

第一章 矩阵分解

题 1.1. (p140.1) 计算下列矩阵的 Doolittle 分解, Crout 三角分解和 LDU 三角分解。

$$(1) \begin{pmatrix} 2 & 4 & 6 \\ 2 & 7 & 12 \\ -2 & -10 & -13 \end{pmatrix} \quad (2) \begin{pmatrix} 4 & 8 & 0 \\ 4 & 11 & 6 \\ -6 & -12 & 10 \end{pmatrix}$$

解. (1) 先对矩阵进行初等行变换变成上三角矩阵

$$\begin{pmatrix} 2 & 4 & 6 \\ 2 & 7 & 12 \\ -2 & -10 & -13 \end{pmatrix} \xrightarrow{r_2-r_1} \begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 6 \\ -2 & -10 & -13 \end{pmatrix} \xrightarrow{r_3+r_1} \begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 6 \\ 0 & -6 & -7 \end{pmatrix} \xrightarrow{r_3+2r_2} \begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{pmatrix}$$

则 \mathbf{R} 为上面所求, 将每一步变换用初等矩阵表示, 则 $\mathbf{P}_3\mathbf{P}_2\mathbf{P}_1\mathbf{A} = \mathbf{R}$, 于是

$$\mathbf{L} = (\mathbf{P}_3\mathbf{P}_2\mathbf{P}_1)^{-1} = \mathbf{P}_1^{-1}\mathbf{P}_2^{-1}\mathbf{P}_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix}$$

Doolittle 分解、Crout 分解、LDU 分解

$$\begin{aligned} \mathbf{LR} &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 6 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{pmatrix} \\ \mathbf{LDU} &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & & \\ & 3 & \\ & & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \\ \mathbf{LU} &= \begin{pmatrix} 2 & 0 & 0 \\ 2 & 3 & 0 \\ -2 & -6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

其中空白位置全是 0。

(2) 先对矩阵进行初等行变换变成上三角矩阵

$$\begin{pmatrix} 4 & 8 & 0 \\ 4 & 11 & 6 \\ -6 & -12 & 10 \end{pmatrix} \xrightarrow{r_2 - r_1} \begin{pmatrix} 4 & 8 & 0 \\ 0 & 3 & 6 \\ -6 & -12 & 10 \end{pmatrix} \xrightarrow{r_3 + \frac{3}{2}r_1} \begin{pmatrix} 4 & 8 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 10 \end{pmatrix}$$

则 \mathbf{R} 为上面所求, 将每一步变换用初等矩阵表示, 则 $\mathbf{P}_2\mathbf{P}_1\mathbf{A} = \mathbf{R}$, 于是

$$\mathbf{L} = (\mathbf{P}_2\mathbf{P}_1)^{-1} = \mathbf{P}_1^{-1}\mathbf{P}_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{pmatrix}$$

Doolittle 分解、Crout 分解、LDU 分解

$$\begin{aligned} \mathbf{LR} &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 8 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 10 \end{pmatrix} \\ \mathbf{LDU} &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & & \\ & 3 & \\ & & 10 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \\ \mathbf{LU} &= \begin{pmatrix} 4 & 0 & 0 \\ 4 & 3 & 0 \\ -6 & 0 & 10 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

其中空白位置全是 0。

□

题 1.2. (p141.2) 计算下列矩阵的 Cholesky 分解。

$$(1) \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -3 \\ -1 & -3 & 6 \end{pmatrix} \qquad (2) \begin{pmatrix} 4 & 4 & -6 \\ 4 & 5 & -6 \\ -6 & -6 & 13 \end{pmatrix}$$

解. (1) 设 $\mathbf{G} = \begin{pmatrix} g_{11} & & \\ g_{21} & g_{22} & \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$, 则 $\mathbf{G}^T = \begin{pmatrix} g_{11} & g_{21} & g_{31} \\ & g_{22} & g_{32} \\ & & g_{33} \end{pmatrix}$, 其中空白位置全是 0。于是

