第一章　矩阵分解

题 1.1.（p140.1）计算下列矩阵的 Doolittle 分解，Crout 三角分解和 LDU 三角分解。

$$(1)\begin{pmatrix} 2 & 4 & 6 \\ 2 & 7 & 12 \\ -2 & -10 & -13 \end{pmatrix} \qquad (2)\begin{pmatrix} 4 & 8 & 0 \\ 4 & 11 & 6 \\ -6 & -12 & 10 \end{pmatrix}$$

解. (1) 先对矩阵进行初等行变换变成上三角矩阵

$$\begin{pmatrix} 2 & 4 & 6 \\ 2 & 7 & 12 \\ -2 & -10 & -13 \end{pmatrix} \xrightarrow{r_2-r_1} \begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 6 \\ -2 & -10 & -13 \end{pmatrix} \xrightarrow{r_3+r_1} \begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 6 \\ 0 & -6 & -7 \end{pmatrix} \xrightarrow{r_3+2r_2} \begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{pmatrix}$$

则 $\boldsymbol{R}$ 为上面所求，将每一步变换用初等矩阵表示，则 $\boldsymbol{P}_3\boldsymbol{P}_2\boldsymbol{P}_1\boldsymbol{A}=\boldsymbol{R}$，于是

$$\boldsymbol{L}=(\boldsymbol{P}_3\boldsymbol{P}_2\boldsymbol{P}_1)^{-1}=\boldsymbol{P}_1^{-1}\boldsymbol{P}_2^{-1}\boldsymbol{P}_3^{-1}=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix}$$

Doolittle 分解、Crout 分解、LDU 分解

$$\boldsymbol{LR}=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{pmatrix}$$

$$\boldsymbol{LDU}=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & & \\ & 3 & \\ & & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{LU}=\begin{pmatrix} 2 & 0 & 0 \\ 2 & 3 & 0 \\ -2 & -6 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

其中空白位置全是 0。

(2) 先对矩阵进行初等行变换变成上三角矩阵

$$\begin{pmatrix} 4 & 8 & 0 \\ 4 & 11 & 6 \\ -6 & -12 & 10 \end{pmatrix} \xrightarrow{r_2-r_1} \begin{pmatrix} 4 & 8 & 0 \\ 0 & 3 & 6 \\ -6 & -12 & 10 \end{pmatrix} \xrightarrow{r_3+\frac{3}{2}r_1} \begin{pmatrix} 4 & 8 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 10 \end{pmatrix}$$

则 $\boldsymbol{R}$ 为上面所求，将每一步变换用初等矩阵表示，则 $\boldsymbol{P}_2\boldsymbol{P}_1\boldsymbol{A} = \boldsymbol{R}$，于是

$$\boldsymbol{L} = (\boldsymbol{P}_2\boldsymbol{P}_1)^{-1} = \boldsymbol{P}_1^{-1}\boldsymbol{P}_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{pmatrix}$$

Doolittle 分解、Crout 分解、LDU 分解

$$\boldsymbol{LR} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 8 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 10 \end{pmatrix}$$

$$\boldsymbol{LDU} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & & \\ & 3 & \\ & & 10 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{LU} = \begin{pmatrix} 4 & 0 & 0 \\ 4 & 3 & 0 \\ -6 & 0 & 10 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

其中空白位置全是 0。

□

题 1.2. （p141.2）计算下列矩阵的 Cholesky 分解。

(1) $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -3 \\ -1 & -3 & 6 \end{pmatrix}$ (2) $\begin{pmatrix} 4 & 4 & -6 \\ 4 & 5 & -6 \\ -6 & -6 & 13 \end{pmatrix}$

解. (1) 设 $\boldsymbol{G} = \begin{pmatrix} g_{11} & & \\ g_{21} & g_{22} & \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$，则 $\boldsymbol{G}^T = \begin{pmatrix} g_{11} & g_{21} & g_{31} \\ & g_{22} & g_{32} \\ & & g_{33} \end{pmatrix}$，其中空白位置全是 0。于是

有：

$$\begin{cases} g_{11}^2 = 1 \\ g_{11}g_{21} = 1 \\ g_{11}g_{31} = -1 \\ g_{21}g_{11} = 1 \\ g_{21}^2 + g_{22}^2 = 2 \\ g_{21}g_{31} + g_{22}g_{32} = -3 \\ g_{31}g_{11} = -1 \\ g_{31}g_{21} + g_{32}g_{22} = -3 \\ g_{31}^2 + g_{32}^2 + g_{33}^2 = 6 \end{cases} \Rightarrow \begin{cases} g_{11} = 1 \\ g_{21} = 1 \\ g_{31} = -1 \\ g_{22} = 1 \\ g_{32} = -2 \\ g_{33} = 1 \end{cases}$$

$$\boldsymbol{G} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix} \quad \boldsymbol{A} = \boldsymbol{G}\boldsymbol{G}^T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) 设 $\boldsymbol{G} = \begin{pmatrix} g_{11} & & \\ g_{21} & g_{22} & \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$，则 $\boldsymbol{G}^T = \begin{pmatrix} g_{11} & g_{21} & g_{31} \\ & g_{22} & g_{32} \\ & & g_{33} \end{pmatrix}$，其中空白位置全是 0。于是有：

$$\begin{cases} g_{11}^2 = 4 \\ g_{11}g_{21} = 4 \\ g_{11}g_{31} = -6 \\ g_{21}g_{11} = 4 \\ g_{21}^2 + g_{22}^2 = 5 \\ g_{21}g_{31} + g_{22}g_{32} = -6 \\ g_{31}g_{11} = -6 \\ g_{31}g_{21} + g_{32}g_{22} = -6 \\ g_{31}^2 + g_{32}^2 + g_{33}^2 = 13 \end{cases} \Rightarrow \begin{cases} g_{11} = 2 \\ g_{21} = 2 \\ g_{31} = -3 \\ g_{22} = 1 \\ g_{32} = 0 \\ g_{33} = 2 \end{cases}$$

$$\boldsymbol{G} = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 2 \end{pmatrix} \quad \boldsymbol{A} = \boldsymbol{G}\boldsymbol{G}^T = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$\square$

