(2,4,1,3)$: If $(s,t)\in\bar{\Gamma}_{31}\setminus\{(\bar{s},\bar{s})\}$, and if $m=(m_1,m_2,m_3,m_4)\in S(q)$ is given by (4.3), then $m(\tau_3)\in S_m(q)$.

Step 1: By setting $m_1=m_{\tau_3(3)}$ and $m_2=m_{\tau_3(1)}$, we solve for u and $u(\tau_3)$ from Eqs. 3.5 and 3.10,

$$\begin{aligned} u&=\frac{\lambda\,(f_2\,f_1+f_{44}\,f_1-f_2\,f_{33}-f_{44}\,f_3)}{f_2{}^2+f_{44}\,f_4},\\ u(\tau_3)&=\frac{\lambda\,(f_2\,f_1+f_4\,f_{33}-f_2\,f_3-f_4\,f_1)}{f_2{}^2+f_{44}\,f_4}. \end{aligned} \tag{4.2}$$

Step 2: Substituting u into Eq. 3.5, we have

$$\begin{aligned} m_1&=\frac{\lambda\,\left(f_1\,f_{44}\,f_4-f_2\,f_{44}\,f_1+f_2{}^2f_{33}+f_2\,f_{44}\,f_3\right)}{\left(f_2{}^2+f_{44}\,f_4\right)\,f_5},\\ m_2&=\frac{\lambda\,\left(f_3\,f_2{}^2+f_4\,f_2\,f_1+f_1\,f_{44}\,f_4-f_4\,f_2\,f_{33}\right)}{\left(f_2{}^2+f_{44}\,f_4\right)\,f_5},\\ m_3&=\frac{\lambda\,\left(f_2{}^2f_{33}+f_{33}\,f_{44}\,f_4-f_2\,f_{44}\,f_1-f_{44}{}^2f_1+f_{44}\,f_2\,f_{33}+f_{44}{}^2f_3\right)}{\left(f_2{}^2+f_{44}\,f_4\right)\,f_5},\\ m_4&=-\frac{\lambda\,\left(-f_{11}\,f_2{}^2-f_{11}\,f_{44}\,f_4-f_{22}\,f_2\,f_1-f_{22}\,f_{44}\,f_1+f_{22}\,f_2\,f_{33}+f_{22}\,f_{44}\,f_3\right)}{\left(f_2{}^2+f_{44}\,f_4\right)\,f_5}, \end{aligned} \tag{4.3}$$

which is the Eq. 2.6. $m_3-m_{\tau_3(4)}=\frac{\lambda g_{31}}{(f_2{}^2+f_{44}\,f_4)f_5}$ and $m_4-m_{\tau_3(2)}=\frac{\lambda g_{32}}{(f_2{}^2+f_{44}\,f_4)f_5}$, where g_{31} is given by (2.2) and

$$\begin{aligned} g_{32}=&-(-f_{11}\,f_2{}^2-f_{11}\,f_{44}\,f_4-f_{22}\,f_2\,f_1-f_{22}\,f_{44}\,f_1+f_{22}\,f_2\,f_{33}\\ &+f_{22}\,f_{44}\,f_3+f_3\,f_2{}^2+f_3\,f_{44}\,f_4-f_4\,f_3\,f_2-f_4{}^2f_1+f_4{}^2f_{33}+f_4\,f_2\,f_1). \end{aligned}$$

$m_3=m_{\tau_3(4)}$ and $m_4=m_{\tau_3(2)}$ are equivalent to $g_{31}=0$ and $g_{32}=0$. Note that

$$\begin{aligned} g_{31}-g_{32}=&-(f_2-f_4+f_{44}-f_{22})\,(f_3\,f_2-f_2\,f_{33}+f_4\,f_1\\ &+f_{44}\,f_1-f_{44}\,f_3-f_4\,f_{33}), \end{aligned}$$

and $(f_2 - f_4 + f_{44} - f_{22}) = 0$ for all $s > 0, t > 0$ by Lemma 3.1. Then $g_{31} = g_{32}$ for all $s > 0, t > 0$. Therefore if (s, t) is a point on the implicit curve $g_{31}(s, t) = 0$ (see Fig. 1) and m given by (4.3) is positive, then $m(\tau_3) \in S_m(q)$. From the properties of Γ_{31} and $\bar{\Gamma}_{31}$ in proposition 2.5, if $(s, t) \in \bar{\Gamma}_{31} \setminus \{(\bar{s}, \bar{s})\}$, then m given by (4.3) is positive and $m(\tau_3) \in S_m(q)$.

Case 4 $\tau_4 = (3, 4, 2, 1)$: $m(\tau_4) \notin S_m(q)$.

Step 1: By setting $m_1 = m_{\tau_4(4)}$ and $m_2 = m_{\tau_4(3)}$, we solve for u and $u(\tau_3)$ from Eqs. 3.5 and 3.10,

$$\begin{aligned} u &= \tfrac{\lambda\,(f_1\,f_{44}-f_{11}\,f_{44}+f_{22}\,f_3-f_{22}\,f_{33})}{f_2\,f_{44}-f_4\,f_{22}}, \\ u(\tau_4) &= -\tfrac{\lambda\,(f_2\,f_3-f_2\,f_{33}+f_4\,f_1-f_4\,f_{11})}{f_2\,f_{44}-f_4\,f_{22}}. \end{aligned} \tag{4.4}$$

Step 2: Substituting u and $u(\tau_3)$ into Eqs. 3.5 and 3.10, $m_3 - m_{\tau_4(1)} = \frac{\lambda g_{41}}{(f_2\,f_{44}-f_4\,f_{22})f_5}$ where

$$\begin{aligned} g_{41} = &-f_{33}\,f_2\,f_{44} + f_4\,f_{22}\,f_{33} + f_1\,f_{44}{}^2 - f_{11}\,f_{44}{}^2 + f_{44}\,f_{22}\,f_3 \\ &-f_{44}\,f_{22}\,f_{33} + f_1\,f_2\,f_{44} - f_1\,f_4\,f_{22} + f_2{}^2 f_3 - f_2{}^2 f_{33} \\ &+f_2\,f_4\,f_1 - f_2\,f_4\,f_{11}. \end{aligned}$$

$m_3 = m_{\tau_4(1)}$ is equivalent $g_{41} = 0$. By direct computation $g_{41} = (t - s)g_{411}$ where g_{411} is a polynomial with all positive terms. So t must be equal to s if $g_{41} = 0$. But when $t = s$, $m(con(\tau_1)) \in S_m(q)$ and then $m = m(\tau_4)$. Therefore $m(\tau_4) \notin S_m(q)$.

