Combinatorial Proofs: How to double count?

(A proof technique for combinatorial identities.)

Types of proofs

There are two main methods to prove combinatorial formulas:

- **algebraic proof:** use $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and possibly induction.
- **combinatorial proof**: interpret $\binom{n}{k}$ as the number of ways of choosing a committee of k people from n people.

Example

Let n be a positive integer and $0 \leq k \leq n$. Prove $\binom{n}{k} = \binom{n}{n-k}$.

Algebraic Proof. Applying $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ with $r = n-k$ and $r = k$ gives

$$\begin{matrix}\text{right}\\\text{side}\end{matrix} = \binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k} = \begin{matrix}\text{left}\\\text{side}\end{matrix}$$

Combinatorial Proof.

- Consider a set of n people.
- Choosing k people to form a committee is equivalent to choosing $n-k$ people to leave out of the committee (**why?**).

Keep your combinatorial proofs organized

Goal: We want to prove a combinatorial identity.

Procedure:

- Determine a question that the identity answers.
- Answer the question in two different ways ("double counting").
- Since both answers count the same thing, they must be equal!

Example

Let n be a positive integer and $0 \leq k \leq n$. Prove $$\binom{n}{k} = \binom{n}{n-k}.$$

Proof. We consider the following problem.
Question: In how many ways can we select k toys from a box of n toys?

Answer 1:

- By definition of binomial coefficient, it is $\binom{n}{k}$.

Answer 2:

- We can pick k toys by choosing which toys we don't want.
- That is, we choose $n-k$ toys to discard and keep the remaining.
- This can be done in $\binom{n}{n-k}$ ways.

Since both answers solve the **same** problem, they must be equal.

Example

Let n be a positive integer. Give a combinatorial proof of $n^2 = (n-1)^2 + 2(n-1) + 1$.

Proof. We consider the following problem.

Problem: Count the number of ordered pairs (i, j) with $1 \leq i, j \leq n$.

Solution 1:

- There are n choices for i and n choices for j.
- Thus, there are n^2 possible ordered pairs.

Solution 2: (**Hint:** Usually the "$+$" means to split into disjoint cases.)

- We partition the pairs according to the number of 1's in it.
  - no 1's: There are $(n-1)^2$ pairs (since there are $n-1$ choices for i and $n-1$ choices for j).
  - one 1: There are $2(n-1)$ pairs (there are 2 positions for the 1 and $n-1$ choices for remaining position).
  - two 1's: There is 1 pair (both i and j must be 1, thus, the pair is $(1, 1)$).
- Hence, there are $(n-1)^2 + 2(n-1) + 1$ possible ordered pairs.

Since both answers solve the **same** problem, they must be equal giving

$$n^2 = (n-1)^2 + 2(n-1) + 1.$$

Example

Let n be a positive integer. Give a combinatorial proof of

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof.

Problem: A pretzel shop offers n toppings (such as sweet glaze, cinnamon sugar, chocolate, white chocolate, fudge, M&M's, caramel dip, mint chip, chocolate chip, marshmallows, nuts, toffee nuts, coconut, peanut butter drizzle, Oreos, sprinkles, cotton candy bits, powdered sugar and so on).

Q: How many pretzels can you make using any number of the n toppings (including no toppings and all toppings) where each topping is used most once?

A1:

- For each of the n toppings you have two choices: **include it** or to **omit it**.
- This gives $\underbrace{\underline{2} \times \underline{2} \times \cdots \underline{2}}_{n} = 2^n$ possible combinations, giving the right-side.

A2: (**Hint:** Usually the "+" means to split into disjoint cases.)

- Fix $0 \leq k \leq n$ and consider a pretzel that has exactly k toppings.
- There are $\binom{n}{k}$ pretzels with exactly k toppings.
- Summing over all possible values of k gives the left-side.

Example

Let $m, n \geq 2$ be positive integers. Give a combinatorial proof of

$$\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + mn.$$

Proof.

Q: How many ways can we select **two** marbles from a collection of marbles that consists of m **red** marbles and n **blue** marbles?

A1: Since there are $m+n$ total marbles, by the definition of binomial coefficient the answer is $\binom{m+n}{2}$, giving the left-side.

A2:

- There are three cases to consider when selecting 2 marbles:
  - Both marbles are **red**. This can be done in $\binom{m}{2}$ ways.
  - Both marbles are **blue**. This can be done in $\binom{n}{2}$ ways.
  - One is **red** and one is **blue**. This can be done in $\binom{m}{1}\binom{n}{1} = mn$ ways.
- Summing over these three cases gives the right-side.

Example

Let $n \geq 1$ be a positive integer. Give a combinatorial proof of

$$n^2 = 1 + 3 + 5 + \cdots + (2n-1).$$

Proof. We consider the following problem.

Problem: Count the number of ordered pairs (i, j) with $1 \leq i, j \leq n$.

Solution 1: There are n choices for i and n choices for j, thus, n^2 possible pairs.

Solution 2: We partition the pairs according to the value of $\max\{i, j\}$.

- $\max\{i, j\} = 1$: There is one pair whose maximum entry is 1, namely, $(1, 1)$.
- $\max\{i, j\} = 2$: There are three pairs whose maximum entry is 2, namely, $(1, 2)$, $(2, 2)$ and $(2, 1)$.

$$\vdots$$

- $\max\{i, j\} = k$: In general, there are $2k - 1$ pairs whose maximum entry is k. There are $k - 1$ of the form (i, k) where $i < k$, $k - 1$ of the form (k, i) where $i < k$, and one of the form (k, k).

$$\vdots$$

- $\max\{i, j\} = n$: There are $2n - 1$ pairs whose maximum entry is n. There are $n - 1$ of the form (i, n) where $i < n$, $n - 1$ of the form (n, i) where $i < n$, and one of the form (n, n).

Hence, there are $1 + 3 + 5 + \cdots + (2n - 1)$ possible ordered pairs.

Since both answers solve the **same** problem, they must be equal.

Example

Let $n \geq 1$ be a positive integer. Give a combinatorial proof of

$$1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n\,2^{n-1}.$$

Proof. We consider the following problem.

Q: Given n people, how many ways can we select a committee (of any size from 1 to n people) that has one person designated as **president**?

A1: We choose the president first. This can be done in $\binom{n}{1} = n$ ways. For the remaining $n-1$ people, we decided whether or not they belong to the committee. This can be done in $2 \times 2 \times \cdots \times 2 = 2^{n-1}$ ways. Thus, there are $n\,2^{n-1}$ such committees.

A2: Fix the size of the committee to be k for some $1 \leq k \leq n$.

- We first select k people to be on the committee. This can be done in $\displaystyle\binom{n}{k}$ ways.
- From these k selected people, we choose one to be president.

  This can be done in $\displaystyle\binom{k}{1} = k$ ways.
- By the multiplication principle, there are $k\displaystyle\binom{n}{k}$ such committees (of size k).
- Summing over the possible values for k gives the left-side.