Sequences and Recurrence Relations

Example

Given the following sequence of real numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

What number comes next?

Solution.

- The answer is 52.
- This is the famous iccanobiF numbers defined by adding the previous term to the reversal of the one before it.
- Thus, the next number is $21 + \text{reversal}(13) = 21 + 31 = 52$.

Another Solution.

- The answer is 34.
- This is the famous Fibonacci numbers defined by adding the previous term to the one before it (that is, $a_n = a_{n-1} + a_{n-2}$).

Actual Solution.

- Many answers are correct (in fact, **any** number could be the next in the sequence!).
- Solving such "problems" requires us to guess the pattern that the person who designed the problem had in mind.
- If the pattern is not clear, then using "$\ldots$" is a bad way to define a sequence.
- It is better to define a sequence with a formula or by using a recurrence relation.

Example: The Fibonacci Sequence

The **Fibonacci sequence** with pattern

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

can be defined using a recurrence:

$$\boxed{\begin{array}{l} F_1 = 1,\ F_2 = 1 \\ F_n = F_{n-1} + F_{n-2},\ (n \geq 3). \end{array}}$$

A closed-form solution is

$$\boxed{F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n, \quad n \geq 1.}$$

This is known as **Binet's formula** (though it was already known by de Moivre, Bernoulli and Euler a century before Binet's derivation of it).

Definitions

- Given a sequence $\{a_0, a_1, a_2, a_3 \ldots\}$ of numbers, a **recurrence relation** for a_n is an equation that relates the nth term a_n to some of its predecessors in the sequence.
- To initiate the computation, we require **initial conditions**.
- The **solution** to a recurrence relation is an expression $a_n = f(n)$ where $f(n)$ is a function satisfying the recurrence and initial conditions.

Example

Solve the recurrence:

$$\boxed{\begin{array}{l} a_1 = 1 \\ a_n = a_{n-1} + 1, \ \ (n \geq 2). \end{array}}$$

Solution.

- The first few terms are $a_1 = 1$, $a_2 = a_1 + 1 = 1 + 1 = 2$, $a_3 = a_2 + 1 = 2 + 1 = 3$, $a_4 = a_3 + 1 = 3 + 1 = 4$ and so on (i.e., $\{1, 2, 3, 4, \ldots\}$).
- The general pattern appears to be $\boxed{a_n = n}$.
- To prove this, we verify the left-hand side (LHS) matches the right-hand side (RHS).
- LHS $= a_n = n$ and RHS $= a_{n-1} + 1 = (n-1) + 1 = n$, thus, LHS $=$ RHS.
- **Note:** We must also check the initial condition: when $n = 1$ we have $a_1 = 1$.

Example

Solve the recurrence:

$$\boxed{\begin{array}{l} a_1 = 1 \\ a_n = na_{n-1}, \ (n \geq 2). \end{array}}$$

Solution.

- This is one way to define the factorial function, that is, $\boxed{a_n = n!}$.
- Often we write this as

$$n! = n \times (n-1)!$$

Example

Solve the recurrence:

$$\boxed{\begin{array}{l} a_0 = 1 \\ a_n = \pi a_{n-1}, \ (n \geq 1). \end{array}}$$

Solution.

- After computing some terms ($\{1, \pi, \pi^2, \pi^3, \ldots\}$) we observe $a_n = \pi^n$.
- To prove this, note that $a_0 = \pi^0 = 1$ so it satisfies the initial condition.
- $\text{LHS} = a_n = \pi^n$ and $\text{RHS} = \pi a_{n-1} = \pi \cdot \pi^{n-1} = \pi^n$.
- Thus, LHS = RHS.

Example

Solve the recurrence:

$$\boxed{\begin{array}{l} a_0 = 1,\ a_1 = 2,\ a_2 = 0 \\ a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3},\ \ (n \geq 3). \end{array}}$$

Solution.

