Data-enabled iterative learning control:
A zero-sum game design for time-scale-varying tasks

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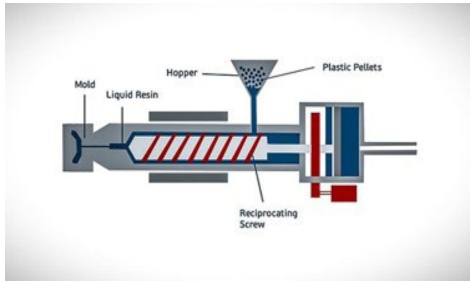
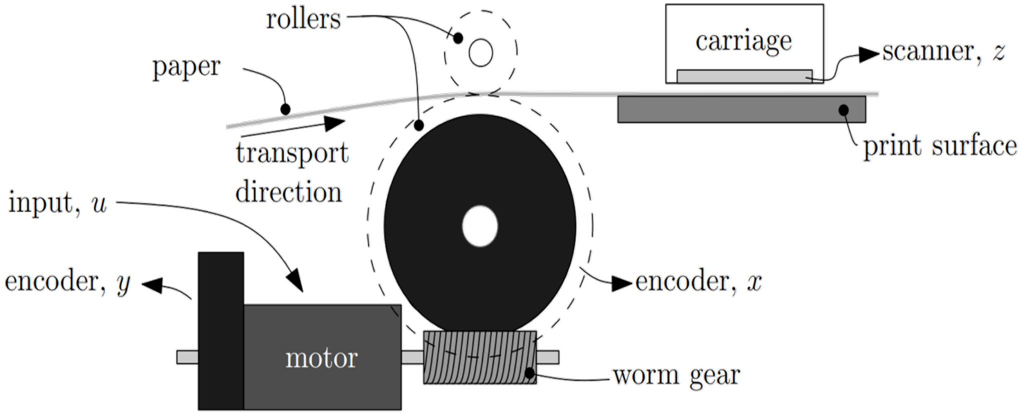
Research background & Motivation

Repetitive systems



Repetitive tracking tasks

Varying tasks in ILC
(iterative learning control)



Adaptability Flexibility VS. **High precision Low cost**

ILC for varying tasks

Goal: Balance between task flexibility and high precision & low cost

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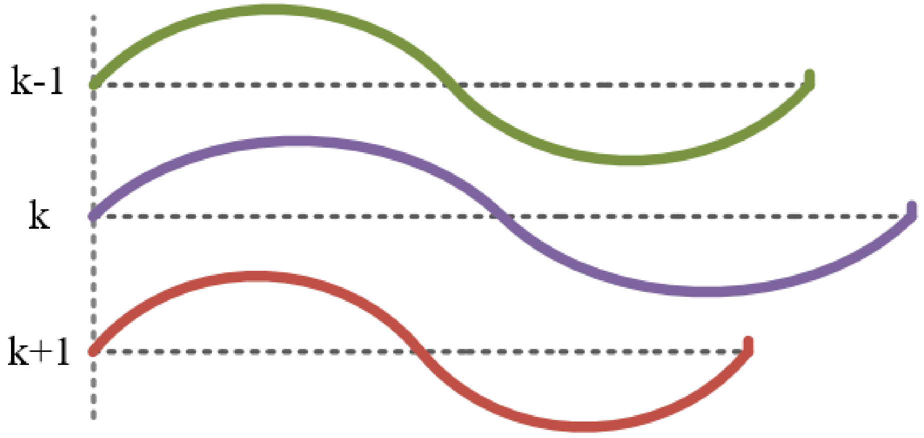

Task definition

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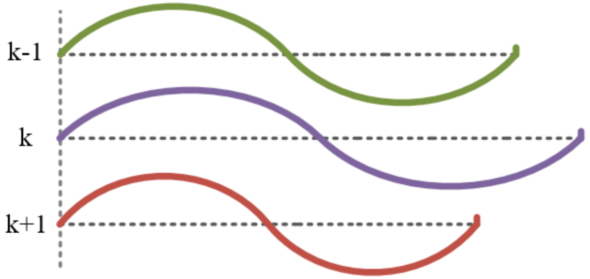
Time-scale-varying tasks

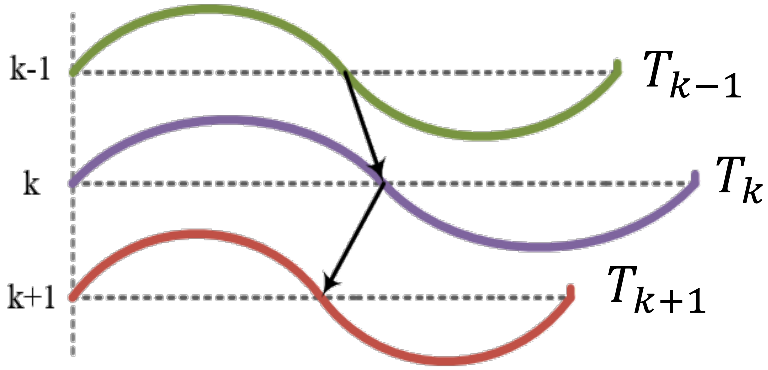
same "shape"

Different time scales

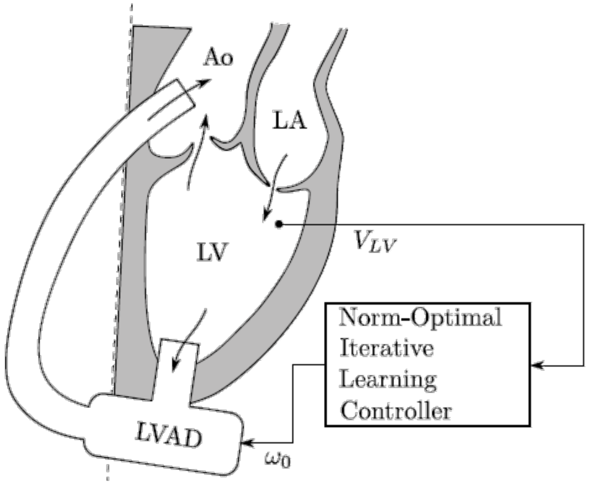


Definition

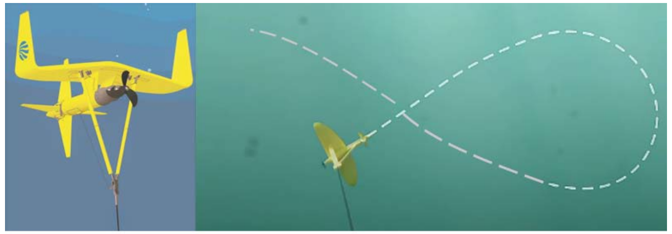
1. **Bijection** between different time scales
2. **Unique** mapping



