Introduction to Graph Theory

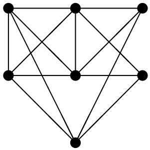
Intuition: What is a graph?

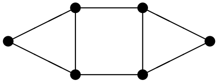
Informally, a $\boxed{\textbf{graph}}$ is an object consisting of

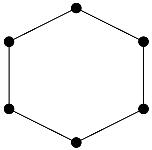
- a collection of dots (called **vertices**), and
- a collection of lines (called **edges**)

where every edge (line) connects two vertices (dots).

Below are **drawings** of graphs:







Definition of graph

Comments:

- We use uppercase letters such as G and H to denote a graph.
- A vertex is a point and is drawn as a dot.
  - The set of vertices of a graph G is denoted by V (or $V(G)$).
  - Vertices are often denoted by lower case letters and sometimes with subscripts: **Examples:** $\{a, b, c, \ldots\}$, or $\{x, y, z, \ldots\}$ or $\{v_1, v_2, v_3 \ldots\}$, or $\{1, 2, 3, \ldots\}$.
- An edge is a line joining two vertices.
  - The set of edges of a graph G is denoted by E (or $E(G)$).
  - Edges are often denoted as ab or $\{a, b\}$. (Sometimes $a \sim b$ or (a, b) is used.)

A more formal definition

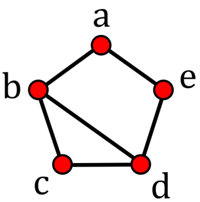
A **graph** is an ordered pair $G = (V, E)$ consisting of

- a nonempty set V (called the **vertices**) and
- a set E (called the **edges**) of two-element subsets of V.

Example of a graph

Problem

Consider the (labelled) graph G drawn below.



What are $V(G)$ and $E(G)$?

Solution.

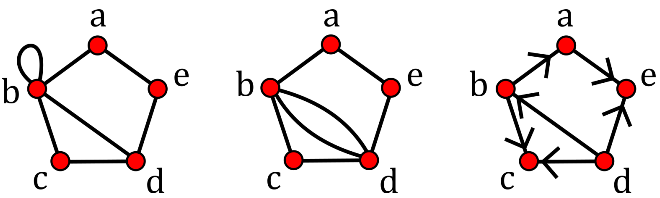
- The vertex set is $V(G) = \{a, b, c, d, e\}$.
- The edge set is $E(G) = \{ab, bc, cd, de, ae, bd\}$.

We only put one of *ab* and *ba* in the edge set (it does not matter which).

Simple graphs

We (mostly) focus on "**simple graphs**":
Neither loops, multiple edges nor directions are allowed.

The following drawings are **NOT** simple graphs:



The first two are **multigraphs**:

- The first graph has a loop "$\{b, b\}$"$= \{b\}$ (so $E(G)$ has a one-element subset).
- The second graph has the edge $\{b, d\}$ twice (so $E(G)$ is a multiset, not a set).

The third drawing shows a **directed graph**:

- We call the lines **arcs** instead of edges ($E(G)$ contains ordered pairs instead of 2-element subsets of V).
- The arc set is $E(G) = \{(b, a), (a, e), (b, c), (d, c), (d, b)\}$.
- The order of the vertices in the pairs matters and gives each edge a direction.

Graph notation and terminology

- ** Moving forward, the term **graph** in this course means a simple and undirected graph on a finite number of vertices (unless specified otherwise). **
- To introduce notation for our problems and theorems, we will often write:

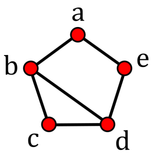
"Let $G = (V, E)$ be a graph..." or **"Let G be a graph..."**

More terminology

- Two vertices are **adjacent** if they are connected by an edge.
- In this case, we say the edge is **incident** to those two vertices.

Example

Let $G = (V, E)$ be the graph drawn below.



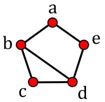
Then
- a and b are adjacent (since $ab \in E$).
- a and c are **not** adjacent (since $ac \notin E$).
- the edge ab is incident to vertices a and b.

Definition

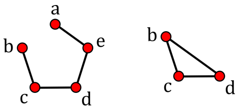
Let $G = (V, E)$ and $H = (V', E')$ be graphs.
Then H is a **subgraph** of G if $V' \subseteq V$ and $E' \subseteq E$.

