
Volume 251 Issue 1 July 1, 2011 ISSN 0022-0396
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J. Differential Equations 251 (2011) 58–72

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

The golden ratio and super central configurations of the n-body problem

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ARTICLE INFO

Article history:
Received 25 March 2010
Revised 20 September 2010
Available online 25 March 2011

MSC:
37N05
70F10
70F15
37N30
70H05

Keywords:
Central configuration
n-Body problem
Super central configuration
Inverse problem
Relative equilibrium
Golden ratio

ABSTRACT

In this paper, we consider the problem of central configurations of the n-body problem with the general homogeneous potential $1/r^\alpha$. A configuration $q = (q_1, q_2, \dots, q_n)$ is called a super central configuration if there exists a positive mass vector $m = (m_1, \dots, m_n)$ such that q is a central configuration for m with m_i attached to q_i and q is also a central configuration for m', where $m' \neq m$ and m' is a permutation of m. The main discovery in this paper is that super central configurations of the n-body problem have surprising connections with the golden ratio φ. Let r be the ratio $\frac{|q_3-q_2|}{|q_2-q_1|}$ of the collinear three-body problem with the ordered positions q_1, q_2, q_3 on a line. q is a super central configuration if and only if $1/r_1(\alpha) < r < r_1(\alpha)$ and $r \neq 1$, where $r_1(\alpha) > 1$ is a continuous function such that $\lim_{\alpha\to 0} r_1(\alpha) = \varphi$, the golden ratio. The existence and classification of super central configurations are established in the collinear three-body problem with general homogeneous potential $1/r^\alpha$. Super central configurations play an important role in counting the number of central configurations for a given mass vector which may decrease the number of central configurations under geometric equivalence.

1. Introduction

In mathematics and the arts, the golden ratio, also known as the divine proportion, golden mean, or golden section, is a special real number. It is often encountered when taking the ratios of two quantities (distances) in simple geometric figures such as the pentagon, pentagram, decagon and dodecahedron. Two quantities are in the golden ratio if the ratio of the sum of the quantities to the

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doi:10.1016/j.jde.2011.03.002

larger one equals the ratio of the larger one to the smaller [38]. The golden ratio is an irrational mathematical constant and it is often denoted by φ, approximately 1.6180.

The golden ratio is known as a mathematical beauty. It has fascinated intellectuals of diverse interests for at least 2400 years [38]. At least since the Renaissance, many artists and architects have proportioned their works to approximate the golden ratio. The golden ratio is also observed in the fields of botany, human body and other natures [23]. We cite a paragraph from the book "The Golden Ratio: The Story of Phi, the World's Most Astonishing Number" by Mario Livio [15].

> Some of the greatest mathematical minds of all ages, from Pythagoras and Euclid in ancient Greece, through the medieval Italian mathematician Leonardo of Pisa aka "Fibonacci" and the Renaissance astronomer Johannes Kepler, to present-day scientific figures such as Oxford physicist Roger Penrose, have spent endless hours over this simple ratio and its properties. But the fascination with the Golden Ratio is not confined just to mathematicians. Biologists, artists, musicians, historians, architects, psychologists, and even mystics have pondered and debated the basis of its ubiquity and appeal. In fact, it is probably fair to say that the Golden Ratio has inspired thinkers of all disciplines like no other number in the history of mathematics.

The golden ratio has surprising connections with super central configurations of the n-body problem. It is amazing that this mathematical beauty is hidden in the action of celestial particles under the universal gravitation.

Consider the n-body problem in the general law of attraction $f(s)$ with Newton's Law of Gravitation ($f(s) = \frac{1}{s^2}$) as a special case:

$$m_k \ddot{q}_k = \sum_{j=1,\, j\neq k}^{n} m_k m_j f(r_{jk}) \frac{(q_j - q_k)}{|q_j - q_k|}, \quad 1 \leqslant k \leqslant n, \tag{1.1}$$

where $m_k > 0$ are the masses of the bodies, $q_k \in \mathbf{R}^d$ (usually with $d = 1$, $d = 2$, or $d = 3$) are their positions respectively, and $r_{jk} = |q_j - q_k|$ is the distance of bodies m_j and m_k. Let $C = m_1 q_1 + \cdots + m_n q_n$, $M = m_1 + \cdots + m_n$, $c = C/M$ be the first moment, total mass and center of mass of the bodies, respectively.

Rather than solving this notoriously difficult system of equations, it would be much easier if it could be reduced to

$$m_k \ddot{q}_k = -\mu(t) m_k (q_k - c),$$

where $\mu(t)$ is a scalar function for all particles. Then the motion at any fixed time must satisfy the following nonlinear algebraic system:

$$\sum_{j=1,\, j\neq k}^{n} m_j f(r_{jk}) \frac{(q_j - q_k)}{|q_j - q_k|} = -\lambda(q_k - c), \quad 1 \leqslant k \leqslant n, \tag{1.2}$$

for a constant λ. In this paper, $f(s)$ takes the general homogeneous form: $f(s) = \frac{1}{s^\alpha}$ with $\alpha > 0$. Because the systems (1.1) and (1.2) are singular when two particles have the same positions, it is natural to assume that the configuration avoids the collision set which is defined by

$$\triangle = \bigcup \bigl\{ q = (q_1, q_2, \ldots, q_n) \in \bigl(\mathbf{R}^d\bigr)^n \bigm| q_i = q_j \text{ for some } i \neq j \bigr\}. \tag{1.3}$$

To avoid singularities we will restrict q to be in $V(n)$:

$$V(n) = \bigl\{ q = (q_1, q_2, \ldots, q_n) \in \bigl(\mathbf{R}^d\bigr)^n \bigr\} \setminus \triangle. \tag{1.4}$$

Definition 1.1 *(Central configuration).* A configuration $q \in V(n)$ is a *central configuration* (CC for short) for a given mass vector $m = (m_1, m_2, \ldots, m_n) \in (\mathbf{R}^+)^n$ if q is a solution of the system (1.2) for some constant $\lambda \in \mathbf{R}$.

The notion of a super central configuration depends on the notion of admissible ordered sets of masses. Given a configuration $q = (q_1, q_2, \ldots, q_n) \in (\mathbf{R}^d)^n \setminus \triangle$, denote $S(q, \alpha)$ the admissible ordered set of masses for fixed $\alpha > 0$ by

$$S(q,\alpha) = \left\{ m = (m_1, \ldots, m_n) \;\middle|\; m_i \in \mathbf{R}^+,\ q \text{ is a CC for } m \text{ with } f(s) = \frac{1}{s^\alpha} \right\}. \tag{1.5}$$

For a given $m \in S(q, \alpha)$, let $S_m(q, \alpha)$ be the permutationally admissible set for m, defined by

$$S_m(q,\alpha) = \left\{ m' \in S(q,\alpha) \;\middle|\; m' \neq m \text{ and } m' \text{ is a permutation of } m \right\}. \tag{1.6}$$

The requirements that $m' \neq m$ and m' is a permutation of m in $S_m(q, \alpha)$ is necessary to exclude some trivial cases. For example, if q is a central configuration for $m = (m_1, m_2, m_3, \ldots, m_n)$ with $m_1 = m_2$, then q is also a central configuration for $m' = (m_2, m_1, m_3, \ldots, m_n)$ but $m' \notin S_m(q, \alpha)$ since $m' = m$. Denote the number of elements in $S_m(q, \alpha)$ by ${}^{\#}S_m(q, \alpha)$. Directly from the definition, ${}^{\#}S_m(q, \alpha)$ is finite and has $n! - 1$ as an upper bound.