题 1.3. 计算矩阵 $\begin{pmatrix} 2 & 4 & 6 & 8 \\ 4 & 12 & 20 & 16 \\ 3 & 10 & 20 & 18 \\ 1 & -4 & -9 & 14 \end{pmatrix}$ 的 Doolittle 分解。

解. 先对矩阵进行初等行变换变成上三角矩阵

$$\begin{pmatrix} 2 & 4 & 6 & 8 \\ 4 & 12 & 20 & 16 \\ 3 & 10 & 20 & 18 \\ 1 & -4 & -9 & 14 \end{pmatrix} \xrightarrow{r_2-2r_1} \begin{pmatrix} 2 & 4 & 6 & 8 \\ 0 & 4 & 8 & 0 \\ 3 & 10 & 20 & 18 \\ 1 & -4 & -9 & 14 \end{pmatrix} \xrightarrow{r_3-\frac{3}{2}r_1} \begin{pmatrix} 2 & 4 & 6 & 8 \\ 0 & 4 & 8 & 0 \\ 0 & 4 & 11 & 6 \\ 1 & -4 & -9 & 14 \end{pmatrix} \xrightarrow{r_4-\frac{1}{2}r_1} \begin{pmatrix} 2 & 4 & 6 & 8 \\ 0 & 4 & 8 & 0 \\ 0 & 4 & 11 & 6 \\ 0 & -6 & -12 & 10 \end{pmatrix}$$

$$\xrightarrow{r_3-r_2} \begin{pmatrix} 2 & 4 & 6 & 8 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 3 & 6 \\ 0 & -6 & -12 & 10 \end{pmatrix} \xrightarrow{r_4+\frac{3}{2}r_2} \begin{pmatrix} 2 & 4 & 6 & 8 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

则 $\boldsymbol{R}$ 为上面所求，将每一步变换用初等矩阵表示，则 $\boldsymbol{P}_5\boldsymbol{P}_4\boldsymbol{P}_3\boldsymbol{P}_2\boldsymbol{P}_1\boldsymbol{A}=\boldsymbol{R}$，于是

$$\begin{aligned} \boldsymbol{L} &= (\boldsymbol{P}_5\boldsymbol{P}_4\boldsymbol{P}_3\boldsymbol{P}_2\boldsymbol{P}_1)^{-1} = \boldsymbol{P}_1^{-1}\boldsymbol{P}_2^{-1}\boldsymbol{P}_3^{-1}\boldsymbol{P}_4^{-1}\boldsymbol{P}_5^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{3}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{3}{2} & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{3}{2} & 1 & 1 & 0 \\ \frac{1}{2} & -\frac{3}{2} & 0 & 1 \end{pmatrix} \end{aligned}$$

$$\boldsymbol{A}=\boldsymbol{L}\boldsymbol{R}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{3}{2} & 1 & 1 & 0 \\ \frac{1}{2} & -\frac{3}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 6 & 8 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

$\square$

题 1.4. 计算矩阵 $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 8 & 10 & 2 \\ 3 & 10 & 14 & 6 \\ 4 & 2 & 6 & 29 \end{pmatrix}$ 的 Cholesky 分解。

解. 设 $\boldsymbol{G} = \begin{pmatrix} g_{11} & & & \\ g_{21} & g_{22} & & \\ g_{31} & g_{32} & g_{33} & \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix}$，则 $\boldsymbol{G}^T = \begin{pmatrix} g_{11} & g_{21} & g_{31} & g_{41} \\ & g_{22} & g_{32} & g_{42} \\ & & g_{33} & g_{43} \\ & & & g_{44} \end{pmatrix}$，其中空白位置全是 0。

于是有：

$$\begin{cases} g_{11}^2 = 1 \\ g_{11}g_{21} = 2 \\ g_{11}g_{31} = 3 \\ g_{11}g_{41} = 4 \\ g_{21}g_{11} = 2 \\ g_{21}^2 + g_{22}^2 = 8 \\ g_{21}g_{31} + g_{22}g_{32} = 10 \\ g_{21}g_{41} + g_{22}g_{42} = 2 \\ g_{31}g_{11} = 3 \\ g_{31}g_{21} + g_{32}g_{22} = 10 \\ g_{31}^2 + g_{32}^2 + g_{33}^2 = 14 \\ g_{31}g_{41} + g_{32}g_{42} + g_{33}g_{43} = 6 \\ g_{41}g_{11} = 4 \\ g_{41}g_{21} + g_{42}g_{22} = 2 \\ g_{41}g_{31} + g_{42}g_{32} + g_{43}g_{33} = 6 \\ g_{41}^2 + g_{42}^2 + g_{43}^2 + g_{44}^2 = 29 \end{cases} \Rightarrow \begin{cases} g_{11} = 1 \\ g_{21} = 2 \\ g_{31} = 3 \\ g_{41} = 4 \\ g_{22} = 2 \\ g_{32} = 2 \\ g_{42} = -3 \\ g_{33} = 1 \\ g_{43} = 0 \\ g_{44} = 2 \end{cases}$$

$$\boldsymbol{G} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & -3 & 0 & 2 \end{pmatrix} \quad \boldsymbol{A} = \boldsymbol{G}\boldsymbol{G}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & -3 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

