BI-PST/24-25 Homework

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Abstract

With this document we present homework for BI-PST. Oleksandr Slyvka was chosen as a represntative for our group. He was born 14.04.2006, so we get $K = 14$ and $L = 6$, then $M = 4$. Such value of M corresponds to case0202 Sleuth dataset, volume of hippocampus with respect to schizofrenia. We will analyse this dataset using numpy, pandas, matplotlib and scipy.stats Python modules.

1 Data and estimating moments

Firstly, we will load data and display it as a table. There are $n = 15$ samples in both categories, values seem to be greater than 1cm^3 and peak arounf 2cm^3, each row repesent a pair of twins, one of whom was affected by schizophrenia.

"Unaffect"	"Affected"
1.94000005722046	1.26999998092651
1.44000005722046	1.62999999523163
1.55999994277954	1.47000002861023
1.58000004291534	1.38999998569489
2.05999994277954	1.92999994754791
1.6599999666214	1.25999999046326
1.75	1.71000003814697
1.76999998092651	1.66999995708466
1.77999997138977	1.27999997138977
1.91999995708466	1.85000002384186
1.25	1.01999998092651
1.92999994754791	1.3400000333786
2.03999996185303	2.01999998092651
1.62000000476837	1.5900000333786
2.07999992370605	1.97000002861023

1.1 Estimations

Let X_{unaff} and X_{aff} denote random variables of hippocampues volume of those who are unaffected and affected by schizofrenia respectively. Then we will compute estimates of mean, median and variance of those random variables. Median estimation will be chosen as 7th value of sorted sequence of data points. Estimated mean and variance will be computed with following formulas:

$$\widehat{\mathbb{E}X} = m_1 = \bar{X} = \frac{1}{n}\sum_{k=1}^{n} X_i$$

$$\widehat{\text{var}X} = s^2 = \frac{1}{n-1}\sum_{k=1}^{n}(X_i - \bar{X})^2$$

Additionaly we will compute uncorrected variance.

$$S^2 = \frac{1}{n}\sum_{k=1}^{n}(X_i - \bar{X})^2$$

All those computations are incapsulated in function estimate.

```
def estimate(X):
    print(f'{'Mean:' :<30}{X.mean():.3f}')
    print(f'{'Median:' :<30}{X.median():.3f}')
    print(f'{'Variance without correction:' :<30}{((X - X.mean()) ** 2).sum() / n :.3f}')
    print(f'{'Variance with correction:' :<30}{((X - X.mean()) ** 2).sum() / (n - 1):.3f}')
```

After plugging values in we obtain such results.

$$\widehat{\mathbb{E}X_{\text{unaff}}} = 1.759 \qquad \widehat{\mathbb{E}X_{\text{aff}}} = 1.560$$

$$\widehat{F^{-1}_{X_{\text{unaff}}}(0.5)} = 1.770 \qquad \widehat{F^{-1}_{X_{\text{aff}}}(0.5)} = 1.590$$

$$\widehat{\text{var}X_{\text{unaff}}} = 0.059 \qquad \widehat{\text{var}X_{\text{aff}}} = 0.091$$

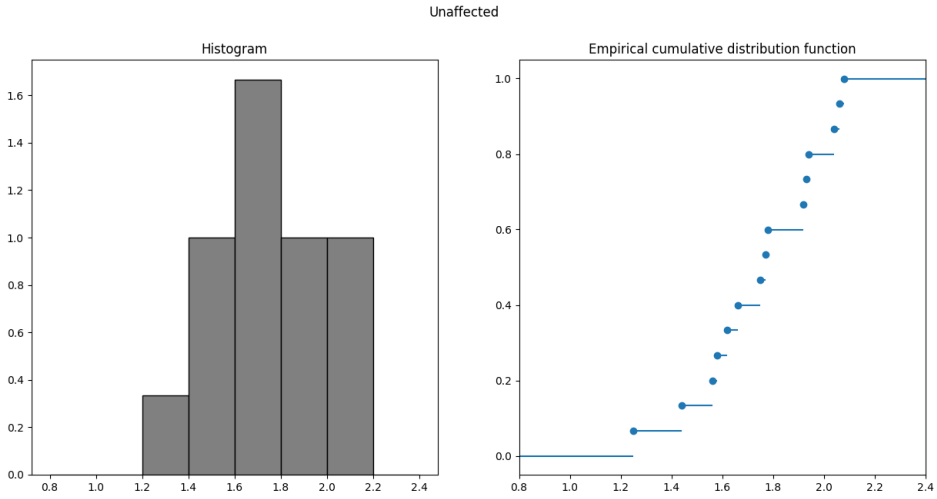
$$S^2_{\text{unaff}} = 0.055 \qquad S^2_{\text{aff}} = 0.085$$

Let's notice that estimated expected value of unaffected distribution is greater than estimated expected value of affected distribution.

