Linear Instability of Kepler Orbits in the Rhombus Four-Body Problem

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Abstract

In this paper we consider the linear stability of the Kepler orbits of the rhombus four-body problem. We show that, for given four proper masses, there exists a family of periodic solutions for which each body with the proper mass is at the vertex of a rhombus and travels along an elliptic Kepler orbit. Instead of studying the 8 degrees of freedom Hamilton system for planar four-body problem, we reduce this number by means of some symmetry to derive a two degrees of freedom system which then can be used to determine the linear instability of the periodic solutions. After making a clever change of coordinates, a two dimensional ordinary differential equation system is obtained, which governs the linear instability of the periodic solutions. The system is surprisingly simple and depends only on the length of the sides of the rhombus and the eccentricity e of the Kepler orbit.

Key word: Central Configuration, N-body Problem, Linear Stability, Kepler Orbits, Hamilton System, Monodromy Matrix.
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1 Introduction

In 1772, Lagrange discovered his remarkable equilateral periodic solutions of the planar three-body problem ([1]). For any choice of the three masses, there exists a family of periodic solutions, each body travelling along a specific Kepler orbit. Contained in the family are two types of periodic orbits: rigid circular motion (choosing a circular Kepler orbit) and homographic motion (choosing an elliptic Kepler orbit).

A crucial first step in analyzing the local behavior near a periodic solution is to compute the characteristic multipliers of the linearized equations. For the circular case, this was first accomplished by Gascheau in 1834 in his thesis ([2]). Recently Roberts ([3])showed that the stability of the family of periodic solution depends on two parameters– the eccentricity e of the orbit and the mass parameter $\beta = 27(m_1m_2 + m_1m_3 + m_2m_3)/(m_1 + m_2 + m_3)^2$. Roberts was able to reduce the dimensions of the problem from 12 to 4 by eliminating 8 standard first integrals and then making a clever change of coordinates. By analyzing the behavior of the characteristic multipliers and how they vary with e and β, he eventually obtained the region of stability and instability of the Kepler periodic solution.

The equilateral triangle is an example of *relative equilibria*, that is, a configuration which becomes an equilibria of Newton's differential equations in uniformly rotating coordinates. The concept of relative equilibrium occurs in Lagrange's work. However, for $n \geq 4$ it is very difficult to find such equilibrium, much less to analyze their stability. The exceptions are the highly symmetrical relative equilibria, like the regular polygon with equal masses. Some progress has been made in finding and analyzing the stability of relative equilibria of the four-body problem ([4],[5],[9]).

In this paper, we study the stability of Kepler orbits for rhombus four body problem. First, we carefully reduce the dimensions of the problem from 16 to 4. This is achieved by means of symmetry and by eliminating the standard integrals. Then we make a change of coordinates which decouples the associated linear system. One of the resulting systems yields two $+1$ multipliers, expected due to the nature of the periodic solution. The other system is two dimensional and governs the linear instability of the periodic solution. This system is a type of Hill Equation. The resulting system is marvellously simple and only depends on the size of the rhombus and the eccentricity e. We then analyze the behavior of the characteristic multipliers and how they vary with e and the size of the rhombus. We prove all the Kepler periodic solutions of the rhombus four body problem are unstable.

2 Kepler Orbit

The N-body problem configuration $\mathbf{q} = (q_1, q_2, \cdots, q_N)$ describes a planar positions of N masses $(m_1, q_1), \cdots, (m_N, q_N)$ where $q_i \in \mathbb{R}^2$. Interaction between the masses is determined by the Newtonian potential function

$$V(\mathbf{q}) = -\sum_{i<j} \frac{m_i m_j}{\|q_i - q_j\|}$$

on the set of noncollision configurations (where $q_i \neq q_j, i \neq j$). The Hamiltonian for the N-body problem is the sum of kinetic plus potential

$$H(q_1, q_2, \cdots, q_N, p_1, p_2, \cdots, p_N) = \sum_{i}^{N} \frac{1}{2m_i}\|p_i\|^2 + V(\mathbf{q}).$$

The Hamiltonian equations of the N-body problem are

$$\dot{q}_i = \frac{1}{m_i} p_i, \qquad \dot{p}_i = \frac{\partial U(\mathbf{q})}{\partial q_i}, \qquad i = 1, 2, \cdots, N \tag{2.1}$$

where the function $U(\mathbf{q}) = -V(\mathbf{q})$ is called the force function. The Hamiltonian equations are equivalent to the second-order ordinary differential equation system

$$m_i \ddot{q}_i = \sum_{i \neq j} \frac{m_i m_j (q_j - q_i)}{\|q_j - q_i\|^3} = \frac{\partial U}{\partial q_j}, \qquad i = 1, 2, \cdots, N. \tag{2.2}$$

We recall the fact that the N-body problem always admits *uniformly rotating* solutions which generalize the circular rotational solutions of the Kepler equation. These are called relative equilibria and their construction is classical([6]). We are looking for solutions of the form

$$q_i(t) = \psi(t) q_{0i}, \qquad i = 1, 2, \cdots, N \tag{2.3}$$

where $\psi(t)$ is a scalar function and q_{0i} is a constant vector. For the moment, identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ so that $q_i(t), \psi(t), q_{0i}$ are complex numbers. Substituting this guess (2.3) into equation (2.2), we have

$$|\psi|^3 \psi^{-1} \ddot{\psi} m_i q_{0i} = \sum_{i \neq j} \frac{m_i m_j (q_{0i} - q_{0j})}{\|q_{0i} - q_{0j}\|^3}.$$

