

第一章 广义逆矩阵

题 1.1. (P162.3) 求下列矩阵的广义逆 \mathbf{A}^+ 。

$$(1) \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \end{pmatrix}$$

$$(2) \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 0 \end{pmatrix}$$

$$(3) \begin{pmatrix} -2 & 0 & 0 & -2 \\ 1 & 2 & -4 & 3 \\ 2 & -1 & 2 & 1 \\ 0 & 2 & -4 & 2 \end{pmatrix}$$

解. (1)

$$\mathbf{A} = \mathbf{B}\mathbf{C} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{C}\mathbf{C}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{B}^T\mathbf{B} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$$

$$\mathbf{A}^+ = \mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$$

(2)

$$\mathbf{A} = \mathbf{B}\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ -1 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$

$$\mathbf{C}\mathbf{C}^T = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}, \mathbf{B}^T\mathbf{B} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{aligned} \mathbf{A}^+ &= \mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{10} & 0 & -\frac{1}{10} & 0 \\ \frac{1}{5} & 0 & \frac{1}{5} & 0 \end{pmatrix} \end{aligned}$$

(3)

$$\mathbf{A} = \mathbf{B}\mathbf{C} = \begin{pmatrix} -2 & 0 \\ 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 1 \end{pmatrix}$$

$$\mathbf{C}\mathbf{C}^T = \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}, \mathbf{B}^T\mathbf{B} = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$$

$$\begin{aligned} \mathbf{A}^+ &= \mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}^{-1} \begin{pmatrix} -2 & 1 & 2 & 0 \\ 0 & 2 & -1 & 2 \end{pmatrix} \\ &= \frac{1}{99} \begin{pmatrix} -12 & 4 & 13 & -2 \\ 2 & 3 & -4 & 4 \\ -4 & -6 & 8 & -8 \\ -10 & 7 & 9 & 2 \end{pmatrix} \end{aligned}$$

□

题 1.2. 验证线性方程组 $\mathbf{A}\mathbf{x} = \mathbf{b}$ 有解, 并求其通解和最小长度解, 其中 $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 2 & 4 \end{pmatrix}, \mathbf{b} =$

$$\begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$$

解. $(\mathbf{A}|\mathbf{b}) = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 2 & 4 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, 则 $r(\mathbf{A}|\mathbf{b}) = r(\mathbf{A})$, 于是方程组有解。

$$\mathbf{A} = \mathbf{B}\mathbf{C} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}$$

$$\mathbf{C}\mathbf{C}^T = \begin{pmatrix} 5 \end{pmatrix}, \mathbf{B}^T\mathbf{B} = \begin{pmatrix} 5 \end{pmatrix}$$

$$\begin{aligned} \mathbf{A}^+ &= \mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 5 \end{pmatrix}^{-1} \begin{pmatrix} 5 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} \\ &= \frac{1}{25} \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \end{pmatrix} \end{aligned}$$

则通解为 $\mathbf{x} = \mathbf{A}^+\mathbf{b} + (\mathbf{E} - \mathbf{A}^+\mathbf{A})\mathbf{y} = \begin{pmatrix} -\frac{1}{5} + \frac{4}{5}y_1 - \frac{2}{5}y_2 \\ -\frac{2}{5} - \frac{2}{5}y_1 + \frac{1}{5}y_2 \end{pmatrix}$, 其中 $\mathbf{y} \in \mathbb{R}^2$ 。最小长度解为

$$\mathbf{x}^* = \begin{pmatrix} -\frac{1}{5} \\ -\frac{2}{5} \end{pmatrix}。$$

□

题 1.3. 验证下列线性方程组 $\mathbf{A}\mathbf{x} = \mathbf{b}$ 为矛盾方程组, 并求其最小二乘解的通解和最小长度二乘解。

$$(1) \mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 2 & 4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$(2) \mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ -3 & -6 & 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(3) \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

