Generating Functions

> A **generating function** encodes a sequence and allows us to solve combinatorial problems algebraically.

Applications include

- finding exact formulas for the terms of a sequence,
- discovering new recurrence relations to give insights into the nature of a sequence,
- find statistical properties of a sequence,
- find asymptotic formulas for a sequence,
- proving combinatorial identities.

Example

Determine the number of integer solutions (a, b, c) satisfying $\boxed{\begin{array}{l} a+b+c=4 \\ 0 \le a \le 2 \\ 0 \le b \le 1 \\ 2 \le c \le 3 \end{array}}$ $(*)$

Solution.

- Consider the function $g(x) = \underbrace{(x^0+x^1+x^2)}_{\boxed{\text{can choose } a \text{ to be } 0, 1 \text{ or } 2}} \cdot \underbrace{(x^0+x^1)}_{\boxed{\text{can choose } b \text{ to be } 0 \text{ or } 1}} \cdot \underbrace{(x^2+x^3)}_{\boxed{\text{can choose } c \text{ to be } 2 \text{ or } 3}}$
- The answer to the problem is the coefficient of x^4 in $g(x)$.
- **Why?** Consider expanding $g(x)$ and note there is a bijection between the ways to form an x^4 term and the solutions (a, b, c) to $(*)$:

$(x^0+x^1+x^2)$	(x^0+x^1)	(x^2+x^3)
x^0	x^1	x^3
x^1	x^0	x^3
x^1	x^1	x^2
x^2	x^0	x^2

a	b	c
0	1	3
1	0	3
1	1	2
2	0	2

Expanding gives $g(x) = x^2 + 3x^3 + \boxed{4}x^4 + 3x^5 + x^6$, thus, there are 4 solutions.

Example

Let $n \geq 0$. Determine the number of integer solutions satisfying

$$\boxed{\begin{array}{l} a+b+c=n \\ 0 \leq a \leq 2 \\ 0 \leq b \leq 1 \\ 2 \leq c \leq 3 \end{array}}$$

Solution.

- Consider the function $g(x) = \underbrace{(x^0 + x^1 + x^2)}_{\boxed{\text{can choose } a \text{ to be } 0, 1 \text{ or } 2}} \cdot \underbrace{(x^0 + x^1)}_{\boxed{\text{can choose } b \text{ to be } 0 \text{ or } 1}} \cdot \underbrace{(x^2 + x^3)}_{\boxed{\text{can choose } c \text{ to be } 2 \text{ or } 3}}$
- The answer to the problem is the coefficient of x^n in $g(x)$.
- Expanding gives $g(x) = x^2 + 3x^3 + 4x^4 + 3x^5 + x^6$.
  - $n = 0$: no solutions
  - $n = 1$: no solutions
  - $n = 2$: one solution
  - $n = 3$: three solutions
  - $n = 4$: four solutions
  - $n = 5$: three solutions
  - $n = 6$: one solution
  - $n \geq 7$: no solutions

Example

Suppose we have several **red**, **green** and **blue** balls. In how many ways can we select n balls if we must have

- at least two **red**,
- at most one **green**, and
- an even number of **blue** balls.

Solution.

- Consider the function

$$g(x) = \underbrace{\left(x^2 + x^3 + x^4 + \cdots\right)}_{\boxed{\text{must choose at least two red}}} \cdot \underbrace{\left(x^0 + x^1\right)}_{\boxed{\text{must choose at most one green}}} \cdot \underbrace{\left(x^0 + x^2 + x^4 + x^6 + \cdots\right)}_{\boxed{\text{must choose an even number of blue}}}$$

- The answer to the problem is the coefficient of x^n in $g(x)$.
- **Why?**
- There is a bijection between
  - combinations of **red**, **green** and **blue** balls satisfying the restrictions, and
  - combinations of terms, one from each of the three factors.
- For example, choosing 3 **red**, no **green** and 4 **blue** corresponds to the term $x^3x^0x^4 = x^7$ in the expansion of $g(x)$, and vice versa.
- **Note:** $g(x)$ is a **power series** and has an infinite number of terms.

How to form the generating function $g(x)$

$g(x)$ is formed by a sequence of $+$'s and $\times$'s corresponding to "**OR**" and "**AND**'.

