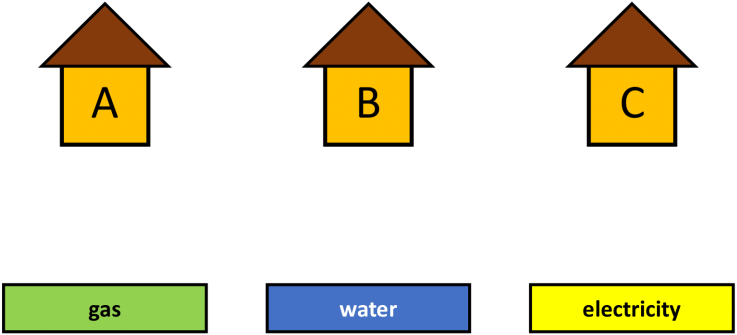
Planar Graphs

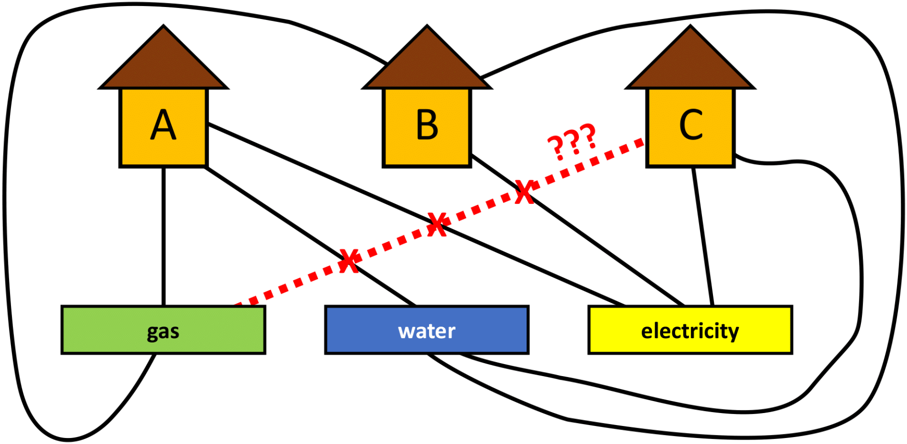
Three Utilities Puzzle

- Three houses each need to be connected to gas, water and electric companies.
- Without using a third dimension or sending any connections through another house or company, is there a way to make all nine connections without crossings?



Is it possible?

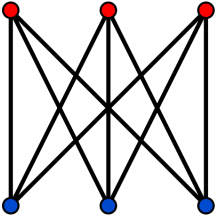
- Three houses each need to be connected to gas, water and electric companies.
- Without using a third dimension or sending any connections through another house or company, is there a way to make all nine connections without crossings?



Is it possible?

Rephrase the utilities puzzle in graph theory terms...

Recall that $\boxed{K_{3,3}}$ is notation for the complete bipartite graph with two parts of size three:



⟵ **ALL** possible edges

The Three Utilities Problem (rephrased)

The utilities puzzle asks the following:

Can the graph $K_{3,3}$ be drawn in the plane without any pair of edges crossing?

This motivates the concept of a **planar graph**.

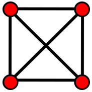
Planar graphs

Definition: Planar graph

A graph G is **planar** if it can be drawn in the plane so that no two edges intersect (except possibly at their endpoints). Such a drawing is called a **plane graph** or a **planar embedding** of G.

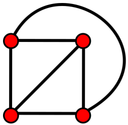
- A **plane graph** depends on the geometry of the points.
- A planar graph might have many "**different**" planar embeddings.
- To prove a graph is planar, we can demonstrate a planar embedding of it.

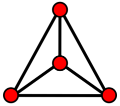
Example

Is the graph K_4 planar? One drawing of K_4 is: 

Solution.

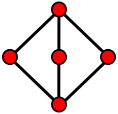
- Here are two planar embeddings (drawings with no edge crossings) of K_4:





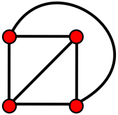
Question

Is $K_{2,3}$ planar?

Solution. Yes, here is a drawing to show this: 

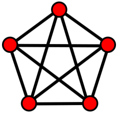
The Three Utilities Problem (rephrased)

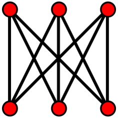
The utilities puzzle asks if $K_{3,3}$ is planar. (We will answer this question later.)

Earlier we demonstrated that K_4 is planar:  What about K_5?

Two Questions

Is K_5 planar? Is $K_{3,3}$ planar?

