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Isosceles trapezoid central configurations of the Newtonian four-body problem

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We use a simple direct and basic method to prove that there is a unique isosceles trapezoid central configuration of the planar Newtonian four-body problem when two pairs of equal masses are located at adjacent vertices of a trapezoid. Such isosceles trapezoid central configurations are an exactly one-dimensional family. Explicit expressions for masses are given in terms of the size of the quadrilateral.

1. Introduction

The classical n-body problem consists of the study of the dynamics of n point masses interacting according to Newtonian gravity. Let $q_1, q_2, \dots, q_n$ represent the positions in Euclidean space $\mathbb{R}^d$ of n particles with respective positive masses $m_1, m_2, \dots, m_n$. The classical n-body motions of the system of particles are determined by

$$m_i\ddot{q}_i = -\sum_{i\neq j} \frac{m_i m_j}{|q_i - q_j|^3}(q_i - q_j), \quad 1 \leqslant i \leqslant n. \tag{1.1}$$

The simplest possible motions are such that the configuration is constant up to rotation and scaling. Such a configuration is called a *central configuration*. A central configuration (cc) in $\mathbb{R}^d$ is a point $(q_1, q_2, \dots, q_n)$ (i.e. a configuration) in $(\mathbb{R}^d)^n$ which is a solution of the following nonlinear algebraic system of equations:

$$\lambda(q_i - c) - \sum_{j=1, j\neq i}^{n} \frac{m_j(q_i - q_j)}{|q_i - q_j|^3} = 0, \quad 1 \leqslant i \leqslant n, \tag{1.2}$$

for a constant λ, where $c = (\sum m_i q_i)/M$ is the centre of mass and $M = m_1 + m_2 + \cdots + m_n$ is the total mass. Equations (1.2) are referred to as the central configuration equations (cc equations). They are invariant under isometries and homotheties of $\mathbb{R}^d$. Thus, when counting ccs, we only count classes of solutions modulo these symmetries. The finiteness problem for central configurations of the n-body problem was proposed by Chazy [5] and Wintner [14] and was listed by Smale [13] as problem 6 on his list of problems for this century. This was proved recently in the case $n = 4$ by Hampton and Moeckel [6]. For any four given positive masses, there exists a convex central configuration with given ordering of the particles [10, 15]. However, the exact number of four-body central configurations is only known for the four-equal-masses case [1,2]. The possible geometrical shape of four body planar central configurations is discussed in [8]. The case for three equal masses is studied in [4, 12]. Long and Sun [9] studied the convex central configurations with equal opposite masses and they proved that such cc must possess a symmetry. Perez-Chavela and Santoprete [11] proved that cc must possess such symmetry when two equal masses are located at opposite vertices of a quadrilateral and, at most, only one of the remaining masses is larger than the equal masses. Albouy *et al.* [3] recently generalized further and obtained the symmetry of convex cc with two equal masses at the opposite vertices. The uniqueness of such symmetric central configurations follows Leandro [7]. Albouy *et al.* also conjectured that the convex central configuration is unique for any given order of four positive masses. These references also give an excellent review.

MacMillan and Bartky [10] proved that there is a unique isosceles trapezoid central configuration of the planar Newtonian four-body problem when two pairs of equal masses are located at adjacent vertices of a trapezoid. Our goal in this paper is to find the explicit expressions for isosceles trapezoid central configurations. The proof for their uniqueness involves a subtle rigorous analysis by using elementary algebra and the implicit function theorem. The expressions also provide a numerical method for computing all isosceles trapezoid central configurations, so that the unique isosceles trapezoid cental configuration can be clearly calculated for given masses. The main results in this paper are as follows.

THEOREM 1.1. *Let four particles form an isosceles trapezoid $q = (q_1, q_2, q_3, q_4)$ with the side $[q_1, q_2]$ parallel to the side $[q_3, q_4]$,*

$$\frac{|q_3 - q_4|}{|q_1 - q_2|} = a \quad \textit{and} \quad \frac{|q_3 - q_2|}{|q_1 - q_2|} = \frac{|q_4 - q_1|}{|q_1 - q_2|} = b.$$

