How to count?

(continued)

Notation: Permutations

- $P(n, k)$ is the number of (ordered) arrangments of k objects of a set of size n

Fact

For $0 \leq k \leq n$, we have $P(n, k) = \underbrace{n \times (n-1) \times \cdots \times (n-k+1)}_{k \text{ factors}} = \dfrac{n!}{(n-k)!}$.

Note: If $k > n$, then $P(n, k) = 0$.

Proof (outline): In an arrangement with k slots/boxes $\square\ \square \cdots \square$, we have n choices to put in the first slot, then $n-1$ choices remain for the second slot, and so on, until we have $n-k+1$ choices for the kth slot: $\boxed{n}\ \boxed{n-1}\ \cdots\ \boxed{n-k+1}$.

Example: Permutations

How many 2-letter strings ("words") can be formed by using each of the letters a, b, c at most once?

Solution. The answer is $P(3, 2) = \dfrac{3!}{(3-2)!} = 6$.

We could also list each possible arrangement (but usually this is not feasible to do):

$$ab,\ ac,\ ba,\ bc,\ ca,\ cb.$$

Definition: Binomial Coefficient

Let n be a nonnegative integer and $0 \leq k \leq n$.

The $\boxed{\textbf{binomial coefficient}}$ is denoted by $\binom{n}{k}$.

Definition 1.

- $\binom{n}{k}$ is the number of ways to choose k objects from a collection of n objects.

Definition 2.

- $\binom{n}{k}$ is the number of k-element subsets of an n-element set.

- The symbol $\binom{n}{k}$ is read as "***n* choose *k***".
- An alternate notation for the binomial coefficient is $C(n, k)$.

Fact

For $0 \leq k \leq n$, we have $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

Proof (outline). One way to show this is to first verify that $P(n,k) = \binom{n}{k} k!$ using the Multiplication Principle, then the formula in the Fact follows from $P(n,k) = \dfrac{n!}{(n-k)!}$.

Example

How many 2-element subsets are there of $S = \{a, b, c\}$?

Solution. The answer is $\binom{3}{2} = \dfrac{3!}{2!(3-2)!} = 3$.

We could also list each possible arrangement (but usually this is not feasible to do):

$$\{a, b\}, \{a, c\}, \{b, c\}.$$

Counting summary

The number of selections of k objects from a set of n objects is given in the table:

	order is significant	order **NOT** significant
repetitions allowed	n^k	$\binom{n+k-1}{k}$
repetitions **NOT** allowed	$n(n-1)\cdots(n-k+1)$	$\binom{n}{k}$

- The entry with $\binom{n+k-1}{k}$ requires a proof (next week)!

Counting tips and how to interpret key words

- The following key words usually indicate problems where order **IS** important:
  - arrangement,
  - queue/line,
  - row,
  - list,
  - tuple,
  - committee positions,
  - ordered,
  - PIN/password,
  - 'word'.

Examples: Order matters

- The 3-tuple $(1,2,3)$ is the **different** from $(2,3,1)$.
- The PIN 0057 is **different** from 5700.
- The row of ninja turtles Leonardo, Raphael, Donatello, Michelangelo is **different** from its reversal Michelangelo, Donatello, Raphael, Leonardo.

- The following key words usually indicate problems where order is **NOT** important:
  - set,
  - group,
  - committee,
  - pile,
  - bag,
  - poker hand.

Examples: Order does not matter

- The set $\{1,2,3\}$ is the **same** set as $\{2,3,1\}$.
- The set $\{\{1,2\},\{3,4\}\}$ is the **same** set as $\{\{3,4\},\{1,2\}\}$.
- The poker hand $7\diamondsuit, 3\clubsuit, A\diamondsuit, K\spadesuit, 8\heartsuit$ is the **same** as $A\diamondsuit, K\spadesuit, 7\diamondsuit, 3\clubsuit, 8\heartsuit$.

- In combinatorics, people are always **distinct**, i.e., unique (as they have names!).

Example: Counting license plates

How many different license plates are there of the following form?

“letter letter letter letter number number number”

What if the letters and numbers must be distinct?

Solution.