Example

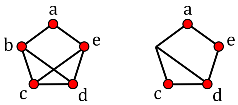
Let $G = (V, E)$ be the graph



The following two graphs **are** subgraphs of G:



But the following two are **NOT** subgraphs of G:



The first has more edges than the G while the second is not a graph.

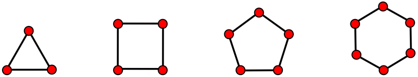
Paths. Let $n \geq 2$.

- Denoted by P_n (the number of edges is the **length** of the path).
- $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$

P_2 P_3 P_4 P_5

Cycles. Let $n \geq 3$.

- Denoted by C_n (the number of edges is the **length** of the cycle).
- $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_1 v_n\}$

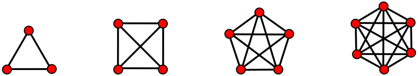


C_3 C_4 C_5 C_6

We typically do not include "C_2" as a cycle.

Complete graphs. Let $n \geq 1$.

- Denoted by K_n (why K?)
- $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(K_n) = \{v_i v_j : 1 \leq i \neq j \leq n\}$



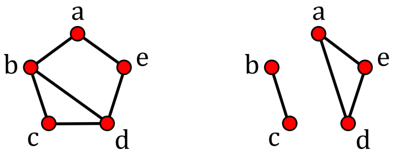
K_3 K_4 K_5 K_6

Connected graphs

Definition

- A graph is **connected** if there is a path between every pair of vertices.
- A graph that is not connected is called **disconnected**.

Example



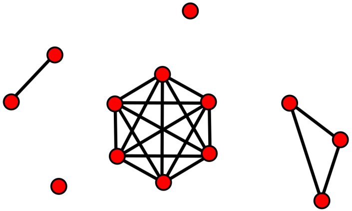
- The graph on the left is connected: every pair of vertices has a path between them.
- The graph on the right is disconnected. There is no path between vertices a and b.

Definition

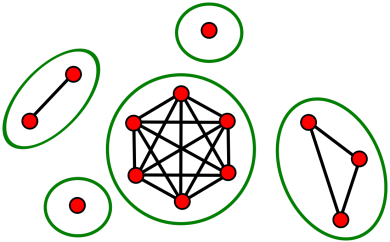
Let G be a graph. A maximal connected subgraph of G is called a **component** of G.

Problem

How many components does the graph drawn below have?



Solution. The answer is 5.



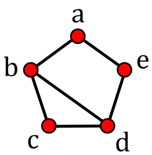
Definition

Let $G = (V, E)$ be a graph.

- The **degree** of a vertex $v \in V$, denoted by $\deg(v)$ (or d_v or $d(v)$), is the number of edges incident to v.
- A vertex of degree zero is called an **isolated vertex**.
- The **minimum degree** in G is denoted by $\delta(G)$.
- The **maximum degree** in G is denoted by $\Delta(G)$.
- The **degree sequence** of G is a list of the degrees of each vertex in V (usually in non-increasing or non-decreasing order).

Example

Let $G = (V, E)$ be the graph drawn below



1. Then $\deg(a) = \deg(c) = \deg(e) = 2$ and $\deg(b) = \deg(d) = 3$.
2. $\delta(G) = 2$ and $\Delta(G) = 3$.
3. G has degree sequence $(2, 2, 2, 3, 3)$.

Results on number of edges and vertex degree

Lemma

For every graph G on n vertices with m edges, we have $0 \leq m \leq \binom{n}{2}$.

Proof.

Note m is a non-negative integer.

As there are a maximum of $\binom{n}{2}$ two-element subsets of an n-set, the upper bound follows by the definition of a graph ($E(G)$ consists of (some) two-element subsets of V).

Lemma

For every vertex v in a graph G on n vertices we have $0 \leq \deg(v) \leq n - 1$.

Proof.

The result follows since each vertex is adjacent to at most $n - 1$ other vertices (loops and multiple/parallel edges are not permitted in a (simple) graph).

A result on distinct vertex degrees

Motivating Question

Is there a graph with degree sequence $(0, 1, 2, 3, 4)$? Try to find one!

The answer is no. We prove a more general result below.

