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Uniqueness and blow-up rate of large solutions for elliptic equation $-\Delta u = \lambda u - b(x)h(u)$

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ABSTRACT

In this paper, we establish the blow-up rate of the large positive solution of the singular boundary value problem

$$\begin{cases} -\Delta u = \lambda u - b(x)h(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$. The weight function $b(x)$ is a non-negative continuous function in the domain. $h(u)$ is locally Lipschitz continuous and $h(u)/u$ is increasing on $(0,\infty)$ and $h(u) \sim Hu^p$ for sufficiently large u with $H > 0$ and $p > 1$. Naturally, the blow-up rate of the problem equals its blow-up rate for the very special, but important, case when $h(u) = Hu^p$. We distinguish two cases: (I) Ω is a ball domain and b is a radially symmetric function on the domain in Theorem 1.1; (II) Ω is a smooth bounded domain and b satisfies some local condition on each boundary normal section assumed in Theorem 1.2. The blow-up rate is explicitly determined by functions b and h. In case (I), the singular boundary value problem has a unique solution u satisfying

$$\lim_{d(x)\to 0} \frac{u(x)}{KH^{-\beta}(b^*(\|x-x_0\|))^{-\beta}} = 1,$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$, $b^*(r)$ and K are defined by

$$b^*(r) = \int_r^R \int_s^R b(t)\,dt\,ds,$$

$$K = \left[\beta\left((\beta+1)C_0 - 1\right)\right]^{\frac{1}{p-1}}, \quad \beta := \frac{1}{p-1}.$$

In case (II), the blow-up rates of the solutions to the boundary value problem are established and the uniqueness is proved.

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1. Introduction and main results

This paper continues the studies of the semilinear elliptic problems with singular boundary value condition in the following form:

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. The boundary condition in (1.1) is understood as $u(x) \to +\infty$ when $d(x) := \operatorname{dist}(x, \partial\Omega) \to 0^+$. The non-negative solutions of (1.1) are called *large* (or *blow-up*) *solutions*.

Singular boundary value problem (1.1) arises naturally from a number of different areas and has a long history. Considerable amounts of study have been attracted by such problems (see, e.g., [2–5,11, 14,18,26–28] and references therein). In 1916 Bieberbach [3] studied the large solutions for the particular case $f(x,u) = -\exp(u)$ and $N = 2$. He showed that there exists a unique solution of (1.1) such that $u(x) - \log(d(x)^{-2})$ is bounded as $x \to \partial\Omega$. Problems of this type arise in Riemannian geometry: if a Riemannian metric of the form $|ds|^2 = \exp(2u(x))|dx|^2$ has constant Gaussian curvature $-c^2$, then $-\Delta u = -c^2 \exp(2u)$. Motivated by a problem in mathematical physics, Rademacher [31] continued the study of Bieberbach on smooth bounded domains in $\mathbb{R}^3$. In 1990s, Bandle and Essèn [2] and Lazer and McKenna [18] extended the results of Bieberbach and Rademacher for bounded domains in $\mathbb{R}^N$ satisfying a uniformal external sphere condition and for nonlinearities $f(x,u) = b(x)\exp(u)$, where b is continuous and strictly positive on $\bar\Omega$. It is shown that the problem exhibits a unique solution in a smooth domain together with an estimate of the form $u = \log d^{-2} + o(d)$ in [18] (where $b(x) \geqslant b_0 > 0$ as $d \to 0$) and in [2] (where $b \equiv 1$).

For $f(x,u) = g(u)$, Lazer and McKenna [19] obtained an asymptotic result for solutions of the above problem under some assumptions on g. Let Ω be a bounded domain in $\mathbb{R}^N$, $N > 1$, which satisfies a uniform internal sphere condition and a uniform external sphere condition. Let g be a C^1-function which is either defined and positive on $(-\infty, \infty)$ or is defined on a ray $[a, \infty)$ with $g(a) = 0$ and $g(s) > 0$ for $s > a$. They further assume that $g'(s) \geqslant 0$ for s in the domain of g, and that there exists a_1 such that $g'(s)$ is non-decreasing for $s \geqslant a_1$. They proved that if $\lim_{s\to\infty} \frac{g'(s)}{\sqrt{G(s)}} = \infty$ where $G'(s) = g(s)$, $G(s) > 0$, then the problem has a unique solution $u(x)$ and moreover,

$$u(x) - Z\big(d(x)\big) \to 0 \quad \text{as } d(x) \to 0,$$

where Z is a solution on an interval $(0,b)$, $b > 0$, of the equation $Z''(r) = g(Z(r))$ and such that $Z(r) \to \infty$ as $r \to 0^+$.

We are interested in large solutions of (1.1) when $f(x,u) = \lambda(x)u - b(x)h(u)$, i.e.

$$-\Delta u = \lambda(x)u - b(x)h(u) \quad \text{in } \Omega \tag{1.2}$$

subject to the singular boundary condition

$$u = +\infty \quad \text{on } \partial\Omega, \tag{1.3}$$

where $h \in C[0,\infty)$ is locally Lipschitz, $\lambda \in L^\infty$ and b is a continuous function in $\bar\Omega$ with positive value in Ω and with non-negative value on $\partial\Omega$.

If $b > 0$ in $\bar\Omega$ and $h(u) = u^p$ $(p > 1)$, then (1.2) is known as the logistic equation. This equation is a basic population model (see, e.g., [11,21,24] and the references therein). Generally speaking, the existence problem is relatively well understood but the uniqueness problem is only partially understood. Assume that $h \geqslant 0$ is non-negative locally Lipschitz continuous and $h(u)/u$ is increasing on $(0,\infty)$. Then, necessarily $h(0) = 0$, and by the strong maximum principle, any non-negative classical solution of (1.2) is positive in Ω unless it is identically zero. Consequently, any blow-up solution of (1.2) is positive. Moreover, it is well known that in this situation, the Keller–Osserman condition

$$\int_1^\infty \mathcal{H}(t)^{-1/2}\,dt < \infty, \quad \text{where } \mathcal{H}(t) = \int_0^t h(s)\,ds$$

is necessary and sufficient for the existence of blow-up solutions of (1.2) (see Keller [17], Osserman [25], J. López-Gómez [21] and Cirstea and Du [7] for a detailed discussion). In particular, the existence of a blow-up solution was established by M. Delgado and J. López-Gómez [12] for a more general class of nonlinearities.

Regarding the uniqueness problem with $h(u) = u^p$ $(p > 1)$, it has received much attention and many papers (see, e.g., [11–31]) have been written about it on a bounded domain or $\mathbb{R}^N$. Under the assumption

$$\lim_{d(x,\partial\Omega)\to 0} \frac{b(x)}{d(x,\partial\Omega)^\gamma} = \zeta$$

with $\gamma > 0$ and $\zeta > 0$, an explicit expression for the blow-up rates of (1.2) has been recently proved in [11] and [15] as $u = (\alpha(\alpha+1)/\zeta)^{1/(p-1)} d^{-\alpha}(1+o(d))$, $\alpha = (\gamma+2)/(p-1)$. Further improvements of this result can be found in [22,24,29,30] and the references therein. A localization method used in [22,30] shows that (1.2) (with $h(u) = u^p$, $p > 1$) has at most one blow-up solution for the case when γ and ζ vary along $\partial\Omega$.

In a different direction, by making use of Karamata's theory for regularly varying functions (which is well known in statistics), it was proved in [7,9] and [10] that uniqueness holds for a class of functions $h(u)$ (including u^p as a special case) and it provided the blow-up rate of large solution of (1.2) if $b(x)$ satisfying some decay conditions. Following [7], a positive measurable function R defined on $[A,\infty)$, for some $A > 0$, is called *regularly varying* with index $q \in \mathbb{R}$, written $R \in RV_q$, provided that

$$\lim_{u\to\infty} \frac{R(\alpha u)}{R(u)} = \alpha^q, \quad \text{for all } \alpha > 0.$$

We denote by $\mathcal{K}$ the set of all positive increasing C^1-functions defined on $[0,R]$ such that

$$l_0 := \lim_{t\to 0^+} \frac{\int_0^r k(s)\,ds}{k(t)} = 0, \qquad l_1 := \lim_{t\to 0^+} \frac{d}{dt}\left(\frac{\int_0^t k(s)\,ds}{k(t)}\right) \in [0,1].$$

If $h \in RV_{\rho+1}$ for some $\rho > 0$ and assume that there exist $k \in \mathcal{K}$ and a positive continuous function $C(x)$ on $\partial\Omega$ such that

$$\lim_{x\to y} \frac{b(x)}{k^2(d(x))} = c(y), \quad \text{uniformly for } y \in \partial\Omega,$$

then the problem (1.2) has a unique positive large solution $u(x)$ and moreover, the blow-up rate of the large solution is given by

$$\lim_{x\to y} \frac{u(x)}{\Psi(d(x))} = \left(\frac{2+\rho l_1}{2c(y)}\right)^{1/\rho}, \quad \text{uniformly for } y \in \partial\Omega,$$

where Ψ is uniquely determined by

$$\int_{\Psi(t)}^\infty \frac{dy}{\sqrt{yh(y)}} = \int_0^t k(s)\,ds, \quad \text{for all } t \in (0,\tau), \text{ for } \tau > 0 \text{ small enough.}$$

The main purpose of this paper is to provide an analysis for a wide range of functions $b(x)$ and $h(u)$. Here we develop the research line in [24,30] to extend the general case $h(u) \sim Hu^p$, $p > 1$, for sufficiently large u. Our approach for the uniqueness is different from that of Lazer and McKenna [19] and Cirstea and Du [7], being based on Karamata's theory and Safonov's iteration technique. Now we turn to state the main results more precisely.

