

# 武汉理工大学研究生考试试卷（A 卷）

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# 1 填空题

1.  $\mathbf{A}^T$ ,  $n - r$

2.  $\dim(\mathbf{V}_1 + \mathbf{V}_2) = \dim(\mathbf{V}_1) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_1 \cap \mathbf{V}_2) = 2 + 3 - 1 = 4$

3. 先对矩阵进行 LR 分解,  $\mathbf{A} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \mathbf{L}\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

接着对矩阵  $\mathbf{L}$  进行列分块  $\mathbf{L} = [\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3]$ , Schmidt 正交化, 得:

$$\boldsymbol{\beta}_1 = \boldsymbol{\alpha}_1 = [1, -1, 1]^T, \boldsymbol{\beta}_2 = \boldsymbol{\alpha}_2 - \frac{(\boldsymbol{\alpha}_2, \boldsymbol{\beta}_1)}{(\boldsymbol{\beta}_1, \boldsymbol{\beta}_1)}\boldsymbol{\beta}_1 = \boldsymbol{\alpha}_2 + \frac{1}{3}\boldsymbol{\beta}_1 = [\frac{1}{3}, \frac{2}{3}, \frac{1}{3}]^T, \boldsymbol{\beta}_3 = \boldsymbol{\alpha}_3 - \frac{(\boldsymbol{\alpha}_3, \boldsymbol{\beta}_1)}{(\boldsymbol{\beta}_1, \boldsymbol{\beta}_1)}\boldsymbol{\beta}_1 - \frac{(\boldsymbol{\alpha}_3, \boldsymbol{\beta}_2)}{(\boldsymbol{\beta}_2, \boldsymbol{\beta}_2)}\boldsymbol{\beta}_2 - \frac{(\boldsymbol{\alpha}_3, \boldsymbol{\beta}_1)}{(\boldsymbol{\beta}_1, \boldsymbol{\beta}_1)}\boldsymbol{\beta}_1 = \boldsymbol{\alpha}_3 - \frac{1}{2}\boldsymbol{\beta}_2 - \frac{1}{3}\boldsymbol{\beta}_1 = [-\frac{1}{2}, 0, \frac{1}{2}]^T$$

对结果单位化,  $[\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3] = [\frac{\boldsymbol{\beta}_1}{\|\boldsymbol{\beta}_1\|}, \frac{\boldsymbol{\beta}_2}{\|\boldsymbol{\beta}_2\|}, \frac{\boldsymbol{\beta}_3}{\|\boldsymbol{\beta}_3\|}] = [\frac{\boldsymbol{\beta}_1}{\sqrt{3}}, \frac{\boldsymbol{\beta}_2}{\sqrt{\frac{2}{3}}}, \frac{\boldsymbol{\beta}_3}{\sqrt{\frac{1}{2}}}] = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

于是,  $[\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3] = [\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3] \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & \sqrt{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$

$$\begin{aligned} \mathbf{LR} &= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & \sqrt{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\sqrt{3} & -\frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ 0 & \frac{2\sqrt{6}}{3} & \frac{\sqrt{6}}{3} \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

4. 7, 6, 3

5.  $(\lambda - 2)^3$

## 2 解答题

### 2.1

1.

$$|\lambda \mathbf{E} - \mathbf{A}| = \begin{vmatrix} \lambda - 3 & 0 & -8 \\ -3 & 1 + \lambda & -6 \\ 2 & 0 & \lambda + 5 \end{vmatrix} = (\lambda + 1)^3$$

所以特征值  $\lambda_1 = \lambda_2 = \lambda_3 = -1$

行列式因子为  $D_1 = 1, D_2 = 1 + \lambda, D_3 = (1 + \lambda)^3$

不变因子为  $d_1 = 1, d_2 = 1 + \lambda, d_3 = (1 + \lambda)^2$

初等因子为  $1 + \lambda, (1 + \lambda)^2$

2. Jordan 标准形  $\mathbf{J} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ , Smith 标准形  $\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + \lambda & 0 \\ 0 & 0 & (1 + \lambda)^2 \end{pmatrix}$

3. 最小多项式  $m_{\mathbf{A}}(\lambda) = (\lambda + 1)^2$

### 2.2

1. 只需验证其满足乘法和加法的八条性质，并且对加法和数乘法封闭

2. 由于存在  $a_0, a_1, a_3$  三个变量，自由度为 3，则  $\dim(F[t]_3) = 3$ ，容易看出  $(e_1, e_2, e_3) = (1, t, t^2)$  构成了  $F[t]_3$  中的一组基。于是：

$$\mathcal{T}(e_1) = 2 = 2e_1$$

$$\mathcal{T}(e_2) = 1 + 2t = e_1 + 2e_2$$

$$\mathcal{T}(e_3) = 2t + 2t^2 = 2e_2 + 2e_3$$

即  $\mathcal{T}(e_1, e_2, e_3) = (e_1, e_2, e_3) \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$