Case 5 $\tau_5 = (2, 3, 4, 1)$: $m(\tau_5) \notin S_m(q)$.

Step 1: By setting $m_1 = m_{\tau_5(4)}$ and $m_2 = m_{\tau_5(1)}$, we solve for u and $u(\tau_5)$ from Eqs. 3.5 and 3.10,

$$\begin{aligned} u &= \tfrac{\lambda\,(f_2\,f_1-f_{22}\,f_1-f_2\,f_{11}+f_{22}\,f_3)}{f_2{}^2-f_{22}\,f_4}, \\ u(\tau_5) &= \tfrac{\lambda\,(-f_3\,f_2-f_4\,f_1+f_4\,f_{11}+f_2\,f_1)}{f_2{}^2-f_{22}\,f_4}. \end{aligned} \tag{4.5}$$

Step 2: Substituting u and $u(\tau_5)$ into Eqs. 3.5 and 3.10, we have

$$\begin{aligned} m_1 &= \frac{\lambda\left(f_1\,f_{22}\,f_4 - f_2\,f_{22}\,f_1 - f_2{}^2 f_{11} + f_2\,f_{22}\,f_3\right)}{\left(-f_2{}^2 + f_{22}\,f_4\right) f_5}, \\ m_2 &= -\frac{\lambda\left(f_3\,f_2{}^2 + f_4\,f_2\,f_1 - f_1\,f_{22}\,f_4 - f_4\,f_2\,f_{11}\right)}{\left(-f_2{}^2 + f_{22}\,f_4\right) f_5}, \\ m_3 &= \frac{\lambda\left(-f_{33}\,f_2{}^2 + f_{33}\,f_{22}\,f_4 + f_{44}\,f_2\,f_1 - f_{44}\,f_{22}\,f_1 - f_{44}\,f_2\,f_{11} + f_{44}\,f_{22}\,f_3\right)}{\left(-f_2{}^2 + f_{22}\,f_4\right) f_5}, \\ m_4 &= \frac{\lambda\left(-f_2{}^2 f_{11} + f_{11}\,f_{22}\,f_4 - f_2\,f_{22}\,f_1 + f_{22}{}^2 f_1 + f_{22}\,f_2\,f_{11} - f_{22}{}^2 f_3\right)}{\left(-f_2{}^2 + f_{22}\,f_4\right) f_5}, \end{aligned} \tag{4.6}$$

and $m_3 - m_{\tau_5(2)} = \frac{\lambda g_{51}}{(-f_2{}^2+f_{22}\,f_4)\,f_5}$ and $m_4 - m_{\tau_5(3)} = \frac{\lambda g_{51}}{(-f_2{}^2+f_{22}\,f_4)\,f_5}$, where

$$\begin{aligned} g_{51} &= f_{33}\,f_2{}^2 - f_{33}\,f_{22}\,f_4 - f_{44}\,f_2\,f_1 + f_{44}\,f_{22}\,f_1 + f_{44}\,f_2\,f_{11} \\ &\quad - f_{44}\,f_{22}\,f_3 - f_3\,f_2{}^2 + f_3\,f_{22}\,f_4 + f_4\,f_3\,f_2 + f_4{}^2\,f_1 - f_4{}^2\,f_{11} - f_4\,f_2\,f_1 \end{aligned} \tag{4.7}$$

$m_3 = m_{\tau_5(2)}$ and $m_4 = m_{\tau_5(3)}$ are equivalent to $g_{51} = 0$. However, m_1 is always negative along the implicit curve Γ_{51} on which $g_{51} = 0$ (see Fig. 6). So $m(\tau_5) \notin S_m(q)$ for any $m \in S(q)$. Further details on properties of the implicit curves Γ_{51} and $m_1 = 0$ are given in Proposition 5.3 below.

Case 6 $\tau_6 = (4, 3, 1, 2)$: $m(\tau_6) \notin S_m(q)$.

Step 1: By setting $m_1 = m_{\tau_6(3)}$ and $m_2 = m_{\tau_6(4)}$, we solve for u and $u(\tau_6)$ from Eqs. 3.5 and 3.10,

$$\begin{aligned} u &= \frac{\lambda\,(f_{44}\,f_3 - f_{22}\,f_{33} - f_{44}\,f_{11} + f_{22}\,f_1)}{-f_{44}\,f_4 + f_{22}\,f_2}, \\ u(\tau_6) &= \frac{\lambda\,(f_1\,f_4 + f_2\,f_3 - f_2\,f_{11} - f_{33}\,f_4)}{-f_{44}\,f_4 + f_{22}\,f_2}. \end{aligned}$$

Step 2: $m_3 - m_{\tau_6(2)} = \frac{\lambda g_{61}}{(-f_{44}\,f_4+f_{22}\,f_2)\,f_5}$, where

$$\begin{aligned} g_{61} + g_{41} = (f_2 + f_{44} - f_{22} - f_4)\,(&-f_{44}\,f_1 + f_{44}\,f_3 \\ &+f_{33}\,f_4 + f_{33}\,f_2 - f_1\,f_4 - f_2\,f_3) \end{aligned}$$

which is identically zero by Lemma 3.1. So $m_3 = m_{\tau_6(2)}$ is equivalent to $g_{41} = 0$ which implies that $s = t$ from Case 4. Then $m(\tau_6) \notin S_m(q)$.

Case 7 $\tau_7 = (4, 1, 2, 3)$: $m(\tau_7) \notin S_m(q)$.

Step 1: By setting $m_1 = m_{\tau_7(2)}$ and $m_2 = m_{\tau_7(3)}$, we solve for u and $u(\tau_7)$ from Eqs. 3.5 and 3.10,

$$\begin{aligned} u &= \frac{\lambda\,(f_1\,f_{44} - f_3\,f_{44} + f_4\,f_3 - f_4\,f_{33})}{f_2\,f_{44} - f_4{}^2}, \\ u(\tau_7) &= -\frac{\lambda\,(f_2\,f_3 - f_2\,f_{33} + f_4\,f_1 - f_4\,f_3)}{f_2\,f_{44} - f_4{}^2}. \end{aligned} \tag{4.8}$$

