University of Toronto Scarborough

CSC B36 | **Final Exam** | **16 August 2022**

Student Number: : : : : : : : : : : :

Last (Family) Name: ____________________

First (Given) Name: ____________________

Do not turn this page until you are told to do so.

In the meantime, complete the above and read the rest of this cover page.

Aids allowed: None.

Duration: 180 minutes.

There are 11 pages in this exam. Each is numbered at the bottom. *When you receive the signal to start, please check that you have all the pages.*

You will be graded on your mastery of course material as taught in class. So you need to demonstrate this. Unless otherwise stated, you must explain or justify every answer.

Answer each question in the space provided. The *second* last page is intentionally left blank in case you need more space for one of your answers. *You must clearly indicate where your answer is and what part should be marked.* **What you write on backs of pages will not be graded.**

Question	Your mark	Out of
1.		10
2.		15
3.		8
4.		7
5.		5
6.		7
7.		8
Total		60

1. We define a set of ordered pairs of integers as follows.

   Let G be the smallest set such that
   BASIS: $(1, 2) \in G$.
   INDUCTION STEP: If $(x, y) \in G$, then $(x + 1, y + 2), (-y, x) \in G$.

   (a) **[2 marks]** Is $(0, 0)$ an element of G? Briefly explain your answer.

   (b) **[6 marks]** Prove that for every $(v, w) \in G$, there exists $k \in \mathbb{Z}$ such that $w - 2v = 5k$ — i.e., $w - 2v$ is a multiple of 5.

   (c) **[2 marks]** Is $(2, 1)$ an element of G? Briefly explain your answer.

2. Consider the program and its specification given on the last page of this exam.

   Use methods from class to prove that this program is correct with respect to its specification.

[more space available on next page . . .]

[. . . additional space for question 2]

3. For each of these implications, state whether it holds for arbitrary regular expressions R and S over alphabet $\{0, 1\}$. Justify your answers.

   (a) [**4 marks**] If $R \equiv S^*$, then $R \equiv R^*$.

   (b) [**4 marks**] If $R1S \equiv S1R$, then $RS \equiv SR$.

4. Let $\Sigma = \{0, 1\}$. Given a language L over Σ, we define $f(L)$ to be:

$$f(L) = \{y : \text{for some } x \in L,\ y = x \text{ with one or more its occurrences of 1 replaced by 10}\}.$$

For example, if $L = \{\epsilon, 0, 1011\}$, then $f(L) = \{10011, 10101, 10110, 100101, 100110, 101010, 1001010\}$.

Let $M = (Q, \Sigma, \delta, s, F)$ be a DFSA.

Describe clearly how to construct an NFSA $M' = (Q', \Sigma, \delta', s', F')$ such that $\mathcal{L}(M') = f(\mathcal{L}(M))$.

No justification is required, but part marks may be given for it if your construction is incorrect.

5. Let $\Sigma = \{0, 1\}$. For arbitrary strings $x, y \in \Sigma^*$, let $\#_y(x) = |\{(u, v) : u, v \in \Sigma^* \text{ and } x = uyv\}|$.

Let $L_5 = \{x \in \Sigma^* : \#_{11}(x) = 2\,\#_{01}(x) + 2\}$.

Nick created a CFG that generates L_5 using the left-to-right method. Here is Nick's design.

- S generates L_5.
- A_{00} generates $\{x \in \Sigma^* : \#_{11}(x) = 2\,\#_{01}(x),\ x \text{ begins with 0 and ends with 0}\}$.
- A_{01} generates $\{x \in \Sigma^* : \#_{11}(x) = 2\,\#_{01}(x),\ x \text{ begins with 0 and ends with 1}\}$.
- A_{10} generates $\{x \in \Sigma^* : \#_{11}(x) = 2\,\#_{01}(x),\ x \text{ begins with 1 and ends with 0}\}$.
- A_{11} generates $\{x \in \Sigma^* : \#_{11}(x) = 2\,\#_{01}(x),\ x \text{ begins with 1 and ends with 1}\}$.

However, before Nick showed his CFG to anyone, the **Grammar Mangler** struck!
All productions were erased. Please restore Nick's CFG. No justification is required.