Lemma

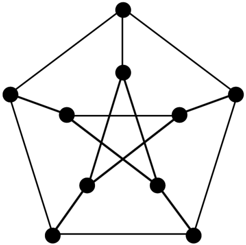
For $n \geq 2$, any (simple) graph on n vertices has at least two vertices of the same degree.

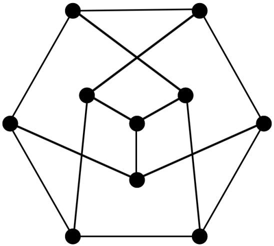
Proof.

- We first prove a graph cannot have both 0 and $n-1$ in its degree sequence.
  - Assume there is a vertex of degree 0 and another vertex of degree $n-1$.
  - Since there is a vertex of degree 0, the graph is disconnected.
  - Since there is a vertex of degree $n-1$, the graph is connected (**why?**).
  - But, a graph cannot be both connected and disconnected, a contradiction.
- Therefore, either every vertex degree is in the set $\{0, 1, 2, \ldots, n-2\}$, or every vertex degree is in the set $\{1, 2, 3, \ldots, n-1\}$.
- Each of these two sets has size $n-1$.
- The result now follows by the **pigeonhole principle** since the graph has n vertices (pigeons) and each vertex has at most $n-1$ possible degrees (pigeonholes).

When are two graphs the same?

Are the two graphs corresponding to the drawings below the same graph?





To formally define by what we mean by “same”, we define **isomorphism**.

For graphs, the geometry (i.e., how you draw a graph) does not matter; only the connections matter.

Definition

Let G and H be graphs. We say G and H are **isomorphic**, written $G \cong H$, if there is a bijection $\sigma : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\sigma(u)\sigma(v) \in E(H)$, that is, the bijection preserves adjacency and non-adjacency. We call σ an **isomorphism**.

Other letters can be used for isomorphisms, such as f and g.

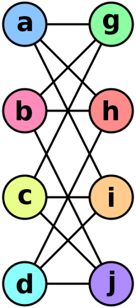
Example

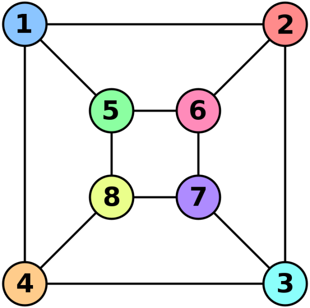
The graphs C_3 and K_3 are isomorphic, while the graphs C_3 and P_3 are not isomorphic.

To prove two graphs are isomorphic, we can write down an isomorphism σ.

Example

Let G represent the graph shown on the left and H represent the graph on the right. Show that $G \cong H$ (i.e., G and H are the "**same**" graph).





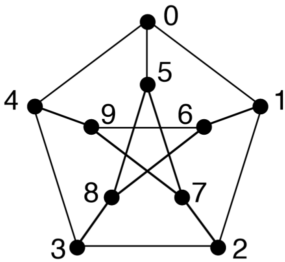
Images by Booyabazooka: left and right.

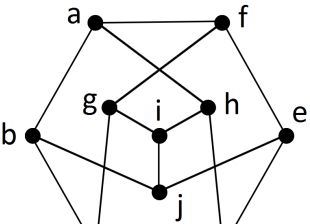
Solution.

An isomorphism is given by the colours of the vertices. In particular, consider $f : V(G) \to V(H)$ where $f(a) = 1$, $f(b) = 6$, $f(c) = 8$, $f(d) = 3$, $f(g) = 5$, $f(h) = 2$, $f(i) = 4$ and $f(j) = 7$. Observe that f maps edges in G to edges in H and non-edges in G to non-edges in H.

Let G be the graph on the left and H the graph on the right as drawn below.

Label the vertices accordingly.





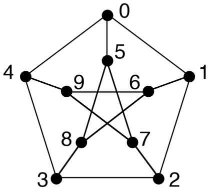
Is there an isomorphism $\sigma : V(G) \to V(H)$ that preserves adjacency and non-adjacency?

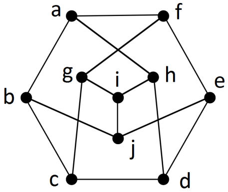
That is, are the two graphs depicted above isomorphic?

Try it out!

The Petersen graph

Yes, the two graphs are isomorphic (i.e., are the **same**). It is the famous Petersen graph.