有：

$$\left\{ \begin{array}{l} g_{11}^2 = 1 \\ g_{11}g_{21} = 1 \\ g_{11}g_{31} = -1 \\ g_{21}g_{11} = 1 \\ g_{21}^2 + g_{22}^2 = 2 \\ g_{21}g_{31} + g_{22}g_{32} = -3 \\ g_{31}g_{11} = -1 \\ g_{31}g_{21} + g_{32}g_{22} = -3 \\ g_{31}^2 + g_{32}^2 + g_{33}^2 = 6 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} g_{11} = 1 \\ g_{21} = 1 \\ g_{31} = -1 \\ g_{22} = 1 \\ g_{32} = -2 \\ g_{33} = 1 \end{array} \right.$$

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix} \quad \mathbf{A} = \mathbf{G}\mathbf{G}^T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) 设 $\mathbf{G} = \begin{pmatrix} g_{11} & & \\ g_{21} & g_{22} & \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$ ，则 $\mathbf{G}^T = \begin{pmatrix} g_{11} & g_{21} & g_{31} \\ & g_{22} & g_{32} \\ & & g_{33} \end{pmatrix}$ ，其中空白位置全是 0。于是有：

$$\left\{ \begin{array}{l} g_{11}^2 = 4 \\ g_{11}g_{21} = 4 \\ g_{11}g_{31} = -6 \\ g_{21}g_{11} = 4 \\ g_{21}^2 + g_{22}^2 = 5 \\ g_{21}g_{31} + g_{22}g_{32} = -6 \\ g_{31}g_{11} = -6 \\ g_{31}g_{21} + g_{32}g_{22} = -6 \\ g_{31}^2 + g_{32}^2 + g_{33}^2 = 13 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} g_{11} = 2 \\ g_{21} = 2 \\ g_{31} = -3 \\ g_{22} = 1 \\ g_{32} = 0 \\ g_{33} = 2 \end{array} \right.$$

$$\mathbf{G} = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 2 \end{pmatrix} \quad \mathbf{A} = \mathbf{G}\mathbf{G}^T = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

□

题 1.3. (p141.3) 计算矩阵 $\begin{pmatrix} 2 & 4 & 6 & 8 \\ 4 & 12 & 20 & 16 \\ 3 & 10 & 20 & 18 \\ 1 & -4 & -9 & 14 \end{pmatrix}$ 的 Doolittle 分解。

解. 先对矩阵进行初等行变换变成上三角矩阵

$$\begin{pmatrix} 2 & 4 & 6 & 8 \\ 4 & 12 & 20 & 16 \\ 3 & 10 & 20 & 18 \\ 1 & -4 & -9 & 14 \end{pmatrix} \xrightarrow{r_2-2r_1} \begin{pmatrix} 2 & 4 & 6 & 8 \\ 0 & 4 & 8 & 0 \\ 3 & 10 & 20 & 18 \\ 1 & -4 & -9 & 14 \end{pmatrix} \xrightarrow{r_3-\frac{3}{2}r_1} \begin{pmatrix} 2 & 4 & 6 & 8 \\ 0 & 4 & 8 & 0 \\ 0 & 4 & 11 & 6 \\ 1 & -4 & -9 & 14 \end{pmatrix} \xrightarrow{r_4-\frac{1}{2}r_1} \begin{pmatrix} 2 & 4 & 6 & 8 \\ 0 & 4 & 8 & 0 \\ 0 & 4 & 11 & 6 \\ 0 & -6 & -12 & 10 \end{pmatrix}$$

$$\xrightarrow{r_3-r_2} \begin{pmatrix} 2 & 4 & 6 & 8 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 3 & 6 \\ 0 & -6 & -12 & 10 \end{pmatrix} \xrightarrow{r_4+\frac{3}{2}r_2} \begin{pmatrix} 2 & 4 & 6 & 8 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