This can be split into an equation for the scalar function $\psi(t)$

$$\ddot{\psi} = -\frac{\mu \psi}{|\psi|^3} \tag{2.4}$$

and an equation for the initial positions $\mathbf{q_0} = (q_{01}, \cdots, q_{0N})$

$$\sum_{i \neq j} \frac{m_i m_j (q_{0i} - q_{0j})}{\|q_{0i} - q_{0j}\|^3} + \mu m_i q_{0i} = 0. \tag{2.5}$$

The motion of our special solution is determined by equation (2.4), which is simply the planar Kepler Problem. Among the solutions to this problem are periodic orbits on circles and ellipses. The initial shape of the solution in position space $\mathbb{R}^{2N}$ is determined by equation (2.5) and the solution is called a *central configuration.* A great deal of effort has gone into understanding their properties (see for example ([8],[9])). The above analysis shows that all central configurations $\mathbf{q}_0$ admit *homothetic solutions* $\mathbf{q}(t) = \psi(t)\mathbf{q}_0$, $\psi(t) \in \mathbb{R}$ satisfies (2.4). Such homothetic solutions end in total collapse. Ejection orbits are the time reversal of collision orbits. Coplanar central configurations admit in addition homographic solutions $\mathbf{q}(t) = \psi(t)\mathbf{q}_0$, $\psi(t) \in \mathbb{C}$ where each of the N-masses executes a similar keplerian ellipse of eccentricity $e, 0 \leq e \leq 1$. When $e = 1$ the homographic solutions degenerate to a homothetic solution which includes total collapse. When $e = 0$, the relative equilibrium solutions are recovered consisting of uniform circular motion for each of the masses about the common center of mass.

Consider the rhombus four body problem, let $\mathbf{q}_0 = (q_{01}, \cdots, q_{04})$ be the position vector as shown in Figure 1.

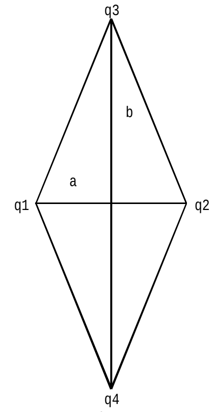


Figure1

Note that summing equation (2.5) over all i, one obtain $\sum_{i=1}^{4} m_i q_{0i} = 0$ so that the center of mass is at the origin. For simplicity, we choose $q_{01} = (-a, 0)$ with $a > 0$ and $q_{03} = (0, b)$ with $b > 0$. Similarly by the symmetry of rhombus, the other two coordinates q_{02}, q_{04} are also determined. Once given a, b, and $\frac{1}{\sqrt{3}}a < b < \sqrt{3}a$, Long and Ouyang ([9]) prove that $m_2 = m_1, m_4 = m_3$ and the masses are determined by the configuration. Furthermore, we can scale the masses so that the parameter $\mu = 1$.

This fixes a pair of unique values of m_1, m_3 as a function of the two parameters a, b. By checking equation (2.5), we find

$$m_1 = 4\,\frac{\left(a^2+b^2\right)^{3/2}\left(8\,b^3-\left(a^2+b^2\right)^{3/2}\right)a^3}{64\,a^3b^3-\left(a^2+b^2\right)^3} \tag{2.6}$$

$$m_3 = 4\,\frac{\left(8\,a^3-\left(a^2+b^2\right)^{3/2}\right)b^3\left(a^2+b^2\right)^{3/2}}{64\,a^3b^3-\left(a^2+b^2\right)^3} \tag{2.7}$$

Kepler's equation (2.4) is solvable up to quadrature ([6] page 100). In polar coordinates (r, θ), the solution with $\mu = 1$ is given by

$$r(t) = \frac{\omega^2}{1 + e\cos\theta(t)}, \qquad \dot{\theta} = \frac{\omega}{r^2}, \qquad \theta(0) = 0, \tag{2.8}$$

where e, the eccentricity of the ellipse, and ω, the angular momentum, are two parameters.We have chosen the argument of the perihelion and $\theta(0)$ both to be zero. This means the true anomaly begins at zero and is measured from the positive horizontal axis. While these choices clearly do not affect the stability of the periodic orbits, the parameters e and ω could. But we will show that ω has no effect on the linear stability of kepler's orbits.