Step 2: Substituting u and $u(\tau_7)$ into Eqs. 3.5 and 3.10, we have

$$\begin{aligned} m_1 &= -\frac{\lambda\left(f_1\,f_4{}^2 - f_2\,f_3\,f_{44} + f_4\,f_2\,f_3 - f_2\,f_4\,f_{33}\right)}{(f_2\,f_{44} - f_4{}^2)\,f_5}, \\ m_2 &= \frac{\lambda\left(f_2\,f_3\,f_{44} + f_1\,f_{44}\,f_4 - f_{44}\,f_4\,f_3 - f_{33}\,f_4{}^2\right)}{(f_2\,f_{44} - f_4{}^2)\,f_5}, \\ m_3 &= -\frac{\lambda\left(-f_{33}\,f_2\,f_{44} + f_{33}\,f_4{}^2 + f_1\,f_{44}{}^2 - f_3\,f_{44}{}^2 + f_{44}\,f_4\,f_3 - f_{44}\,f_4\,f_{33}\right)}{(f_2\,f_{44} - f_4{}^2)\,f_5}, \\ m_4 &= \frac{\lambda\left(f_{11}\,f_2\,f_{44} - f_{11}\,f_4{}^2 + f_{22}\,f_1\,f_{44} - f_{22}\,f_3\,f_{44} + f_{22}\,f_4\,f_3 - f_{22}\,f_4\,f_{33}\right)}{(f_2\,f_{44} - f_4{}^2)\,f_5}, \end{aligned} \tag{4.9}$$

and $m_3 - m_{\tau_7(4)} = \frac{\lambda g_{71}}{(f_2\, f_{44} - f_4{}^2) f_5}$ and $m_4 - m_{\tau_7(1)} = \frac{\lambda g_{71}}{(f_2\, f_{44} - f_4{}^2) f_5}$, where

$$\begin{aligned} g_{71} &= f_{33}\, f_2\, f_{44} - f_{33}\, f_4{}^2 - f_1\, f_{44}{}^2 + f_3\, f_{44}{}^2 - f_{44}\, f_4\, f_3 + f_{44}\, f_4\, f_{33} \\ &\quad - f_{11}\, f_2\, f_{44} + f_{11}\, f_4{}^2 + f_{22}\, f_2\, f_3 - f_{22}\, f_2\, f_{33} + f_{22}\, f_4\, f_1 - f_{22}\, f_4\, f_3. \end{aligned}$$

Furthermore, we notice that

$$\begin{aligned} g_{51} + g_{71} = (f_2 + f_{44} - f_4 - f_{22})\,(&-f_3\, f_2 + f_2\, f_{33} \\ &- f_{44}\, f_1 - f_4\, f_1 + f_{44}\, f_3 + f_{33}\, f_4). \end{aligned}$$

$m_3 = m_{\tau_7(4)}$ and $m_4 = m_{\tau_7(1)}$ are equivalent to $g_{51} = 0$. However, m_4 is always negative along the implicit curve Γ_{51} (see Fig. 6). So $m(\tau_7) \notin S_m(q)$ for any $m \in S(q)$. Further details on properties of the implicit curves Γ_{51} and $m_4 = 0$ are given in proposition 5.3 below.

Case 8 $con(\tau_2) = (3, 4, 1, 2)$: $m(con(\tau_2)) \notin S_m(q)$.

Step 1: By setting $m_1 = m_{con(\tau_2)(3)}$ and $m_2 = m_{con(\tau_2)(4)}$, we solve for u and $u(\tau_8)$ from Eqs. 3.5 and 3.10,

$$u = \frac{\lambda\, g_{81}}{-f_{44}\, f_4 + f_{22}\, f_2}, \quad u(con(\tau_2)) = \frac{\lambda\, g_{82}}{-f_{44}\, f_4 + f_{22}\, f_2}.$$

where $g_{81} = (f_{44}\, f_3 - f_{22}\, f_{33} - f_{44}\, f_{11} + f_{22}\, f_1)$,
$g_{82} = (f_1\, f_4 + f_2\, f_3 - f_2\, f_{11} - f_{33}\, f_4)$.

Step 2: $m_3 - m_{con(\tau_2)(1)} = \frac{\lambda (f_2 + f_{44})(g_{81} - g_{82})}{(-f_{44}\, f_4 + f_{22}\, f_2) f_5}$. $m_3 = m_{con(\tau_2)(1)}$ is equivalent to $g_{81} = g_{82}$ which implies $u(con(\tau_2)) = u$. Therefore $m(con(\tau_2)) = m$ and $m(con(\tau_2)) \notin S_m(q)$.

Case 9 $con(\tau_3) = (3, 1, 4, 2)$: If $(s, t) \in \bar{\Gamma}_{31} \setminus \{(\bar{s}, \bar{s})\}$ (see Fig. 1), and if $m = (m_1, m_2, m_3, m_4) \in S(q)$ is given by (4.11), then $m(con(\tau_3)) \in S_m(q)$.

Step 1: By setting $m_1 = m_{con(\tau_3)(2)}$ and $m_2 = m_{con(\tau_3)(4)}$, we solve for u and $u(con(\tau_3))$ from Eqs. 3.5 and 3.10,

$$\begin{aligned} u &= \frac{\lambda\,(f_1\, f_{22} - f_3\, f_{22} - f_4\, f_3 + f_4\, f_{11})}{f_2\, f_{22} + f_4{}^2}, \\ u(con(\tau_3)) &= \frac{\lambda\,(f_2\, f_3 - f_2\, f_{11} + f_4\, f_1 - f_4\, f_3)}{f_2\, f_{22} + f_4{}^2}. \end{aligned} \tag{4.10}$$

Step 2: Substituting u into Eq. 3.5, we have

$$\begin{aligned} m_1 &= \frac{\lambda\left(f_1\, f_4{}^2 + f_2\, f_3\, f_{22} + f_2\, f_4\, f_3 - f_2\, f_4\, f_11\right)}{(f_2\, f_{22} + f_4{}^2)\, f_5}, \\ m_2 &= \frac{\lambda\left(f_2\, f_3\, f_{22} + f_4\, f_1\, f_{22} - f_4\, f_3\, f_{22} + f_4{}^2 f_{11}\right)}{(f_2\, f_{22} + f_4{}^2)\, f_5}, \\ m_3 &= \frac{\lambda\left(f_{33}\, f_2\, f_{22} + f_{33}\, f_4{}^2 - f_{44}\, f_1\, f_{22} + f_{44}\, f_3\, f_{22} + f_{44}\, f_4\, f_3 - f_{44}\, f_4\, f_{11}\right)}{(f_2\, f_{22} + f_4{}^2)\, f_5}, \\ m_4 &= \frac{\lambda\left(f_{11}\, f_2\, f_{22} + f_4{}^2 f_{11} + f_1\, f_{22}{}^2 - f_3\, f_{22}{}^2 - f_4\, f_3\, f_{22} + f_{22}\, f_4\, f_{11}\right)}{(f_2\, f_{22} + f_4{}^2)\, f_5}, \end{aligned} \tag{4.11}$$

