University of Toronto Scarborough

CSC B36 **Final Examination** **7 December 2018**

NAME: ____________________________________
(circle your last name)

STUDENT NUMBER: ______________________

Do not begin until you are told to do so. In the meantime, put your name and student number on this cover page and read the rest of this page.

Aids allowed: None. .... **Duration:** 3 hours.

There are 9 pages and each is numbered at the bottom. Make sure you have all of them.

Write legibly in the space provided. Use the backs of pages for rough work; they will not be graded.

1.	______	/ 10
2.	______	/ 20
3.	______	/ 10
4.	______	/ 10
5.	______	/ 14
6.	______	/ 6
7.	______	/ 10
Total	______	/ 80

1. **[10 marks]** Let $f : \mathbb{N} \to \mathbb{N}$ be the function defined as follows.

$$f(n) = \begin{cases} n^2 & \text{if } 0 \le n \le 16; \\ f(\lfloor \frac{2n}{3} \rfloor) + 4\ f(\lceil \frac{n}{3} \rceil) & \text{if } n > 16. \end{cases}$$

Prove that $f(n) \le n^2$ for every $n \in \mathbb{N}$.

Advice: You may use the following helpful facts without proof.

Fact 1: For any $n \in \mathbb{N}$, $\lceil \frac{n}{3} \rceil \le \frac{n+2}{3}$.

Fact 2: For any $n \in \mathbb{N}$, $(n-1)(n+1) < n^2$.

2. **[20 marks]**
Use methods from class to prove the program below correct with respect to its given specification.

▷ Precondition: L is a list of integers, $p \in \mathbb{Z}$.
▷ Postcondition: The items of L are rearranged and a number i is returned so that
▷ • $0 \leq i \leq \text{len}(L)$,
▷ • every item in $L[: i]$ is less than p,
▷ • every item in $L[i :]$ is greater than or equal to p.