则 \mathbf{R} 为上面所求, 将每一步变换用初等矩阵表示, 则 $\mathbf{P}_5\mathbf{P}_4\mathbf{P}_3\mathbf{P}_2\mathbf{P}_1\mathbf{A} = \mathbf{R}$, 于是

$$\begin{aligned} \mathbf{L} &= (\mathbf{P}_5\mathbf{P}_4\mathbf{P}_3\mathbf{P}_2\mathbf{P}_1)^{-1} = \mathbf{P}_1^{-1}\mathbf{P}_2^{-1}\mathbf{P}_3^{-1}\mathbf{P}_4^{-1}\mathbf{P}_5^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{3}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{3}{2} & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{3}{2} & 1 & 1 & 0 \\ \frac{1}{2} & -\frac{3}{2} & 0 & 1 \end{pmatrix} \end{aligned}$$

$$\mathbf{A} = \mathbf{LR} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{3}{2} & 1 & 1 & 0 \\ \frac{1}{2} & -\frac{3}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 6 & 8 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

□

题 1.4. (p141.4) 计算矩阵 $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 8 & 10 & 2 \\ 3 & 10 & 14 & 6 \\ 4 & 2 & 6 & 29 \end{pmatrix}$ 的 Cholesky 分解。

解. 设 $\mathbf{G} = \begin{pmatrix} g_{11} & & & \\ g_{21} & g_{22} & & \\ g_{31} & g_{32} & g_{33} & \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix}$, 则 $\mathbf{G}^T = \begin{pmatrix} g_{11} & g_{21} & g_{31} & g_{41} \\ & g_{22} & g_{32} & g_{42} \\ & & g_{33} & g_{43} \\ & & & g_{44} \end{pmatrix}$, 其中空白位置全是 0。

于是有:

$$\left\{ \begin{array}{l} g_{11}^2 = 1 \\ g_{11}g_{21} = 2 \\ g_{11}g_{31} = 3 \\ g_{11}g_{41} = 4 \\ g_{21}g_{11} = 2 \\ g_{21}^2 + g_{22}^2 = 8 \\ g_{21}g_{31} + g_{22}g_{32} = 10 \\ g_{21}g_{41} + g_{22}g_{42} = 2 \\ g_{31}g_{11} = 3 \\ g_{31}g_{21} + g_{32}g_{22} = 10 \\ g_{31}^2 + g_{32}^2 + g_{33}^2 = 14 \\ g_{31}g_{41} + g_{32}g_{42} + g_{33}g_{43} = 6 \\ g_{41}g_{11} = 4 \\ g_{41}g_{21} + g_{42}g_{22} = 2 \\ g_{41}g_{31} + g_{42}g_{32} + g_{43}g_{33} = 6 \\ g_{41}^2 + g_{42}^2 + g_{43}^2 + g_{44}^2 = 29 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} g_{11} = 1 \\ g_{21} = 2 \\ g_{31} = 3 \\ g_{41} = 4 \\ g_{22} = 2 \\ g_{32} = 2 \\ g_{42} = -3 \\ g_{33} = 1 \\ g_{43} = 0 \\ g_{44} = 2 \end{array} \right.$$

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & -3 & 0 & 2 \end{pmatrix} \quad \mathbf{A} = \mathbf{G}\mathbf{G}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & -3 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

□

题 1.5. (p141.5) 计算下列矩阵的满秩分解。

$$(1) \begin{pmatrix} 1 & 2 & 3 & 3 \\ 4 & 5 & 9 & 6 \\ 7 & 8 & 15 & 9 \\ 2 & 5 & 7 & 8 \end{pmatrix}$$

$$(2) \begin{pmatrix} 1 & 3 & -3 & 4 \\ 3 & 5 & -5 & 8 \\ 6 & -1 & 1 & 5 \\ 8 & -6 & 6 & 2 \end{pmatrix}$$

解. (1) 先对矩阵进行初等行变换变成行阶梯型矩阵

$$\begin{pmatrix} 1 & 2 & 3 & 3 \\ 4 & 5 & 9 & 6 \\ 7 & 8 & 15 & 9 \\ 2 & 5 & 7 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{于是 } \mathbf{A} = \mathbf{BC} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

(2) 先对矩阵进行初等行变换变成行阶梯型矩阵

$$\begin{pmatrix} 1 & 3 & -3 & 4 \\ 3 & 5 & -5 & 8 \\ 6 & -1 & 1 & 5 \\ 8 & -6 & 6 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & -3 & 4 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{于是 } \mathbf{A} = \mathbf{BC} = \begin{pmatrix} 1 & 3 \\ 3 & 5 \\ 6 & -1 \\ 8 & -6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