Definition 1.2 *(Super central configuration).* A configuration q is called a *super central configuration* (SCC for short) for some $\alpha > 0$ if there exists a positive mass vector m such that $S_m(q, \alpha)$ is not empty.

The motivation to study the set $S_m(q, \alpha)$ emanates from the example of the equilateral triangle configuration [7] in the planar three-body problem. If q is the equilateral triangle configuration and $m = (m_1, m_2, m_3)$, then q is also a central configuration for each permutation of m. No configuration is a super central configuration for planar four-body problem under the Newtonian potential as an immediate consequence of a theorem proved by W. Macmillan and W. Bartky (see [14, p. 872]). The existence and classification of the super central configurations were studied in the collinear three-body problem [35] and in the collinear four-body problem [36] in Newton's law of gravitation ($f(s) = \frac{1}{s^2}$). They provide examples of super central configurations other than the equilateral triangle configuration for three-body problem.

Super central configurations play an important role in counting the number of central configurations for a given mass vector m. The number of central configurations refers to the number of equivalent classes.

Definition 1.3 *(Geometric equivalence).* Two configurations q and $p \in V$ are *geometrically equivalent*, denoted by $q \sim p$, if and only if q and p are similar modulo translations, dilations, rotations and permutations of the configuration. From now on, the number of central configurations refers to the number of the equivalent classes.

In history, there are different understandings of the equivalence and one is called *permutation equivalence*, which is the same definition as geometric equivalence but without permutation. Under the definition of permutation equivalence of central configurations, collinear central configurations are one of a few families of central configurations with given positive masses which are sort of completely understood. For each way the particles can be ordered along a line, it is well known that there is a unique position that causes a central configuration. In this case, Euler discovered the collinear configurations for the three-body problem. For the Newtonian case ($f(s) = \frac{1}{s^2}$), Moulton [19] analyzed the general n-body case and proved that the number of central configurations in the collinear n-body problem is $n!/2$ for any $m \in \mathbf{R}^+$ in 1910 and Smale [29] reconfirmed the result by a different variational approach in 1970. In 2009, Woodlin and Xie [32] generalized Moulton's results in the general homogeneous case ($f(s) = \frac{1}{s^\alpha}$, $\alpha > 0$). For a given mass vector

$m = (m_1, m_2, \ldots, m_n)$, the question concerning the number of central configurations is still a challenging problem for 21st century mathematicians (see Smale [28]). The finiteness under Newtonian potential was proved for $n = 4$ by Hampton and Moeckel [20], and it is still open for general n. In fact, an exact count is known only for the equal four masses case [1,2]. Some partial results of central configurations are given in [3,10,12,25,26,30,34] for the four-body problem with some equal masses, in [8,9,24,31] for the five-body problem, and in [5,32] for general homogeneous or quasi-homogeneous potentials. Existence of different types of central configurations can be found in [4,6, 22,37] and the references therein. For the importance and additional properties of central configurations and related topics, we refer to the works of R. Moeckel [18], D. Saari [27], and the books [16,17].

By Moulton's Theorem, there are $3!/2 = 3$ collinear central configurations under permutation equivalence for three equal positive masses but it is counted one collinear central configuration under geometric equivalence. More significantly, if a configuration q is a super central configuration for mass m, the number of central configurations for m may decrease under geometric equivalence. For example: as a consequence of Theorem 1.7 below, the number of collinear central configurations are less than or equal to two for three positive distinct masses given by (1.9) because the central configuration for (m_1, m_2, m_3) is the same as the central configuration for (m_3, m_1, m_2). It would be interesting to know the exact number of central configurations for any mass distribution in the general homogeneous potential $f(s) = \frac{1}{s^\alpha}$. The existence and uniqueness of central configuration (up to translation and scaling) is proved in [19,32] for any given ordered positive masses $m_1, m_2, \ldots, m_n$ in the collinear n-body problem with the general homogeneous potential $f(s) = \frac{1}{s^\alpha}$.

In the first part of this paper, we study the existence and classifications of the super central configurations in the collinear three-body problem with general homogeneous potential $f(s) = \frac{1}{s^\alpha}$. In the second part of this paper, we give the exact number of central configurations by using the property of SCC. The golden ratio arises as a certain limit for these super central configurations.

Central configurations are invariant up to translation and scaling since $f(s) = \frac{1}{s^\alpha}$ is homogeneous. We can choose the coordinate system so that all the three bodies are on the x-axis with positions $q_1 = 0$, $q_2 = 1$, and $q_3 = 1 + r$, where $r > 0$ is the ratio $\frac{|q_3 - q_2|}{|q_2 - q_1|}$. This is a general form of the collinear 3-body configuration up to translation and scaling.

Definition 1.4. For each $\alpha > 0$, let

$$\bar{r}(\alpha) = \sup\left\{ r > 0 \;\middle|\; q = (0, 1, 1 + r) \text{ is an SCC for } f(s) = \frac{1}{s^\alpha} \right\}$$

and

$$\underline{r}(\alpha) = \inf\left\{ r > 0 \;\middle|\; q = (0, 1, 1 + r) \text{ is an SCC for} f(s) = \frac{1}{s^\alpha} \right\}.$$

Theorem 1.5 *(Golden ratio).*

$$\lim_{\alpha \to 0^+} \bar{r}(\alpha) = \varphi \approx 1.6180$$

and

$$\lim_{\alpha \to 0^+} \underline{r}(\alpha) = \frac{1}{\varphi} \approx 0.6180.$$

For any $r \notin (\frac{1}{\varphi}, \varphi)$, $q = (0, 1, 1 + r)$ is not a super central configuration.