Let Γ be the smooth curve implicitly defined by $\Gamma = \{(a, b) \mid g(a, b) = 0,\ 0 < a < 1\}$, where

$$g(a, b) = (b^2 + a)^{3/2}(2b^3 - a^3 - 1) - b^3 - a^3b^3 + 2a^3. \tag{1.3}$$

If $m_1 = m_2 \neq m_3 = m_4$ and $m_3/m_1 = m$, there is a unique isosceles trapezoid central configuration. Moreover, if $0 < m < 1$, then $(a, b) \in \Gamma$. Conversely, if $(a, b) \in \Gamma$, then there exists a unique m such that q is a central configuration and m is given by

$$m = \frac{(1 - b^3)a^2}{b^3 - a^3}. \tag{1.4}$$

2. Uniqueness of isosceles trapezoid central configurations

First, we observe that if (q_1, q_2, q_3, q_4) is a central configuration with parameter λ and positive masses (m_1, m_2, m_3, m_4), then $\mu^{-1/3}(q_1, q_2, q_3, q_4)$ is the same central configuration with masses μ^{-1} (m_1, m_2, m_3, m_4) and the same value of λ. Second, we observe that central configurations are invariant under rotation and translation. Third, we observe that the shortest and the longest exterior sides have to face each other [9, proposition 2.5], and $m_1 = m_2$ and $m_3 = m_4$ if q is an isosceles trapezoid. Without loss of generality, let $r_{ij} = |q_i - q_j|$ and we assume $r_{12} = 1$ is

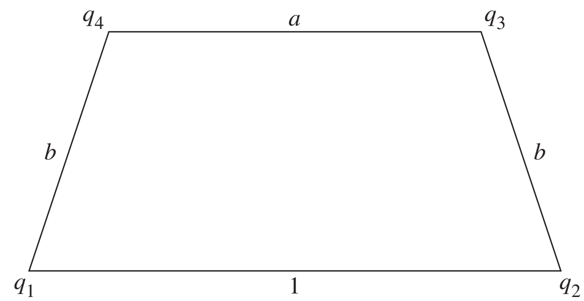

Figure 1. Isosceles trapezoidal configuration

the longest exterior side of the quadrilateral. We consider the isosceles trapezoid central configurations with masses

$$m_1 = m_2 = 1, \qquad m_3 = m_4 = m,$$

and

$$\left.\begin{aligned} q_1 &= (-\tfrac{1}{2}, 0), & q_2 &= (\tfrac{1}{2}, 0), \\ q_3 &= \left(\frac{a}{2}, \sqrt{b^2 - \left(\frac{1-a}{2}\right)^2}\right), & q_4 &= \left(-\frac{a}{2}, \sqrt{b^2 - \left(\frac{1-a}{2}\right)^2}\right), \end{aligned}\right\} \tag{2.1}$$

where $0 < a, b < 1$. Here $q = (q_1, q_2, q_3, q_4)$ forms an isosceles trapezoid in counter-clockwise order, and the side $[q_1, q_2]$ is parallel to the side $[q_3, q_4]$ (see figure 1).

We have

$$r_{12} = 1, \qquad r_{14} = r_{23} = b, \qquad r_{34} = a, \qquad r_{13} = r_{24} = \sqrt{b^2 + a},$$

and the centre of mass is

$$\left(0, \frac{m\sqrt{4b^2 - 1 + 2a - a^2}}{2 + 2m}\right).$$

By using the symmetry of the configuration, the central configuration equations (1.2) are reduced to the following three equations:

$$2 + \frac{m(a+1)}{(a+b^2)^{3/2}} + \frac{m(1-a)}{b^3} - \lambda = 0, \tag{2.2}$$

$$-\frac{a+1}{(a+b^2)^{3/2}} + \frac{1-a}{b^3} - \frac{2m}{a^2} + 2\lambda a = 0, \tag{2.3}$$

$$\frac{1}{(a+b^2)^{3/2}} + \frac{1}{b^3} - \frac{\lambda}{1+m} = 0. \tag{2.4}$$

Solving (2.4) for λ and then substituting into (2.2) and (2.3), after some subtle simplifications and rearrangements, yields