In the last example,

- we could choose 2 red balls **OR** 3 **OR** 4 **OR** $\cdots$
  giving $(x^2 + x^3 + x^4 + \cdots)$

AND

- we could choose 0 **OR** 1 green balls
  giving $(x^0 + x^1)$

AND

- we could choose 0 blue balls **OR** 2 **OR** 4 **OR** 6 **OR** $\cdots$
  giving $(x^0 + x^2 + x^4 + x^6 + \cdots)$

Thus,

$$g(x) = \left(x^2 \underbrace{+}_{\text{OR}} x^3 \underbrace{+}_{\text{OR}} x^4 \underbrace{+}_{\text{OR}} \cdots \right) \underbrace{\times}_{\text{AND}} \left(x^0 \underbrace{+}_{\text{OR}} x^1 \right) \underbrace{\times}_{\text{AND}} \left(x^0 \underbrace{+}_{\text{OR}} x^2 \underbrace{+}_{\text{OR}} x^4 \underbrace{+}_{\text{OR}} \cdots \right)$$

Question

Since the coefficient of x^n in the expansion of $g(x)$ gives us the answer to our problem, how can we rewrite $g(x)$ in the explicit form $g(x) = a_0 + a_1 x + a_2 x^2 + \cdots$?

Notation

The coefficient of x^n in $g(x)$ is denoted by $[x^n]g(x)$.

To write $g(x)$ in a closed form (i.e., no Σ or $\cdots$), we can use known power series.

Geometric series/sequences

Recall $\sum_{k=0}^{n} ar^k = a\left(\frac{1-r^{n+1}}{1-r}\right)$ and $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$.

In particular, we have the following

$$\boxed{1+x+x^2+x^3+\cdots = \frac{1}{1-x}}.$$

$$1+x^2+x^4+x^6+\cdots = \frac{1}{1-x^2}$$

$$x^2+x^3+x^4+\cdots = x^2\left(1+x+x^2+\cdots\right) = \frac{x^2}{1-x}$$

$$1+x+x^2+x^3+x^4 = \frac{1-x^5}{1-x}$$

Example (revisited)

Suppose we have several **red**, **green** and **blue** balls. In how many ways can we select n balls if we must have

- at least two **red**,
- at most one **green**,
- even number of **blue**.

Solution.

- Consider the function $g(x) = (x^2 + x^3 + x^4 + \cdots)(x^0 + x^1)(x^0 + x^2 + x^4 + x^6 + \cdots)$.
- The answer to the problem is $[x^n]g(x)$.
- Using $\boxed{1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}}$ $(*)$:

$$1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2} \quad \text{and} \quad x^2 + x^3 + x^4 + \cdots = x^2\left(1 + x + x^2 + \cdots\right) = \frac{x^2}{1-x}$$

- Thus,

$$g(x) = \left(\frac{x^2}{1-x}\right)(1+x)\left(\frac{1}{1-x^2}\right) = x^2\frac{1}{(1-x)^2} = x^2\left(1 + 2x + 3x^2 + 4x^3 + \cdots\right)$$

  since the derivative of $(*)$ gives $1 + 2x + 3x^2 + \cdots = \dfrac{1}{(1-x)^2}$.

- Therefore, $g(x) = x^2 + 2x^3 + 3x^4 + 4x^5 + \cdots = 0 + 0 + \sum_{n=2}^{\infty}(n-1)x^n$.
- Thus, the number of ways is $[x^n]g(x) = \boxed{(n-1)}$ if $n \geq 2$ (and 0 if $n = 0, 1$).

Example

How many ways can we fill a box with n snacks if

- the # of chocolate bars is even,
- the # of cookies is a multiple of five,
- there is at most four pies,
- there is at most one mooncake.

Solution.

- The generating function is

$$g(x) = \underbrace{\left(1 + x^2 + x^4 + \cdots\right)}_{\boxed{\text{even \# of chocolate bars}}} \cdot \underbrace{\left(1 + x^5 + x^{10} + \cdots\right)}_{\boxed{\text{cookies is a multiple of 5}}} \cdot \underbrace{\left(1 + x + x^2 + x^3 + x^4\right)}_{\boxed{\text{at most four pies}}} \cdot \underbrace{(1 + x)}_{\boxed{\text{0 or 1 mooncakes}}}$$

- The answer to the problem is $[x^n]g(x)$. **Why?**
- Using $1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x}$ $(*)$ and $1 + x + x^2 + x^3 + x^4 = \dfrac{1-x^5}{1-x}$ gives:

$$g(x) = \left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1-x^5}{1-x}\right)(1+x) = \frac{1}{(1-x)^2} = \left(1 + 2x + 3x^2 + \cdots\right)$$

since the derivative of $(*)$ gives $1 + 2x + 3x^2 + \cdots = \dfrac{1}{(1-x)^2}$.

- Therefore, $g(x) = \sum_{n=0}^{\infty} (n+1)x^n$.
- Thus, the number of ways is $[x^n]g(x) = \boxed{(n+1)}$ for $n \geq 0$.