解. (1) 则 $r(\mathbf{A}|\mathbf{b}) = 2, r(\mathbf{A}) = 1, r(\mathbf{A}) \neq r(\mathbf{A}|\mathbf{b})$, 于是为矛盾方程组。

$$\mathbf{A} = \mathbf{B}\mathbf{C} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}$$

$$\mathbf{C}\mathbf{C}^T = \begin{pmatrix} 5 \end{pmatrix}, \mathbf{B}^T\mathbf{B} = \begin{pmatrix} 5 \end{pmatrix}$$

$$\begin{aligned} \mathbf{A}^+ &= \mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 5 \end{pmatrix}^{-1} \begin{pmatrix} 5 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} \\ &= \frac{1}{25} \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \end{pmatrix} \end{aligned}$$

最小二乘解为 $\mathbf{x} = \mathbf{A}^+\mathbf{b} + (\mathbf{E} - \mathbf{A}^+\mathbf{A})\mathbf{y} = \begin{pmatrix} \frac{1}{5} + \frac{4}{5}y_1 - \frac{2}{5}y_2 \\ \frac{2}{5} - \frac{2}{5}y_1 + \frac{1}{5}y_2 \end{pmatrix}$, 其中 $\mathbf{y} \in \mathbb{R}^2$ 。最小长度

二乘解为 $\mathbf{x}^* = \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \end{pmatrix}$ 。

(2) 则 $r(\mathbf{A}|\mathbf{b}) = 2, r(\mathbf{A}) = 1, r(\mathbf{A}) \neq r(\mathbf{A}|\mathbf{b})$, 于是为矛盾方程组。

$$\mathbf{A} = \mathbf{B}\mathbf{C} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \end{pmatrix}$$

$$\mathbf{C}\mathbf{C}^T = \begin{pmatrix} 6 \end{pmatrix}, \mathbf{B}^T\mathbf{B} = \begin{pmatrix} 10 \end{pmatrix}$$

$$\begin{aligned}
\mathbf{A}^+ &= \mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T \\
&= \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} (6)^{-1} (10)^{-1} \begin{pmatrix} 1 & -3 \end{pmatrix} \\
&= \frac{1}{60} \begin{pmatrix} 1 & -3 \\ 2 & -6 \\ -1 & 3 \end{pmatrix}
\end{aligned}$$

最小二乘解为 $\mathbf{x} = \mathbf{A}^+\mathbf{b} + (\mathbf{E} - \mathbf{A}^+\mathbf{A})\mathbf{y} = \begin{pmatrix} -\frac{1}{30} + \frac{5}{6}y_1 - \frac{1}{3}y_2 + \frac{1}{6}y_3 \\ -\frac{1}{15} - \frac{1}{3}y_1 + \frac{1}{3}y_2 + \frac{1}{3}y_3 \\ \frac{1}{30} + \frac{1}{6}y_1 + \frac{1}{3}y_2 + \frac{5}{6}y_3 \end{pmatrix}$, 其中 $\mathbf{y} \in \mathbb{R}^3$ 。最

小长度二乘解为 $\mathbf{x}^* = \begin{pmatrix} -\frac{1}{30} \\ -\frac{1}{15} \\ \frac{1}{30} \end{pmatrix}$ 。

(3) 则 $r(\mathbf{A}|\mathbf{b}) = 3, r(\mathbf{A}) = 2, r(\mathbf{A}) \neq r(\mathbf{A}|\mathbf{b})$, 于是为矛盾方程组。

$$\mathbf{A} = \mathbf{B}\mathbf{C} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{C}\mathbf{C}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{B}^T\mathbf{B} = \begin{pmatrix} 6 & -2 \\ -2 & 10 \end{pmatrix}$$

$$\begin{aligned}
\mathbf{A}^+ &= \mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 6 & -2 \\ -2 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 3 \end{pmatrix} \\
&= \frac{1}{14} \begin{pmatrix} 3 & 5 & -1 \\ 2 & 1 & 4 \end{pmatrix}
\end{aligned}$$

最小二乘解为 $\mathbf{x} = \mathbf{A}^+\mathbf{b} + (\mathbf{E} - \mathbf{A}^+\mathbf{A})\mathbf{y} = \begin{pmatrix} \frac{1}{14} \\ \frac{10}{14} \\ \frac{1}{14} \end{pmatrix}$, 其中 $\mathbf{y} \in \mathbb{R}^3$ 。最小长度二乘解为

$\mathbf{x}^* = \begin{pmatrix} \frac{1}{14} \\ \frac{10}{14} \\ \frac{1}{14} \end{pmatrix}$ 。