$$m = \frac{(1-b^3)a^2}{b^3 - a^3}, \tag{2.5}$$

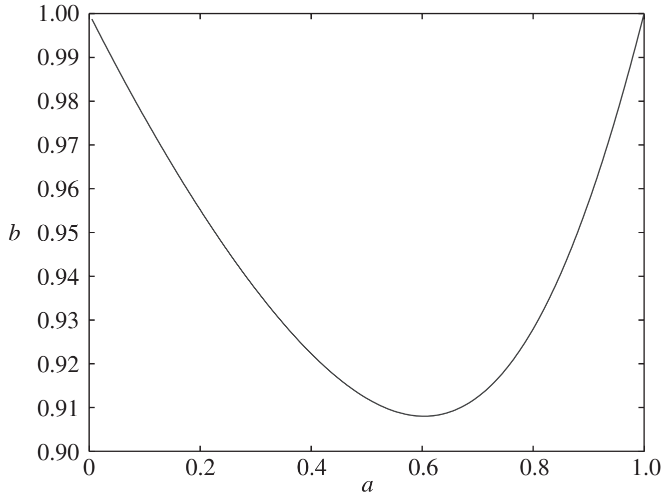

Figure 2. Graph $g(a, b) = 0$.

with the constraint

$$g(a, b) = (b^2 + a)^{3/2}(2b^3 - a^3 - 1) - b^3 - a^3b^3 + 2a^3 = 0. \tag{2.6}$$

We first show that $\Gamma = \{(a, b) \mid g(a, b) = 0,\ 0 < a < 1,\ 0 < b < 1\}$ is a smooth curve.

LEMMA 2.1. *Suppose that $g(a, b)$ is defined as in (2.6). Then, for any $0 < a < 1$, there exists a unique $b = b_0(a)$ such that*

(i) $g(a, b) < 0$ *for* $0 < b < b_0(a)$,

(ii) $g(a, b_0(a)) = 0$ *and*

(iii) $g(a, b) > 0$ *for* $b_0(a) < b < 1$.

Moreover, $\Gamma_b = \{(a, b_0(a))\colon 0 < a < 1\}$ is a smooth curve in $(0, 1) \times (0, 1)$, as shown in figure 2.

Proof. $g(a, b)$ is a continuous function on the first quadrant $a > 0$, $b > 0$. By direct computation, we have

$$g(a, 0) = -a^{3/2}(1 - a^{3/2})^2 < 0,$$
$$g(a, 1) = (\sqrt{1 + a} - 1)(2 + a + \sqrt{1 + a})(1 - a)(a^2 + a + 1) > 0,$$

for any $0 < a < 1$. By the intermediate value theorem, there exists at least one $b \in (0, 1)$ such that $g(a, b) = 0$. We write (2.6) as

$$g(a, b) = 2b^3(b^2 + a)^{3/2} + 2a^3 - (1 + a^3)((b^2 + a)^{3/2} + b^3), \tag{2.7}$$
$$\frac{\partial g}{\partial b} = 6b^2(b^2 + a)^{3/2} + 6b^4\sqrt{b^2 + a} - 3b(1 + a^3)(\sqrt{b^2 + a} + b). \tag{2.8}$$

Then above two equations can be rewritten as

$$\begin{pmatrix} g \\ \dfrac{\partial g}{\partial b} \end{pmatrix} = A \begin{pmatrix} 2b^2\sqrt{b^2+a} \\ 1 \end{pmatrix}, \tag{2.9}$$

where

$$A := (A_{ij}) := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{2.10}$$

and

$$A_{11} = b(b^2+a), \qquad A_{12} = 2a^3 - (1+a^3)((b^2+a)^{3/2} + b^3),$$
$$A_{21} = 3(2b^2+a), \qquad A_{22} = -3b(1+a^3)(\sqrt{b^2+a} + b).$$

A simple calculation shows that

$$\begin{aligned} \det(A) &= -3b^2(2b^2+a)(1+a^3)(\sqrt{b^2+a}+b) + 3b^4(1+a^3)(\sqrt{b^2+a}+b) \\ &\qquad - 6a^3(2b^2+a) + 3(1+a^3)((b^2+a)^{3/2}+b^3)(2b^2+a) \\ &= 3b^4(1+a^3)(\sqrt{b^2+a}+b) + 3a(2b^2+a)(-2a^2 + (1+a^3)(b^2+a)^{1/2}) \\ &\geqslant 3b^4(1+a^3)(\sqrt{b^2+a}+b) + 3(1-a^{3/2})^2 a^{3/2}(2b^2+a) \\ &> 0. \end{aligned}$$

Then

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}.$$

Multiplying both sides of (2.9) by A^{-1} implies that, for any $0 < a, b < 1$,

$$-A_{21}g(a,b) + A_{11}\frac{\partial g}{\partial b}(a,b) = \det(A) > 0. \tag{2.11}$$

