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COLLINEAR CENTRAL CONFIGURATION IN FOUR-BODY PROBLEM

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Abstract. In the n-body problem a central configuration is formed if the position vector of each particle with respect to the center of mass is a common scalar multiple of its acceleration vector. We consider the problem: given a collinear configuration of four bodies, under what conditions is it possible to choose positive masses which make it central. We know it is always possible to choose three positive masses such that the given three positions with the masses form a central configuration. However for an arbitrary configuration of four bodies, it is not always possible to find positive masses forming a central configuration. In this paper, we establish an expression of four masses depending on the position x and the center of mass u, which gives a central configuration in the collinear four body problem. Specifically we show that there is a compact region in which no central configuration is possible for positive masses. Conversely, for any configuration in the complement of the compact region, it is always possible to choose positive masses to make the configuration central.

Key words: central configuration, n-body problem, inverse problem of central configuration, Descartes' rule

1. Introduction

Consider the Newtonian n-body problem:

$$\ddot{q}_k = \sum_{j=1, j\neq k}^{n} \frac{m_j(q_j - q_k)}{|q_j - q_k|^3} \quad 1 \leqslant k \leqslant n. \tag{1.1}$$

Here $m_k > 0$ are the masses of the bodies, and $q_k \in \mathbb{R}^d$, $(d = 1, 2, 3)$ are their positions. Let

$$C = m_1 q_1 + \cdots + m_n q_n, \qquad M = m_1 + \cdots + m_n, \quad c = C/M$$

be the first moment, total mass and center of mass of the bodies, respectively.

The configuration $q=(q_1,\ldots,q_n)$ is called a *central configuration* if the acceleration vectors of the bodies satisfy:

$$\sum_{j=1,j\neq k}^{n} \frac{m_j(q_j-q_k)}{|q_j-q_k|^3} = -\lambda(q_k-c) \quad 1\leqslant k\leqslant n. \tag{1.2}$$

for a constant λ. Furthermore $\lambda=U/2I>0$, where U is the potential function $U=\sum_{1\leqslant k<j\leqslant n} m_k m_j/|q_k-q_j|$ and I is the moment of inertial of the system, i.e. $I=(1/2)\sum_{i=1}^{n} m_i\|q_i\|^2$. An effort has gone into understanding central configuration properties (see for example Saari, 1980; Moeckel, 1990 and Smale, 1970).

If we let

$$a_{jk}=\frac{(q_k-q_j)}{|q_k-q_j|^3} \text{ if } j\neq k, \quad a_{jj}=0, \quad m=(m_1,m_2,\ldots,m_n)^T,$$
$$A=\left(a_{jk}\right)_{1\leqslant j,k\leqslant n}.$$

Then the central configuration equations becomes

$$Am=-\lambda q+L\bigotimes\mu=\bar{b}, \quad \text{where } L=(1,1,\ldots,1)\in\mathbb{R}^n \tag{1.3}$$

for some constant $\mu\in\mathbb{R}^d$, where $\bigotimes$ denotes the tensor product, i.e. $L\bigotimes\mu=(\mu,\ldots,\mu)^T$. Comparing μ in (1.3) with c in (1.2), we have $\mu=\lambda c$.

A configuration $q=(q_1,\ldots,q_n)$ is *collinear*, if all the q_is are located on a line. A collinear central configuration is called a *Moulton configuration* after F.R. Moulton who proved that for a fixed mass vector $m=(m_1,\ldots,m_n)$ and a fixed ordering of the bodies along the line, there exists a unique collinear central configuration (up to translation and scaling) (Moulton, 1910). In this paper we consider the inverse problem: given a collinear configuration, find the positive mass vectors, if any, for which it is a central configuration. Recall for any given three collinear positions, it is always possible to choose three positive masses making it central (Albouy and Moeckel, 2000). This result is generally not true for $n\geqslant 4$. The main result in this paper is to prove that for $n=4$, there exist a compact region E in configuration space such that within E it is not possible to choose a positive mass vector to make the configuration central. Furthermore, on the complement G of the compact region E, there always exist positive mass vector to make it central.

Due to the fact that central configuration is invariant up to translation and rescaling, we can choose coordinates for the collinear four bodies as follows. Let $x_1=-s-1, x_2=-1, x_3=1, x_4=t+1$, where $s,t>0$. If we let $r=-\lambda, u=\mu$ (for ease of notation), then central configuration equation

(1.3) in collinear four bodies case is

$$Am = rx + uL, \tag{1.4}$$

where

$$A = \begin{bmatrix} 0 & s^{-2} & (s+2)^{-2} & (s+t+2)^{-2} \\ -s^{-2} & 0 & 1/4 & (t+2)^{-2} \\ -(s+2)^{-2} & -1/4 & 0 & t^{-2} \\ -(s+t+2)^{-2} & -(t+2)^{-2} & -t^{-2} & 0 \end{bmatrix} \tag{1.5}$$

and $\bar{b} = rx + uL = (r(-s-1) + u, -r + u, r + u, r(t+1) + u)^T$.

In this paper, the following results are obtained:

THEOREM 1. *Let* $x = (x_1, x_2, x_3, x_4) = (-s-1, -1, 1, t+1)$ *be a collinear configuration with positive mass vector* $m = (m_1, m_2, m_3, m_4)$ *and assume that the center of mass* $c = u/\lambda = \sum_{i=1}^{4} m_i x_i = 0$. *Then there exist two constants* t_0, s_0 *and two implicit functions* $pm_{02}(t,s) = 0$ *and* $pm_{03}(t,s) = 0$, *which are defined by* (3.1) *and* (3.2) *respectively, such that*

(1) $pm_{02}(t,s) = pm_{03}(s,t)$.
(2) *The equation* $pm_{02}(t,s) = 0$ *can be globally solved for* t *to get a smooth monotone increasing function* $t_2 = f(s_2)$. *Furthermore,* $\lim_{s_2\to\infty} f(s_2) = \infty$. *Similarly, we can get a smooth monotone increasing function* $s_3 = f(t_3)$ *from* $pm_{03}(t,s) = 0$, *such that* $\lim_{t_3\to\infty} f(t_3) = \infty$.

Then there exist an unbounded stripe-like region B *bounded below by* $s = s_0$, *bounded to the left by* $t = t_0$, *between* $(f(s_2), s_2)$ *and* $(t_3, f(t_3))$. *For any point* $(t,s) \in B$, *with the center of mass at origin, the configuration* x *can be a central configuration with a positive mass* $m = (m_1, m_2, m_3, m_4)$ *which is unique up to a scalar.*

Remark 1. Numerically, the region B is the one shown in Figure 1. Here $t_0 = s_0 = 1.396812289$.