Consider the singular boundary value problem:

$$\begin{cases} -\Delta u = \lambda u - b(x)h(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\lambda \in \mathbb{R}$, Ω is a bounded smooth domain in $\mathbb{R}^N$, and the weight function $b(x) > 0$ in Ω. The nonlinear function $h(u)$ satisfies:

($\mathcal{A}$) $h \geqslant 0$ is locally Lipschitz continuous on $[0,\infty)$ and $h(u)/u$ is increasing on $(0,\infty)$; and, for some $p > 1$,

$$H := \lim_{u\to\infty} \frac{h(u)}{u^p} > 0. \tag{1.5}$$

Note that (1.5) implies that h satisfies Keller–Osserman condition. Indeed, according to (1.5), there exists large $\zeta > 0$ such that

$$h(u) \geqslant \frac{H}{2}u^p, \quad u \geqslant \zeta.$$

Therefore

$$\begin{aligned} \int_1^\infty \left[\mathcal{H}(t)\right]^{-1/2} dt &= \int_1^\infty \left[\int_0^t h(u)\,du\right]^{-1/2} dt \\ &= \int_1^\zeta \left[\int_0^t h(u)\,du\right]^{-1/2} dt + \int_\zeta^\infty \left[\int_0^t h(u)\,du\right]^{-1/2} dt \\ &= \int_1^\zeta \left[\int_0^t h(u)\,du\right]^{-1/2} dt + \int_\zeta^\infty \left[\int_0^\zeta h(u)\,du + \int_\zeta^t h(u)\,du\right]^{-1/2} dt \\ &\leqslant M^{-1/2}(\zeta - 1) + \int_\zeta^\infty \left[\frac{Ht^{p+1}}{2p} - \frac{H\zeta^{p+1}}{2p}\right]^{-1/2} dt < \infty \end{aligned}$$

because $p > 1$ and $0 < M = \int_0^1 h(u)\,du < \int_0^t h(u)\,du$ for $t \geqslant 1$. So the existence of large solutions of (1.5) is guaranteed.

The next theorems collect the main findings of this paper.

Theorem 1.1. *Suppose that $\Omega = B_R(x_0)$ is a ball in $\mathbb{R}^N$ of radius R centered at x_0 and $h(u)$ satisfies ($\mathcal{A}$). $\lambda \in \mathbb{R}$ and $b(x) = b(\|x - x_0\|)$ is a radially symmetric function on the ball. $b \in C([0,R]; [0,\infty))$ satisfies $b > 0$ in $[0,R)$, $\lim_{r\to R} \frac{B(r)}{b(r)} = 0$, and*

$$C_0 := \lim_{r\to 0} \frac{(B(r))^2}{b^*(r)b(r)} \geqslant 1, \tag{1.6}$$

where $B(r)=\int_r^R b(s)\,ds$ *and* $b^*(r)=\int_r^R B(s)\,ds$. *Then the problem* (1.4) *has a unique solution* u *satisfying*

$$\lim_{d(x)\to 0}\frac{u(x)}{KH^{-\beta}(b^*(\|x-x_0\|))^{-\beta}}=1,$$

where $d(x)=\operatorname{dist}(x,\partial\Omega)$ *and* K *is a constant defined by*

$$K=\left[\beta\big((\beta+1)C_0-1\big)\right]^{\frac{1}{p-1}},\quad \beta:=\frac{1}{p-1}.$$

Remark 1.1. Recall that function $h(u)$ satisfies the condition $(\mathcal{A})$: $h\geqslant 0$ is locally Lipschitz continuous on $[0,\infty)$ and $h(u)/u$ is increasing on $(0,\infty)$; and, for some $p>1$, $\lim_{u\to\infty}h(u)/u^p=H$. It is easy to prove the following properties of h. (1) $h(0)=0$. (2) $h(\varsigma u)/\varsigma\leqslant h(u)$ if $0<\varsigma\leqslant 1$ and $h(\varsigma u)/\varsigma\geqslant h(u)$ if $\varsigma\geqslant 1$. (3) For any λ, $b>0$, there exists a large N such that $\lambda u-bh(u)\geqslant\lambda(u+M)-bh(u+M)$ when $M\geqslant N$.

Remark 1.2. S. Cano-Casanova and J. López-Gómez [6] published their very recent uniqueness results, which dealt with the same problem under different assumptions on $b(x)$ and $h(u)$. If Ω is a ball or an annulus, $b(x)=f(d(x))$ for some $f\in\mathcal{C}[0,\infty)$ is positive and non-decreasing, $h(u)=V(u)u$ and $V\in\mathcal{C}[0,\infty)\cap\mathcal{C}^2(0,\infty)$ satisfies $V(0)$, $V'(u)>0$, $V''(u)>0$ and $V(u)\sim Hu^{p-1}$ as $u\to\infty$, then in [6] they proved that (1.4) possesses a unique positive large solution and the exact blow-up rate of the large solution is estimated. If the results in [6] and [20] are combined, they would only require the monotonicity of f and the concavity of $V(u)$ and would not require that $V(u)\sim Hu^{p-1}$ as $u\to\infty$.

Remark 1.3. We note that this result extends Theorem 1 in [29], where $h(u)=u^p$, $p>1$. We also note that $b(x)$ has a wider range in the above theorem than in [9], where b is a positive increasing C^1-function.

The following example illustrates that b can be an oscillating function. Let $b(r)=r(2+3r\cos r^{-1})$, then $b'(r)=2+6r\cos(r^{-1})+3\sin(r^{-1})$. As $r\to 0^+$, $b'(r)$ oscillates between -1 and 1. On the other hand,

$$0\leqslant\frac{\int_0^r s(2+3s\cos\frac{1}{s})\,ds}{r(2+3r\cos\frac{1}{r})}\leqslant\frac{\int_0^r s(2+3s)\,ds}{r(2+3r\cos\frac{1}{r})}\leqslant\frac{r(r(2+3r))}{r(2+3r\cos\frac{1}{r})}\to 0\quad\text{as } r\to 0^+.$$

Similarly we can check that $b(r)$ satisfies the conditions above and $C_0=\frac{3}{2}$.

The second example illustrates that b can decay to zero faster than any power function as x approaches the boundary. Let $b(r)=\exp(-\frac{1}{(R-r)^2})$, then there does not exist a constant γ such that $\lim_{d(x)\to 0}\frac{b(x)}{d(x)^\gamma}$ exists and is greater than zero. $\lim_{r\to R}\frac{B(r)}{b(r)}=\lim_{r\to R}2(R-r)^3=0$. Similarly we can check that $b(r)$ satisfies the conditions above and $C_0=1$.

Thus Theorem 1.1 works for the oscillating and fast decay weight function $b(r)$ which is not previously covered in [8,9] and [23].

As an immediate consequence of Theorem 1.1, combining a translation together with a reflection about

$$r_0:=\frac{R_1+R_2}{2},$$

it readily follows that the corresponding result can be proved in each of the annuli

$$A_{R_1,R_2}(x_0):=\left\{x\in\mathbb{R}^N:\ 0<R_1<\|x-x_0\|<R_2\right\}.$$

Corollary 1.1. *Consider the problem*

$$\begin{cases} -\Delta u = \lambda u - b(r)h(u) & \text{in } A_{R_1,R_2}(x_0), \\ u = \infty & \text{on } \partial A_{R_1,R_2}(x_0), \end{cases} \tag{1.7}$$

where $\lambda \in \mathbb{R}$, $r = \operatorname{dist}(x, \partial A_{R_1,R_2}(x_0))$, *and* $R = \frac{R_2 - R_1}{2}$. $b \in C([0, R]; [0, \infty))$ *satisfying* $b > 0$ *in* $(0, R]$. *Let* $B(r) = \int_0^r b(s)\,ds$ *and* $b^*(r) = \int_0^r B(s)\,ds$ *and assume* $C_0 := \lim_{r\to 0} \frac{(B(r))^2}{b^*(r)b(r)} \geqslant 1$. *Then there exists a unique solution* u *satisfying*

$$\lim_{x\to\partial A_{R_1,R_2}(x_0)} \frac{u(x)}{KH^{-\beta}(b^*(r))^{-\beta}} = 1,$$

where K *and* β *are defined in Theorem* 1.1.