Fix $a \in (0,1)$. Let $b = b_0(a)$ be the smallest $b \in (0,1)$ such that $g(a,b) = 0$. Note that $A_{11} > 0$, $A_{21} > 0$ for any $0 < a, b < 1$. Then $\partial g(a, b_0(a))/\partial b > 0$ from (2.11). In fact, whenever $g(a,b) \geqslant 0$, $\partial g(a, b_0(a))/\partial b > 0$ from (2.11). This implies that $g(a,b) > 0$ for all $b > b_0(a)$. Finally, since $\partial g(a, b_0(a))/\partial b > 0$ for any $a \in (0,1)$, the set $\mathit{\Gamma}_b = \{(a, b_0(a)) \colon 0 < a < 1\}$ is a smooth curve in the first quadrant of the ab-plane from the implicit function theorem. □

LEMMA 2.2. *Suppose that $h(m,a)$ is defined as*

$$h(m,a) = \left(a^{1/3}\left(\frac{1+ma}{a^2+m}\right)^{2/3} + 1\right)^{3/2}(a^2 - m) + a^{1/2}(ma - 1). \tag{2.12}$$

Then, for any $0 < m < 1$, there exists a unique $a = a_0(m)$ such that

(i) *$h(m,a) < 0$ for $0 < a < a_0(m)$,*

(ii) *$h(m, a_0(m)) = 0$,*

(iii) $h(m,a) > 0$ *for* $a_0(m) < a < 1$,

(iv) $\lim_{m\to 0} a_0(m) = 0$ *and* $\lim_{m\to 1} a_0(m) = 1$.

Moreover, $\Gamma_a = \{(m, a_0(m)) \colon 0 < m < 1\}$ *is a smooth curve in* $(0,1)\times(0,1)$.

Proof. By direct computation we have

$$h(m,0) = -m < 0, \qquad h(m,1) = (2\sqrt{2}-1)(1-m) > 0$$

for any $0 < m < 1$. Note that $h(m,a)$ is a continuous function. By the intermediate value theorem, there exists at least one $a \in (0,1)$ such that $h(m,a) = 0$:

$$\begin{pmatrix} h \\ \dfrac{\partial h}{\partial a} \end{pmatrix} = M \begin{pmatrix} \sqrt{a^{1/3}\left(\dfrac{ma+1}{m+a^2}\right)^{2/3} + 1} \\ a^{-1/2} \end{pmatrix}, \tag{2.13}$$

where

$$M := (M_{ij}) := \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \tag{2.14}$$

and

$$\begin{aligned}
M_{11} &= \left(\sqrt[3]{a}\left(\frac{ma+1}{m+a^2}\right)^{2/3} + 1\right)(a^2 - m), \\
M_{12} &= a(ma-1), \\
M_{21} &= \tfrac{1}{2}a^{-2/3}\left(\frac{ma+1}{m+a^2}\right)^{-1/3}(m+a^2)^{-2} \\
&\quad \times \Bigg[m(a^2-m) + 3ma(a-m^2) \\
&\qquad \times +4ma^2 + 4a^{5/3}\sqrt[3]{\frac{ma+1}{m+a^2}}(m+a^2)^2 3a^5 m + a^4 + 8m^2a^3\Bigg], \\
M_{22} &= \tfrac{3}{2}ma - \tfrac{1}{2}.
\end{aligned}$$

We consider the region $D = \{(m,a) \mid a^2 - m > 0,\ 0 < m < 1,\ 0 < a < 1\}$. Note that $h(m,a) < 0$ if $a^2 - m \leqslant 0$, because $ma - 1 < 0$ for any $0 < m, a < 1$. So (m,a) should be in D if $h(m,a) = 0$. A simple calculation shows that

$$\begin{aligned}
\det(M) &= \tfrac{1}{2}a^{-2/3}\left(\frac{ma+1}{m+a^2}\right)^{-1/3}(m+a^2)^{-2} \\
&\quad \times \Bigg[4ma^2((a^2+m)(a^2-m) + 2a(1-m^2a^2)) \\
&\qquad \times +a^{2/3}\sqrt[3]{\frac{ma+1}{m+a^2}}(3a(a-m^2) + m(1-a^3))(m+a^2)^2\Bigg] \\
&> 0
\end{aligned}$$

