STAB57: An Introduction to Statistics

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Week 4 (Sufficiency & Consistency of an estimator, Score & Fisher Information)

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Recap of Week 2-3

- Learned two formal ways of defining an estimator
  - Method of Moments Estimator.
  - Maximum Likelihood Estimator (MLE).
- Sampling distribution of an estimator
  - Sampling distribution of $(\bar{X})$ (*under Normal and non-Normal dist*)
  - Sampling distribution of (S^2) (*only under Normal dist*)
- Properties of an estimator
  - Unbiasedness

Learning goals for this week

- Sufficient statistic (Rice page 305, E&R page 302)
- Consistent estimator (Rice page 266, ER page 325)
- Score and Fisher Information (Rice page-276, E&R page 365)

These are selected topics from Evans and Rosenthal: chapter 6 and John A. Rice: Chap 8

Section 1

Sufficient Statistic

An example to explain the intuition

- Suppose I give these following two sets of information to the two sections of this course.
- It involves estimating the parameter (θ) of a Bernoulli distribution.

Info-1 for morning section

I have tossed a fair coin 5 times and the outcomes are (1,0,1,1,0).
can you calculate the MLE of θ?

Info-2 for afternoon section

I have tossed a fair coin 5 times and got 3 Heads.
can you calculate the MLE of θ?

An example to explain the intuition (cont...)

Note:

- For info-1, $L(\theta) = \theta^3(1-\theta)^2$
- For info-2, $L(\theta) = \binom{5}{3}\theta^3(1-\theta)^2$
- Both sections will give me the same answer $[\hat{\theta} = 0.6]$
- In info-2, I calculated a summary of the sample observations.
- This summary (Let's call it $T(x_1, x_2, ...x_n)$) contains the same info about θ as it is contained in the entire sample $(x_1, x_2, ...x_n)$.
- So we say $T(x_1, x_2, ...x_n)$ is sufficient for θ.
- This is considered as a <u>data reduction</u>.
- Sufficient statistic is parameter specific.

Definition of sufficient statistic (Rice-P305)

A statistic $T(X_1, X_2, ...X_n)$ is said to be **sufficient** for θ if the conditional distribution of $X_1, X_2, ...X_n$, given $T = t$, does not depend on θ.

In other words: once we have the value of the sufficient statistic, the actual sample observations don't add any more information about the parameter.

For example, for the afternoon section where I have told the total number of heads already, giving them the actual sequence (1,0,1,1,0) won't add anything new.

Example using $Poisson(\lambda)$

Suppose $X_1, X_2, X_3 \overset{iid}{\sim} Poisson(\lambda)$. Verify that $T = \sum_{i=1}^{3} X_i$ is a sufficient statistic for λ.

Note: a similar example is given in the Rice text book (page 306) for Bernoulli distribution.

Factorization theorem - An easier way of finding sufficient statistic

$T(X_1, X_2, ...X_n)$ is said to be **sufficient** for θ if the joint probability function factors in the form

$$f(x_1, x_2, ...x_n|\theta) = g[T(x_1, x_2, ...x_n), \theta] * h(x_1, x_2, ...x_n)$$

where,

- $h(x_1, x_2, ...x_n)$ is a function of sample observations only
- $g[T(x_1, x_2, ...x_n), \theta]$ involves θ and the sufficient statistic T

Note: By now we know $f(x_1, x_2, ...x_n|\theta)$ is just the likelihood function. Proof of this theorem is available on page 307 of Rice book (not needed for the course)

Factorization theorem applied on $Poisson(\lambda)$

$$
\begin{aligned}
L(\lambda) &= e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i} * \frac{1}{\prod_{i=1}^{n} x_i!} \\
&= g[\sum_{i=1}^{n} x_i, \lambda] * h(x_1, x_2, ...x_n)
\end{aligned}
$$

Therefore, according to the factorization theorem, $T = \sum_{i=1}^{n} x_i$ is a sufficient statistic for λ

Note: when we maximize likelihood, we maximize $g[T(x_1, x_2, ...x_n), \theta]$. Hence,

MLE is a function of sufficient statistic $T(x_1, x_2, ...x_n)$.

Section 2

Consistent Estimator

Definition of consistent estimator (E&R-P325)

- Let T_n be an estimator of parameter θ
- T_n is said to be consistent(in probability) if $T_n \xrightarrow{P} \theta$
- In words, T_n converges to θ in probability.

Note:

- There are multiple forms of consistency which depends on the type of convergence used.
- In this course we will only talk about consistent(in probability)

Proving consistency using LLN

- LLN tells us, $\bar{X} = \frac{1}{n}\sum X_i \xrightarrow{P} E[X_i]$ for any distribution.
- Immediately that tells us:
  - If $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$ then $\bar{X}$ is a consistent estimator of μ
  - If $X_i \overset{iid}{\sim} Poisson(\lambda)$ then $\bar{X}$ is a consistent estimator of λ
  - And we can say this for few other known distributions (do it yourself)

- How can we prove consistency when the estimator is not simply $\bar{X}$?
  - We can still use LLN but with the help of a well known Lemma and the continuous mapping theorem.