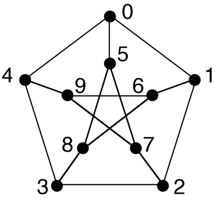


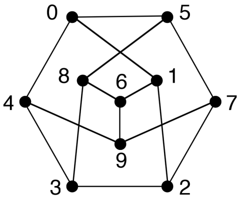


Consider $\sigma : V(G) \to V(H)$ defined by

$$\sigma(0) = a, \quad \sigma(1) = h, \quad \sigma(2) = d, \quad \sigma(3) = c, \quad \sigma(4) = b,$$
$$\sigma(5) = f, \quad \sigma(6) = i, \quad \sigma(7) = e, \quad \sigma(8) = g, \quad \sigma(9) = j.$$

To see (graphically) that σ is an isomorphism, we label H by using σ^{-1}:





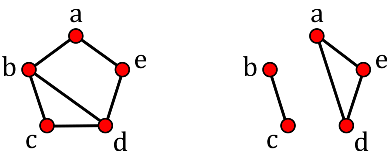
It is now simple to verify that both labelled graphs have the same edge set, thus, $G \cong H$.

How to prove two graphs are not isomorphic?

- To prove two graphs are **isomorphic**, we find an **isomorphism** between them.
- To prove two graphs are **not** isomorphic can be a bit harder.
- **Note:** if G and H are isomorphic, they must have the same structural properties.
  - Thus: **If we can find a property where the graphs differ, then the two graphs must be different graphs (i.e., are not isomorphic)!**
- Here are some common properties you could check (this is not a complete list!):
  - Do they have the same number of vertices? the same number of edges?
  - Do they have the same degree sequence? minimum degree? maximum degree?
  - Do they have the same cycle structure? (e.g., both contain cycles of length 3.)
  - Are they both planar? bipartite?
  - Do their adjacency matrices have the same eigenvalues?
  - Do they have the same chromatic number?
- We can also check the above list for the complements (defined later) of the graphs.

Example

Are the following two graphs isomorphic?



Solution.

- The graph on the left is connected but the graph on the right is disconnected.
- Thus, the two graphs have different connectivity properties and cannot be isomorphic.
- Alternatively, we can verify they have different degree sequences, namely $(2, 2, 2, 3, 3)$ and $(1, 1, 2, 2, 2)$ respectively, so they cannot be isomorphic.

Graph complements

Definition

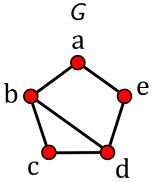
Let G be a graph.
The **complement** of G, denoted by $\overline{G}$, has vertex set $V(\overline{G}) = V(G)$ and edge set

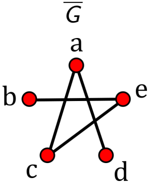
$$E(\overline{G}) = \{xy : xy \notin E(G)\}.$$

In the above definition, you "**flip**" the edges and non-edges to draw the complement.

Example

- The graph G is drawn on the left.
  - $V(G) = \{a, b, c, d, e\}$
  - $E(G) = \{ab, bc, cd, de, ae, bd\}$
- Its complement $\overline{G}$ is drawn on the right.
  - $V(\overline{G}) = \{a, b, c, d, e\}$
  - $E(\overline{G}) = \{ac, ad, be, ce\}$





Graph complements

Fact

Let G be a graph. Then $E(G) + E(\overline{G}) = \binom{n}{2}$.

Fact

Let G and H be graphs. Then $G \cong H$ if and only if $\overline{G} \cong \overline{H}$.

Proof (outline).
Use the fact that if $\sigma : V(G) \to V(H)$ is an isomorphism then the same function can be used to give an isomorphism between complements (i.e., $\sigma : V(\overline{G}) \to V(\overline{H})$ is an isomorphism). This is because isomorphisms must map "edges to edges", and "non-edges to non-edges" (i.e., preserve adjacency and non-adjacency).

More generally, the "automorphism group" of a graph is the "automorphism group" of its complement.

Example

Let G have vertex set $V(G) = \{a, b, c, d, e, f, g, h\}$ and edge set

$$E(G) = \{ab, ac, ae, ag, ah, bc, bd, bf, bh, cd, ce, cg, de, df, dh, ef, eg, fg, fh, gh\}$$

and H have vertex set $V(H) = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and edge set

$$E(H) = \{12, 14, 15, 16, 18, 23, 25, 26, 27, 34, 36, 37, 38, 45, 47, 48, 56, 58, 67, 78\}.$$

(a) Draw the graphs G and H.