THEOREM 2. *Let* $x = (x_1, x_2, x_3, x_4) = (-s-1, -1, 1, t+1)$ *be a collinear configuration with positive mass vector* $m = (m_1, m_2, m_3, m_4)$. *Then there exist an unbounded region* G *in the first quadrant* $(t > 0, s > 0)$ *bounded away from the origin by an implicit function* $h(t,s) = 0$ *defined by* (4.2), *such that for any* $(t,s) \in G$, *there exist positive masses* (m_1, m_2, m_3, m_4) *making* x *as a central configuration. Conversely for any* (t,s) *in* $E = R^{2+} \setminus G$, *there is no positive mass* $m = (m_1, m_2, m_3, m_4)$ *making* $x = (-s-1, -1, 1, t+1)$ *central, where* R^{2+} *is the first quadrant in the plane.*

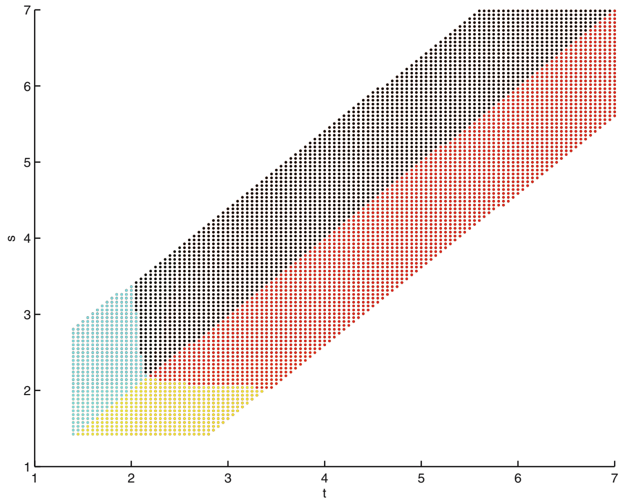

Figure 1.

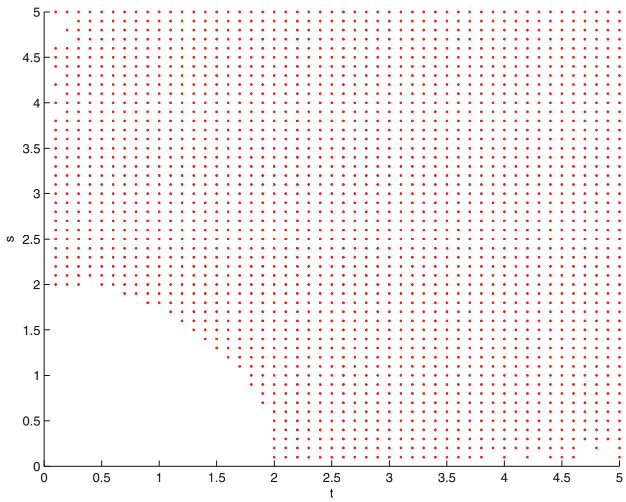

Figure 2.

Remark 2. Numerically the region G is the one shown in Figure 2.

The mass vector $m=(m_1,m_2,m_3,m_4)=m(x,u)$ depends on the position and center of mass up to a scalar. For fixed x,u, there is a unique solution $m(x,u)$ making x central with center of mass at u. Define

$$\underline{u}(x):=\underline{u}(t,s):=\min\{u|x \text{ forms a central configuration centered at } u \text{ with positive mass } m(x,u)\},$$

$$\bar{u}(x):=\bar{u}(t,s):=\max\{u|x \text{ forms a central configuration centered at } u \text{ with positive mass } m(x,u)\}.$$

If the set $\{u|x$ forms a central configuration centered at u with positive mass $m(x,u)\}$ is empty, let $\underline{u}(x)=\bar{u}(x)=0$. Defining

$$\mathrm{d}(t,s):=\bar{u}(t,s)-\underline{u}(t,s),$$

we have the following.

THEOREM 3. (1) *For each point* $(t_0,s_0)\in E$, $\mathrm{d}(t_0,s_0)=0$.
(2) *For each point* $(t_0,s_0)\in G$, $d(t_0,s_0)>0$ *and*

$$\lim_{t_0\to\infty,s_0\to 0}\mathrm{d}(t_0,s_0)=0,$$
$$\lim_{t_0\to 0,s_0\to\infty}\mathrm{d}(t_0,s_0)=0.$$

This paper is distributed as follows. In Section 2, we find the general solutions for 4-body collinear central configuration and simplify the solutions so that they can be analyzed. We prove Theorems 1–3 in Sections 3–5, respectively.

2. General Solutions for 4-body Collinear Central Configuration

Given $s,t>0$, we will find the general solution of masses $m_1,\dots,m_4$ with two parameters r,u for the 4-body collinear central configuration, i.e. a solution of (1.4).

Because the matrix A is skew symmetric, the determinant of A is the square of its Pfaffian, that is $det(A)=(PfA)^2=[a_{12}a_{34}-a_{13}a_{24}+a_{14}a_{23}]^2=(st)^{-2}-((s+2)(t+2))^{-2}+(1/4)(s+t+2)^{-2}>0$. So the matrix has full rank. Therefore, the solution is uniquely determined. Albouy and Mockel (2000) proved that the given 4-body collinear configuration determines a two-parameter family of masses making it central. Here we can find a solution of masses explicitly by standard row reduction:

$$m_3=\left(\left(-\frac{r\,(t+1)+u}{s^2}+\frac{-r+u}{(s+t+2)^2}\right)s^{-2}-\frac{r\,(-s-1)+u}{s^2\,(t+2)^2}\right)$$
$$\left(\left(\frac{1}{s^2t^2}+1/4\,(s+t+2)^{-2}\right)s^{-2}-\frac{1}{s^2\,(t+2)^2\,(s+2)^2}\right)^{-1},$$
$$m_4=\left(\left(-\frac{r+u}{s^2}+\frac{-r+u}{(s+2)^2}\right)s^{-2}-\frac{1}{4}\,\frac{r\,(-s-1)+u}{s^2}\right)$$
$$\left(\left(-\frac{1}{s^2t^2}+\frac{1}{(s+2)^2\,(t+2)^2}\right)s^{-2}-\frac{1}{4}\,\frac{1}{s^2\,(s+t+2)^2}\right)^{-1}.$$

If we write the central configuration equation from right to left, i.e. from m_4 to m_1, we have $m'=(m_4,m_3,m_2,m_1)$, $x'=(x_4,x_3,x_2,x_1)$, r,u the same constants, then the coefficient matrix is

$$B=\begin{bmatrix} 0 & -t^{-2} & -(t+2)^{-2} & -(s+t+2)^{-2} \\ t^{-2} & 0 & -1/4 & -(s+2)^{-2} \\ (t+2)^{-2} & 1/4 & 0 & -s^{-2} \\ (s+t+2)^{-2} & (s+2)^{-2} & s^{-2} & 0 \end{bmatrix}.$$