which is Eq. 2.7. By direct computation, $m_3 - m_{con(\tau_3)(1)} = \frac{\lambda g_{91}}{(f_2 f_{22}+f_4{}^2) f_5}$ and $m_4 - m_{con(\tau_3)(3)} = \frac{\lambda g_{92}}{(f_2 f_{22}+f_4{}^2) f_5}$, where

$$
\begin{aligned}
g_{91} = {} & f_{33} f_2 f_{22} + f_{33} f_4{}^2 - f_{44} f_1 f_{22} + f_{44} f_3 f_{22} + f_{44} f_4 f_3 - f_{44} f_4 f_{11} \\
& - f_1 f_2 f_{22} - f_1 f_4{}^2 + f_2{}^2 f_3 - f_2{}^2 f_{11} + f_2 f_4 f_1 - f_2 f_4 f_3, \\
g_{92} = {} & - f_{11} f_2 f_{22} - f_4{}^2 f_{11} - f_1 f_{22}{}^2 + f_3 f_{22}{}^2 + f_4 f_3 f_{22} - f_{22} f_4 f_{11} \\
& + f_{33} f_2 f_{22} + f_{33} f_4{}^2 - f_{44} f_2 f_3 + f_{44} f_2 f_{11} - f_{44} f_4 f_1 + f_{44} f_4 f_3.
\end{aligned}
$$

$m_3 = m_{con(\tau_3)(1)}$ and $m_4 = m_{con(\tau_3)(3)}$ are equivalent to $g_{91} = 0$ and $g_{92} = 0$. Note that

$$
\begin{aligned}
g_{91} - g_{92} = {} & (f_2 - f_4 + f_{44} - f_{22})(f_2 f_3 - f_2 f_{11} \\
& - f_1 f_{22} + f_4 f_1 + f_3 f_{22} - f_4 f_{11}),
\end{aligned}
$$

and $(f_2 - f_4 + f_{44} - f_{22}) = 0$ for all $s > 0, t > 0$. Then $g_{91} = g_{92}$ for all $s > 0, t > 0$. We further note that

$$
\begin{aligned}
g_{31} - g_{91} = {} & (f_2 - f_4 + f_{44} - f_{22}) \\
& \times (-f_2 f_3 + f_2 f_{33} - f_{44} f_1 - f_4 f_1 + f_{44} f_3 + f_{33} f_4) \quad (4.12)
\end{aligned}
$$

Then $g_{91} = g_{31}$ for all $s > 0, t > 0$ because $f_2 - f_4 + f_{44} - f_{22} \equiv 0$. Therefore if (s, t) is a point on the implicit curve Γ_{31} (see Fig. 1) and m given by (4.11) is positive, then $m(con(\tau_3)) \in S_m(q)$. From the properties of Γ_{31} and $\bar{\Gamma}_{31}$ in Proposition 2.5 below, if $(s, t) \in \bar{\Gamma}_{31} \setminus \{(\bar{s}, \bar{s})\}$, then m given by (4.11) is positive and $m(\tau_3) \in S_m(q)$.

The proof of Theorem 2.2 is complete.

5 Properties of singular curve $\bar{\Gamma}_{31}$

In this section, we study the implicit curves $g_{31}(s, t) = 0$, $g_{51}(s, t) = 0$, Γ_{51}, Γ_{31}, and $\bar{\Gamma}_{31}$ in more details. We need Descartes' rule of signs for polynomials:

Proposition 5.1 cf. p. 300 of Jacobson 1974. *Let $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n_1} x + a_n$ with $a_0 a_n \neq 0$, and m be the number of variations of sign in the sequence $\{a_0, a_1, \ldots, a_n\}$, i.e., the number of times the a_is change sign when counted from a_0 to a_n. Then m exceeds the number of positive roots of $f(x)$, counting multiplicity, by a non-negative even integer.*

The proof of Proposition 2.5 (i) By using the Eq. 2.1 and $g_{31}(s, t)$ in Eq. 2.2, we have

$$
\begin{aligned}
g_{31}(t, s) = {} & f_3 f_{22}{}^2 + f_3 f_4 f_{44} - f_4 f_{22} f_{11} - f_4{}^2 f_{11} + f_4 f_{22} f_3 + f_4{}^2 f_{33} - f_1 f_{22}{}^2 \\
& - f_1 f_{44} f_4 + f_{22} f_{33} f_2 + f_2 f_{44} f_{11} - f_2 f_{44} f_3 - f_{22} f_2 f_{11}.
\end{aligned}
$$

Then

$$
\begin{aligned}
g_{31}(s, t) - g_{31}(t, s) = {} & -(f_{22} + f_4 - f_{44} - f_2)(f_{22} f_3 - f_{22} f_1 + f_4 f_{33} - f_4 f_{11} \\
& + f_{33} f_2 - f_2 f_{11} + f_3 f_{44} - f_1 f_{44})
\end{aligned}
$$

is identically zero by Lemma 3.1. So the points on $g_{31}(s, t) = 0$ are symmetric about $s = t$.

(ii) This follows the direct computation of $g_{31}(s,s)=0$ and Descartes' rule.
(iii) We first rewrite $g_{31}(s,t)$ (here it is the possible factor which has positive zeros by dropping all other positive factors) as a Taylor polynomial in t:

$$g_{31}(s,t)=a_0+a_1t+a_2t^2+\cdots+a_{15}t^{15},$$

where $a_i=a_i(s)(i=0,1,2,\ldots,15)$ are polynomials of s:

