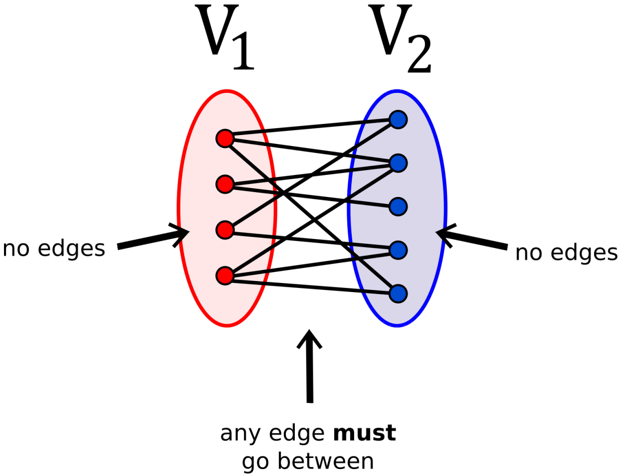
Bipartite Graphs

Intuition: What is a bipartite graph?

- "**bi**" is a prefix meaning "**two**".
- "**partite**" means **divided into parts**.

- Informally, a **bipartite** graph is a graph divided into **two parts** with the property that edges must go between the parts.



Definition of a bipartite graph

Write down a formal definition of a bipartite graph...

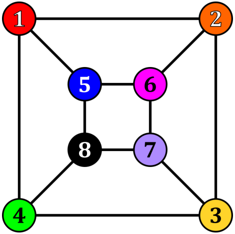
Definition: Bipartite graph

A graph is **bipartite** if the vertex set can be partitioned into **two** disjoint sets V_1 and V_2 such that every edge has one endpoint in V_1 and the other endpoint in V_2.

- **Note:** A **partition** of a set V is a set of non-empty subsets of V such that every element $v \in V$ is in exactly one of these subsets.
- This means $V(G) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.

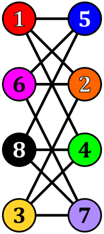
Example

Is the graph drawn below a bipartite graph?



Solution.

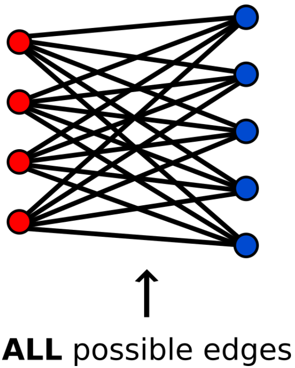
Redrawing the graph with the partition $V_1 = \{1, 3, 6, 8\}$ and $V_2 = \{2, 4, 5, 7\}$ gives:



All edges have one endpoint in V_1 and the other in V_2, thus, the graph **is** bipartite.

Complete bipartite graphs

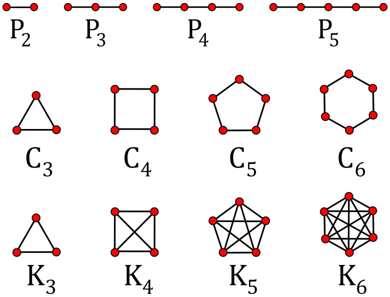
- A **complete bipartite graph** is a special type of bipartite graph where every vertex of the first set V_1 is connected to every vertex of the second set V_2.
- **Notation:** $K_{m,n}$ where $m = |V_1|$ and $n = |V_2|$.
- For example, $K_{4,5}$ is drawn below:



Examples and non-examples

Example

Which of the paths, cycles and complete graphs are bipartite?



Solution.

- For $n \geq 2$, P_n is bipartite.
- For $n \geq 3$, C_n is bipartite if and only if n is even (**why?**).
- For $n \geq 3$, K_n is not bipartite.
- **Note:** P_2, P_3 and C_4 are also **complete bipartite** (they are $K_{1,1}$, $K_{1,2}$ and $K_{2,2}$).

Subgraphs of bipartite graphs

Example

True or false?

Any subgraph of a bipartite graph is also bipartite.

Solution.

We prove this is **true**, i.e., "If G is bipartite, then every subgraph H of G is bipartite."

- Let $G = (V, E)$ be bipartite with vertex partition $V = X \cup Y$.
- Let $H = (V', E')$ be a subgraph of G.
- Define $X' = X \cap V'$ and $Y' = Y \cap V'$.
- Then $V' = X' \cup Y'$ is a bipartition of H (otherwise, there is an edge of H with both endpoints in X' (or Y') and hence this is also an edge of G with both endpoints in X (or Y) contradicting that $X \cup Y$ is a bipartition of G).

Example

True or false?

If a graph G has a subgraph that is not bipartite, then G is not bipartite.

Solution. This is **True** as it is the **contrapositive** of the first statement.

A handshaking type of lemma for bipartite graphs

Lemma

If $G = (V, E)$ be a bipartite graph with bipartition (X, Y) (i.e., $V = X \cup Y$). Then

$$|E(G)| = \sum_{x \in X} \deg(x) = \sum_{y \in Y} \deg(y).$$

Proof.

- Every edge has exactly one endpoint in X, thus the number of edges is

$$\sum_{x \in X} \deg(x).$$

- Every edge has exactly one endpoint in Y, thus the number of edges is

$$\sum_{y \in Y} \deg(y).$$

- Hence, the statement follows.

König (1936) proved the following characterization of bipartite graphs.

Theorem

A graph is bipartite if and only if it does not contain an odd cycle.

Proof.

See Mike or the proof in this video: https://www.youtube.com/watch?v=YiGFhWxtHjQ.

Later we (might) see other characterizations of bipartite graphs.

Theorem

A graph is bipartite if and only if its **chromatic number** is less than or equal to two.

Theorem

A graph is bipartite if and only if the **spectrum** of the graph is symmetric.