If we write our central configuration in polar coordinates, $q_{0i} = \bar{r}_i(\cos\bar{\theta}_i, \sin\bar{\theta}_i)$, where $\bar{r}_1 = \bar{r}_2 = a$, $\bar{r}_3 = \bar{r}_4 = b$ and $\bar{\theta}_1 = \pi, \bar{\theta}_2 = 2\pi, \bar{\theta}_3 = \frac{\pi}{2}, \bar{\theta}_4 = \frac{3\pi}{2}$ then the position component of the periodic orbit is written as

$$q_i(t) = \bar{r}_i r(t)(\cos(\theta(t) + \bar{\theta}_i), \sin(\theta(t) + \bar{\theta}_i)). \tag{2.9}$$

In order to study the stability of the periodic solution, one has to compute a fundamental matrix solution $X(t)$ to the equations of motion linearized about the periodic orbit. The monodromy matrix is the matrix C satisfying $X(t+T) = X(t)C$ (for example see ([10])). Stability is governed by the eigenvalues of the monodromy matrix, called the characteristic multipliers. Since we are dealing with a Hamiltonian system, C is symplectic and the multipliers are symmetric about the unit circle. In order to have linear stability, it is necessary that all the multipliers have modulus one. For the planar four body problem, it is a 16 dimensional ODE system (2.1). As is well known, the N-body problem is a Hamiltonian system with several first integrals, therefore we can reduce the dimension by eliminating all 8 standard first integrals. But the remaining system still has 8 dimensions even after eliminating all first integrals. For this reason, it is still hard to analyze the stability of the Kepler periodic orbits.

Here we employ a new approach to reduce the degree of freedom of the Hamilton system by means of symmetry constraint. The constrained Hamilton system has 2 degrees of freedom and the corresponding ODE system which is a type of Hill equation has dimension 2 after eliminating the first integrals. The instability of the original system is governed by the two dimensional ODE system. However the stability of the new ODE system may not corresponds to the stability of the original ODE system. But we will show the Kepler orbits of rhombus four body problem are unstable in the reduced system, therefore, Kepler orbits of rhombus four body problem are unstable.

3 Constrained Hamilton System on the Rhombus Four-Body Problem

Let us turn to the variational method ([11]) to construct the constrained Hamilton system on rhombus four-body problem. We define the *Lagrangian* $L(\mathbf{q}(t), \dot{\mathbf{q}}(t)) = L(q_1(t), \cdots, q_4(t), \dot{q}_1(t), \cdots, \dot{q}_4(t))$ to be the Kinetic energy *minus* the potential energy of the system and $\dot{q}_i = \frac{dq}{dt}$ to be the velocity.

$$L(q_1(t), \cdots, q_4(t), \dot{q}_1(t), \cdots, \dot{q}_4(t)) = \sum_{i=1}^{4} \frac{m_i}{2} \|\dot{q}_i\|^2 + \sum_{i<j} \frac{m_i m_j}{\|q_i - q_j\|}.$$

Then the action functional

$$I[\mathbf{q}(t)] := \int_0^T \sum_{i=1}^{4} \frac{m_i}{2} \|\dot{q}_i\|^2 + \sum_{i<j} \frac{m_i m_j}{\|q_i - q_j\|} dt, \qquad \mathbf{q}(t) \in M$$

is defined for absolutely continuous T-periodic curves $\mathbf{q}(t)$ in the configuration manifold $M = \{\mathbf{q}(t) \in C^2([0,T]; \mathbb{R}^{2\times 4}) | \mathbf{q}(t+T) = \mathbf{q}(t)\}$. For the particular rhombus four body problem, we will look for *symmetric solutions* of the equations of the motion. Consider the following symmetric function space in polar coordinates

$$M = \begin{Bmatrix} q_1(t) = r_1(t)\exp(i\theta_1), q_2(t) = \exp(i\pi)q_1(t), \\ q_3(t) = \frac{r_3(t)}{r_1(t)}\exp(-i\frac{\pi}{2})q_1(t), q_4(t) = \exp(i\pi)q_3(t) \end{Bmatrix}.$$

Under these constraints, $\dot{q}_1 = (\dot{r}_1\cos(\theta_1) - r_1\sin(\theta_1)\dot{\theta}_1, \dot{r}_1\sin(\theta_1) + \dot{r}_1\cos(\theta_1)\dot{\theta}_1)$, we have a new Lagrangian in polar coordinates

$$L(r_1, r_3, \theta_1, \dot{r}_1, \dot{r}_3, \dot{\theta}_1) = \frac{1}{2}[m_1(\dot{r}_1^2 + r_1^2\dot{\theta}_1^2) + m_2(\dot{r}_1^2 + r_1^2\dot{\theta}_1^2) + m_3(\dot{r}_3^2 + r_3^2\dot{\theta}_1^2) + m_4(\dot{r}_3^2 + r_3^2\dot{\theta}_1^2)]$$

$$+\frac{m_1m_2}{2r_1}+\frac{m_1m_3}{k}+\frac{m_1m_4}{k}+\frac{m_2m_3}{k}+\frac{m_2m_4}{k}+\frac{m_3m_4}{2r_3},$$

where $k=\sqrt{r_1^2+r_3^2}$. Then the corresponding conjugate variables R_1, R_3, Θ_1 with respect to r_1, r_3, θ_1 are

$$R_1=(m_1+m_2)\dot r_1, R_3=(m_3+m_4)\dot r_3, \Theta_1=(m_1r_1^2+m_2r_1^2+m_3r_3^2+m_4r_3^2)\dot\theta_1.$$