- Order matters (ABCD123 is different from BACD123).
- If repetition is allowed (e.g., AABP888) then the number of license plates is

$$\underbrace{\underline{26} \times \underline{26} \times \underline{26} \times \underline{26}}_{\text{any four letters}} \times \underbrace{\underline{10} \times \underline{10} \times \underline{10}}_{\text{any three numbers}} = 26^4 \times 10^3.$$

- If repetition is allowed (e.g., AABP888) then the number of license plates is

$$\underbrace{\underline{26} \times \underline{25} \times \underline{24} \times \underline{23}}_{\text{four letters (none repeated)}} \times \underbrace{\underline{10} \times \underline{9} \times \underline{8}}_{\text{three numbers (none repeated)}} = \frac{26!}{22!} \times \frac{10!}{7!}.$$

Example: Form a committee

From nine people, how many different committees of size four can be formed?

Solution.

- **Note:** People are distinct (they have names).
- Order of selection does not matter.
- Repetition is not allowed (Mike cannot count as two of the four people).
- The number of different committees we can form is $\binom{9}{4} = 126$.

Example: Form a committee with ranks

From nine people, we must select a committee of four with one president, one vice president, one secretary and one treasurer. How many ways can this be done if no person can serve more than one position?

Solution.

- Order of selection matters.
- Repetition is not allowed.
- We arrange the positions as follows:

$$\underbrace{\underline{\quad ? \quad}}_{\text{president}} \times \underbrace{\underline{\quad ? \quad}}_{\text{vice president}} \times \underbrace{\underline{\quad ? \quad}}_{\text{secretary}} \times \underbrace{\underline{\quad ? \quad}}_{\text{treasurer}} = \; ?$$

- By the Multiplication Principle, the number of different committees we can form is:

$$\underbrace{\underline{\quad 9 \quad}}_{\text{president}} \times \underbrace{\underline{\quad 8 \quad}}_{\text{vice president}} \times \underbrace{\underline{\quad 7 \quad}}_{\text{secretary}} \times \underbrace{\underline{\quad 6 \quad}}_{\text{treasurer}} = \frac{9!}{5!} = 3024.$$

Example: A "gluing" technique

There are seven professors and three students in a gathering.
How many ways can they be arranged in a row so that the three students form a single block (i.e., there is no professor between any two students)?

Solution.

- Arranged in a row means that the order of selection matters.
- The ten people are distinct (they have names).
- As the three students must be together, we can treat them as a single entity (block).
- We first arrange the seven professors together with this block of students.
- For example, one possible arrangement is

  $\boxed{\textbf{Prof 6}}, \boxed{\textbf{Prof 2}}, \boxed{\textbf{Prof 7}}, \boxed{\textbf{Students}}, \boxed{\textbf{Prof 1}}, \boxed{\textbf{Prof 5}}, \boxed{\textbf{Prof 3}}, \boxed{\textbf{Prof 4}}.$
- The number of ways to arrange seven professors with this block of students is $\boxed{8!}$.
- But the three students can permute among themselves in $\boxed{3!}$ ways, for example,

  $\boxed{\textbf{Student 2}}, \boxed{\textbf{Student 1}}, \boxed{\textbf{Student 3}}.$
- Thus, the desired answer is $\boxed{8! \times 3!}$.

Example: Student's Nightmare?

There are seven professors and three students in a gathering.
How many ways can they be arranged in a row so that the two end-positions are occupied by professors and **no** two students are adjacent?

Solution.

- We first consider arrangements of professors.
- There are $\boxed{7!}$ ways to arrange the 7 professors in a row.
- Fix an arbitrary one of these arrangements:

  $\boxed{\textbf{Prof 6}}, \boxed{\textbf{Prof 2}}, \boxed{\textbf{Prof 7}}, \boxed{\textbf{Prof 1}}, \boxed{\textbf{Prof 5}}, \boxed{\textbf{Prof 3}}, \boxed{\textbf{Prof 4}}.$
- Since the end-positions are occupied by professors, there are only six spaces available for the students:

  $\boxed{\textbf{P6}}, \_\_, \boxed{\textbf{P2}}, \_\_, \boxed{\textbf{P7}}, \_\_, \boxed{\textbf{P1}}, \_\_, \boxed{\textbf{P5}}, \_\_, \boxed{\textbf{P3}}, \_\_, \boxed{\textbf{P4}}.$
- **Student 1** has 6 choices for a spot.
- Since no two students are adjacent, **Student 2** has 5 choices for a spot.
- Finally, **Student 3** has 4 choices for a spot.
- Thus, the number of arrangements is $\boxed{7! \times 6 \times 5 \times 4}$.
- Alternatively, we can choose 3 spots from the 6 available for the students and then arrange the students in 3! ways giving $\boxed{7! \times \binom{6}{3} \times 3!}$ possible arrangements.

