试卷答案模板

作者名字

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一、填空题

1. $\boldsymbol{A}^T$，$n-r$

2. $\dim(\boldsymbol{V}_1 + \boldsymbol{V}_2) = \dim(\boldsymbol{V}_1) + \dim(\boldsymbol{V}_2) - \dim(\boldsymbol{V}_1 \cap \boldsymbol{V}_2) = 2 + 3 - 1 = 4$

3. 先对矩阵进行 LR 分解，$\boldsymbol{A} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \boldsymbol{L}\boldsymbol{R} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

   接着对矩阵 $\boldsymbol{L}$ 进行列分块 $\boldsymbol{L} = [\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha_3}]$，Schmidt 正交化，得：

   $\boldsymbol{\beta}_1 = \boldsymbol{\alpha}_1 = [1, -1, 1]^T, \boldsymbol{\beta}_2 = \boldsymbol{\alpha}_2 - \frac{(\boldsymbol{\alpha_2}, \boldsymbol{\beta}_1)}{(\boldsymbol{\beta}_1, \boldsymbol{\beta}_1)} \boldsymbol{\beta}_1 = \boldsymbol{\alpha}_2 + \frac{1}{3} \boldsymbol{\beta}_1 = [\frac{1}{3}, \frac{2}{3}, \frac{1}{3}]^T, \boldsymbol{\beta}_3 = \boldsymbol{\alpha}_3 - \frac{(\boldsymbol{\alpha_3}, \boldsymbol{\beta}_2)}{(\boldsymbol{\beta}_2, \boldsymbol{\beta}_2)} \boldsymbol{\beta}_2 -$ $\frac{(\boldsymbol{\alpha_3}, \boldsymbol{\beta}_1)}{(\boldsymbol{\beta}_1, \boldsymbol{\beta}_1)} \boldsymbol{\beta}_1 = \boldsymbol{\alpha}_3 - \frac{1}{2} \boldsymbol{\beta}_2 - \frac{1}{3} \boldsymbol{\beta}_1 = [-\frac{1}{2}, 0, \frac{1}{2}]^T$

   对结果单位化，$[\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3] = [\frac{\boldsymbol{\beta}_1}{||\boldsymbol{\beta}_1||}, \frac{\boldsymbol{\beta}_2}{||\boldsymbol{\beta}_2||}, \frac{\boldsymbol{\beta}_3}{||\boldsymbol{\beta}_3||}] = [\frac{\boldsymbol{\beta}_1}{\sqrt{3}}, \frac{\boldsymbol{\beta}_2}{\sqrt{\frac{2}{3}}}, \frac{\boldsymbol{\beta}_3}{\sqrt{\frac{1}{2}}}] = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

   于是，$[\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3] = [\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3] \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & \sqrt{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$

$$\boldsymbol{LR} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & \sqrt{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\sqrt{3} & -\frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ 0 & \frac{2\sqrt{6}}{3} & \frac{\sqrt{6}}{3} \\ 0 & 0 & 0 \end{pmatrix}$$

4. 7，6，3

5. $(\lambda - 2)^3$

二、解答题

1.

$$|\lambda \boldsymbol{E} - \boldsymbol{A}| = \begin{vmatrix} \lambda - 3 & 0 & -8 \\ -3 & 1 + \lambda & -6 \\ 2 & 0 & \lambda + 5 \end{vmatrix} = (\lambda + 1)^3$$

所以特征值 $\lambda_1 = \lambda_2 = \lambda_3 = -1$

行列式因子为 $D_1 = 1, D_2 = 1 + \lambda, D_3 = (1 + \lambda)^3$

不变因子为 $d_1 = 1, d_2 = 1 + \lambda, d_3 = (1 + \lambda)^2$

初等因子为 $1 + \lambda, (1 + \lambda)^2$

2. Jordan 标准形 $\boldsymbol{J} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$，Smith 标准形 $\boldsymbol{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + \lambda & 0 \\ 0 & 0 & (1 + \lambda)^2 \end{pmatrix}$

3. 最小多项式 $m_{\boldsymbol{A}}(\lambda) = (\lambda + 1)^2$

1. 只需验证其满足乘法和加法的八条性质，并且对加法和数乘法封闭

2. 由于存在 a_0, a_1, a_3 三个变量，自由度为 3，则 $\dim(F[t]_3) = 3$，容易看出 $(e_1, e_2, e_3) =$

$(1, t, t^2)$ 构成了 $F[t]_3$ 中的一组基。于是:

$$\mathcal{T}(e_1) = 2 = 2e_1$$

$$\mathcal{T}(e_2) = 1 + 2t = e_1 + 2e_2$$

$$\mathcal{T}(e_2) = 2t + 2t^2 = 2e_2 + 2e_3$$

即 $\mathcal{T}(e_1, e_2, e_3) = (e_1, e_2, e_3) \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$