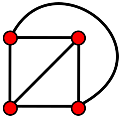
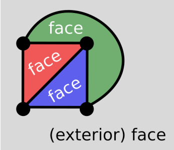




That is, can we redraw the two pictures above without edge crossings?

Definition: Face

A plane graph (planar embedding of G) divides the plane into regions called **faces**. Every plane graph has an unbounded region called the **exterior face**.

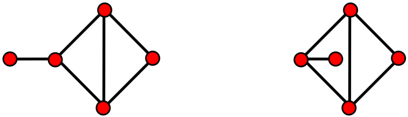
The **plane graph**  divides the plane into four regions: 

Example

Give an example of two "**different**" plane graphs that are isomorphic as graphs.

Solution.

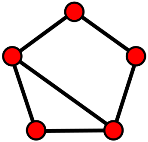
- Here are two planar embeddings of the **same** graph (call the graph G):



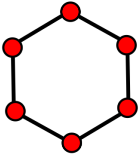
- They are "**different**" plane graphs (embeddings) since the number of **boundary** edges of the exterior faces is different in each one.

Question: Is there a planar embedding of G with a different number of faces?

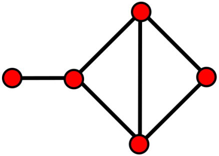
Let $\boxed{v = \text{number of vertices}}$ $\boxed{e = \text{number of edges}}$ $\boxed{f = \text{number of faces}}$



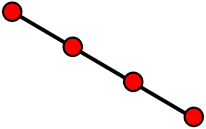
v =
e =
f =



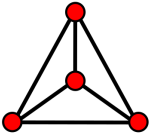
v =
e =
f =



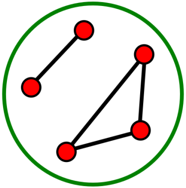
v =
e =
f =



v =
e =
f =

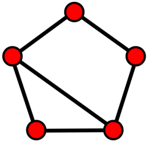


v =
e =
f =

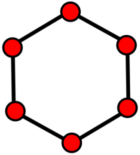


v =
e =
f =

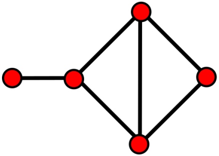
Let $\boxed{v = \text{number of vertices}}$ $\boxed{e = \text{number of edges}}$ $\boxed{f = \text{number of faces}}$



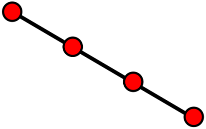
v = 5
e = 6
f = 3



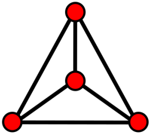
v = 6
e = 6
f = 2



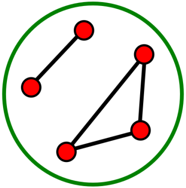
v = 5
e = 6
f = 3



v = 4
e = 3
f = 1



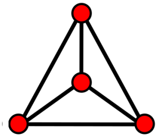
v = 4
e = 6
f = 4



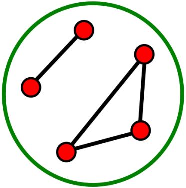
v = 5
e = 4
f = 2

Euler's formula

Let $\boxed{v = \text{number of vertices}}$ $\boxed{e = \text{number of edges}}$ $\boxed{f = \text{number of faces}}$



$v = 4$
$e = 6$
$f = 4$



$v = 5$
$e = 4$
$f = 2$

Euler noticed the following formula seems to hold for plane graphs:

$$v - e + f = 1 + (\textbf{number of components})$$

When the plane graph is **connected** (i.e., has one component), then this means

$$\boxed{v - e + f = 2}$$

Theorem: Euler's Formula

If G is a connected **plane graph** with v vertices, e edges and f faces, then $v - e + f = 2$.

Proof. See separate notes.

A corollary to Euler's formula

Theorem: Euler's Formula

If G is a connected **plane graph** with v vertices, e edges and f faces, then $v - e + f = 2$.

Corollary

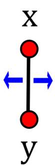
Every planar embedding of a connected planar graph has the **same number** of faces.

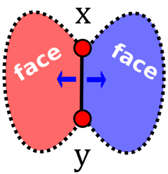
Proof.