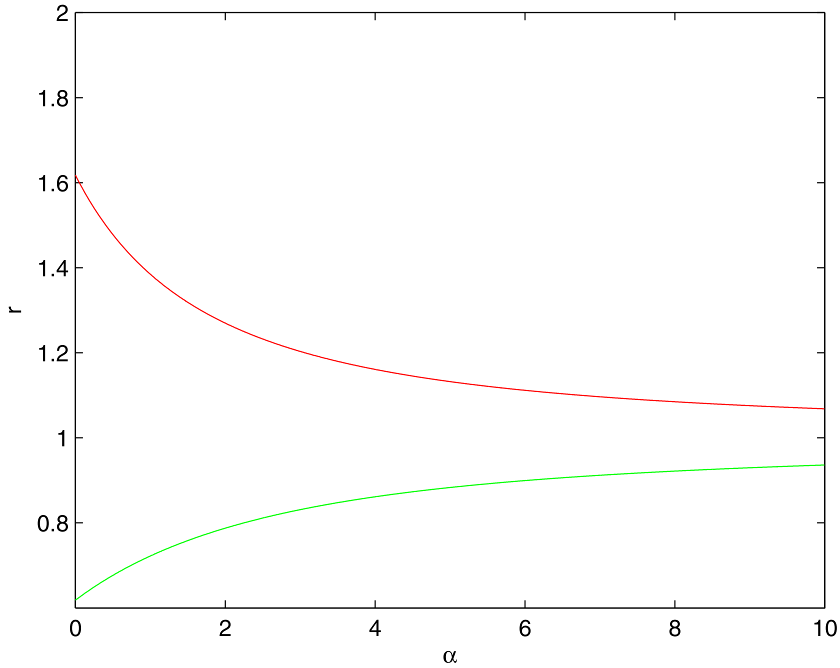

Fig. 1. Graph of $\bar{r}(\alpha)$ (upper curve) and $\underline{r}(\alpha)$ (lower curve).

Theorem 1.6. *Assume* $f(s) = \frac{1}{s^\alpha}$ *with* $\alpha > 0$.

(1) $\bar{r}(\alpha)$ *and* $\underline{r}(\alpha)$ *are continuous functions in* $\alpha \in (0, \infty)$ (*see Fig.* 1).
(2) $0 < \underline{r}(\alpha) < 1 < \bar{r}(\alpha) < \infty$ *and* $\bar{r}(\alpha) = \frac{1}{\underline{r}(\alpha)}$.
(3) *For any* $\alpha > 0$, $r \in (\underline{r}(\alpha), \bar{r}(\alpha))$ *and* $r \neq 1$, $q = (0, 1, 1 + r)$ *is a super central configuration.*

To state the existence and classification of super central configurations, let us define $h_i(r, \alpha)$ $(i = 1, 2, 3)$, and $g(r, \alpha)$ by

$$\begin{aligned}
h_1(r, \alpha) &= 2rf(r) + f(r)r^2 + f(r) - r^2 f(1) - r^2 f(1 + r) - rf(1),\\
h_2(r, \alpha) &= -rf(r) + r^2 f(1) + 2rf(1) - f(1 + r) - f(r) + f(1),\\
h_3(r, \alpha) &= f(r) - rf(1 + r) + r^2 f(1),\\
g(r, \alpha) &= \bigl(r + r^2 + 1\bigr)\bigl(f(1) - f(1 + r) + f(r)\bigr).
\end{aligned} \tag{1.7}$$

We also define $\Gamma(\alpha)$ by

$$\Gamma(\alpha) = \bigl\{r \bigm| h_1(r, \alpha) > 0,\ h_2(r, \alpha) > 0,\ r > 0\bigr\} \quad \text{for each fixed } \alpha > 0. \tag{1.8}$$

Theorem 1.7. *Fix* $\alpha > 0$, $r > 0$, *and* $q = (0, 1, 1 + r)$.

(1) *For any* $m = (m_1, m_2, m_3) \in S(q, \alpha)$, $S_m(q, \alpha)$ *has at most one element, i.e. either* ${}^{\#}S_m(q, \alpha) = 0$ *or* ${}^{\#}S_m(q, \alpha) = 1$.
(2) $q = (0, 1, 1 + r)$ *is a super central configuration if and only if* $r \in \Gamma(\alpha) \setminus \{1\}$. *If* $r \notin \Gamma(\alpha)$ *or* $r = 1$, $S_m(q, \alpha)$ *is empty for any* $m \in S(q, \alpha)$.
(3) *If* $r \in \Gamma(\alpha) \setminus \{1\}$, *then* ${}^{\#}S_m(q, \alpha) = 1$ *only in the following two cases*:
 (i) $S_m(q, \alpha) = \{(m_3, m_1, m_2)\}$ *and* m *is given by*

$$m_1 = \frac{Mh_1(r, \alpha)}{g(r, \alpha)}, \qquad m_2 = \frac{Mh_2(r, \alpha)}{g(r, \alpha)}, \qquad m_3 = \frac{Mh_3(r, \alpha)}{g(r, \alpha)}. \tag{1.9}$$

(ii) $S_m(q,\alpha) = \{(m_2, m_3, m_1)\}$ *and m is given by*

$$m_1 = \frac{Mh_3(r,\alpha)}{g(r,\alpha)}, \qquad m_2 = \frac{Mh_1(r,\alpha)}{g(r,\alpha)}, \qquad m_3 = \frac{Mh_2(r,\alpha)}{g(r,\alpha)}. \tag{1.10}$$

Remark 1.8. The properties of $h_i(r,\alpha)$ are shown in Section 3. The masses m_1, m_2, m_3 given by (1.9) and (1.10) are mutually distinct. More precisely, $m_1 > m_3 > m_2$ in (1.9) for $\underline{r}(\alpha) < r < 1$, $m_1 = m_2 = m_3$ for $r = 1$, and $m_2 > m_3 > m_1$ in (1.9) for $1 < r < \bar{r}(\alpha)$. Some methods used in [11,36] for the Newtonian case $\alpha = 2$ are directly applied to the general homogeneous potential. But some new methods have to be employed to overcome the difficulty due to the general power α. For instance, the Descartes' rule works for the polynomials in the Newtonian potential $\alpha = 2$, and it does not work any more for the general power when we study the property of $\bar{r}(\alpha)$ and $\underline{r}(\alpha)$. A method involving some analysis skills of some elementary algebra and calculus is presented to study the central configurations in the general homogeneous potential $f(s) = 1/s^\alpha$.

Permutations of the masses play a particular role in the classification of collinear central configurations. For any $n \in \mathbf{N}$ (the set of integers), we denote by $P(n)$ the set of all permutations of $\{1, 2, \ldots, n\}$. For any element $\tau \in P(n)$, we use $\tau = (\tau(1), \tau(2), \ldots, \tau(n))$ to denote the permutation τ. We also denote a permutation of $(m_1, m_2, \ldots, m_n)$ by $m(\tau) = (m_{\tau(1)}, m_{\tau(2)}, \ldots, m_{\tau(n)})$ for $\tau \in P(n)$. We define the converse permutation of τ by $con(\tau) = (\tau(n), \ldots, \tau(1))$ and we define the converse position of $p = (p_1, p_2, \ldots, p_n)$ by $con(p) = (p_n, p_{n-1}, \ldots, p_1)$.

