University of Toronto Scarborough
Department of Computer and Mathematical Sciences
MATC44H3F (LEC01) - Fall 2022 - Final Exam - Practice 3

Date: Tuesday, December 20, 2022 from 9:00 to 12:00 (IC 200 & IC 204)

Instructor: Michael Cavers

First name (please write as legibly as possible within the boxes)

Last name

Student ID number

Signature: ________________________________

- **Time:** 180 minutes
- Write your solutions in this booklet (only those pages with a QR code will be graded).
- Use the back of each page for **rough** work.
- This is a closed-book exam. No aids are allowed for this exam other than those provided by the instructor. Calculators and the use of personal electronic or communication devices is prohibited.
- This exam has 12 pages with the last page being blank.
- There are 9 problems with the number of points indicated by each problem.
- The total number of points possible is 100.
- The University of Toronto's Code of Behaviour on Academic Matters (July 2019) applies to all University of Toronto Scarborough students. The Code prohibits all forms of academic dishonesty including, but not limited to, cheating, plagiarism, and the use of unauthorized aids. Students violating the Code may be subject to penalties up to and including suspension or expulsion from the University.

1. (20 points) For the following problem, you only need to provide your final answers. Correct answers are 2 points each and incorrect answers are 0 points each. Part marks is possible.

   (a) A container contains 15 red, 12 orange, 10 yellow, 7 green and 5 blue balls. If we randomly select balls from the container, what is the minimum number of balls we must select to guarantee we get 9 balls of the same colour?

   Solution: 37 (by pigeonhole principle). If 36, we could have chosen 8 red, 8 orange, 8 yellow, 7 green and 5 blue.

   (b) In how many ways can we partition 5 people into two sets of sizes 2 and 3?

   Solution: 10

   (c) In how many ways can we partition 4 people into two sets of equal sizes?

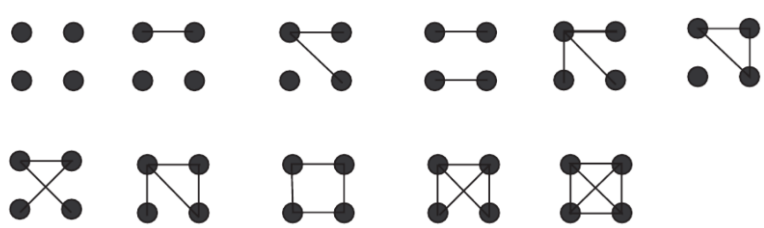
   Solution: 3

   (d) There are n **types** of coins in a bag (with an unlimited number of each type of coin). In how many ways can we choose $2k$ coins and place them into two equal-sized unlabelled rows?

   Solution: $n^k + \frac{1}{2}(n^k)(n^k - 1)$ (this comes from analyzing identical rows and non-identical rows).

   (e) How many non-isomorphic graphs are there having 4 vertices?

   Solution: There are 11 as drawn below:



(f) Let A and B be sets with $|A| = 4$ and $|B| = 9$. Determine $|A \cup B| + |A \cap B|$.

Solution: By PIE, we have $|A \cup B| = 4 + 9 - |A \cap B|$, thus, $|A \cup B| + |A \cap B| = 13$.

(g) Find a closed form formula for the nth term of the sequence (say a_n) with ordinary generating function $g(x) = \dfrac{1}{1+x}$.

Solution: $a_n = (-1)^n$.

(h) Consider the sequence $a_k = 2$ for all integers $k \geq 0$. Determine the (ordinary) generating function for the sequence.

Solution: We have $g(x) = 2 + 2x + 2x^2 + \cdots = 2(1 + x + x^2 + \cdots)$. Thus, $g(x) = \dfrac{2}{1-x}$.

(i) Find a closed form formula for the nth term of the sequence with exponential generating function $G(x) = xe^x$.

Solution: $a_n = n$

(j) The local pizza restaurant is having a special on three-topping pizzas with seven choices for the toppings. Your friend declares that two of the three toppings must be what they want otherwise they will throw a tantrum. They also won't tell you which two toppings they want. What is the minimum number of three-topping pizzas you must order to guarantee a happy friend?

Solution: The answer is seven. There are $\binom{7}{2} = 21$ distinct pairs of toppings. Thus, we must order at least seven pizzas since six three-topping pizzas contain at most $3 \times 6 = 18$ distinct pairs of toppings. The Fano plane (or STS(7)) can be used to show seven is sufficient. Number the toppings from 1 to 7 and consider

$$\mathcal{B} = \{123, 145, 167, 246, 257, 347, 356\}.$$

Then every pair ij is in exactly one block (i.e., pizza) of $\mathcal{B}$. Thus, no matter which two toppings your friend wants, one of the pizzas will have those two toppings.

2. (10 points)
Let $n \geq 1$. Two players are playing a Nim game with n heaps of objects of sizes $1, 2, 3, \ldots, n$, respectively. Prove that if $n = 4k$, $n = 4k + 1$ or $n = 4k + 2$, for some non-negative integer k, then the first player can guarantee a win.