Definition

Let $a_0, a_1, a_2, \ldots$ be a sequence. The **generating function** of the sequence is

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

- The previous examples motivates us to study formal power series in more details.
- We can perform operations on them (e.g., derivatives, multiple by x, etc.) to turn a combinatorial problem into an algebraic problem.
- Generating functions also allow us to **derive** formulas.
- In combinatorics, we deal with **formal** power series where we only care about the coefficient of x^n and we ignore any convergence or divergence issues.
- Since it is useful to go between closed-form expressions for $g(x)$ and explicit expressions, we present a list of known power series on the next slide.

Some Helpful Power Series

- $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ (**Geometric Series**)
- $\dfrac{1-x^{m+1}}{1-x} = \sum_{k=0}^{m} x^k$ (**Geometric Sequence**)
- $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, where $n \in \mathbb{Z}^+$ (**Binomial Theorem**)
- $(1-x^m)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{mk}$, $n \in \mathbb{Z}^+$ (substitute $-x^m$ into Binomial Theorem)
- $\dfrac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{k+n-1}{k} x^k$, (special case of **Generalized Binomial Theorem**)
- $\frac{1}{2}\left(e^x + e^{-x}\right) = \sum_{k=0}^{\infty} \dfrac{x^{2k}}{(2k)!} = \cosh x$ and $\frac{1}{2}\left(e^x - e^{-x}\right) = \sum_{k=0}^{\infty} \dfrac{x^{2k+1}}{(2k+1)!} = \sinh x$
- The coefficient of x^r in $\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right)$ is $\sum_{k=0}^{r} a_k b_{r-k}$.

Example

Find the generating function for the sequence $1, 1, 1, \ldots, 1, \ldots$.

Solution. $g(x) = 1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x}$.

Example

Find the generating function for the sequence $\binom{2022}{0}, \binom{2022}{1}, \binom{2022}{2}, \ldots, \binom{2022}{2022}, 0, 0, \ldots$.

Solution.

$$g(x) = \binom{2022}{0} + \binom{2022}{1}x + \binom{2022}{2}x^2 + \cdots + \binom{2022}{2022}x^{2022} = (1+x)^{2022}.$$

Example

Find the generating function for the sequence $1, 2, 3, 4, 5, \ldots, n, \ldots$.

Solution.

$$g(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots = \text{derivative of } (1 + x + x^2 + x^3 + \cdots)$$

$$= \text{derivative of } \left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

Example

Let $g(x) = (1 + x^2 + x^4)^5$. Give a combinatorial problem that $[x^n]g(x)$ represents.

Solution.

- A term in the expansion of $g(x)$ has the form

$$x^a x^b x^c x^d x^e \quad = x^{a+b+c+d+e}$$

  where $a, b, c, d, e \in \{0, 2, 4\}$.
- Hence, $[x^n]g(x)$ represents the number of integer solutions to

$$a + b + c + d + e = n$$

  where $a, b, c, d, e \in \{0, 2, 4\}$.

The Generalized Binomial Theorem

Question

Can we generalize the binomial coefficient to make sense of $\binom{-1/2}{5}$ or $\binom{\pi}{2}$?

Is there a reason to?

Recall the Binomial Theorem (rewritten with $y = 1$).

Binomial Theorem

For any integer $n \geq 0$, we have $\boxed{(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k}$.

- Isaac Newton ($\sim$1665) generalized the Binomial Theorem to allow for n to take on any real number (in fact, it can be generalized to complex values of n).
- Instead of a finite sum, we get an infinite series.
- However, we must also generalize the notion of a binomial coefficient.

The Generalized Binomial Theorem

For any nonzero **real number** $a \in \mathbb{R}$, we have $\boxed{(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k}$.

Notation

For $a \in \mathbb{R}$ and $k \in \mathbb{Z}^+$, define

$$\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!} \qquad \left(= \frac{a}{k} \cdot \frac{(a-1)}{(k-1)} \cdots \frac{a-k+1}{1} \right).$$

Notation

For $a \in \mathbb{R}$ and $k \in \mathbb{Z}^+$, define

$$\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!} \qquad \left(= \frac{a}{k} \cdot \frac{(a-1)}{(k-1)} \cdots \frac{a-k+1}{1}\right).$$

We set $\binom{a}{0} = 1$.

Example

Evaluate $\binom{-2}{5}$ and $\binom{1/3}{3}$.

Solution.