Examples



Left ventricular assist device (LVAD)



Marine hydrokinetic energy system

M. Ketelhut et al. 2019
Control Engineering Practice.

M. Cobb et al. 2022
IEEE Transactions on Control Systems Technology.

- **Library-based learning control,**

J. Xu 1998, *IEEE Transactions on Automatic Control.*

- **ILC with basis function,**

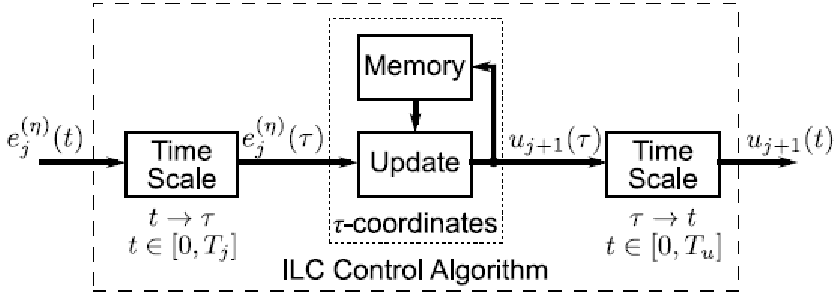
Bolder &Oomen 2015, *IEEE Transactions on Control Systems Technology.*

- **Path optimization ILC,**

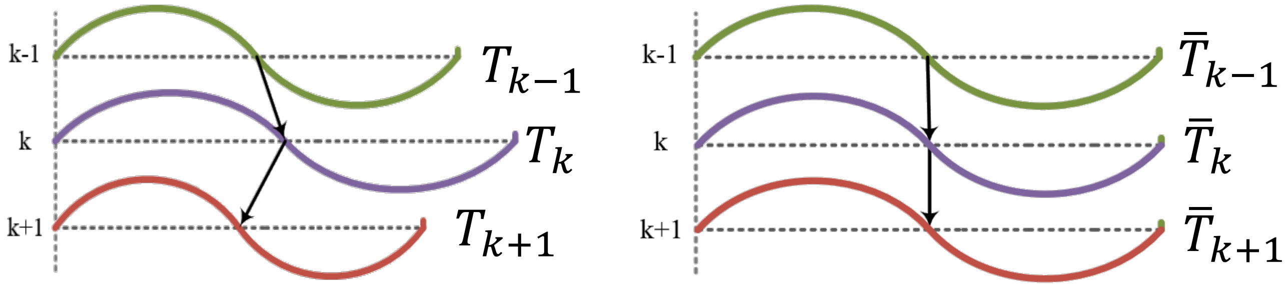
Cobb et al. 2021, *IEEE Transactions on Control Systems Technology.*

Failure of corresponding learning

Normalization



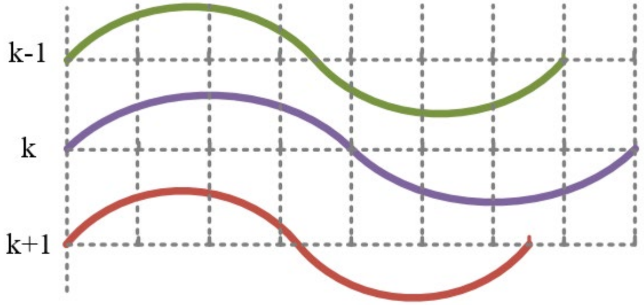
M. Cobb et al. 2022
IEEE Transactions on Control Systems Technology.

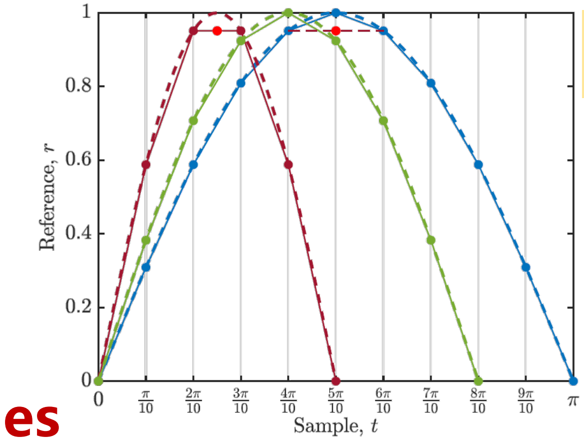


Bijection in continuous-time domain

Discrete-time ILC

with sampling behavior?





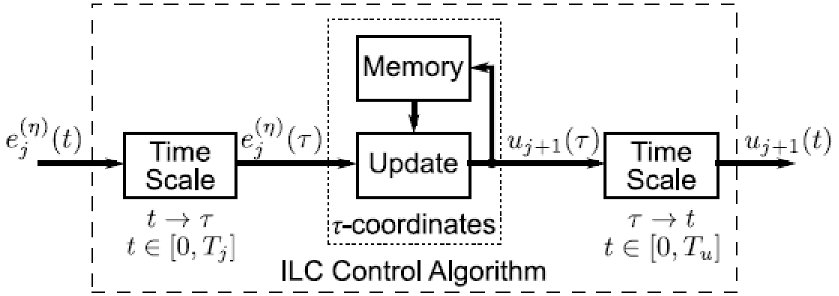
Sine example

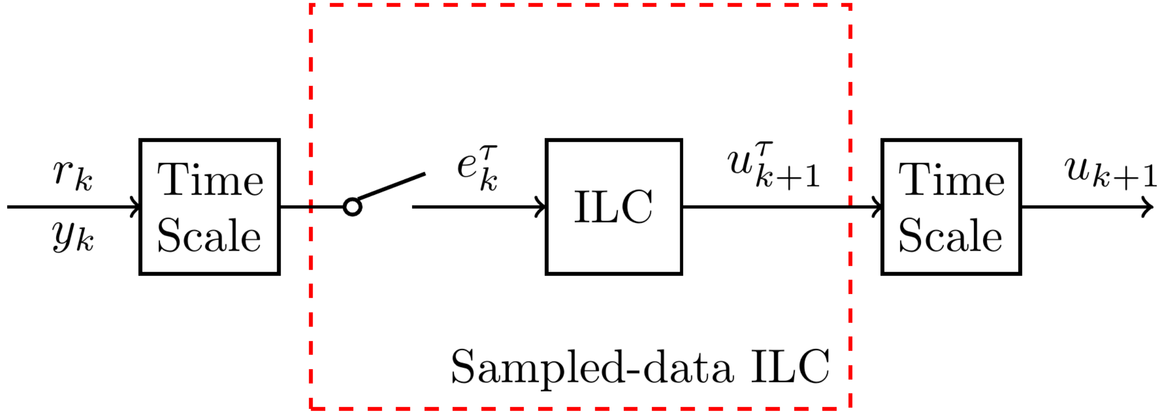
Issues

1. **Varying set points at a sampling instant**
2. **Missing information for learning**