$$\begin{aligned}
a_0 &= 256\,s^2\left(s^5+4\,s^4+4\,s^3-32\,s^2-80\,s-64\right)(s+2)^8\,,\\
a_1 &= 256\,s^2\left(s^6+12\,s^5+36\,s^4-36\,s^3-504\,s^2-976\,s-672\right)(s+2)^7\,,\\
a_2 &= 128\,(s^9+20\,s^8+126\,s^7+176\,s^6-1400\,s^5-6816\,s^4-11104\,s^3\\
&\quad -7424\,s^2-1152\,s-512)\,(s+2)^6\,,\\
a_3 &= 64\,(3\,s^{10}+45\,s^9+306\,s^8+968\,s^7-640\,s^6-16640\,s^5-55680\,s^4-83136\,s^3\\
&\quad -63104\,s^2-26368\,s-10752)\,(s+2)^5\,,\\
a_4 &= 16\,(7\,s^{11}+162\,s^{10}+1404\,s^9+6224\,s^8+11808\,s^7-26240\,s^6-232640\,s^5\\
&\quad -649728\,s^4-967168\,s^3-887808\,s^2-543744\,s-204800)\,(s+2)^4\,,\\
a_5 &= 16\,(-579584-1655808\,s-2407936\,s^2-2291968\,s^3-1470592\,s^4-565184\,s^5\\
&\quad -83328\,s^6+30304\,s^7+21528\,s^8+6284\,s^9+1028\,s^{10}+88\,s^{11}+3\,s^{12})\,(s+2)^3\,,\\
a_6 &= 8\,(-2138112-6541312\,s-9443328\,s^2-8499200\,s^3-5181440\,s^4\\
&\quad -1975936\,s^5-277120\,s^6+155904\,s^7+119056\,s^8+42200\,s^9+9236\,s^{10}\\
&\quad +1242\,s^{11}+93\,s^{12}+3\,s^{13})\,(s+2)^2\,,\\
a_7 &= 8\,(-2625536-8583168\,s-12837888\,s^2-11473408\,s^3-6662400\,s^4\\
&\quad -2288000\,s^5-86400\,s^6+421312\,s^7+294320\,s^8+119672\,s^9+32704\,s^{10}\\
&\quad +5924\,s^{11}+666\,s^{12}+41\,s^{13}+s^{14})\,(s+2)\,,\\
a_8 &= -16252928-57081856\,s-90357760\,s^2-81248256\,s^3-42610688\,s^4\\
&\quad -8468480\,s^5+6581760\,s^6+8079616\,s^7+5068800\,s^8+2231552\,s^9\\
&\quad +704032\,s^{10}+153360\,s^{11}+21776\,s^{12}+1844\,s^{13}+78\,s^{14}+s^{15},\\
a_9 &= -2785280-9748480\,s-14667776\,s^2-9236480\,s^3+999424\,s^4\\
&\quad +6513664\,s^5+6407424\,s^6+4269312\,s^7+2231552\,s^8+884096\,s^9\\
&\quad +246864\,s^{10}+45488\,s^{11}+5124\,s^{12}+308\,s^{13}+7\,s^{14},\\
a_{10} &= 802816+1642496\,s+1548288\,s^2+2806784\,s^3+4241920\,s^4\\
&\quad +3889664\,s^5+2598400\,s^6+1480640\,s^7+704032\,s^8+246864\,s^9\\
&\quad +57808\,s^{10}+8344\,s^{11}+658\,s^{12}+21\,s^{13},\\
a_{11} &= 708608+1519616\,s+1499136\,s^2+1607680\,s^3+1609472\,s^4\\
&\quad +1156352\,s^5+672896\,s^6+356416\,s^7+153360\,s^8+45488\,s^9\\
&\quad +8344\,s^{10}+840\,s^{11}+35\,s^{12},\\
a_{12} &= 237568+449536\,s+390144\,s^2+388352\,s^3+345088\,s^4+216512\,s^5\\
&\quad +116608\,s^6+58048\,s^7+21776\,s^8+5124\,s^9+658\,s^{10}+35\,s^{11},\\
a_{13} &= (s^4+4\,s^3+4\,s^2+16)(21\,s^6+224\,s^5+864\,s^4\\
&\quad +1632\,s^3+2688\,s^2+4608\,s+2880),
\end{aligned}$$

$$a_{14} = \left(s^4 + 4\,s^3 + 4\,s^2 + 16\right)\left(7\,s^5 + 50\,s^4 + 116\,s^3 + 176\,s^2 + 416\,s + 320\right),$$
$$a_{15} = \left(s^4 + 4\,s^3 + 4\,s^2 + 16\right)\left(s^4 + 4\,s^3 + 4\,s^2 + 16\,s + 16\right).$$

Note that $a_i(s) > 0$ for all $s > 0$ when $i = 10, 11, \ldots, 15$. $a_i(0) < 0$ for $i = 0, 1, 2, \ldots, 9$. Because the signs of $a_i(s)$ for $i = 0, 1, \ldots, 9$ change exactly once for $s > 0$ and $a_j(\bar{s}) < 0$ for $j = 0, 1, \ldots, 6$. So $a_i(s) < 0$ for all $0 < s < \bar{s}$ when $i = 0, 1, 2, \ldots, 6$. $\bar{s} > s_{a7} > s_{a8} > s_{a9} > 0$ where $a_i(s_{ai}) = 0$ for $i = 7, 8, 9$. In fact, $1 < s_{a9} < 1.2$ because $a_9(1) < 0$ and $a_9(1.2) > 0$ by the intermediate value theorem. $1.2 < s_{a8} < 1.7$ because $a_8(1.2) < 0$ and $a_8(1.7) > 0$. $1.7 < s_{a7} < 2.16 < \bar{s}$ because $a_7(1.7) < 0$ and $a_7(2.16) > 0$.

The change of sign of a_is is exactly once for each given $s (0 < s < \bar{s})$. So for any given $0 < s < \bar{s}$, there exists one and only one $t > 0$ such that $g_{31}(s, t) = 0$ by Descartes' rule. By the continuity of algebraic solutions for polynomials, t is a continuous function of s.

Furthermore $a_i(4) > 0$ for all $i = 1, 2, \ldots, 15$, then $a_i(s) > 0$ when $s > 4$. So $g_{31}(s, t) > 0$ when $s > 4$. Symmetrically, $g_{31}(s, t) > 0$ when $t > 4$. The curve $\Gamma_{31} = \{(s, t) | g_{31}(s, t) = 0\}$ is in the region $\{(s, t) | 0 < s < 4, 0 < t < 4\}$ (see Fig. 1).

(iv) m_1 is positive along $\bar{\Gamma}_{31}$.
The possible negative factor of m_1 is