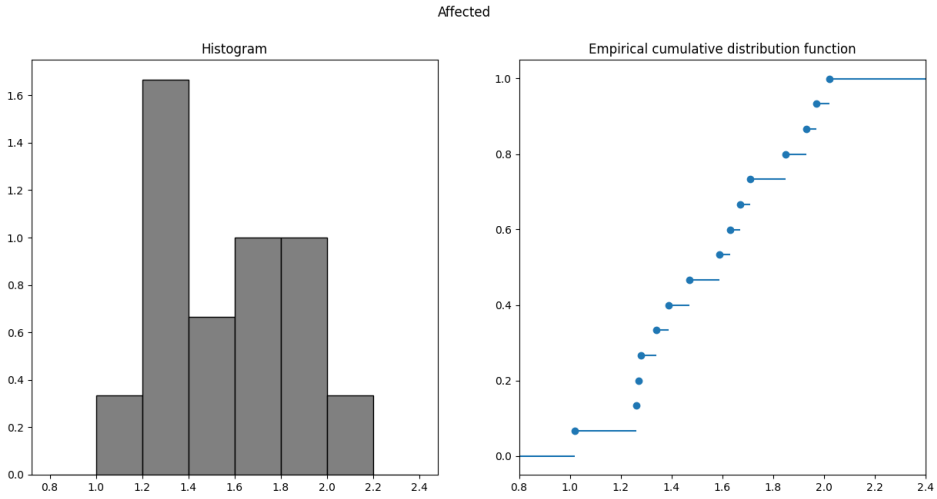
2 Histograms and empirical cumulitive distribution functions

We will plot histograms, bins' width will be 0.2cm^3. Near them we will plot empirical cumuitive distribution functions, that will have a step of $1/n$ at each sample value.

Let's plot histogram and ecdf for distribution of unaffected twins.



Here is the same plot for distibution of schizophrenic twins.



We can see that later histogram is a little wider, that corresponds with its variance being greater than variance of unaffected distributions. Another observation is that it is altogether slightly shifted towards zero.

3 Choosing between normal, uniform and exponential distributions

We will try to model our data as one of the distributions listed above. To do that we need to find parameters of distributions, we will calculate them using method of moments.

3.1 Calculating parameters

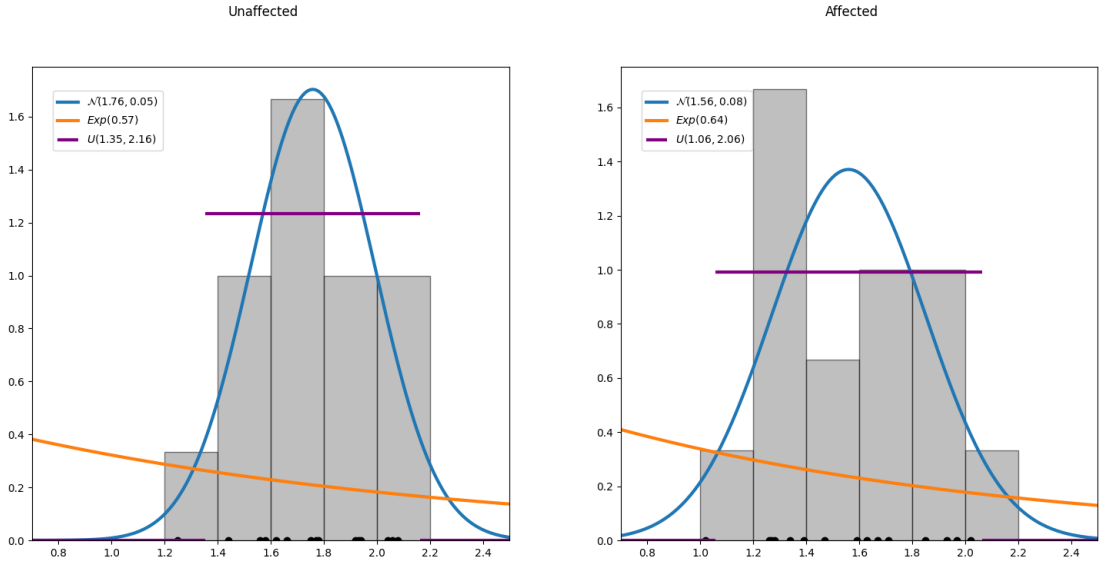
Normal distribution is parametrized by μ and σ^2, uniform distribution is parametrized by its lower and upper bounds resp. a and b and exponential one is parametrized by the rate λ. All of them can be expressed as functions of at least two moments (exponential can be derived using single moment).