- By definition we have

$$\binom{-2}{5} = \frac{(-2)(-3)(-4)(-5)(-6)}{5!} = -6$$

and

$$\binom{1/3}{3} = \frac{(1/3)(1/3-1)(1/3-2)}{3!} = \frac{5}{81}.$$

- Note that $\binom{a}{k}$ agrees with the usual definition when $a \in \mathbb{Z}^+$.
- We next find a simple expression for $\binom{-n}{k}$ when $n \in \mathbb{Z}^+$.

Lemma

If $n \in \mathbb{Z}^+$, then $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$.

Proof. By definition we have

$$\begin{aligned}
\binom{-n}{k} &= \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!} \\
&= \frac{(-1)^k n(n+1)\cdot(n+k-1)}{k!} \\
&= (-1)^k \binom{n+k-1}{k}
\end{aligned}$$

Corollary

If $n \in \mathbb{Z}^+$, then $\dfrac{1}{(1+x)^n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k$.

Extracting coefficients of generating functions

In using generating functions to solve combinatorial problems, we often switch between closed-form formulas and explicit forms.

Example: Extracting coefficients using the Generalized Binomial Theorem

Let $g(x) = \dfrac{1}{(1-x)^4}$ be a generating function. What is the coefficient of x^6 in its expansion? That is, find $[x^6]\dfrac{1}{(1-x)^4}$.

Solution.

- From the Corollary we have if $n \in \mathbb{Z}^+$, then

$$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k.$$

- Replacing x with $-x$ and setting $n = 4$ gives

$$\frac{1}{(1-x)^4} = \sum_{k=0}^{\infty} (-1)^k \binom{3+k}{k} (-x)^k = \sum_{k=0}^{\infty} (-1)^{2k} \binom{3+k}{k} x^k = \sum_{k=0}^{\infty} \binom{3+k}{k} x^k.$$

- Therefore, $[x^6]\dfrac{1}{(1-x)^4} = \dbinom{3+6}{6} = \dbinom{9}{6}$.

Extracting coefficients of generating functions

Example: Extracting coefficients using partial fractions

Find the number of integer solutions to $x_1 + x_2 + x_3 = n$ where $x_1 \geq 0$, $0 \leq x_2 \leq 2$, $x_3 \geq 0$ and x_3 must be even.

Solution.

- The generating function is $g(x) = (1 + x + x^2 + \cdots)(1 + x + x^2)(1 + x^2 + x^4 + \cdots)$.
- Simplifying gives

$$g(x) = \frac{1}{1-x}(1 + x + x^2)\frac{1}{1-x^2} = \frac{1 + x + x^2}{(1-x)(1-x)(1+x)} = \frac{1 + x + x^2}{(1-x)^2(1+x)}.$$

- We can extract $[x^n]g(x)$ by using partial fractions (see Slides 26-38 for a review):

$$\frac{1 + x + x^2}{(1+x)(1-x)^2} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}.$$

- Then $1 + x + x^2 = A(1-x)^2 + B(1+x)(1-x) + C(1+x)$.
  - $x = 1$ gives $3 = 2C$, hence, $C = 3/2$.
  - $x = -1$ gives $1 = 4A$, hence, $A = 1/4$.
  - $x = 0$ gives $1 = A + B + C$, hence, $B = -3/4$.

Solution (continued).

$$\begin{aligned} g(x) &= \frac{1/4}{1+x} + \frac{-3/4}{1-x} + \frac{3/2}{(1-x)^2} \\ &= \frac{1}{4} \cdot \frac{1}{1+x} - \frac{3}{4} \cdot \frac{1}{1-x} + \frac{3}{2} \cdot \frac{1}{(1-x)^2} \\ &= \frac{1}{4} \sum_{k=0}^{\infty} (-x)^k - \frac{3}{4} \sum_{k=0}^{\infty} x^k + \frac{3}{2} \sum_{k=0}^{\infty} (-1)^k \binom{2+k-1}{k} (-x)^k \\ &= \frac{1}{4} \sum_{k=0}^{\infty} (-1)^k x^k - \frac{3}{4} \sum_{k=0}^{\infty} x^k + \frac{3}{2} \sum_{k=0}^{\infty} (k+1) x^k \quad \text{since } \binom{k+1}{k} = k+1 \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{4}(-1)^k - \frac{3}{4} + \frac{3}{2}(k+1) \right) x^k. \end{aligned}$$

Therefore, $[x^n]g(x) = \dfrac{(-1)^n}{4} - \dfrac{3}{4} + \dfrac{3}{2}(n+1)$.

Example: Solving recurrences with generating functions

Use generating functions to derive a closed-form expression for the Fibonacci sequence:

$$\boxed{\begin{array}{l} a_0 = 0,\ a_1 = 1 \\ a_n = a_{n-1} + a_{n-2},\ (n \geq 2). \end{array}}$$

Solution.

