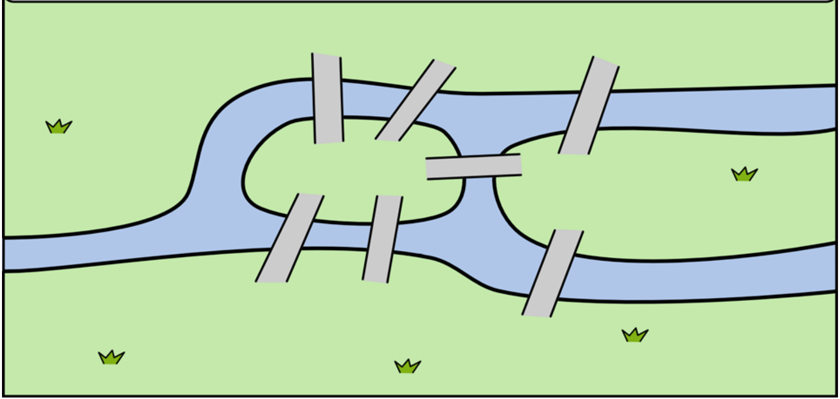
Eulerian Graphs

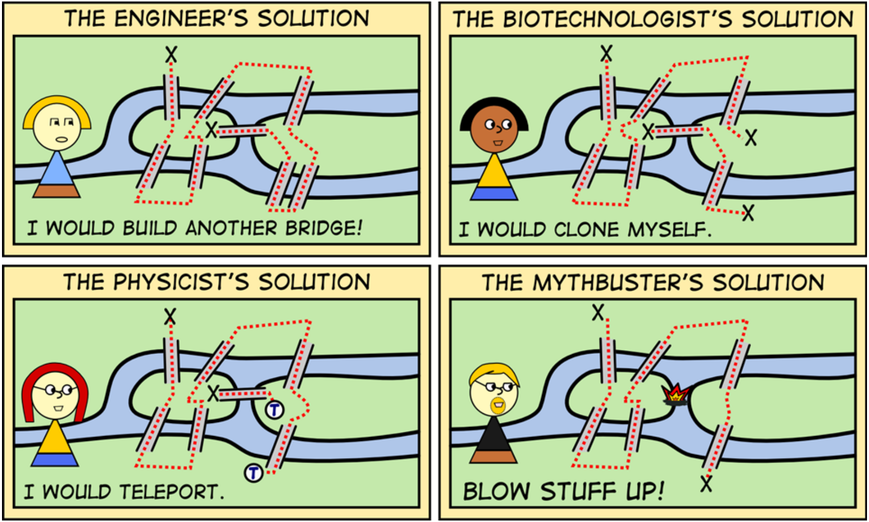
Puzzle: The Seven Bridges of Königsberg

The Seven Bridges of Königsberg

Below is the city of Königsberg with four land masses and seven bridges connecting the various land masses. Can you find a walk through the city of Königsberg that crosses each bridge exactly once? You may start at any land mass you wish but may only travel between land masses by using a bridge.



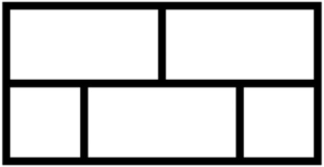
Puzzle: The Seven Bridges of Königsberg



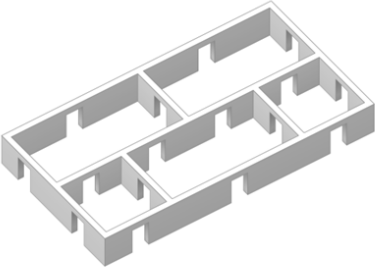
The above are jokes. Following the given rules, graph theory can show it is **impossible**.

The Five Rooms Puzzle

Question: In the diagram, can you draw a path that goes through each wall exactly once? The path cannot go through any intersections! (There are 16 walls.)



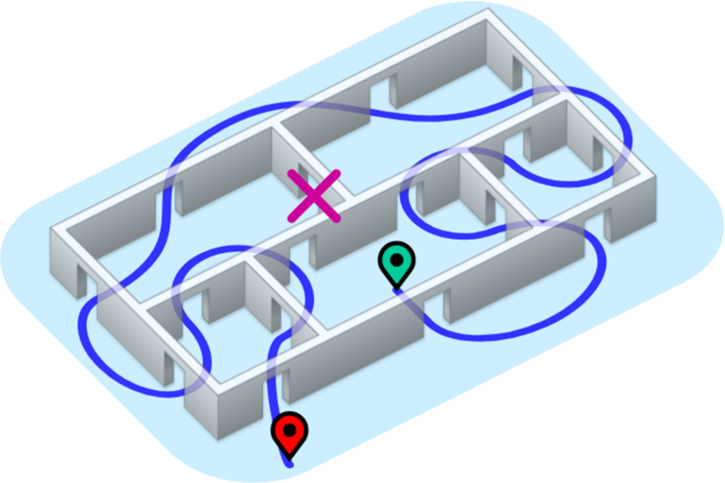
Alternate formulation with doors:
Starting in any room, can you walk through every door exactly once?



Source: Image by CMG Lee (Cmglee)

The Five Rooms Puzzle

The below shows an attempted solution, but it misses a door.

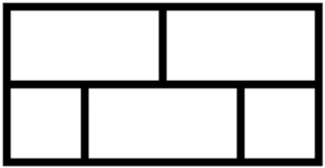


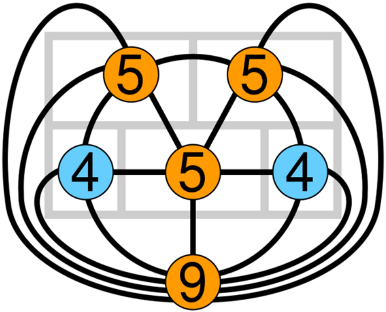
Source: Image by CMG Lee (Cmglee)

Side Note: The link above shows how it is possible if the rooms are on a torus (donut).