The central configuration equation changes to

$$Bm' = rx' + uL = (r\,(t+1)+u,\, r+u,\, -r+u,\, r\,(-s-1)+u)^T. \tag{2.1}$$

If both sides of (2.1) are multiplied by -1, s and t are exchanged, and u is switched to $-u$, then Equation (2.1) will be Same as Equation (1.4). Therefore, m_1, m_2 are symmetrical to m_3, m_4 respectively, in the sense that m_1 is equal to m_4 by exchanging s and t, and switching u to $-u$ in m_4 (similarly for m_2 and m_3). So m_1, m_2 have the following expressions:

$$m_1 = \left(\left(-\frac{r-u}{t^2} + \frac{-r-u}{(t+2)^2}\right)t^{-2} - \frac{1}{4}\,\frac{r\,(-t-1)-u}{t^2}\right)$$
$$\left(\left(-\frac{1}{s^2t^2} + \frac{1}{(s+2)^2\,(t+2)^2}\right)t^{-2} - \frac{1}{4}\,\frac{1}{t^2\,(s+t+2)^2}\right)^{-1},$$
$$m_2 = \left(\left(-\frac{r\,(s+1)-u}{t^2} + \frac{-r-u}{(s+t+2)^2}\right)t^{-2} - \frac{r\,(-t-1)-u}{t^2\,(s+2)^2}\right)$$
$$\left(\left(\frac{1}{s^2t^2} + 1/4\,(s+t+2)^{-2}\right)t^{-2} - \frac{1}{(t+2)^2\,t^2\,(s+2)^2}\right)^{-1}.$$

Note that $r=-\lambda$ is negative if $m=(m_1, m_2, m_3, m_4)$ is a positive solution of (1.4). Also $m/|r|$ is a positive solution of

$$Am = -x + (u/|r|)L. \tag{2.2}$$

Because we are only concerned with the sign of the mass functions, we can assume $r=-1$. Under this assumption $u=\lambda c=-rc$ becomes the center of mass. In the following two sections, we will analyze the mass functions and find the possible region in $ts-$plane such that the mass functions are positive.

3. Symmetrical Collinear Central Configuration

Again, in this section, we will fix the center of mass at the origin (i.e. $u=0$) and let $r=-1$. Then x_2 is symmetric to x_3. By using symbolic computation of Matlab, we find the numerators and denominators of the masses. The numerators of masses are:

$$
\begin{aligned}
nm_1 =& (s+2)^2\,(s+t+2)^2\,s^2\left(-4\,t^2-16\,t-16+t^5+5\,t^4+8\,t^3\right),\\
nm_2 =& 4\,(t+2)^2\left(16+36\,s^2t+48\,s+32\,s^3+s^5+9\,s^4+40\,st+8\,st^2\right.\\
&+2\,s^4t+14\,s^3t+s^3t^2+5\,s^2t^2+56\,s^2+4\,t^2+16\,t-t^3s^2\\
&\left.-2\,t^4s-6\,t^3s-t^5-5\,t^4-8\,t^3\right)s^2,\\
nm_3 =& -4\left(-16-8\,s^2t-32\,t^3-t^5-9\,t^4-t^3s^2-16\,s-40\,st-36\,st^2\right.\\
&-5\,s^2t^2-2\,t^4s-14\,t^3s-4\,s^2-56\,t^2-48\,t+s^5+2\,s^4t+5\,s^4\\
&\left.+s^3t^2+6\,s^3t+8\,s^3\right)t^2\,(s+2)^2,\\
nm_4 =& (t+2)^2\,(s+t+2)^2\,t^2\left(-4\,s^2-16\,s-16+s^5+5\,s^4+8\,s^3\right).
\end{aligned}
$$

They have the same positive denominator

$$
\begin{aligned}
dm_0 =& 256+512\,t+384\,t^2+16\,t^4+128\,t^3+384\,s^2+16\,s^4+128\,s^3\\
&+512\,s+4\,s^4t^2+s^4t^4+16\,t^3s^3+4\,t^3s^4+896\,st+576\,st^2\\
&+16\,s^4t+160\,s^3t+64\,s^3t^2+304\,s^2t^2+16\,t^4s+160\,t^3s+4\,s^3t^4\\
&+64\,t^3s^2+4\,s^2t^4+576\,s^2t.
\end{aligned}
$$

Then $m_i = (nm_i/dm_0), 1 \leqslant i \leqslant 4$. Because the denominator is bigger than 256 for $s, t > 0$, the masses can not go to infinity if s, t are bounded. The configuration fails to be a central configuration if only if some of masses become negative. The possible negative terms in numerators are :

$$
\begin{aligned}
pm_{01} =& t^5+5\,t^4+8\,t^3-4\,t^2-16\,t-16,\\
pm_{02} =& 16+48\,s+56\,s^2+32\,s^3+9\,s^4+s^5\\
&+\left(16+40\,s+36\,s^2+14\,s^3+2\,s^4\right)t+\left(4+8\,s+5\,s^2+s^3\right)t^2\\
&+\left(-8-6\,s-s^2\right)t^3+(-5-2\,s)\,t^4-t^5, \qquad (3.1)\\
pm_{03} =& 16+48\,t+56\,t^2+32\,t^3+9\,t^4+t^5\\
&+\left(16+40\,t+36\,t^2+14\,t^3+2\,t^4\right)s+\left(4+8\,t+5\,t^2+t^3\right)s^2\\
&+\left(-8-6\,t-t^2\right)s^3+(-5-2\,t)\,s^4-s^5, \qquad (3.2)\\
pm_{04} =& s^5+5\,s^4+8\,s^3-4\,s^2-16\,s-16.
\end{aligned}
$$

It is clear that the sign of the coefficients in the polynomial pm_{01} changes only once. By Descartes' rule of sign, there is exactly one positive root t_0 of pm_{01}. Because pm_{01} does not depend on s, the equation $pm_{01}=0$ implicitly gives rise to a straight line $t=t_0$ in the ts-plane (Figure 3, $m_1=0$). Also m_1 is positive on the right of this line because pm_{01} goes to infinity as t goes to infinity. Similarly, the equation pm_{04} implicitly gives rise to a straight line $s=s_0$ in the ts-plane (Figure 3, $m_4=0$). m_4 is positive above this line because pm_{04} goes to infinity as s goes to infinity. So the region of $m_1>0$ and $m_4>0$ is a nonempty open set indicated in Figure 3.