The new Lagrangian is a projection of the 12-dimensional Euler-Lagrange flow on a non-invariant 6-dimensional submanifold (the tangent space of the space of rhombi). The Hamiltonian is

$$\begin{aligned} H1 = & \frac{{R_1}^2}{2(m_1+m_2)}+\frac{{R_3}^2}{2(m_3+m_4)}+\frac{{\Theta_1}^2}{2(m_1\,{r_1}^2+m_2\,{r_1}^2+m_3\,{r_3}^2+m_4\,{r_3}^2)} \\ & -\frac{m_1\,m_2}{2r_1}-\frac{m_1\,m_3}{k}-\frac{m_1\,m_4}{k}-\frac{m_2\,m_3}{k}-\frac{m_2\,m_4}{k}-\frac{m_3\,m_4}{2r_3}. \end{aligned} \tag{3.1}$$

The Hamiltonian will be independent of θ_1 which means that Θ_1 (angular momentum) is a first integral, and θ_1 is an ignorable variable. Setting $\Theta_1=c$ and substituting into equation (3.1) gives

$$\begin{aligned} H = & \frac{{R_1}^2}{2(m_1+m_2)}+\frac{{R_3}^2}{2(m_3+m_4)}+\frac{c^2}{2(m_1\,{r_1}^2+m_2\,{r_1}^2+m_3\,{r_3}^2+m_4\,{r_3}^2)} \\ & -\frac{m_1\,m_2}{2r_1}-\frac{m_1\,m_3}{k}-\frac{m_1\,m_4}{k}-\frac{m_2\,m_3}{k}-\frac{m_2\,m_4}{k}-\frac{m_3\,m_4}{2r_3}. \end{aligned} \tag{3.2}$$

This reduces the system to four dimensions, in the variables (r_1, r_3, R_1, R_3). The equations of motion in these new variables are

$$\dot r_1=\frac{R_1}{m_1+m_2},$$

$$\dot r_3=\frac{R_3}{m_3+m_4},$$

$$\begin{aligned} \dot R_1 = & \frac{c^2\,(m_1\,r_1+m_2\,r_1)}{(m_1\,{r_1}^2+m_2\,{r_1}^2+m_3\,{r_3}^2+m_4\,{r_3}^2)^2}-\frac{m_1\,m_2}{2{r_1}^2}-\frac{m_1\,m_3\,r_1}{({r_1}^2+{r_3}^2)^{3/2}} \\ & -\frac{m_1\,m_4\,r_1}{({r_1}^2+{r_3}^2)^{3/2}}-\frac{m_2\,m_3\,r_1}{({r_1}^2+{r_3}^2)^{3/2}}-\frac{m_2\,m_4\,r_1}{({r_1}^2+{r_3}^2)^{3/2}}, \end{aligned}$$

$$\begin{aligned} \dot R_3 = & \frac{c^2\,(\,m_3\,r_3+\,m_4\,r_3)}{(m_1\,{r_1}^2+m_2\,{r_1}^2+m_3\,{r_3}^2+m_4\,{r_3}^2)^2}-\frac{m_1\,m_3\,r_3}{({r_1}^2+{r_3}^2)^{3/2}}-\frac{m_1\,m_4\,r_3}{({r_1}^2+{r_3}^2)^{3/2}} \\ & -\frac{m_2\,m_3\,r_3}{({r_1}^2+{r_3}^2)^{3/2}}-\frac{m_2\,m_4\,r_3}{({r_1}^2+{r_3}^2)^{3/2}}-\frac{m_3\,m_4}{2{r_3}^2}. \end{aligned}$$

Although the submanifold M is non-invariant, we still have lemma 1.
Lemma 1 If $\gamma_1(t)=(q_1(t),\cdots,q_4(t),p_1(t),\cdots,p_4(t))$ is a critical point of the original system and $\gamma_1(t)=(q_1(t),\cdots,q_4(t),p_1(t),\cdots,p_4(t))$ is also in the constrained space M, then $\gamma_2(t)=(r_1(t),r_3(t),R_1(t),R_3(t))$ is a periodic solution of the constrained Hamiltonian system, where $\gamma_2(t)$ is from $\gamma_1(t)$ by relation M.
Proof. By the construction of the new Lagrangian, it is easy to prove Lemma 1. The following

solution is an example. ♯

Using (2.8) and (2.9), a short calculation shows that the Kepler periodic solution, denoted in general as $\gamma(t)$, is written as

$$\begin{aligned} r_1(t) &= ar(t), & R_1(t) &= (m_1+m_2)aR(t), \\ r_3(t) &= br(t), & R_3(t) &= (m_3+m_4)bR(t), \end{aligned} \tag{3.3}$$

where $m_2 = m_1, m_4 = m_3$ and m_1, m_3 satisfy the equations (2.6), (2.7). Recall that

$$r(t) = \frac{\omega^2}{1+e\cos\theta(t)}, \qquad \dot{r}(t) = R(t), \qquad \dot{\theta} = \frac{\omega}{r^2}, \qquad \theta(0) = 0$$

is the periodic solution to Kepler's problem mentioned earlier. In addition to the size of the rhombus, the two parameters in this solution are eccentricity e and the angular momentum ω of the elliptic orbits. The total angular momentum for the full problem has the value $\Theta_1 = c = (m_1a^2 + m_2a^2 + m_3b^2 + m_4b^2)\omega$.