$S \rightarrow$

$A_{00} \rightarrow$

$A_{01} \rightarrow$

$A_{10} \rightarrow$

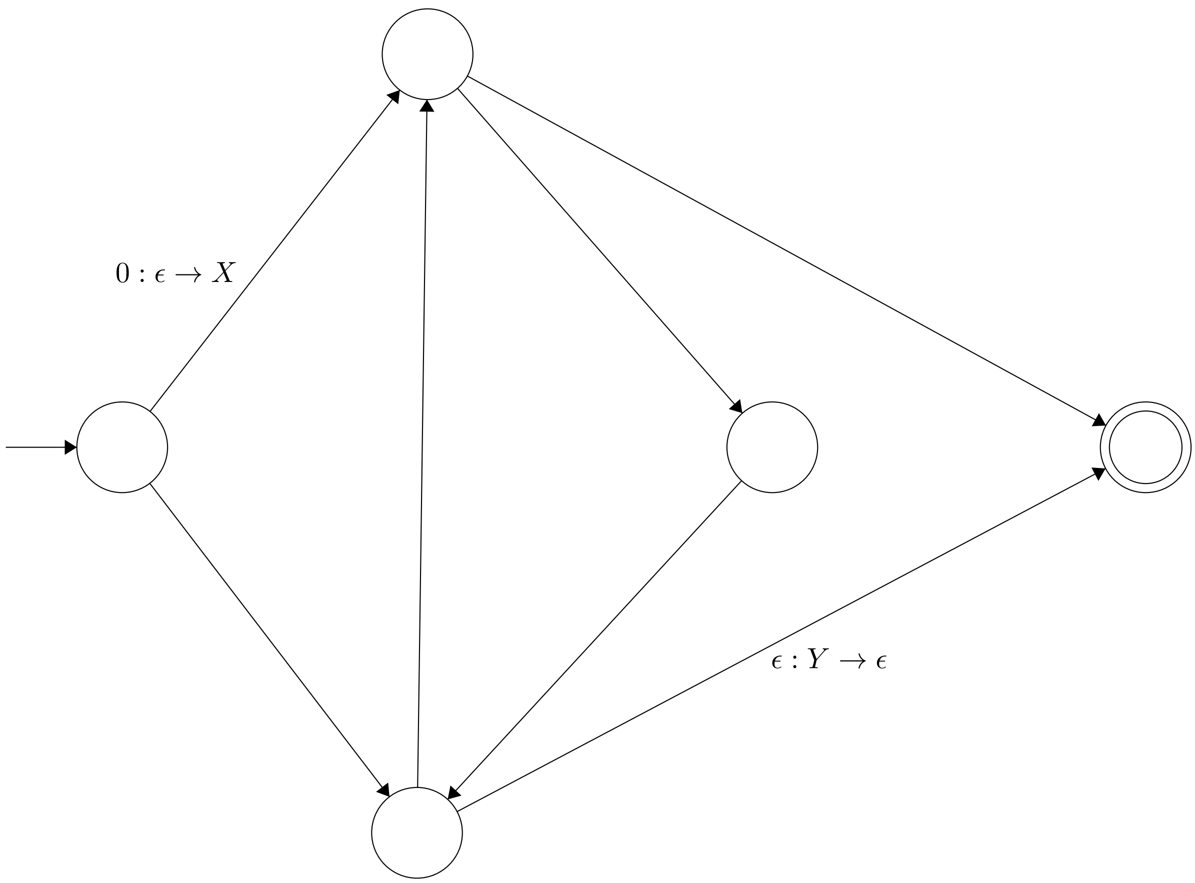
$A_{11} \rightarrow$

6. Let $\Sigma = \{0, 1\}$. For arbitrary strings $x, y \in \Sigma^*$, let $\#_y(x) = |\{(u, v) : u, v \in \Sigma^* \text{ and } x = uyv\}|$.

   Let $L_6 = \{x \in \Sigma^* : \#_{11}(x) = 2\,\#_{01}(x) + 2\}$ (*same language as in the previous question*).

   Nick created a PDA that accepts L_6.

   However, before he showed it to anyone, the **Automaton Mangler** struck!
   Two transition arrows and ten transition labels were erased.
   Here is what remains of Nick's PDA.



   Please reconstruct Nick's PDA by adding the missing transition arrows and labels.
   No justification is required.

7. Consider a first-order language with binary predicate symbol L, unary predicate symbol M, and constant symbol $\mathbf{c}$. Furthermore, consider the following sentences from our language.

$F : M(\mathbf{c})$ $\qquad$ $G : \exists x \forall y \Big(M(y) \rightarrow L(x,y) \Big)$ $\qquad$ $H : \neg \exists x \Bigg(M(x) \wedge \forall y \Big(M(y) \rightarrow L(x,y) \Big) \Bigg)$

Let $\mathcal{R}$ be a partial[1] structure for our language with domain $\mathbb{R}$,

$$L^{\mathcal{R}} = \{(x,y) \in \mathbb{R} : x \leq y\} \quad \text{and} \quad \mathbf{c}^{\mathcal{R}} = 1.$$

Using each of the following definitions of $M^{\mathcal{R}}$ to complete our definition of $\mathcal{R}$, state whether each of F, G, H is true or false in $\mathcal{R}$. Briefly explain your answers.

(a) **[2 marks]** $M^{\mathcal{R}} = \{x \in \mathbb{R} : x > 0\}$.

(b) **[2 marks]** $M^{\mathcal{R}} = \{1\}$.

(c) **[2 marks]** $M^{\mathcal{R}} = \mathbb{R}$.

(d) **[2 marks]** $M^{\mathcal{R}} = \emptyset$.

[1]The structure is *partial* because $M^{\mathcal{R}}$ is left undefined (until later).

This page is intentionally left blank in case you need more space for one of your answers.

You may detach this page if you find it convenient to do so.

There is no need to submit this page, as nothing on it will be graded.

<u>Program and specification for question 2</u>

We say that an integer n is a *cube* iff there exists $k \in \mathbb{Z}$ such that $n = k^3$.

▷ Precondition: $n \in \mathbb{N}$.
▷ Postcondition: Return **True** if n is a cube; otherwise return **False**.
isCube(n)
    $b = 0; \quad c = 0; \quad i = 0$
    while $c < n$:
        $b = b + 6i$
        $c = c + b + 1$
        $i = i + 1$
    if $c == n$: **return True**   **else**: **return False**