第一章　广义逆矩阵

题 1.1. （P162.3）求下列矩阵的广义逆 $\boldsymbol{A}^{+}$。

(1) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \end{pmatrix}$　(2) $\begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 0 \end{pmatrix}$　(3) $\begin{pmatrix} -2 & 0 & 0 & -2 \\ 1 & 2 & -4 & 3 \\ 2 & -1 & 2 & 1 \\ 0 & 2 & -4 & 2 \end{pmatrix}$

解. (1)

$$\boldsymbol{A} = \boldsymbol{B}\boldsymbol{C} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\boldsymbol{C}\boldsymbol{C}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{B}^T\boldsymbol{B} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$$

$$\begin{aligned} \boldsymbol{A}^{+} &= \boldsymbol{C}^T(\boldsymbol{C}\boldsymbol{C}^T)^{-1}(\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

(2)

$$\boldsymbol{A} = \boldsymbol{B}\boldsymbol{C} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ -1 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$

$$\boldsymbol{C}\boldsymbol{C}^T = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}, \boldsymbol{B}^T\boldsymbol{B} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{aligned}
\boldsymbol{A}^+ &= \boldsymbol{C}^T(\boldsymbol{C}\boldsymbol{C}^T)^{-1}(\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{10} & 0 & -\frac{1}{10} & 0 \\ \frac{1}{5} & 0 & \frac{1}{5} & 0 \end{pmatrix}
\end{aligned}$$

(3)

$$\boldsymbol{A} = \boldsymbol{B}\boldsymbol{C} = \begin{pmatrix} -2 & 0 \\ 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 1 \end{pmatrix}$$

$$\boldsymbol{C}\boldsymbol{C}^T = \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}, \boldsymbol{B}^T\boldsymbol{B} = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$$

$$\begin{aligned}
\boldsymbol{A}^+ &= \boldsymbol{C}^T(\boldsymbol{C}\boldsymbol{C}^T)^{-1}(\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}^{-1} \begin{pmatrix} -2 & 1 & 2 & 0 \\ 0 & 2 & -1 & 2 \end{pmatrix} \\
&= \frac{1}{99} \begin{pmatrix} -12 & 4 & 13 & -2 \\ 2 & 3 & -4 & 4 \\ -4 & -6 & 8 & -8 \\ -10 & 7 & 9 & 2 \end{pmatrix}
\end{aligned}$$

$\square$

题 1.2. （P163.9）验证线性方程组 $\boldsymbol{Ax}=\boldsymbol{b}$ 有解，并求其通解和最小长度解，其中 $\boldsymbol{A}=\begin{pmatrix}1&2\\0&0\\2&4\end{pmatrix},\boldsymbol{b}=\begin{pmatrix}-1\\0\\-2\end{pmatrix}$

解. $(\boldsymbol{A}|\boldsymbol{b})=\begin{pmatrix}1&2&-1\\0&0&0\\2&4&-2\end{pmatrix}\to\begin{pmatrix}1&2&-1\\0&0&0\\0&0&0\end{pmatrix}$，则 $\mathrm{r}(\boldsymbol{A}|\boldsymbol{b})=\mathrm{r}(\boldsymbol{A})$，于是方程组有解。

$$\boldsymbol{A}=\boldsymbol{BC}=\begin{pmatrix}1\\0\\2\end{pmatrix}\begin{pmatrix}1&2\end{pmatrix}$$

$$\boldsymbol{CC}^T=\begin{pmatrix}5\end{pmatrix},\boldsymbol{B}^T\boldsymbol{B}=\begin{pmatrix}5\end{pmatrix}$$

$$\begin{aligned}\boldsymbol{A}^+&=\boldsymbol{C}^T(\boldsymbol{CC}^T)^{-1}(\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\\&=\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}5\end{pmatrix}^{-1}\begin{pmatrix}5\end{pmatrix}^{-1}\begin{pmatrix}1&0&2\end{pmatrix}\\&=\frac{1}{25}\begin{pmatrix}1&0&2\\2&0&4\end{pmatrix}\end{aligned}$$

则通解为 $\boldsymbol{x}=\boldsymbol{A}^+\boldsymbol{b}+(\boldsymbol{E}-\boldsymbol{A}^+\boldsymbol{A})\boldsymbol{y}=\begin{pmatrix}-\frac{1}{5}+\frac{4}{5}y_1-\frac{2}{5}y_2\\-\frac{2}{5}-\frac{2}{5}y_1+\frac{1}{5}y_1\end{pmatrix}$，其中 $\boldsymbol{y}\in\mathbb{R}^2$。最小长度解为 $\boldsymbol{x}^*=\begin{pmatrix}-\frac{1}{5}\\-\frac{2}{5}\end{pmatrix}$。

□

题 1.3. （P163.10）验证下列线性方程组 $\boldsymbol{Ax}=\boldsymbol{b}$ 为矛盾方程组，并求其最小二乘解的通解和最小长度二乘解。

(1)$\boldsymbol{A}=\begin{pmatrix}1&2\\0&0\\2&4\end{pmatrix},\boldsymbol{b}=\begin{pmatrix}1\\1\\2\end{pmatrix}$　　(2)$\boldsymbol{A}=\begin{pmatrix}1&2&-1\\-3&-6&3\end{pmatrix},\boldsymbol{b}=\begin{pmatrix}1\\1\end{pmatrix}$

(3)$\boldsymbol{A}=\begin{pmatrix}1&1\\2&0\\-1&3\end{pmatrix},\boldsymbol{b}=\begin{pmatrix}1\\0\\2\end{pmatrix}$

