Independence number

Hamilton paths/cycles

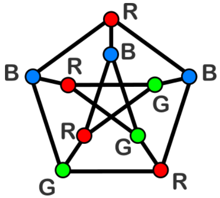
Trees (time permitting)

Recall: Graph Colouring

Let G be a graph.

- The **chromatic number** of G, denoted $\boxed{\chi(G)}$, is the smallest number of colours required to colour the vertices so that no two adjacent vertices share the same color.

- **Example.** The Petersen graph P has $\chi(P) = 3$.



- **Thm.** Suppose G has at least one edge. Then $\chi(G) = 2$ iff G is bipartite.

- **Thm.** $\chi(G) \geq \omega(G)$.

  Recall: $\omega(G)$ is the clique number (maximum size of a complete subgraph).

- **Thm.** $\chi(G) \leq \Delta(G) + 1$.

  Recall: $\Delta(G)$ is the maximum degree of G.

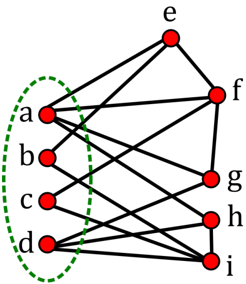
- **The Four Colour Theorem:** Every planar graph is 4-colourable (i.e., $\chi(G) \leq 4$).

Independent Sets

Definition

An **independent set** of a graph G is a set of vertices in which **no** two are adjacent.

Example



The set $S = \{a, b, c, d\}$ is an independent set of the graph shown above because there are no edges between any pair of vertices in S.

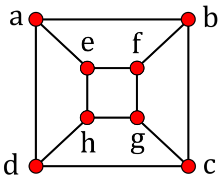
Maximal Versus Maximum

Definition

- A **maximal** independent set is one that is not a subset of any other independent set.
- Of all maximal independent sets, the ones with the largest size are called **maximum** independent sets.

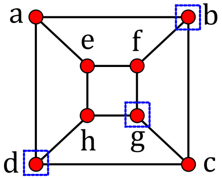
Example

Let G be the graph shown (this is a planar drawing of a cube).

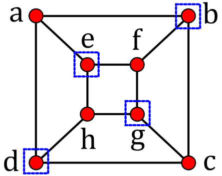


(a) Find an independent set in G that is **not** **maximal**.
(b) Find an independent set in G that is **both** **maximal** and **maximum**.
(c) Find an independent set in G that is **maximal** but **not** **maximum**.

Solution.

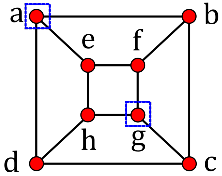


(a) $S = \{b, d, g\}$ is an independent set but **not maximal** since we can make it larger by including e.



(b) $S = \{b, d, e, g\}$ is an independent set that is **maximal** (we cannot include any other vertices).

It is also **maximum** since no independent sets can have size greater than 4 (**why? see the next slide**).



(c) $S = \{a, g\}$ is an independent set that is **maximal** (it cannot be made larger).

It is **not maximum** (it does not have largest size).

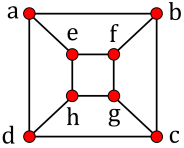
Independence Number

Definition

The size of a maximum independent set of a graph G is the **independence number** of G and denoted by $\boxed{\alpha(G)}$.

Example

Determine $\alpha(G)$ where G is the graph of a cube:



Solution.

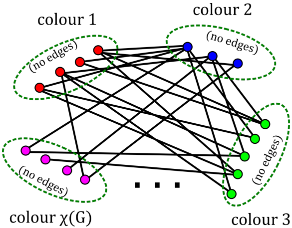
- Since $S = \{a, c, f, h\}$ is an independent set of size 4, we have $\boxed{\alpha(G) \geq 4}$.
- Each set $\{a, b\}$, $\{c, d\}$, $\{e, f\}$, $\{g, h\}$ induces a clique of size 2 (i.e., K_2).
- Observe that an independent set S can have at most one element from each of these four cliques (e.g., S cannot have both a and b), thus, $\boxed{\alpha(G) \leq 4}$.
- Combining the two inequalities above gives $\alpha(G) = 4$.

Theorem

Let G be a graph. Then $\chi(G) \geq \dfrac{|V(G)|}{\alpha(G)}$.

Proof.