(b) Compute the degree sequences of G and H.

(c) Are G and H isomorphic?

Solution.

(a) For drawings, see Mike.

(b) The degree sequence for both graphs is the same:

$$(5, 5, 5, 5, 5, 5, 5, 5)$$

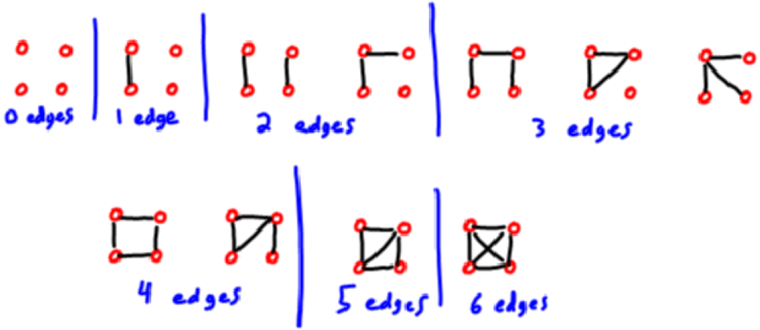
(c) G and H are **NOT** isomorphic.

- One method is to analyze the complements of G and H.
- The graph $\overline{G}$ is isomorphic to C_8, the cycle on 8 vertices (i.e., $\overline{G} \cong C_8$).
- The graph $\overline{H}$ is isomorphic to two disjoint copies of C_4, that is, it is the union of two cycles of length 4 (we write this as $\overline{H} \cong 2C_4$).
- Hence, $\overline{G}$ is connected and $\overline{H}$ is disconnected, thus, $\overline{G} \ncong \overline{H}$ implying $G \ncong H$.

Example

How many (non-isomorphic) graphs are there on 4 vertices?

Solution. There are 11 as drawn below:

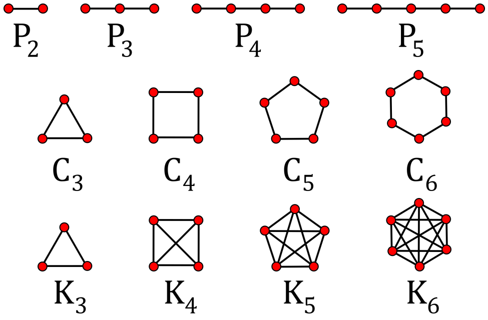


Edge Counting

Example

- How many edges does P_n have?
- How many edges does C_n have?
- How many edges does K_n have?

Solution. Recall the graphs of the path, cycle and complete graph:



We observe that $|E(P_n)| = n - 1$, $|E(C_n)| = n$ and $|E(K_n)| = \binom{n}{2}$ (to prove the last one we can use induction or a combinatorial argument).

The handshaking lemma

Euler (1736) proved the following degree sum formula in his landmark paper on graph theory (where Euler solved the Seven Bridges of Königsberg problem).

Theorem (The handshaking lemma)

Let G be a graph. Then

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|.$$

Proof.

- We try a double counting argument (as done by Euler).
- Euler counted pairs (v, e) in two different ways where e is incident to v.
- The first way fixes v and notices there are $\deg(v)$ such pairs.
- Now summing over v gives $\displaystyle\sum_{v \in V(G)} \deg(v)$ pairs.
- The second way fixes e and notices that there are 2 such pairs (one for each endpoint of e).
- Now summing over e gives $\displaystyle\sum_{e \in E(G)} 2 = \underbrace{2 + 2 + \cdots + 2}_{|E(G)| \text{ times}} = 2|E(G)|$.
- Since both ways count the number of pairs, they must be equal.

Problems: The handshaking lemma

Applications of the handshaking lemma:

- We can use it to count the number of edges.
- Sometimes it tells us when a graph with certain properties may **not** exist.

Example

Does there exist with 5 vertices and every vertex having degree equal to 3?

Solution.