Theorem 1.2. *Suppose that* Ω *is a bounded smooth domain in* $\mathbb{R}^N$, *and* $h(u)$ *satisfies* $(\mathcal{A})$. *Further, suppose* $b(x) > 0$ *in* Ω *has the following properties. For each* $x \in \partial\Omega$, *we define the boundary normal sections* $b_x(r)$ *as*

$$b_x(r) = b(x - r\mathbf{n}_x), \quad r \geqslant 0,\ r \sim 0. \tag{1.8}$$

For any $x_0 \in \partial\Omega$, *suppose there exists* $\tau > 0$, *such that* $b(x) \in C^1(\bar{B}_\tau(x_0) \cap \bar{\Omega})$ *and*

$$b_{x_0}(r) \in C^1(0, \tau), \qquad b'_{x_0}(r) > 0, \quad \text{for each } t \in (0, \tau), \tag{1.9}$$

and

$$\lim_{x\in\partial\Omega,\ x\to x_0,\ r\to 0^+} \frac{b_x(r)}{b_{x_0}(r)} = 1. \tag{1.10}$$

Let $B_{x_0}(r) = \int_0^r b_{x_0}(s)\,ds$, $b^*_{x_0}(r) = \int_0^r B_{x_0}(s)\,ds$. *We assume that* $\lim_{r\to 0} \frac{B_{x_0}(r)}{b_{x_0}(r)} = 0$ *and*

$$C_{x_0} := \lim_{r\to 0} \frac{(B_{x_0}(r))^2}{b^*_{x_0}(r)b_{x_0}(r)} \geqslant 1.$$

Then we have the following results:

(1) *For each* $x_0 \in \partial\Omega$, *any positive solution* u *of* (1.4) *satisfies*

$$\lim_{r\to 0} \frac{u(x_0 - r\mathbf{n}_{x_0})}{KH^{-\beta}(b^*_{x_0}(r))^{-\beta}} = 1, \tag{1.11}$$

where K *is a constant defined by*

$$K = \left[\beta\left((\beta+1)C_{x_0} - 1\right)\right]^{\frac{1}{p-1}}, \quad \beta := \frac{1}{p-1}.$$

(2) *If the conditions* (1.9) *and* (1.10) *are uniformly satisfied on* $\partial\Omega$, *then for any positive solution* u *of* (1.4),

$$\lim_{r\to 0} \frac{u(x_0 - r\mathbf{n}_{x_0})}{KH^{-\beta}(b^*_{x_0}(r))^{-\beta}} = 1, \quad \text{uniformly for } x_0 \in \partial\Omega.$$

(3) *The problem* (1.4) *possesses a unique positive solution* $u(x)$ *in* Ω.

Remark 1.4. We want to point out that the function $b(r)$ in Theorem 1.1 is defined for $r = \|x - x_0\|$, where x_0 is the center of the ball $B_R(x_0)$. The function $b(r)$ in Corollary 1.1 and $b_{x_0}(r)$ in Theorem 1.2 are defined differently, i.e. $r = \operatorname{dist}(x, \partial\Omega)$. The integrals $B(r)$ and $b^*(r)$ are also different.

Remark 1.5. We use the localization method of [22] and Theorem 1.1 under the radially symmetric case to prove Theorem 1.2. Actually, this result gives the first step to generalize it up to cover the general case when $(\mathcal{A})$ is imposed instead of $h(u) = u^p$. It improves the main results in [22,24] and [30].

Remark 1.6. Our results in this paper can be extended easily to more general situations. For example, with a little more effort, our arguments can be carried out when constant λ is replaced by a continuous function $\lambda(x)$ on $\bar{\Omega}$. Moreover, as in [8] and [22], instead of a complete boundary blow-up problem, we can consider the following case:

$$\begin{cases} -\Delta u = \lambda(x)u - b(x)h(u) & \text{in } \Omega, \\ u = \infty & \text{on } \Gamma_\infty, \\ \mathcal{B}u = 0 & \text{on } \Gamma_{\mathcal{B}}, \end{cases} \tag{1.12}$$

where Γ_∞ is a non-empty open and closed subset of $\partial\Omega$, $\Gamma_{\mathcal{B}} := \partial\Omega \setminus \Gamma_\infty$, and $\mathcal{B}$ denotes either the Dirichlet boundary operator $\mathcal{D}u = u$ or the Neumann/Robin boundary operator $\mathcal{R}u = \partial u/\partial\nu + \beta(x)u$. Here ν is the outward unit normal to $\partial\Omega$ and $\beta \geqslant 0$ is in $C^{1,\mu}(\partial\Omega)$, for $0 < \mu < 1$. Furthermore, we can allow $b(x)$ to vanish in certain subsets of Ω. Problems of the form (1.12) arise naturally in certain population models such as the logistic equation (see, e.g., [11,14] and [16]).

The rest of the paper is organized as follows. In Section 2, we summarize some comparison results which are needed throughout this paper, and carry out the proof of Theorem 1.1. In Section 3, we make use of Theorem 1.1 and localization method to prove Theorem 1.2.

2. Proof of Theorem 1.1

In this section we first summarize some important comparison results from the papers [1,11,13,15,29] and [30], that we are going to use throughout the paper.

Consider the problem

$$\begin{cases} -\Delta u = \lambda u - b(x)h(u) & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded smooth domain, $\phi \in C(\partial\Omega)$, h satisfies $(\mathcal{A})$ and $b \in C(\Omega, \mathbb{R}^+)$.

Lemma 2.1. *Let $\underline{u}, \bar{u} \in C^2(\bar{\Omega})$ be both positive in $\bar{\Omega}$ such that*

$$-\Delta \underline{u} \leqslant \lambda \underline{u} - b(x)h(\underline{u}) \quad \text{in } \Omega,$$

$$-\Delta \bar{u} \geqslant \lambda \bar{u} - b(x)h(\bar{u}) \quad \text{in } \Omega.$$

If $\underline{u} \leqslant \phi \leqslant \bar{u}$ on $\partial\Omega$, then $\underline{u}(x) \leqslant \bar{u}(x)$ on $\bar{\Omega}$.

Lemma 2.2. *Let $\phi = \infty$ in (2.1). Let $\underline{u}, \bar{u} \in C^2(\bar{\Omega})$ be both positive in $\bar{\Omega}$ such that $\underline{u} = +\infty$ on $\partial\Omega$ and $\bar{u} = +\infty$ on $\partial\Omega$. If*

$$-\Delta \underline{u} \leqslant \lambda \underline{u} - b(x)h(\underline{u}) \quad \text{in } \Omega,$$

$$-\Delta \bar{u} \geqslant \lambda \bar{u} - b(x)h(\bar{u}) \quad \text{in } \Omega,$$

and $\underline{u} \leqslant \bar{u}$ *in* Ω, *then there exists at least one classical solution* u *such that* $\underline{u} \leqslant u \leqslant \bar{u}$ *and* $u(x) \to \infty$ *as* $x \to \partial\Omega$.

Definition. A function $\underline{u} \in C^2(\Omega)$ is a (classical) subsolution to problem (1.4) if $\underline{u} = +\infty$ on $\partial\Omega$ and

$$-\Delta\underline{u} \leqslant \lambda\underline{u} - b(x)h(\underline{u}) \quad \text{in } \Omega.$$

Similarly, $\bar{u}$ is a (classical) supersolution to problem (1.4) if $\bar{u} = +\infty$ on $\partial\Omega$ and

$$-\Delta\bar{u} \geqslant \lambda\bar{u} - b(x)h(\bar{u}) \quad \text{in } \Omega.$$

Lemma 2.3. *Let* $b(r) : [0, R] \mapsto [0, \infty)$ *be continuous function such that* $b(r) > 0$ *for* $r \in [0, R)$. *Define*

$$B(r) = \int_r^R b(s)\,ds, \qquad b^*(r) = \int_r^R B(s)\,ds.$$

If $g(r) = \frac{B(r)}{b(r)}$ *is differentiable in* $[0, R]$ *and* $\lim_{r\to R} g(r) = 0$, $\lim_{r\to R} g'(r) \leqslant 0$, *then we have*

(1) $$\frac{B(r)}{b(r)} \to 0 \quad \text{as } r \to R,$$

(2) $$\frac{b^*(r)}{B(r)} \to 0 \quad \text{as } r \to R,$$

(3) $$\frac{(B(r))^2}{b^*(r)b(r)} \to C_0 \geqslant 1 \quad \text{as } r \to R.$$

Remark 2.1. Lemma 2.1 is a very classical result that should be attributed to [1]. Lemma 2.3 is from [29]. The assumption on $g(r)$ is sufficient but not necessary for our main theorems. Indeed, we only need the limits of (1), (2) and (3) in Lemma 2.3 and the continuity of potential function $b(r)$ to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We first consider the corresponding singular problem (1.4) in one dimension

$$\begin{cases} -\psi'' - \dfrac{N-1}{r}\psi' = \lambda\psi - b(r)h(\psi) \quad \text{in } (0, R), \\ \lim\limits_{r\to R} \psi(r) = \infty, \\ \psi'(0) = 0. \end{cases} \tag{2.2}$$

We claim that for each $\epsilon > 0$, the problem (2.2) possesses a positive large solution ψ_ϵ such that

$$1 - \epsilon \leqslant \liminf_{r\to R} \frac{\psi_\epsilon(r)}{KH^{-\beta}(b^*(r))^{-\beta}} \leqslant \limsup_{r\to R} \frac{\psi_\epsilon(r)}{KH^{-\beta}(b^*(r))^{-\beta}} \leqslant 1 + \epsilon, \tag{2.3}$$

where we have denoted

$$\beta := \frac{1}{p-1}, \qquad b^*(r) := \int_r^R \int_s^R b(t)\,dt\,ds, \qquad K = \big[\beta\big((\beta+1)C_0 - 1\big)\big]^{\frac{1}{p-1}}, \tag{2.4}$$

and C_0 is given by (1.6) and H is constant as in (1.5).

