A note on Fourier Weak SINDy

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1 Weak SINDy with sinusoidal test functions

Consider an ODE model

$$\frac{dx(t)}{dt} = f(x(t)) \tag{1}$$

and its weak form

$$\phi(T)x(T) - \phi(0)x(0) - \int_0^T \phi'(t)x(t)dt = \int_0^T \phi(t)f(x(t))dt \tag{2}$$

where $\phi(t)$ is a compactly supported test function. Disregarding the compact support requirement, consider orthogonal test functions of the form

$$\phi_c^n(t) = \cos\left(\frac{2\pi n}{T}t\right), \tag{3a}$$

$$\phi_s^n(t) = \sin\left(\frac{2\pi n}{T}t\right), \tag{3b}$$

and observe that their derivatives are

$$\phi_c^{n\prime}(t) = -\frac{2\pi n}{T}\sin\left(\frac{2\pi n}{T}t\right) = -\frac{2\pi n}{T}\phi_s^n(t), \tag{4a}$$

$$\phi_s^{n\prime}(t) = \frac{2\pi n}{T}\cos\left(\frac{2\pi n}{T}t\right) = \frac{2\pi n}{T}\phi_c^n(t). \tag{4b}$$

For the two choices of test functions, (2) reads

$$x(T) - x(0) + \frac{2\pi n}{T}\int_0^T \phi_s^n(t)x(t)dt = \int_0^T \phi_c^n(t)f(x(t))dt, \tag{5a}$$

$$-\frac{2\pi n}{T}\int_0^T \phi_c^n(t)x(t)dt = \int_0^T \phi_s^n(t)f(x(t))dt. \tag{5b}$$

Recall that sufficiently well-behaved (e.g. continuously differentiable) functions $g(t)$ on the interval $[0,T]$ can be expressed as a Fourier series

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i2\pi n}{T}t} = a_0 + \sum_{n=1}^{\infty}\left(a_n\phi_c^n(t) + b_n\phi_s^n(t)\right) \tag{6}$$

where the real-form Fourier coefficients are

$$a_0 = \frac{1}{T}\int_0^T g(t)dt, \quad a_n = \frac{2}{T}\int_0^T g(t)\phi_c^n(t)dt, \quad b_n = \frac{2}{T}\int_0^T g(t)\phi_s^n(t)dt \tag{7}$$

and they are related to the exponential form coefficient as

$$c_n = \begin{cases} \frac{1}{2}(a_n - ib_n) & n > 0 \\ a_n & n = 0 \\ \frac{1}{2}(a_n + ib_n) & n < 0 \end{cases} \tag{8}$$

with inverse relationships

$$a_0 = c_0 \tag{9a}$$
$$a_n = c_n + c_{-n} \tag{9b}$$
$$b_n = i(c_n - c_{-n}). \tag{9c}$$

Let $x(t)$ and $f(x(t))$ defined over $[0, T]$ be expanded as Fourier series

$$x(t) = a_0^x + \sum_{n=1}^{\infty} \left(a_n^x \phi_c^n(t) + b_n^x \phi_s^n(t)\right), \tag{10a}$$
$$f(x(t)) = a_0^f + \sum_{n=1}^{\infty} \left(a_n^f \phi_c^n(t) + b_n^f \phi_s^n(t)\right). \tag{10b}$$

Then the weak form expressions (5) are equivalent to

$$x(T) - x(0) + \frac{2\pi n}{T} b_n^x = a_n^f \tag{11a}$$
$$-\frac{2\pi n}{T} a_n^x = b_n^f. \tag{11b}$$

Note that the a_n and b_n coefficients can be quickly and efficiently computed using FFT.

1.1 Dictionary function expansion

Next, suppose we do not know the exact form of f and instead can expand it in a dictionary of functions $\Theta(x(t)) = [\theta_1(x(t)), \ldots, \theta_m(x(t))]$,

$$f(x) = \sum_{i=1}^{m} K_i \theta_i(x) \tag{12}$$

and let the Fourier series expansion of $\theta_i(x(t))$ defined over $[0, T]$ be

$$\theta_i(x(t)) = a_0^{\theta_i} + \sum_{n=1}^{\infty} \left(a_n^{\theta_i} \phi_c^n(t) + b_n^{\theta_i} \phi_s^n(t)\right) \tag{13}$$

By linearity of the Fourier operator,

$$a_n^f = \sum_{i=1}^{m} K_i a_n^{\theta_i}, \qquad a_n^f = \sum_{i=1}^{m} K_i a_n^{\theta_i}. \tag{14}$$

Let $K = [K_1, \ldots, K_m]^T$ be the vector of unknown coefficients, and define

$$A_n^\Theta = \begin{bmatrix} a_n^{\theta_1} & \ldots & a_n^{\theta_m} \end{bmatrix}, \qquad B_n^\Theta = \begin{bmatrix} b_n^{\theta_1} & \ldots & b_n^{\theta_m} \end{bmatrix}. \tag{15}$$

Then $a_n^f = A_n^\Theta K$, $b_n^f = B_n^\Theta K$, and the weak form expressions (2) become

$$x(T) - x(0) + \frac{2\pi n}{T} b_n^x = A_n^\Theta K \tag{16a}$$

$$-\frac{2\pi n}{T} a_n^x = B_n^\Theta K. \tag{16b}$$

Using one or both of the expansions (16) we can set up a linear regression problem to learn the coefficients K. Note that since the coefficients a_n, b_n can be computed directly from data using FFT, this method will not require any integration by quadrature and avoids accumulation of integration error altogether, with results being precise to floating point precision.