□

题 1.5. 计算下列矩阵的满秩分解。

(1) $\begin{pmatrix} 1 & 2 & 3 & 3 \\ 4 & 5 & 9 & 6 \\ 7 & 8 & 15 & 9 \\ 2 & 5 & 7 & 8 \end{pmatrix}$ (2) $\begin{pmatrix} 1 & 3 & -3 & 4 \\ 3 & 5 & -5 & 8 \\ 6 & -1 & 1 & 5 \\ 8 & -6 & 6 & 2 \end{pmatrix}$

解. (1) 先对矩阵进行初等行变换变成行阶梯型矩阵

$$\begin{pmatrix} 1 & 2 & 3 & 3 \\ 4 & 5 & 9 & 6 \\ 7 & 8 & 15 & 9 \\ 2 & 5 & 7 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

于是 $\boldsymbol{A} = \boldsymbol{BC} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$。

(2) 先对矩阵进行初等行变换变成行阶梯型矩阵

$$\begin{pmatrix} 1 & 3 & -3 & 4 \\ 3 & 5 & -5 & 8 \\ 6 & -1 & 1 & 5 \\ 8 & -6 & 6 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -3 & 4 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

于是 $\boldsymbol{A} = \boldsymbol{BC} = \begin{pmatrix} 1 & 3 \\ 3 & 5 \\ 6 & -1 \\ 8 & -6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$

□

题 1.6. 计算下列矩阵的谱分解。

(1) $\begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ (2) $\begin{pmatrix} 5 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 7 \end{pmatrix}$

解. (1) $\lambda_1=\lambda_2=2,\lambda_3=4$ 所以

$$
\begin{aligned}
\boldsymbol{A}&=\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^T\\
&=\begin{pmatrix}0&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\1&0&0\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}2&&\\&2&\\&&4\end{pmatrix}\begin{pmatrix}0&1&0\\-\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\end{pmatrix}\\
&=2\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0&1&0\end{pmatrix}+2\begin{pmatrix}-\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}-\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\end{pmatrix}+4\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\end{pmatrix}
\end{aligned}
$$

其中空白位置全是 0。

(2) $\lambda_1=3,\lambda_2=6,\lambda_3=9$ 所以

$$
\begin{aligned}
\boldsymbol{A}&=\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^T\\
&=\begin{pmatrix}\frac{2}{3}&-\frac{2}{3}&\frac{1}{3}\\\frac{2}{3}&\frac{1}{3}&-\frac{2}{3}\\\frac{1}{3}&\frac{2}{3}&\frac{2}{3}\end{pmatrix}\begin{pmatrix}3&&\\&6&\\&&9\end{pmatrix}\begin{pmatrix}\frac{2}{3}&\frac{2}{3}&\frac{1}{3}\\-\frac{2}{3}&\frac{1}{3}&\frac{2}{3}\\\frac{1}{3}&-\frac{2}{3}&\frac{2}{3}\end{pmatrix}\\
&=3\begin{pmatrix}\frac{2}{3}\\\frac{2}{3}\\\frac{1}{3}\end{pmatrix}\begin{pmatrix}\frac{2}{3}&\frac{2}{3}&\frac{1}{3}\end{pmatrix}+6\begin{pmatrix}-\frac{2}{3}\\\frac{1}{3}\\\frac{2}{3}\end{pmatrix}\begin{pmatrix}-\frac{2}{3}&\frac{1}{3}&\frac{2}{3}\end{pmatrix}+9\begin{pmatrix}\frac{1}{3}\\-\frac{2}{3}\\\frac{2}{3}\end{pmatrix}\begin{pmatrix}\frac{1}{3}&-\frac{2}{3}&\frac{2}{3}\end{pmatrix}
\end{aligned}
$$

其中空白位置全是 0。

□

题 1.7. 计算下列矩阵的 QR 分解。

(1) $\begin{pmatrix}2&1\\1&1\\2&1\end{pmatrix}$ (2) $\begin{pmatrix}1&0\\0&1\\1&1\end{pmatrix}$ (3) $\begin{pmatrix}0&1&1\\1&1&0\\1&0&0\end{pmatrix}$ (4) $\begin{pmatrix}2&2&1\\0&2&2\\2&1&2\end{pmatrix}$

解. 把矩阵记作 $\boldsymbol{A}_{m\times n}$，并进行列分块 $(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n)$。

(1) 对 $\boldsymbol{A}$ 进行 Schmidt 正交化，得：

$$
\begin{aligned}
\boldsymbol{\beta}_1&=\boldsymbol{\alpha}_1=(2,1,2)^T\\
\boldsymbol{\beta}_2&=\boldsymbol{\alpha}_2-\frac{(\boldsymbol{\alpha}_2,\boldsymbol{\beta}_1)}{(\boldsymbol{\beta}_1,\boldsymbol{\beta}_1)}\boldsymbol{\beta}_1=\boldsymbol{\alpha}_2-\frac{5}{9}\boldsymbol{\beta}_1=(-\frac{1}{9},\frac{4}{9},-\frac{1}{9})^T
\end{aligned}
$$