Without loss of generality, we suppose simply that the n bodies are located on the x-axis if q is collinear. Let the collinear configuration space for m be defined by

$$W(n) = \{q = (q_1, q_2, \ldots, q_n) \in \mathbf{R}^n \mid q_1 < q_2 < \cdots < q_n\}.$$

Because in $W(n)$ we do not allow q_is to change their order, we now allow m_is to change their order. Note that when we say by $q = (q_1, \ldots, q_n) \in W(n)$ is a collinear CC for $m(\zeta) \equiv (m_{\zeta(1)}, \ldots, m_{\zeta(n)})$ with some $\zeta \in P(n)$, we always mean that $m_{\zeta(i)}$ is put on q_i for all $i = 1, \ldots, n$. Using this notation, for collinear CCs Definition 1.3 becomes

Definition 1.9 *(Geometric equivalence).* For a given $m \in (\mathbf{R}^+)^n$, let $q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n) \in W(n)$ be two collinear CCs for $m(\zeta)$ and $m(\eta)$ with ζ and $\eta \in P(n)$ respectively. Then q and p are *geometrically equivalent*, denoted by $q \sim p$, if and only if either $q = a(p - b)$ or $q = a(con(p) - b)$ for some $a \in \mathbf{R} \setminus \{0\}$ and $b = (b_0, b_0, \ldots, b_0) \in \mathbf{R}^n$. We denote by $L(n,m)$ the set of all geometrical equivalent classes of n-body collinear central configurations for any given mass vector $m \in (\mathbf{R}^+)^n$.

Historically, there also exist two other ways to define the equivalence classes among collinear CCs. Because of these different understandings, the number of CCs were counted differently in different papers. A good review and discussion can be found in [11] and the references therein.

Definition 1.10 *(Permutation equivalence).* For a given $m \in (\mathbf{R}^+)^n$, let $q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n) \in W(n)$ be two collinear CCs for $m(\zeta)$ and $m(\eta)$ with ζ and $\eta \in P(n)$ respectively. Then q and p are *permutation equivalent*, denoted by $q \sim_P p$, if and only if either $\zeta = \eta$ and $q \sim p$, or $\zeta = con(\eta)$ and $q \sim p$. We denote by $L_P(n,m)$ the set of all permutation equivalent classes of n-body collinear central configurations for any given mass vector $m \in (\mathbf{R}^+)^n$.

Definition 1.11 *(Mass equivalence).* For a given $m \in (\mathbf{R}^+)^n$, let $q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n) \in W(n)$ be two collinear CCs for $m(\zeta)$ and $m(\eta)$ with ζ and $\eta \in P(n)$ respectively. Then q and p are *mass equivalent*, denoted by $q \sim_M p$, if and only if either $m(\zeta) = m(\eta)$ and $q \sim p$, or $m(\zeta) = m(con(\eta))$ and $q \sim p$. We denote by $L_M(n,m)$ the set of all mass equivalent classes of n-body collinear central configurations for any given mass vector $m \in (\mathbf{R}^+)^n$.

Directly from the definitions, we can deduce that ${}^{\#}L(n,m) \leqslant {}^{\#}L_M(n,m) \leqslant {}^{\#}L_P(n,m)$. That ${}^{\#}L_P(n,m) = n!/2$ for any positive masses is proved in [19,29,32]. The quantity ${}^{\#}L_M(n,m)$ has been studied in the papers [11,12,33]. The quantity ${}^{\#}L(3,m)$ has been studied by Long and Sun in [11,13] and ${}^{\#}L(4,m)$ has been studied by Ouyang and Xie in [21] for Newtonian gravitational law $\alpha = 2$. We provide the number of central configurations for the general homogeneous potential $\alpha > 0$ in the collinear 3-body problem.

Fix $\alpha > 0$. Let

$$\mathcal{F}(\alpha) = \left\{ m(\tau) \;\middle|\; \begin{array}{l} \tau \in P(3),\ r \in \big(\underline{r}(\alpha), \bar{r}(\alpha)\big),\ \text{and } r \neq 1, \\ m_1, m_2, m_3 \text{ given by (1.9)} \end{array} \right\}. \tag{1.11}$$

Theorem 1.12. *Fix $\alpha > 0$. For any mass vector $m = (m_1, m_2, m_3) \in (\mathbf{R}^+)^3$, one and only one of the following four cases must apply*:

(i) ${}^{\#}L_M(3,m) = 3$ *and* ${}^{\#}L(3,m) = 3$, *if* $(m_1, m_2, m_3) \notin \mathcal{F}(\alpha)$ *and* m_1, m_2, *and* m_3 *are mutually distinct*;
(ii) ${}^{\#}L_M(3,m) = 3$ *but* ${}^{\#}L(3,m) = 2$, *if* $(m_1, m_2, m_3) \in \mathcal{F}(\alpha)$;
(iii) ${}^{\#}L_M(3,m) = 2$ *and* ${}^{\#}L(3,m) = 2$, *if two of* m_1, m_2, *and* m_3 *are equal to each other but not the third*;
(iv) ${}^{\#}L_M(3,m) = 1$ *and* ${}^{\#}L(3,m) = 1$, *if* $m_1 = m_2 = m_3$.

Example 1.13. Let $\alpha = 2$, $r = 0.8$ and $M = 1$. Then $m_1 = 0.6227880638$, $m_2 = 0.02161285437$, and $m_3 = 0.3555990819$ by Eq. (1.9). It is easy to check that $q = (0, 1, 1 + r)$ is a central configuration for $m = (m_1, m_2, m_3)$ with $\lambda = 0.1985301558$ and the center of mass $c = 0.6616912016$ from Eq. (1.2). The central configuration $q = (0, 1, 1 + r)$ is also a central configuration for $m = (m_3, m_1, m_2)$ with $\lambda = 0.9512876948$ and the same center of mass $c = 0.6616912016$ from Eq. (1.2). So $(m_3, m_1, m_2) \in S_{(m_1,m_2,m_3)}(q)$ and $q = (0, 1, 1 + r)$ is a super central configuration. By Theorem 1.12, the number of central configurations is decreased by one, i.e. ${}^{\#}L(3,m) = 2$.

Our paper is organized as follows. The structure of the set $S_m(q,\alpha)$ for given $\alpha > 0$ and the proof of Theorem 1.7 are provided in Section 2. In Section 3, we study the properties of the functions $h_i(r,\alpha)$, $\bar{r}(\alpha)$, $\underline{r}(\alpha)$, and the proofs of Theorems 1.5 and 1.6 are carried out. Theorem 1.12 is proved in Section 4.

2. The structure of $S_m(q,\alpha)$ for given $\alpha > 0$

Fix $\alpha > 0$ and $r > 0$. Let $q = (0, 1, 1 + r)$ be a central configuration for some appropriately chosen positive mass $m = (m_1, m_2, m_3)$ where m_i is attached to q_i. So $S(q,\alpha)$ is not an empty set. Conversely, r is uniquely determined by the mass $m = (m_1, m_2, m_3)$ [32]. Since $S_m(q,\alpha)$ is a subset of $\{m(\tau) \mid \tau \in P(3)\}$, we only need to check whether the other five permutations of mass m are also in $S(q,\alpha)$. Because $q = (0, 1, 1 + r)$ is fixed, when we say $m(\tau) \equiv (m_{\tau(1)}, m_{\tau(2)}, m_{\tau(3)}) \in S(q,\alpha)$ for some $\tau \in P(3)$, we always mean that $m_{\tau(i)}$ is attached to q_i for all $i = 1, 2, 3$.