□

题 1.6. (p141.6) 计算下列矩阵的谱分解。

$$(1) \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

$$(2) \begin{pmatrix} 5 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 7 \end{pmatrix}$$

解. (1) $\lambda_1 = \lambda_2 = 2, \lambda_3 = 4$ 所以

$$\begin{aligned} \mathbf{A} &= \mathbf{P}\mathbf{\Lambda}\mathbf{P}^T \\ &= \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & & \\ & 2 & \\ & & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} + 2 \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} + 4 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \end{aligned}$$

其中空白位置全是 0。

(2) $\lambda_1 = 3, \lambda_2 = 6, \lambda_3 = 9$ 所以

$$\begin{aligned} \mathbf{A} &= \mathbf{P}\mathbf{\Lambda}\mathbf{P}^T \\ &= \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 3 & & \\ & 6 & \\ & & 9 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix} \\ &= 3 \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} + 6 \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} + 9 \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix} \end{aligned}$$

其中空白位置全是 0。

□

题 1.7. (p141.7) 计算下列矩阵的 QR 分解。

$$\begin{aligned} (1) & \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} & (2) & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} & (3) & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} & (4) & \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 2 \\ 2 & 1 & 2 \end{pmatrix} \end{aligned}$$

解. 把矩阵记作 $\mathbf{A}_{m \times n}$, 并进行列分块 $(\mathbf{a}_1, \dots, \mathbf{a}_n)$ 。

(1) 对 \mathbf{A} 进行 Schmidt 正交化, 得:

$$\begin{aligned} \boldsymbol{\beta}_1 &= \boldsymbol{\alpha}_1 = (2, 1, 2)^T \\ \boldsymbol{\beta}_2 &= \boldsymbol{\alpha}_2 - \frac{(\boldsymbol{\alpha}_2, \boldsymbol{\beta}_1)}{(\boldsymbol{\beta}_1, \boldsymbol{\beta}_1)} \boldsymbol{\beta}_1 = \boldsymbol{\alpha}_2 - \frac{5}{9} \boldsymbol{\beta}_1 = \left(-\frac{1}{9}, \frac{4}{9}, -\frac{1}{9}\right)^T \end{aligned}$$

于是

$$\begin{aligned}
 \mathbf{A} &= (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) \begin{pmatrix} 1 & \frac{5}{9} \\ 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 2 & -\frac{1}{9} \\ 1 & \frac{4}{9} \\ 2 & -\frac{1}{9} \end{pmatrix} \begin{pmatrix} 1 & \frac{5}{9} \\ 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{2}{3} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{3} & \frac{4}{3\sqrt{2}} \\ \frac{2}{3} & -\frac{1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & \\ & \frac{\sqrt{2}}{3} \end{pmatrix} \begin{pmatrix} 1 & \frac{5}{9} \\ 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{2}{3} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{3} & \frac{4}{3\sqrt{2}} \\ \frac{2}{3} & -\frac{1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & \frac{5}{3} \\ 0 & \frac{\sqrt{2}}{3} \end{pmatrix}
 \end{aligned}$$

(2) 对 \mathbf{A} 进行 Schmidt 正交化, 得:

$$\begin{aligned}
 \boldsymbol{\beta}_1 &= \boldsymbol{\alpha}_1 = (1, 0, 1)^T \\
 \boldsymbol{\beta}_2 &= \boldsymbol{\alpha}_2 - \frac{(\boldsymbol{\alpha}_2, \boldsymbol{\beta}_1)}{(\boldsymbol{\beta}_1, \boldsymbol{\beta}_1)} \boldsymbol{\beta}_1 = \boldsymbol{\alpha}_2 - \frac{1}{2} \boldsymbol{\beta}_1 = \left(-\frac{1}{2}, 1, -\frac{1}{2}\right)^T
 \end{aligned}$$

于是

$$\begin{aligned}
 \mathbf{A} &= (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{\sqrt{6}}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \\ & \frac{\sqrt{3}}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{\sqrt{6}}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{6}}{2} \end{pmatrix}
 \end{aligned}$$