- The first few terms are $\{1, 2, 0, 0, -4, -8, -20, -40, \dots\}$.
- Let us ignore the initial conditions for now.
- Based on the last example, we might guess a solution $a_n = 2^n$.
- LHS $= a_n = 2^n$ and RHS $= 2(2^{n-1}) + 2^{n-2} - 2(2^{n-3}) = 2^n + 2^{n-2} - 2^{n-2} = 2^n$.
- Thus, LHS $=$ RHS, so $a_n = 2^n$ solves the recurrence but does **not** satisfy the initial conditions.
- **Observation 1:** $\boxed{a_n = c2^n}$ for any constant c also solves the recurrence: LHS $= a_n = c2^n$ and RHS $= 2(c2^{n-1}) + c2^{n-2} - 2(c2^{n-3}) = c2^n$.
- **Observation 2:** $\boxed{a_n = c(-1)^n}$ for any constant c also solves the recurrence: LHS $= a_n = c(-1)^n$ and RHS $= 2(c(-1)^{n-1}) + c(-1)^{n-2} - 2(c(-1)^{n-3}) = c(-1)^n$.
- **Observation 3:** $\boxed{a_n = c(1)^n = c}$ for any constant c also solves the recurrence: LHS $= a_n = c$ and RHS $= 2(c) + c - 2(c) = c$.
- **Question:** Are there other solutions of the form x^n for some number x?

Solution (continued). $(a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3})$

- **Question:** Are there other solutions of the form x^n?
- Try it out! Assuming $\boxed{a_n = x^n}$ solves the recurrence we must have:

$$x^n = 2x^{n-1} + x^{n-2} - 2x^{n-3} \quad \rightarrow \quad x^n - 2x^{n-1} - x^{n-2} + 2x^{n-3} = 0$$
$$\rightarrow \quad x^{n-3}(x^3 - 2x^2 - x + 2) = 0 \quad \rightarrow \quad x^{n-3}(x-1)(x+1)(x-2) = 0$$

- We get $x = 0$, $x = 1$, $x = -1$ and $x = 2$.
- This shows $(1)^n$, $(-1)^n$ and 2^n are solutions to the recurrence (ignore 0^n).
- **Fact.** We can take linear combinations of solutions to form new ones (**why?**):

$$\boxed{a_n = C_1(1)^n + C_2(-1)^n + C_3(2)^n}$$

- **Question:** Which solutions are correct?
- **Answer:** All of them! But only some satisfy the initial conditions.
- Plugging in initial conditions will give the constants C_1, C_2, C_3 that work:

$$\begin{aligned} n = 0: &\quad 1 = C_1 + C_2 + C_3 \\ n = 1: &\quad 2 = C_1 - C_2 + 2C_3 \\ n = 2: &\quad 0 = C_1 + C_2 + 4C_3 \end{aligned}$$

- Solving this system gives $C_1 = 2$, $C_2 = -2/3$ and $C_3 = -1/3$.
- The solution to the original recurrence with the specified initial conditions is:

$$\boxed{a_n = 2 - \frac{2}{3}(-1)^n - \frac{1}{3}(2)^n.}$$

Definition

Let $\{a_n\}$ be a sequence. Then

$$\boxed{c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r} = 0} \quad (*)$$

where c_i is constant ($c_0, c_r \neq 0$) and r is fixed $1 \leq r \leq n$ is an "**rth order constant coefficient linear homogeneous recurrence relation.**"

Example

$a_n = a_{n-1} + a_{n-2}$ is 2nd order, constant coefficient, linear, homogeneous.
$a_n - 2a_{n-1} + 3a_{n-2} - a_{n-3} = 0$ is 3rd order, constant coefficient, linear, homogeneous.
$a_n = a_{n-1} + (a_{n-2})^2$ is **not** linear.
$a_n = a_{n-1} + a_{n-2} + 2$ is **not** homogeneous.
$a_n = na_{n-1}$ is **not** constant coefficient.

In what follows, we outline solutions to recurrence relations of the form $(*)$.

Definition: Constant coefficient linear homogeneous recurrence relations

$$\boxed{c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r} = 0} \quad (*)$$

Since geometric solutions of the form $a_n = x^n$ show up in such recurrences, we plug this into the recurrence to form an equation.

Definition

Replacing a_i with x^i and factoring out x^{n-r} gives

$$\boxed{c_0 x^r + c_1 x^{r-1} + \cdots + c_{r-1} x + c_r = 0}$$

called the $\boxed{\textbf{characteristic equation}}$. Its roots are called the $\boxed{\textbf{characteristic roots}}$.