Therefore, for each $x_0 \in \mathbb{R}^N$, the function

$$u_\epsilon(x) := \psi_\epsilon(r); \quad r := \|x - x_0\|$$

provides us with a radially symmetric positive large solution of (1.4) with the assumptions in Theorem 1.1 and the solution satisfies

$$1 - \epsilon \leqslant \liminf_{d(x)\to 0} \frac{u_\epsilon(x)}{KH^{-\beta}(b^*(\|x - x_0\|))^{-\beta}} \leqslant \limsup_{d(x)\to 0} \frac{u_\epsilon(x)}{KH^{-\beta}(b^*(\|x - x_0\|))^{-\beta}} \leqslant 1 + \epsilon. \tag{2.5}$$

To prove the claim, we first construct a supersolution of (2.2) for each $\epsilon > 0$. Let

$$\overline{\psi}_\epsilon(r) = A + B_+\left(\frac{r}{R}\right)^2 \big(b^*(r)\big)^{-\beta}, \tag{2.6}$$

where $A > 0$ and $B_+ > 0$ have to be determined later. Then

$$\overline{\psi}'_\epsilon(r) = 2B_+\frac{r}{R^2}\big(b^*(r)\big)^{-\beta} - \beta B_+\left(\frac{r}{R}\right)^2\big(b^*(r)\big)^{-\beta-1}\big(b^*(r)\big)',$$

$$\begin{aligned}\overline{\psi}''_\epsilon(r) = {} & 2B_+\frac{1}{R^2}\big(b^*(r)\big)^{-\beta} - 4\beta B_+\frac{r}{R^2}\big(b^*(r)\big)^{-\beta-1}\big(b^*(r)\big)' \\ & + \beta(\beta+1)B_+\left(\frac{r}{R}\right)^2\big(b^*(r)\big)^{-\beta-2}\big[\big(b^*(r)\big)'\big]^2 - \beta B_+\left(\frac{r}{R}\right)^2\big(b^*(r)\big)^{-\beta-1}\big(b^*(r)\big)''.\end{aligned}$$

$\overline{\psi}_\epsilon(r) \to \infty$ as $r \to R$ because $b^*(r) \to 0$ as $r \to R$ and $\beta > 0$. Also $\overline{\psi}'_\epsilon(r) \to 0$ as $r \to 0$. Then $\overline{\psi}_\epsilon(r)$ is a supersolution if

$$-\overline{\psi}''_\epsilon(r) - \frac{N-1}{r}\overline{\psi}'_\epsilon(r) \geqslant \lambda\overline{\psi}_\epsilon(r) - b(r)h\big(\overline{\psi}_\epsilon(r)\big). \tag{2.7}$$

By the assumption $(\mathcal{A})$ on h, it is easy to see that for the same $\epsilon > 0$,

$$(1-\epsilon)H\overline{\psi}^p_\epsilon(r) \leqslant h\big(\overline{\psi}_\epsilon(r)\big) \leqslant (1+\epsilon)H\overline{\psi}^p_\epsilon(r) \tag{2.8}$$

for all $r \in [0, R)$ by choosing A sufficiently large, say $A \geqslant A_0$. The inequality (2.7) holds if

$$-\overline{\psi}''_\epsilon(r) - \frac{N-1}{r}\overline{\psi}'_\epsilon(r) \geqslant \lambda\overline{\psi}_\epsilon(r) - b(r)(1-\epsilon)H\overline{\psi}^p_\epsilon(r). \tag{2.9}$$

That is,

$$\begin{aligned}& -2N\frac{B_+}{R^2}\big(b^*(r)\big)^{-\beta} + [N+3]\beta B_+\frac{r}{R^2}\big(b^*(r)\big)^{-\beta-1}\big(b^*(r)\big)' \\ & \quad - \beta(\beta+1)B_+\left(\frac{r}{R}\right)^2\big(b^*(r)\big)^{-\beta-2}\big[\big(b^*(r)\big)'\big]^2 + \beta B_+\left(\frac{r}{R}\right)^2\big(b^*(r)\big)^{-\beta-1}\big(b^*(r)\big)'' \\ & \geqslant \lambda\big(b^*(r)\big)^{-\beta}\left[A\big(b^*(r)\big)^\beta + B_+\left(\frac{r}{R}\right)^2\right] - b(r)(1-\epsilon)H\big(b^*(r)\big)^{-p\beta}\left[A\big(b^*(r)\big)^\beta + B_+\left(\frac{r}{R}\right)^2\right]^p.\end{aligned}$$

Multiplying both sides of this inequality by $\frac{(b^*(r))^{p\beta}}{b(r)}$ and taking into consideration that $p\beta = \beta + 1$,

$$\begin{aligned}
&-2N\frac{B_+}{R^2}\frac{b^*(r)}{b(r)} + [N+3]\beta B_+ \frac{r}{R^2}\frac{(b^*(r))'}{b(r)} \\
&\quad - \beta(\beta+1)B_+\left(\frac{r}{R}\right)^2 \frac{[(b^*(r))']^2}{b^*(r)b(r)} + \beta B_+\left(\frac{r}{R}\right)^2\frac{(b^*(r))''}{b(r)} \\
&\quad \geqslant \lambda\frac{b^*(r)}{b(r)}\left[A\big(b^*(r)\big)^\beta + B_+\left(\frac{r}{R}\right)^2\right] - (1-\epsilon)H\left[A\big(b^*(r)\big)^\beta + B_+\left(\frac{r}{R}\right)^2\right]^p.
\end{aligned}$$

Since when $r \to R$, $\frac{b^*(r)}{b(r)} \to 0$, $\frac{(b^*(r))'}{b(r)} \to 0$, $\frac{[(b^*(r))']^2}{b^*(r)b(r)} \to C_0 \geqslant 1$ and $\frac{(b^*(r))''}{b(r)} \to 1$ by Lemma 2.3, then the above inequality becomes into

$$-\beta(\beta+1)B_+C_0 + \beta B_+ \geqslant -(1-\epsilon)H(B_+)^p$$

as $r \to R$, which is

$$B_+ \geqslant \frac{[\beta((\beta+1)C_0-1)]^{\frac{1}{p-1}}}{[(1-\epsilon)H]^{\frac{1}{p-1}}}.$$

Let $B_+ = (1+\epsilon)(1-\epsilon)^{-\beta}H^{-\beta}[\beta((\beta+1)C_0-1)]^\beta = (1+\epsilon)(1-\epsilon)^{-\beta}H^{-\beta}K$. Therefore, by making the choice B_+, the inequality (2.9) is satisfied in a left neighborhood of $r = R$, say $(R-\delta, R]$, for some $\delta = \delta(\epsilon) > 0$. Finally, by choosing A sufficiently large (larger than A_0) it is clear that the inequality is satisfied in the whole interval $[0, R]$ since $p > 1$ and $b^*(r)$ is bounded away from zero in $[0, R-\delta]$. Then $\overline{\psi}_\epsilon$ is our required supersolution of problem (2.2).

Next, we construct a subsolution with the same blow-up rate as the above supersolution. For doing this we shall distinguish two different cases according to the sign of the parameter λ. First, we assume $\lambda \geqslant 0$. Due to the assumption $(\mathcal{A})$ on h, for $u \geqslant A_0$ large,

$$(1-\epsilon)Hu^p \leqslant h(u) \leqslant (1+\epsilon)Hu^p.$$

For each $A_0 > 0$ and $0 < R_0 < R$, we consider the auxiliary problem

$$\begin{cases} -\psi'' - \dfrac{N-1}{r}\psi' = \lambda\psi - b(r)h(\psi) & \text{in } (0, R_0), \\ \psi(R_0) = A_0, \\ \psi'(0) = 0. \end{cases} \tag{2.10}$$

By the assumption on b and h, we have

$$\min_{r\in[0,R_0]} b(r) > 0, \quad h(0) = 0, \quad \text{and} \quad h(u)/u \to \infty \quad \text{as } u \to \infty.$$

Then it is easy to know that

$$\underline{\psi}_{A_0} := 0, \qquad \overline{\psi}_{A_0} := A_0$$

provides us with an ordered sub-supersolution pair of (2.10). Thus (2.10) possesses a solution ψ_{A_0} such that $\psi_{A_0}(r) \in [0, A_0]$ for all $r \in [0, R_0]$.

For each $\epsilon > 0$ sufficiently small, we claim that there exists $0 < C < A_0$ for which the function

$$\underline{\psi}_\epsilon(r) = \begin{cases} \psi_{A_0}(r), & r \in [0, R_0], \\ \max\{A_0, C + B_-(\frac{r}{R})^2 (b^*(r))^{-\beta}\}, & r \in (R_0, R], \end{cases}$$

provides a subsolution, where R_0 and C are to be determined later and

$$B_- = (1-\epsilon)(1+\epsilon)^{-\beta} H^{-\beta} \big[\beta\big((\beta+1)C_0 - 1\big)\big]^\beta = (1-\epsilon)(1+\epsilon)^{-\beta} H^{-\beta} K.$$

In fact, denoting $f_C(r) = C + B_-(\frac{r}{R})^2 (b^*(r))^{-\beta}$ we have

$$\begin{aligned} f_C'(r) &= 2B_- \frac{r}{R^2} \big(b^*(r)\big)^{-\beta} - \beta B_- \left(\frac{r}{R}\right)^2 \big(b^*(r)\big)^{-\beta-1} \big(b^*(r)\big)' \\ &= 2B_- \frac{r}{R^2} \big(b^*(r)\big)^{-\beta} + \beta B_- \left(\frac{r}{R}\right)^2 \big(b^*(r)\big)^{-\beta-1} \int_r^R b(s)\, ds \end{aligned}$$

which is strictly bigger than zero in $(0, R)$. It follows that $f_C(r)$ is increasing and

$$\lim_{r \to R} f_C(r) = +\infty, \qquad \lim_{r \to 0} f_C(r) = C < A_0.$$

By the continuity of $f_C(r)$ and the intermediate-value theorem, there exists a unique $Z = Z(C) \in (0, R)$ such that

$$\begin{aligned} &C + B_-\left(\frac{r}{R}\right)^2 \big(b^*(r)\big)^{-\beta} < A_0 \quad \text{when } r \in \big[0, Z(C)\big), \\ &C + B_-\left(\frac{r}{R}\right)^2 \big(b^*(r)\big)^{-\beta} \geqslant A_0 \quad \text{when } r \in \big[Z(C), R\big]. \end{aligned}$$

