♢ **Best before:** September 26.

1. Read and understand chapter 0 of the course notes. Try any of the exercises at the end of the chapter.

2. Do exercises 3-6 on page 44 of the course notes (about validity of induction proofs). It suffices to give informal arguments (like the explanations on why induction works in the additional notes).

3. (a) Use induction to prove that for every $n \in \mathbb{N}$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

   (b) Use induction to prove that for every $n \in \mathbb{N}$, $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

   (c) Find a similar equation for the sum $\sum_{i=0}^{n} i^3$, and prove that it holds for every $n \in \mathbb{N}$.

   (d) This question is about finding lower and upper bounds. It is not about induction.
   Recall the definition of big-Oh.
   Given two functions $f, g : \mathbb{N} \to \mathbb{N}$, we say that $f \in O(g)$ iff there are constants $n_0 \in \mathbb{N}$ and $c \in \mathbb{R}^+$ such that for every $n \geq n_0$, $f(n) \leq c \cdot g(n)$.
   Prove that for every $k \in \mathbb{N}$, $\sum_{i=0}^{n} i^k \in O\left(n^{k+1}\right)$ and $n^{k+1} \in O\left(\sum_{i=0}^{n} i^k\right)$.

4. Recall the formula for a geometric series,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1},$$

   which holds for any constant $r > 1$ and $n \in \mathbb{N}$. We explore ways of defining recurrences for r^n.

   (a) Consider the following recurrences defining functions $f_2, g_2 : \mathbb{N} \to \mathbb{N}$.

$$f_2(n) = \begin{cases} 1 & \text{if } n = 0; \\ 1 + \sum_{i=0}^{n-1} f_2(i) & \text{if } n > 0. \end{cases}$$

$$g_2(n) = \begin{cases} n+1 & \text{if } n = 0 \text{ or } n = 1; \\ n+1+\sum_{i=1}^{n-1} (\ i \cdot g_2(n-1-i)\ ) & \text{if } n > 1. \end{cases}$$

   Prove that $f_2(n) = 2^n$ and $g_2(n) = 2^n$ for all $n \in \mathbb{N}$.

   (b) Analogous to f_2 and g_2 from part (a), find recurrences defining functions $f_3, g_3 : \mathbb{N} \to \mathbb{N}$ so that $f_3(n) = 3^n = g_3(n)$ for all $n \in \mathbb{N}$. Prove that your recurrences are correct.
   Alert: Finding g_3 is crunchy!

5. Let $n \in \mathbb{N}$. Let S be a set of n integers.
   Let $\langle t_1, \cdots, t_n \rangle$ be a permutation of the integers in S, with the following property.

   If (p, q) is a pair of integers with $1 \leq p < q \leq n$, then there is an integer k such that
   - $p \leq k \leq q$,
   - for any j in the interval $(p, k]$, $t_j < t_p$,
   - for any j in the interval $(k, q]$, $t_j > t_p$.

   Put another way, we can partition $\{t_{p+1}, \cdots, t_q\}$ into $L = \{t_{p+1}, \cdots, t_k\}$ and $R = \{t_{k+1}, \cdots, t_q\}$, so that any integer in L is less than t_p and any integer in R is greater than t_p.
   Notice that $L = \emptyset$ if $k = p$ and $R = \emptyset$ if $k = q$.

(a) Prove that there is a unique binary search tree whose pre-order traversal is $t_1, \cdots, t_n$. Be sure to prove both existence and uniqueness.

(b) State the analogous problem with post-order traversal. Specifically, how must the property of S change for it to work with post-order traversals? Give the corresponding proof.

(c) Is there an analogous problem with breadth-first traversal? Explain your answer.

6. (a) Do exercise 16 on page 46 of the course notes (about sets of binary strings that do not differ from each other in exactly one position).

   Note: We can conclude from this exercise that the maximum size of a set of n-bit strings where no two strings differ in exactly one position is 2^{n-1}. Do you see why?

   (b) There are two ways to generalize part (a).

   - Binary strings use digits from $\{0, 1\}$. *Ternary* strings use digits from $\{0, 1, 2\}$. In general k-ary strings use digits from $\{0, 1, \cdots, k-1\}$.
   - "*... differ in exactly one position*" can be restated as "*... differ in at most one position*". In general we consider strings that differ in at most p positions.

   For integers k, p, n where $k > 2$ and $1 < p \leq n$, what is the maximum size of a set of k-ary strings of length n where no two strings differ in at most p positions?