Each characteristic root α gives a solution $a_n = \alpha^n$ of the recurrence.

Lemma: Linear Combinations of Solutions

If $a_n = f(n)$ and $a_n = g(n)$ are both solutions to $(*)$ then so is $a_n = C_1 f(n) + C_2 g(n)$.

Proof.

LHS $= c_0(C_1 f(n) + C_2 g(n)) + c_1(C_1 f(n-1) + C_2 g(n-1)) + \cdots + c_r(C_1 f(n-r) + C_2 g(n-r))$

$= \cdots = C_1(c_0 f(n) + c_1 f(n-1) + \cdots + c_r f(n-r)) + C_2(c_0 g(n) + c_1 g(n-1) + \cdots + c_r g(n-r))$

$= 0 + 0 = 0 =$ RHS since $f(n)$ and $b(n)$ are solutions to (*).

Case 1 - Distinct Roots

$$\boxed{c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r} = 0} \qquad (*)$$

Theorem

If $\alpha_1, \alpha_2, \ldots, \alpha_r$ are **distinct** characteristic roots, then the general solution to $(*)$ is

$$a_n = C_1(\alpha_1)^n + C_2(\alpha_2)^n + \cdots + C_r(\alpha_r)^n.$$

where $C_1, C_2, \ldots, C_r$ are constants dependant on the initial conditions.

- Each $(\alpha_i)^n$ solves $(*)$.
- Any linear combination of $\{(\alpha_1)^n, (\alpha_2)^n, \ldots, (\alpha_r)^n\}$ solves $(*)$.
- We get a set of r linearly independent solutions to $(*)$.

Example

Try solving for a closed-form formula for the Fibonacci sequence.

$f_0 = 0,\ f_1 = 1$
$f_n = f_{n-1} + f_{n-2},\ (n \geq 2)$

Example

Solve the recurrence:

$$\boxed{\begin{array}{l} a_0 = 7,\ a_1 = 14 \\ a_n = 5a_{n-1} + 6_{n-2},\ (n \geq 2). \end{array}}$$

Solution.

- The characteristic equation (plug in $a_i = x^i$ and factor out x^{n-r}) is

$$x^2 = 5x + 6 \qquad \to \qquad x^2 - 5x - 66 = 0 \qquad \to \qquad (x+1)(x-6) = 0$$

- There are two distinct characteristic roots $\alpha_1 = -1$ and $\alpha_2 = 6$.
- The general solution of the recurrence is

$$a_n = C_1(-1)^n + C_2(6)^n$$

  for some constants C_1 and C_2.
- Plugging in initial conditions gives

$$\begin{array}{rl} n = 0\ (a_0 = 1): & 7 = C_1 + C_2 \\ n = 1\ (a_1 = 3): & 14 = -C_1 + 6C_2 \end{array}$$

- Solving this system gives $C_1 = 4$ and $C_2 = 3$.
- The solution to the original recurrence with the specified initial conditions is:

$$\boxed{a_n = 4(-1)^n + 3(6)^n.}$$

Case 2 - Repeated Roots

Assume $c_0, c_r \neq 0$.

$$\boxed{c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r} = 0} \qquad (*)$$

Idea: Keep multiplying by n to generate new (linearly independent) solutions.

Theorem

If $\alpha_1, \alpha_2, \ldots, \alpha_k$ $(1 \leq k \leq r)$ are the **distinct** characteristic roots each with multiplicity m_i, then the general solution to $(*)$ is

$$\begin{aligned} a_n =& (C_{11} + C_{12}n + C_{13}n^2 + \cdots + C_{1m_1}n^{m_1-1})(\alpha_1)^n \\ &+ (C_{21} + C_{22}n + C_{23}n^2 + \cdots + C_{2m_2}n^{m_2-1})(\alpha_2)^n \\ &+ \cdots + (C_{k1} + C_{k2}n + C_{k3}n^2 + \cdots + C_{km_k}n^{m_k-1})(\alpha_k)^n. \end{aligned}$$

- That is
$$a_n = P_1(n)(\alpha_1)^n + \cdots + P_k(n)(\alpha_k)^n,$$
where $P_i(n)$ is a polynomial with degree less than m_i.
- If α is a root with multiplicity 3, it contributes
$$C_1\alpha^n + C_2 n\alpha^n + C_3 n^2\alpha^n$$
to the general solution.