Solution: The first player can guarantee a win iff the nim-sum of the sizes of the heaps is nonzero. When $n = 4k + 1$ or $n = 4k + 2$, there are an odd number of heaps with odd size, hence, the units digit in the nim-sum is 1. Thus, in these two cases, the nim-sum is non-zero.

Next assume $n = 4k$. We first write $n = 2^r + s$ where $0 \leq s \leq 2^{r-1}$. Since n is even so is s. In binary, each of the heaps of sizes $2^r, 2^r + 1, \ldots, 2^r + s$ have a 1 in the most significant bit (2^t), whereas the heaps of sizes $1, 2, \ldots, 2^r - 1$ have a 0 in this bit. Since s is even, there are an odd number of 1's contributing to this bit in the nim-sum, thus, the nim-sum will be non-zero.

[For an added challenge, can prove that the nim-sum is zero when $n = 4k + 3$ where k is any non-negative integer?]

3. (10 points) How many arrangements of the letters of the word FLIBBERTIGIBBET have the second T appearing after the last vowel? Justify your answer.

Solution: First, place and arrange the letters that are **not** T's or vowels: F, G, L, R, and four B's. There are

$$\binom{15}{8}\binom{8}{4,1,1,1,1}$$

ways to do this. Then place a T in the last empty position: there is only one way to do this. The six remaining letters, one T, two E's, and three I's, must be arranged in the remaining positions, and there are

$$\binom{6}{1,2,3}$$

ways to do this. Therefore, the number of arrangements is

$$\binom{15}{8}\binom{8}{4,1,1,1,1}\binom{6}{1,2,3}.$$

4. (10 points) Let $n \geq 3$. Give a combinatorial proof of $\displaystyle\sum_{k=0}^{3} \binom{n}{k}\binom{n-k}{3-k} = 2^3\binom{n}{3}$.

Solution: We count the number of ways to choose a committee of three people who may or may not be wearing hats. Since each of the three has two choices, this gives the right-side. Alternatively, let k be the number of people wearing hats on the committee and from n choose k people to be the hat wearers on the committee. Now, from $n-k$ remaining people, choose $3-k$ to be the non-hat wearing people on the committee giving a total committee size of three.

5. (10 points) Use the binomial theorem to prove the following identity for any integer $n \geq 1$:

$$\frac{3^{n+1}-1}{n+1} = \sum_{k=0}^{n} \frac{2^{k+1}}{k+1}\binom{n}{k}.$$

Hint: Write out the binomial theorem for $(x+1)^n$ and consider the definite integral of both sides over a conveniently chosen interval.

Solution: We take the definite integral on the interval $[0, 2]$ of the polynomial function

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

The definite integral of the left side gives

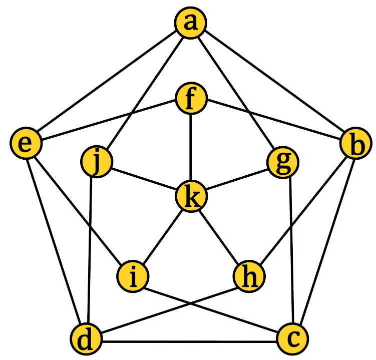
$$\int_0^2 (x+1)^n \, dx = \frac{1}{n+1}\, (x+1)^{n+1}\bigg|_0^2 = \frac{(2+1)^{n+1} - (0+1)^{n+1}}{n+1} = \frac{3^{n+1}-1}{n+1}.$$

The definite integral of the right side gives

$$\int_0^2 \left[\sum_{k=0}^{n} \binom{n}{k} x^k\right] dx = \sum_{k=0}^{n} \binom{n}{k} \left[\int_0^2 x^k \, dx\right] = \sum_{k=0}^{n} \binom{n}{k} \left[\frac{x^{k+1}}{k+1}\right]\bigg|_0^2 = \sum_{k=0}^{n} \frac{2^{k+1}}{k+1}\binom{n}{k}.$$

This proves the required identity.

6. (10 points) Let G be the Grötzsch graph drawn below.



(a) Show that G is non-planar.

(b) Show that G is not bipartite.

(c) Does G have a Hamilton cycle?

Solution:

(a) Delete the vertex labelled k. The graph that remains is a subdivsion of K_5 having vertex set $\{a, b, c, d, e\}$ where the edge ad is subdivided into the path ajd, the edge ac is subdivided into the path agc, the edge be is subdivided into the path bfe, the edge bd is subdivided into the path bhd and the edge ce is subdivided into the path cie. Therefore, G contains a subgraph that is a subdivision of K_5. By Kuratowski's theorem, G is non-planar.

(b) G is not bipartite since it has an odd length cycle, namely, $C = abcdea$.