Time-scale transformation scheme

Modified scheme



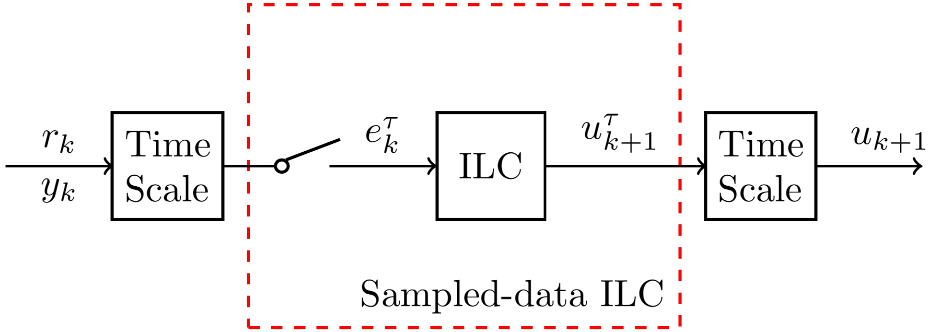


$[0, T_k] \longrightarrow [0,1]$

Continuous-time reference ⟷ **Discrete-time ILC input**

$[0,1] \longrightarrow [0, N_k]$

ILC problem



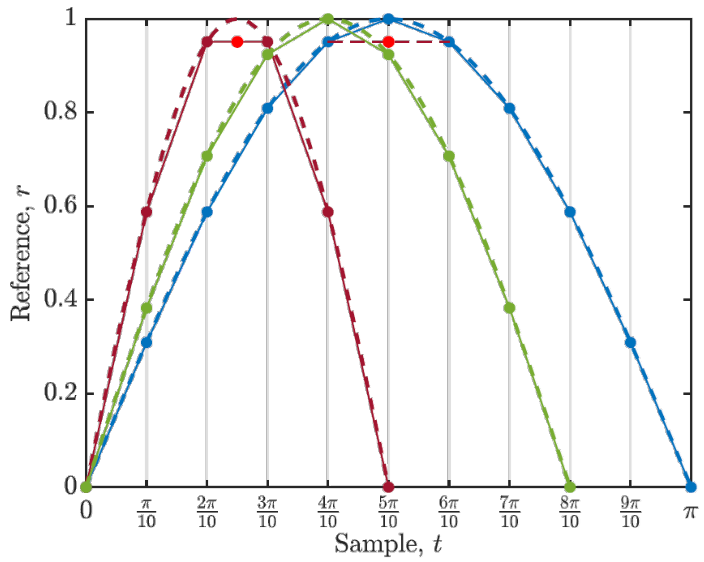
Sampled-data ILC problem

$$u_{k+1} = \arg\min_{u} J_{k+1}^S(u, r^c, y_{k+1}^c)$$

suitable sampling time

Discrete-time ILC problem

$$u_{k+1}^\tau = \arg\min_{u^\tau} J_{k+1}(u^\tau, e^\tau)$$



ILC problem:

1. define a suitable **cost function**
2. solve the **optimization** problem
3. Ensure **trial convergence**
4. Handle **trial-varying disturbances**

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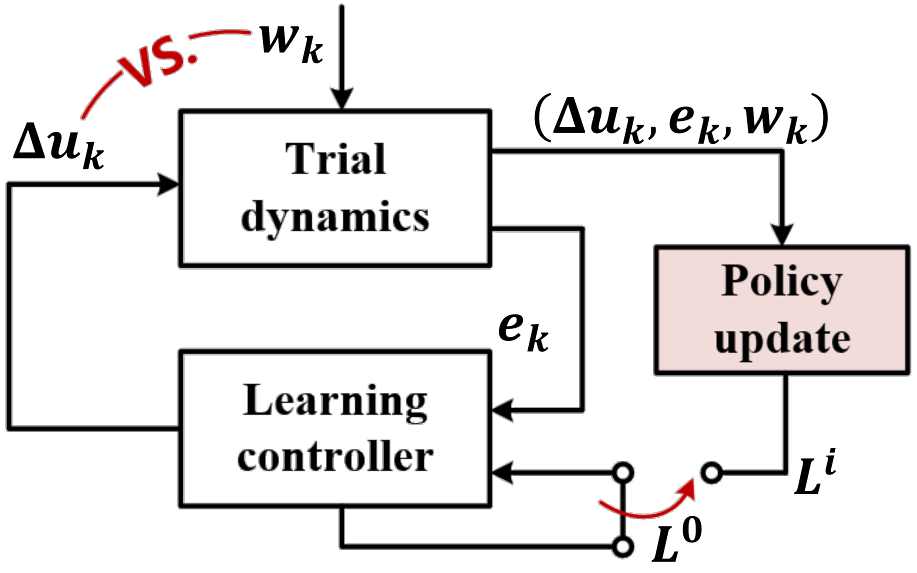

VS.
w_k
Δu_k
$(\Delta u_k, e_k, w_k)$
Trial dynamics
Policy update
e_k
Learning controller
L^i
L^0

System dynamics

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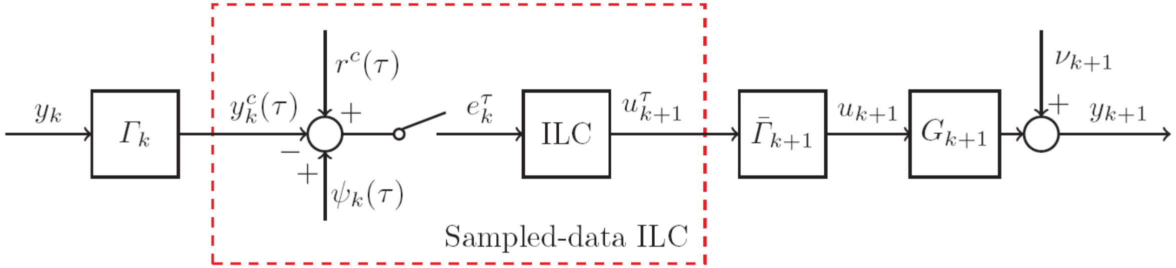
Data-driven representation

$$y_k = G_k u_k + v_k$$

y_k: Output data sequence
u_k: Input data sequence
G_k: Data transfer matrix
v_k: External disturbance

+

Modified time-scale transformation scheme



Trial error dynamics

with trial-varying disturbances

$$e_{k+1} = e_k - G\Delta u_{k+1} + \boxed{w_{k+1}}$$

1. **Process disturbance ψ_k from varying time scales**
2. **External unknown noise v_k**

Finite energy assumption:

$$\sum_{k=0}^{\infty} w_k^\top w_k < \infty$$

Issue

Why trial dynamics?