Example

Solve the recurrence:

$$\boxed{\begin{array}{l} a_0 = 2,\ a_1 = 6 \\ a_n = 4a_{n-1} - 4a_{n-2},\ (n \geq 2). \end{array}}$$

Solution.

- The characteristic equation (plug in $a_i = x^i$ and factor out x^{n-r}) is

$$x^2 = 4x - 4 \qquad \rightarrow \qquad x^2 - 4x + 4 = 0 \qquad \rightarrow \qquad (x-2)^2 = 0$$

- There is a repeated characteristic root $\alpha_1 = 2$.
- This gives a solution $C_1(2)^n$ but we need two linearly independent solutions.
- One tip: **If in doubt, multiply by n** (i.e., multiply by n to generate new solutions).
- The general solution of the recurrence is then

$$a_n = C_1(2)^n + C_2 n(2)^n$$

  for some constants C_1 and C_2.
- Plugging in initial conditions gives $C_1 = 2$ and $2C_1 + 2C_2 = 6$.
- Solving this system gives $C_1 = 2$ and $C_2 = 1$.
- The solution to the original recurrence with the specified initial conditions is:

$$\boxed{a_n = 2(2)^n + n(2)^n = (n+2)2^n.}$$

Example

Solve the recurrence:

$$\boxed{\begin{array}{l} h_0 = 1,\ h_1 = 0,\ h_2 = 1,\ h_3 = 2 \\ h_n = -h_{n-1} + 3h_{n-2} + 5h_{n-3} + 2h_{n-4},\ (n \geq 4). \end{array}}$$

Solution.

- The characteristic equation is

$$x^4 = -x^3 + 3x^2 + 5x + 2 \quad \to \quad x^4 + x^3 - 3x^2 - 5x - 2 = 0 \quad \to \quad (x+1)^3(x-2) = 0$$

- The factoring can be done via long division after guessing a root.
- The roots are $-1, -1, -1, 2$.
- The general solution of the recurrence is

$$h_n = C_1(-1)^n + C_2 n(-1)^n + C_3 n^2(-1)^n + C_4(2)^n.$$

  That is $h_n = (C_1 + C_2 n + C_3 n^2)(-1)^n + C_4(2)^n$ for some constants C_1, C_2, C_3, C_4.
- Plugging in initial conditions $n = 0, 1, 2, 3$ and solving the system gives $C_1 = 7/9$, $C_2 = -3/9,\ C_3 = 0,\ C_4 = 2/9$.
- The solution to the original recurrence with the specified initial conditions is:

$$\boxed{h_n = \frac{7}{9}(-1)^n - \frac{n}{3}(-1)^n + \frac{2^{n+1}}{9}.}$$

Example

Find a recurrence relation that has a solution $a_n = (-2)^n + (2+n)(1)^n$.

Solution.

- The numbers -2 and 1 are characteristic roots of multiplicities 1 and 2 (since $(1)^n$ has a degree one polynomial associated with it).
- A possible characteristic equation is then $(x+2)(x-1)^2 = 0$.
- Thus, $x^3 = 3x - 2$ implying $a_n = 3a_{n-2} - 2a_{n-3}$.
- Since this is order 3, we need three initial conditions to get started.
- From the original equation in the question, $a_0 = 3$, $a_1 = 1$ and $a_2 = 8$.
- Thus, a recurrence with solution $a_n = (-2)^n + (2+n)(1)^n$ is

$$\boxed{\begin{array}{l} a_0 = 3,\ a_1 = 1,\ a_2 = 8 \\ a_n = 3a_{n-2} - 2a_{n-3},\ (n \geq 3). \end{array}}$$

Constant coefficient linear non-homogeneous recurrence relations

Let $\{a_n\}$ be a sequence, assume $c_0, c_r \neq 0$ and r is fixed $1 \leq r \leq n$. Then if $f(n) \neq 0$,

$$\boxed{c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r} = f(n)} \quad (**)$$

an **rth order constant coefficient linear non-homogeneous recurrence relation.**