The Five Rooms Puzzle

We can use graph theory to show the puzzle is impossible.





- For each room (including the outer room), draw a vertex.
- Connect two vertices if there is a door between them.
- This produces a **multigraph** with some pairs of vertices having multiple edges between them. (The top right shows the degree of each vertex.)

Before proving any theorems about impossibility, we require some terminology!

Terminology: Walks, trails, paths, closed, tours, Eulerian

Definitions

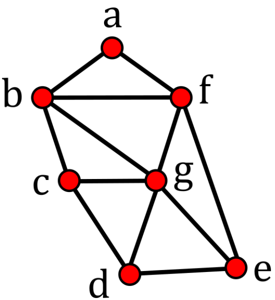
- A **walk** in a (multi)-graph G is a sequence

$$W = v_0 e_1 v_1 e_2 v_2 \cdots v_{k-1} e_k v_k$$

  whose terms alternate between vertices and edges (**not** necessarily distinct) such that $e_i = v_{i-1}v_i$ for $1 \leq i \leq k$. When G is a simple graph, we write $W = v_0 v_1 \cdots v_k$.
- The **length** of a walk is the number of edges it contains.
- A **trail** is a walk such that all of its edges are distinct.
- A **path** is a walk such that all of its vertices and edges are distinct.
- A walk is **closed** if the initial and terminal vertices are the same (i.e., $v_0 = v_k$).
- A walk from vertex x to vertex y is called an ***xy*-walk**.
- An **Euler trail** is a trail that visits every edge exactly once.
- An **Euler tour** is a closed Euler trail.
- A graph that has an Euler tour is called **Eulerian**.

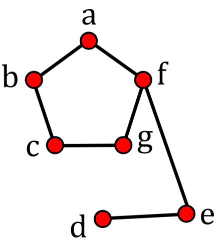
Examples

Consider the graph G:



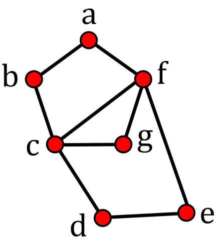
- $W = abcgbaf$ is a **walk** of length 6. (It is an af-walk.)
  - It is **not** a trail (edge ab is used twice) and **not** closed.
- $W = abcgb$ is a **trail** of length 4.
  - It is also a **walk** (and has distinct edges). It is **not** closed.
- $W = abgfabfa$ is a **closed walk** of length 7.
  - It is **not** a closed trail (edges ab and af are each repeated twice).
- $W = bcgbfab$ is a **closed trail** of length 6.
  - It is also a **closed walk**.
- $P = abcdefg$ is a **path** of length 6.

Example



An **Euler trail** (every edge used once) is $W = defabcgf$.

Example



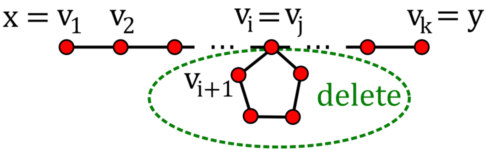
An **Euler tour** (closed Euler trail) is $W = abcdefgcfa$.

Theorem

Let G be a connected graph with $x, y \in V(G)$ and W be an xy-walk. Then W contains a path between x and y.

Proof.

- Let $W = v_1 v_2 \cdots v_k$ be an xy-walk where $v_1 = x$ and $v_k = y$.
- If W has no repeated vertices, then W is in fact a path and we are done.
- Otherwise, suppose W has a repeated vertex, say $v_i = v_j$ for some $i < j$.



- Since $v_i = v_j$, the walk W looks like

$$W = v_1 v_2 \cdots v_{i-1} {\color{red} v_i} v_{i+1} \cdots {\color{red} v_j} v_{j+1} \cdots v_k .$$

- Then by deleting part of W we get a walk W' of smaller length:

$$W = v_1 v_2 \cdots v_{i-1} {\color{red} v_i} \underbrace{v_{i+1} \cdots v_{j-1} {\color{red} v_i}}_{\textit{delete}} v_{j+1} \cdots v_k \quad \rightarrow \quad W' = v_1 v_2 \cdots v_{i-1} v_i v_{j+1} \cdots v_k .$$

- If W' has no repeated vertices, then it is a path between x and y, otherwise, repeat the above process.

Theorem

Let G be a connected graph (or multigraph). Then
G is Eulerian if and only if every vertex of G has even degree.

Proof. See Mike.

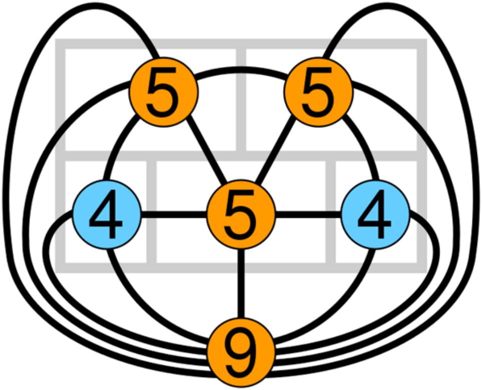
Theorem

Let G be a connected graph (or multigraph).
Then G has an Euler trail if and only if G has at most two vertices of odd degree.

Proof. See Mike.

The Five Rooms Puzzle

Recall the multigraph for the five rooms puzzle:



By Euler's theorem, this graph does not have an Euler trail or Euler tour since it has four vertices of odd degree.

Therefore, the puzzle is **impossible**.