Zero-sum game design

Cost design via “Learning from the delayed rewards”

$$J_{k+1}(\Delta u_{k+1}) = \sum_{j=k}^{\infty} \rho^{j-k}\left(e_{j+1}^{\top} Q e_{j+1} + \Delta u_{j+1}^{\top} R \Delta u_{j+1} - \gamma^2 w_{j+1}^{\top} w_{j+1}\right)$$

Discounted factor

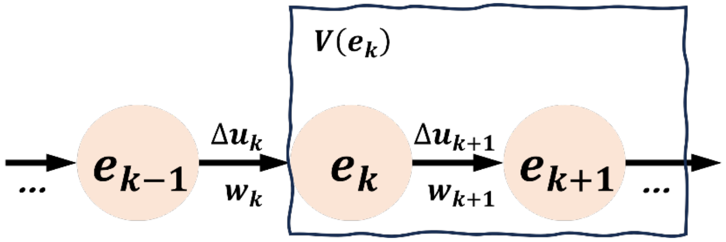
Disturbance attenuation goal:

$$\sum_{k=0}^{\infty}\left(e_k^{\top} Q e_k + \Delta u_k^{\top} R \Delta u_k\right) \leq \gamma^2 \sum_{k=0}^{\infty} w_k^{\top} w_k$$

Reinforcement learning perspective

$$J_{k+1}(\Delta u_{k+1}) \Rightarrow V(e_k) = \underbrace{c(e_k, \Delta u_{k+1}, w_{k+1})}_{\text{current state}} + \underbrace{\rho V(e_{k+1})}_{\text{future cost}}$$

Bellman equation



Zero-sum game design

$$e_{k+1} = e_k - G\Delta u_{k+1} + w_{k+1}$$

Player 1 Δu_k

Minimize $V(e_k)$

Player 2 w_k

Maximize $V(e_k)$

Min-max optimization problem

$$\min_{\Delta u_{k+1}} \max_{w_{k+1}} V(e_k)$$

$$\text{s.t. } e_{k+1} = e_k - G\Delta u_{k+1} + w_{k+1}$$

Model-based solution

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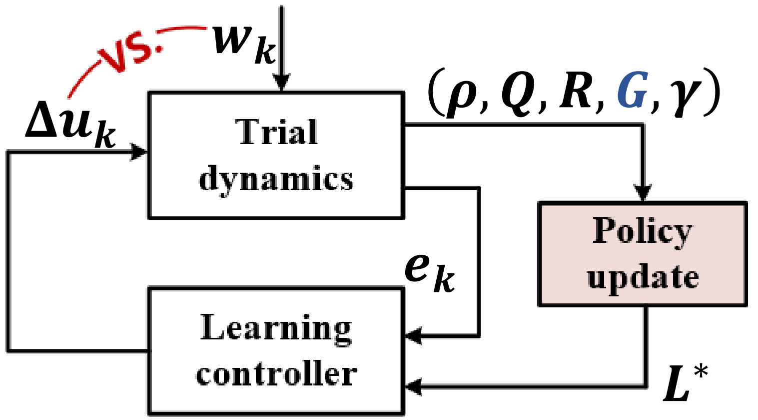
If known G

$$V(e_k) = c(e_k, \Delta u_{k+1}, w_{k+1}) + \rho V(e_{k+1})$$



$$\Delta u_{j+1} = L_u^* e_k$$
$$w_{j+1} = L_w^* e_k$$

$$L_u^* = \left[R + G^\top PG + G^\top P\left(\gamma^2 I - P\right)^{-1} PG\right]^{-1} \cdot \left[G^\top P + G^\top P\left(\gamma^2 I - P\right)^{-1} P\right]$$

$$L_w^* = \left[\gamma^2 I - P + PG(R + G^\top PG)^{-1} G^\top P\right]^{-1} \cdot \left[P - PG(R + G^\top PG)^{-1} G^\top P\right]$$

Saddle point

$$\min_{\Delta u_{k+1}} \max_{w_{k+1}} V(e_k)$$
$$s.t.\ e_{k+1} = e_k - G\Delta u_{k+1} + w_{k+1}$$

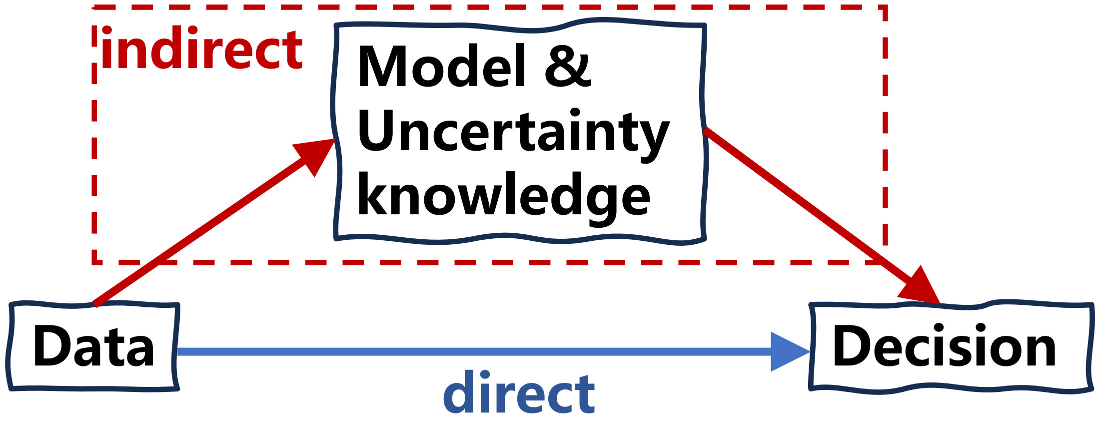
Discounted game algebraic Riccati equation (DGARE)

$$P = \rho P + Q - \rho\begin{bmatrix} -PG & P \end{bmatrix} \begin{bmatrix} R + G^\top PG & -G^\top P \\ -PG & P - \gamma^2 I \end{bmatrix}^{-1} \begin{bmatrix} -G^\top P \\ P \end{bmatrix}$$

