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Nested regular polygon solutions of $2N$-body problem ☆

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Abstract

In this Letter we study the necessary conditions for the masses of the nested regular polygon solutions of the planar $2N$-body problem. © 2001 Published by Elsevier Science B.V.

Keywords: $2N$-body problems; Nested regular polygon periodic solutions; Circulant matrix

1. Main results

This Letter uses the same notations as paper [5]. For $n \geqslant 2$ the equations of motion of the planar n-body problem [1,2,5,6] can be written in the form

$$\ddot{z}_k = -\sum_{j \neq k} m_j \frac{z_k - z_j}{|z_k - z_j|^3}, \tag{1.1}$$

where z_k is the complex coordinate of the kth mass m_k in an inertial coordinate system. In Eq. (1.1) and throughout this Letter, unless otherwise restricted, all indices and summations will range from 1 to N.

Let ρ_k denote the N complex kth roots of unity, i.e.,

$$\rho_k = \exp(2\pi i k/N). \tag{1.2}$$

This equation will also serve to define ρ_k for any number k. We assume mass m_k $(k = 1, \ldots, N)$ locate at the vertices ρ_k of a regular polygon inscribed on the unit circle, and $\bar{m}_k = bm_k$ ($b > 0$, $k = 1, \ldots, N$) locate at

$$\bar{\rho}_k = a\rho_k \quad (a > 0,\ a \neq 1). \tag{1.3}$$

Then the center of masses $m_1, \ldots, m_N, \bar{m}_1, \ldots, \bar{m}_N$ is

$$z_0 = \sum_j \big(m_j \rho_j + \bar{m}_j \bar{\rho}_j\big)/M, \tag{1.4}$$

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where $M=\sum_j(m_j+\bar{m}_j)$. The functions describing their rotation about z_0 with angular velocity ω are then given by

$$z_k(t)=(\rho_k-z_0)\exp(i\omega t),\quad k=1,\ldots,N, \tag{1.5}$$

and

$$\bar{z}_k(t)=(a\rho_k-z_0)\exp(i\omega t),\quad k=1,\ldots,N. \tag{1.6}$$

Then the equations of motion of the planar $2N$-body problem can be written in the form

$$\ddot{z}_k=\sum_{j\neq k}m_j\frac{z_j-z_k}{|z_j-z_k|^3}+\sum_j\bar{m}_j\frac{\bar{z}_j-z_k}{|\bar{z}_j-z_k|^3} \tag{1.7}$$

and

$$\ddot{\bar{z}}_k=\sum_j m_j\frac{z_j-\bar{z}_k}{|z_j-\bar{z}_k|^3}+\sum_{j\neq k}\bar{m}_j\frac{\bar{z}_j-\bar{z}_k}{|\bar{z}_j-\bar{z}_k|^3}. \tag{1.8}$$

Moeckel and Simo [4] proved the following result:

Theorem (Moeckel–Simo). *For every mass ratio b, there are exactly two planar central configurations consisting of two nested regular N-gons. For one of these, the ratio of the sizes of the two polygons is less than* 1, *and for the other it is greater than* 1.

In this Letter, we study the inverse problem of the theorem (Moeckel–Simo) and the following results are established.

Theorem 1. *If, for $N\geqslant 2$, the functions $z_k(t)$ and $\bar{z}_k(t)$ given by* (1.5) *and* (1.6) *are solutions of the $2N$-body problem* (1.7) *and* (1.8), *it follows that $\omega^2=M\gamma/N$, where*

$$\gamma=\frac{1}{1+b}\left(\sum_{j=1}^{N-1}\frac{1-\rho_j}{|1-\rho_j|^3}+\sum_{j=0}^{N-1}\frac{b(1-a\rho_j)}{|1-a\rho_j|^3}\right), \tag{1.9}$$

b and a has the following relationship:

$$b=\frac{\left(\sum_{j=0}^{N-1}(a-\rho_j)/|a-\rho_j|^3-\sum_{j=1}^{N-1}a(1-\rho_j)/|1-\rho_j|^3\right)}{\left(\sum_{j=0}^{N-1}a(1-a\rho_j)/|1-a\rho_j|^3-\sum_{j=1}^{N-1}a(1-\rho_j)/a^3|1-\rho_j|^3\right)}. \tag{1.10}$$

Theorem 2. *For $N\geqslant 2$, $m_k>0$, and $\bar{m}_k=bm_k$, $b>0$, the functions $z_k(t)$ and $\bar{z}_k(t)$ given by* (1.5) *and* (1.6) *with $\omega^2=M\gamma/N$ and γ given by* (1.9) *are solutions of the $2N$-body problem* (1.7) *and* (1.8) *if and only if $m_1=m_2=\cdots=m_N$ and b is determined uniquely by* (1.10).