Denote the six permutations in $P(3)$ by

$$\begin{array}{lll} \tau_1 = (1,2,3), & \tau_2 = (3,1,2), & \tau_3 = (2,3,1), \\ \tau_4 = (1,3,2), & \tau_5 = (2,1,3), & \tau_6 = (3,2,1). \end{array}$$

Proof of Theorem 1.7. For $q = (0, 1, 1 + r)$ with $r > 0$, the central configuration equations (1.2) are equivalent to

$$\begin{cases} m_2 f(1) + m_3 f(1+r) - \lambda c = 0, \\ -m_1 f(1) + m_3 f(r) + \lambda(1 - c) = 0, \\ -m_1 f(1+r) - m_2 f(r) + \lambda(1 + r - c) = 0, \end{cases} \tag{2.1}$$

where $c = \frac{m_2+m_3(1+r)}{M}$ is the center of mass. The quantity M should be regarded as a parameter. For the sake of concise computation, we use $f(s)$ rather than $\frac{1}{s^\alpha}$. But we should keep in mind that $f(s) = \frac{1}{s^\alpha}$. Substituting c into above equations (2.1), they become a system of linear equations of the masses m_1, m_2, m_3. By Gaussian Elimination, we have

$$\begin{cases} m_1 = \dfrac{f(r)M - \lambda r}{f(1) - f(1+r) + f(r)}, \\ m_2 = \dfrac{\lambda r - f(1+r)M + \lambda}{f(1) - f(1+r) + f(r)}, \\ m_3 = \dfrac{-\lambda + f(1)M}{f(1) - f(1+r) + f(r)}. \end{cases} \tag{2.2}$$

By direct computation and simplification, we have the total mass M:

$$M = m_1 + m_2 + m_3,$$

and the center of mass c:

$$c = \frac{m_2 + m_3(1+r)}{M} = \frac{-f(1+r) + f(1) + f(1)r}{f(1) - f(1+r) + f(r)}, \tag{2.3}$$

which is independent of the choice of masses. Because $f(1) - f(1+r) + f(r) = 1 - \frac{1}{(1+r)^\alpha} + \frac{1}{r^\alpha} > 0$ for any $r > 0$, $\alpha > 0$, all three masses m_1, m_2, m_3 are positive if and only if the parameters satisfy

$$\max\left\{\frac{r}{f(r)}, \frac{1}{f(1)}\right\} < \frac{M}{\lambda} < \frac{1+r}{f(1+r)},$$

which is

$$\max\left\{r^{\alpha+1}, 1\right\} < \frac{M}{\lambda} < (1+r)^{\alpha+1}.$$

Through the following series of claims we will show that ${}^{\#}S_m(q,\alpha)$ is 0 or 1.

Claim 1. If $r = 1$, i.e. $q = (0,1,2)$, then $S_m(q,\alpha)$ is an empty set for any $m \in S(q,\alpha)$.

In fact, if $r = 1$, then $m_1 = \frac{(M-\lambda)2^\alpha}{2^{1+\alpha}-1}$, $m_2 = \frac{2^{1+\alpha}\lambda - M}{2^{1+\alpha}-1}$, $m_3 = \frac{(M-\lambda)2^\alpha}{2^{1+\alpha}-1}$ from Eq. (2.2). So m_1 must be equal to m_3. Therefore (m_1, m_3, m_2) is not in $S_m(q,\alpha)$, for otherwise $m_1 = m_2$ from the above equations for m_1 and m_2. Similarly, $(m_2, m_1, m_3) \notin S_m(q,\alpha)$. This proves the claim.

Because of Claim 1 and the uniqueness of central configuration for a given order of masses m, $S_m(q,\alpha)$ is an empty set if m_1, m_2, m_3 are not mutually distinct.

We assume $r \neq 1$ and m_1, m_2, m_3 are mutually distinct from now on. Suppose $m = (m_1, m_2, m_3) \in S(q,\alpha)$.

Claim 2. $m(\tau_4)$, $m(\tau_5)$, and $m(\tau_6)$ are not in $S_m(q,\alpha)$.

Note that the center of mass c is fixed for a given r by (2.3). The center of mass is $(m_2 + m_3(1+r))/M$ for $m(\tau_1)$ and the center of mass is $(m_3 + m_2(1+r))/M$ for $m(\tau_4)$. If $m(\tau_4) \in S_m(q,\alpha)$, we have $m_2 + m_3(1+r) = m_3 + m_2(1+r)$ which implies $m_2 = m_3$. This contradiction proves that $m(\tau_4)$ is not in $S_m(q,\alpha)$. Similar arguments prove that $m(\tau_5)$ and $m(\tau_6)$ are not in $S_m(q,\alpha)$.

Claim 3. $m(\tau_2)$ and $m(\tau_3)$ cannot be in $S_m(q,\alpha)$ simultaneously.

If so, the center of mass for $m = m(\tau_1)$, the center of mass for $m(\tau_2)$, and the center of mass for $m(\tau_3)$ should be same, i.e.

$$m_2 + m_3(1+r) = m_1 + m_2(1+r) = m_3 + m_1(1+r),$$

which implies $m_1 = m_2 = m_3$. This contradiction proves the claim.

The three claims prove that either ${}^{\#}S_m(q,\alpha) = 0$ or ${}^{\#}S_m(q,\alpha) = 1$ must hold for any $m \in S(q,\alpha)$ and any $q = (0, 1, 1+r)$. When $S_m(q,\alpha) = 1$, either $m(\tau_2) \in S_m(q,\alpha)$ or $m(\tau_3) \in S_m(q,\alpha)$.

Case 1. $m(\tau_2) = (m_3, m_1, m_2) \in S_m(q,\alpha)$ if and only if $m = (m_1, m_2, m_3)$ is given by (1.9).