$$pm_1 = b_0 + b_1 t + b_2 t^2 + \cdots + b_{12} t^{12},$$

where

$$\begin{aligned}
b_0 &= 131072 + 458752\,s + 688128\,s^2 + 557056\,s^3 \\
&\quad + 253952\,s^4 + 61440\,s^5 + 6144\,s^6, \\
b_1 &= 720896 + 2359296\,s + 3293184\,s^2 + 2473984\,s^3 \\
&\quad + 1044480\,s^4 + 233472\,s^5 + 21504\,s^6, \\
b_2 &= 1802240 + 5488640\,s + 7086080\,s^2 + 4902912\,s^3 \\
&\quad + 1898496\,s^4 + 387072\,s^5 + 32256\,s^6, \\
b_3 &= 2719744 + 7749632\,s + 9277440\,s^2 + 5894144\,s^3 + 2069504\,s^4 \\
&\quad + 368640\,s^5 + 18432\,s^6 - 3840\,s^7 - 768\,s^8 - 64\,s^9, \\
b_4 &= 2793472 + 7606272\,s + 8570880\,s^2 + 5030912\,s^3 + 1593344\,s^4 \\
&\quad + 233984\,s^5 - 4480\,s^6 - 7360\,s^7 - 1216\,s^8 - 80\,s^9, \\
b_5 &= 2109440 + 5705728\,s + 6197248\,s^2 + 3406848\,s^3 + 981248\,s^4 \\
&\quad + 116992\,s^5 - 10944\,s^6 - 5504\,s^7 - 704\,s^8 - 32\,s^9, \\
b_6 &= 1245184 + 3487744\,s + 3696128\,s^2 + 1905152\,s^3 + 493184\,s^4 \\
&\quad + 41472\,s^5 - 10848\,s^6 - 3536\,s^7 - 416\,s^8 - 20\,s^9, \\
b_7 &= 596992 + 1756160\,s + 1783808\,s^2 + 838400\,s^3 + 180096\,s^4 \\
&\quad + 576\,s^5 - 8928\,s^6 - 2096\,s^7 - 212\,s^8 - 8\,s^9, \\
b_8 &= 230912 + 696576\,s + 652032\,s^2 + 265536\,s^3 + 38656\,s^4 \\
&\quad - 8160\,s^5 - 4464\,s^6 - 764\,s^7 - 54\,s^8 - s^9, \\
b_9 &= 68096 + 202496\,s + 167104\,s^2 + 54976\,s^3 + 2256\,s^4 \\
&\quad - 3936\,s^5 - 1212\,s^6 - 140\,s^7 - 5\,s^8,
\end{aligned}$$

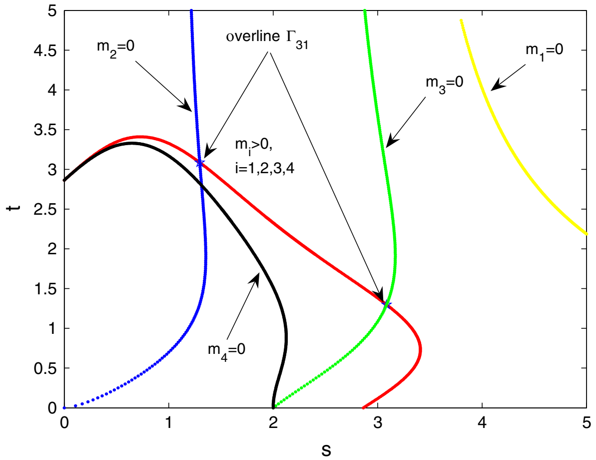


Fig. 5 Curves of $\bar{\Gamma}_{31}, m_1 = 0, m_2 = 0, m_3 = 0$ and $m_4 = 0$ in Case 3 in st-plane. $m_i > 0, i = 1, 2, 3, 4$ along singular curve $\bar{\Gamma}_{31}$

$$
\begin{aligned}
b_{10} &= 13952 + 39616\,s + 27520\,s^2 + 6544\,s^3 - 872\,s^4 - 840\,s^5 - 166\,s^6 - 10\,s^7,\\
b_{11} &= 1728 + 4608\,s + 2544\,s^2 + 336\,s^3 - 192\,s^4 - 84\,s^5 - 9\,s^6,\\
b_{12} &= 96 + 240\,s + 96\,s^2 - 12\,s^4 - 3\,s^5.
\end{aligned}
$$

Note that $b_0(s) > 0, b_1(s) > 0$, and $b_2(s) > 0$ for all $s > 0$. The unique root of $b_i(s) = 0 (i = 3, 4, \ldots, 12)$ are decreasing with respect to i. Numerically the root s_{b12} of $b_{12}(s) = 0$ is 2.987485248. Then $m_1(s, t) > 0$ for $s \le s_{b12}$ and any $t > 0$. The change of sign of b_is is exactly once for each given $s > s_{b12}$. By Descartes' rule, $m_1(s, t) = 0$ defines an implicit function t with respect to s. The curve $m_1(s, t) = 0$ has no intersections with $g_{31}(s, t) = 0$ by using the resultant of the two polynomials to solve the common zeros. In fact the curve $m_1(s, t) = 0$ is to the right of $g_{31}(s, t) = 0$ (see Fig. 5). So m_1 is positive along $\bar{\Gamma}_{31}$. Similarly we can prove m_4 is positive along $\bar{\Gamma}_{31}$.

We are going to describe the procedures of the proof without writing out the polynomials. By using Descartes' rule, $m_2(s, t) = 0$ defines an implicit function $s = s_{m_2}(t)$ and $m_2(s, t) > 0$ with $s > s_{m_2}(t)$. The unique intersection point (s_0, t_0) is found by using the resultant of the two polynomials $m_2(s, t)$ and $g_{31}(s, t)$. So m_2 is positive along $\bar{\Gamma}_{31}$. Similarly, $m_3(s, t) = 0$ defines an implicit function $s = s_{m_3}(t)$ and $m_3(s, t) > 0$ with $0 < s < s_{m_3}(t)$. The unique intersection point (s_1, t_1) is found by using the resultant of the two polynomials $m_3(s, t)$ and $g_{31}(s, t)$. m_3 is positive along $\bar{\Gamma}_{31}$.

We only give the proof of $m_3 > m_1$ for $s_0 < s < \bar{s}$ and $m_3 < m_1$ for $\bar{s} < s < s_1$ along $\bar{\Gamma}_{31}$ (see Fig. 3). Let $m_{p31}(s, t)$ be the factor of $m_3 - m_1$ which is not positive for all (s, t). Then $m_{p31}(s, t) = b_{m0}(t) + b_{m1}(t)s + \cdots + b_{m_9}(t)s^9$ is a ninth degree polynomial of s. For any $t > 0$, the change of sign of b_{mi} is exactly once. By Descartes' rule, $m_{p31}(s, t) = 0$ defines an implicit curve $s = s_{mp31}(t)$. The curve has exactly one intersection point with $g_{31}(s, t) = 0$ at $(\bar{s}, \bar{s})$. By checking a sample point, $m_{p31}(s, t) > 0$ for $s < s_{mp31}(t)$ which implies $m_3 > m_1$ for $s_0 < s < \bar{s}$ along $\bar{\Gamma}_{31}$. $m_{p31}(s, t) < 0$ for $s > s_{mp31}(t)$ which implies $m_3 < m_1$ for $\bar{s} < s < s_1$ along $\bar{\Gamma}_{31}$.

(v) We prove this by showing that m (say $m = (m_{91}, m_{92}, m_{93}, m_{94})$) given by (4.11) in Case 9 is a permutation of m (say $m = (m_{31}, m_{32}, m_{33}, m_{34})$) given by (4.3) in Case 3 along $\bar{\Gamma}_{31}$. Specifically, $m_{31} = m_{93}, m_{32} = m_{91}, m_{33} = m_{94}$, and $m_{34} = m_{92}$.