Corollary. *For $N=2$ and $a>1$ the functions $z_k(t)$ and $\bar{z}_k(t)$ given by* (1.5) *and* (1.6) *with*

$$\omega^2=M\gamma/N\quad\textit{and}\quad\gamma=\frac{1}{1+b}\left(\frac{1}{4}-\frac{4ab}{(a^2-1)^2}\right)$$

are solutions of the (2×2)-body problem (1.7) *and* (1.8) *if and only if*

$$m_1=m_2\quad\textit{and}\quad b=\frac{a^7-2a^5-8a^4+a^3-8a^2}{17a^4-2a^2+1}.$$

This is a collinear periodic solution for 4*-body problem. When* $0 < a < 1$*,*

$$\gamma = \frac{1}{1+b}\left(\frac{1}{4} + \frac{2b(a^2+1)}{(a^2-1)^2}\right) \quad \textit{and} \quad b = \frac{a^7 - 2a^5 + 17a^3}{-8a^5 + a^4 - 8a^3 - 2a^2 + 1}.$$

2. The proof of the main results

For two nested regular polygons, we have defined

$$\rho_k = \exp(2\pi i k/N), \tag{2.1}$$

$$\bar\rho_k = a\exp(2\pi i k/N), \quad a \neq 1, \tag{2.2}$$

$$z_0 = \sum_j \left(m_j\rho_j + \bar m_j\bar\rho_j\right)/M, \tag{2.3}$$

where

$$M = \sum_j \left(m_j + \bar m_j\right), \tag{2.4}$$

$$z_k(t) = (\rho_k - z_0)\exp(i\omega t), \quad k = 1, \ldots, N, \tag{2.5}$$

and

$$\bar z_k(t) = (a\rho_k - z_0)\exp(i\omega t), \quad k = 1, \ldots, N. \tag{2.6}$$

Proof of Theorem 1. Direct substitution into the differential equations (1.7), (1.8) shows that the $z_k(t)$ and $\bar z_k(t)$ are the solution of (1.7) and (1.8) if and only if

$$(\rho_k - z_0)\omega^2\exp(i\omega t) = \left(\sum_{j\neq k} m_j\frac{\rho_k - \rho_j}{|\rho_k - \rho_j|^3} + \sum_j \bar m_j\frac{\rho_k - \bar\rho_j}{|\rho_k - \bar\rho_j|^3}\right)\exp(i\omega t) \tag{2.7}$$

and

$$\left(\bar\rho_k - z_0\right)\omega^2\exp(i\omega t) = \left(\sum_j m_j\frac{\bar\rho_k - \rho_j}{|\bar\rho_k - \rho_j|^3} + \sum_{j\neq k} \bar m_j\frac{\bar\rho_k - \bar\rho_j}{|\bar\rho_k - \bar\rho_j|^3}\right)\exp(i\omega t), \tag{2.8}$$

or if and only if

$$\sum_{j\neq k} m_j\left(\frac{1}{|\rho_k - \rho_j|^3} - \frac{\omega^2}{M}\right)(\rho_k - \rho_j) + \sum_j \bar m_j\left(\frac{1}{|\rho_k - \bar\rho_j|^3} - \frac{\omega^2}{M}\right)\left(\rho_k - \bar\rho_j\right) = 0 \tag{2.9}$$

and

$$\sum_j m_j\left(\frac{1}{|\bar\rho_k - \rho_j|^3} - \frac{\omega^2}{M}\right)\left(\bar\rho_k - \rho_j\right) + \sum_{j\neq k} \bar m_j\left(\frac{1}{|\bar\rho_k - \bar\rho_j|^3} - \frac{\omega^2}{M}\right)\left(\bar\rho_k - \bar\rho_j\right) = 0. \tag{2.10}$$

Multiplying both sides by ρ_{N-k} and noting that $|\rho_k - \rho_j| = |\rho_k||1 - \rho_{j-k}| = |1 - \rho_{j-k}|$ and using $\bar\rho_k = a\rho_k$,

$$\sum_{j\neq k} m_j\left(\frac{1}{|1 - \rho_{j-k}|^3} - \frac{\omega^2}{M}\right)(1 - \rho_{j-k}) + \sum_j \bar m_j\left(\frac{1}{|1 - a\rho_{j-k}|^3} - \frac{\omega^2}{M}\right)(1 - a\rho_{j-k}) = 0 \tag{2.11}$$

and

$$\sum_j m_j\left(\frac{1}{|a-\rho_{j-k}|^3}-\frac{\omega^2}{M}\right)(a-\rho_{j-k})+\sum_{j\neq k}\bar{m}_j\left(\frac{1}{|a-a\rho_{j-k}|^3}-\frac{\omega^2}{M}\right)(a-a\rho_{j-k})=0. \tag{2.12}$$