Moreover, $Z(C)$ is decreasing and

$$\lim_{C \to -\infty} Z(C) = R, \qquad \lim_{C \to A_0} Z(C) = 0.$$

Let $R_0 = Z(C)$. From the definition of $\underline{\psi}_\epsilon(r)$ and R_0, $\underline{\psi}_\epsilon(r) \equiv \psi_{A_0}(r)$ in $[0, Z(C)]$, and then the inequality $-\underline{\psi}_\epsilon'' - \frac{N-1}{r}\underline{\psi}_\epsilon' \leqslant \lambda \underline{\psi}_\epsilon - b(r)h(\underline{\psi}_\epsilon)$ holds in $[0, Z(C)]$. So $\underline{\psi}_\epsilon(r)$ is a subsolution if the following inequality is satisfied in $[Z(C), R]$

$$-\underline{\psi}_\epsilon''(r) - \frac{N-1}{r}\underline{\psi}_\epsilon'(r) \leqslant \lambda \underline{\psi}_\epsilon(r) - b(r) h\big(\underline{\psi}_\epsilon(r)\big). \tag{2.11}$$

By direct computation and by using the fact $h(\underline{\psi}_\epsilon(r)) \leqslant (1+\epsilon) H \underline{\psi}_\epsilon(r)^p$ in $[Z(C), R]$, (2.11) holds if

$$\begin{cases} -2N \dfrac{B_-}{R^2} \dfrac{b^*(r)}{b(r)} + [N+3]\beta B_- \dfrac{r}{R^2} \dfrac{(b^*(r))'}{b(r)} \\ \quad - \beta(\beta+1) B_- \left(\dfrac{r}{R}\right)^2 \dfrac{[(b^*(r))']^2}{b^*(r) b(r)} + \beta B_- \left(\dfrac{r}{R}\right)^2 \dfrac{(b^*(r))''}{b(r)} \\ \quad \leqslant \lambda \dfrac{b^*(r)}{b(r)} \left[C\big(b^*(r)\big)^\beta + B_-\left(\dfrac{r}{R}\right)^2\right] - (1+\epsilon) H \left[C\big(b^*(r)\big)^\beta + B_-\left(\dfrac{r}{R}\right)^2\right]^p. \end{cases} \tag{2.12}$$

Since $\lambda > 0$, for each $r \in [Z(C), R)$,

$$-2N\frac{B_-}{R^2}\frac{b^*(r)}{b(r)} + [N+3]\beta B_-\frac{r}{R^2}\frac{(b^*(r))'}{b(r)} = -\left(2N\frac{B_-}{R^2}\frac{b^*(r)}{b(r)} + [N+3]\beta B_-\frac{r}{R^2}\frac{\int_r^R b(s)\,ds}{b(r)}\right)$$
$$\leqslant 0 \leqslant \lambda\frac{b^*(r)}{b(r)}\left[C\big(b^*(r)\big)^\beta + B_-\left(\frac{r}{R}\right)^2\right].$$

Then the inequality (2.11) holds if

$$-\beta(\beta+1)B_-\left(\frac{r}{R}\right)^2\frac{[(b^*(r))']^2}{b^*(r)b(r)} + \beta B_-\left(\frac{r}{R}\right)^2 \leqslant -(1+\epsilon)H\left[C\big(b^*(r)\big)^\beta + B_-\left(\frac{r}{R}\right)^2\right]^p$$

for each $r \in [Z(C), R]$. At $r = R$, it becomes

$$-\beta(\beta+1)B_-C_0 + \beta B_- \leqslant -(1+\epsilon)HB_-^p.$$

That is

$$B_- \leqslant \big[(1+\epsilon)H\big]^{-\frac{1}{p-1}}\big[\beta\big((\beta+1)C_0 - 1\big)\big]^{\frac{1}{p-1}}.$$

By making the choice $B_- = (1-\epsilon)[(1+\epsilon)H]^{-\frac{1}{p-1}}[\beta((\beta+1)C_0-1)]^{\frac{1}{p-1}}$ and using the continuity, it is easy to see that a constant $\delta = \delta(\epsilon) > 0$ exists for which the inequality is satisfied in $[R-\delta, R)$, then we choose C such that $Z(C) = R - \delta(\epsilon)$ (therefore $R_0 = R - \delta(\epsilon)$). For this choice of C, it readily follows that $\underline{\psi}_\epsilon$ is a subsolution to the problem.

In the case of $\lambda < 0$, let $\mu = \frac{N-1}{R}\sqrt{-\lambda} > 0$ and $\underline{\psi}_\epsilon$ be a subsolution constructed as above for

$$\begin{cases} -\psi'' - \dfrac{N-1}{r}\psi' = \mu\psi - b(r)h(\psi) \quad \text{in } (0,R), \\ \lim\limits_{r\to R}\psi(r) = \infty, \\ \psi'(0) = 0. \end{cases} \tag{2.13}$$

Then

$$\underline{\tilde\psi}_\epsilon(r) = \exp\big(\sqrt{-\lambda}(r-R)\big)\cdot\underline{\psi}_\epsilon(r)$$

is a subsolution to the problem (2.2) for $\lambda < 0$. In fact,

$$\underline{\tilde\psi}'_\epsilon(r) = \sqrt{-\lambda}\exp\big(\sqrt{-\lambda}(r-R)\big)\cdot\underline{\psi}_\epsilon(r) + \exp\big(\sqrt{-\lambda}(r-R)\big)\cdot\underline{\psi}'_\epsilon(r),$$
$$\underline{\tilde\psi}''_\epsilon(r) = -\lambda\exp\big(\sqrt{-\lambda}(r-R)\big)\cdot\underline{\psi}_\epsilon(r) + 2\sqrt{-\lambda}\exp\big(\sqrt{-\lambda}(r-R)\big)\cdot\underline{\psi}'_\epsilon(r) + \exp\big(\sqrt{-\lambda}(r-R)\big)\cdot\underline{\psi}''_\epsilon(r).$$

Similarly $\underline{\tilde\psi}_\epsilon(r) \to \infty$ as $r \to R$. $\underline{\tilde\psi}'_\epsilon(r) \to 0$ as $r \to 0$. Let $\varsigma = \exp(\sqrt{-\lambda}(r-R))$. Therefore $\underline{\tilde\psi}'_\epsilon(r)$ is a subsolution if the following inequality is satisfied

$$\lambda\varsigma\underline{\psi}_\epsilon(r) - 2\sqrt{-\lambda}\varsigma\underline{\psi}'_\epsilon(r) - \varsigma\underline{\psi}''_\epsilon(r) - \frac{N-1}{r}\varsigma\underline{\psi}'_\epsilon(r) - \frac{N-1}{r}\sqrt{-\lambda}\varsigma\underline{\psi}_\epsilon(r)$$
$$\leqslant \lambda\varsigma\underline{\psi}_\epsilon(r) - b(r)h\big(\varsigma\underline{\psi}_\epsilon(r)\big).$$

Multiplying both sides by $1/\varsigma$, the inequality simplifies to

$$-\underline{\psi}_\epsilon''(r) - \frac{N-1}{r}\underline{\psi}_\epsilon'(r) - 2\sqrt{-\lambda}\underline{\psi}_\epsilon'(r) \leqslant \frac{N-1}{r}\sqrt{-\lambda}\underline{\psi}_\epsilon(r) - b(r)\frac{h(\varsigma\underline{\psi}_\epsilon(r))}{\varsigma}. \tag{2.14}$$

By the fact of $\underline{\psi}_\epsilon' \geqslant 0$, $\lambda < 0$ and $\underline{\psi}_\epsilon(r)$ is a subsolution to the problem (2.13),

$$-\underline{\psi}_\epsilon''(r) - \frac{N-1}{r}\underline{\psi}_\epsilon'(r) - 2\sqrt{-\lambda}\underline{\psi}_\epsilon'(r) \leqslant \mu\underline{\psi}_\epsilon(r) - b(r)h\big(\underline{\psi}_\epsilon(r)\big).$$

Because $h(u)/u$ is increasing and $\varsigma = \exp(\sqrt{-\lambda}(r-R)) \leqslant 1$ for $r \in [0, R]$,

$$\frac{h(\varsigma\underline{\psi}_\epsilon(r))}{\varsigma\underline{\psi}_\epsilon(r)} \leqslant \frac{h(\underline{\psi}_\epsilon(r))}{\underline{\psi}_\epsilon(r)}, \quad \text{i.e.,} \quad \frac{h(\varsigma\underline{\psi}_\epsilon(r))}{\varsigma} \leqslant h\big(\underline{\psi}_\epsilon(r)\big).$$