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PseudoSort(L, p)
    i = 0;    j = len(L) − 1
    while i ≤ j:
        if L[i] < p:
            i = i + 1
        else:
            L[i], L[j] = L[j], L[i]    ▷ swap L[i] and L[j]
            j = j − 1
    return i
```

[more space available on next page . . .]

[... additional space for question 2]

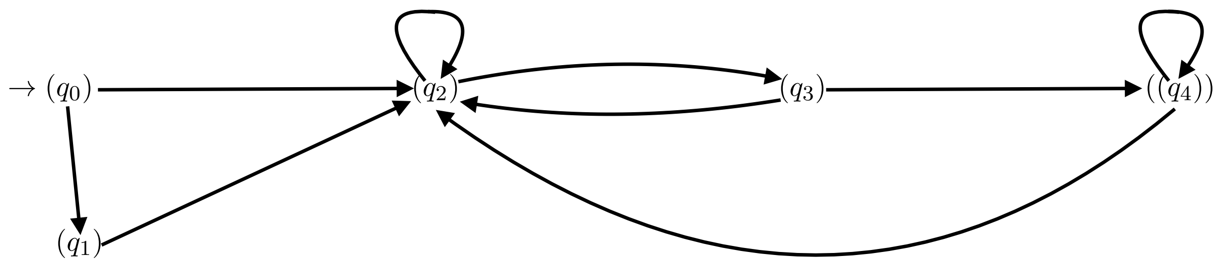
3. **[10 marks total; 5 for each part]** For each implication below, state whether it holds for arbitrary regular expressions R, S and T, and justify your answer.

   (a) If $R^*S^* \equiv S^*R^*$, then $RS \equiv SR$.

   (b) If $RST \equiv TSR$ and $RS \equiv SR$, then $RRSSTT \equiv TTSSRR$.

4. **[10 marks total; 5 for each part]** Let $\Sigma = \{0, 1\}$.
For strings x, y, we define $\#_y(x)$ to be $|\{(u, v) : x = uyv\}|$.
Informally $\#_y(x)$ is the number of places in x where y appears as a substring.

(a) Let $L_{4a} = \{x \in \Sigma^* : \ \#_{100}(x) > \#_{001}(x)\}$.
Nick designed a DFSA that accepts L_{4a}, but before he could show it to anyone, the Label Stealer (an associate of the Automaton Mangler) struck! All transition labels were removed.
Please restore Nick's DFSA by labeling the transitions in the state diagram below.



(b) Let $L_{4b} = \{x \in \Sigma^* : \ \#_{110}(x) > \#_{001}(x)\}$. Prove that L_{4b} is **not** regular.

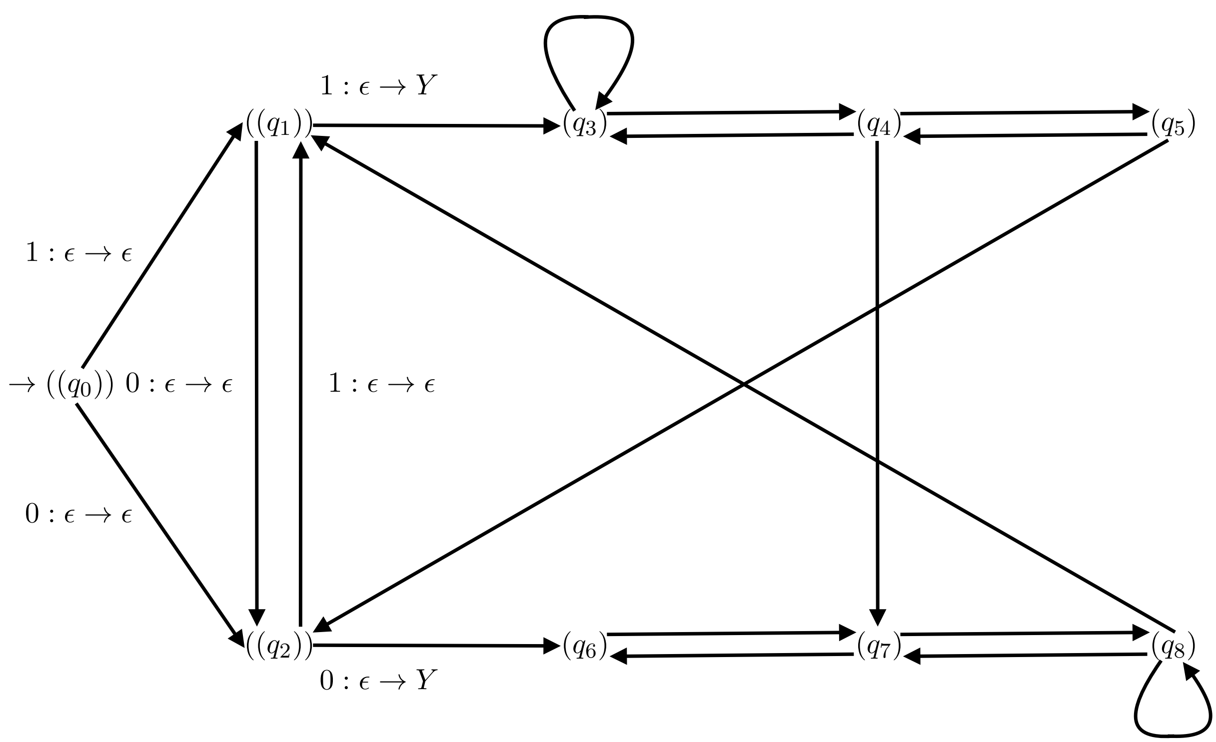
5. **[14 marks total; 7 for each part]** Let $\Sigma = \{0, 1\}$.
For strings x, y, we define $\#_y(x)$ to be $|\{(u, v) : x = uyv\}|$.
Informally $\#_y(x)$ is the number of places in x where y appears as a substring.

Let $L_5 = \{x \in \Sigma^* : \#_{11}(x) = 2 \cdot \#_{00}(x)\}$.

(a) Nick created a PDA that accepts L_5. Its state diagram has 9 states.
States q_0, q_1, q_2 are for strings $x \in \Sigma^*$ where $\#_{11}(x) = 2 \cdot \#_{00}(x)$.
States q_3, q_4, q_5 are for strings $x \in \Sigma^*$ where $\#_{11}(x) > 2 \cdot \#_{00}(x)$.
States q_6, q_7, q_8 are for strings $x \in \Sigma^*$ where $\#_{11}(x) < 2 \cdot \#_{00}(x)$.
Before Nick could show off his PDA to anyone, the Label Stealer struck — *oh no ... not again!*
All transition labels, except those for transitions coming out of states q_0, q_1 and q_2, were erased.
Here are the stolen labels.

$$\epsilon : \epsilon \to X \quad \epsilon : X \to \epsilon \quad \epsilon : Y \to \epsilon$$
$$1 : \epsilon \to \epsilon \quad 1 : \epsilon \to X \quad 1 : X \to \epsilon \quad 1 : Y \to \epsilon$$
$$0 : \epsilon \to \epsilon \quad 0 : \epsilon \to X \quad 0 : X \to \epsilon \quad 0 : Y \to Y$$

Please restore Nick's PDA below by adding one label to each transition that is missing a label.
Some labels are used more than once, since there are more transitions than labels.