Notice that every step from (2.7)–(2.12) can be conversed respectively. Now we define the $N\times N$ circulant matrix [3] $C=[c_{k,j}]$, $A=[a_{k,j}]$, $B=[b_{k,j}]$, $D=[d_{k,j}]$ as follows:

$$c_{k,j}=0,\quad \text{for } k=j,\qquad c_{k,j}=\left(\frac{1}{|1-\rho_{j-k}|^3}-\frac{\omega^2}{M}\right)(1-\rho_{j-k}),\quad \text{for } k\neq j, \tag{2.13}$$

$$a_{k,j}=\left(\frac{1}{|1-a\rho_{j-k}|^3}-\frac{\omega^2}{M}\right)(1-a\rho_{j-k}), \tag{2.14}$$

$$b_{k,j}=\left(\frac{1}{|a-\rho_{j-k}|^3}-\frac{\omega^2}{M}\right)(a-\rho_{j-k}), \tag{2.15}$$

$$d_{k,j}=0,\quad \text{for } k=j,\qquad d_{k,j}=\left(\frac{1}{|a-a\rho_{j-k}|^3}-\frac{\omega^2}{M}\right)(a-a\rho_{j-k}),\quad \text{for } k\neq j. \tag{2.16}$$

Then (2.11) and (2.12) hold if and only if the matrix equation

$$\begin{bmatrix} C & A\\ B & D\end{bmatrix}\begin{pmatrix} m_1\\ \vdots\\ m_N\\ \bar{m}_1\\ \vdots\\ \bar{m}_N\end{pmatrix}=0 \tag{2.17}$$

has a positive solution.

When $\bar{m}_j=bm_j$, $b>0$, (2.17) is equivalent to

$$\begin{bmatrix} C & bA\\ B & bD\end{bmatrix}\begin{pmatrix} m_1\\ \vdots\\ m_N\\ m_1\\ \vdots\\ m_N\end{pmatrix}=0 \tag{2.18}$$

$$\Leftrightarrow\quad \begin{bmatrix} C+bA\\ B+bD\end{bmatrix}\begin{pmatrix} m_1\\ \vdots\\ m_N\end{pmatrix}=0 \tag{2.19}$$

$$\Leftrightarrow\quad (C+bA)\begin{pmatrix} m_1\\ \vdots\\ m_N\end{pmatrix}=0 \tag{2.20}$$

and

$$(B+bD)\begin{pmatrix} m_1\\ \vdots\\ m_N\end{pmatrix}=0. \tag{2.21}$$

We notice that Eqs. (2.11) and (2.12) are equivalent to (2.20) and (2.21). According to (2.11), (2.12), (2.20) and (2.21), similar to the proof of Theorem 1 in [6], we have

$$\left(\sum_k m_k\right)\left[\sum_{j\neq N}\left(\frac{1}{|1-\rho_j|^3}-\frac{\omega^2}{M}\right)(1-\rho_j)+\sum_j b\left(\frac{1}{|1-a\rho_j|^3}-\frac{\omega^2}{M}\right)(1-a\rho_j)\right]=0 \tag{2.22}$$

$$\Leftrightarrow\quad \frac{\omega^2}{M}\sum_j\left[(1-\rho_j)+b(1-a\rho_j)\right]=\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3}+\sum_j\frac{b(1-a\rho_j)}{|1-a\rho_j|^3} \tag{2.23}$$

and

$$\left(\sum_k m_k\right)\left[\sum_j\left(\frac{1}{|a-\rho_j|^3}-\frac{\omega^2}{M}\right)(a-\rho_j)+\sum_{j\neq N} b\left(\frac{1}{|a-a\rho_j|^3}-\frac{\omega^2}{M}\right)(a-a\rho_j)\right]=0 \tag{2.24}$$

$$\Leftrightarrow\quad \frac{\omega^2}{M}\sum_j\left[(a-\rho_j)+b(a-a\rho_j)\right]=\sum_j\frac{a-\rho_j}{|a-\rho_j|^3}+\sum_{j\neq N}\frac{b(a-a\rho_j)}{|a-a\rho_j|^3}. \tag{2.25}$$

We note that

$$\sum_j(1-\rho_j)=N, \tag{2.26}$$

$$\sum_j b(1-a\rho_j)=Nb, \tag{2.27}$$

$$\sum_j(a-\rho_j)=Na, \tag{2.28}$$

$$\sum_j b(a-a\rho_j)=Nba, \tag{2.29}$$

then (2.23) and (2.25) are equivalent to the following equations, respectively:

$$\frac{\omega^2}{M}=\frac{1}{N(1+b)}\left[\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3}+\sum_j\frac{b(1-a\rho_j)}{|1-a\rho_j|^3}\right], \tag{2.30}$$

$$\frac{\omega^2}{M}=\frac{1}{Na(1+b)}\left[\sum_j\frac{a-\rho_j}{|a-\rho_j|^3}+\sum_{j\neq N}\frac{b(a-a\rho_j)}{|a-a\rho_j|^3}\right]. \tag{2.31}$$

So γ is determined by (1.9) and b is determined uniquely by (1.10). The proof is completed. □

Proof of Theorem 2.