(3) 对 \mathbf{A} 进行 Schmidt 正交化, 得:

$$\begin{aligned}\beta_1 &= \alpha_1 = (0, 1, 1)^T \\ \beta_2 &= \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)}\beta_1 = \alpha_2 - \frac{1}{2}\beta_1 = (1, \frac{1}{2}, -\frac{1}{2})^T \\ \beta_3 &= \alpha_3 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)}\beta_2 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)}\beta_1 = \alpha_3 - \frac{2}{3}\beta_2 = (\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})^T\end{aligned}$$

于是

$$\begin{aligned}\mathbf{A} &= (\alpha_1, \alpha_2, \alpha_3) = (\beta_1, \beta_2, \beta_3) \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & \frac{1}{3} \\ 1 & \frac{1}{2} & -\frac{1}{3} \\ 1 & -\frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{\sqrt{3}}{3} \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{3}} & -\frac{\sqrt{3}}{3} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{3}} & \frac{\sqrt{3}}{3} \end{pmatrix} \begin{pmatrix} \sqrt{2} & & \\ & \frac{\sqrt{6}}{2} & \\ & & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{\sqrt{3}}{3} \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{3}} & -\frac{\sqrt{3}}{3} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{3}} & \frac{\sqrt{3}}{3} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{3} \\ 0 & 0 & \frac{\sqrt{3}}{3} \end{pmatrix}\end{aligned}$$

(4) 对 \mathbf{A} 进行 Schmidt 正交化, 得:

$$\begin{aligned}\beta_1 &= \alpha_1 = (2, 0, 2)^T \\ \beta_2 &= \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)}\beta_1 = \alpha_2 - \frac{3}{4}\beta_1 = (\frac{1}{2}, 2, -\frac{1}{2})^T \\ \beta_3 &= \alpha_3 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)}\beta_2 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)}\beta_1 = \alpha_3 - \frac{7}{9}\beta_2 - \frac{3}{4}\beta_1 = (-\frac{8}{9}, \frac{4}{9}, \frac{8}{9})^T\end{aligned}$$

于是

$$\begin{aligned}
 \mathbf{A} &= (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) = (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3) \begin{pmatrix} 1 & \frac{3}{4} & \frac{3}{4} \\ 0 & 1 & \frac{7}{9} \\ 0 & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 2 & \frac{1}{2} & -\frac{8}{9} \\ 0 & 2 & \frac{4}{9} \\ 2 & -\frac{1}{2} & \frac{8}{9} \end{pmatrix} \begin{pmatrix} 1 & \frac{3}{4} & \frac{3}{4} \\ 0 & 1 & \frac{7}{9} \\ 0 & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 2\sqrt{2} & & \\ & \frac{3}{\sqrt{2}} & \\ & & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 1 & \frac{3}{4} & \frac{3}{4} \\ 0 & 1 & \frac{7}{9} \\ 0 & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 2\sqrt{2} & \frac{3\sqrt{2}}{2} & \frac{3\sqrt{2}}{2} \\ 0 & \frac{3}{\sqrt{2}} & \frac{7}{3\sqrt{2}} \\ 0 & 0 & \frac{4}{3} \end{pmatrix}
 \end{aligned}$$

□

题 1.8. (p141.8) 计算下列矩阵的奇异值分解。

$$(1) \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \qquad (2) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

解. (1) 不妨令 $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$, 其中 $\boldsymbol{\Sigma} = \begin{pmatrix} \mathbf{D} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}$ 。先计算 \mathbf{V}^T , 即利用相似对角化计算

$$\mathbf{V}^T \mathbf{A}^T \mathbf{A} \mathbf{V} = \begin{pmatrix} \mathbf{D}^2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \text{ 其中 } \mathbf{A}^T \mathbf{A} = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}, \mathbf{V}^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} 2 & 0 \\ 0 & \sqrt{6} \\ 0 & 0 \end{pmatrix},$$

$$\text{此时 } \mathbf{V}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & \sqrt{6} \end{pmatrix}.$$

$$\mathbf{U}_1 = \mathbf{A} \mathbf{V}_1 \mathbf{D}^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & \sqrt{6} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$$