7. Consider a video game in which there is a 2 by n grid of rooms, where n is some positive integer. Each room has a door on each of its walls. If a wall is shared by adjacent rooms, then going through its door takes you from one room to the other. Each room contains a treasure. Once you leave a room, you must never enter that room again — for you'll angered the beast that was guarding it.

```
+----------+----------+----------+ ... +----------+
| room 1-1 | room 1-2 | room 1-3 | ... | room 1-n |
+----------+----------+----------+ ... +----------+
| room 2-1 | room 2-2 | room 2-3 | ... | room 2-n |
+----------+----------+----------+ ... +----------+
        place where the love of your life awaits
            your gift of many treasures
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To play the game, you must start in room 1-1, get every treasure, then exit through any one of the doors in the second row of rooms to give your loot of treasures to your loved one.

(a) Let $B(n)$ be the number of different ways you can go about visiting rooms and getting treasures. Explain why the following recurrence correctly describes $B(n)$.

$$B(n) = \begin{cases} 1, & \text{if } 1 \leq n \leq 2; \\ 1 + B(n-2), & \text{if } n > 2. \end{cases}$$

(b) Find a closed (non-recursive) form of the function $B(n)$ — call it $B'(n)$ — and use induction to prove that $B'(n) = B(n)$ for all $n > 0$.

Note: It is possible to do part (b) without having done part (a).

(c) **Food for thought:** Suppose we modify the game to have 3 rows of rooms, and you must start in room 1-1 and exit from room 3-n. Let $C(n)$ be the number of different ways you can go about visiting rooms and getting treasures in this modified game. Find a recurrence for $C(n)$.

8. Do exercise 4 on page 93 of the course notes (an example of when we need to prove a stronger predicate). Try doing the induction proof without the "-3^n" and explain how it fails to work.

9. Let $\mathbb{Z}^+$ be the set of all positive integers. For an unspecified number $m \in \mathbb{Z}^+$, consider the following recurrence defining a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$.

$$f(n) = \begin{cases} 1, & \text{if } n = 1; \\ m\, f(\lceil \frac{n}{2} \rceil) + 5n^2, & \text{if } n > 1. \end{cases}$$

   (a) For arbitrary positive integers a, b, n, prove that $a^{\log_b n} = n^{\log_b a}$.

   (b) Let $m = 3$ and suppose n is some large integer power of 2.
   Unwind the recurrence to obtain a closed-form formula for $f(n)$ by using repeated substitution.

      i. Apply the recursive definition of $f(n)$ at least 3 times.

      ii. For $1 \leq i \leq \log_2 n$, give the formula for $f(n)$ after i substitutions.

      iii. Get the final result by substituting $\log_2 n$ for i in what you got for part ii.
      Express your answer with no exponents that contain the term $\log_2 n$.

   (c) Repeat part (b) with $m = 4$.

   (d) Repeat part (b) with $m = 5$.

10. Recall from chapter 0 that $\mathbb{Z}^2$ is the set of all ordered pairs of integers. We want to use structural induction to define the subset $T \subset \mathbb{Z}^2$ of all integer pairs (x, y) such that $x^2 + y^2$ is a power of 2 — i.e., $x^2 + y^2$ equals one of $1, 2, 4, 8, 16, \cdots$.

   (a) Without reading the rest of this question, try to find a structural induction definition for T. Spend at least 10 minutes before going on to part (b).

   If you have spent at least 10 minutes on part (a), then turn/scroll to the next page, where you will find part (b).

(b) We define the set $T_1 \subset \mathbb{Z}^2$ as follows.

Let T_1 be the smallest set such that
BASIS: $(0,1), (1,1) \in T_1$.
INDUCTION STEP: If $(p,q), (r,s) \in T_1$, then $(pr - qs, ps + qr) \in T_1$.

Prove that $T_1 \subseteq T$.
I.e., prove that $P(x,y)$ holds for every $(x,y) \in T_1$, where P is a predicate on $\mathbb{Z}^2$ defined by

$P(x,y)$: $x^2 + y^2$ is a power of 2.

(c) Prove that $T \subseteq T_1$.
I.e., prove that for any pair $(x,y) \in \mathbb{Z}^2$, if $x^2 + y^2$ is a power of 2, then $(x,y) \in T_1$.

Alert: This one's crunchy! A standard direct proof won't work.
Think about what kind of induction, if any, is needed.
Also think about what predicates are needed.