$$\begin{aligned}
\hat{\mu} &= m_1 \\
\hat{\sigma}^2 &= m_2 - m_1^2 \\
\hat{a} &= m_1 - \sqrt{3(m_2 - m_1^2)} \\
\hat{b} &= m_1 + \sqrt{3(m_2 - m_1^2)} \\
\hat{\lambda} &= \frac{1}{m_1}
\end{aligned}$$

Those computations are calculated in this code:

```
m1 = X.mean()
m2 = (X ** 2).mean()
mu_hat= m1
sigma2_hat= m2 - m1 ** 2
lambda_hat = 1 / m1
a_hat = m1 - np.sqrt(3 * (m2 - m1**2))
b_hat = m1 + np.sqrt(3 * (m2 - m1**2))
print(f"Normal: {mu_hat=}, {sigma2_hat=}")
print(f"Exp: {lambda_hat=}")
print(f"Uniform: {a_hat=}, {b_hat=}")
```

3.2 Comparing histogram and probability distribution functions



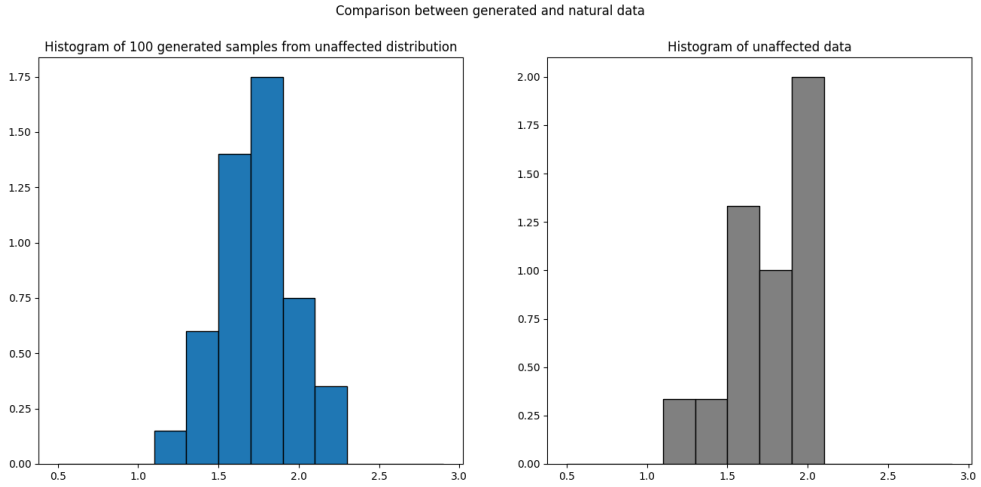
In both cases exponential distribution performs horribly, so we will reject that immideatly. At both plots there are data points, that lay in zero-probability area for uniform distribution, so it will be rejected too. We are left only with normal distribution, so we naturally choose it, additionaly it matches histogram peaks quite nicely.

4 Comparing histograms of 100 generated samples and data

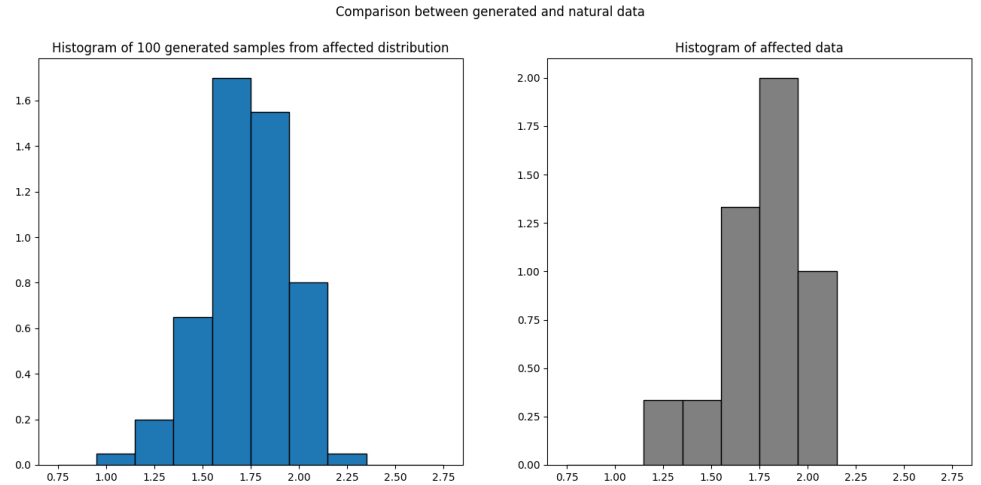
We have chosen parameters $\hat{\mu} = 1.76$ and $\hat{\sigma} = 0.05$ for distribution of those, who were not affected, and $\hat{\mu} = 1.56$ and $\hat{\sigma} = 0.08$ for those who were. We will generate a 100 random samples using following code:

```
samples = stats.norm.rvs(
    loc=mu_hat, scale=np.sqrt(sigma2_hat]),
    size=100, random_state=RND_SEED
)
```

Now let's plot histograms and compare them.