3. $\dim(\mathrm{Ker}\mathcal{T}) = 0$，所以其基为 0

$\dim(\mathrm{Im}\mathcal{T}) = 3$，所以其基为 $(2e_1, e_1 + 2e_2, 2e_2 + 2e_3)$

$$\boldsymbol{A} - \lambda\boldsymbol{E} = \begin{pmatrix} 3 - \lambda & 0 & 8 \\ 3 & -1 - \lambda & 6 \\ -2 & 0 & -5 - \lambda \end{pmatrix}$$

特征值为 $\lambda_1 = \lambda_2 = \lambda_3 = -1$，由于其最小多项式 $m_{\boldsymbol{A}}(\lambda) = (\lambda+1)^2$，不妨设 $p(\lambda) = a_0 + a_1\lambda$，可得下列方程组:

$$p(-1) = a_0 - a_1 = e^{-t}$$

$$p'(-1) = a_1 = te^{-t}$$

解得 $a_0 = (t+1)e^{-t}, a_1 = te^{-t}$。则:

$$e^{\boldsymbol{A}t} = p(\boldsymbol{A}) = (t+1)e^{-t}\boldsymbol{E} + te^{-t}\boldsymbol{A} = \begin{pmatrix} (4t+1)e^{-t} & 0 & 8te^{-t} \\ 3te^{-t} & te^{-t} & 6te^{-t} \\ -2te^{-t} & 0 & (1-4t)e^{-t} \end{pmatrix}$$

微分方程的解为:

$$\boldsymbol{x} = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + e^{\boldsymbol{A}t}\int_0^t e^{-\boldsymbol{A}u}\boldsymbol{b}(t)du = \begin{pmatrix} (4t+1)e^{-t} & 0 & 8te^{-t} \\ 3te^{-t} & te^{-t} & 6te^{-t} \\ -2te^{-t} & 0 & (1-4t)e^{-t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (4t+1)e^{-t} \\ 3te^{-t} \\ -2te^{-t} \end{pmatrix}$$

1. 只需验证子空间 $\boldsymbol{W}$ 对加法和数乘封闭。

2. 显然，$\dim(\boldsymbol{W}) = 2$，则 $\dim(\boldsymbol{W}^\perp) = 4 - 2 = 2$。

   取 $\boldsymbol{e}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \boldsymbol{e}_2 = \frac{1}{\sqrt{33}}\begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix}$，显然这构成了在 $\boldsymbol{W}$ 上的标准正交基。

取 $\boldsymbol{e}_3 = \frac{1}{\sqrt{10}}\begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}, \boldsymbol{e}_4 = \frac{1}{\sqrt{110}}\begin{pmatrix} 4 & 2 \\ 6 & -5 \end{pmatrix}$，显然这构成了在 $\boldsymbol{W}^{\perp}$ 上的标准正交基。

1.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -2 & 0 & -2 \end{pmatrix} = \boldsymbol{BC} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

2.

$$\boldsymbol{A}^{+} = \boldsymbol{C}^{T}(\boldsymbol{C}\boldsymbol{C}^{T})^{-1}(\boldsymbol{B}^{T}\boldsymbol{B})^{-1}\boldsymbol{B}^{T}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{15} & -\frac{1}{3} & -\frac{4}{15} \\ -\frac{1}{15} & \frac{2}{3} & \frac{2}{15} \\ \frac{1}{15} & \frac{1}{3} & -\frac{2}{15} \end{pmatrix}$$

3. 若有解，则 $\boldsymbol{x}_0 = \boldsymbol{A}^{+}\boldsymbol{b}$ 显然是方程 $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ 的一个解，且必满足 $\boldsymbol{A}\boldsymbol{A}^{+}\boldsymbol{b} = \boldsymbol{b}$，经验证，$\boldsymbol{A}\boldsymbol{A}^{+}\boldsymbol{b} = \boldsymbol{b}$ 这样的等式不成立，则 $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ 不相容。

4.

$$\boldsymbol{x} = \boldsymbol{A}^{+}\boldsymbol{b} + (\boldsymbol{E} - \boldsymbol{A}^{+}\boldsymbol{A})\boldsymbol{y}$$

$$= \begin{pmatrix} \frac{2}{15} & -\frac{1}{3} & -\frac{4}{15} \\ -\frac{1}{15} & \frac{2}{3} & \frac{2}{15} \\ \frac{1}{15} & \frac{1}{3} & -\frac{2}{15} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{8}{15} + \frac{1}{3}y_1 + \frac{1}{3}y_2 - \frac{1}{3}y_3 \\ \frac{19}{15} + \frac{1}{3}y_1 + \frac{1}{3}y_2 - \frac{1}{3}y_3 \\ \frac{11}{15} - \frac{1}{3}y_1 - \frac{1}{3}y_2 + \frac{1}{3}y_3 \end{pmatrix}$$

其中 $\boldsymbol{y} = [y_1, y_2, y_3]^{T} \in \mathbb{R}^3$。

5. 极小范数二乘解为：

$$\boldsymbol{x}_0 = \boldsymbol{A}^+ \boldsymbol{b}$$

$$= \begin{pmatrix} -\frac{8}{15} \\ \frac{19}{15} \\ \frac{11}{15} \end{pmatrix}$$