Because $m(\tau_2)$ is a permutation of m, $M = m_1 + m_2 + m_3 = m_{\tau_2(1)} + m_{\tau_2(2)} + m_{\tau_2(3)}$ is a constant. If $m(\tau_2)$ is in $S_m(q,\alpha)$, then $m(\tau_2)$ should be given by Eq. (2.2) with different λ, say $\lambda(\tau_2)$, i.e.

$$\begin{cases} m_{\tau_2(1)} = \dfrac{f(r)M - \lambda(\tau_2)r}{f(1) - f(1+r) + f(r)}, \\ m_{\tau_2(2)} = \dfrac{\lambda(\tau_2)r - f(1+r)M + \lambda(\tau_2)}{f(1) - f(1+r) + f(r)}, \\ m_{\tau_2(3)} = \dfrac{-\lambda(\tau_2) + f(1)M}{f(1) - f(1+r) + f(r)}. \end{cases} \tag{2.4}$$

By setting $m_1 = m_{\tau_2(2)}$ and $m_2 = m_{\tau_2(3)}$ for the corresponding equations in (2.2), we get two linear equations in $\lambda = \lambda(\tau_1)$ and $\lambda(\tau_2)$, which solve as λ and $\lambda(\tau_2)$,

$$\begin{aligned} \lambda &= \frac{M(-f(r) + rf(1) + rf(1+r) + f(1))}{r + r^2 + 1}, \\ \lambda(\tau_2) &= \frac{M(-f(1)r + f(1+r) + f(r) + rf(r))}{r + r^2 + 1}. \end{aligned} \tag{2.5}$$

By direct computation, we have $m_3 = m_{\tau_2(1)}$ if λ and $\lambda(\tau_2)$ are taken as in (2.5). So $m(\tau_2)$ is a permutation of m for the above $\lambda(\tau_2)$. Substituting λ into (2.2), we have

$$m_1 = \frac{Mh_1(r,\alpha)}{g(r,\alpha)}, \qquad m_2 = \frac{Mh_2(r,\alpha)}{g(r,\alpha)}, \qquad m_3 = \frac{Mh_3(r,\alpha)}{g(r,\alpha)} \tag{2.6}$$

which is the same as Eq. (1.9). By Lemma 3.5 in Section 3, $m_1 > m_3 > m_2$ for $\underline{r}(\alpha) < r < 1$, $m_1 = m_2 = m_3 = \frac{M}{3}$ when $r = 1$, and $m_2 > m_3 > m_1$ for $1 < r < \bar{r}(\alpha)$.

Conversely, if the positive mass m is given by (2.6), then m satisfies Eq. (2.1) with λ given by (2.5) and $m(\tau_2)$ satisfies Eq. (2.1) with $\lambda(\tau_2)$ given by (2.5). So q is a central configuration for both m and $m(\tau_2)$ and $S_m(q,\alpha) = \{(m_3, m_1, m_2)\}$. This completes the proof of case 1.

Case 2. $m(\tau_3) = (m_2, m_3, m_1) \in S_m(q,\alpha)$ if and only if $m = (m_1, m_2, m_3)$ is given by (1.10).

The proof for $m(\tau_3) = (m_2, m_3, m_1) \in S_m(q,\alpha)$ is very similar to the proof for $m(\tau_2) \in S_m(q,\alpha)$ and thus the proof is omitted. We only give the results. We have $m(\tau_3) \in S_m(q,\alpha)$ if and only if the parameters λ and $\lambda(\tau_3)$ are

$$\lambda = \frac{M(f(1+r) + f(r) + rf(r) - rf(1))}{1+r+r^2},$$

$$\lambda(\tau_3) = \frac{M(-f(r) + rf(1+r) + f(1) + rf(1))}{1+r+r^2},$$

and m is given by

$$m_1 = \frac{Mh_3(r,\alpha)}{g(r,\alpha)}, \qquad m_2 = \frac{Mh_1(r,\alpha)}{g(r,\alpha)}, \qquad m_3 = \frac{Mh_2(r,\alpha)}{g(r,\alpha)}. \tag{2.7}$$

Because $f(r) = \frac{1}{r^\alpha}$ and $\alpha > 0$, the functions $g(r,\alpha)$ and $h_3(r,\alpha)$ are always positive for any $r > 0$. If $r \in \Gamma(\alpha) \setminus \{1\}$, then $m_i > 0$ $(i = 1,2,3)$ given by (1.9) and (1.10) are mutually distinct. So $q = (0, 1, 1+r)$ is a super central configuration if and only if $r \in \Gamma(\alpha) \setminus \{1\}$. $\square$

3. Golden ratio and properties of $\bar{r}(\alpha)$ and $\underline{r}(\alpha)$

Let

$$H_i(r,\alpha) = r^\alpha(1+r)^\alpha h_i(r,\alpha), \quad i = 1,2,3.$$

Then $h_i(r,\alpha)$ has the same zeros and positiveness for $\alpha > 0, r > 0$ as $H_i(r,\alpha)$ has. Here

$$H_1(r,\alpha) = (1+r)^{\alpha+2} - r^{\alpha+2} - r^{\alpha+1}(1+r)^{\alpha+1}, \tag{3.1}$$

$$H_2(r,\alpha) = r^\alpha(1+r)^{\alpha+2} - r^\alpha - (1+r)^{\alpha+1}, \tag{3.2}$$

$$H_3(r,\alpha) = (1+r)^\alpha - r^{\alpha+1} + r^{\alpha+2}(1+r)^\alpha. \tag{3.3}$$

Note that $H_3(r,\alpha) > 0$ for any $\alpha > 0, r > 0$.

Lemma 3.1. *For any $\alpha > 0$, there exists a unique $r_1 = r_1(\alpha)$ such that*

(i) $H_1(r,\alpha) > 0$ *for* $0 < r < r_1(\alpha)$,
(ii) $H_1(r_1(\alpha),\alpha) = 0$,
(iii) $H_1(r,\alpha) > 0$ *for* $r > r_1(\alpha)$, *and*
(iv) $r_1(\alpha) > 1$.

Moreover, $\Psi_1 = \{(\alpha, r_1(\alpha))\colon 0 < \alpha < \infty\}$ is a continuous curve in the first quadrant of αr-plane.

Proof. $H_1(r,\alpha)$ is a smooth function on the first quadrant $\alpha > 0$, $r > 0$. By direct computation we have

$$(1) \quad H_1(0,\alpha) = 1 > 0, \qquad (2) \quad \lim_{r\to\infty} H_1(r,\alpha) = -\infty. \tag{3.4}$$

By the Intermediate Value Theorem, there exists at least one $r \in (0,\infty)$ such that $H_1(r,\alpha) = 0$ for each given $\alpha > 0$. We compute $\partial H_1/\partial r$:

$$\begin{aligned} \frac{\partial H_1}{\partial r}(r,\alpha) &= (\alpha+2)(1+r)^{\alpha+1} - (\alpha+2)r^{\alpha+1} \\ &\quad - (\alpha+1)r^\alpha(1+r)^{\alpha+1} - (\alpha+1)r^{\alpha+1}(1+r)^\alpha. \end{aligned} \tag{3.5}$$

Then (3.1) and (3.5) can be rewritten as

$$\begin{pmatrix} H_1 \\ \frac{\partial H_1}{\partial r} \end{pmatrix} = A \begin{pmatrix} (1+r)^{\alpha+1} \\ -r^{\alpha} \end{pmatrix}, \tag{3.6}$$

where

$$A = (A_{ij}) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{3.7}$$

and

$$A_{11} = 1+r, \qquad A_{12} = r^2 + r(1+r)^{\alpha+1},$$
$$A_{21} = (\alpha+2), \qquad A_{22} = (\alpha+2)r + (\alpha+1)(1+r)^{\alpha+1} + (\alpha+1)r(1+r)^{\alpha}.$$