于是

$$
\begin{aligned}
\boldsymbol{A} = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) \begin{pmatrix} 1 & \frac{5}{9} \\ 0 & 1 \end{pmatrix} \\
= \begin{pmatrix} 2 & -\frac{1}{9} \\ 1 & \frac{4}{9} \\ 2 & -\frac{1}{9} \end{pmatrix} \begin{pmatrix} 1 & \frac{5}{9} \\ 0 & 1 \end{pmatrix} \\
= \begin{pmatrix} \frac{2}{3} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{3} & \frac{4}{3\sqrt{2}} \\ \frac{2}{3} & -\frac{1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & \\ & \frac{\sqrt{2}}{3} \end{pmatrix} \begin{pmatrix} 1 & \frac{5}{9} \\ 0 & 1 \end{pmatrix} \\
= \begin{pmatrix} \frac{2}{3} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{3} & \frac{4}{3\sqrt{2}} \\ \frac{2}{3} & -\frac{1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & \frac{5}{3} \\ 0 & \frac{\sqrt{2}}{3} \end{pmatrix}
\end{aligned}
$$

(2) 对 $\boldsymbol{A}$ 进行 Schmidt 正交化，得：

$$
\begin{aligned}
\boldsymbol{\beta}_1 &= \boldsymbol{\alpha}_1 = (1, 0, 1)^T \\
\boldsymbol{\beta}_2 &= \boldsymbol{\alpha}_2 - \frac{(\boldsymbol{\alpha}_2, \boldsymbol{\beta}_1)}{(\boldsymbol{\beta}_1, \boldsymbol{\beta}_1)} \boldsymbol{\beta}_1 = \boldsymbol{\alpha}_2 - \frac{1}{2} \boldsymbol{\beta}_1 = (-\frac{1}{2}, 1, -\frac{1}{2})^T
\end{aligned}
$$

于是

$$
\begin{aligned}
\boldsymbol{A} = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \\
= \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \\
= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{\sqrt{6}}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \\ & \frac{\sqrt{3}}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \\
= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{\sqrt{6}}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{6}}{2} \end{pmatrix}
\end{aligned}
$$

(3) 对 $\boldsymbol{A}$ 进行 Schmidt 正交化，得：

$$\begin{aligned}
\boldsymbol{\beta}_1 &= \boldsymbol{\alpha}_1 = (0,1,1)^T \\
\boldsymbol{\beta}_2 &= \boldsymbol{\alpha}_2 - \frac{(\boldsymbol{\alpha}_2,\boldsymbol{\beta}_1)}{(\boldsymbol{\beta}_1,\boldsymbol{\beta}_1)}\boldsymbol{\beta}_1 = \boldsymbol{\alpha}_2 - \frac{1}{2}\boldsymbol{\beta}_1 = (1,\frac{1}{2},-\frac{1}{2})^T \\
\boldsymbol{\beta}_3 &= \boldsymbol{\alpha}_3 - \frac{(\boldsymbol{\alpha}_3,\boldsymbol{\beta}_2)}{(\boldsymbol{\beta}_2,\boldsymbol{\beta}_2)}\boldsymbol{\beta}_2 - \frac{(\boldsymbol{\alpha}_3,\boldsymbol{\beta}_1)}{(\boldsymbol{\beta}_1,\boldsymbol{\beta}_1)}\boldsymbol{\beta}_1 = \boldsymbol{\alpha}_3 - \frac{2}{3}\boldsymbol{\beta}_2 = (\frac{1}{3},-\frac{1}{3},\frac{1}{3})^T
\end{aligned}$$

于是

$$\begin{aligned}
\boldsymbol{A} = (\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3) &= (\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\boldsymbol{\beta}_3)\begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & \frac{1}{3} \\ 1 & \frac{1}{2} & -\frac{1}{3} \\ 1 & -\frac{1}{2} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{\sqrt{3}}{3} \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{3}} & -\frac{\sqrt{3}}{3} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{3}} & \frac{\sqrt{3}}{3} \end{pmatrix}\begin{pmatrix} \sqrt{2} & & \\ & \frac{\sqrt{6}}{2} & \\ & & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{\sqrt{3}}{3} \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{3}} & -\frac{\sqrt{3}}{3} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{3}} & \frac{\sqrt{3}}{3} \end{pmatrix}\begin{pmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{3} \\ 0 & 0 & \frac{\sqrt{3}}{3} \end{pmatrix}
\end{aligned}$$