- Consider the generating function

$$\begin{aligned}
g(x) &= g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\
&= a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \\
&= a_0 + a_1 x + \sum_{n=2}^{\infty} (a_{n-1} + a_{n-2}) x^n \\
&= a_0 + a_1 x + \sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n
\end{aligned}$$

- We now write these two series in terms of $g(x)$.
- In particular, $\sum_{n=2}^{\infty} a_{n-1} x^n = x g(x)$ (since $a_0 = 0$) and $\sum_{n=2}^{\infty} a_{n-2} x^n = x^2 g(x)$.

Solution (continued).

- Thus, $g(x) = a_0 + a_1 x + xg(x) + x^2 g(x) = 0 + x + xg(x) + x^2 g(x)$ giving

$$g(x) = \frac{x}{1 - x - x^2} = \frac{-x}{x^2 + x - 1}.$$

- We now use partial fractions to extract the coefficient of x^n.
- The roots of $x^2 + x - 1 = 0$ are $x = \frac{-1 \pm \sqrt{5}}{2}$.
- Denote $\alpha = \frac{-1+\sqrt{5}}{2}$ and $\beta = \frac{-1-\sqrt{5}}{2}$.
- Then $g(x) = \dfrac{-x}{(x-\alpha)(x-\beta)} = \dfrac{A}{x-\alpha} + \dfrac{B}{x-\beta}$.
- Thus, $-x = A(x-\beta) + B(x-\alpha)$.
  - Let $x = \alpha$ to get $A = \frac{-\alpha}{\alpha - \beta}$.
  - Let $x = \beta$ to get $B = \frac{-\beta}{\beta - \alpha}$.
- Note that $\alpha - \beta = \sqrt{5}$, thus, $A = -\frac{1}{\sqrt{5}}\alpha$ and $B = \frac{1}{\sqrt{5}}\beta$.
- Therefore,

$$g(x) = \frac{\alpha}{\sqrt{5}} \frac{1}{\alpha - x} - \frac{\beta}{\sqrt{5}} \frac{1}{\beta - x}$$

$$= \frac{\alpha}{\sqrt{5}} \frac{1}{\alpha} \sum_{k=0}^{\infty} \left(\frac{x}{\alpha}\right)^k - \frac{\beta}{\sqrt{5}} \frac{1}{\beta} \sum_{k=0}^{\infty} \left(\frac{x}{\beta}\right)^k$$

Solution (continued).

- From the last slide:

$$g(x) = \frac{\alpha}{\sqrt{5}} \frac{1}{\alpha} \sum_{k=0}^{\infty} \left(\frac{x}{\alpha}\right)^k - \frac{\beta}{\sqrt{5}} \frac{1}{\beta} \sum_{k=0}^{\infty} \left(\frac{x}{\beta}\right)^k$$

- Thus,

$$a_n = [x^n]g(x) = \frac{1}{\sqrt{5}} \frac{1}{\alpha^n} - \frac{1}{\sqrt{5}} \frac{1}{\beta^n}$$

- This can be rewritten as

$$a_n = \frac{1}{\sqrt{5}} \left(\left(\frac{2}{-1+\sqrt{5}} \right)^n - \left(\frac{2}{-1-\sqrt{5}} \right)^n \right)$$

$$= \frac{1}{\sqrt{5}} \left(\left(\frac{2(-1-\sqrt{5})}{(-1+\sqrt{5})((-1-\sqrt{5}))} \right)^n - \left(\frac{2(-1+\sqrt{5})}{(-1-\sqrt{5})(-1+\sqrt{5})} \right)^n \right)$$

$$= \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

Example

Let k, n be fixed nonnegative integers. Use a generating function to count the number of integer solutions to

$$t_1 + t_2 + \cdots + t_k = n$$

where $t_i \geq i$ for $i = 1, 2, \ldots k$.

Solution.

$$g(x) = (x + x^2 + x^3 + \cdots)(x^2 + x^3 + x^4 + \cdots) \cdots (x^k + x^{k+1} + x^{k+2} + \cdots)$$

$$= \prod_{i=1}^{k} x^i (1 + x + x^2 + \cdots) = \prod_{i=1}^{k} x^i \left(\frac{1}{1-x} \right)$$

$$= \left(x \frac{1}{1-x} \right) \left(x^2 \frac{1}{1-x} \right) \cdots \left(x^k \frac{1}{1-x} \right)$$

$$= x^{1+2+\cdots+k} (1-x)^{-k} = x^{\binom{k+1}{2}} \sum_{i=0}^{\infty} \binom{-k}{i} (-x)^i$$

$$= x^{\binom{k+1}{2}} \sum_{i=0}^{\infty} (-1)^i \binom{k+i-1}{i} (-1)^i x^i = \sum_{i=0}^{\infty} \binom{k+i-1}{i} x^{i+\binom{k+1}{2}}$$

Therefore, $[x^n] g(x) = \binom{k + n - \binom{k+1}{2} - 1}{n - \binom{k+1}{2}} = \binom{n - \binom{k}{2} - 1}{n - \binom{k+1}{2}}$.