A simple calculation shows that for any $\alpha > 0$, $r > 0$,

$$\det(A) = (\alpha+2)r + \alpha(1+r)^{\alpha+2} + (1+r)^{\alpha} > 0.$$

This implies that for any $\alpha > 0$ and $r > 0$,

$$A_{21}H_1(r,\alpha) - A_{11}\frac{\partial H_1}{\partial r}(r,\alpha) = \det(A)r^{\alpha} > 0. \tag{3.8}$$

Fix $\alpha \in (0,\infty)$. Let $r_1 = r_1(\alpha)$ be the smallest $r \in (0,\infty)$ such that $H_1(r_1,\alpha) = 0$. Then $\partial H(r_1(\alpha),\alpha)/\partial r < 0$ from (3.8). In fact whenever $H_1(r,\alpha) \leqslant 0$, $\partial H_1(r(\alpha),\alpha)/\partial r < 0$ from (3.8). This implies that $H_1(r,\alpha) < 0$ for all $r > r_1(\alpha)$ and that $r_1(\alpha)$ is unique. Furthermore, we have $H_1(r,\alpha) > 0$ for all $r < r_1(\alpha)$. Because $H_1(1,\alpha) = 2^{\alpha+1} - 1 > 0$, $r_1(\alpha) > 1$ for any $\alpha \in (0,\infty)$. Finally since $\partial H_1(r_1(\alpha),\alpha)/\partial r < 0$ for any $\alpha \in (0,\infty)$, then the set $\Psi_1 = \{(\alpha, r_1(\alpha))\colon\ 0 < \alpha < \infty\}$ is a smooth curve in the first quadrant of αr-plane by the Implicit Function Theorem. □

Remark 3.2. For the Newtonian potential $\alpha = 2$,

$$H_1(r,2) = 1 + 4r + 6r^2 + 3r^3 - 3r^4 - 3r^5 - r^6$$

is a polynomial in r (as in [35]) and the sign of its coefficients only changes once. By Descartes' rule, $H_1(r,2) = 0$ has exact one positive solution at $r = r_1(2)$ and numerically $r_1(2) = 1.269815222$. But this method is not suitable for the general case $\alpha \in (0,\infty)$.

Lemma 3.3. *For any $\alpha > 0$, there exists a unique $r_2 = r_2(\alpha)$ such that*

(i) $H_2(r,\alpha) < 0$ *for* $0 < r < r_2(\alpha)$,
(ii) $H_2(r_2(\alpha),\alpha) = 0$,
(iii) $H_2(r,\alpha) > 0$ *for* $r > r_2(\alpha)$, *and*
(iv) $r_2(\alpha) < 1$.

Moreover, $\Psi_2 = \{(\alpha, r_2(\alpha))\colon\ 0 < \alpha < \infty\}$ is a continuous curve in the first quadrant of αr-plane.

Proof. The proof can be carried out in a similar way as in Lemma 3.1. □

Lemma 3.4. *For $r_1 = r_1(\alpha)$ in Lemma 3.1 and $r_2 = r_2(\alpha)$ in Lemma 3.3, r_1 is the reciprocal of r_2, i.e.,*

$$r_2(\alpha) = \frac{1}{r_1(\alpha)}, \quad \alpha \in (0, \infty). \tag{3.9}$$

Proof. Note that

$$H_1(r, \alpha) = r^{2\alpha+2} H_2\left(\frac{1}{r}, \alpha\right).$$

Then

$$H_1\big(r_1(\alpha), \alpha\big) = r_1(\alpha)^{2\alpha+2} H_2\left(\frac{1}{r_1(\alpha)}, \alpha\right) = 0.$$

By the uniqueness of the solution of $H_2(r, \alpha) = 0$ in Lemma 3.3,

$$r_2(\alpha) = \frac{1}{r_1(\alpha)}, \quad \alpha \in (0, \infty). \qquad \square$$

Lemma 3.5. *For any $r > 0$, $\alpha > 0$, we have*

(1) $0 < H_2(r, \alpha) < H_3(r, \alpha) < H_1(r, \alpha)$ *when* $r_2(\alpha) < r < 1$.
(2) $0 < H_1(r, \alpha) < H_3(r, \alpha) < H_2(r, \alpha)$ *when* $1 < r < r_1(\alpha)$.
(3) $\{(r, \alpha) \mid H_i(r, \alpha) = H_j(r, \alpha)\} = \{(1, \alpha) \mid \alpha > 0\}$, *where* $1 \leqslant i \neq j \leqslant 3$.

Proof. Let $H_{13}(r, \alpha) = H_1(r, \alpha) - H_3(r, \alpha)$ and $H_{32}(r, \alpha) = H_3(r, \alpha) - H_2(r, \alpha)$. Then we have

$$\begin{aligned} H_{13}(r, \alpha) &= (1+r)^\alpha\big((1+r)^2 - r^{\alpha+1}(1+r) - 1 - r^{\alpha+2}\big) + (1-r)r^{\alpha+1} \\ &= (1+r)^\alpha r\big((1+r)\big(1-r^\alpha\big) + \big(1-r^{\alpha+1}\big)\big) + (1-r)r^{\alpha+1}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} H_{32}(r, \alpha) &= (1+r)^\alpha\big(1 + r^{\alpha+2} - r^\alpha(1+r)^2 + (1+r)\big) + (1-r)r^\alpha \\ &= (1+r)^\alpha\big((1+r)\big(1-r^\alpha\big) + \big(1-r^{\alpha+1}\big)\big) + (1-r)r^\alpha. \end{aligned} \tag{3.11}$$

Because for any $\beta > 0$, $1 - r^\beta$ has only one zero at $r = 1$ and $1 - r^\beta > 0$ for $0 < r < 1$ and $1 - r^\beta < 0$ for $1 < r$, then $H_{13}(r, \alpha) > 0$, $H_{32}(r, \alpha) > 0$ for $0 < r < 1$ and $H_{13}(r, \alpha) < 0$, $H_{32}(r, \alpha) < 0$ for $1 < r$. Note that $H_{13}(1, \alpha) \equiv 0$ and $H_{32}(1, \alpha) \equiv 0$ for any $\alpha > 0$. Combining this with Lemmas 3.1 and 3.3, we obtain Lemma 3.5. $\square$

We are ready to prove our main theorem now.