We can see that artificial one is more gravited towards mean, while given one is skewed. Besides that they share number of similarities, their ranges match almost perfectly and they both have similar probabilities for peak and tail values.



This histograms follows shape of data really closely, though in this case it tends to be more spread out, with it peak sitting 0.4 points lower and samples reaching further.

5 Computing two-sided confidence intervals for expected value at 95% level

We do not have theoretical variance, so we will need to use t distribution to derive our intervals. From lectures we know that they will be of them form $\text{CI} = \left(\bar{X}_n - \frac{t_{n-1}(\frac{\alpha}{2})\cdot s}{\sqrt{n}}, \bar{X}_n + \frac{t_{n-1}(\frac{\alpha}{2})\cdot s}{\sqrt{n}}\right)$, we calulate them with this Python code:

```
def compute_interval(X, alpha=0.05, n=n):
    m = X.mean()
    s = np.sqrt(((X - X.mean()) ** 2).sum() / (n - 1))
    t_alpha_2 = stats.t.isf(alpha/2, df=n-1)
    L = m - t_alpha_2 * s / np.sqrt(n)
    R = m + t_alpha_2 * s / np.sqrt(n)
    return L, R
```

As a result we obtain these confidence intervals at 95% level for X_{unaff} and X_{aff}:

$$\text{CI}_{\text{unaff}} = (1.6244165793298755, 1.89291672157853)$$
$$\text{CI}_{\text{aff}} = (1.3931681784763261, 1.7268318183447595)$$

6 Testing two-sided hypothesis that expected values equal K at 5% level

Let's formulate hypothesis for unaffected distribution. Null hypothesis will be that expected value of X_{unaff} equals $K = 14$, hypothesis for affected distribution can be formulated analogically.

$$H_0 : \mathbb{E}X_{\text{unaff}} = K$$
$$H_A : \mathbb{E}X_{\text{unaff}} \neq K$$

To test this hypothesis we would need two-sided confidence intervals for mean. Luckily they were already computed, so we can reject null hypothesis for both affected and unaffected cases, because $K = 14 \notin (1.39, 1.72)$ and $K = 14 \notin= (1.39, 1.72)$, rounding was made for readability purposes.

7 Testing if means of distributions of volumes affected and unaffected people equal at 5% level

It is clear that null hypothesis is that expected values are equal. We choose left-sided alternative hypothesis, because estimated mean of affected distribution is lesser than mean of unaffected. We have no tests for comparing two different random variables, so we will need to transform our hypothesis in terms of difference of random variables. Let $Y = X_{\text{aff}} - X_{\text{unaff}}$ be such difference, then we can write hypothesis as follows:

$$H_0 : \mathbb{E}X_{\text{aff}} = \mathbb{E}X_{\text{unaff}} \iff \mathbb{E}Y = 0$$
$$H_A : \mathbb{E}X_{\text{unaff}} < \mathbb{E}X_{\text{unaff}} \iff \mathbb{E}Y < 0$$

To test this hypothesis we need to compute left confidence interval for Y. We do not know variance, so we will use formula with t-distribtion $CI_L = (-\infty, \bar{Y} - \frac{t_{n-1}(0.95)*s}{\sqrt{n}})$. We will use almost the same code, that we have used for computing two-sided intervals.

```
X = aff - una
m = X.mean()
s = np.sqrt((X ** 2).sum() / (n - 1))
alpha = 0.05

R = m + stats.t.isf(alpha, df=n-1) * s / np.sqrt(n)
print(f"Left CI is: (-inf, {R})")
```

After plugging values in we obtain left confidence interval $(-\infty, -0.055525533052551174)$. Zero does not belong to it, so we reject null hypothesis.

8 Conclusion

We have conducted analysis of Sleuth case0202 dataset, derived estimates for moments and tried to model both random variables as normal distribution. Lastly, we computed confidence intervals for them and tested if their expected values are equal by one-sided t-test.

References

[1] BI-PST 24/25, lectures