University of Toronto at Scarborough
CSCC37—Numerical Algorithms for Computational Mathematics, Fall 2022

Assignment #3: NonLinear Equations / FP Iteration

Due: November 17, 2022 at 11:45 p.m.
This assignment is worth 10% of your final grade.

Warning: Your electronic submission on *MarkUs* affirms that this assignment is your own work and no one else's, and is in accordance with the University of Toronto Code of Behaviour on Academic Matters, the Code of Student Conduct, and the guidelines for avoiding plagiarism in CSCC37.

This assignment is due by 11:45 p.m. November 17. If you haven't finished by then, you may hand in your assignment late with a penalty as specified in the course information sheet.

[15] 1. In lecture we introduced the *Fixed Point Theorem (FPT)*, which can be used to identify a guaranteed interval of convergence for the iteration $x_{k+1} = g(x_k), k = 0, 1, \ldots$. We started the proof of this theorem, but did not complete it. Now you will complete the proof.

(a) State the theorem, carefully including the conditions and claim.

(b) Repeat the first part of the proof, given in lecture, where we show that under the conditions of the theorem the iteration $x_{k+1} = g(x_k)$ converges to *some* point $\tilde{x}$ in the sink.

(c) Prove that $\tilde{x}$ in (b) is indeed a fixed point of the iteration.

(d) Complete the proof by showing $\tilde{x}$ in (b) is *unique* in the sink.

(**Hint:** As with part (b), both parts (c) and (d) use the Mean Value Theorem (MVT). You just need to choose your variables carefully.)

[15] 2. Consider the functions $f(x) = 1 - 1/(2x)$ and $g(x) = 2x(1-x)$.

(a) How many roots does f have? Are the roots of f fixed-points of g? Are there more fixed points of g than roots of f? **Justify your answers.**

(b) Using the FPT, determine the region of local convergence of the fixed-point iteration $x_{k+1} = g(x_k),\ k = 0, 1, \ldots$, with $g(x)$ as defined above. In other words, find the largest interval on the x-axis for which the iteration is *guaranteed* to converge.

[10] 3. Consider the equation $x + \ln(x) = 0$, whose root is $\alpha \approx 0.5$, and the three iteration formulae:

$$(1)\ \ x_{k+1} = -\ln(x_k), \qquad (2)\ \ x_{k+1} = e^{-x_k}, \qquad (3)\ \ x_{k+1} = \frac{x_k + e^{-x_k}}{2}.$$

(a) Which of the formulae can be used to solve this equation?

(b) Which formula *should* be used?

(c) Give an even better formula.

[5] 4. What is the purpose of the following fixed-point iteration? (**Hint:** Identify it as the Newton iteration for a certain function.)

$$x_{k+1} = 2x_k - x_k^2 y$$

[15] 5. Steffensen's method is an alternative to Newton's method for solving the single equation $f(x) = 0$. In Steffensen's method, we attempt to find the roots of f as the fixed points of

$$g(x) = x - \frac{f(x)^2}{f(x + f(x)) - f(x)}$$

(a) Are the roots of f fixed-points of g? Are there more fixed points of g than roots of f? **Justify your answers.**

(b) Show that Steffensen's method can be derived from Newton's method with an appropriate approximation for $f'(x)$.

(c) For $f(x) = x^2 - 10x + 24$, prove that both Newton's method and Steffensen's method are at least quadratically convergent near the roots $x = 4$ and $x = 6$.

(d) What is the (obvious) advantage of Steffensen's method over Newton's method?

[total: 60 marks]