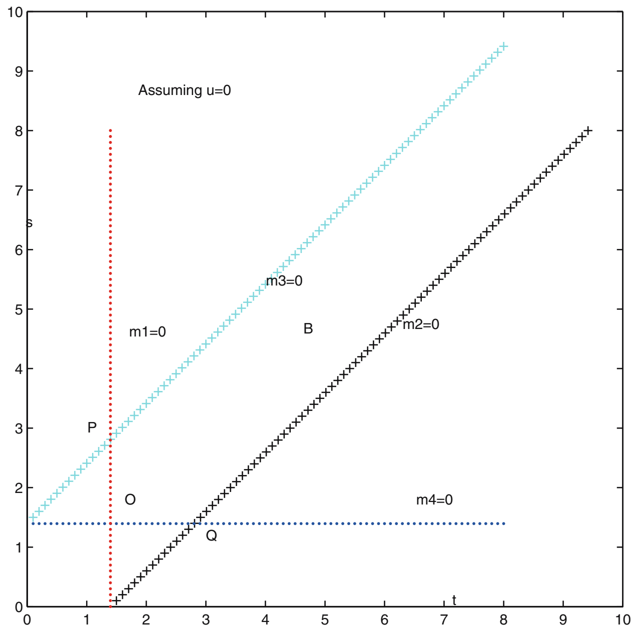

Figure 3.

If we consider pm_{02} to be a polynomial in the variable t,then the coefficients are $c_0(s) = 16 + 48s + 56s^2 + 32s^3 + 9s^4 + s^5$, $c_1(s) = 16 + 40s + 36s^2 + 14s^3 + 2s^4$, $c_2(s) = 4 + 8s + 5s^2 + s^3$, $c_3(s) = -8 - 6s - s^2$, $c_4(s) = -5 - 2s$, $c_5(s) = -1$ with respect to the increasing order of the variable t, i.e. $pm_{02} = c_0(s) + c_1(s)t + c_2(s)t^2 + c_3(s)t^3 + c_4(s)t^4 + c_5(s)t^5$. For $s > 0$, c_0, c_1, c_2 are positive and c_3, c_4, c_5 are negative. So the sign of the polynomial pm_{02} changes only once implying there is exactly one positive root t for any given $s > 0$. Therefore, $pm_{02} = 0$ implicitly determines a smooth monotone increasing function $t = f(s)$ with the domain $s > 0$. Because the degree of the positive coefficients $c_0(s), c_1(s), c_2(s)$ are larger than the degree of $c_3(s), c_4(s), c_5(s)$, $t = f(s)$ must go to infinity as s goes to infinity. Similarly, $pm_{03} = 0$ implicitly determines a smooth monotone increasing function $s = g(t)$ with the domain $t > 0$. Moreover, the functions f and g are the same by the symmetry of s, t in pm_{02} and pm_{03}.

Now, we want to show that the implicit curves $pm_{02} = 0$ and $pm_{03} = 0$ have no intersecting points. That means the system of equations

$$\begin{aligned} &pm_{02} = 0, \\ &pm_{03} = 0, \\ &pm_{02} = pm_{03} \end{aligned} \tag{3.3}$$

has no solution. Considering the difference of pm_{02} and pm_{03}, we have

$$pm_{02} - pm_{03} = 2\,(t-s)\left(t^4 + 7t^3 + 3st^3 + 20t^2 + 4s^2t^2 + 17st^2 + 3s^3t + 26t + 34st + 17s^2t + 16 + 20s^2 + s^4 + 7s^3 + 26s\right)$$

Then for $s, t > 0$, $pm_{02} - pm_{03} = 0$ if only if $s = t$. But for $s = t > 0$, $pm_{02} = pm_{03} = (16 + 64s + 100s^2 + 68s^3 + 17s^4)$, which has no zeros. So $pm_{02} = 0$ and $pm_{03} = 0$ can not be satisfied simultaneously, i.e. the two curves given by the two implicit functions have no intersecting points. Furthermore, the curve $(t, f(t))$ is above the curve $(f(s), s)$. Therefore, the four implicit curves give rise to a region (Figure 3, B) described in Theorem 1. The region bounded by the four implicit curves in first quadrant is called *central configuration region.* For any point in the central configuration region, there are four unique positive masses which make it central with the center of mass at origin. (Note, the uniqueness is not true if the center of mass is not fixed. The mass vector will admit one parameter u.) This completes the proof of Theorem 1.

Our investigation will now go into the change of the masses with respect to the positions. The intersecting point O of the line $m_1 = 0$ and $m_4 = 0$ is $(1.396812289, 1.396812289)$. The intersecting point P of the line $m_1 = 0$ and the curve $m_3 = 0$ is $(1.396812289, 2.807744118)$ with the same first coordinate as O. By symmetry, the point Q is $(2.807744118, 1.396812289)$. For example, the point $(1, 1)$ is out of the central configuration region for fixing center of mass at origin. It gives the configuration $x = (-2, -1, 1, 2)$. By solving equation (1.4) with $r = -1, u = 0$, the unique solution is $[-3168/5201, 9540/5201, 9540/5201, -3168/5201]$, which is not positive. That is the configuration $x = (-2, -1, 1, 2)$ could not be a central configuration by fixing the center of mass at origin. We will also show in the next section that the configuration $x = (-2, -1, 1, 2)$ could not be a central configuration even without fixing the center of mass.

Note that the central configuration region is symmetric along $s = t$. If $s = t$, we have $m_1 = m_4$ and $m_2 = m_3$. The different grey levels in Figure 1 show that the heaviest mass of the four bodies changes as the change of the point in central configuration region. For example, when the point is in the triangle $\triangle$POQ with $s > t$, m_3 becomes the heaviest mass. As the point moves up while t becomes larger with $s > t$, the heaviest mass m_3 decreases and will equal the mass m_1 on some curve, and eventually m_1 will become the heaviest mass.

We may intuitively think of m_1, m_4 as going to zero and m_2, m_3 going to infinity, while $s = t$ goes to infinity because the outer two bodies travel much further than the inner two bodies do. Hence the smaller the masses of the outer two bodies the faster they travel. However, this is not the case.

Both m_1, m_4 go to infinity as $s=t$ goes to infinity, but the limit of m_2, m_3 goes to a finite number 68 which is unexpected. Here we fix the center at the origin and get the symmetric region in which the central configuration of collinear four bodies lies. It is natural to ask how the center of mass affects the central configuration region. We will answer this question in next section.