- Consider a colouring of G using the minimum number of colours (i.e., $\chi(G)$ colours).
- Suppose the colours are $\{1, 2, \ldots, \chi(G)\}$.
- Let S_i be the set of vertices of G with colour i:

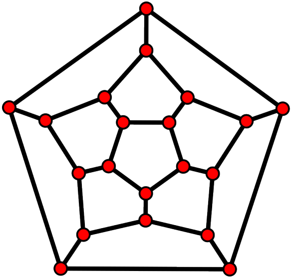


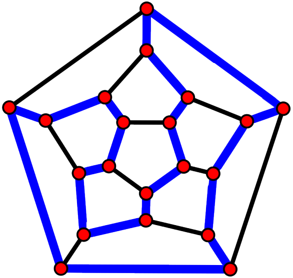
- Each set S_i is an independent set (by definition of a colouring).
- Thus, $|S_i| \leq \alpha(G)$ (since $\alpha(G)$ is the size of a maximum independent set).
- The result now follows from: $|V(G)| = \sum_{i=1}^{\chi(G)} |S_i| \leq \sum_{i=1}^{\chi(G)} \alpha(G) = \chi(G)\,\alpha(G)$.

The Icosian Game

(The Puzzle Museum article has pictures of the Icosian Game.)

- The **icosian game** was invented by William Hamilton in 1857.
- The game involves tracing the edges of a dodecahedron.





- In particular, nails were placed at the vertices of the dodecahedron (which represented cities).
- To play the game, you were tasked to wrap string around the nails (following along the edges) so that each nail is visited once and so you also end where you started.
- The Toy & Games Makers "**Jaques of London**" paid £25 for the rights.
- The game was not successful.

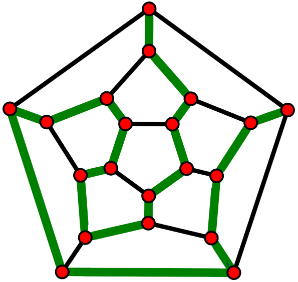
Terminology: Hamilton paths and cycles

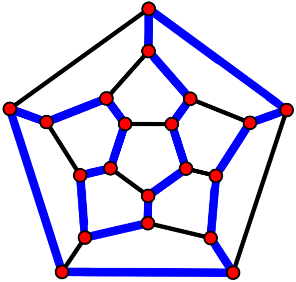
Definitions

- A **Hamilton path** of a graph G is a path that uses every vertex exactly once.
- A **Hamilton cycle** of a graph G is a cycle that uses every vertex exactly once.
- A graph G that contains a Hamilton cycle is called **Hamiltonian**.

Example

- A Hamilton **path** in a planar drawing of the dodecahedron is shown on the **left**.
- A Hamilton **cycle** in a planar drawing of the dodecahedron is shown on the **right**.





Note: A Hamilton cycle can be converted into a Hamilton path by deleting an edge.

Example

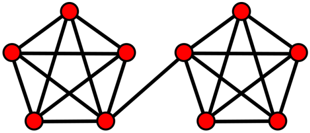
Give an example of a connected graph that has no Hamilton path and no Hamilton cycle.

Solution: The star graph $K_{1,n}$ for $n \geq 3$ has no Hamilton path or Hamilton cycle.

Example

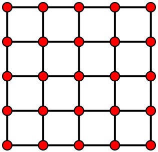
Give an example of a connected graph that has a Hamilton path but no Hamilton cycle.

Solution:



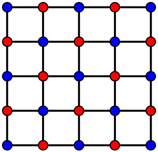
Example

Does the grid graph below have: (a) A Hamilton path? (b) A Hamilton cycle?



Solution. (a) It has a Hamilton path (start at the top left, go right as far as possible, go down one vertex, go left as far as possible, etc.).

(b) It does **not** have a Hamilton cycle. The graph is bipartite (below is a 2-colouring):



Lemma: Let G be a bipartite graph with bipartition (V_1, V_2). If G has a Hamilton cycle, then $|V_1| = |V_2|$ (**why?**).

The grid graph has a bipartition with $|V_1| = 13$ and $|V_2| = 12$, thus, it cannot have a Hamilton cycle by the Lemma.

Proving existence of Hamilton cycles

- If a graph G has every vertex with large degree, then intuitively, you could argue that G must contain a Hamilton cycle (i.e., G is Hamiltonian).
- This is formalized in the following theorem due to Dirac.
- Recall that $\delta(G)$ is the minimum degree of a vertex in G.

Dirac's Theorem

Let G be a graph on $n \geq 3$ vertices. If $\delta(G) \geq n/2$, then G is Hamiltonian.