(1) Proof of the necessary conditions of Theorem 2: The proof hinges on showing that certain eigenvalues of the circulant matrix $C+bA$ are zero. This is accomplished by using the general formulas (2.32), (2.33) for the eigenvalues λ_k and the eigenvectors $\vec{v}_k$ of a circulant matrix $C+bA=[c_{k,j}+ba_{k,j}]$:

$$\lambda_k=\sum_j(c_{1,j}+ba_{1,j})\rho_{k-1}^{j-1}, \tag{2.32}$$

$$\vec{v}_k=\left(\rho_{k-1},\rho_{k-1}^2,\ldots,\rho_{k-1}^N\right)^{\mathrm{T}}. \tag{2.33}$$

Since

$$(c_{1,1}+ba_{1,1})=b\left(\frac{1}{|1-a|^3}-\frac{\omega^2}{M}\right)(1-a)\neq 0,$$

so $C + bA$ is not a zero matrix. According to the relations between the eigenvalues λ_k and the eigenvectors $\vec{v}_k$, one has

$$(C + bA)\vec{v}_k = \lambda_k \vec{v}_k, \quad 1 \leqslant k \leqslant N.$$

Then, in order to find the solution of (2.20), it is enough to find the zero eigenvalue λ_k with the positive real eigenvectors $\vec{v}_k$.

(i) N is an odd number. Only $\vec{v}_1 = (1, 1, \dots, 1)^{\mathrm{T}}$ is the positive real eigenvector. At the same time

$$\lambda_1 = \sum_j (c_{1,j} + ba_{1,j})\rho_{1-1}^{j-1} = \sum_j (c_{1,j} + ba_{1,j}) = 0$$

(see (2.22) in the proof of Theorem 1). That is, $(C + bA)\vec{v}_1 = 0$ and $\vec{m} = (m_1, m_1, \dots, m_1)^{\mathrm{T}}$, $m_1 > 0$, is the unique solution of (2.20).

(ii) N is an even number. Only $\vec{v}_1 = (1, 1, \dots, 1)^{\mathrm{T}}$ and $\vec{v}_{N/2+1} = (-1, 1, \dots, -1, 1)^{\mathrm{T}}$ are real eigenvector, but only $\vec{v}_1$ is a positive real eigenvector. The corresponding eigenvalue is

$$\lambda_1 = \sum_j (c_{1,j} + ba_{1,j})\rho_{1-1}^{j-1} = \sum_j (c_{1,j} + ba_{1,j}) = 0$$

(see (2.22) in the proof of Theorem 1). That is, $(C + bA)\vec{v}_1 = 0$ and $\vec{m} = (m_1, m_1, \dots, m_1)^{\mathrm{T}}$, $m_1 > 0$, is the unique solution of (2.20).

Similar to the above proof, the matrix equation (2.21) have the same result as the matrix equation (2.20).

(2) Proof of sufficient conditions of Theorem 2: Assume $m_1 = m_2 = \cdots = m_N > 0$, γ is determined by (1.9) and a, b are determined by (1.10), then $(m_1, \dots, m_1)^{\mathrm{T}}$ is a solution of (2.21) and (2.22) or (1.7) and (1.8). Since

$$\begin{aligned}
&\sum_{j \neq k} m_j \left(\frac{1}{|1 - \rho_{j-k}|^3} - \frac{\omega^2}{M} \right)(1 - \rho_{j-k}) + \sum_j b m_j \left(\frac{1}{|1 - a\rho_{j-k}|^3} - \frac{\omega^2}{M} \right)(1 - a\rho_{j-k}) \\
&\quad = m_1 \left[\sum_{j \neq k} \left(\frac{1}{|1 - \rho_{j-k}|^3} - \frac{\omega^2}{M} \right)(1 - \rho_{j-k}) + \sum_j b \left(\frac{1}{|1 - a\rho_{j-k}|^3} - \frac{\omega^2}{M} \right)(1 - a\rho_{j-k}) \right] \\
&\quad = m_1 \left[\sum_{j \neq N} \left(\frac{1}{|1 - \rho_j|^3} - \frac{\omega^2}{M} \right)(1 - \rho_j) + \sum_j b \left(\frac{1}{|1 - a\rho_j|^3} - \frac{\omega^2}{M} \right)(1 - a\rho_j) \right] = 0.
\end{aligned} \tag{2.34}$$

Similar to the above proof, the same results for (2.22) are easily obtained. Thus the proof is completed. □

The corollary is obvious from Theorem 2.

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