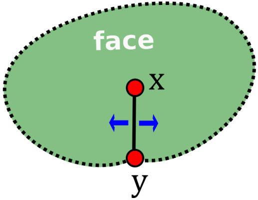
- Let G be a planar graph with v vertices and e edges.
- Let G_1 and G_2 be plane graphs that are both planar representations of G and suppose G_1 has v_1 vertices, e_1 edges and f_1 faces,
  and G_2 has v_2 vertices, e_2 edges and f_2 faces.
- Since G_1 and G_2 are both drawings of G, they are isomorphic as graphs implying $v_1 = v_2 = v$ and $e_1 = e_2 = e$.
- By **Euler's formula** applied to G_1 we have $f_1 = 2 + e_1 - v_1 = 2 + e - v$.
- By **Euler's formula** applied to G_2 we have $f_2 = 2 + e_2 - v_2 = 2 + e - v$.
- Therefore, $f_1 = f_2$ as required to show.

- Informally, the **degree** of a face F in a plane graph is the **length of its boundary**.
- **Notation:** $\deg(F)$

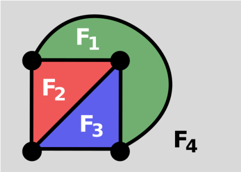
- Think of each edge having two "**sides**".
- Edges that are entirely in one face (i.e., do not belong to any cycles) are counted twice to the degree of that face.



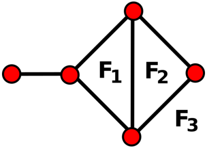




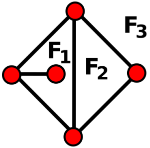
Example



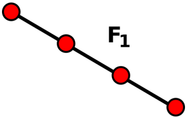
$$\deg(F_1) = \deg(F_2) = \deg(F_3) = \deg(F_4) = 3.$$



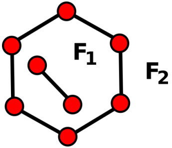
$\deg(F_1) = 3, \quad \deg(F_2) = 3, \quad \deg(F_3) = 6.$



$\deg(F_1) = 5, \quad \deg(F_2) = 3, \quad \deg(F_3) = 4.$



$\deg(F_1) = 6.$



$\deg(F_1) = 8, \quad \deg(F_2) = 6.$

The handshaking lemma for faces

Recall the **handshaking lemma**: Let G be a graph. Then $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$.

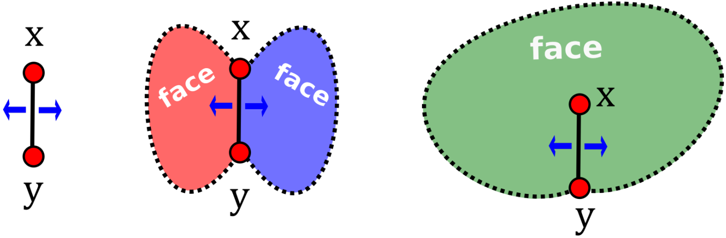
For planar graphs, we also have an analogous lemma for faces.

Theorem (The handshaking lemma for faces)

Let G be a plane graph. Then $\sum_{F \text{ a face}} \deg(F) = 2|E(G)|$.

Proof.

- Every edge either forms part of the boundary of two faces or appears twice on the boundary of a single face:



- In both cases, the edge contributes 2 to the total sum of degrees of the faces.

Another corollary to Euler's formula

Question: How many (max) edges can a connected planar graph with v vertices have?

Corollary to Euler's formula

Let G be a connected planar graph with $v \geq 3$ vertices and e edges. Then $\boxed{e \leq 3v - 6}$.
Furthermore, if G is bipartite then $\boxed{e \leq 2v - 4}$.

Proof.

- Suppose a planar embedding of G has f faces.
- Since G is connected and $v \geq 3$, every face must have degree at least 3.
- By the handshaking lemma (for faces) we have $2e = \sum_{F \text{ a face}} \deg(F) \geq 3f$.
- Thus, $f \leq \frac{2}{3}e$.
- By Euler's formula ($v - e + f = 2$), we have

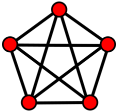
$$f = 2 - v + e \leq \frac{2}{3}e \quad \rightarrow \quad \frac{1}{3}e \leq v - 2 \quad \rightarrow \quad e \leq 3v - 6.$$

- If G is also bipartite, then G has no odd cycles implying that every face must have degree at least 4.
- Then $2e \geq 4f$ and a similar argument (with Euler's formula) implies $e \leq 2v - 4$.

Another consequence to Euler's formula

Theorem

The graph K_5 is not planar.

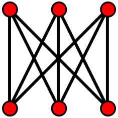


Proof.