In fact,

$$m_{31} - m_{93} = \frac{\lambda\, f_{44}\, f_4\, g_{m31}}{\left(f_2{}^2 + f_{44}\, f_4\right) f_5\, \left(f_2\, f_{22} + f_4{}^2\right)},$$

where

$$\begin{aligned} g_{m31} &= f_{22}\, f_2\, f_1 + f_1\, f_4{}^2 - f_4\, f_2\, f_1 + f_2\, f_4\, f_3 - f_{33}\, f_2\, f_{22} - f_{33}\, f_4{}^2 \\ &\quad + f_{44}\, f_1\, f_{22} - f_{44}\, f_3\, f_{22} - f_3\, f_2{}^2 - f_{44}\, f_4\, f_3 + f_{11}\, f_2{}^2 + f_{44}\, f_4\, f_{11}. \\ g_{31} + g_{m31} &= (f_{22} + f_4 - f_2 - f_{44})\,(f_4\, f_1 - f_4\, f_{33} \\ &\quad - f_{44}\, f_3 + f_2\, f_3 + f_1\, f_{44} - f_{33}\, f_2) \end{aligned}$$

which is zero for all $s > 0, t > 0$ by Lemma 3.1. So $g_{m31} = g_{31} = 0$ along Γ_{31} which implies that $m_{31} = m_{93}$. Similarly, we can prove the other three equalities. □

Remark 5.2 The proof of (v) could be done by showing that u and $u(\tau_3)$ in (4.2) are same as $u(con(\tau_3))$ and u in (4.10), respectively, along Γ_{31}. In fact, the difference between u in (4.2) and $u(con(\tau_3))$ in (4.10) is $(\lambda f_2 u_{93})/((f_2 f_{22} + f_4^2)(f_2^2 + f_{44} f_4))$, and the difference between $u(\tau_3)$ in (4.2) and u in (4.10) is $-(\lambda f_4 u_{93})/((f_2 f_{22} + f_4^2)(f_2^2 + f_{44} f_4))$. Note that

$$\begin{aligned} g_{31} - u_{93} = -(f_2 - f_4 - f_{22} + f_{44})(f_2 f_3 - f_2 f_{33} \\ + f_{44} f_1 + f_4 f_1 - f_{44} f_3 - f_4 f_{33}), \end{aligned}$$

which is zero for all s, t. So m and $m(\tau_3)$ in Case 3 are same as $m(con(\tau_3))$ and m in Case 9, respectively. That means $m(\tau_3) \in S_m(q)$ in Case 3 are equivalent to $m \in S_{m(con(\tau_3))}(q)$ in Case 9.

Let g_{p51} be the polynomial such that $g_{51}(s,t) = -4s^2 t^2 (s-t)(s+t+2)^2 g_{p51}$. Let

$$\Gamma_{51} = \{(s,t) \in (\mathbf{R}^+)^2 | g_{p51}(s,t) = 0\}. \tag{5.1}$$

Proposition 5.3 (i) $g_{51}(s,t) = -g_{51}(t,s)$ *holds for all* $(s,t) \in (\mathbf{R}^+)^2$. *Then the points on* $g_{51}(s,t) = 0$ *are symmetric about* $s = t$ *and* $g_{51}(s,s) = 0$ *for all* $s > 0$.

(ii) m_1 *given by* (4.6) *in Case* 5 *is non-positive along* Γ_{51}.

(iii) m_4 *given by* (4.9) *in Case* 7 *is non-positive along* Γ_{51}.

The proof of Proposition 5.3 (i) By using the Eq. 2.1 and $g_{51}(s,t)$ in Eq. 4.7, we have

$$\begin{aligned} g_{51}(t,s) &= f_3\, f_{22}{}^2 - f_3\, f_2\, f_{44} - f_4\, f_{22}\, f_{11} + f_4\, f_2\, f_{11} + f_4\, f_{22}\, f_1 \\ &\quad - f_4\, f_2\, f_{33} - f_{33}\, f_{22}{}^2 + f_{33}\, f_2\, f_{44} + f_{44}\, f_{33}\, f_{22} + f_{44}{}^2 f_{11} \\ &\quad - f_{44}{}^2 f_1 - f_{44}\, f_{22}\, f_{11}. \end{aligned}$$

Then

$$\begin{aligned} g_{51}(s,t) + g_{51}(t,s) = (f_2 - f_{22} - f_4 + f_{44})(f_2 f_{33} - f_2 f_3 - f_4 f_1 - f_{44} f_1 \\ + f_4 f_{11} + f_{44} f_{11} + f_{33} f_{22} - f_3 f_{22}) \end{aligned}$$

which is zero for all s, t. So $g_{51}(s,s) = 0$ for all s. The set of the points such that $g_{51}(s,t) = 0$ is symmetric about $s = t$ and including the points of (s,s). Factoring

$g_{51}(s,t)$, we have $g_{51}(s,t) = -4s^2t^2(s-t)(s+t+2)^2 g_{p51}$, where

$$
\begin{aligned}
g_{p51}(s,t) &= c_0 + c_1 t + c_2 t^2 + \cdots + c_{14} t^{14}. \qquad (5.2)\\
c_0(s) &= 256\, s \left(s^5 + 4\, s^4 + 4\, s^3 - 32\, s^2 - 80\, s - 64\right)(s+2)^8,\\
c_1(s) &= 256\,(3\, s^7 + 23\, s^6 + 58\, s^5 - 56\, s^4 - 608\, s^3\\
&\quad -1184\, s^2 - 896\, s - 128)\,(s+2)^7,\\
c_2(s) &= 128\,(7\, s^8 + 86\, s^7 + 382\, s^6 + 464\, s^5 - 2152\, s^4 - 9696\, s^3\\
&\quad -16096\, s^2 - 11904\, s - 2688)\,(s+2)^6,\\
&\quad \cdots \qquad \cdots\\
c_{14}(s) &= \left(s^2 + 6\, s + 4\right)\left(s^4 + 4\, s^3 + 4\, s^2 + 16\, s + 16\right)\left(s^2 + 2\, s + 4\right).
\end{aligned}
$$

Let $\tilde{s}$ be the unique positive root of $g_{p51}(s,s) = 0$ which is

$$
\begin{aligned}
&-32768 - 245760s - 827392s^2 - 1638400s^3 - 2095104s^4 - 1773056s^5\\
&-947200s^6 - 231424s^7 + 85632s^8 + 121632s^9 + 72320s^{10} + 29888s^{11}\\
&+9184s^{12} + 2046s^{13} + 286s^{14} + 17s^{15} = 0.
\end{aligned}
$$

Numerically $\tilde{s} = 1.868932005$.
For $0 < s \leq \tilde{s}$, the change of sign of c_is is exactly once. By Descartes' rule, $g_{p51}(s,t) = 0$ defines an implicit function $t = t_{p51}(s)$ for $0 < s < \tilde{s}$ and $t_{p51}(\tilde{s}) = \tilde{s}$ (see Fig. 6). Γ_{51} can be characterized as

$$
\Gamma_{51} = \{(s, t_{p51}(s)) | 0 < s \leq \tilde{s}\} \bigcup \{(t_{p51}(s), s) | 0 < s \leq \tilde{s}\}. \qquad (5.3)
$$

by symmetry of the set of Γ_{51}.