Note that $\mu = \frac{N-1}{R}\sqrt{-\lambda} \leqslant \frac{N-1}{r}\sqrt{-\lambda}$. Therefore

$$\mu\underline{\psi}_\epsilon(r) - b(r)h\big(\underline{\psi}_\epsilon(r)\big) \leqslant \frac{N-1}{r}\sqrt{-\lambda}\underline{\psi}_\epsilon(r) - b(r)\frac{h(\varsigma\underline{\psi}_\epsilon(r))}{\varsigma},$$

which proves that inequality (2.14) holds. So we have constructed a subsolution and a supersolution with the same blow-up rate of the problem (2.2). By Lemma 2.2, there exists a solution $\psi_\epsilon(r)$ of (2.2) such that

$$1-\epsilon \leqslant \liminf_{r\to R}\frac{\psi_\epsilon(r)}{KH^{-\beta}(b^*(r))^{-\beta}} \leqslant \limsup_{r\to R}\frac{\psi_\epsilon(r)}{KH^{-\beta}(b^*(r))^{-\beta}} \leqslant 1+\epsilon.$$

Proof of uniqueness. Let u be an arbitrary solution of (1.4) with assumptions on the domain and weight function b as in Theorem 1.1. We first show that

$$\lim_{d(x)\to 0}\frac{u(x)}{KH^{-\beta}(b^*(\|x-x_0\|))^{-\beta}} = 1.$$

Consequently, for any pair of solutions u, v of (1.4)

$$\lim_{d(x)\to 0}\frac{u(x)}{v(x)} = 1.$$

In doing so, for any $\epsilon > 0$, there exists a radially symmetric positive large solution u_ϵ of (1.4) satisfying (2.5). Choose $0 < \delta < \frac{R}{3}$ small, fix $0 < \tau < \frac{\delta}{4}$ and introduce the region

$$Q_\tau := \left\{x \,\middle|\, \tau < d\big(x, \partial B_R(x_0)\big) < \frac{\delta}{2}\right\}.$$

Let $M \geqslant \max_{\|x-x_0\|\leqslant(R-\frac{\delta}{4})} u(x)$ be large. Thus for every $\tau \in (0, \frac{\delta}{4})$,

$$\bar{V}_\epsilon(x) = u_\epsilon\left(x + \tau\frac{(x-x_0)}{\|x-x_0\|}\right) + M = u_\epsilon\big(\|x-x_0\| + \tau\big) + M$$

is a supersolution to

$$\begin{cases} -\Delta v = \lambda v - bh(v) & \text{in } Q_\tau, \\ v = u & \text{on } \partial Q_\tau \end{cases} \tag{2.15}$$

with u an arbitrary fixed solution to (1.4) since $\bar{V}_\epsilon(x) \geqslant u$ for $x \in \partial Q_\tau$, $\tau \in (0, \frac{\delta}{4})$. Note that $\bar{V}_\epsilon(x) \to \infty$ as $x \to \partial B_{R-\tau}(x_0)$. $\bar{V}_\epsilon(x) \geqslant M \geqslant u$ as $x \to \partial B_{R-\frac{\delta}{2}}(x_0)$. In addition, the auxiliary problem (2.15) has $v = u$ as its unique solution. Since 0 is a subsolution ($h(0) = 0$ by the assumption $h(u)/u$ is increasing), we conclude $u(x) \leqslant \bar{V}_\epsilon(x) = u_\epsilon(\|x - x_0\| + \tau) + M$ for every $x \in Q_\tau$, $0 < \tau < \frac{\delta}{4}$. Letting $\tau \to 0^+$, we arrive at $u(x) \leqslant u_\epsilon(x) + M$ for every $x \in A_{\frac{\delta}{2},R}(x_0)$ and we obtain

$$\limsup_{d(x)\to 0} \frac{u(x)}{KH^{-\beta}(b^*(\|x - x_0\|))^{-\beta}} \leqslant \limsup_{d(x)\to 0} \frac{u_\epsilon(x)}{KH^{-\beta}(b^*(\|x - x_0\|))^{-\beta}} \leqslant 1 + \epsilon.$$

We now prove

$$1 - \epsilon \leqslant \liminf_{d(x)\to 0} \frac{u_\epsilon(x)}{KH^{-\beta}(b^*(\|x - x_0\|))^{-\beta}} \leqslant \liminf_{d(x)\to 0} \frac{u}{KH^{-\beta}(b^*(\|x - x_0\|))^{-\beta}}.$$

For the same $\epsilon > 0$ and the radially symmetric positive large solution u_ϵ of (1.4) satisfying (2.5), we choose $0 < \delta < \frac{R}{3}$ small, fix $0 < \tau < \frac{\delta}{4}$ and introduce the annuli region

$$A_{R-\delta,R-\tau} = \{x\colon\ R - \delta < \|x - x_0\| < R - \tau\}.$$

Let $M_1 \geqslant \max_{\|x-x_0\| \leqslant R - \frac{3\delta}{4}} u_\epsilon(x)$ be large. Thus for every $\tau \in (0, \frac{\delta}{4})$,

$$\tilde{V}_\epsilon(x) = u\left(x + \tau \frac{(x - x_0)}{\|x - x_0\|}\right) + M_1$$

is a supersolution to

$$\begin{cases} -\Delta v = \lambda v - bh(v) & \text{in } A_{R-\delta,R-\tau}, \\ v = u_\epsilon & \text{on } \partial A_{R-\delta,R-\tau} \end{cases} \tag{2.16}$$

for all $\tau \in (0, \frac{\delta}{4})$. It readily implies the above inequalities. As a result, we obtain

$$1 - \epsilon \leqslant \lim_{d(x)\to 0} \frac{u}{KH^{-\beta}(b^*(\|x - x_0\|))^{-\beta}} \leqslant 1 + \epsilon.$$

Letting $\epsilon \to 0^+$, we have

$$\lim_{d(x)\to 0} \frac{u}{KH^{-\beta}(b^*(\|x - x_0\|))^{-\beta}} = 1.$$

Now let u and v be large positive solutions to (1.4). By virtue of above, u and v satisfy $\lim_{d(x)\to 0} \frac{u}{v} = 1$. Thus, for every $\epsilon > 0$, we can find $\delta > 0$ (as small as we want) such that

$$(1 - \epsilon)v(x) \leqslant u(x) \leqslant (1 + \epsilon)v(x)$$

when $0 < d(x) \leqslant \delta$. On the other hand $\underline{w} = (1 - \epsilon)v(x)$ and $\overline{w} = (1 + \epsilon)v(x)$ are sub- and supersolutions to

$$\begin{cases} -\Delta w = \lambda w - bh(w) & \text{in } B_{R-\delta}(x_0), \\ w = u & \text{on } \partial B_{R-\delta}(x_0), \end{cases} \tag{2.17}$$

where we use the property $\frac{h((1-\epsilon)v)}{1-\epsilon} \leqslant h(v)$ and $\frac{h((1+\epsilon)v)}{1+\epsilon} \geqslant h(v)$. The unique solution to this problem is $w = u$. Then by Lemma 2.1

$$(1-\epsilon)v(x) \leqslant u(x) \leqslant (1+\epsilon)v(x)$$

holds in $B_{R-\delta}(x_0)$, therefore it is true in $B_R(x_0)$. Letting $\epsilon \to 0$ we arrive at $u = v$. $\square$

3. Proof of Theorem 1.2

Let u be a large positive solution of the problem (1.4) with the assumptions of Theorem 1.2. We first construct a large supersolution locally for each $x_0 \in \partial\Omega$.

For a sufficiently small $\epsilon > 0$, thanks to (1.10), there exist $\rho = \rho(\epsilon) \in (0, \tau)$ and $\mu = \mu(\epsilon) > 0$ such that

$$1 - \epsilon < \frac{b_x(r)}{b_{x_0}(r)} = \frac{b(x - r\mathbf{n}_x)}{b(x_0 - r\mathbf{n}_{x_0})} < 1 + \epsilon, \tag{3.1}$$

for any $x \in \partial\Omega \cap \bar{B}_\rho(x_0)$, $r \in [0, \mu]$. Let

$$\mathcal{B} = \left\{x - r\mathbf{n}_x \;\middle|\; (x, r) \in \left[\partial\Omega \cap \bar{B}_\rho(x_0)\right] \times [0, \mu]\right\}. \tag{3.2}$$

Because $\partial\Omega$ is smooth, ρ, μ can be shortened, if necessary, so that for each $y \in \mathcal{B}$ there exists a unique $(\pi(y), r(y)) \in [\partial\Omega \cap \bar{B}_\rho(x_0)] \times [0, \mu]$ and $y = \pi(y) - r(y)\mathbf{n}_{\pi(y)}$ and $r(y) = |y - \pi(y)| = \operatorname{dist}(y, \partial\Omega)$. From now on, we assume that ρ, μ satisfy these requirements. Furthermore, there exists $R_0 \in (0, \min\{\frac{\rho}{2}, \frac{\mu}{2}\})$, such that

$$B_{R_0}(x_0 - R_0\mathbf{n}_{x_0}) \subset \Omega \cap \operatorname{Int}\mathcal{B} \quad \text{and} \quad \bar{B}_{R_0}(x_0 - R_0\mathbf{n}_{x_0}) \cap \partial\Omega = \{x_0\}. \tag{3.3}$$

Then for any $\delta \in [0, \delta_0]$ (δ_0 is chosen to be very small), the family of the small ball is in $\Omega \cap \operatorname{Int}\mathcal{B}$ and

$$B_{R_0-\delta}(x_0 - R_0\mathbf{n}_{x_0}) \subset \bar{B}_{R_0}(x_0 - R_0\mathbf{n}_{x_0}).$$