Lemma 2 If the Kepler solution $\gamma_2(t)$ is linearly unstable in constrained Hamiltonian system, then the Kepler solution $\gamma_1(t)$ is also linearly unstable in original Hamiltonian system, where $\gamma_2(t)$ is constructed from $\gamma_1(t)$ by relation M.

Proof. We'll prove it by contradiction. Assume $\gamma_1(t)$ is linearly stable in original Hamiltonian system. For any initial condition v_{10} with $|v_{10}|$ very small, the solution $v_1(t, v_{10})$ of the original linearized Hamiltonian system along $\gamma_1(t)$ is also very small, where $v_1(0, v_{10}) = v_{10}$. Because the constrained linearized Hamiltonian system along $\gamma_2(t)$ is the projection of 12 dimensional original system, for any initial condition v_{20} with $|v_{20}|$ very small, the solution $v_2(t, v_{20})$ of the constrained linearized Hamiltonian system along $\gamma_2(t)$ is also very small, where $v_2(0, v_{20}) = v_{20}$. So $\gamma_2(t)$ is also stable. ♯

In order to study the linear stability of the Kepler solution $\gamma(t)$ given by (3.3), we will linearize the reduced Hamilton equation along the Kepler solution. Then we compute a fundamental matrix solution $X(t)$ to the linearized equation and calculate its eigenvalues. But we will decouple the system before we compute the fundamental solution.

Linearizing the four-dimensional system about the periodic solution $\gamma(t)$ gives the time-dependent periodic linear Hamiltonian system

$$\dot{X}(t) = J_2 D^2 H(\gamma(t))X = A(t)X$$

where $A(t) = J_2 D^2 H(\gamma(t))$ and J_2 is the canonical matrix

$$J_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

After a good deal of calculation and simplification, we have

$$A(t) = \begin{bmatrix} 0 & 0 & \frac{1}{2m_1} & 0 \\ 0 & 0 & 0 & \frac{1}{2m_3} \\ D_{31} & D_{32} & 0 & 0 \\ D_{32} & D_{42} & 0 & 0 \end{bmatrix}$$

where

$$D_{31} = \frac{-8\omega^2 m_1{}^2 a^2}{r^4\,(m_1\,a^2 + m_3\,b^2)} + \frac{2\omega^2 m_1}{r^4} + \frac{m_1{}^2}{r^3 a^3} + \frac{12 m_1\, m_3\, a^2}{(a^2+b^2)^{5/2}\, r^3} - \frac{4 m_1\, m_3}{(a^2+b^2)^{3/2}\, r^3},$$

$$D_{32} = \frac{-8\omega^2 m3\, bm1\, a}{r^4\,(m1\,a^2 + m3\,b^2)} + \frac{12 m1\, m3\, ba}{(a^2+b^2)^{5/2}\, r^3},$$

$$D_{42} = \frac{-8\omega^2 m3^2 b^2}{r^4\,(m1\,a^2 + m3\,b^2)} + \frac{2\omega^2 m3}{r^4} + \frac{12 m1\, m3\, b^2}{(a^2+b^2)^{5/2}\, r^3} - \frac{4 m1\, m3}{(a^2+b^2)^{3/2}\, r^3} + \frac{m3^2}{r^3 b^3}.$$

Now we follow the Gareth E. Roberts' idea in ([3]) to decouple the linear system. For the convenience of the reader we give a complete proof and develop the method to decouple the linear system.

4 Decoupling the Linear System

A linear, time-dependent periodic Hamiltonian system is one of the form

$$\dot{X}(t) = JD^2H(\gamma(t))X \tag{4.1}$$

where J is the canonical matrix

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

and $D^2H(t+T) = D^2H(t)$.

When such a system results from linearizing about a periodic solution, it can be shown that there are at least two $+1$ characteristic multipliers. One of these is attributable to the periodic orbit and another arise from the existence of an integral, which in this case is the Hamiltonian H. This fact is easily proved via differentiation (see [6] or [10]). Indeed, given a periodic solution $\gamma(t)$ to a Hamiltonian system $\dot{x} = J\nabla H(x)$, plugging in $\gamma(t)$ and differentiating with respect to t yields

$$\ddot{\gamma}(t) = J_2 D^2 H(\gamma(t))\dot{\gamma}(t). \tag{4.2}$$

Thus, $\dot{\gamma}(t)$ is a solution of the associated linear system. Since $\gamma(t)$ is periodic, so is its derivative. If we choose coordinates so that $\gamma(0) = (1, 0, \cdots, 0)$, the first column of the monodromy matrix is $(1, 0, \cdots, 0)$ and $+1$ is an eigenvalue. But relation (4.2) is important for another reason. That is, it suggests a useful change of coordinates. Choosing variables so that the periodic orbit is easily represented helps decouple the system. This follows from a standard result in the theory of Hamiltonian system.

Define the skew-inner product of two vectors $v, w \in \mathbb{C}^{4n}$ as

$$\Omega(v,w) = v^T J w.$$

Note that $J^T = -J = J^{-1}$ so that J is orthogonal and skew-symmetric. A key trait of linear Hamiltonian systems is that the skew-orthogonal complement of an invariant subspace is also invariant.