4. Proof of Theorem 2

In this section, we will find the central configuration region without fixing the center of mass in advance. For the reason previously given at the end of Section 2, we can let $r=-1$. By using symbolic computation of Matlab, we find the numerators and denominators of the masses. They have the same positive denominators as given by

$$\begin{aligned}
\mathrm{d}m \;=\; & 256+512\,s+512\,t+s^4t^4+384\,s^2+16\,s^4+128\,s^3+384\,t^2+16\,t^4\\
&+128\,t^3+64\,s^3t^2+64\,s^2t^3+4\,s^4t^3+4\,s^4t^2+4\,s^3t^4+16\,s^3t^3+4\,s^2t^4\\
&+576\,s^2t+576\,st^2+896\,st+16\,s^4t+160\,s^3t+16\,st^4+160\,st^3\\
&+304\,s^2t^2.
\end{aligned}$$

Furthermore, we find that the possible negative terms in each mass solution are the following.

$$\begin{aligned}
pm_1(t,u) =& -16\,u-16+(-16\,u-16)\,t+(-4-4\,u)\,t^2\\
&+(-4\,u+8)\,t^3+(-u+5)\,t^4+t^5,\\
pm_2(t,s,u) =& 16+16\,u+48\,s+us^4+56\,s^2+s^5\\
&+32\,s^3+24\,us^2+32\,us+8\,us^3+9\,s^4\\
&+\left(16+40\,s+2\,us^3+24\,us+36\,s^2+2\,s^4+14\,s^3+12\,us^2+16\,u\right)t\\
&+\left(us^2+4+4\,us+5\,s^2+4\,u+8\,s+s^3\right)t^2\\
&+\left(-s^2-6\,s+2\,us+4\,u-8\right)t^3+(-5+u-2\,s)\,t^4-t^5,\\
pm_3(t,s,u) =& 16-16\,u-24\,ut^2+48\,t+t^5+9\,t^4+32\,t^3-t^4u-32\,ut-8\,t^3u+56\,t^2\\
&+\left(-2\,t^3u+14\,t^3-12\,ut^2-24\,ut+16+40\,t+36\,t^2+2\,t^4-16\,u\right)s\\
&+\left(-ut^2+t^3-4\,ut+5\,t^2+8\,t+4-4\,u\right)s^2+\left(-t^2-4\,u-8-6\,t-2\,ut\right)s^3\\
&+(-u-5-2\,t)\,s^4-s^5,\\
pm_4(s,u) =& 16\,u-16+(16\,u-16)\,s+(-4+4\,u)\,s^2\\
&+(4\,u+8)\,s^3+(u+5)\,s^4+s^5.
\end{aligned}$$

Let $k_1(t)=(t+2)^2, k_2(t,s)=(t+s+2)^2$. The mass solutions of (1.4) are

$$\begin{aligned} m_1(t,s,u) &= \frac{k_1(s)k_2(t,s)s^2 pm_1(t,u)}{\mathrm{d}m}, \\ m_2(t,s,u) &= \frac{4k_1(t)s^2 pm_2(t,s,u)}{\mathrm{d}m}, \\ m_3(t,s,u) &= \frac{4k_1(s)t^2 pm_3(t,s,u)}{\mathrm{d}m}, \\ m_4(t,s,u) &= \frac{k_1(t)k_2(t,s)t^2 pm_4(s,u)}{\mathrm{d}m}, \end{aligned} \tag{4.1}$$

which have the following relations:

$$m_1(t,s,u)=m_4(s,t,-u), \qquad m_2(t,s,u)=m_3(s,t,-u).$$

Then the central configuration region is the region on which $pm_1,\ldots,pm_4$ are all positive.We prove Theorem 2 by using the following Lemmas.

LEMMA 1. *The region in which* $m_1>0, m_4>0$ *for* $s>0, t>0$ *by choosing the proper* u *is the infinite region* G *in Figure 6 bounded by an implicit function* $h(t,s)=0$ *far away from the origin.*

Proof. Because pm_1 is independent on s, the positivity of pm_1 only depends on u and t. For each fixed u, the number of sign changes of the coefficients of pm_1 is at most one. More precisely, if $u>-1$, the sign of polynomial pm_1 changes only once. If $u<-1$, the coefficients of pm_1 are all positive. Therefore, pm_1 is always positive for $t>0$ while $u<-1$. When $u=-1$, we have $pm_1=12t^3+6t^4+t^5$ which is zero at $t=0$ and positive for $t>0$. By Descartes' rule of sign, there is exactly one positive root for any given $u>-1$. The equation $pm_1=0$ implicitly defines t as a function of u on $u>-1$. The curve is a smooth monotonically increasing curve by the property of polynomial functions as shown in Figure 4. Another way show this by considering the sign of its first derivative. From $pm_1=0$, it is easy to solve for u, which is

$$\begin{aligned} u &= \frac{-4t^2-16t-16+8t^3+t^5+5t^4}{16t+16+t^4+4t^3+4t^2}, \\ \frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{t^2\left(768t+112t^3+576+416t^2+t^6+8t^5+24t^4\right)}{\left(16t+16+t^4+4t^3+4t^2\right)^2}, \end{aligned}$$

which is always positive for $t>0$.

Because u is the center of mass, u is a real value between $x_1=-s-1$ and $x_4=t+1$. In this graph, m_1 is positive for the points above the curve.

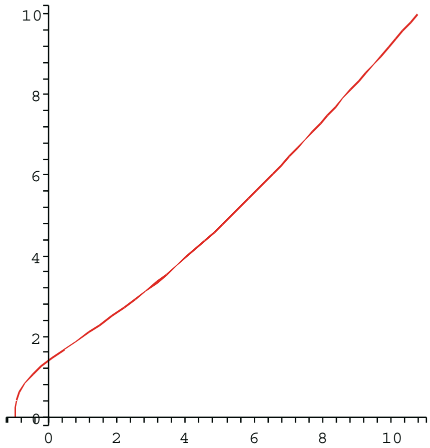

Figure 4.

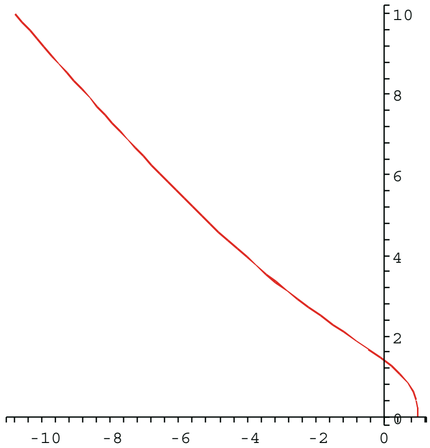

Figure 5.

For example, for $u=0$, $t \geqslant 1.396812289$. For the same reasons, the implicit function $pm_4=0$ has similar properties and the implicit graph is given by Figure 5.