(4) 对 $\boldsymbol{A}$ 进行 Schmidt 正交化，得：

$$\begin{aligned}
\boldsymbol{\beta}_1 &= \boldsymbol{\alpha}_1 = (2,0,2)^T \\
\boldsymbol{\beta}_2 &= \boldsymbol{\alpha}_2 - \frac{(\boldsymbol{\alpha}_2,\boldsymbol{\beta}_1)}{(\boldsymbol{\beta}_1,\boldsymbol{\beta}_1)}\boldsymbol{\beta}_1 = \boldsymbol{\alpha}_2 - \frac{3}{4}\boldsymbol{\beta}_1 = (\frac{1}{2},2,-\frac{1}{2})^T \\
\boldsymbol{\beta}_3 &= \boldsymbol{\alpha}_3 - \frac{(\boldsymbol{\alpha}_3,\boldsymbol{\beta}_2)}{(\boldsymbol{\beta}_2,\boldsymbol{\beta}_2)}\boldsymbol{\beta}_2 - \frac{(\boldsymbol{\alpha}_3,\boldsymbol{\beta}_1)}{(\boldsymbol{\beta}_1,\boldsymbol{\beta}_1)}\boldsymbol{\beta}_1 = \boldsymbol{\alpha}_3 - \frac{7}{9}\boldsymbol{\beta}_2 - \frac{3}{4}\boldsymbol{\beta}_1 = (-\frac{8}{9},\frac{4}{9},\frac{8}{9})^T
\end{aligned}$$

于是

$$
\boldsymbol{A}=(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)=(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\boldsymbol{\beta}_3)\begin{pmatrix}1&\frac{3}{4}&\frac{3}{4}\\0&1&\frac{7}{9}\\0&0&1\end{pmatrix}
$$

$$
=\begin{pmatrix}2&\frac{1}{2}&-\frac{8}{9}\\0&2&\frac{4}{9}\\2&-\frac{1}{2}&\frac{8}{9}\end{pmatrix}\begin{pmatrix}1&\frac{3}{4}&\frac{3}{4}\\0&1&\frac{7}{9}\\0&0&1\end{pmatrix}
$$

$$
=\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{3\sqrt{2}}&-\frac{2}{3}\\0&\frac{4}{3\sqrt{2}}&\frac{1}{3}\\\frac{1}{\sqrt{2}}&-\frac{1}{3\sqrt{2}}&\frac{2}{3}\end{pmatrix}\begin{pmatrix}2\sqrt{2}&&\\&\frac{3}{\sqrt{2}}&\\&&\frac{4}{3}\end{pmatrix}\begin{pmatrix}1&\frac{3}{4}&\frac{3}{4}\\0&1&\frac{7}{9}\\0&0&1\end{pmatrix}
$$

$$
=\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{3\sqrt{2}}&-\frac{2}{3}\\0&\frac{4}{3\sqrt{2}}&\frac{1}{3}\\\frac{1}{\sqrt{2}}&-\frac{1}{3\sqrt{2}}&\frac{2}{3}\end{pmatrix}\begin{pmatrix}2\sqrt{2}&\frac{3\sqrt{2}}{2}&\frac{3\sqrt{2}}{2}\\0&\frac{3}{\sqrt{2}}&\frac{7}{3\sqrt{2}}\\0&0&\frac{4}{3}\end{pmatrix}
$$

□

题 1.8. 计算下列矩阵的奇异值分解。

(1) $\begin{pmatrix}2&0\\0&2\\1&1\end{pmatrix}$ (2) $\begin{pmatrix}1&0&1\\0&1&-1\end{pmatrix}$

解. (1) 不妨令 $\boldsymbol{A}=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$，其中 $\boldsymbol{\Sigma}=\begin{pmatrix}\boldsymbol{D}&\boldsymbol{O}\\\boldsymbol{O}&\boldsymbol{O}\end{pmatrix}$。先计算 $\boldsymbol{V}^T$，即利用相似对角化计算

$\boldsymbol{V}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{V}=\begin{pmatrix}\boldsymbol{D}^2&\boldsymbol{O}\\\boldsymbol{O}&\boldsymbol{O}\end{pmatrix}$，其中 $\boldsymbol{A}^T\boldsymbol{A}=\begin{pmatrix}5&1\\1&5\end{pmatrix}$，$\boldsymbol{V}^T=\begin{pmatrix}\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}$，$\boldsymbol{\Sigma}=\begin{pmatrix}2&0\\0&\sqrt{6}\\0&0\end{pmatrix}$，

此时 $\boldsymbol{V}_1=\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}$，$\boldsymbol{D}=\begin{pmatrix}2&0\\0&\sqrt{6}\end{pmatrix}$。

$$
\boldsymbol{U}_1=\boldsymbol{A}\boldsymbol{V}_1\boldsymbol{D}^{-1}=\begin{pmatrix}2&0\\0&\sqrt{6}\\0&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{2}&0\\0&\frac{1}{\sqrt{6}}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}\\0&\frac{1}{\sqrt{3}}\end{pmatrix}
$$

把 $\boldsymbol{U}_1$ 扩充成正交矩阵 $\boldsymbol{U}=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}}\end{pmatrix}$，于是：

$$\boldsymbol{A}=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & \sqrt{6}\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

(2) 不妨令 $\boldsymbol{A}=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$，其中 $\boldsymbol{\Sigma}=\begin{pmatrix}\boldsymbol{D} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{O}\end{pmatrix}$。先计算 $\boldsymbol{V}^T$，即利用相似对角化计算 $\boldsymbol{V}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{V}=\begin{pmatrix}\boldsymbol{D}^2 & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{O}\end{pmatrix}$，其中 $\boldsymbol{A}^T\boldsymbol{A}=\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 1 & -1 & 2\end{pmatrix}$，$\boldsymbol{V}^T=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}\\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{pmatrix}$，$\boldsymbol{\Sigma}=\begin{pmatrix}1 & 0 & 0\\ 0 & \sqrt{3} & 0\end{pmatrix}$，此时 $\boldsymbol{V}_1=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ 0 & \frac{2}{\sqrt{6}}\end{pmatrix}$，$\boldsymbol{D}=\begin{pmatrix}1 & 0\\ 0 & \sqrt{3}\end{pmatrix}$。

$$\boldsymbol{U}_1=\boldsymbol{A}\boldsymbol{V}_1\boldsymbol{D}^{-1}=\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ 0 & \frac{2}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \frac{1}{\sqrt{3}}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}$$