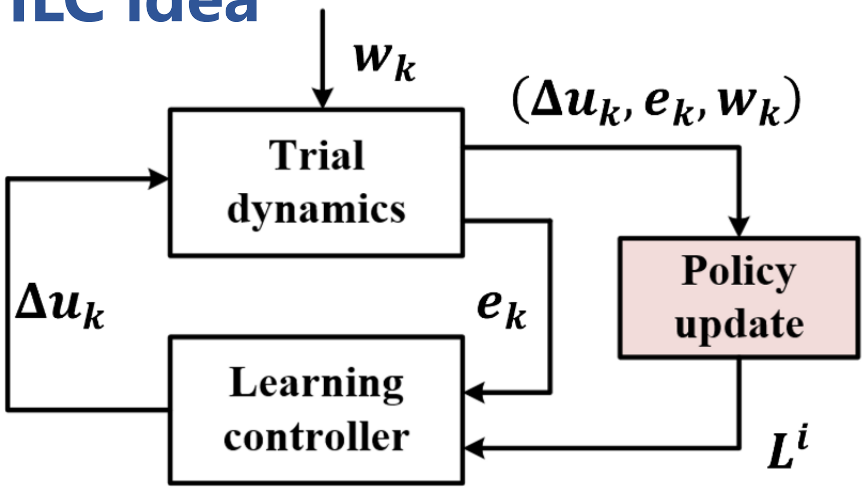
Robust optimal policy

Data-based solution

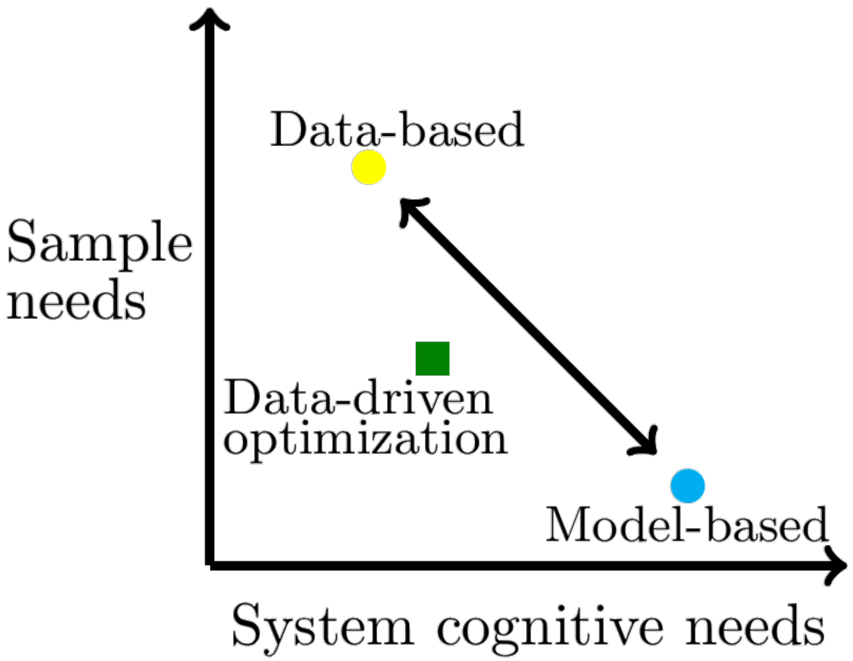
Data-driven control



Standard ILC idea



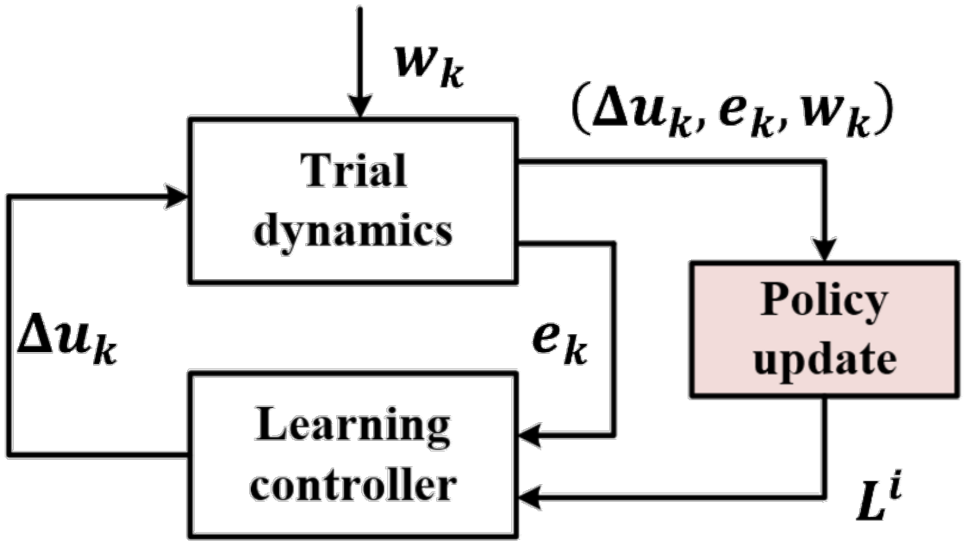
Purpose

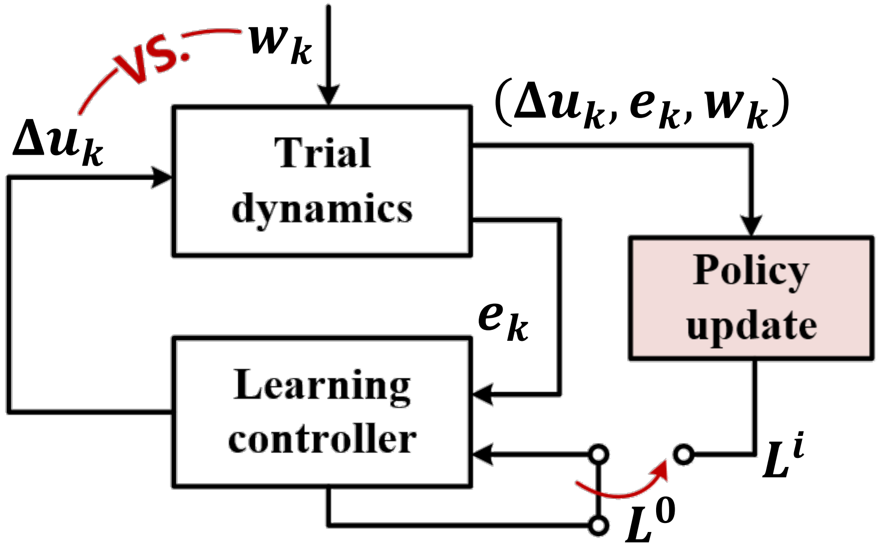


1. **Sample efficiency improvement**
2. **Disturbance attenuation**

Off-policy ILC

On-policy vs. Off-policy

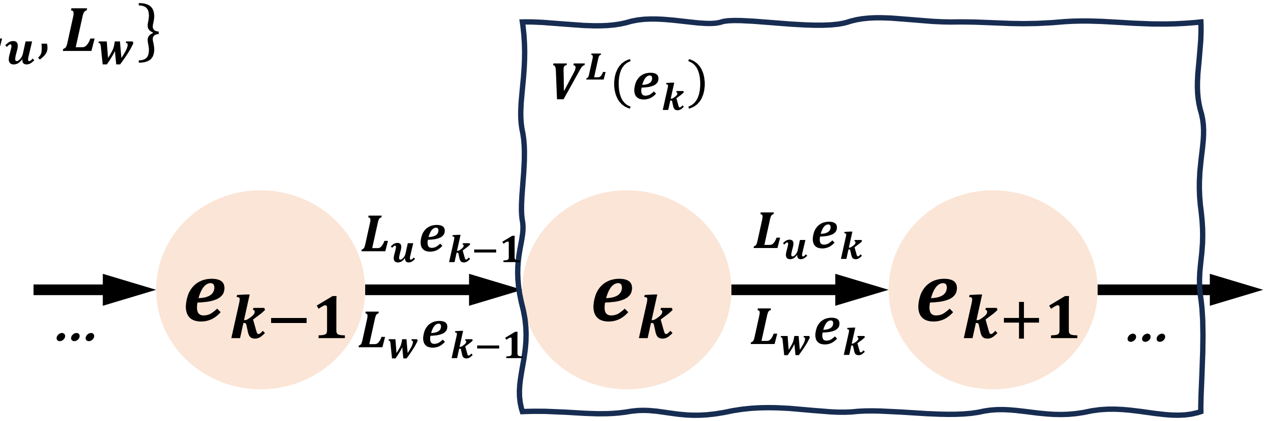




Data-based solution

Value function if given policy $L = \{L_u, L_w\}$

$$V^L(e_k) = \sum_{j=k}^{\infty} \rho^{j-k} c(e_j, L_u e_j, L_w e_j)$$



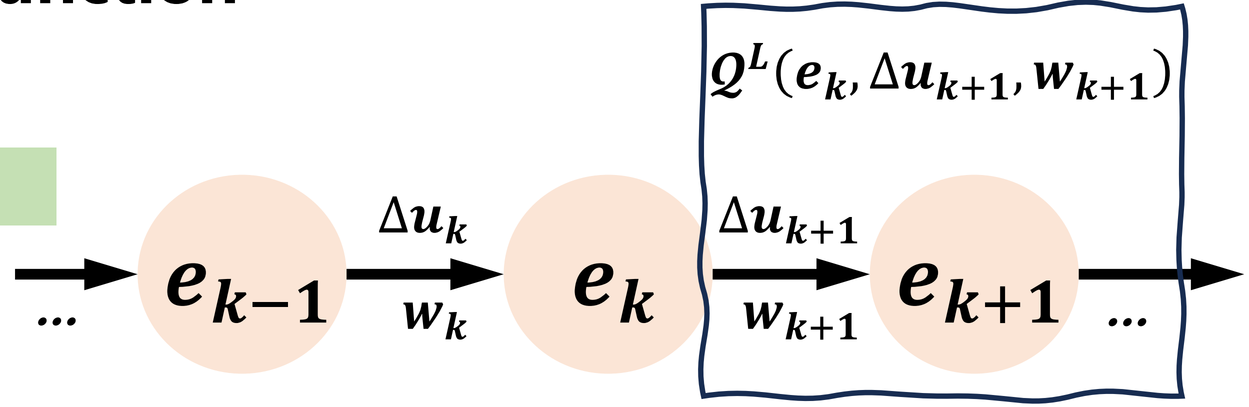
$$Q^L(e_k, L_u e_k, L_w e_k) = V^L(e_k)$$

Define the Q-function: state-action function

state

action

$$Q^L(e_k, \Delta u_{k+1}, w_{k+1})$$
$$= c(e_k, \Delta u_{k+1}, w_{k+1}) + \rho V^L(e_{k+1})$$



Data-based solution

Goal: find the target policy $L = \{L_u, L_w\}$