- To derive a contradiction, assume K_5 is planar.
- Then $e \leq 3v - 6$ by the **Corollary to Euler's formula**.
- But $v = 5$ and $e = 10$, a contradiction since $3v - 6 = 9 < 10 = e$.

Theorem

The graph $K_{3,3}$ is not planar.



Proof.

- To derive a contradiction, assume $K_{3,3}$ is planar.
- Then $e \leq 2v - 4$ by the **Corollary to Euler's formula** since $K_{3,3}$ is bipartite.
- But $v = 6$ and $e = 9$, a contradiction since $2v - 4 = 8 < 9 = e$.

- Kuratowski proved a characterization of planar graphs in 1930.
- Informally, it says that either K_5 or $K_{3,3}$ "**show up**" in every nonplanar graph!
- By "show up", we formally mean as a subdivision (defined on the next slide).

Kuratowski's Theorem (1930)

A graph G is planar if and only if no subgraph of G is a subdivision of K_5 or $K_{3,3}$.

Proof.

We can prove one direction but the other direction is outside of the scope of this course.

Restating the theorem for nonplanar graphs gives:

Kuratowski's Theorem (1930)

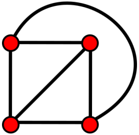
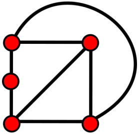
A graph G is nonplanar if and only if G has a subgraph that is a subdivision of K_5 or $K_{3,3}$.

Tips.

- To prove a graph is **planar**, we exhibit a drawing of it with no edge crossings.
- To prove a graph is **nonplanar**, we find a subgraph that is a subdivision of K_5 or $K_{3,3}$.

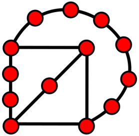
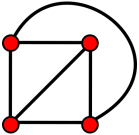
Definition: Subdividing an edge

An edge xy of a graph can be **subdivided** by placing a vertex somewhere along its length.

Subdividing the leftmost edge of the graph  gives the graph 

Definition: Subdivision

A graph which has been derived from G by a sequence of edge subdivision operations is called a **subdivision** of G.

The graph  is a **subdivision** of 

Fact

Every subgraph of a planar graph is also planar (i.e., if a graph G contains a nonplanar subgraph, then G is not planar).

Proof.

If we represent G as a plane graph, then its subgraphs are also plane graphs.

Fact

Every subdivision of a planar graph is also planar (i.e., if a graph G is a subdivision of a nonplanar graph, then G is not planar).

Proof.

- Represent G as a plane graph.
- Then subdividing edges does not produce edge crossings, thus are also plane graphs.

Since we previously proved K_5 and $K_{3,3}$ are not planar, the above facts imply:

Theorem (one direction of Kuratowski's theorem)

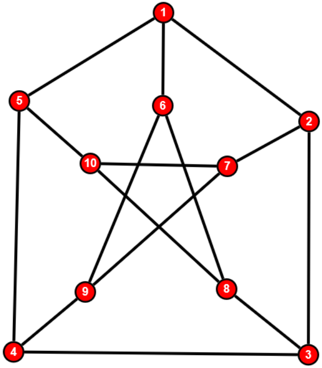
If G has a subgraph that is a subdivision of K_5 or $K_{3,3}$, then G is nonplanar.

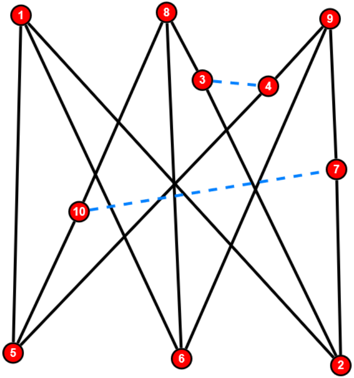
Example

Is the Petersen graph planar?

Solution.

- We exhibit a $K_{3,3}$ subdivsion of the Petersen graph:





- Therefore, the Petersen graph is **not** planar by Kuratowski's theorem.

Example

Let G be a connected planar graph where every vertex has degree 3. If in a plane representation of G every face has degree either 5 or 6, and there are 20 faces of degree 6, then how many faces are there of degree 5?

Solution.

- Assume we have a planar representation with v vertices, e edges and f faces.
- Suppose x faces have degree 5. Then $f = x + 20$.
- By the handshaking lemma (for vertices) we have $3v = 2e$.
- By the handshaking lemma (for faces) we have $120 + 5x = 2e$.
- By Euler's formula we have $v - e + f = 2$.
- Solving gives $f = 32$, $x = 12$, $e = 90$ and $v = 60$.

Therefore, there are 12 faces of degree five.