由于 $\boldsymbol{U}_1$ 已经是正交矩阵，无需扩充，令 $\boldsymbol{U}=\boldsymbol{U}_1$，于是：

$$\boldsymbol{A}=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \sqrt{3} & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}\\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{pmatrix}$$

□

题 1.9. 证明：对任意实（复）非退化方阵 $\boldsymbol{A}$，存在唯一的正交（酉）矩阵 $\boldsymbol{Q}$ 和正定矩阵 $\boldsymbol{H}_1$ 和 $\boldsymbol{H}_2$，使得 $\boldsymbol{A}=\boldsymbol{Q}\boldsymbol{H}_1=\boldsymbol{H}_2\boldsymbol{Q}$，该分解称为矩阵的**极分解**，若去掉矩阵的非退化条件，结论改如何修正？

证明. 先给出需要用到的引理：

引理 1.1. 任意一个正定矩阵 $\boldsymbol{A}$，一定存在唯一的一个正定矩阵 $\boldsymbol{S}$ 使得 $\boldsymbol{A}=\boldsymbol{S}^2$

证明. **存在性**:由于 $\boldsymbol{H}\succ\boldsymbol{0}$,则存在正交矩阵 $\boldsymbol{P}$,使得 $\boldsymbol{H}=\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^T$,其中 $\boldsymbol{\Lambda}=\operatorname{diag}\{\lambda_1,\ldots,\lambda_n\}$, $\lambda_i>0$。令 $\boldsymbol{Z}=\boldsymbol{\Lambda}^{\frac{1}{2}}=\operatorname{diag}\{\sqrt{\lambda_1},\ldots,\sqrt{\lambda_n}\}$，则

$$\boldsymbol{H}=\boldsymbol{P}\boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{P}^T=(\boldsymbol{P}\boldsymbol{Z}\boldsymbol{P}^T)(\boldsymbol{P}\boldsymbol{Z}\boldsymbol{P}^T)=(\boldsymbol{P}\boldsymbol{Z}\boldsymbol{P}^T)^2=\boldsymbol{S}^2$$

而任取 $\boldsymbol{x}\in\mathbb{R}^n$，$\boldsymbol{x}^T\boldsymbol{S}\boldsymbol{x}=\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{Z}\boldsymbol{P}^T\boldsymbol{x}=(\boldsymbol{P}^T\boldsymbol{x})^T\boldsymbol{Z}(\boldsymbol{P}^T\boldsymbol{x})>0$，这说明了 $\boldsymbol{S}$ 是正定阵。

唯一性：记矩阵 $\boldsymbol{A}$ 的特征值与对应的特征向量为 $\lambda,\boldsymbol{\nu}$。若存在两个正定阵 $\boldsymbol{S}_0,\boldsymbol{S}_1$，使得 $\boldsymbol{H}=\boldsymbol{S}_0^2=\boldsymbol{S}_1^2$，显然有 $\boldsymbol{S}_0\boldsymbol{\nu}=\boldsymbol{S}_1\boldsymbol{\nu}=\sqrt{\lambda}\boldsymbol{\nu}$。于是 $\boldsymbol{S}_0\boldsymbol{S}_1\boldsymbol{\nu}=\sqrt{\lambda}\boldsymbol{S}_0\boldsymbol{\nu}=\lambda\boldsymbol{\nu},\boldsymbol{S}_0\boldsymbol{S}_1\boldsymbol{\nu}=\sqrt{\lambda}\boldsymbol{S}_0\boldsymbol{\nu}=\lambda\boldsymbol{\nu},(\boldsymbol{S}_0\boldsymbol{S}_1-\boldsymbol{S}_1\boldsymbol{S}_0)\boldsymbol{\nu}=\boldsymbol{0}$，由于 $\boldsymbol{H}$ 是对称的，一定存在 n 个线性无关的特征向量，即特征子空间的维数 $\dim(\boldsymbol{V})=n$，则 $\mathrm{r}(\boldsymbol{S}_0\boldsymbol{S}_1-\boldsymbol{S}_1\boldsymbol{S}_0)=n-\dim(\boldsymbol{V})=0$，即 $\boldsymbol{S}_0\boldsymbol{S}_1-\boldsymbol{S}_1\boldsymbol{S}_0=\boldsymbol{O}$。此时 $(\boldsymbol{S}_0+\boldsymbol{S}_1)(\boldsymbol{S}_0-\boldsymbol{S}_1)=\boldsymbol{S}_0^2-\boldsymbol{S}_0\boldsymbol{S}_1+\boldsymbol{S}_1\boldsymbol{S}_0-\boldsymbol{S}_1^2=\boldsymbol{O}$，于是 $\mathrm{r}(\boldsymbol{S}_0+\boldsymbol{S}_1)+\mathrm{r}(\boldsymbol{S}_0-\boldsymbol{S}_1)=n$，而 $\boldsymbol{S}_0+\boldsymbol{S}_1$ 是正定矩阵，即 $\mathrm{r}(\boldsymbol{S}_0-\boldsymbol{S}_1)=n$，所以 $\mathrm{r}(\boldsymbol{S}_0-\boldsymbol{S}_1)=0$，$\boldsymbol{S}_0=\boldsymbol{S}_1$。