把 U_1 扩充成正交矩阵 $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{pmatrix}$, 于是:

$$A = U\Sigma V^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & \sqrt{6} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(2) 不妨令 $A = U\Sigma V^T$, 其中 $\Sigma = \begin{pmatrix} D & O \\ O & O \end{pmatrix}$. 先计算 V^T , 即利用相似对角化计算

$$V^T A^T A V = \begin{pmatrix} D^2 & O \\ O & O \end{pmatrix}, \text{ 其中 } A^T A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}, V^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{pmatrix}, \text{ 此时 } V_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{pmatrix}.$$

$$U_1 = A V_1 D^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

由于 U_1 已经是正交矩阵, 无需扩充, 令 $U = U_1$, 于是:

$$A = U\Sigma V^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

□

题 1.9. (p142.9) 证明: 对任意实 (复) 非退化方阵 A , 存在唯一的正交 (酉) 矩阵 Q 和正定矩阵 H_1 和 H_2 , 使得 $A = QH_1 = H_2Q$, 该分解称为矩阵的极分解, 若去掉矩阵的非退化条件, 结论改如何修正?

证明. 先给出需要用到的引理:

引理 1.1. 任意一个正定矩阵 A ，一定存在唯一的一个正定矩阵 S 使得 $A = S^2$

证明. 存在性: 由于 $H \succ 0$, 则存在正交矩阵 P , 使得 $H = P\Lambda P^T$, 其中 $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}$, $\lambda_i > 0$ 。令 $Z = \Lambda^{\frac{1}{2}} = \text{diag}\{\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}\}$, 则

$$H = P\Lambda^{\frac{1}{2}}\Lambda^{\frac{1}{2}}P^T = (PZP^T)(PZP^T) = (PZP^T)^2 = S^2$$

而任取 $x \in \mathbb{R}^n$, $x^T S x = x^T P Z P^T x = (P^T x)^T Z (P^T x) > 0$, 这说明了 S 是正定阵。

唯一性: 记矩阵 A 的特征值与对应的特征向量为 λ, ν 。若存在两个正定阵 S_0, S_1 , 使得 $H = S_0^2 = S_1^2$, 显然有 $S_0 \nu = S_1 \nu = \sqrt{\lambda} \nu$ 。于是 $S_0 S_1 \nu = \sqrt{\lambda} S_0 \nu = \lambda \nu$, $S_0 S_1 \nu = \sqrt{\lambda} S_0 \nu = \lambda \nu$, $(S_0 S_1 - S_1 S_0) \nu = 0$, 由于 H 是对称的, 一定存在 n 个线性无关的特征向量, 即特征子空间的维数 $\dim(V) = n$, 则 $r(S_0 S_1 - S_1 S_0) = n - \dim(V) = 0$, 即 $S_0 S_1 - S_1 S_0 = O$ 。此时 $(S_0 + S_1)(S_0 - S_1) = S_0^2 - S_0 S_1 + S_1 S_0 - S_1^2 = O$, 于是 $r(S_0 + S_1) + r(S_0 - S_1) = n$, 而 $S_0 + S_1$ 是正定矩阵, 即 $r(S_0 - S_1) = n$, 所以 $r(S_0 - S_1) = 0$, $S_0 = S_1$ 。

□

下面给出两种证明方法。

- (1) **存在性:** 由于 $A^T A$ 是一个正定矩阵, 由引理??可知, $A^T A = H_1^2$, 则有 $E = H_1^{-1} A^T A H_1^{-1} = (H_1^T)^{-1} A^T A H_1^{-1} = (A H_1^{-1})^T A H_1^{-1}$, 令 $Q_1 = A H_1^{-1}$, 有 $Q_1^T Q_1 = E$, 故 Q_1 是正交矩阵, 同时 $A = Q_1 H_1$ 。

同理: $A A^T = H_2^2$, 则有 $E = H_2^{-1} A A^T H_2^{-1} = H_2^{-1} A A^T (H_2^T)^{-1} = H_2^{-1} A (H_2^{-1} A)^T$, 令 $Q_2 = H_2^{-1} A$, 有 $Q_2 Q_2^T = E$, 故 Q_2 是正交矩阵, 同时 $A = H_2 Q_2$ 。