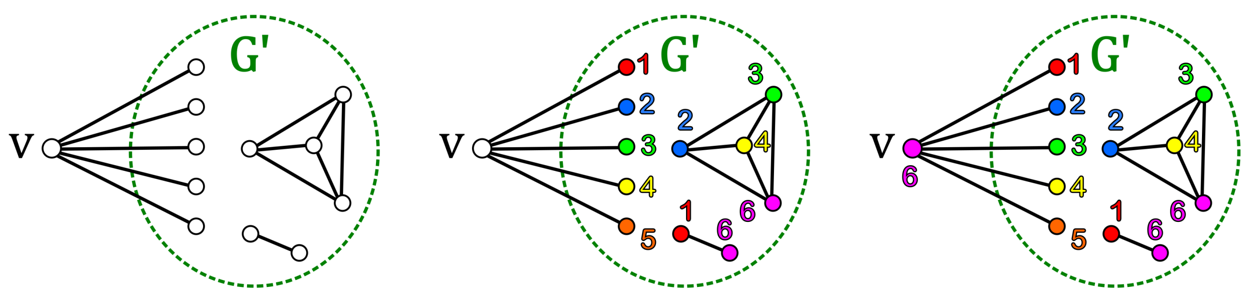
(c) Yes, here is one example of a cycle that uses every vertex: $C = ajdhbfkgciea$.

7. (10 points) Use induction to prove that every planar graph is 6-colourable.

   (You may not use the four or five colour theorems in your proof, however, you may assume that every planar graph has a vertex of degree at most five.)

Solution:

- We use induction on the number of vertices in the graph.
- Base Case: It is certainly true for graphs with at most 6 vertices.
- Induction Hypothesis: Assume it holds for planar graphs with less than n vertices.
- Let G be a planar graph with n vertices. WTS the statement holds for G.
- By the Lemma, G has a vertex v with $\deg(v) \leq 5$.



- Delete the vertex v (and all incident edges) to form the graph $G' = G - v$.
- By induction, we can colour the vertices of G' with at most six colours.
- Since $\deg(v) \leq 5$, the neighbours of v use at most 5 colours.
- Thus, there is an unused colour that we may use to colour v which gives rise to a 6-colouring of G. This shows that $\chi(G) \leq 6$.

8. (10 points) Suppose $a_0 = 4$ and $a_n = a_{n-1} + n$ for $n \geq 1$.

   (a) Solve the recurrence relation by using a characteristic equation and a particular solution.

   (b) Find a closed form formula for the ordinary generating function $g(x) = \sum_{n=0}^{\infty} a_n x^n$, then solve the recurrence relation by computing $[x^n]g(x)$.

Solution: (a) We have $x = 1$, thus $a_n^{(h)} = A$. For a particular solution, we try $a_n^{(p)} = (Bn + C)n$ since part of a polynomial of degree 1 (i.e., $(Bn + C)$) solves the homogoenous part of the recurrence. Then

$$(Bn+C)n = (B(n-1)+C)(n-1)+n \quad \to \quad 0 = -2Bn+B-C+n \quad \to \quad B = C = 1/2.$$

Therefore, $a_n^{(p)} = \dfrac{n(n+1)}{2}$ is a particular solution implying $a_n = A + \dfrac{n(n+1)}{2}$ is a general solution. Using $a_0 = 4$ we have $A = 4$, thus, the solution to the given recurrence relation is $a_n = 4 + \dfrac{n(n+1)}{2}$.

(b) Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$. We have

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} (a_{n-1} + n)x^n = a_0 + \sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} nx^n$$

$$= a_0 + x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + \frac{x}{(1-x)^2} = a_0 + xg(x) + \frac{x}{(1-x)^2}$$

Using $a_0 = 4$ gives

$$g(x) = \frac{4}{1-x} + \frac{x}{(1-x)^3}.$$

Then

$$g(x) = 4\sum_{n=0}^{\infty} x^n + x\sum_{n=0}^{\infty} \binom{n+2}{n} x^n = 4\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^{n+1}$$

$$= 4\sum_{n=0}^{\infty} x^n + \sum_{n=1}^{\infty} \frac{(n+1)n}{2} x^n = \sum_{n=0}^{\infty} \left(4 + \frac{n(n+1)}{2}\right) x^n.$$

Thus,

$$a_n = [x^n]g(x) = 4 + \frac{n(n+1)}{2}.$$

9. (10 points) There is an unlimited number of A's, B's and C's. Determine the number of ordered strings of length n using A's, B's and C's if the string must contain an even number of A's and at least one C.

Recall: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ and $\frac{1}{2}\left(e^x + e^{-x}\right) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$.

Solution: The answer (call it h_n) is the coefficient of $\frac{x^n}{n!}$ in the expansion of

$$G(x) = \left(\frac{x^0}{0!} + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots\right)\left(\frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$\begin{aligned} \text{Thus,}\quad G(x) &= \left(\frac{1}{2}\left(e^x + e^{-x}\right)\right) \cdot e^x \cdot (e^x - 1) \\ &= \frac{1}{2}\left(e^{3x} - e^{2x} + e^x - 1\right) \\ &= \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{(3x)^k}{k!}\right) - \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{(2x)^k}{k!}\right) + \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right) - \frac{1}{2} \\ &= -\frac{1}{2} + \sum_{k=0}^{\infty} \frac{3^k - 2^k + 1}{2} \frac{x^k}{k!} \end{aligned}$$

Therefore $h_0 = -\frac{1}{2} + \frac{3^0 - 2^0 + 1}{2} = 0$ and $h_n = \frac{3^n - 2^n + 1}{2}$ $(n \geq 1)$.

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