Example

How many ways can you tile a $2 \times n$ board (containing 1×1 squares) completely using dominoes, i.e., tiles of size 1×2 and 2×1?

Solution.

- Let a_n be the number of perfect domino tilings of a $2 \times n$ board.
- Observe $a_1 = 1$, $a_2 = 2$ and $a_3 = 3$.
- In general for a_n, we could have the bottom left square covered by a vertical domino giving a_{n-1} tilings, or we could have it covered by a horizontal domino in which case there is another horizontal domino above it. This second scenario gives a_{n-2} such tilings.
- Thus, $a_n = a_{n-1} + a_{n-2}$ with $a_1 = 1$ and $a_2 = 2$.
- This is the Fibonacci sequence whose closed-form formula can be found by either solving the linear recurrence relation using a characteristic equation, or using generating functions.

Review of Partial Fractions

Recap: Partial Fraction Decomposition

Example

Show that $\dfrac{2}{x} + \dfrac{1}{x+1} = \dfrac{3x+2}{x^2+x}$.

Solution.

- This is an elementary problem where you find a **common denominator**:

$$
\begin{aligned}
\frac{2}{x} + \frac{1}{x+1} &= \frac{2}{x} \cdot \frac{(x+1)}{(x+1)} + \frac{1}{(x+1)} \cdot \frac{x}{x} \\
&= \frac{2(x+1)}{x(x+1)} + \frac{x}{(x+1)x} \\
&= \frac{2(x+1)+x}{x(x+1)} \\
&= \frac{3x+2}{x^2+x}
\end{aligned}
$$

- We are interested in the reverse problem, i.e., given the right hand side expression, how can we rewrite it as the left hand side expression?
- **Question:** Starting from $\dfrac{3x+2}{x^2+x}$, how would you break it up into two fractions?
- **Answer:** You perform a technique call **partial fraction decomposition**.

- Partial fractions can be done if the degree of the top is strictly less than the bottom.
- Factor the bottom as much as possible then use the following table to write down the partial fraction decomposition of $P(x)/Q(x)$:

Type	**Factor in $Q(x)$**	**Term in partial fraction decomposition**
Linear factor appearing once	$ax+b$	$\dfrac{A}{ax+b}$
Linear factor appearing twice	$(ax+b)^2$	$\dfrac{A}{ax+b}+\dfrac{B}{(ax+b)^2}$
$\vdots$	$\vdots$	$\vdots$
Linear factor appearing m times	$(ax+b)^m$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_m}{(ax+b)^m}$

Quadratic factor appearing once	ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
Quadratic factor appearing twice	$(ax^2+bx+c)^2$	$\dfrac{Ax+B}{ax^2+bx+c}+\dfrac{Cx+D}{(ax^2+bx+c)^2}$
$\vdots$	$\vdots$	$\vdots$
Quadratic factor appearing m times	$(ax^2+bx+c)^m$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_mx+B_m}{(ax^2+bx+c)^m}$

- The general pattern is if a factor appears m times in $Q(x)$, then it should also appear m times in the decomposition (**once for each power**).
- For linear factors you put constants on top (i.e., "A")
- For quadratic factors, you put linear polynomials on top (i.e., "$Ax+B$")

Example

Write down the partial fraction decomposition of $\dfrac{2x^2+1}{x^2(x-1)^3(x^2+1)^2}$.

Solution.

- The decomposition comes from looking at the bottom $Q(x) = x^2(x-1)^3(x^2+1)^2$.
- x is a **linear** factor appearing **twice**.
  - This contributes $\dfrac{A}{x} + \dfrac{B}{x^2}$
- $(x-1)$ is a **linear** factor appearing **three** times.
  - This contributes $\dfrac{C}{x-1} + \dfrac{D}{(x-1)^2} + \dfrac{E}{(x-1)^3}$
  - Note that we used different constants since A and B are already used.
- (x^2+1) is a **quadratic** factor appearing **twice**.
  - This contributes $\dfrac{Fx+G}{x^2+1} + \dfrac{Hx+I}{(x^2+1)^2}$
- Combining the above gives:

$$\frac{2x^2+1}{x^2(x-1)^3(x^2+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3} + \frac{Fx+G}{x^2+1} + \frac{Hx+I}{(x^2+1)^2}$$

Example

Below, write the appropriate form of the partial fraction decomposition (do not solve for the coefficients).