Lemma 4.1. Suppose W is an invariant subspace of the matrix $JD^2H(t)$, then the skew-orthogonal complement of W, defined as $W^\perp = \{v \in \mathbb{C}^{4b} : \Omega(v,w) = 0 \ \ \forall w \in W\}$, is also an invariant subspace of $JD^2H(t)$.

Proof. Suppose $v \in W^\perp$. Then, for any $w \in W$ we have

$$\Omega(JD^2H(t)v, w) = v^T D^2H(t) J^T J w = v^T D^2 H(t) w = -v^T J \hat{w} = 0,$$

where $\hat{w} = JD^2H(t)w \in W$. Thus $JD^2H(t)v \in W^\perp$. ♯

Given an invariant subspace, Lemma 4.1 shows that a simple linear change of variables will decouple the system. The characteristic multipliers remain the same since the transformation is linear. To apply these ideas to our problem, we need to find an invariant subspace for $A(t) = J_2 D^2 H(\gamma(t))$. As mentioned before, the periodic orbit itself provides an excellent suggestion. We make use of the fact that the Kepler periodic solution $r(t)$ satisfies

$$\ddot{r}(t) = \frac{\omega^2}{r^3} - \frac{1}{r^2}. \tag{4.3}$$

$$\dddot{r}(t) = \left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right)\dot{r}. \tag{4.4}$$

From these we have

$$\gamma = \begin{bmatrix} r_1 \\ r_3 \\ R_1 \\ R_3 \end{bmatrix} = \begin{bmatrix} ar(t) \\ br(t) \\ (m_1+m_2)aR(t) \\ (m_3+m_4)bR(t) \end{bmatrix} = \begin{bmatrix} ar(t) \\ br(t) \\ (m_1+m_2)a\dot{r}(t) \\ (m_3+m_4)b\dot{r}(t) \end{bmatrix},$$

and

$$\dot{\gamma} = \begin{bmatrix} a\dot{r}(t) \\ b\dot{r}(t) \\ (m_1+m_2)a\ddot{r}(t) \\ (m_3+m_4)b\ddot{r}(t) \end{bmatrix} \quad \text{and} \quad \ddot{\gamma} = \begin{bmatrix} a\ddot{r}(t) \\ b\ddot{r}(t) \\ (m_1+m_2)a\left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right)\dot{r}(t) \\ (m_3+m_4)b\left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right)\dot{r}(t) \end{bmatrix},$$

as expressions for the first and second derivatives of the periodic orbit. A short calculation gives

$$A(t)\dot{\gamma}(t) = J_2 D^2 H(\gamma(t))\dot{\gamma}(t) = \ddot{\gamma}(t),$$

$$A(t)\ddot{\gamma}(t) = J_2 D^2 H(\gamma(t))\ddot{\gamma}(t) = \left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right)\dot{\gamma}(t).$$

Then the vectors $W_1 := [a, b, 0, 0], W_2 := [0, 0, 2\, m_1\, a, 2\, m_3\, b]$ will span an invariant subspace W for $J_2 D^2 H(\gamma(t))$. We then apply the equalities $m_2 = m_1, m_4 = m_3$ as well as other relations such as (2.6) to (2.8) as needed. Consider the change of variables determined by

$$\begin{pmatrix} r_1 \\ r_3 \\ R_1 \\ R_3 \end{pmatrix} = \begin{bmatrix} a & 0 & 2\, m_3\, b & 0 \\ b & 0 & -2\, m_1\, a & 0 \\ 0 & 2\, m_1\, a & 0 & b \\ 0 & 2\, m_3\, b & 0 & -a \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

The determinant of the linear transformation matrix is $-4\, {m_1}^2 a^4 - 8\, m_3\, b^2 m_1\, a^2 - 4\, {m_3}^2 b^4$ which is nonzero. The last two columns of the above matrix are chosen to form a basis for the skew-orthogonal complement of W. Consequently, this change of variables will decouple our linear system into two 2×2 system. The new coordinates are

$$x_1 = \frac{m_1\, a r_1}{m_1\, a^2 + m_3\, b^2} + \frac{m_3\, b r_3}{m_1\, a^2 + m_3\, b^2},$$

$$x_2 = \frac{a R_1}{2(m_1\, a^2 + m_3\, b^2)} + \frac{b R_3}{2(m_1\, a^2 + m_3\, b^2)},$$

$$x_3 = \frac{b r_1}{2(m_1\, a^2 + m_3\, b^2)} - \frac{a r_3}{2(m_1\, a^2 + m_3\, b^2)},$$

$$x_4 = \frac{m_3\, b R_1}{m_1\, a^2 + m_3\, b^2} - \frac{m_1\, a R_3}{m_1\, a^2 + m_3\, b^2},$$

and the new differential equation system is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{34} \\ 0 & 0 & D_{43} & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

where $D_{34} = \frac{1}{4 m_1\, m_3}$, and

$$D_{43} = -\frac{8\left(a^4 {m_1}^2 - 2\, a^2 {m_1}^2 b^2 + 6\, a^2 m_3\, b^2 m_1 - 2\, a^2 {m_3}^2 b^2 + {m_3}^2 b^4\right) m_1\, m_3}{r^3\left(m_1\, a^2 + m_3\, b^2\right)\left(a^2 + b^2\right)^{5/2}} +$$

$$\frac{4 m_1\, m_3\, \omega^2}{r^4} + \frac{2 {m_3}^2 {m_1}^2\left(a^5 + b^5\right)}{r^3 b^3\left(m_1\, a^2 + m_3\, b^2\right) a^3}.$$

Note that along the periodic orbit $\gamma(t)$, $x_1 = r, x_2 = R, x_3 = x_4 = 0$. Thus, we expect the first 2×2 system in the x_1, x_2 variables to identify the two $+1$ multipliers, leaving the remaining two variables x_3, x_4 to decide the linear instability of the Kepler solution.