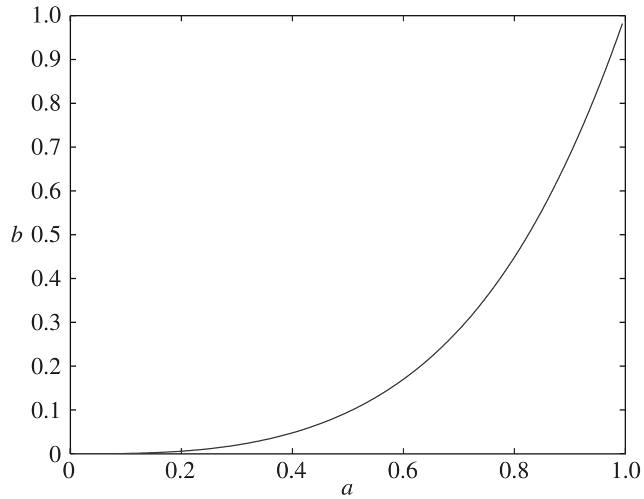

Figure 3. m is an increasing function of a.

in region D. This implies that, for any $0 < m, a < 1$,

$$-M_{21}h(m,a) + M_{11}\frac{\partial h}{\partial a}(m,a) = \det(M)a^{-1/2} > 0. \tag{2.15}$$

Note that $M_{11} > 0$ and $M_{21} > 0$ in D. Fix $m \in (0,1)$. Let $a = a_0(m)$ be the smallest $a \in (0,1)$ such that $h(m,a) = 0$. Then $\partial h(m, a_0(m))/\partial a > 0$ from (2.15). In fact, whenever $h(m,a) \geqslant 0$, $\partial h(m, a_0(m))/\partial a > 0$ from (2.15). This implies that $h(m,a) > 0$ for all $a > a_0(m)$. Finally, since $\partial h(m, a_0(m))/\partial a > 0$ for any $m \in (0,1)$, the set $\Gamma_a = \{(m, a_0(m))\colon 0 < m < 1\}$ is a smooth curve in the region D from the implicit function theorem. By a simple direct computation, $\lim_{m\to 0} a_0(m) = 0$ and $\lim_{m\to 1} a_0(m) = 1$. □

LEMMA 2.3. *The function $m = m(a, b_0(a))$ given by (2.5) is an increasing function of a ranging from 0 to 1 (see figure 3).*

Proof. By lemma 2.1, b is a function of a implicitly defined by (2.6). Then m is a function of a given by (2.5). Now we prove that a is also a function of m when $0 < m < 1$. To do this, solving (2.5) for b, we have

$$b = \sqrt[3]{\frac{a^2(1+ma)}{a^2+m}}.$$

Substitute b into (2.6), we have

$$g = \frac{(1-a^3)a^{3/2}}{a^2+m}h(m,a) = 0,$$

where $h(m,a)$ is defined by (2.12). $g = 0$ is equivalent to $h = 0$ for $0 < m, a < 1$. By lemma 2.2, a is a function of m implicitly defined by $h(m,a) = 0$. So $a = a_0(m)$ is a one-to-one and onto function from $(0,1)$ to $(0,1)$. Note that it is increasing, as $0 = \lim_{m\to 0} a_0(m) < \lim_{m\to 1} a_0(m) = 1$. Therefore, $m = m(a, b_0(a))$ is also an increasing function. □

Proof of theorem 1.1. In order to become a central configuration, the isosceles trapezoid must satisfy (2.6). By lemma 2.1, there is exactly a one-dimensional family of isosceles trapezoid central configurations. By lemma 2.3, the isosceles trapezoid central configuration is unique. The longest side has the larger masses and the shortest side has the smaller masses, which are $0 < a < 1$ and $0 < m < 1$. □

REMARK 2.4. The one-dimensional family of isosceles trapezoid central configurations has the bound of the square configuration when the smallest side assumes its maximum ($a = 1$ and $m = 1$, i.e. four equal masses). It has the bound of the equilateral triangle when the smallest side assumes its minimum ($a = 0$ and $m = 0$). Geometrically, the larger interior angle is between 90° and 120°. The smaller interior angle is between 60° and 90°.

REMARK 2.5. Numerically, we can find the unique isosceles trapezoid central configuration for any given $0 < m < 1$. For example, for a given $m = \frac{1}{2}$, $h(\frac{1}{2}, a) = 0$ has a unique solution $a = 0.8251535579$; $g(0.8251535579, b) = 0$ has a unique solution $b = 0.9338823045$.

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