Therefore, the two equations $pm_1=0$ and $pm_4=0$ give us an implicit function of (t,s) by eliminating u.

$$\frac{-4t^2-16t-16+8t^3+t^5+5t^4}{16+16t+t^4+4t^3+4t^2}+\frac{-4s^2-16s-16+8s^3+s^5+5s^4}{16+16s+s^4+4s^3+4s^2}=0$$

Because its denominator is always positive for positive s, t, the equation is equivalent to $h(t,s)=0$ which defines an implicit function, where

$$\begin{aligned} h(t,s) \;=\; & (-512-512\,t-128\,t^2+64\,t^3+64\,t^4+16\,t^5) \\ & +\left((-512-512\,t-128\,t^2+64\,t^3+64\,t^4+16\,t^5)\right)s \\ & +\left((-128-128\,t-32\,t^2+16\,t^3+16\,t^4+4\,t^5)\right)s^2 \\ & +\left((64+64\,t+16\,t^2+64\,t^3+28\,t^4+4\,t^5)\right)s^3 \\ & +\left((64+64\,t+16\,t^2+28\,t^3+10\,t^4+t^5)\right)s^4 \\ & +\left((16+16\,t+4\,t^2+4\,t^3+t^4)\right)s^5. \end{aligned} \tag{4.2}$$

The function is symmetric on s,t, i.e. $h(t,s)=h(s,t)$. Note that

$$\begin{aligned} & (-512-512\,t-128\,t^2+64\,t^3+64\,t^4+16\,t^5) \\ & \quad =4(-128-128\,t-32\,t^2+16\,t^3+16\,t^4+4\,t^5) \\ & \quad =16\,(t-2)\left(t^2+2\,t+4\right)(t+2)^2 \end{aligned}$$

The coefficients of s^3, s^4, s^5 are all positive. By Descartes' rule, for $t\in(0,2)$, the equation $h(t,s)=0$ gives rise to an implicit curve Γ. But for $t\in(2,\infty)$, equation has no positive solutions because all terms in $h(t,s)$ are positive. The implicit curve Γ of (t,s) is the curve in Figure 6 which divide the first quadrant into two parts, the region E and the unbounded region G. Given any point $(t,s)\in\Gamma$, there is a unique u such that $pm_1=pm_4=0$ because the implicit curve $pm_1=0$ is smooth and monotonically increasing. Now for any point $(t_0,s_0)\in G$, there exists $(t,s)\in\Gamma$ such that $t<t_0, s<s_0$. We can choose u to make $pm_1(t,u)=pm_4(s,u)=0$. Therefore $pm_1(t_0,u)>0$ and $pm_4(s_0,u)>0$. At any point (t_0,s_0) in open region G, pm_1, pm_4 could

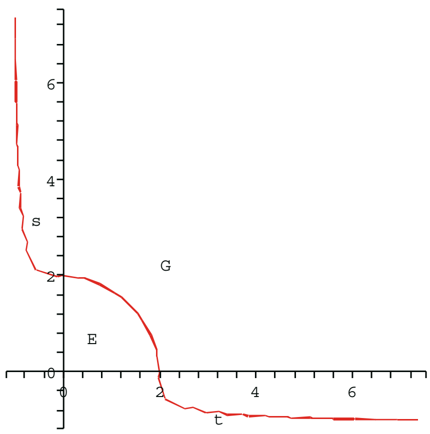

Figure 6.

be positive simultaneously for some u. By a similar argument, we can show that pm_1, pm_4 could not be positive simultaneously in the region E for any u. This completes the proof of Lemma 1.

Remark 3. We give here an example of Lemma 1. Let the configuration $x = (-2, -1, 1, 2)$ correspond to $(t, s) = (1, 1) \in E$. In this case, we get $m = [-3168/5201 - 5904/5201\, u, 9540/5201 + 5436/5201\, u, 9540/5201 - 5436/5201\, u, -3168/5201 + 5904/5201\, u]$ by solving equation (1.4) under $r = -1$. Note that $m_1 > 0$ implies $u < 0$ while $m_4 > 0$ implies $u > 0$. Lemma 1 tells us the region E is not a central configuration region but that region G could be. Here we restrict our regions in the first quadrant of ts-plane. In fact, we will show that the open region G is a central configuration region. We achieve this by use of the following lemmas.

LEMMA 2. *The implicit curve $pm_2 = 0$ intersects the vertical line $pm_1 = 0$ only at $s = 0$ for any $u \geqslant -1$ in the first quadrant of ts-plane. The implicit curve $pm_3 = 0$ intersects the horizontal line $pm_4 = 0$ only at $t = 0$ for any $u \leqslant -1$ in the first quadrant of ts-plane.*

Proof. In the proof of Lemma 1, we know that $pm_1 = 0$ gives rise to a vertical line for $u > -1$ and $pm_1 > 0$ for $u < -1$. $pm_1 = 0$ gives rise to the s-axis when $u = -1$. Solving for u from $pm_1 = 0$, we have

$$u = \frac{-4t^2 - 16t - 16 + 8t^3 + t^5 + 5t^4}{16 + 16t + t^4 + 4t^3 + 4t^2}. \tag{4.3}$$

Substituting u into $pm_2 = 0$ and simplifying, we have

$$\frac{s\left(256 + 16s^4 t + \cdots + 144 st^5\right)}{16 + 16t + t^4 + 4t^3 + 4t^2} = 0. \tag{4.4}$$

The only solution is $s = 0$. So $pm_2 = 0$ intersects $pm_1 = 0$ only at $s = 0$ for any $u > -1$. We complete the proof of the first part. We can similarly prove the second part.

LEMMA 3. *Given any $u \leqslant -1$, the three equations $pm_2 = 0$, $pm_3 = 0$, $pm_4 = 0$ give rise to three implicit curves and the three implicit curves enclose a nonempty open central configuration region B_1 (shown in Figure 8). Given any $-1 \leqslant u \leqslant 1$, the four equations $pm_1 = 0$, $pm_2 = 0$, $pm_3 = 0$, $pm_4 = 0$ give rise to four implicit curves and the four implicit curves enclose a nonempty open central configuration region B_2 (shown in Figure 7). Given any $1 \leqslant u$, the three equations $pm_1 = 0$, $pm_2 = 0$, $pm_3 = 0$ give rise to three implicit curves and the three implicit curves enclose a nonempty open central configuration region B_3 (shown in Figure 8).*

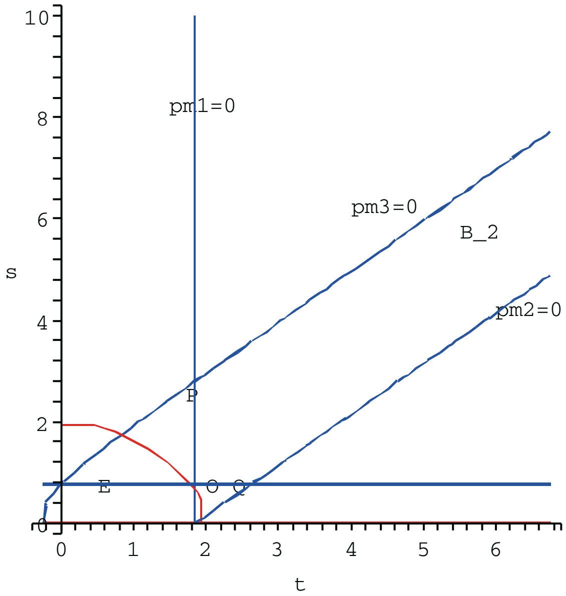

Figure 7.