The equations for the x_1 and x_2 variables give a simple 2×2 periodic, linear Hamiltonian system:

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = \left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right) x_1.$$

For the initial condition $x_1(0) = 0, x_2(0) = 1$, making use of (4.3), (4.4), we have as a solution $x_1 = h\dot{r}, x_2 = h\ddot{r}$, where $h = \omega^4/(e(1+e)^2)$ is chosen so that $x_2(0) = 1$. Since this is a periodic solution with the same period as the system itself, the second column of the monodromy matrix for this system will be (0,1). Since we have a Hamiltonian system, the monodromy matrix is symplectic, with determinant one, and must have the form

$$\begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix}.$$

The equations for the remaining two variables give the following 2×2 periodic, linear Hamiltonian system:

$$\dot{x}_3 = D_{34} x_4,$$

$$\dot{x}_4 = D_{43} x_3.$$

We now make a scaling of the variables using the transformation $\hat{x}_3 = \omega^{-3/2} x_3, \hat{x}_4 = \omega^{3/2} x_4$. Since this is a linear transformation, it will not change the characteristic multipliers. Next, we change the independent variable from t to θ. In other words, use

$$\dot{x}_3 = \frac{dx_3}{dt} = \frac{dx_3}{d\theta}\frac{d\theta}{dt} = x_3' \frac{\omega}{r^2}$$

and similar expressions for $\dot{x}_4$ and $\dot{r}$. Dropping the hats off the variables and letting $'$ represent the derivative with respect to θ, our final two-dimensional system for the linearization about the relative periodic orbit $\gamma(t)$ is

$$\begin{bmatrix} x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & f(a,b,e,\theta) \\ g(a,b,e,\theta) & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \tag{4.5}$$

where $f(a,b,e,\theta) = \frac{1}{4 m_3\, m_1\, (1+e\cos(\theta))^2}$ and

$$g(a,b,e,\theta) = 4 m_3\, m_1\, (1+e\cos(\theta))^2 + \frac{2 {m_3}^2 {m_1}^2 \left(a^5+b^5\right)(1+e\cos(\theta))}{b^3 \left(m_1\, a^2 + m_3\, b^2\right) a^3}$$

$$-\frac{8\left(a^4 {m_1}^2 - 2\, a^2 {m_1}^2 b^2 + 6\, a^2 m_3\, b^2 m_1 - 2\, a^2 {m_3}^2 b^2 + {m_3}^2 b^4\right) m_1\, m_3\, (1+e\cos(\theta))}{\left(m_1\, a^2 + m_3\, b^2\right)\left(a^2+b^2\right)^{5/2}}.$$

m_1, m_3 satisfy the equation (2.6), (2.7). The differential equation system (4.5) could be regarded as Hill's equation. One crucial fact about this system is that the masses are determined by the parameters a, b of the rhombus size through (2.6), (2.7). Moreover, since ω is not present, the angular momentum of the elliptic Kepler orbit does not affect the linear stability. So the stability depends on a, b and e the eccentricity of the elliptic Kepler orbit.

5 Linear Stability Analysis

In this section, we analyze the linear stability of the elliptic Kepler orbits in terms of the parameters a, b and e through system (4.5). This entails computing the fundamental matrix solution $X(\theta)$ to the system (4.5) with initial conditions $X(0) = I_2$. The monodromy matrix, subsequently denoted by M, is then $X(2\pi)$ and the eigenvalues of this matrix are the characteristic multipliers. The map $x \mapsto Mx$ can be interpreted as the linearization of the Poincare map on our reduced space. Since system (4.5) has been derived from a Hamiltonian system with coordinate changes, which do not alter the multipliers, the characteristic polynomial of M will be reciprocal ([6]). In other words, λ is an eigenvalue of M if and only if $1/\lambda$ is also an eigenvalue. Thus, to have linear stability, we require that the eigenvalues reside on the unit circle.

Theorem 4.1. For all possible values of the three parameters a, b and e, the elliptic periodic orbits are linearly unstable.

Proof. The characteristic polynomial of M has the form

$$p(\lambda) = det(\lambda I - M) = \lambda^2 + q\lambda + 1, \tag{5.1}$$

where $q = -\mathrm{tr}(M)$ and $det(M) = 1$.