3.  $\dim(\text{Ker } \mathcal{T}) = 0$ ，所以其基为 0

$\dim(\text{Im } \mathcal{T}) = 3$ ，所以其基为  $(2e_1, e_1 + 2e_2, 2e_2 + 2e_3)$

## 2.3

$$\mathbf{A} - \lambda \mathbf{E} = \begin{pmatrix} 3 - \lambda & 0 & 8 \\ 3 & -1 - \lambda & 6 \\ -2 & 0 & -5 - \lambda \end{pmatrix}$$

特征值为  $\lambda_1 = \lambda_2 = \lambda_3 = -1$ , 由于其最小多项式  $m_{\mathbf{A}}(\lambda) = (\lambda+1)^2$ , 不妨设  $p(\lambda) = a_0 + a_1\lambda$ , 可得下列方程组:

$$p(-1) = a_0 - a_1 = e^{-t}$$

$$p'(-1) = a_1 = te^{-t}$$

解得  $a_0 = (t+1)e^{-t}$ ,  $a_1 = te^{-t}$ 。则:

$$e^{\mathbf{A}t} = p(\mathbf{A}) = (t+1)e^{-t}\mathbf{E} + te^{-t}\mathbf{A} = \begin{pmatrix} (4t+1)e^{-t} & 0 & 8te^{-t} \\ 3te^{-t} & te^{-t} & 6te^{-t} \\ -2te^{-t} & 0 & (1-4t)e^{-t} \end{pmatrix}$$

微分方程的解为:

$$\mathbf{x} = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t} \int_0^t e^{-\mathbf{A}u}\mathbf{b}(u)du = \begin{pmatrix} (4t+1)e^{-t} & 0 & 8te^{-t} \\ 3te^{-t} & te^{-t} & 6te^{-t} \\ -2te^{-t} & 0 & (1-4t)e^{-t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (4t+1)e^{-t} \\ 3te^{-t} \\ -2te^{-t} \end{pmatrix}$$

## 2.4

1. 只需验证子空间  $\mathbf{W}$  对加法和数乘封闭。

2. 显然,  $\dim(\mathbf{W}) = 2$ , 则  $\dim(\mathbf{W}^\perp) = 4 - 2 = 2$ 。

取  $\mathbf{e}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $\mathbf{e}_2 = \frac{1}{\sqrt{33}} \begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix}$ , 显然这构成了在  $\mathbf{W}$  上的标准正交基。

取  $\mathbf{e}_3 = \frac{1}{\sqrt{10}} \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix}$ ,  $\mathbf{e}_4 = \frac{1}{\sqrt{110}} \begin{pmatrix} 4 & 2 \\ 6 & -5 \end{pmatrix}$ , 显然这构成了在  $\mathbf{W}^\perp$  上的标准正交基。

## 2.5

1.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -2 & 0 & -2 \end{pmatrix} = \mathbf{B}\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

2.

$$\begin{aligned} \mathbf{A}^+ &= \mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{2}{15} & -\frac{1}{3} & -\frac{4}{15} \\ -\frac{1}{15} & \frac{2}{3} & \frac{2}{15} \\ \frac{1}{15} & \frac{1}{3} & -\frac{2}{15} \end{pmatrix} \end{aligned}$$

3. 若有解, 则  $\mathbf{x}_0 = \mathbf{A}^+\mathbf{b}$  显然是方程  $\mathbf{A}\mathbf{x} = \mathbf{b}$  的一个解, 且必满足  $\mathbf{A}\mathbf{A}^+\mathbf{b} = \mathbf{b}$ , 经验证,  $\mathbf{A}\mathbf{A}^+\mathbf{b} = \mathbf{b}$  这样的等式不成立, 则  $\mathbf{A}\mathbf{x} = \mathbf{b}$  不相容。

4.

$$\begin{aligned} \mathbf{x} &= \mathbf{A}^+\mathbf{b} + (\mathbf{E} - \mathbf{A}^+\mathbf{A})\mathbf{y} \\ &= \begin{pmatrix} \frac{2}{15} & -\frac{1}{3} & -\frac{4}{15} \\ -\frac{1}{15} & \frac{2}{3} & \frac{2}{15} \\ \frac{1}{15} & \frac{1}{3} & -\frac{2}{15} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{8}{15} + \frac{1}{3}y_1 + \frac{1}{3}y_2 - \frac{1}{3}y_3 \\ \frac{19}{15} + \frac{1}{3}y_1 + \frac{1}{3}y_2 - \frac{1}{3}y_3 \\ \frac{11}{15} - \frac{1}{3}y_1 - \frac{1}{3}y_2 + \frac{1}{3}y_3 \end{pmatrix} \end{aligned}$$

其中  $\mathbf{y} = [y_1, y_2, y_3]^T \in \mathbb{R}^3$ 。

5. 极小范数二乘解为:

$$\begin{aligned}\boldsymbol{x}_0 &= \boldsymbol{A}^+ \boldsymbol{b} \\ &= \begin{pmatrix} -\frac{8}{15} \\ \frac{19}{15} \\ \frac{11}{15} \end{pmatrix}\end{aligned}$$