Slutsky's Lemma and Continuous mapping theorem

- **Slutsky's Lemma:**
  - We have two different sequences X_n and Y_n
  - $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y \implies X_n + Y_n \xrightarrow{P} X + Y$
  - $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y \implies X_n Y_n \xrightarrow{P} XY$

- **Continuous mapping theorem:**
  - Let $X_n \xrightarrow{P} X$ and $g()$ be a continuous function
  - then $g(X_n) \xrightarrow{P} g(X)$

Proving S^2 is a consistent estimator of σ^2

$$
\begin{aligned}
S^2 = \frac{1}{n-1}\sum_i (X_i - \bar{X})^2 &= \left(\frac{n}{n-1}\right)\left(\frac{1}{n}\sum_i \left(X_i - \bar{X}\right)^2\right) \\
&= \left(\frac{n}{n-1}\right)\left(\frac{1}{n}\left[\sum_i X_i^2 - n\bar{X}^2\right]\right) \\
&= \left(\frac{n}{n-1}\right)\left(\frac{1}{n}\sum_i X_i^2 - \bar{X}^2\right) \\
&= \left(\frac{n}{n-1}\right)\left(\frac{1}{n}\sum_i X_i^2 - (\bar{X})^2\right) \\
\implies S^2 \xrightarrow{p} (1)\left(E[X^2] - (E[X])^2\right) &= \sigma^2
\end{aligned}
$$

Note: Using continuous mapping theorem, we can say, S is a consistent estimator of σ

MSE consistent (often used in practice)

- An estimator T_n is called MSE consistent if

$$MSE(T_n) \to 0 \text{ as } n \to \infty$$

- **Example:** for $N(\mu, \sigma^2)$
  - $MSE(\bar{X}) = \sigma^2/n \to 0 \text{ as } n \to \infty$
  - Therefore $\bar{X}$ is a MSE consistent estimator of μ

- In naive words, after you have calculated the MSE of an estimator, just check if it goes to zero for large n...

MLE is consistent

Before making this claim, Let us introduce a new notation and revisit few of the old ones

- θ_0 : The TRUE value of the parameter which produced the data. (which is a unknown constant)
- Suppose $(X_1, X_2, ..., X_n) \overset{iid}{\sim} f(x|\theta_0)$
- $\hat{\theta}$ is MLE

Claim: $\hat{\theta}$ converges to θ_0 in probability ($\hat{\theta} \xrightarrow{P} \theta_0$)

MLE is consistent (illustration of the proof)

- Let us consider the situation where we haven't observed the samples yet.
- log-likelihood, $l(\theta) = \sum_{i=1}^{n} \log f(X_i|\theta)$
- In naive words, the log-likehood function varies from one set of sample to the other.
- Dividing both side by the sample size n

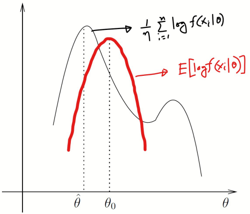
$$\frac{1}{n} l(\theta) = \frac{1}{n} \sum_{i=1}^{n} \log f(X_i|\theta)$$

- The right hand side of the above equation is a sample mean!
- Applying LLN,

$$\frac{1}{n} \sum_{i=1}^{n} \log f(X_i|\theta) \xrightarrow{P} E[\log f(X_i|\theta)]$$

MLE is consistent (illustration of the proof cont...)

- The main idea:
  since $\frac{1}{n}l(\theta)$ gets closer to $E[\log f(X_i|\theta)]$, the θ that maximizes $\frac{1}{n}l(\theta)$ should be close to the θ that maximizes $E[\log f(X_i|\theta)]$



$$E[\log f(X_i|\theta)] = \int_x \log f(x|\theta) f(x|\theta_0) dx$$

- Show that $E[\log f(X_i|\theta)]$ is maximized at θ_0 (Rice page 276)

source of the graph: `https://ocw.mit.edu/courses/mathematics/18-443-statistics-for-applications-fall-2006/lecture-notes/lecture3.pdf`

Section 3

Score and Fisher Information

Score

- **Score function,** $S(\theta)$**:**
  - it's the derivative of the log-likelihood

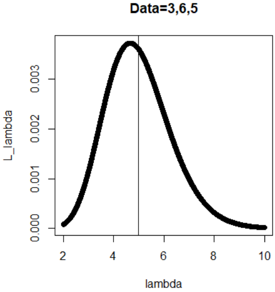
$$S(\theta) = \frac{\partial l(\theta)}{\partial \theta} = l'(\theta)$$