1. $\dfrac{3x^2+8}{(x-1)(x-2)(x-7)} =$

2. $\dfrac{x^4+x^2+1}{(x+4)^2(x^2+4)^2} =$

3. $\dfrac{3x^3-2x+1}{(x+5)(x^2+4)(x^2+9)} =$

4. $\dfrac{7x^6-4x^3+2x}{(x-1)^2(x^2+x+1)^3} =$

5. $\dfrac{x^4+x^2+1}{x^2(x+3)(x-4)(x-7)^3} =$

Example

Below, write the appropriate form of the partial fraction decomposition (do not solve for the coefficients).

1. $$\frac{3x^2+8}{(x-1)(x-2)(x-7)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-7}.$$

2. $$\frac{x^4+x^2+1}{(x+4)^2(x^2+4)^2} = \frac{A}{x+4} + \frac{B}{(x+4)^2} + \frac{Cx+D}{x^2+4} + \frac{Ex+F}{(x^2+4)^2}.$$

3. $$\frac{3x^3-2x+1}{(x+5)(x^2+4)(x^2+9)} = \frac{A}{x+5} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{x^2+9}.$$

4. $$\frac{7x^6-4x^3+2x}{(x-1)^2(x^2+x+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{(x^2+x+1)^2}.$$

5. $$\frac{x^4+x^2+1}{x^2(x+3)(x-4)(x-7)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3} + \frac{D}{x-4} + \frac{E}{x-7} + \frac{F}{(x-7)^2} + \frac{G}{(x-7)^3}.$$

Steps for the Partial Fraction Decomposition

To write the partial fraction decomposition of $\dfrac{P(x)}{Q(x)}$:

1. If **degree(TOP) $\geq$ degree(BOTTOM)** then use **long division**.
2. **Factor the bottom** $Q(x)$ as much as possible.
3. Write down the **partial fraction decomposition** based on the previous table.
4. **Solve** for the unknown constants by either:
   (i) Comparing coefficients after finding a common denominator on the right side.
   (ii) Plugging in values of x to get a system of equations.

Example

Rewrite $\dfrac{x^3 + 3x^2 + 2}{x + 1}$.

Solution.

- Since the degree of the top is larger than or equal to the bottom, we must use long division:

$$
\begin{array}{r|lllll}
 & & x^2 & + 2x & \text{-}2 \\
\cline{2-5}
x+1 & x^3 & + 3x^2 & & +2 \\
 & x^3 & + x^2 & & \\
\cline{2-3}
 & & 2x^2 & & \\
 & & 2x^2 & + 2x & \\
\cline{3-4}
 & & & - 2x & + 2 \\
 & & & - 2x & -2 \\
\cline{4-5}
 & & & & 4
\end{array}
$$

- Hence $\dfrac{x^3 + 3x^2 + 2}{x + 1} = (x^2 + 2x - 2) + \dfrac{4}{x + 1}$.

Example

Write the partial fraction decomposition of $\dfrac{1}{x^2-4}$.

Solution.

- Since the degree of the top is strictly smaller than the bottom, we skip long division.
- Factoring the bottom gives: $x^2 - 4 = (x-2)(x+2)$.
- We have two **linear factors** each appearing **once**.
- The decomposition is:

$$\frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} = \frac{A(x+2)}{(x-2)(x+2)} + \frac{B(x-2)}{(x+2)(x-2)}$$

- This gives us the equation: $1 = A(x+2) + B(x-2)$.
- **Method 2:** Sub in values of x (**Tip:** Plug in values that give 0 somewhere).
  - $x = 2$: $\quad 1 = A(2+2) + B(2-2) \quad \rightarrow \quad 1 = 4A \quad \rightarrow \quad$ **A=1/4**.
  - $x = -2$: $\quad 1 = A(-2+2) + B(-2-2) \quad \rightarrow \quad 1 = -4B \quad \rightarrow \quad$ **B=-1/4**.
- The decomposition is $\dfrac{1}{x^2-4} \quad = \quad \dfrac{1/4}{x-2} + \dfrac{-1/4}{x+2}$.

Example

With reference to the last example, solve for A and B in $\frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$ by comparing coefficients (**Method 1**).

Solution.