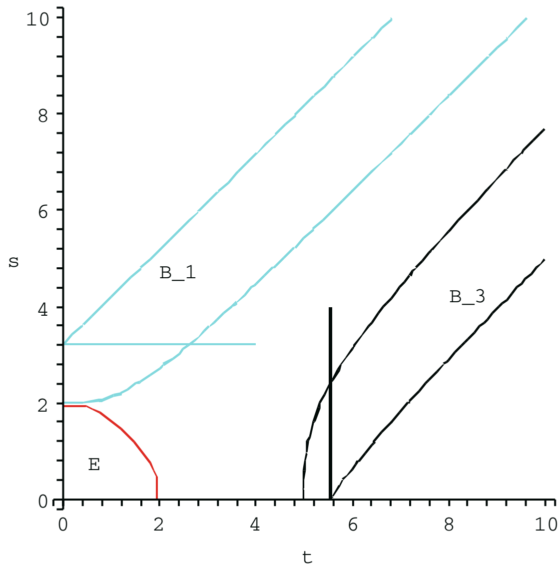

Figure 8.

Proof. First, we show that $pm_2=0$, $pm_3=0$ give rise two smooth monotone increasing curves enclosing an open region in which $pm_2>0$, $pm_3>0$ with the curve $pm_3=0$ above the curve $pm_2=0$.

In fact, let

$$\begin{aligned}
c_0(s,u) &= 16+16u+48s+us^4+56s^2+s^5+32s^3+24us^2+32us+8us^3+9s^4 \\
&= (s+2)^4(s+1+u), \\
c_1(s,u) &= 16+40s+2us^3+24us+36s^2+2s^4+14s^3+12us^2+16u \\
&= 2(s+2)^3(s+1+u), \\
c_2(s,u) &= \left(us^2+4+4us+5s^2+4u+8s+s^3\right) = (s+2)^2(s+1+u), \\
c_3(s,u) &= \left(-s^2-6s+2us+4u-8\right) = -(s+2)(s-2u+4), \\
c_4(s,u) &= (-5+u-2s).
\end{aligned}$$

then $pm_2(t,s,u)=c_0(s,u)+c_1(s,u)t+c_2(s,u)t^2+c_3(s,u)t^3+c_4(s,u)t^4-t^5$.

The zeros of the coefficients are $s=-u-1$, $s=-u-1$, $s=-u-1$, $s=2u-4$, $s=-5/2+1/2u$ respectively. In $us-$plane, these linear functions of u and s divide the half plane $(s>0)$ into 4 parts, $A_1,\ldots,A_4$ as indicated in Figure 9.

For given u,s in any of $A_1,\ldots,A_4$, the signs of the coefficients change at most once. More precisely, all the coefficients have negative signs in region

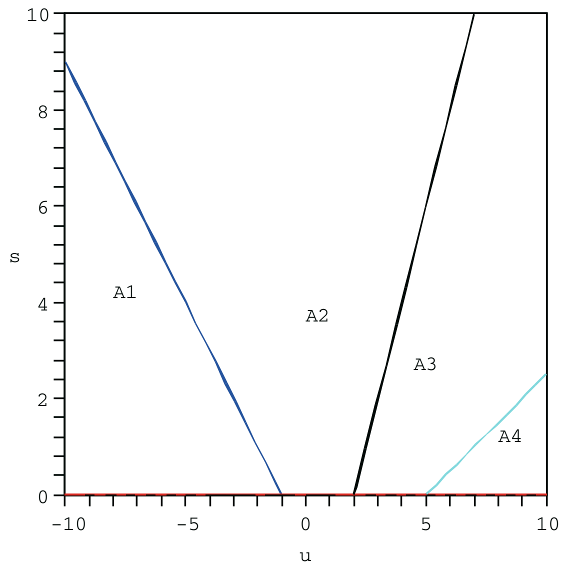

Figure 9.

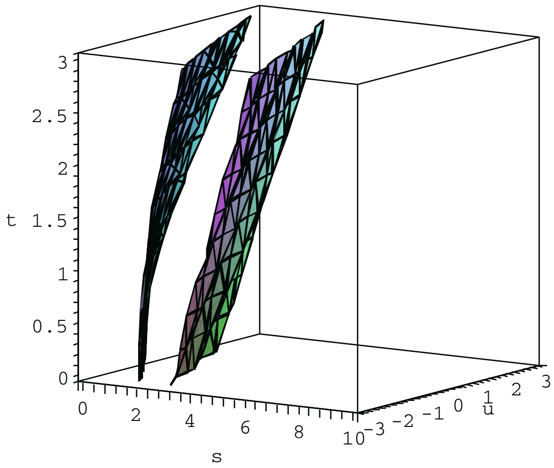

Figure 10.

A_1 and the coefficients only change sign once in the other regions. By Descartes' rule, $pm_2(t, s, u) = 0$ has at most one positive zero for given $s > 0$, and any u. Then $pm_2(t, s, u) = 0$ gives rise to an implicit surface (the left surface in Figure 10) with $s > 0, t > 0$. Therefore $pm_2(t, s, u) = 0$ gives rise to the implicit curve in Figure 7 for given u (the curve $pm_2 = 0$ in Figure 7 is for $u = 3/4$). Using similar arguments for $pm_3 = 0$ we can conclude that given u, $pm_2 = 0$ and $pm_3 = 0$ define two implicit curves in first quadrant of ts-plane. Now we want to show the curve $pm_3 = 0$ is always above $pm_2 = 0$ for any given u (or equivalently show that the surface $pm_3 = 0$ is above the surface pm_2).

From the proof of Lemma 1, for given $u = 0$, the curve $pm_3 = 0$ is above the curve $pm_2 = 0$. The surface $pm_3 = 0$ is above the surface $pm_2 = 0$ if the surface $pm_3 = 0$ doesn't intersect with the surface $pm_2 = 0$ i.e., if the equation system

$$\begin{aligned} &pm_2(t, s, u) = 0, \\ &pm_3(t, s, u) = 0, \\ &pm_2(t, s, u) = pm_3(t, s, u) \end{aligned} \tag{4.5}$$

has no solution for $t > 0, s > 0$. In fact, from $pm_3 - pm_2 = 0$ we have $u = (-16t + \cdots + s^3t^2)/(24t + \cdots + 8st^2)$. Substituting u into pm_2, we have $pm_2 = (256 + 896s + \cdots + 9s^2t^6)/(16 + 24s + \cdots + t^4)$ in which all terms are positive. So it is impossible that $pm_2 = 0$ and $pm_3 = 0$ simultaneously.