解. (1) 则 $\mathrm{r}(\boldsymbol{A}|\boldsymbol{b})=2,\mathrm{r}(\boldsymbol{A})=1,\mathrm{r}(\boldsymbol{A})\neq\mathrm{r}(\boldsymbol{A}|\boldsymbol{b})$，于是为矛盾方程组。

$$\boldsymbol{A}=\boldsymbol{BC}=\begin{pmatrix}1\\0\\2\end{pmatrix}\begin{pmatrix}1&2\end{pmatrix}$$

$$\boldsymbol{CC}^T=\begin{pmatrix}5\end{pmatrix},\boldsymbol{B}^T\boldsymbol{B}=\begin{pmatrix}5\end{pmatrix}$$

$$\begin{aligned}\boldsymbol{A}^+&=\boldsymbol{C}^T(\boldsymbol{CC}^T)^{-1}(\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\\&=\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}5\end{pmatrix}^{-1}\begin{pmatrix}5\end{pmatrix}^{-1}\begin{pmatrix}1&0&2\end{pmatrix}\\&=\frac{1}{25}\begin{pmatrix}1&0&2\\2&0&4\end{pmatrix}\end{aligned}$$

最小二乘解为 $\boldsymbol{x}=\boldsymbol{A}^+\boldsymbol{b}+(\boldsymbol{E}-\boldsymbol{A}^+\boldsymbol{A})\boldsymbol{y}=\begin{pmatrix}\frac{1}{5}+\frac{4}{5}y_1-\frac{2}{5}y_2\\\frac{2}{5}-\frac{2}{5}y_1+\frac{1}{5}y_1\end{pmatrix}$，其中 $\boldsymbol{y}\in\mathbb{R}^2$。最小长度二乘解为 $\boldsymbol{x}^*=\begin{pmatrix}\frac{1}{5}\\\frac{2}{5}\end{pmatrix}$。

(2) 则 $\mathrm{r}(\boldsymbol{A}|\boldsymbol{b})=2,\mathrm{r}(\boldsymbol{A})=1,\mathrm{r}(\boldsymbol{A})\neq\mathrm{r}(\boldsymbol{A}|\boldsymbol{b})$，于是为矛盾方程组。

$$\boldsymbol{A}=\boldsymbol{BC}=\begin{pmatrix}1\\-3\end{pmatrix}\begin{pmatrix}1&2&-1\end{pmatrix}$$

$$\boldsymbol{CC}^T=\begin{pmatrix}6\end{pmatrix},\boldsymbol{B}^T\boldsymbol{B}=\begin{pmatrix}10\end{pmatrix}$$

$$\begin{aligned}
\boldsymbol{A}^{+} &= \boldsymbol{C}^{T}(\boldsymbol{C}\boldsymbol{C}^{T})^{-1}(\boldsymbol{B}^{T}\boldsymbol{B})^{-1}\boldsymbol{B}^{T} \\
&= \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 6 \end{pmatrix}^{-1} \begin{pmatrix} 10 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -3 \end{pmatrix} \\
&= \frac{1}{60} \begin{pmatrix} 1 & -3 \\ 2 & -6 \\ -1 & 3 \end{pmatrix}
\end{aligned}$$

最小二乘解为 $\boldsymbol{x} = \boldsymbol{A}^{+}\boldsymbol{b} + (\boldsymbol{E} - \boldsymbol{A}^{+}\boldsymbol{A})\boldsymbol{y} = \begin{pmatrix} -\frac{1}{30} + \frac{5}{6}y_1 - \frac{1}{3}y_2 + \frac{1}{6}y_3 \\ -\frac{1}{15} - \frac{1}{3}y_1 + \frac{1}{3}y_2 + \frac{1}{3}y_3 \\ \frac{1}{30} + \frac{1}{6}y_1 + \frac{1}{3}y_2 + \frac{5}{6}y_3 \end{pmatrix}$，其中 $\boldsymbol{y} \in \mathbb{R}^3$。最小长度二乘解为 $\boldsymbol{x}^* = \begin{pmatrix} -\frac{1}{30} \\ -\frac{1}{15} \\ \frac{1}{30} \end{pmatrix}$。

(3) 则 $\mathrm{r}(\boldsymbol{A}|\boldsymbol{b}) = 3, \mathrm{r}(\boldsymbol{A}) = 2, \mathrm{r}(\boldsymbol{A}) \neq \mathrm{r}(\boldsymbol{A}|\boldsymbol{b})$，于是为矛盾方程组。

$$\boldsymbol{A} = \boldsymbol{B}\boldsymbol{C} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boldsymbol{C}\boldsymbol{C}^{T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{B}^{T}\boldsymbol{B} = \begin{pmatrix} 6 & -2 \\ -2 & 10 \end{pmatrix}$$

$$\begin{aligned}
\boldsymbol{A}^{+} &= \boldsymbol{C}^{T}(\boldsymbol{C}\boldsymbol{C}^{T})^{-1}(\boldsymbol{B}^{T}\boldsymbol{B})^{-1}\boldsymbol{B}^{T} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 6 & -2 \\ -2 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 3 \end{pmatrix} \\
&= \frac{1}{14} \begin{pmatrix} 3 & 5 & -1 \\ 2 & 1 & 4 \end{pmatrix}
\end{aligned}$$

最小二乘解为 $\boldsymbol{x} = \boldsymbol{A}^{+}\boldsymbol{b} + (\boldsymbol{E} - \boldsymbol{A}^{+}\boldsymbol{A})\boldsymbol{y} = \begin{pmatrix} \frac{1}{14} \\ \frac{10}{14} \end{pmatrix}$，其中 $\boldsymbol{y} \in \mathbb{R}^3$。最小长度二乘解为 $\boldsymbol{x}^* = \begin{pmatrix} \frac{1}{14} \\ \frac{10}{14} \end{pmatrix}$。

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