HW3

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Problem 1.

$$f(x_1, x_2) = [x_1, x_2]\begin{bmatrix}1 & \frac{3}{2} \\ \frac{3}{2} & 2\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + [2,0]\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + 6$$

Problem 2.

$$f(X) = \sum_{i=1}^{n}\sum_{j=1}^{n} Q_{i,j}x_i x_j + \sum_{i=1}^{n} b_j x_i + c$$

$$\frac{\partial f(x)}{\partial x_k} = \sum_{\substack{j=1 \\ j\neq k}}^{n} Q_{kj}\, x_k + \sum_{\substack{i=1 \\ i\neq k}}^{n} Q_{ik}\, x_k + 2Q_{kk}\, x_k + b_k$$

Because Q is symmetric, so $Q_{kj} = Q_{ik}$ for $i=j$

Thus $\frac{\partial f(x)}{\partial x_k} = 2\sum_{i=1}^{n} Q_{ik}\, x_k + b_k$

$$\nabla f(x) = \sum_{k=1}^{n} \frac{\partial f(x)}{\partial x_k} = 2\sum_{k=1}^{n}\left[\sum_{i=1}^{n} Q_{ik}\, x_i + b_k\right]$$

$$= 2QX + b$$

Problem 3.

Let $\nabla f(x) = 0$, we have

$$X = -\frac{b}{2}Q^{-1}$$

Problem 4.

a) $$MSE = \frac{1}{N}\sum_{i=1}^{N}\left(y_i - \sum_{j=0}^{M} w_j x_{ij}\right)^2$$

$$= \frac{1}{N}(Y - XW)^T(Y - XW)$$

$$= \frac{1}{N}(Y Y^T - 2X^T Y W + W^T X^T X W)$$

$$= \frac{1}{N}(YY^T - 2X^T Y' W + W^T X^T X W)$$

b) $Q = X^T X / N$, $b = -2X^T Y / N$, $C = YY^T / N$

c) $W = -\frac{b}{2} Q^{-1} = X^T Y (X^T X)^{-1}$

d) $W \leftarrow W - \alpha \cdot MSE'(W)$

$$MSE'(W) = 2QW + b = (2X^T X W - 2X^T Y)/N$$

$$W \leftarrow W - \alpha (2X^T X W - 2X^T Y)/N$$

Problem 5.

$E(Y) = 3E(X) + 2 = 3 \times 0.5 + 2 = 3.5$

$Var(Y) = E(Y^2) - E(Y)^2$

$= E[(3X + 2)^2] - 3.5^2$

$= E[9X^2 + 12X + 4] - 12.25$

$= 9E(X^2) + 12E(X) - 8.25$

$E(X^2) = \int_{x=0}^{1} x^2 dx = \frac{1}{3}$

$Var(Y) = 3 + 6 - 8.25 = 0.75$

Problem 6.

We define the following event:

A: $a\ woman\ has\ cancer$

B: $a\ woman\ is\ tested\ positive\ for\ mammogram$

Then according to the information

$P(A) = 0.01$

$P(B) = 0.08 \Rightarrow P(\neg B) = 0.92$

$P(B|A) = 0.9 \Rightarrow P(\neg B|A) = 0.1$

For the two problem:

$P(a\ woman\ has\ cancer\ if\ she\ has\ a\ positive\ mammogram\ result) = P(A|B) = \frac{P(B|A)P(A)}{P(B)} = 0.1125$

$P(a\ woman\ has\ cancer\ if\ she\ has\ a\ negative\ mammogram\ result) = P(A|\neg B) = \frac{P(\neg B|A)P(A)}{P(\neg B)} = 0.001$