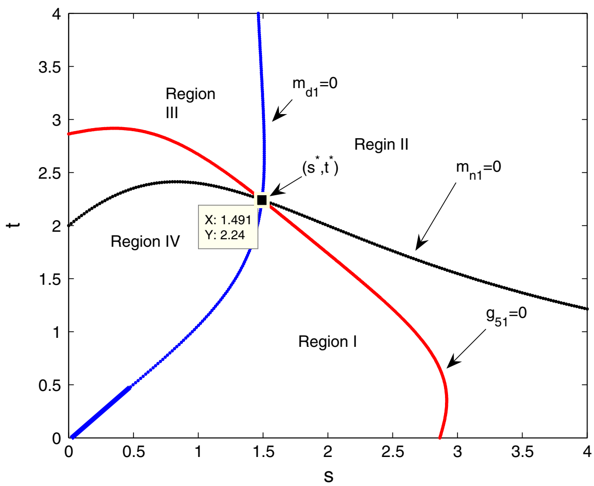


Fig. 6 Implicit curve Γ_{51} is in region I and region III. m_1 in Case 5 is negative along Γ_{51}

(ii) We investigate the sign of m_1 given by (4.6) by studying the possible negative factor $m_{n1}(s,t)$ and $m_{d1}(s,t)$ in the numerator and denominator of m_1.

$$m_1 = \frac{\lambda t^2 s^2 (s+2)^2 (t+2)^2 (s+t+2)^2}{f_5(s,t)} \frac{m_{n1}(s,t)}{m_{d1}(s,t)}.$$
$$m_{n1}(s,t) = d_{n0} + d_{n1}t + d_{n2}t^2 + \cdots + d_{n11}t^{11},$$

where

$$\begin{aligned}
d_{n0} &= -2048\,\left(3\,s^2+6\,s+4\right)(s+2)^4\,,\\
d_{n1} &= -4096\,(s+3)\left(3\,s^2+6\,s+4\right)(s+2)^3\,,\\
d_{n2} &= -1024\,\left(3\,s^2+6\,s+4\right)\left(3\,s^2+20\,s+30\right)(s+2)^2\,,\\
d_{n3} &= 64\,(s+2)\,(s^8+10\,s^7+40\,s^6+4\,s^5\\
&\quad -824\,s^4-3920\,s^3-7968\,s^2-7360\,s-2688),\\
&\quad \cdots \qquad \cdots,\\
d_{n10} &= \left(3\,s^5+26\,s^4+68\,s^3+80\,s^2+224\,s+192\right)(s+2)^2\,,\\
d_{n11} &= \left(s^4+4\,s^3+4\,s^2+16\,s+16\right)(s+2)^2\,.
\end{aligned}$$

For any given $0 < s < \infty$, $d_{ni} < 0 (i = 0, 1, \ldots, 5)$ and $d_{nk} > 0 (k = 7, 8, \ldots, 11)$. So the change of sign of d_{ni}s is exactly once for $s > 0$. By Descartes' rule, $m_{n1}(s,t) = 0$ defines an implicit function $t = t_{mn1}(s)$. Furthermore, $m_{n1}(s,t) > 0$ for $t > t_{mn1}(s)$, $m_{n1}(s,t) < 0$ for $t < t_{mn1}(s)$, and $m_{n1}(s,t) = 0$ for $t = t_{mn1}(s)$ (see Fig. 6). Similarly, by checking the change of sign of polynomial $m_{d1}(s,t)$ and by Descartes' rule, $m_{d1}(s,t) = 0$ defines an implicit function $s = s_{md1}(t)$ for $0 < t < \infty$. Furthermore $m_{d1}(s,t) > 0$ for $s > s_{md1}(t)$, $m_{d1}(s,t) < 0$ for $s < s_{md1}(t)$, and $m_{d1}(s,t) = 0$ for $s = s_{md1}(t)$ (see Fig. 6).
Solving $m_{n1}(s,t) = 0$ and $m_{d1}(s,t) = 0$, we have a unique solution $(s^*, t^*) = (1.491364438, 2.242204256)$. So the two curves $t = t_{mn1}(s)$ and $s = s_{md1}(t)$ divide the first quadrant into four subregions. Define

$$\begin{aligned}
\text{Region I} &= \{(s,t)|t < t_{mn1}(s),\ s > s_{md1}(t)\},\\
\text{Region II} &= \{(s,t)|t > t_{mn1}(s),\ s > s_{md1}(t)\},\\
\text{Region III} &= \{(s,t)|t > t_{mn1}(s),\ s < s_{md1}(t)\},\\
\text{Region IV} &= \{(s,t)|t < t_{mn1}(s),\ s < s_{md1}(t)\}.
\end{aligned}$$

So $m_1 > 0$ when (s,t) is in region II and IV and $m_1 < 0$ when (s,t) is in region I and III.
Solving $m_{n1}(s,t) = 0$ and $g_{51}(s,t) = 0$, and solving $m_{d1}(s,t) = 0$ and $g_{51}(s,t) = 0$, they give the same solution $(s^*, t^*) = (1.491364438, 2.242204256)$. This implies that the three curves intersect at the same point (s^*, t^*). Note that $\tilde{s} < s^*$. The intersection point (s^*, t^*) is the unique intersection point of three graphs of functions $t = t_{mn1}(s)$, $s = s_{md1}(t)$ and $t = g_{p51}(s)$. So the curve $g_{51}(s,t)$ can only be in two regions and they are in region I and III by checking some sample points. Therefore m_1 is negative along the curve Γ_{51}.

(iii) We prove this by showing that m_4 given by (4.9) in Case 7 is equal to m_1 given by (4.6) in Case 5 along Γ_{51}. In fact,

$$m_4 - m_1 = \frac{\lambda f_{22} f_4 g_{51}}{(f_2 f_{44} - f_4^2)(f_2^2 - f_{22} f_4) f_5},$$

which implies that $m_4 - m_1 = 0$ along $g_{51} = 0$. $\square$

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