University of Toronto Scarborough
Department of Computer and Mathematical Sciences

December Examinations 2015

CSC C37H3 F

Duration—2 hours

Aids allowed: A single $8\frac{1}{2} \times 11$-inch crib sheet. No calculators.

Make sure that your examination booklet has 11 pages (including this one). Write your answers in the spaces provided. You will be rewarded for concise, well-thought-out answers, rather than long rambling ones. Please write legibly.

This exam was designed so that you have plenty of time to write it. Take a few minutes before you begin to read each question, and then start with the question you are most comfortable with.

Name: ______________________________
(Circle your family name.)

Student #: ________________________

YOU MUST SIGN THE FOLLOWING:

I declare that this exam was written by the person whose name and student # appear above.

Signature: ______________________________

Your grade

1.	____________	/ 10
2.	____________	/ 15
3.	____________	/ 15
4.	____________	/ 15
5.	____________	/ 20
Total	____________	/ 75

Question 1

[10 marks]

When each of the following expressions is evaluated using floating-point arithmetic, poor results are obtained for a certain range of values of x. In each instance, identify this range and provide an alternate expression that can be used for such values of x.

a. $\sqrt{1+x} - \sqrt{1-x}$

b. $e^x - 1$

Question 2

[15 marks]

Consider calculating the LU-factorization of $A \in \mathcal{R}^{5\times 5}$, using Gaussian Elimination with partial pivoting. After stage 4 of the elimination we have

$$\mathcal{L}_4\,\mathcal{P}_4\,\mathcal{L}_3\,\mathcal{P}_3\,\mathcal{L}_2\,\mathcal{P}_2\,\mathcal{L}_1\,\mathcal{P}_1\,A = U \tag{1}$$

where $\mathcal{P}_i$, $\mathcal{L}_i$ are, respectively, the permutation and Gauss transform used in the i-th stage of the elimination, and U is the upper-triangular factor of the factorization.

The final form of the factorization is

$$P\,A = L\,U$$

where

$$P = \mathcal{P}_4\,\mathcal{P}_3\,\mathcal{P}_2\,\mathcal{P}_1$$

and

$$L = \tilde{\mathcal{L}}_1^{-1}\,\tilde{\mathcal{L}}_2^{-1}\,\tilde{\mathcal{L}}_3^{-1}\,\mathcal{L}_4^{-1}$$

a. Express $\tilde{\mathcal{L}}_1^{-1}$ in terms of the original $\mathcal{P}_i$ and $\mathcal{L}_i$ appearing in (1). **Show all of your work.**

b. Given that

$$\mathcal{L}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 & 0 \\ m_{31} & 0 & 1 & 0 & 0 \\ m_{41} & 0 & 0 & 1 & 0 \\ m_{51} & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned} \mathcal{P}_1 &\equiv \mathcal{P}_{14} \\ \mathcal{P}_2 &\equiv \mathcal{P}_{25} \\ \mathcal{P}_3 &\equiv \mathcal{P}_{34} \\ \mathcal{P}_4 &\equiv \mathcal{P}_{45} \end{aligned}$$

($\mathcal{P}_{ij}$ interchanges rows i and j for $j > i$), and considering your answer in part (**a**), write out the matrix representation of $\tilde{\mathcal{L}}_1^{-1}$ showing precisely the sign and position of the four multipliers m_{i1}.

Question 3

[15 marks]

Let $A \in \mathcal{R}^{n \times n}$ be a non-singular matrix.

a. Show how Gaussian elimination with partial pivoting can be used to solve the k linear systems

$$Ax_1 = t_1,\ Ax_2 = t_2,\ \ldots,\ Ax_k = t_k$$

in $n^3/3 + \mathcal{O}(kn^2)$ flops. You must show precisely how the matrix components of the $PA = LU$ factorization are used in each stage of your algorithm, and justify the final operation count.

(*Note*: You do **not** need to give details of the Gaussian elimination algorithm—you may assume the factorization is available.)

b. For $k = n$ and a suitable choice for $t_1, t_2, \ldots, t_n$, your algorithm in (**a**) can be used to compute A^{-1}. Explain how this can be done.

c. A possible scheme for solving $Ax = b$ is to first compute A^{-1} using (**b**), and then compute $x = A^{-1}b$. Is this scheme preferable to the Gaussian elimination algorithm as discussed in lecture? Give operation counts to justify your answer.

Question 4

[15 marks]

(When answering this question, you may use $\sqrt{2} \approx 1.4$.)

We wish to find the positive root of the nonlinear equation $f(x) = x^2 - 2$, using a fixed-point iteration $x_{k+1} = g(x_k), k = 0, 1, \ldots$.

Consider the following four variations of g:

$$
\begin{aligned}
g_1(x) &= x(1-x)+2 \\
g_2(x) &= x(1-x)+4 \\
g_3(x) &= 2/x \\
g_4(x) &= x-1+2/x^2
\end{aligned}
$$

a. Which variation(s) of g *could* be used, and why?

b. Which variation of g *should* be used, and why?

c. Derive an even *better* variation of g for use in this problem, and explain why it is better.

Question 5

[20 marks]

Consider the data points $\{(-1, 4), (0, 6), (1, 12)\}$.

a. Derive the divided-difference form of the polynomial that interpolates these data points. Show all of your work, including the divided-difference table.

b. Derive the Lagrange form.

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c. Verify that the polynomials in (**a**) and (**b**) are indeed the same polynomial by converting both to monomial basis form (i.e., write both as $p(x) = \sum_{i=0}^{2} a_i x^i$).

d. Append $(2, 16)$ to the data. Using either the divided-difference or Lagrange form, construct the single polynomial that interpolates all four data points and convert it to monomial basis form.

e. Construct the linear spline (i.e., piecewise linear interpolant) that interpolates all four data points $\{(-1,4),(0,6),(1,12),(2,16)\}$.

END OF EXAM