下面证明 $Q_1 = Q_2 = Q$, 即证 $A H_1^{-1} = H_2^{-1} A \Leftrightarrow H_2 A = A H_1$,

唯一性: 假设存在另外一个正交矩阵 U 与正定矩阵 W , 使得 $A = U W$ 由引理??唯一性可知, $W = H_1$; 而 $U = A W^{-1} = A H_1^{-1} = Q$, 同理可以说明 $A = H_2 Q$ 分解的唯一性。

- (2) **存在性:** 由 SVD 分解可知, $A = U \Sigma V^T$, 其中 U, V^T 是正交矩阵, $\Sigma = \text{diag}\{\lambda_1, \dots, \lambda_n\}$, $\lambda_i > 0$ 。又因为 $A = U \Sigma V^T = U (V^T V) \Sigma V^T = (U V^T) V \Sigma V^T$, 令 $Q = U V^T$, $H_1 = V \Sigma V^T$, 容易验证 Q 为正交矩阵, H_1 为正定矩阵。

同理: $A = U \Sigma V^T = U \Sigma (U^T U) V^T = U \Sigma U^T (U V^T)$, 令 $Q = U V^T$, $H_2 = U \Sigma U^T$ 。

唯一性: 假设存在另外一个正交矩阵 U 与正定矩阵 W , 使得 $A = UW$, $A^T A = (QH_1)^T QH_1 = H_1^2$, $A^T A = (UW)^T UW = W^2$, 由引理??唯一性可知, $W = H_1$, 而 $AH_1^{-1} = Q$, $AW^{-1} = U$, 于是 $U = Q$ 。同理可以说明 $A = H_2 Q$ 分解的唯一性。

结论修正为: 存在唯一的酉矩阵 Q 与半正定矩阵 H_1 与 H_2 使得 $A = QH_1 = H_2 Q$ 。

□

题 1.10. (p142.10) 证明: 对任何正定矩阵 H , 存在唯一的正定矩阵 S , 使得 $H = S^2$ 。若将正定矩阵改为半正定矩阵, 结论如何?

证明. **存在性:** 由于 $H \succ 0$, 则存在正交矩阵 P , 使得 $H = P\Lambda P^T$, 其中 $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}$, $\lambda_i > 0$ 。令 $Z = \Lambda^{\frac{1}{2}} = \text{diag}\{\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}\}$, 则

$$H = P\Lambda^{\frac{1}{2}}\Lambda^{\frac{1}{2}}P^T = (PZP^T)(PZP^T) = (PZP^T)^2 = S^2$$

而任取 $x \in \mathbb{R}^n$, $x^T Sx = x^T PZP^T x = (P^T x)^T Z(P^T x) > 0$, 这说明了 S 是正定阵。

唯一性: 记矩阵 A 的特征值与对应的特征向量为 λ, ν 。若存在两个正定阵 S_0, S_1 , 使得 $H = S_0^2 = S_1^2$, 显然有 $S_0\nu = S_1\nu = \sqrt{\lambda}\nu$ 。于是 $S_0 S_1 \nu = \sqrt{\lambda} S_0 \nu = \lambda \nu$, $S_0 S_1 \nu = \sqrt{\lambda} S_0 \nu = \lambda \nu$, $(S_0 S_1 - S_1 S_0)\nu = 0$, 由于 H 是对称的, 一定存在 n 个线性无关的特征向量, 即特征子空间的维数 $\dim(V) = n$, 则 $r(S_0 S_1 - S_1 S_0) = n - \dim(V) = 0$, 即 $S_0 S_1 - S_1 S_0 = O$ 。此时 $(S_0 + S_1)(S_0 - S_1) = S_0^2 - S_0 S_1 + S_1 S_0 - S_1^2 = O$, 于是 $r(S_0 + S_1) + r(S_0 - S_1) = n$, 而 $S_0 + S_1$ 是正定矩阵, 即 $r(S_0 + S_1) = n$, 所以 $r(S_0 - S_1) = 0$, $S_0 = S_1$ 。

结论修正为: 存在唯一的半正定矩阵 S , 使得 $H = S^2$ 。

□