For any given $u \leqslant -1$, $pm_1 > 0$ for $t > 0, s > 0$ by Lemma 1. Also $pm_4 = 0$ gives rise to a horizontal line $s = s_0$ with $s_0 \geqslant 2$ by Lemma 1. From Lemma 2, $pm_3 = 0$ intersects $pm_4 = 0$ at $t = 0$. In addition, $pm_3 = 0$ is above

$pm_2=0$. So the three implicit curves $pm_2=0, pm_3=0, pm_4=0$ enclose an open unbounded strip-like central configuration region B_1 as indicated in Figure 8. The region slides from infinity to $s=2$ with one vertex on the s-axis while u changes from negative infinity to -1.

For any given $-1<u<1$, $pm_1=0, pm_4=0$ give rise to two implicit straight line $t=t_0, s=s_0$, respectively, with $(t_0,s_0)\in\Gamma$ from Lemma 1. Then the four implicit curves $pm_1=0, pm_2=0, pm_3=0, pm_4=0$ enclose an open unbounded strip-like central configuration region B_2 as indicated in Figure 7. The region slides with one vertex on Γ.

For any given $u\geqslant 1$, $pm_4>0$ for $t>0, s>0$ by Lemma 1. Also $pm_1=0$ gives rise to a vertical line $t=t_0$ with $t_0\geqslant 2$ by Lemma 1. From Lemma 2, $pm_2=0$ intersects $pm_1=0$ at $s=0$. In addition $pm_3=0$ is above $pm_2=0$. So the three implicit curves $pm_1=0, pm_2=0, pm_3=0$ enclose an open unbounded strip-like central configuration region B_3 as indicated in Figure 8. The region slides from $t=2$ to infinity with one vertex on the t-axis while u changes from 1 to infinity.

Remark 4. From Lemmas 2 and 3, for given $0<u<1$, the unique intersecting point between $pm_1=0$ and $pm_3=0$ is above the unique intersecting point between $pm_1=0$ and $pm_4=0$ because $pm_3=0$ intersects $pm_4=0$ at $t=0$ and $pm_3=0$ is monotonically increasing (for example: P is above O in Figure 7 with $u=3/4$). Similarly, the unique intersecting point between $pm_4=0$ and $pm_2=0$ is to the right of the unique intersecting point between $pm_4=0$ and $pm_1=0$ (for example: Q is at the right of O in Figure 7 with $u=3/4$). It follows that as u changes continuously, the central configuration region is swept out continuously.

LEMMA 4. *For any point (t_0,s_0) in region G in Figure 6, there exists at least one u such that the corresponding configuration $(-s_0-1,-1,1,t_0+1)$ could become a central configuration centered at u. Therefore, region G is a central configuration region.*

Proof. Lemma 4 can be obtained from the proof of Lemma 3 because the central configuration region B obtained in lemma 3 sweeps all the region G. Once (t_0,s_0) falls in a central configuration region B which is obtained for a fixed u, then the configuration $(-s_0-1,-1,1,t_0+1)$ is a central configuration by choosing proper positive masses centered at u.

Remark 5. The four lemmas complete the proof of theorem 2. For any given point (t_0,s_0) in region G, the configuration $x=(-s_0-1,-1,1,t_0+1)$ could be a central configuration for some center of mass u.

5. Proof of Theorem 3

For any given configuration $x = (-s-1, -1, 1, t+1)$ and center of mass u, we get the unique mass solution $m = m(x, u)$ in Section 2. Given any x, if there exist u such that $\underline{u}(x) < u < \bar{u}(x)$, then the configuration x is a central configuration with mass $m(x, u)$ which is positive and centered at u. Now we turn to prove Theorem 3. From Lemma 1 in Section 4 we know that there does not exist a positive mass making configuration $x = (-s_0-1, -1, 1, t_0+1)$ central at any point $(t_0, s_0) \in E$. Then the set $\{u|x$ forms a central configuration centered at u with positive mass $m(x, u)\}$ is empty implying $d(t_0, s_0) = 0$.

From Lemma 4 in Section 4, for each point (t_0, s_0) in the region G, there exist at least one u_0 such that $(t_0, s_0) \in B$, which is an open central configuration region corresponding to u_0. Because B is open, (t_0, s_0) is an interior point of B. Then for u in a small neighborhood of u_0, (t_0, s_0) is still in the central configuration region for those u. Note that the central configuration changes continuously w.r.t. the change of center of mass u. So the small neighborhood is in the set $\{u|x$ forms a central configuration centered at u with positive mass m(x,u)$\}$. So $\mathrm{d}(t_0, s_0) > 0$.

In order to show $\lim_{t_0\to\infty, s_0\to 0}\mathrm{d}(t_0, s_0) = 0$, we need to show the central configuration region moves with the almost the same speed as u moves. We also need to show that the slope of $pm_2 = 0$ goes to infinity as t_0 goes to infinity and as s_0 goes to zero. In fact, from $pm_1 = 0$, it is easy to solve for u, which is

$$u = \frac{-4t^2 - 16t - 16 + 8t^3 + t^5 + 5t^4}{16t + 16 + t^4 + 4t^3 + 4t^2}$$
$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{t^2\left(768t + 112t^3 + 576 + 416t^2 + t^6 + 8t^5 + 24t^4\right)}{\left(16t + 16 + t^4 + 4t^3 + 4t^2\right)^2}.$$

As t goes to infinity, $\mathrm{d}u/\mathrm{d}t$ goes to 1.
The implicit derivative $\mathrm{d}s/\mathrm{d}t$ in $pm_2 = 0$ is

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -\frac{16 + \cdots - 5t^4 + 4t^3u - 18st^2 + 10s^2t + 2s^4 + 36s^2 - 3s^2t^2}{48 + \cdots + 48us + 4us^3 + 8s^3t - 2st^3 + 3s^2t^2}.$$

In order to consider the change of the slope of the curve $pm_2 = 0$ as t goes to infinity and as s goes to zero, we substitute u obtained above from $pm_1 = 0$ into $\mathrm{d}s/\mathrm{d}t$ and let s go to zero which gives us

$$\left.\frac{\mathrm{d}s}{\mathrm{d}t}\right|_{s\to 0} = 1/4\,\frac{\left(t^5 + 6t^4 + 12t^3 + 88t^2 + 240t + 288\right)t^2}{9t^5 + 30t^4 + 44t^3 + 24t^2 + 48t + 32} \approx \frac{t^2}{36} \text{ for large t.}$$

So for small s_0 and large t_0, $\mathrm{d}(t_0, s_0) \approx 36s_0/t_0^2$. Therefore $\lim_{t_0\to\infty, s_0\to 0} \mathrm{d}(t_0, s_0) = 0$.

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