Proof.

- The proof is on the next slide.
- During the lecture we drew the corresponding pictures for the statements (see Mike for these).
- I also described the proof of **Claim 1** in the proof more intuitively by counting the number of vertices in P as $1 + \deg(v_k) + \deg(v_1)$ since $V(P)$ includes v_k, all the vertices adjacent to v_k (of which there are $\deg(v_k)$) and all the non-adjacent vertices to v_k (of which there are at least $\deg(v_1)$ by the assumption) giving $k \geq n + 1$, a contradiction.

Proof of Dirac's Theorem.

- Let $P = v_1 v_2 \cdots v_{k-1} v_k$ be a longest path in G.
- All the neighbours of v_1 are on the path P (otherwise, there is a vertex w adjacent to v_1 and not on the path P; then $P' = wv_1 v_2 \cdots v_k$ is longer contradicting that P is a longest path). Note this implies that $k \geq 1 + n/2$.
- Similarly, all neighbours of v_k are on the path P.
- Since $\delta(G) \geq n/2$, each of v_1 and v_k has at least $n/2$ neighbours on P.
- **Claim 1:** There is a j (with $1 \leq j \leq k-1$) such that v_j is adjacent to v_k and v_{j+1} is adjacent to v_1.
  - **Proof of Claim 1:** Suppose (to derive a contradiction) this is not true.
  - Then for every neighbour v_i of v_k on P (there are at least $n/2$ of them), v_{i+1} is **NOT** adjacent to v_1. Thus, $\deg(v_1) \leq k - 1 - \frac{n}{2} < n - \frac{n}{2} = \frac{n}{2}$ contradicting that $\delta(G) \geq n/2$.
  - Thus, such a j exists.
- Let $C = v_1 v_2 \cdots v_j v_k v_{k-1} \cdots v_{j+1} v_1$ and observe C is a cycle.
- **Claim 2:** C is a Hamilton cycle (i.e., $k = n$).
  - **Proof of Claim 2:** Suppose not. Then there is a vertex w not on P.
  - Since $\deg(w) \geq n/2$ and $k \geq 1 + n/2$, it must be that w is adjacent to some vertex v_i on the path P.
  - But now the path $P' = wv_i \cdots$ (see picture) is a path on $k+1$ vertices and thus a longer path than P contradicting that P is a longest path.
- Hence, C is a Hamilton cycle showing that G is Hamiltonian.

By analyzing the proof of Dirac's theorem, Ore was able to improve the result.

Ore's Theorem

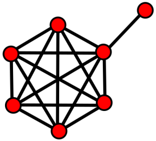
Let G be a graph on $n \geq 3$ vertices. If for all non-adjacent pairs $x, y \in V(G)$ the inequality $\deg(x) + \deg(y) \geq n$ holds, then G is Hamiltonian.

Exercise

Use Ore's theorem to prove that the maximum number of edges in a non-Hamiltonian graph on n vertices is $\binom{n-1}{2} + 1$.

Solution.

- We first give an example of a graph with $\binom{n-1}{2} + 1$ edges that has no Hamilton cycle.
- Consider K_{n-1} and attach a "pendant" vertex to get a graph with n vertices and $\binom{n-1}{2} + 1$ edges. For example, below is a K_6 with a vertex attached:



- This graph is non-Hamiltonian and has $\binom{n-1}{2} + 1$ edges.

Solution (continued).

- Next we assume G is a graph with at least $\binom{n-1}{2} + 2$ edges and we prove that G must have a Hamilton cycle.
- Consider a pair of non-adjacent vertices x and y.
- Form a graph G' by deleting x, y and any incident edges to x and y.
- Since G' has $n-2$ vertices, we have $|E(G')| \le \binom{n-2}{2}$.
- On the other hand,

$$|E(G')| = |E(G)| - \deg(x) - \deg(y) \ge \binom{n-1}{2} + 2 - (\deg(x) + \deg(y)).$$

- Thus,

$$\binom{n-2}{2} \ge |E(G')| \ge \binom{n-1}{2} + 2 - (\deg(x) + \deg(y))$$

  implying

$$\deg(x) + \deg(y) \ge \frac{(n-1)(n-2)}{2} + 2 - \frac{(n-2)(n-3)}{2} = n.$$

- Since x and y were arbitrarily chosen as any pair of non-adjacent vertices, the hypothesis of Ore's theorem holds.
- Therefore, G contains a Hamilton cycle by Ore's theorem.