DS 102 Discussion 10

Wednesday, November 11th, 2020

Examples on Nash Equilibrium

In class, we discussed the Nash equilibrium in two-player games. Denote the action space of player $i (i \in \{1, 2\})$ as $\mathcal{A}_i$. The payoff function (outcome) for player i is a function that maps the vector of actions taken by player $1, 2$ to some real value $u_i : \mathcal{A}_1 \times \mathcal{A}_2 \mapsto \mathbb{R}$. Each player i would like to maximize their own payoff function $u_i(a_1, a_2)$. We say the action pair (a_1^*, a_2^*) for the two players is a Nash equilibrium if $\forall a_1' \in \mathcal{A}_1, u_1(a_1^*, a_2^*) \geq u_1(a_1', a_2^*)$ and $\forall a_2' \in \mathcal{A}_2, u_2(a_1^*, a_2^*) \geq u_2(a_1^*, a_2')$.

The definition of Nash Equilibrium can be extended to multi-player setting. Assume we have n players in total. Denote the payoff function for player i as $u_i : \mathcal{A}_1 \times \mathcal{A}_2 \times \cdot \times \mathcal{A}_n \mapsto \mathbb{R}$. Denote $a_{-i} = (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$ as the vector of actions of all players except for player i. We say the action pair $(a_1^*, a_2^*, \cdots, a_n^*)$ is a Nash equilibrium if for all $i \in [n], a_i' \in \mathcal{A}_i$, we have $u_i(a_i^*, a_{-i}^*) \geq u_i(a_i', a_{-i}^*)$.

Consider the following games and provide one Nash equilibrium for each of them.

(a) A two-player non-zero-sum game with payoff matrix as below.

		Player 1	
		0	1
Player 2	0	$(3, 3)$	$(1, 0)$
	1	$(2, 2)$	$(5, 5)$

Solution: Both $(0, 0)$ and $(1, 1)$ are Nash equilibrium in this case. $(0, 0)$ is a Nash equilibrium since $u_1(0, 0) = 3 \geq u_1(1, 0) = 1$ and $u_2(0, 0) = 3 \geq u_2(0, 1) = 2$. Similarly, $(1, 1)$ is a Nash equilibrium since $u_1(1, 1) = 5 \geq u_1(0, 1) = 2$ and $u_2(1, 1) = 5 \geq u_2(1, 0) = 0$.

(b) A two-player zero-sum game. The action space for both players is $\mathcal{A}_i = \mathbb{R}$. We use $X, Y \in \mathbb{R}$ to denote the action of player 1 and 2, separately. The payoff function for player 1 is $u_1(X, Y) = Y^2 - X^2 + 2XY + 2X$. The payoff function for player 2 is $u_2(X, Y) = -u_1(X, Y) = -Y^2 + X^2 - 2XY - 2X$.

Solution: Assume that (X_*, Y_*) is the Nash equilibrium. From the definition, we shall have $u_1(X_*, Y_*) \geq u_1(X, Y_*)$ for any $X \in \mathbb{R}$, i.e. X_* is the maximizer of function $u_1(X, Y_*) = Y_*^2 - X^2 + 2XY_* + 2X$ for fixed Y_*. Note that this is a quadratic function of X when Y_* is a fixed constant. We know that it is maximized at the point $X_* = Y_* + 1$. This gives the first relationship between X_* and Y_*.

On the other hand, we shall also have $u_2(X_*, Y_*) \geq u_2(X_*, Y)$ for any $Y \in \mathbb{R}$, i.e. Y_* is the maximizer of the function $u_2(X_*, Y) = -Y^2 + X_*^2 - 2X_*Y - 2X_*$ for fixed X_*. Note that this is a quadratic function of Y when X_* is a fixed constant. We know that it is maximized at the point $Y_* = -X_*$. This gives the second relationship between X_* and Y_*.

Now by solving $Y_* = -X_*$ and $X_* = Y_* + 1$ altogether, we see that $X_* = 0.5, Y_* = -0.5$.

In optimization, the point (X_*, Y_*) is called a sadlle point. In general, equilibrium situations in a two-person zero-sum game are saddle points of the payoff function. Hence they are also called saddle points of the game itself.

(c) (Optional) A n-player single-item second-price auction. Denote the private valuation of the i-th bidder as $v_i \in \mathbb{R}^+$, the bid of the i-th bidder as $b_i \in \mathbb{R}^+$. The payoff function for bidder i is $u_i(b_1, b_2 \cdots, b_n) = (v_i - \max_{j \neq i} b_j) \cdot 1(b_i \geq \max_{j \neq i} b_j)$. (Take some time to convince yourself that this payoff function is exactly the gain of bidder i from second-price auction.)

Solution: We can show that the Nash equilibrium is $(v_1, v_2, \cdots, v_t)$, i.e. each bidder bids their true private valuation of the item.

We first show that the payoff of bidder 1 satisfies $u_1(v_1, v_2, \cdots, v_t) \geq u_1(v', v_2, \cdots, v_t)$ for any $v' \in \mathbb{R}^+$. Denote $m = \max_{j \neq 1} b_j$. We prove the inequality holds under $v_1 \geq m$ and $v_1 < m$, separately.

When $v_1 \geq m$, the payoff function is $u_1(v_1, v_2, \cdots, v_n) = v_1 - m > 0$. If we increase our bidding price b_1 to larger than v_1, then it will not make any difference to the payoff since we only need to pay the amount that is equal to the second highest bid. If we decrease our bidding price b_1 to smaller than v_1, then we might lose (i.e. $b_1 < m$), and the payoff can be reduced to 0. The payoff will never be larger than $v_1 - m$.

When $v_1 < m$, the payoff function is 0. If we increase our bidding price b_1 to larger than v_1, then we might win (i.e. $b_1 \geq m$), where we need to pay more price than v_1 and the corresponding payoff function is $v_1 - m < 0$. If we decrease our bidding price b_1, then we will still lose, and the payoff is still 0.

By combining the two cases together, we have shown that if we bid any value other than v_1, the payoff function will not be increased. Using the identical proof, we can show that for any bidder i, $u_i(v_1, v_2, \cdots, v_t) \geq u_i(v_1, v_2, \cdots, v', \cdots v_t)$ for any $v' \in \mathbb{R}^+$. Thus we can claim that $(v_1, v_2, \cdots, v_n)$ is a Nash equilibrium. Note that the proof here is exactly the same as is introduced in lecture on the right thing to do in second-price auction.