□

下面给出两种证明方法。

(1) **存在性**:由于 $\boldsymbol{A}^T\boldsymbol{A}$ 是一个正定矩阵,由引理1.1可知,$\boldsymbol{A}^T\boldsymbol{A}=\boldsymbol{H}_1^2$,则有 $\boldsymbol{E}=\boldsymbol{H}_1^{-1}\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{H}_1^{-1}=(\boldsymbol{H}_1^T)^{-1}\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{H}_1^{-1}=(\boldsymbol{A}\boldsymbol{H}_1^{-1})^T\boldsymbol{A}\boldsymbol{H}_1^{-1}$，令 $\boldsymbol{Q}_1=\boldsymbol{A}\boldsymbol{H}_1^{-1}$，有 $\boldsymbol{Q}_1^T\boldsymbol{Q}_1=\boldsymbol{E}$，故 $\boldsymbol{Q}_1$ 是正交矩阵，同时 $\boldsymbol{A}=\boldsymbol{Q}_1\boldsymbol{H}_1$。

同理：$\boldsymbol{A}\boldsymbol{A}^T=\boldsymbol{H}_2^2$，则有 $\boldsymbol{E}=\boldsymbol{H}_2^{-1}\boldsymbol{A}\boldsymbol{A}^T\boldsymbol{H}_2^{-1}=\boldsymbol{H}_2^{-1}\boldsymbol{A}\boldsymbol{A}^T(\boldsymbol{H}_2^T)^{-1}=\boldsymbol{H}_2^{-1}\boldsymbol{A}(\boldsymbol{H}_2^{-1}\boldsymbol{A})^T$，令 $\boldsymbol{Q}_2=\boldsymbol{H}_2^{-1}\boldsymbol{A}$，有 $\boldsymbol{Q}_2\boldsymbol{Q}_2^T=\boldsymbol{E}$，故 $\boldsymbol{Q}_2$ 是正交矩阵，同时 $\boldsymbol{A}=\boldsymbol{H}_2\boldsymbol{Q}_2$。

下面证明 $\boldsymbol{Q}_1=\boldsymbol{Q}_2=\boldsymbol{Q}$，即证 $\boldsymbol{A}\boldsymbol{H}_1^{-1}=\boldsymbol{H}_2^{-1}\boldsymbol{A}\Leftrightarrow\boldsymbol{H}_2\boldsymbol{A}=\boldsymbol{A}\boldsymbol{H}_1$，

唯一性：假设存在另外一个正交矩阵 $\boldsymbol{U}$ 与正定矩阵 $\boldsymbol{W}$，使得 $\boldsymbol{A}=\boldsymbol{U}\boldsymbol{W}$ 由引理1.1唯一性可知，$\boldsymbol{W}=\boldsymbol{H}_1$；而 $\boldsymbol{U}=\boldsymbol{A}\boldsymbol{W}^{-1}=\boldsymbol{A}\boldsymbol{H}_1^{-1}=\boldsymbol{Q}$，同理可以说明 $\boldsymbol{A}=\boldsymbol{H}_2\boldsymbol{Q}$ 分解的唯一性。

(2) **存在性**: 由 SVD 分解可知, $\boldsymbol{A}=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$,其中 $\boldsymbol{U},\boldsymbol{V}^T$ 是正交矩阵,$\boldsymbol{\Sigma}=\operatorname{diag}\{\lambda_1,\ldots,\lambda_n\}$, $\lambda_i>0$。又因为 $\boldsymbol{A}=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T=\boldsymbol{U}(\boldsymbol{V}^T\boldsymbol{V})\boldsymbol{\Sigma}\boldsymbol{V}^T=(\boldsymbol{U}\boldsymbol{V}^T)\boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{V}^T$，令 $\boldsymbol{Q}=\boldsymbol{U}\boldsymbol{V}^T,\boldsymbol{H}_1=\boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{V}^T$，容易验证 $\boldsymbol{Q}$ 为正交矩阵，$\boldsymbol{H}_1$ 为正定矩阵。

同理：$\boldsymbol{A}=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T=\boldsymbol{U}\boldsymbol{\Sigma}(\boldsymbol{U}^T\boldsymbol{U})\boldsymbol{V}^T=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{U}^T(\boldsymbol{U}\boldsymbol{V}^T)$，令 $\boldsymbol{Q}=\boldsymbol{U}\boldsymbol{V}^T,\boldsymbol{H}_2=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{U}^T$。

唯一性：假设存在另外一个正交矩阵 $\boldsymbol{U}$ 与正定矩阵 $\boldsymbol{W}$，使得 $\boldsymbol{A}=\boldsymbol{U}\boldsymbol{W}$，$\boldsymbol{A}^T\boldsymbol{A}=(\boldsymbol{Q}\boldsymbol{H}_1)^T\boldsymbol{Q}\boldsymbol{H}_1=\boldsymbol{H}_1^2, \boldsymbol{A}^T\boldsymbol{A}=(\boldsymbol{U}\boldsymbol{W})^T\boldsymbol{U}\boldsymbol{W}=\boldsymbol{W}^2$，由引理1.1唯一性可知，$\boldsymbol{W}=\boldsymbol{H}_1$，而 $\boldsymbol{A}\boldsymbol{H}_1^{-1}=\boldsymbol{Q}, \boldsymbol{A}\boldsymbol{W}^{-1}=\boldsymbol{U}$，于是 $\boldsymbol{U}=\boldsymbol{Q}$。同理可以说明 $\boldsymbol{A}=\boldsymbol{H}_2\boldsymbol{Q}$ 分解的唯一性。