- The answer is no.
- To derive a contradiction, assume such a graph exists:
  Let G have 5 vertices and $\deg(v) = 3$ for every $v \in V(G)$.
- Then, by the **handshaking lemma**, the number of edges in G is:

$$|E(G)| = \frac{\sum_{v \in V(G)} \deg(v)}{2} = \frac{\sum_{v \in V(G)} (3)}{2} = \frac{3(5)}{2} = 7.5$$

- But it is impossible to have 7.5 edges in a graph, giving a contradiction.
- Therefore, no such graph can exist.

Corollary

A graph has an even number of vertices of odd degree.

Example

Let G be a graph with 31 edges and every vertex having degree at least 4. That is, $|E(G)| = 31$ and $\delta(G) \geq 4$).

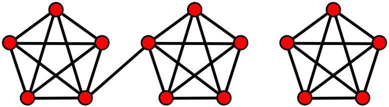
(a) What is the maximum number of vertices that G can have?

(b) What is the minimum number of vertices that G can have?

Prove your answer is correct.

(a) The answer is 15.

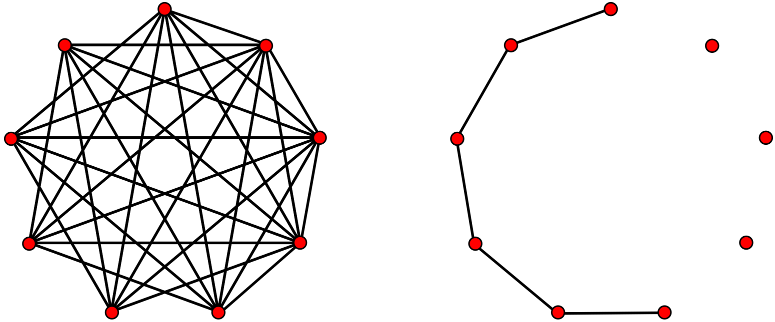
- By the handshaking lemma, $\sum_{v \in V(G)} \deg(v) = 2|E(G)| = 62$.
- Since $\deg(v) \geq 4$ for every $v \in V(G)$, we have
$$62 = \sum_{v \in V(G)} \deg(v) \geq \sum_{v \in V(G)} 4 = 4|V(G)|$$
implying that $|V(G)| \leq 15$ (since $|V(G)|$ is an integer).
- The graph below has the required properties with $|V(G)| = 15$.



- Therefore, the maximum number of vertices that G can have is 15.

(b) The answer is 9.

- The maximum number of edges in a graph is at most $\binom{n}{2}$.
- Thus, we must have $n \geq 9$ (otherwise $n \leq 8$ impying $|E(G)| \leq 28$).
- This shows that $|V(G)| \leq 9$.
- The graph below shown on the left has the required properties with $|V(G)| = 9$ (its complement is shown on the right).



- Therefore, the minimum number of vertices that G can have is 9.

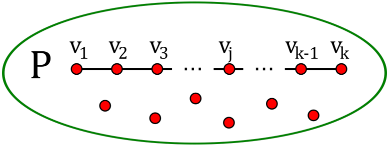
Problems: The extreme principle

Theorem

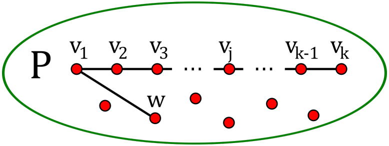
Let G be a graph in which every vertex has degree at least two. Prove G contains a cycle.

Proof.

- We apply the extremal principle.
- Let $P = v_1 v_2 \cdots v_{k-1} v_k$ be a longest path in G.

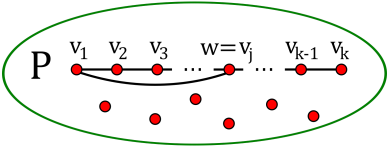


- But v_1 is adjacent to another vertex (call it w) other than v_2 since $\deg(v_1) \geq 2$.
- If w is not on the path P, then $P' = w\, v_1 v_2 \cdots v_k$ is a longer path in G contradicting that P is a longest path:



Proof (continued).

- Thus, all the neighbours of v_1 are vertices on the path P.
- This implies that $w \in \{v_3, v_4, \ldots, v_k\}$ (since $w \neq v_2$).
- Thus $w = v_j$ for some $j \in \{3, 4, \ldots, k\}$.



- Since $v_1 v_j \in E(G)$ (as $v_1 w \in E(G)$ and $w = v_j$), $C = v_1 v_2 \cdots v_j v_1$ is a cycle in G implying that G contains a cycle.