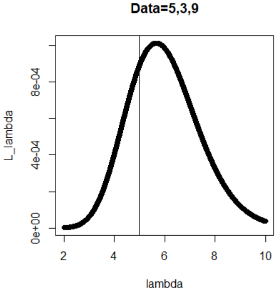
  - When we say "score function" we mean it's a function of θ
- **Score equation:**
  - $S(\theta) = 0$
  - the solution of this equation is the the MLE
  - we can say $S(\theta)|_{\theta=\hat{\theta}} = 0$
- **Score as a random variable:**
  - For the random variable X_i, $S(\theta|X_i) = \frac{\partial}{\partial\theta}\log f(X_i|\theta)$
  - For *iid* $(X_1, X_2, ..., X_n)$,

$$S(\theta|X_1, ..., X_n) = \frac{\partial}{\partial\theta}\sum_i \log f(X_i|\theta) = \sum_i \frac{\partial}{\partial\theta}\log f(X_i|\theta) = \sum_i S(\theta|X_i)$$

A plot showing the randomness of $S(\theta)$

- Both of these likelihood plots are for $Poisson(\lambda)$ distribution.
- In both cases we have 3 observations (n=3) generated from a Poisson with $\lambda = 5$ (true value, $\lambda_0 = 5$)
- The likelihood function looks different for different data!
- The slopes at $\lambda = 5$ differs $\implies$ Score evaluated at $\lambda = 5$ is a random variable





Randomness of Score using Poisson distribution

- Let $X_1, X_2, .., X_n \overset{iid}{\sim} Pois(\lambda)$ with true value of λ being λ_0
- We can show, $S(\lambda|X_1, ..., X_n) = -n + \frac{\sum_i X_i}{\lambda}$
- The score contains $\sum_i X_i$ which will change value from one set of sample to the other.
- It's the sample obs. $(X_1, X_2, .., X_n)$ in the score func. which makes it a random variable

One important property of $S(\theta)$

Under some assumptions,

- $E[S(\theta|X)]\big|_{\theta=\theta_0} = 0$ (proof...)
- This expectation is taken over X.

Example: Let $X_1, X_2, .., X_n \overset{iid}{\sim} Pois(\lambda)$ with true value of λ being λ_0

$$
\begin{aligned}
E[S(\lambda|X_1, ..., X_n)] &= E[-n + \frac{\sum_i X_i}{\lambda}] \\
&= -n + \frac{1}{\lambda} E[\sum_i X_i] \\
&= -n + \frac{1}{\lambda} n E[X_i] \\
&= -n + \frac{1}{\lambda} n \lambda_0
\end{aligned}
$$

at $\lambda = \lambda_0$, $E[S(\lambda|X_1, ..., X_n)] = 0$

Fisher Information, $I(\theta_0)$

Definition:

$$I(\theta_0) = var[S(\theta|X)\big|_{\theta=\theta_0}] = E[\frac{\partial}{\partial\theta}\log f(X|\theta)\big|_{\theta=\theta_0}]^2$$

- It's the amount of "information" that each observable random variable X contains about θ

For $Pois(\lambda_0)$

$$I(\lambda_0) = var[S(\lambda|X)\big|_{\lambda=\lambda_0}] = var[-1 + \frac{X}{\lambda_0}] = \frac{1}{\lambda_0^2}var[X] = \frac{1}{\lambda_0}$$

Fisher Information from a sample of size n

Fisher Information of a set of sample of size n,

$$
\begin{aligned}
& var[S(\theta|X_1, X_2, ..., X_n)|_{\theta=\theta_0}] \\
=& var[\sum_i S(\theta|X_i)|_{\theta=\theta_0}] \\
=& \sum_i var[S(\theta|X_i)|_{\theta=\theta_0}] \\
=& nI(\theta_0)
\end{aligned}
$$

An easier way of calculating $nI(\theta_0)$

- It can be shown that

$$I(\theta_0) = E[\frac{\partial}{\partial\theta}\log f(X|\theta)\Big|_{\theta=\theta_0}]^2 = -E[\frac{\partial^2}{\partial\theta^2}\log f(X|\theta)\Big|_{\theta=\theta_0}]$$

- Often in practice, we use the second derivative of the log-likelihood to calculate $nI(\theta_0)$

Summarizing the steps:

- Write down the log-likelihood in terms of random variable X
- Differentiate twice with respect to θ and then put $\theta = \theta_0$
- Take Expectation over X and finally multiply by (-1)

For $X_1, X_2, .., X_n \overset{iid}{\sim} Pois(\lambda_0)$

$$nI(\lambda_0) = -E[\frac{\partial^2}{\partial\lambda^2}(-n\lambda + \sum_i X_i log\lambda)|_{\lambda=\lambda_0}] = -E[-\frac{\sum_i X_i}{\lambda_0^2}] = \frac{n}{\lambda_0}$$

Homework (Non-credit)

Evans and Rosenthal

6.5.1-6.5.3

John A. Rice

Exercise 8: 16(d), 17(e), 18(d), 21(c), 47(d), 52(d), 69-72

R home work

1. Write a function that generates 30 random samples from $Poisson(\lambda = 5)$ dist and calculates the score function for $\lambda = 5$
2. Run this function 100K times and calculate the mean and variance of the output (mean should be ≈ 0 and var $\approx 30/5 = 6$)