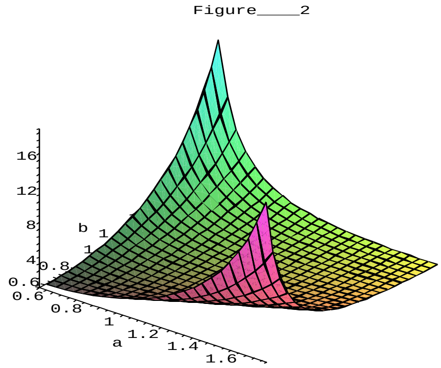
Given that the multipliers are on the unit circle, there are two ways in which stability can be lost:
1. period-doubling bifurcation (two -1 eigenvalues), occurring when $q = 2$,
2. two $+1$ eigenvalues, occurring when $q = -2$.
In these two cases, a pair of eigenvalues meets and then breaks off onto the real line yielding an eigenvalue with modulus greater than one and an eigenvalue with modulus less than one. As $q < -2$, we have a pair of reciprocal real positive eigenvalues i.e. λ and $1/\lambda$ and one of them greater than positive one. As q goes to -2 from the left, the pair of reciprocal real eigenvalues approach each other at 1. The characteristic multipliers move continuously on the unit circle from $(1, 0)$ to $(-1, 0)$ as q increases from -2 to 2. As $q > 2$, we have a pair of reciprocal real negative eigenvalues i.e. λ and $1/\lambda$ and one of them less than negative one. As q goes to 2 from the right, the pair of reciprocal real eigenvalues approach each other at -1.

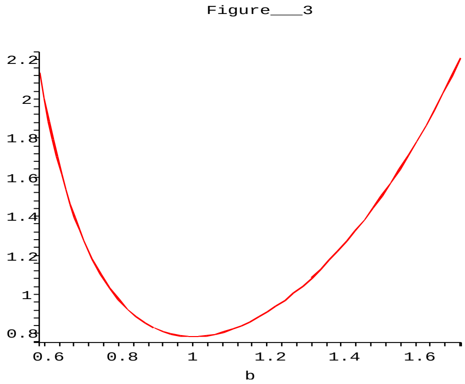
To have linear stability, we need the roots of $p(\lambda)$ on the unit circle, i.e. $q^2 \leq 4$. We begin by analyzing the behavior of the multipliers for the circular case $e = 0$. In this case, the matrix in system (4.5) is constant and therefore, the multipliers can be explicitly computed. The characteristic polynomial of the coefficient matrix of (4.5) is give by

$$\rho^2 - g(a, b, 0, \theta)f(a, b, 0, \theta),$$

where g, f only depend on the parameters a, b as $e = 0$. If ρ is a root of this polynomial, then $e^{2\pi\rho}$ is a characteristic multiplier. In order to have stability, the root ρ must be purely imaginary. It requires that $h(a, b) = g(a, b, 0, \theta)f(a, b, 0, \theta)$ is negative.

Using Maple, we have $h(a, b) > 0$ in all possible a, b values as shown in figure 2.





Because $h(a,b) = h(b,a)$, $a/\sqrt{3} < b < a\sqrt{3}$, we also can determine the sign of $h(a,b)$ by fixing $a = 1$ as shown in figure 3. The minimum of $h(1,b)$ is 0.7836116249 while b=1. This proves that the Kepler orbit for the circular case is unstable.

We now investigate the linear stability of the periodic orbits which are truly elliptic ($e \neq 0$). Since the stability is determined by $q = -tr(M)$, the trace of monodromy matrix, we do not need to calculate the eigenvalues explicitly. If $-2 \leq tr(M) \leq 2$, then the periodic solution is linear stable. If $|tr(M)| > 2$, it is unstable. Writing a Maple program to calculate the trace of monodromy matrix M, we find all the traces are significantly greater than 2. For example, when a=1, b=1, e varying from 0 to 1 by 1/20, the corresponding traces are

260.3442854,	261.0797976,	263.3111371,	267.1146285,	272.6248514,
280.0463890,	289.6736785,	301.9214558,	317.3731802,	336.8583635,
361.5791520,	393.3279113,	434.8785301,	490.7344415,	568.6778938,
683.3152296,	865.3720827,	1191.302564,	1911.784759,	4478.934516

When $a = 1, b = 1/\sqrt{3} + 1/100$, e varying from 0 to 1 by 1/20, the corresponding traces are

9684.871521,	9728.151790,	9859.880060,	10085.91046,	10416.67040,
10868.29203,	11464.49530,	12239.64874,	13243.73696,	14550.63459,
16272.39134,	18585.00314,	21777.62330,	26353.37645,	33254.88426,
44429.81158,	64486.29034,	106749.3980,	225015.7447,	861056.0827

Thus, the periodic solution of the rhombus four body problem is unstable.

Remark: We apply this method to study the linear stability of Kepler orbits for the regular polygon N-body problem with one body in the center of the polygon. Because for N equal masses m at the vertices of regular polygon and arbitrary mass μ at the center, the N+1 bodies form a central configuration. We use the symmetry to constraint our solution on regular polygon and get a similar reduced Hamilton system. The corresponding Kepler orbit is unstable if $\mu \leq m$. Although we can not infer that the original Kepler orbit is stable if $\mu > m$, the possible stable Kepler orbit occurs only if $\mu > m$. R. Moeckel ([7]) studied the linear stability of Kepler orbits with a dominant mass. The regular one is one of his particular cases.

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