By (1.9) and (1.10), for each $\delta \in [0, \delta_0]$ and $y \in \bar{B}_{R_0-\delta}(x_0 - R_0)$, we have

$$\begin{aligned} b(y) &= b\big(\pi(y) - r(y)\mathbf{n}_{\pi(y)}\big) > (1-\epsilon)b\big(x_0 - r(y)\mathbf{n}_{x_0}\big) \\ &= (1-\epsilon)b_{x_0}\big(r(y)\big) = (1-\epsilon)b_{x_0}\big(\operatorname{dist}(y, \partial\Omega)\big) \\ &\geqslant (1-\epsilon)b_{x_0}\big(\operatorname{dist}\big(y, \partial B_{R_0-\delta}(x_0 - R_0\mathbf{n}_{x_0})\big)\big). \end{aligned} \tag{3.4}$$

Therefore, for all $y \in \bar{B}_{R_0-\delta}(x_0 - R_0\mathbf{n}_{x_0})$,

$$b(y) \geqslant (1-\epsilon)b_{x_0}(r_\delta), \tag{3.5}$$

where $r_\delta = (R_0 - \delta) - \|y - x_0 + R_0\mathbf{n}_{x_0}\| = \operatorname{dist}(y, \partial B_{R_0-\delta}(x_0 - R_0\mathbf{n}_{x_0}))$, which entails that, for any $\delta \in [0, \delta_0]$, the restriction

$$\underline{u}_\delta := u|_{B_{R_0-\delta}}(x_0 - R_0\mathbf{n}_{x_0}) \tag{3.6}$$

is a positive smooth subsolution of

$$\begin{cases} -\Delta u = \lambda u - (1-\epsilon) b_{x_0}(r) h(u) & \text{in } B_{R_0-\delta}(x_0 - R_0 \mathbf{n}_{x_0}), \\ u = \infty & \text{on } \partial B_{R_0-\delta}(x_0 - R_0 \mathbf{n}_{x_0}). \end{cases} \tag{3.7}$$

By Theorem 1.1, for each $\delta \in [0, \delta_0]$, there exists a unique solution Φ_δ of (3.7) satisfying

$$\lim_{x \to \partial B_{R_0-\delta}(x_0 - R_0 \mathbf{n}_{x_0})} \frac{\Phi_\delta(x)}{K H^{-\beta}((1-\epsilon) b^*_{x_0}(r_\delta))^{-\beta}} = 1, \tag{3.8}$$

where $r_\delta = \operatorname{dist}(x, \partial B_{R_0-\delta}(x_0 - R_0 \mathbf{n}_{x_0}))$ and

$$b^*_{x_0}(r_\delta) = \int_0^{r_\delta} \int_s^{r_\delta} b_{x_0}(t)\, dt\, ds.$$

K and β are given by

$$K = \left[\beta\big((\beta+1)C_0 - 1\big)\right]^{\frac{1}{p-1}}, \qquad \beta = \frac{1}{p-1},$$

and C_0 is defined as

$$C_0 = \lim_{r_\delta \to R_0} \frac{(B_{x_0}(r_\delta))^2}{b^*_{x_0}(r_\delta) b_{x_0}(r_\delta)}.$$

Then for each $\delta \in [0, \delta_0]$,

$$\underline{u}_\delta = u|_{B_{R_0-\delta}}(x_0 - R_0 \mathbf{n}_{x_0}) \leqslant \Phi_\delta$$

for x in $B_{R_0-\delta}(x_0 - R_0 \mathbf{n}_{x_0})$. Thus

$$\limsup_{x \to \partial B_{R_0-\delta}(x_0 - R_0 \mathbf{n}_{x_0})} \frac{u_\delta(x)}{K((1-\epsilon) b^*_{x_0}(r_\delta))^{-\beta}} \leqslant 1, \tag{3.9}$$

passing to the limit as $\delta \to 0$ gives

$$\limsup_{r \to 0} \frac{u(x_0 - r\mathbf{n}_{x_0})}{K H^{-\beta}((1-\epsilon) b^*_{x_0}(r))^{-\beta}} \leqslant 1. \tag{3.10}$$

In particular, (3.10) is valid for any sufficiently small $\epsilon > 0$, then

$$\limsup_{r \to 0} \frac{u(x_0 - r\mathbf{n}_{x_0})}{K H^{-\beta}(b^*_{x_0}(r))^{-\beta}} \leqslant 1. \tag{3.11}$$

To prove (1.11) in Theorem 1.2, we build up a large subsolution having the adequate growth at $x_0 \in \partial\Omega$ so that we can show

$$\liminf_{r \to 0} \frac{u(x_0 - r\mathbf{n}_{x_0})}{K H^{-\beta}(b^*_{x_0}(r))^{-\beta}} \geqslant 1. \tag{3.12}$$

Since Ω has a smooth boundary, for any $x_0 \in \partial\Omega$, there exist $R_2 > R_1 > 0$ and $\delta_0 > 0$ such that

$$\Omega \subset \bigcap_{\delta \in [0,\delta_0]} A_{R_1-\delta, R_2-\delta}(x_0 + R_1 \mathbf{n}_{x_0})$$

and

$$\partial\Omega \cap \bar{A}_{R_1,R_2}(x_0 + R_1 \mathbf{n}_{x_0}) = \{x_0\}.$$

Moreover, R_2 can be taken arbitrarily large. We suppose R_2 has been chosen to satisfy

$$\Omega \subset A_{R_1, \frac{R_2}{3}}(x_0 + R_1 \mathbf{n}_{x_0}),$$

and R_1 can be taken arbitrarily small.

Fix a sufficiently small $\epsilon > 0$ and $x_0 \in \partial\Omega$. Using (3.1), we have

$$1 - \epsilon < \frac{b_x(r)}{b_{x_0}(r)} = \frac{b(x - r\mathbf{n}_x)}{b(x_0 - r\mathbf{n}_{x_0})} < 1 + \epsilon, \tag{3.13}$$

for any $x \in \partial\Omega \cap \bar{B}_\rho(x_0)$, $r \in [0, \mu]$. Choosing a small $2\eta < \min\{\rho, \mu\}$, for each $y \in B_{2\eta}(x_0) \cap \bar{\Omega}$, we have

$$\begin{aligned} b(y) = b\big(\pi(y) - r(y)\mathbf{n}_{\pi(y)}\big) &\leqslant (1+\epsilon) b_{x_0}\big(r(y)\big) \\ &= (1+\epsilon) b_{x_0}\big(\mathrm{dist}(y, \partial\Omega)\big) \\ &\leqslant (1+\epsilon) b_{x_0}\big(\mathrm{dist}\big(y, \partial B_{R_1}(x_0 + R_1\mathbf{n}_{x_0})\big)\big) \\ &= (1+\epsilon) b_{x_0}\big(\mathrm{dist}\big(y, \partial A_{R_1,R_2}(x_0 + R_1\mathbf{n}_{x_0})\big)\big). \end{aligned} \tag{3.14}$$

Thus we can construct a radially symmetric function

$$\hat{b} : A_{R_1,R_2}(x_0 + R_1\mathbf{n}_{x_0}) \mapsto [0, \infty),$$

such that

$$\hat{b} \geqslant b \quad \text{in } \Omega, \tag{3.15}$$

by extending the function

$$\hat{b}(y) = \hat{b}(r) = (1+\epsilon) b_{x_0}(r), \tag{3.16}$$

where

$$r = \mathrm{dist}\big(y, \partial A_{R_1,R_2}(x_0 + R_1\mathbf{n}_{x_0})\big) \quad \text{and} \quad y \in B_{2\eta}(x_0) \cap \bar{\Omega}.$$

Further, for $y \in \bar{\Omega}$, $\hat{b}$ also satisfies

$$\hat{b}(r_\delta) = \hat{b}\big(\mathrm{dist}\big(y, \partial A_{R_1-\delta,R_2-\delta}(x_0 + R_1\mathbf{n}_{x_0})\big)\big) \geqslant b(y)$$

for a sufficiently small δ_0 and any $0 \leqslant \delta \leqslant \delta_0$.

For each sufficiently small $\delta > 0$, $\delta \in (0, \delta_0]$, consider the auxiliary problem

$$\begin{cases} -\Delta u = \lambda u - \hat{b}(r)h(u) & \text{in } A_{R_1-\delta, R_2-\delta}(x_0 + R_1 \mathbf{n}_{x_0}), \\ u = \infty & \text{on } \partial A_{R_1-\delta, R_2-\delta}(x_0 + R_1 \mathbf{n}_{x_0}), \end{cases} \tag{3.17}$$

where $r = \operatorname{dist}(x, \partial A_{R_1-\delta, R_2-\delta}(x_0 + R_1 \mathbf{n}_{x_0}))$.