$$Q^L(e_k, \Delta u_{k+1}, w_{k+1}) = \underbrace{\begin{bmatrix} e_k \\ \Delta u_{k+1} \\ w_{k+1} \end{bmatrix}^\top}_{z_k^\top} \Phi^L \underbrace{\begin{bmatrix} e_k \\ \Delta u_{k+1} \\ w_{k+1} \end{bmatrix}}_{z_k}, \quad \Phi^L = \begin{bmatrix} \phi_{ee} & \phi_{eu} & \phi_{ew} \\ \phi_{ue} & \phi_{uu} & \phi_{uw} \\ \phi_{we} & \phi_{wu} & \phi_{ww} \end{bmatrix}$$

unknown G

known G

$$L_u = \left(\phi_{uu} - \phi_{uw}\phi_{ww}^{-1}\phi_{wu}\right)^{-1} \cdot \left(\phi_{uw}\phi_{ww}^{-1}\phi_{we} - \phi_{ue}\right)$$

$$L_w = \left(\phi_{ww} - \phi_{wu}\phi_{uu}^{-1}\phi_{uw}\right)^{-1} \cdot \left(\phi_{wu}\phi_{uu}^{-1}\phi_{ue} - \phi_{we}\right)$$

$$\Phi^{L^*} = \begin{bmatrix} Q + \rho P & -\rho P G & \rho P \\ -\rho G^\top P & \rho(G^\top P G + R) & -\rho G^\top P \\ \rho P & -\rho P G & \rho\left(P - \gamma^2 I\right) \end{bmatrix}$$