- As before, we find a common denominator and cancel to get the equation:
$$1 = A(x+2) + B(x-2).$$
- The left side should not be thought of as a single number! It is a **constant** polynomial (of degree 0).
- We can think of the left side as $0x + 1$.
- The right side is a **linear polynomial** (of degree 1).
- Expand the right side to get:
$$Ax + 2A + Bx - 2B \qquad \to \qquad (A+B)x + (2A-2B)$$
- Thus, we have:
$$0x + 1 = (A+B)x + (2A-2B)$$
- We now **compare coefficients**:
  - Coefficient of x^1: $\quad 0 = A + B$
  - Coefficient of x^0: $\quad 1 = 2A - 2B$
- We now solve this system of equations.
- The first equation is $A = -B$. Plugging this into the second equation gives:
$$1 = 2(-B) - 2B \qquad \to \qquad 1 = -4B \qquad \to \qquad \mathbf{B=-1/4}$$
- Now plugging in $B = -1/4$ into the first equation gives **A=1/4**.

Example

Write the partial fraction decomposition of $\dfrac{x+2}{x^3-x}$.

Solution.

- Since the degree of the top is strictly smaller than the bottom, we skip long division.
- Factoring the bottom gives: $x^3 - x = x(x^2-1) = x(x-1)(x+1)$.
- We have three **linear factors** each appearing **once**.
- The decomposition is: $\dfrac{x+2}{x(x-1)(x+1)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+1} \qquad (*)$
- To find A, B and C we do a common denominator on the right side:

$$\frac{x+2}{x(x-1)(x+1)} = \frac{A(x-1)(x+1)}{x(x-1)(x+1)} + \frac{Bx(x+1)}{x(x-1)(x+1)} + \frac{Cx(x-1)}{x(x-1)(x+1)}$$

- This gives us the equation: $x+2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$.
- **Method 2:** Sub in values of x (**Hint:** Plug in values that give 0 somewhere).
  - $x=0$: $\quad 2 = A(0-1)(0+1)+0+0 \quad \rightarrow \quad 2=-A \quad \rightarrow \quad$ **A=-2**.
  - $x=1$: $\quad 3 = 0+B(1)(1+1)+0 \quad \rightarrow \quad 3=2B \quad \rightarrow \quad$ **B=3/2**.
  - $x=-1$: $\quad 1 = 0+0+C(-1)(-1-1) \quad \rightarrow \quad 1=2C \quad \rightarrow \quad$ **C=1/2**.
- The decomposition is:

$$\frac{x+2}{x^3-x} = \frac{-2}{x} + \frac{3/2}{x-1} + \frac{1/2}{x+1}.$$

Example

Write the partial fraction decomposition of $\dfrac{x^2-3x+5}{(x+1)(x-1)^2}$.

Solution.

- Since the degree of the top is strictly smaller than the bottom, we skip long division.
- In the bottom, we have two **linear factors**:
  - $(x+1)$ is a linear factor appearing **once**.
  - $(x-1)$ is a linear factor appearing **twice**.
- The decomposition is: $\dfrac{x^2-3x+5}{(x+1)(x-1)^2} = \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2} \qquad (*)$
- To find A, B and C we do a common denominator on the right side:

$$\frac{x^2-3x+5}{(x+1)(x-1)^2} = \frac{A(x-1)^2}{(x+1)(x-1)^2} + \frac{B(x+1)(x-1)}{(x+1)(x-1)^2} + \frac{C(x+1)}{(x+1)(x-1)^2}$$

- This gives us the equation: $x^2-3x+5 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$.
- **Method 2:** Sub in values of x (**Tip:** Plug in values that give 0 somewhere).

Continued on next slide...

Solution (continued).

- $x = 1$: $\quad 1^2 - 3(1) + 5 = 0 + 0 + C(1+1) \quad \to \quad 3 = 2C \quad \to \quad C = 3/2$.
- $x = -1$: $\quad (-1)^2 - 3(-1) + 5 = A(-2)^2 + 0 + 0 \quad \to \quad 9 = 4A \quad \to \quad A = 9/4$.
- No other values of x give zero, so arbitrarily choose another value.
- $x = 0$: $\quad 5 = A(-1)^2 + B(1)(-1) + C(1) \quad \to \quad 5 = A - B + C$
- But $A = 9/4$ and $C = 3/2$:
  $5 = 9/4 - B + 3/2 \quad \to \quad B = 9/4 + 3/2 - 5 \to B = -5/4$.

The decomposition is:

$$\frac{x^2 - 3x + 5}{(x+1)(x-1)^2} \quad = \quad \frac{9/4}{x+1} + \frac{-5/4}{x-1} + \frac{3/2}{(x-1)^2}$$