By Corollary 1.1, there exists a unique large positive solution $\Phi_{\epsilon,\delta}$ such that

$$\lim_{x \to \partial A_{R_1-\delta, R_2-\delta}(x_0 + R_1 \mathbf{n}_{x_0})} \frac{\Phi_{\epsilon,\delta}(x)}{K H^{-\beta}((1+\epsilon) b^*_{x_0}(r))^{-\beta}} = 1,$$

where $b^*_{x_0}(r)$ and K are defined as before. Moreover, by construction, the restriction $\Phi_{\epsilon,\delta}|_\Omega$ provides us with a subsolution of (1.4). Thus, for each $\delta \in (0, \delta_0]$, we have

$$\Phi_{\epsilon,\delta}(x) \leqslant u(x)$$

for each $x \in A_{R_1-\delta, R_2-\delta}(x_0 + R_1 \mathbf{n}_{x_0}) \cap \Omega$, then

$$\lim_{r \to 0} \frac{u(x_0 - r \mathbf{n}_{x_0})}{K H^{-\beta}((1+\epsilon) b^*_{x_0}(r))^{-\beta}} \geqslant 1. \tag{3.18}$$

Taking $\mu \to 0$, it readily gives

$$\liminf_{x \to x_0,\ x \in A_{R_1, R_2}(x_0 + R_1 \mathbf{n}_{x_0}) \cap \Omega} \frac{u(x)}{K H^{-\beta}(b^*_{x_0}(r))^{-\beta}} \geqslant 1. \tag{3.19}$$

(3.11) and (3.19) readily imply (1.11) of Theorem 1.2. So the first part of Theorem 1.2 is proved.

If the conditions (1.9) and (1.10) are satisfied uniformly on $\partial\Omega$, we prove that (1.12) of Theorem 1.2 is satisfied uniformly on $\partial\Omega$ by checking the proof above whether it is true uniformly. It is clear that ρ, τ can be chosen small enough so that (3.1) is true for each $x, x_0 \in \partial\Omega$ sufficiently close to each other. R_0 and R_1 can be chosen to be the same for each $x_0 \in \partial\Omega$. Also note that the limit of $b^*_{x_0}(r)$ is independent of the choice of δ in (3.9) and (3.18). Therefore

$$\lim_{r \to 0} \frac{u(x_0 - r \mathbf{n}_{x_0})}{K H^{-\beta}(b^*_{x_0}(r))^{-\beta}} = 1, \quad \text{uniformly in } x_0 \in \partial\Omega.$$

Proof of the uniqueness. The proof of the uniqueness is similar to the proof of the uniqueness in Theorem 1.1. For any pair of solutions u, v of (1.2),

$$\lim_{\operatorname{dist}(y, \partial\Omega) \to 0} \frac{u(y)}{v(y)} = \lim_{r(y) \to 0} \frac{u(\pi(y) - r(y)\mathbf{n}_{\pi(y)})}{v(\pi(y) - r(y)\mathbf{n}_{\pi(y)})} = 1, \quad \text{uniformly on } \partial\Omega.$$

Thus, for every $\epsilon > 0$, we can find $\delta > 0$ (as small as we want) such that

$$(1-\epsilon)v(x) \leqslant u(x) \leqslant (1+\epsilon)v(x), \quad x \in \Omega \setminus \Omega_\delta,$$

where, for each small enough $\delta > 0$, we have denoted

$$\Omega_\delta := \bigl\{ x \in \Omega\colon\ \operatorname{dist}(x, \partial\Omega) > \delta \bigr\}.$$

On the other hand $\underline{w} = (1-\epsilon)v(x)$ and $\overline{w} = (1+\epsilon)v(x)$ are sub- and supersolutions to

$$\begin{cases} -\Delta w = \lambda w - ah(w) & \text{in } \Omega_\delta, \\ w = u & \text{on } \partial\Omega_\delta. \end{cases} \tag{3.20}$$

The unique solution to this problem is $w = u$. Then by Theorem 2.1

$$(1-\epsilon)v(x) \leqslant u(x) \leqslant (1+\epsilon)v(x)$$

holds in Ω_δ, therefore it is true in Ω. Letting $\epsilon \to 0$ we arrive at $u = v$.

Finally, by the uniqueness above and the abstract existence theory of [13] and [21], the problem (1.4) possesses a unique positive solution. This concludes the proof of Theorem 1.2.

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References

[1] H. Amann, On the existence of positive solutions of nonlinear elliptic boundary value problems, Indiana Univ. Math. J. 21 (1971) 125–146.
[2] C. Bandle, M. Essèn, On the solutions of quasilinear elliptic problems with boundary blow-up, Sympos. Math. 35 (1994) 93–111.
[3] L. Bieberbach, $\Delta u = e^u$ und die automorphen Funktionen, Math. Ann. 77 (1916) 173–212.
[4] C. Bandle, M. Marcus, On second order effects in the boundary behaviour of large solutions of semilinear elliptic problems, Differential Integral Equations 11 (1) (1998) 23–34.
[5] P. Bates, J. Shi, Existence and instability of spike layer solutions to singular perturbation problems, J. Funct. Anal. 196 (2002) 429–482.
[6] S. Cano-Casanova, J. López-Gómez, Blow-up rates of radially symmetric large solutions, J. Math. Anal. Appl. 352 (2009) 166–174.
[7] F.C. Cirstea, Y. Du, General uniqueness results and variation speed for blow-up solutions of elliptic equations, Proc. London Math. Soc. (3) 91 (2005) 459–482.
[8] F.C. Cirstea, V. Radulescu, Existence and uniqueness of blow-up solutions for a class of logistic equations, Commun. Contemp. Math. 4 (2002) 559–586.
[9] F.C. Cirstea, V. Radulescu, Uniqueness of the blow-up boundary solution of logistic equations with absorbtion, C. R. Acad. Sci. Paris Ser. I 335 (2002) 447–452.
[10] F.C. Cirstea, V. Radulescu, An extreme variation phenomenon for some nonlinear elliptic problems with boundary blow-up, C. R. Acad. Sci. Paris Ser. I 339 (2004) 689–694.
[11] Y. Du, Q. Huang, Blow-up solutions for a class of semilinear elliptic and parabolic equations, SIAM J. Math. Anal. 31 (1) (1999) 1–18.
[12] M. Delgado, J. López-Gómez, A. Suarez, Singular bvp's of a porous media logistic equation, Hiroshima Math. J. 34 (2004) 57–80.
[13] J. García-Melián, A remark on the existence of large solutions via sub and supersolutions, Electron. J. Differential Equations 2003 (110) (2003) 1–4.
[14] J. García-Melián, R. Gómez-Reñasco, J. López-Gómez, J.C. Sabina de Lis, Point-wise growth and uniqueness of positive solutions for a class of sublinear elliptic problems where bifurcation from infinity occurs, Arch. Ration. Mech. Anal. 145 (1998) 261–289.
[15] J. García-Melián, R. Letelier-Albornoz, J. Sabina de Lis, Uniqueness and asymptotic behaviour for solutions of semilinear problems with boundary blow-up, Proc. Amer. Math. Soc. 129 (12) (2001) 3593–3602.
[16] R. Gómez-Reñasco, J. López-Gómez, On the existence and numerical computation of classical and non-classical solutions for a family of elliptic boundary value problems, Nonlinear Anal. 48 (2002) 567–605.
[17] J.B. Keller, On solutions of $\Delta u = f(u)$, Comm. Pure Appl. Math. 10 (1957) 503–510.
[18] A.C. Lazer, P.J. McKenna, On a problem of Bieberbach and Rademacher, Nonlinear Anal. 21 (1993) 327–335.
[19] A.C. Lazer, P.J. McKenna, Asymptotic behavior of solutions of boundary blowup problems, Differential Integral Equations 7 (4) (1994) 1001–1019.
[20] J. López-Gómez, Uniqueness of radially symmetric large solutions, Supplement volume 2007 dedicated to the 6th AIMS conference, Poitiers, France, Discrete Contin. Dyn. Syst. (2007) 677–686.
[21] J. López-Gómez, Large solutions, metasolutions, and asymptotic behaviour of the regular positive solutions of sublinear parabolic problems, Electron. J. Differ. Equ. Conf. 05 (2000) 135–171.
[22] J. López-Gómez, The boundary blow-up rate of large solutions, J. Differential Equations 195 (2003) 25–45.
[23] J. López-Gómez, Uniqueness of large solutions for a class of radially symmetric elliptic equations, in: S. Cano-Casanova, J. López-Gómez, C. Moracorral (Eds.), Spectral Theory and Nonlinear Analysis with Applications to Spatial Ecology, World Scientific, Singapore, 2005, pp. 75–110.

[24] J. López-Gómez, Optimal uniqueness theorems and exact blow-up rates of large solutions, J. Differential Equations 224 (2006) 385–439.
[25] R. Osserman, On the inequality $\Delta u \geqslant f(u)$, Pacific J. Math. 7 (1957) 1641–1647.
[26] T. Ouyang, On the positive solutions of semilinear equations $\Delta u + \lambda u - hu^p = 0$ on compact manifolds, Trans. Amer. Math. Soc. 331 (1992) 503–527.
[27] T. Ouyang, J. Shi, Exact multiplicity of positive solutions for a class of semilinear problem, J. Differential Equations 146 (1998) 121–156.
[28] T. Ouyang, J. Shi, Exact multiplicity of positive solutions for a class of semilinear problem, II, J. Differential Equations 158 (1999) 94–151.
[29] T. Ouyang, Z. Xie, The uniqueness of blow-up solution for radially symmetric semilinear elliptic equation, Nonlinear Anal. 64 (9) (2006) 2129–2142.
[30] T. Ouyang, Z. Xie, The exact boundary blow-up rate of large solutions for semilinear elliptic problems, Nonlinear Anal. 68 (2008) 2791–2800.
[31] H. Rademacher, Einige besondere Probleme der partiellen Differentialgleichungen, in: P. Frank, R. Von Mises (Eds.), Die Differential und Integralgleichungen der Mechanik und Physik, I, second ed., Rosenberg, New York, 1943, pp. 838–845.