Data-based solution

Solve the **Bellman equation** in a model-free way

$$Q^L(e_k, \Delta u_{k+1}, w_{k+1}) = c(e_k, \Delta u_{k+1}, w_{k+1}) + \rho Q^L(e_{k+1}, L_u e_{k+1}, L_w e_{k+1})$$

$$\Downarrow$$

$$z_k^\top \Phi^L z_k = z_k^\top \begin{bmatrix} Q & & \\ & \rho R & \\ & & \gamma^2 I \end{bmatrix} z_k + \rho \underbrace{\begin{bmatrix} e_{k+1} \\ L_u e_{k+1} \\ L_w e_{k+1} \end{bmatrix}^\top}_{\chi_k^\top} \Phi^L \underbrace{\begin{bmatrix} e_{k+1} \\ L_u e_{k+1} \\ L_w e_{k+1} \end{bmatrix}}_{\chi_k}$$

Data-based solution

Data collection

$$z_k^\top \Phi^L z_k = z_k^\top W z_k + \rho \chi_k^\top \Phi^L \chi_k$$

$$Z = \begin{bmatrix} e_{k_1} & \cdots & e_{k_\eta} \\ \Delta u_{k_1+1} & \cdots & \Delta u_{k_\eta+1} \\ w_{k_1+1} & \cdots & w_{k_\eta+1} \end{bmatrix} \qquad E_i = \begin{bmatrix} e_{k_1+1} & \cdots & e_{k_\eta+1} \\ L_u^i e_{k_1+1} & \cdots & L_u^i e_{k_\eta+1} \\ L_w^i e_{k_1+1} & \cdots & L_w^i e_{k_\eta+1} \end{bmatrix}$$

$$\begin{array}{ccc} \underbrace{\qquad}_{z_{k_1}} & & \underbrace{\qquad}_{z_{k_\eta}} \end{array} \qquad \begin{array}{ccc} \underbrace{\qquad}_{\chi_{k_1+1}^i} & & \underbrace{\qquad}_{\chi_{k_\eta+1}^i} \end{array}$$

$$Z^\top \Phi^{L^{i+1}} Z = Z^\top W Z + \rho E_i^\top \Phi^{L^{i+1}} E_i$$

Generalized Sylvester Equation: MATLAB command *dlyap*

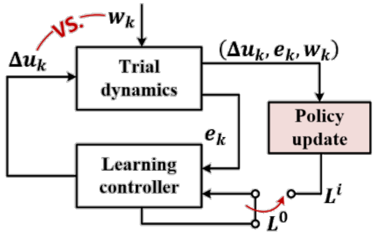
Data-based solution

Off-policy direct data-based ILC

Behavioral systems theory

LQR learning

$$H_J(u_{[0,K-1]}) = \begin{bmatrix} u_0 & \cdots & u_{K-J} \\ \vdots & \ddots & \vdots \\ u_{J-1} & \cdots & u_{K-1} \end{bmatrix}$$



Step 1 Run initial policy to collect data

Step 2 Sort the data to construct matrix

Step 3 Learn the optimal policy

Step 4 Switch to the optimal policy

Issues:

1. How many data should we collect?
2. How to ensure to learn the optimal policy?

Contents

Willems' Fundamental Lemma:
All trajectories of controllable linear systems are constructible from finitely many previous trajectories

Sample efficiency

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To estimate

$$\Phi^L = \begin{bmatrix} \phi_{ee} & \phi_{eu} & \phi_{ew} \\ \phi_{ue} & \phi_{uu} & \phi_{uw} \\ \phi_{we} & \phi_{wu} & \phi_{ww} \end{bmatrix}$$

$$Z^\top \Phi^{L^{i+1}} Z = Z^\top W Z + \rho E_i^\top \Phi^{L^{i+1}} E_i$$

Generalized Sylvester equation solvable

Z, E_i **are full row rank**

How many trials needed?

Willems' Fundamental Lemma:

All trajectories of controllable linear systems are constructible from finitely many previous trajectories

Design a persistently exciting (PE) input of sufficient order

Hankel matrix with order J:

$$H_J(u_{[0,K-1]}) = \begin{bmatrix} u_0 & \cdots & u_{K-J} \\ \vdots & \ddots & \vdots \\ u_{J-1} & \cdots & u_{K-1} \end{bmatrix}$$

full row rank

Sample efficiency

How about disturbance?

$$e_{k+1} = e_k - G\Delta u_{k+1} + w_{k+1}$$

$$e_{k+1} = e_k + [-G \quad I]\,\bar{u}_k$$

$$\bar{u}_k = \begin{bmatrix} \Delta u_{k+1} \\ w_{k+1} \end{bmatrix}$$

Augmented input vector

$$\bar{u}_{[0,K-1]} = [\bar{u}_0^\top \quad \cdots \quad \bar{u}_{K-1}^\top]^\top$$

PE of sufficient order

$\bar{u}_{[0,K-1]}$ is PE of order J

$\xrightarrow{\text{Willems' Fundamental Lemma}}$

$$\begin{bmatrix} H_1(e_{[0,K-J+1]}) \\ H_J(\bar{u}_{[0,K-1]}) \end{bmatrix}$$

full row rank

$\xrightarrow{J=1}$

$$\begin{bmatrix} H_1(e_{[0,K+1]}) \\ H_1(u_{[1,K]}) \\ H_1(w_{[1,K]}) \end{bmatrix}$$

Enough data for Z

full row rank: $K \geq ?$

Convergence analysis

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The policy iteration is stabilizing

$$Z^\top \Phi^{L^{i+1}} Z = Z^\top W Z + \rho E_i^\top \Phi^{L^{i+1}} E_i$$

$\chi_{k+1}^i = \Theta_i z_k$

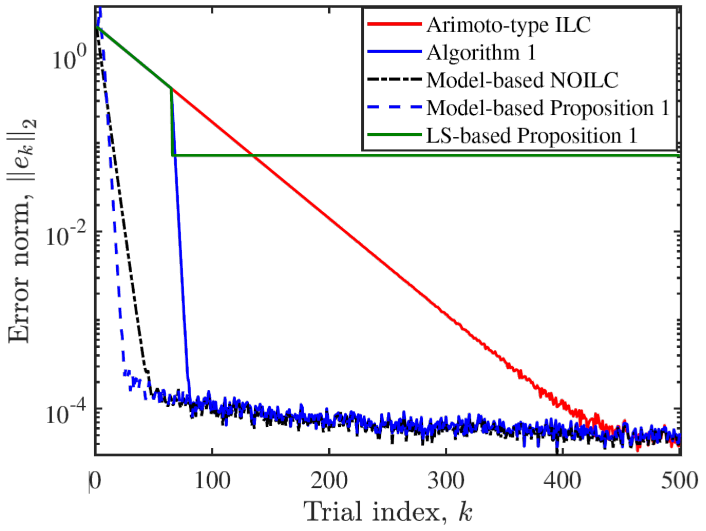
Z, E_i are full row rank

$$\Phi^{L^{i+1}} = W + \rho \Theta_i^\top \Phi^{L^{i+1}} \Theta_i$$

Model-based solution

$$\Phi^{L^*} = \begin{bmatrix} Q + \rho P & -\rho P G & \rho P \\ -\rho G^\top P & \rho(G^\top P G + R) & -\rho G^\top P \\ \rho P & -\rho P G & \rho\left(P - \gamma^2 I\right) \end{bmatrix}$$