结论修正为：存在唯一的酉矩阵 $\boldsymbol{Q}$ 与半正定矩阵 $\boldsymbol{H}_1$ 与 $\boldsymbol{H}_2$ 使得 $\boldsymbol{A}=\boldsymbol{Q}\boldsymbol{H}_1=\boldsymbol{H}_2\boldsymbol{Q}$。

□

题 1.10. 证明：对任何正定矩阵 $\boldsymbol{H}$，存在唯一的正定矩阵 $\boldsymbol{S}$，使得 $\boldsymbol{H}=\boldsymbol{S}^2$。若将正定矩阵改为半正定矩阵，结论如何？

证明. **存在性:** 由于 $\boldsymbol{H}\succ\boldsymbol{0}$，则存在正交矩阵 $\boldsymbol{P}$，使得 $\boldsymbol{H}=\boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^T$，其中 $\boldsymbol{\Lambda}=\operatorname{diag}\{\lambda_1,\ldots,\lambda_n\}$，$\lambda_i>0$。令 $\boldsymbol{Z}=\boldsymbol{\Lambda}^{\frac{1}{2}}=\operatorname{diag}\{\sqrt{\lambda_1},\ldots,\sqrt{\lambda_n}\}$，则

$$\boldsymbol{H}=\boldsymbol{P}\boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{P}^T=(\boldsymbol{P}\boldsymbol{Z}\boldsymbol{P}^T)(\boldsymbol{P}\boldsymbol{Z}\boldsymbol{P}^T)=(\boldsymbol{P}\boldsymbol{Z}\boldsymbol{P}^T)^2=\boldsymbol{S}^2$$

而任取 $\boldsymbol{x}\in\mathbb{R}^n$，$\boldsymbol{x}^T\boldsymbol{S}\boldsymbol{x}=\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{Z}\boldsymbol{P}^T\boldsymbol{x}=(\boldsymbol{P}^T\boldsymbol{x})^T\boldsymbol{Z}(\boldsymbol{P}^T\boldsymbol{x})>0$，这说明了 $\boldsymbol{S}$ 是正定阵。

唯一性：记矩阵 $\boldsymbol{A}$ 的特征值与对应的特征向量为 $\lambda,\boldsymbol{\nu}$。若存在两个正定阵 $\boldsymbol{S}_0,\boldsymbol{S}_1$，使得 $\boldsymbol{H}=\boldsymbol{S}_0^2=\boldsymbol{S}_1^2$，显然有 $\boldsymbol{S}_0\boldsymbol{\nu}=\boldsymbol{S}_1\boldsymbol{\nu}=\sqrt{\lambda}\boldsymbol{\nu}$。于是 $\boldsymbol{S}_0\boldsymbol{S}_1\boldsymbol{\nu}=\sqrt{\lambda}\boldsymbol{S}_0\boldsymbol{\nu}=\lambda\boldsymbol{\nu}, \boldsymbol{S}_0\boldsymbol{S}_1\boldsymbol{\nu}=\sqrt{\lambda}\boldsymbol{S}_0\boldsymbol{\nu}=\lambda\boldsymbol{\nu}, (\boldsymbol{S}_0\boldsymbol{S}_1-\boldsymbol{S}_1\boldsymbol{S}_0)\boldsymbol{\nu}=\boldsymbol{0}$，由于 $\boldsymbol{H}$ 是对称的，一定存在 n 个线性无关的特征向量，即特征子空间的维数 $\dim(\boldsymbol{V})=n$，则 $\mathrm{r}(\boldsymbol{S}_0\boldsymbol{S}_1-\boldsymbol{S}_1\boldsymbol{S}_0)=n-\dim(\boldsymbol{V})=0$，即 $\boldsymbol{S}_0\boldsymbol{S}_1-\boldsymbol{S}_1\boldsymbol{S}_0=\boldsymbol{O}$。此时 $(\boldsymbol{S}_0+\boldsymbol{S}_1)(\boldsymbol{S}_0-\boldsymbol{S}_1)=\boldsymbol{S}_0^2-\boldsymbol{S}_0\boldsymbol{S}_1+\boldsymbol{S}_1\boldsymbol{S}_0-\boldsymbol{S}_1^2=\boldsymbol{O}$，于是 $\mathrm{r}(\boldsymbol{S}_0+\boldsymbol{S}_1)+\mathrm{r}(\boldsymbol{S}_0-\boldsymbol{S}_1)=n$，而 $\boldsymbol{S}_0+\boldsymbol{S}_1$ 是正定矩阵，即 $\mathrm{r}(\boldsymbol{S}_0-\boldsymbol{S}_1)=n$，所以 $\mathrm{r}(\boldsymbol{S}_0-\boldsymbol{S}_1)=0$，$\boldsymbol{S}_0=\boldsymbol{S}_1$。

结论修正为：存在唯一的半正定矩阵 $\boldsymbol{S}$，使得 $\boldsymbol{H}=\boldsymbol{S}^2$。

□