Steps to solve $(**)$

1. Find a general solution $a_n^{(h)}$ of the homogeneous recurrence obtained from $(**)$:

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r} = 0$$

2. Find a particular solution $a_n^{(p)}$ of $(**)$.
3. The general solution of $(**)$ is then

$$a_n = a_n^{(h)} + a_n^{(p)}.$$

For step 2, we usually guess a particular solution of the same type

- if $f(n)$ is a polynomial of degree t, try $a_n^{(p)}$ to be a polynomial of degree t [unless 1 is a characteristic root of multiplicty m, then try $a_n^{(p)} = n^m \cdot$ (polynomial of degree t)].
- if $f(n)$ is exponential (e.g., $f(n) = 4k^n$), then try letting $a_n^{(p)}$ be an exponential function (e.g., $a_n^{(p)} = Ck^n$). An exception is if k is a characteristic root of multiplicity m, then try $a_n^{(p)} = n^m \cdot (Ck^n)$.

Example

Solve the recurrence: $\boxed{\begin{array}{l} a_0 = 3 \\ a_n = 3a_{n-1} + 2 - 2n^2, \ (n \geq 1). \end{array}}$

- We first solve the associated **homogeneous system** $a_n = 3a_{n-1}$ whose characteristic equation is $x = 3$ giving a solution $\boxed{a_n^{(h)} = A \cdot 3^n}$ for some constant A.
- We next find a **particular solution** $a_n^{(p)}$.
- Since $f(n) = 2 - 2n^2$ is a polynomial of degree 2, we try the same type:

$$a_n^{(p)} = Bn^2 + Cn + D.$$

- Plugging into the recurrence gives

$$\left(Bn^2 + Cn + D\right) = 3\left(B(n-1)^2 + C(n-1) + D\right) + 2 - 2n^2$$

- We equate coefficients on both sides to get a system:

$$\boxed{B = 3B - 2} \quad \boxed{C = -6B + 3C} \quad \boxed{D = 3B - 3C + 3D + 2}$$

- Thus, $B = 1$, $C = 3$ and $D = 2$ giving $\boxed{a_n^{(p)} = n^2 + 3n + 2}$.
- The general solution to the original recurrence is:

$$a_n = a_n^{(h)} + a_n^{(p)} = A \cdot 3^n + (n^2 + 3n + 2).$$

Using the initial conditions we can solve for A to get $3 = A + 2$ or $A = 1$. Thus,

$$\boxed{a_n = 3^n + n^2 + 3n + 2.}$$

Example

Solve the recurrence:
$$\boxed{\begin{array}{l} a_0 = 3,\ a_1 = 8 \\ a_n = 3a_{n-1} - 2a_{n-2} + 2^n,\ (n \geq 2). \end{array}}$$

- We first solve the associated **homogeneous system** $a_n = 3a_{n-1} - 2a_{n-2}$ whose characteristic equation is $x^2 = 3x - 2$ giving a solution $\boxed{a_n^{(h)} = A \cdot (1)^n + B \cdot (2)^n}$.
- We next find a **particular solution** $a_n^{(p)}$.
- Since $f(n) = 2^n$ this suggests trying $a_n^{(p)} = C2^n$ but this is already a solution to the homogeneous part! We use our tip: $\boxed{\textbf{If in doubt, multiply by } n}$ (i.e., multiply by n to generate new solutions). Thus, try $a_n^{(p)} = Cn2^n$ instead.
- Plugging into the recurrence gives $Cn2^n = 3C(n-1)2^{n-1} - 2C(n-2)2^{n-2} + 2^n$.
- Dividing by 2^{n-2} gives $4Cn = 6C(n-1) - 2C(n-2) + 4$.
- We equate coefficients on both sides to get a system:

$$\boxed{4C = 6C - 2C} \qquad \boxed{0 = -6C + 4C + 4}$$

- Thus, $C = 2$ giving $a_n^{(p)} = 2n2^n = n2^{n+1}$.
- The general solution to the original recurrence is:

$$a_n = a_n^{(h)} + a_n^{(p)} = A \cdot (1)^n + B \cdot (2)^n + n2^{n+1}.$$

Using the initial conditions we can solve for A, B to get $A = 2,\ B = 1$. Thus,

$$\boxed{a_n = 2 + 2^n + n2^{n+1}.}$$