Contents



Arimoto-type ILC
Algorithm 1
Model-based NOILC
Model-based Proposition 1
LS-based Proposition 1
10^0
10^{-2}
10^{-4}
Error norm, $\|e_k\|_2$
0
100
200
300
400
500
Trial index, k

Numerical simulation

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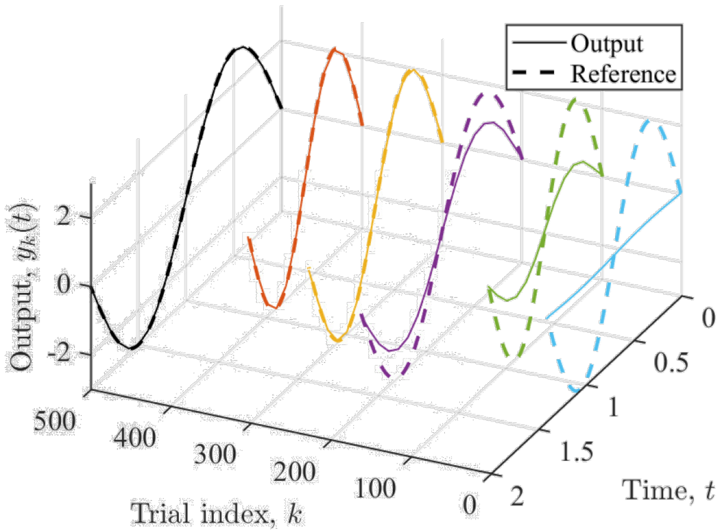
Linear time-invariant system

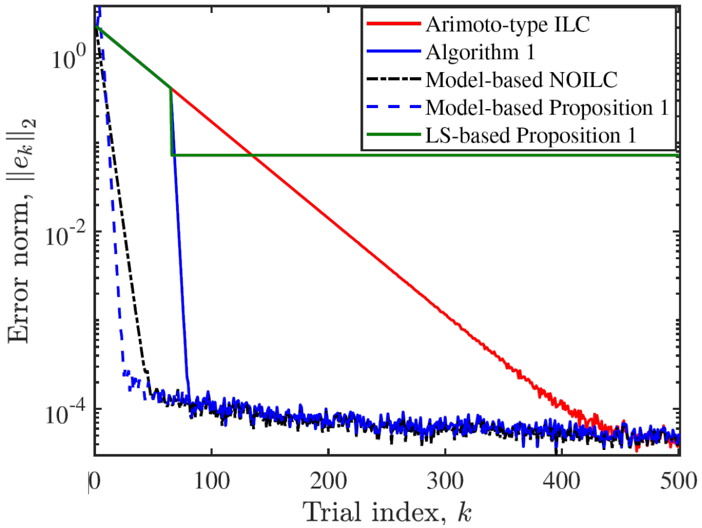
$$A = \begin{bmatrix} 0.90644 & -0.0816012 & -0.0005 \\ 0.074349 & 0.90121 & -0.000708383 \\ 0 & 0 & 0.132655 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.00150808 \\ -0.0096 \\ 0.867345 \end{bmatrix}, C = [0 \quad 0 \quad 0.5], D = [0].$$

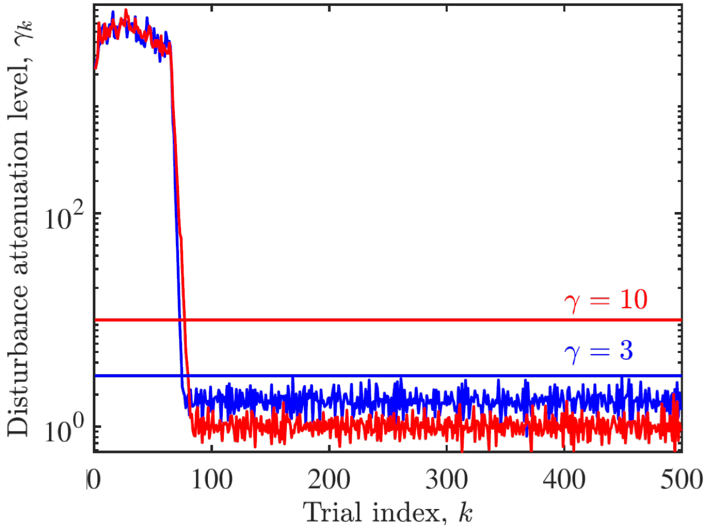
$$\gamma_k = \sqrt{\sum_{k=0}^{\infty} (e_k^\top Q e_k + \Delta u_k^\top R \Delta u_k) / \sum_{k=0}^{\infty} w_k^\top w_k}$$



Tracking outputs



Trial error convergence



Disturbance attenuation level

Conclusion and Future work

Conclusion

1. Discrete-time ILC for time-scale-varying tasks
2. Robust direct data-based ILC design:
    1) zero-sum game design for general trial-varying disturbances
    2) Off-policy design in the trial domain
3. Sample efficiency assessment and convergence analysis

Future work

1. Sample needs prediction
2. Extension to nonlinear systems
3. ……

Acknowledgements

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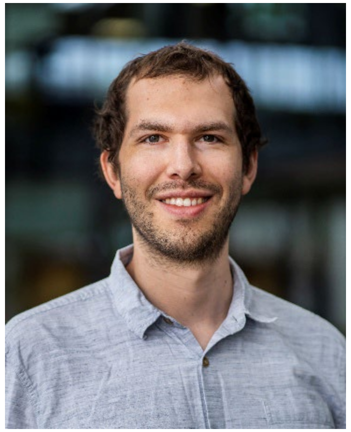

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