(b) Nick created a CFG that generates L_5. It has 5 variables and 17 productions. Here is his design.

S generates x iff $x \in L_5$.
A_{00} generates x iff $x \in L_5$ and x starts with 0 and ends with 0.
A_{01} generates x iff $x \in L_5$ and x starts with 0 and ends with 1.
A_{10} generates x iff $x \in L_5$ and x starts with 1 and ends with 0.
A_{11} generates x iff $x \in L_5$ and x starts with 1 and ends with 1.

Nick wanted to show off his CFG, but before he could do that, the Grammar Mangler struck! The right hand sides (RHS, part after $\to$) of 14 productions were scrambled (its symbols were permuted), and these scrambled RHSs were thrown haphazardly into a list. The RHSs for variable A_{00} were left untouched. Here is the list of scrambled RHSs.

$$A_{00}, \quad A_{00}1, \quad A_{01}, \quad A_{01}1, \quad A_{10}, \quad A_{11}, \quad A_{11}0, \quad 1, \quad \epsilon,$$
$$A_{00}A_{10}A_{11}1, \quad A_{01}A_{10}A_{10}1, \quad A_{01}A_{10}A_{11}1, \quad A_{01}A_{10}A_{11}1, \quad A_{01}A_{11}A_{11}0$$

Please help Nick restore his CFG by unscrambling each of the 14 deleted strings and placing it on the appropriate right hand side below.

$S \to$

$A_{00} \to 0, \quad 0A_{10}, \quad 0A_{01}A_{11}A_{10}$

$A_{01} \to$

$A_{10} \to$

$A_{11} \to$

6. **[6 marks total; 2 for each part]** Recall the binary connective $\oplus$, which is defined by the following truth table.

P	Q	$P \oplus Q$
0	0	0
0	1	1
1	0	1
1	1	0

Let $n \geq 2$, and let $P_1, P_2, \cdots, P_n$ be arbitrary propositional formulas.

For each of the formulas below, describe exactly when it is satisfied. No justification is required.

(a) $P_1 \oplus (P_2 \oplus (\cdots \oplus (P_{n-1} \oplus P_n) \cdots))$

(b) $P_1 \to (P_2 \to (\cdots \to (P_{n-1} \to P_n) \cdots))$

(c) $P_1 \leftrightarrow (P_2 \leftrightarrow (\cdots \leftrightarrow (P_{n-1} \leftrightarrow P_n) \cdots))$

7. **[10 marks]** This question consists of 12 parts, and each part requires a Yes/No answer.
One mark will be deducted for every incorrect or blank answer, up to a maximum of 10 deducted marks.

Consider the following predicate formulas.

F_1: $\forall x \ (\ P(x) \rightarrow Q(x) \ )$

F_2: $\forall x \ P(x) \rightarrow Q(x)$

F_3: $\exists x \ (\ P(x) \rightarrow Q(x) \ )$

F_4: $\exists x \ P(x) \rightarrow Q(x)$

For each of the following questions, answer Yes or No.
No justification is required.

(a) Does F_1 logically imply F_2?

(b) Does F_1 logically imply F_3?

(c) Does F_1 logically imply F_4?

(d) Does F_2 logically imply F_1?

(e) Does F_2 logically imply F_3?

(f) Does F_2 logically imply F_4?

(g) Does F_3 logically imply F_1?

(h) Does F_3 logically imply F_2?

(i) Does F_3 logically imply F_4?

(j) Does F_4 logically imply F_1?

(k) Does F_4 logically imply F_2?

(l) Does F_4 logically imply F_3?