Example: Labelled sets? Unlabelled sets?

Fix $n \geq 2$.

(a) How many ways can we choose 2 people from among n people?

(b) How many ways can we partition n people into a set of size 2 and a set of size $n-2$?

Solution.

(a) The answer is $\binom{n}{2}$.

- The set of chosen people is a special set (it is the "**chosen set**").
- That is, 2 people are **chosen**, and $n-2$ people are **not chosen**.

(b) The answer depends on n.

- Here we must be careful since in a partition, we might not be able to tell the two sets apart (neither part of the partition is "the chosen set").
- Consider $n = 4$ and label the people by $\{1, 2, 3, 4\}$.
- Observe that there is no way to distinguish between the partition $\{\{1, 2\}, \{3, 4\}\}$ and $\{\{3, 4\}, \{1, 2\}\}$ where people 1 and 2 are in one part of the partition and people 3 and 4 are in the other part of the partition.
- Thus, when $n = 4$, there are $\frac{1}{2}\binom{4}{2} = 3$ partitions:

$$\{\{1, 2\}, \{3, 4\}\}, \{\{1, 3\}, \{2, 4\}\}, \{\{1, 4\}, \{2, 3\}\}.$$

- When $n \neq 4$, then the answer is $\binom{n}{2}$ because the two parts of the partition are different sizes and we are able to distinguish between them.

Common types of examples to think about

We will solve these next week (some require a new technique).

(1) There are 11 **distinct** students in a classroom. In how many ways can we...

  (a) choose 8 students and arrange them in a row.
  (b) choose 8 students and place them into two equal-sized rows that are labelled as "Row 1" and "Row 2".
  (c) choose 8 students and place them into two equal-sized unlabelled rows.
  (d) choose 8 students to create a group.
  (e) choose 8 students and place them into two equal-sized groups that are labelled as "Group 1" and "Group 2".
  (f) choose 8 students and place them into two equal-sized unlabelled groups.

(2) There are 11 **identical** dimes in a coin bag. In how many ways can we...

  (a) choose 8 dimes and arrange them in a row.
  (b) choose 8 dimes to create a group.
  (c) choose 8 dimes and give them to three people where some people might not get any.
  (d) choose 8 dimes and give them to three people where each person gets at least one.

(3) There are 11 **types** of coins in a coin bag (with an unlimited number of each type of coin). In how many ways can we...

  (a) choose 8 coins and arrange them in a row.
  (b) choose 8 coins and place them into two equal-sized rows labelled as "Row 1" and "Row 2".
  (c) choose 8 coins and place them into two equal-sized unlabelled rows.
  (d) choose 8 coins to create a group.
  (e) choose 8 coins and place them into two equal-sized groups that are labelled as "Group 1" and "Group 2".
  (f) choose 8 coins and place them into two equal-sized unlabelled groups.

Answers

(i) Answers

(a) $11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4$.
(b) $(11 \times 10 \times 9 \times 8) \times (7 \times 6 \times 5 \times 4)$.
(c) $\frac{1}{2} \times (11 \times 10 \times 9 \times 8) \times (7 \times 6 \times 5 \times 4)$.
(d) $\binom{11}{8}$.
(e) $\binom{11}{4}\binom{7}{4}$ or $\binom{11}{8}\binom{8}{4}$.
(f) $\frac{1}{2}\binom{11}{4}\binom{7}{4}$ or $\frac{1}{2}\binom{11}{8}\binom{8}{4}$.

(ii) Answers

(a) 1.
(b) 1.
(c) $\binom{10}{2}$ (by stars and bars).
(d) $\binom{7}{2}$ (by stars and bars; place bars in 7 slots between 8 stars).

(iii) Answers

(a) 11^8.
(b) 11^8.
(c) $11^4 + \frac{1}{2}(11^4)(11^4 - 1)$ (rows are identical plus rows are not identical).
(d) $\binom{18}{10}$ or $\binom{18}{8}$ (10 bars separate stars/coins into 11 types).
(e) $\binom{14}{10}\binom{14}{10}$ (stars and bars for each separate group)
(f) $\binom{14}{10} + \frac{1}{2}\binom{14}{10}\left[\binom{14}{10} - 1\right]$ (groups are identical plus groups are not identical).