Proofs of Theorems 1.5 and 1.6. For any given $\alpha > 0$, by Theorem 1.7, $q = (0, 1, 1+r)$ is a super central configuration if and only if $r \in \Gamma(\alpha) \setminus \{1\}$, where $\Gamma(\alpha) = \{r \mid h_1(r, \alpha) > 0,\ h_2(r, \alpha) > 0,\ r > 0\}$ given by (1.8). By Lemmas 3.1 and 3.3, $\Gamma(\alpha)$ is an open set and it is equal to $(r_2(\alpha), r_1(\alpha))$. Then

$$\bar{r}(\alpha) = \sup\big\{r > 0 \,\big|\, q = (0, 1, 1+r) \text{ is a super central configuration}\big\} = r_1(\alpha)$$

and

$$\underline{r}(\alpha) = \inf\big\{r > 0 \,\big|\, q = (0, 1, 1+r) \text{ is a super central configuration}\big\} = r_2(\alpha).$$

Because $H_1(r,\alpha)$ is a smooth function and $r_1(\alpha)$ is continuous for $\alpha \in (0,\infty)$, $\lim_{\alpha\to 0}\bar{r}(\alpha) = \lim_{\alpha\to 0} r_1(\alpha)$ is the nonnegative solution of $H_1(r,0)=0$, which is

$$1 + r - r^2 = 0.$$

So $\lim_{\alpha\to 0}\bar{r}(\alpha) = \varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180$.

Similarly, because $H_2(r,\alpha)$ is a smooth function and $r_2(\alpha)$ is continuous for $\alpha \in (0,\infty)$, $\lim_{\alpha\to 0}\underline{r}(\alpha) = \lim_{\alpha\to 0} r_2(\alpha)$ is the nonnegative solution of $H_2(r,0)=0$, which is

$$r^2 + r - 1 = 0.$$

So $\lim_{\alpha\to 0}\underline{r}(\alpha) = \frac{1}{\varphi} = \frac{-1+\sqrt{5}}{2} \approx 0.6180$.

Theorem 1.6 is an immediate consequence of Lemmas 3.1, 3.3, 3.4 and Theorem 1.7. □

4. Number of central configurations $^{\#}L(3,m)$

We only need to compute $^{\#}L(3,m)$ as $^{\#}L_M(3,m)$ is already known in [11,12,33]. To do this, we will need the following result characterizing geometric equivalence of collinear three-body central configurations.

Proposition 4.1 *(Geometric equivalence for collinear CC in 3-body problem). For a given $m \in (\mathbf{R}^+)^3$, let $q = (q_1,q_2,q_3) = (0,1,1+r_1)$ and $p = (p_1,p_2,p_3) = (0,1,1+r_2) \in W(3,m)$ be two collinear CCs for $m(\zeta)$ and $m(\eta)$ with ζ and $\eta \in P(3)$ respectively. Then q and p are equivalent, if and only if $r_2 = r_1$ or $r_2 = 1/r_1$.*

Proof. By Definition 1.9, q and p are geometrically equivalent if and only if either $q = a(p-b)$ or $q = a(con(p)-b)$ for some $a \in \mathbf{R}\setminus\{0\}$ and $b = (b_0,b_0,b_0) \in \mathbf{R}^3$. If $q = a(p-b)$, we have $(0,1,1+r_1) = a(-b_0, 1-b_0, 1+r_2-b_0)$. Then b_0 must be zero and $a=1$. Therefore we have $r_2 = r_1$. If $q = a(con(p)-b)$, we have $(0,1,1+r_1) = a(1+r_2-b_0, 1-b_0, -b_0)$, which implies that $b_0 = 1+r_2$. Then $a = -\frac{1}{r_2}$. Therefore we have $r_2 = 1/r_1$. □

Proof of Theorem 1.12. Fix $\alpha > 0$ and $m = (m_1,m_2,m_3) \in (\mathbf{R}^+)^3$. To get all the 3-body collinear CCs, we use Eq. (1.2) or (2.1) and choose $q_1 = 0$, $q_2 = 1$, and $q_3 = 1+r$ with $r > 0$. For each permutation of (m_1,m_2,m_3), there is a unique solution $(r,M,\lambda) \in (\mathbf{R}^+)^3$ to Eq. (2.1) and this r produces one collinear central configuration $q = (q_1,q_2,q_3) = (0,1,1+r)$ [32, Theorem 1.5]. It is well known that two central configurations are equivalent for the permutations $\zeta = (1,2,3)$ and $\eta = (3,2,1)$ because $r(\zeta) = 1/r(\eta)$. Also $^{\#}L(3,m) \leqslant 3 = \frac{3!}{2}$, and so the possible three nonequivalent central configurations correspond to the three permutations $\tau_1 = (1,2,3)$, $\tau_2 = (3,1,2)$, and $\tau_3 = (2,3,1)$, and the corresponding solutions to Eq. (2.1) are defined by (r_1,M,λ_1), (r_2,M,λ_2), and (r_3,M,λ_3) respectively.

Case 1. Three equal masses: $m_1 = m_2 = m_3$.

The unique solution to Eq. (2.1) for three equal masses is $(r,M,\lambda) = (1, 3m_1, m_1(1+2^{-\alpha}))$. So $^{\#}L(3,m) = 1$.

Case 2. Two equal masses: Without loss of generality, assume $m_1 = m_2 \neq m_3$.

Let $q(\tau_1) = (0,1,1+r_1)$, $q(\tau_2) = (0,1,1+r_2)$, and $q(\tau_3) = (0,1,1+r_3)$. By the existence and uniqueness of the solutions to Eqs. (2.1) for any given mass (m_1,m_2,m_3), r_1 must be equal to $1/r_2$ because $m(\tau_1) = m(con(\tau_2)) = (m_1,m_1,m_3)$. So by Proposition 4.1, we have $q(\tau_1) \sim q(\tau_2)$, from which it follows that $^{\#}L(3,m) \leqslant 2$.

It is easy to check that $r_3 = 1$, $M = 2m_1 + m_3$, and $\lambda_3 = m_3 + \frac{m_1}{2^\alpha}$ for $m(\tau_3)$. If $r_1 = r_3 = 1$, we must have $m_1 = m_3$ since the center of mass is at 1. This contradiction implies by Proposition 4.1 that $q(\tau_1)$ is not equivalent to $q(\tau_3)$. Therefore, $^{\#}L(3,m) = 2$.

Case 3. m_1, m_2, and m_3 are mutually distinct.

First we note that $r_i \neq 1$ for $i = 1, 2, 3$, otherwise $m_{\tau_i(1)} = m_{\tau_i(3)}$ by (2.2).

Second we note that $^{\#}L(3, m)$ must be strictly bigger than 1 for three mutually distinct masses. Otherwise, three of the six permutations of m give same configurations, i.e. $^{\#}S_m(q, \alpha) = 2$ which is contradiction to either $^{\#}S_m(q, \alpha) = 0$ or $^{\#}S_m(q, \alpha) = 1$ by Theorem 1.7. By the definition of $\mathcal{F}(\alpha)$ and Theorem 1.7, we have $^{\#}L(3, m) = 2$ if $m \in \mathcal{F}(\alpha)$, and $^{\#}L(3, m) = 3$ if $m \notin \mathcal{F}(\alpha)$.

We completed the proof of Theorem 1.12. $\square$

Acknowledgments

The author would like to thank the editors and the anonymous referees for their comments and suggestions regarding this work. The referees' suggestions helped to improve the clarity and the overall presentation of the results.

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