University of Toronto Scarborough
Department of Computer and Mathematical Sciences

December Examinations 2021

CSC C37H3 F

Duration—3 hours

This is a take-home exam. Complete your solutions, and submit no later than the Registrar's scheduled finish time for the exam, **following the instructions given on the final exam page of the course website.**

Aids allowed: Open-book. All aids are allowed.

This exam consists of 7 questions. **Make sure the copy you have downloaded has 14 pages (including this one).** Write your answers in the spaces provided. You will be rewarded for concise, well-thought-out answers, rather than long rambling ones. Please write legibly.

Take a few minutes before you begin the exam to read through each question, and then start with the question(s) you find easiest.

Name: ____________________ UTORid: ____________________
(Circle your family name.)

Student #: ____________________ Tutorial section: ____________________

YOU MUST SIGN THE FOLLOWING:

I declare that this exam was written by the person whose name and student # appear above.

Signature: ____________________

Your grade

1.	______ / 10	*5.*	______ / 20
2.	______ / 10	*6.*	______ / 10
3.	______ / 15	*7.*	______ / 15
4.	______ / 10		
		Total	______ / 90

Question 1

[10 marks]

Write a function (pseudo-code will suffice) that takes as input $n \in \mathbb{N}$ and outputs the sum of the first n reciprocals $\sum_{k=1}^{n} 1/k$. Pretty simple, except that your function must be *accurate* and protected against performing unnecessary floating point operations in any floating point number system in which it is implemented. You cannot assume you know the mantissa length or base of the floating point system.

Question 2

[10 marks]

In lecture we saw that Gaussian elimination with partial pivoting usually, but not always, leads to a stable factorization of $A \in \mathbb{R}^{n \times n}$. A stable factorization is guaranteed if we use *full* pivoting, which employs both row and column interchanges before the k-th stage of the elimination to ensure that the largest element in magnitude in the $(n-k) \times (n-k)$ submatrix finds its way to the pivot position.

Full pivoting leads to a $PAQ = LU$ factorization, where P and Q are permutation matrices. Show how this factorization can be used to solve $Ax = b$.

Question 3

[15 marks]

Consider the linear system $Ax = b$ where

$$A = \begin{bmatrix} 3 & 5 & 9 \\ 4 & 4 & 4 \\ 1 & 5 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 40 \\ 24 \\ 26 \end{bmatrix}.$$

a. Compute the $PA = LU$ factorization of A. Use exact arithmetic. Show all intermediate calculations, including Gauss transforms and permutation matrices.

b. Use the factorization computed in (a) to solve the system.

c. Instead of first computing the $PA = LU$ factorization, we could have solved the system above by processing the left and right-hand sides simultaneously with Gauss transforms and permutations. Would this alternate approach incur any extra cost? **Explain.** We are solving one system only.

Question 4

[10 marks]

Recall the fixed-point framework for solving the nonlinear equation $f(x) = 0, f : \mathbb{R} \to \mathbb{R}$.

a. Describe two methods for converting the "root-finding problem" into a "fixed-point problem". For each conversion, is the number of roots in the root-finding problem equal to the number of fixed points in the fixed-point problem? Explain. Illustrate both conversions for the function $f(x) = x - e^{-x} = 0$. (Choose the auxiliary function in the second conversion so that the fixed-point iteration is Newton's method.)

b. **Why** do we convert the root-finding problem into a fixed-point problem? Why not solve the root-finding problem directly?

Question 5

[20 marks]

Consider the data points $\{(-1, 4), (0, 6), (1, 12)\}$.

a. Using the method of undetermined coefficients, derive the monomial basis form of the quadratic polynomial interpolating this data. You **do not** need to first compute the $PA = LU$ factorization of the Vandermonde matrix. Just solve the Vandermonde system for the coefficients of the monomial basis quadratic interpolant.

b. Derive the divided-difference form of the quadratic interpolant. Show all of your work, including the divided-difference table.

c. Derive the Lagrange form of the quadratic interpolant.

d. Verify that the polynomials in (a), (b) and (c) are indeed the same polynomial. (**Hint:** You *could* convert (b) and (c) to monomial-basis form and then compare to (a), but there is an easier way by using an appropriate theorem discussed in lecture. If you choose the latter approach, you must apply the theorem correctly.)

e. Append $(2, 16)$ to the data. Using the *most appropriate* method from (a), (b), or (c), construct the single cubic polynomial interpolant over all four data points and, if you have used method (b) or (c), convert the cubic interpolant to monomial basis form. (**Hint:** By "most appropriate" method, we mean the method which requires the *least amount of extra work* given the quadratic interpolant you have already computed.)

f. Construct the linear spline (i.e., piecewise linear interpolant) that interpolates all four data points $\{(-1,4),(0,6),(1,12),(2,16)\}$.

