How to Count?

Addition Example

A Trivial Example

You decide to purchase a snack from one food outlet located in either the **Student Centre** or the **Market Place**.

- If you visit the **Student Centre**, you can purchase a snack from one of **eight** food outlets.
- If you visit the **Market Place**, you can purchase a snack from one of **twelve** food outlets.

How many possible food outlets could you purchase a snack from?

Solution.
There are $8 + 12 = 20$ possible food outlets you could purchase a snack from.

Addition

Theorem: The Addition Principle (AP)

Suppose there are:

- n_1 ways for event E_1 to occur,
- n_2 ways for event E_2 to occur,
- $\vdots$
- n_k ways for event E_k to occur.

If the ways the different events can occur are pairwise disjoint, then the number of ways for at least one of the events $E_1, E_2, \ldots, E_k$ to occur is $\displaystyle\sum_{i=1}^{k} n_i = n_1 + n_2 + \cdots + n_k$.

The Addition Principle can be restated using sets:

Theorem: The Addition Principle (AP)

Let $A_1, A_2, \ldots, A_k$ be any k finite sets. If $A_i \cap A_j = \emptyset$ for all $1 \leq i, j \leq k$ with $i \neq j$, then

$$\left|\bigcup_{i=1}^{k} A_i\right| = |A_1 \cup A_2 \cup \cdots \cup A_k| = \sum_{i=1}^{k} |A_i| .$$

Multiplication Example

A Trivial Example

You visit a restaurant for dinner that has a three-course fixed price menu. You are asked to choose **one appetizer**, **one entrée** and **one dessert** from the menu below.

Appetizer:

- Potato leek soup
- Ceasar salad
- Egg rolls

Entrée:

- Grilled beef tenderloin
- Vegan lasagna
- Enchiladas

Dessert:

- Gulab jamun
- Chocolate cake

How many different meal possibilities are there?

Solution. There are $3 \times 3 \times 2 = 18$ possible meals.

Theorem: The Multiplication Principle (MP)

Suppose that an event E can be decomposed into k ordered events $E_1, E_2, \ldots, E_k$, and that there are:

- n_1 ways for event E_1 to occur,
- n_2 ways for event E_2 to occur,
- $\vdots$
- n_k ways for event E_k to occur.

Then the total number of ways for the event E to occur is $\displaystyle\prod_{i=1}^{k} n_i = n_1 \times n_2 \times \cdots \times n_k$.

The Multiplication Principle can be restated using sets:

Theorem: The Multiplication Principle (MP)

Let $A_1, A_2, \ldots, A_k$ be any k finite sets and

$$\prod_{i=1}^{k} A_i := \{(a_1, a_2, \ldots, a_k) \, : \, a_1 \in A_1,\ a_2 \in A_2, \ldots, a_k \in A_k\}$$

be the **Cartesian product** of $A_1, A_2, \ldots, A_k$. Then $\left|\prod_{i=1}^{r} A_i\right| = \prod_{i=1}^{r} |A_i|$.

Permutations

Definition: Permutation

A $\boxed{\textbf{permutation}}$ is an arrangement of distinct objects.

Definition: Factorial

The $\boxed{\textbf{factorial}}$ of n is defined as

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$$

with the convention that $0! = 1$.

Facts

(i) There are $n!$ permutations of a set of n objects.

(ii) Fix k so that $0 \leq k \leq n$.
The number of permutations of k objects of a set of size n is:

$$n \cdot (n-1) \cdot (n-2) \cdots (n-k+1).$$

Note: We can rewrite this as $\dfrac{n!}{(n-k)!}$.

Problems

Example

There are

- 2 ways to travel from city A to city B,
- 4 ways to travel from city B to city C, and
- 3 ways to travel from city C to city D.

If to reach city D from city A one must first pass through city B followed by city C, then how many ways are there from city A to city D?

Example

(a) Find the number of positive divisors of 600 (including 1 and 600).

(b) Let n be a positive integer and $\sigma(n)$ denote the number of positive divisors of n. Give a formula for $\sigma(n)$ and prove it is correct.

Example

Let $X = \{1, 2, \ldots, 100\}$ and $S = \{(a, b, c) : a, b, c \in X \text{ and } a < \min\{b, c\}\}$. Find $|S|$.

Example

How many ways can we arrange 3 objects from $\{a, b, c, d\}$?