Question 6

[10 marks]

Consider the Newton Polynomial Theorem (NPT) discussed on pages 6.2.1-6.2.4 of the lecture notes. In this theorem we prove that divided-differences, as defined on the top of page 6.2.1, are the coefficients of the Newton-basis polynomial $p(x) \in \mathcal{P}_n$ interpolating $\{(x_i, y_i), i = 0, \ldots, n\}$ with $x_i, i = 0, \ldots, n$ distinct.

a. In the induction step of the proof, we construct an auxiliary polynomial $r(x)$ from a linear combination of the induction hypotheses polynomials $p_{n-1}(x), q(x) \in \mathcal{P}_{n-1}$. (See part (b) of this question for a precise definition of $r(x)$.) We then argue that $r(x)$ and $p_n(x)$ ($p_n(x) \equiv p(x)$ is the final Newton-basis polynomial interpolating all $n+1$ data points) must be the **same** polynomial because $r(x)$ and $p_n(x)$ interpolate the **same** $n+1$ data points.

Explain how the Fundamental Theorem of Algebra (FTA) allows us to conclude that $r(x)$ and $p_n(x)$ are the same polynomial.

b. The auxiliary polynomial $r(x)$ used in the induction step of the NPT proof is defined as:

$$r(x) = \frac{(x - x_0)q(x) - (x - x_n)p_{n-1}(x)}{x_n - x_0}$$

where, by our induction hypothesis,

$$\begin{aligned} p_{n-1}(x) &= y[x_0] + (x - x_0)y[x_1, x_0] + (x - x_0)(x - x_1)y[x_2, x_1, x_0] \\ &\quad + \ldots + (x - x_0)(x - x_1)\cdots(x - x_{n-2})y[x_{n-1}, \ldots, x_0] \end{aligned}$$

and

$$\begin{aligned} q(x) &= y[x_1] + (x - x_1)y[x_2, x_1] + (x - x_1)(x - x_2)y[x_3, x_2, x_1] \\ &\quad + \ldots + (x - x_1)(x - x_2)\cdots(x - x_{n-1})y[x_n, \ldots, x_1] \end{aligned}$$

interpolate $\{(x_i, y_i),\ i = 0, 1, \ldots, n-1\}$ and $\{(x_i, y_i),\ i = 1, 2, \ldots, n\}$, respectively.

Circle either TRUE or FALSE to indicate the accuracy of each of the statements below. One mark will be given for each correct answer, **and one mark will be deducted for each incorrect answer**. (The minimum mark you can get is 0, and the maximum is 5.)

(a) $r(x) \in \mathcal{P}_n$.

TRUE FALSE

(b) $r(x_i) = y_i,\ i = 0, 1, \ldots, n-1$.

TRUE FALSE

(c) $r(x_i) = y_i,\ i = 1, 2, \ldots, n$.

TRUE FALSE

(d) When $r(x)$ is rewritten in monomial-basis form, the coefficient of x^0 in $r(x)$ is $y[x_0]$.

TRUE FALSE

(e) When $r(x)$ is rewritten in monomial-basis form, the coefficient of x^n in $r(x)$ is

$$\frac{y[x_n, \ldots, x_1] - y[x_{n-1}, \ldots, x_0]}{x_n - x_0} \equiv y[x_n, \ldots, x_0].$$

TRUE FALSE

Question 7

[15 marks]

Consider the definite integral $I(f) = \int_0^2 f(x)\,dx$.

a. Construct the interpolatory quadrature rule for this integral based on nodes $0, 1, 2$. **You must do this <u>without</u> first computing the interpolating polynomial.**

b. What is the precision of the quadrature rule derived in (a)? **Justify your answer.**

c. If possible, use the quadrature rule derived in (a) to compute the exact value of $\int_0^2 (2x^3 + 4x^2)\, dx$. If this is not possible, explain why not.

END OF EXAM