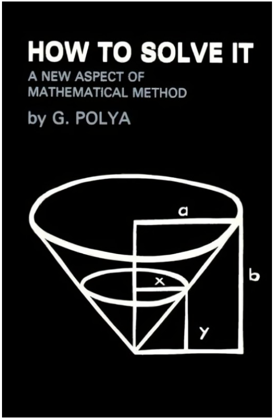
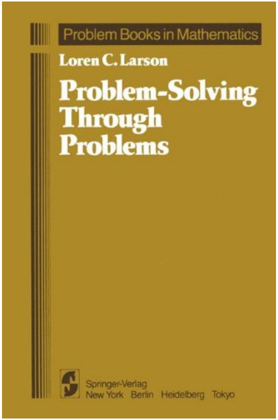
Problem Solving Strategies (in Combinatorics)

Some classic books on problem solving:



(1945)



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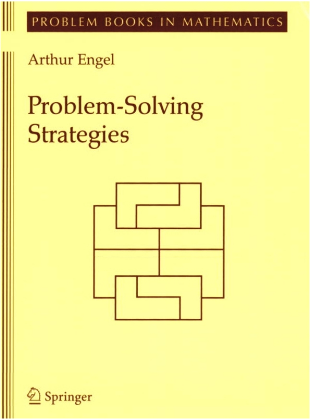


(1998)

Some strategies and tips

Tips:

- try small cases or examples
- plug in numbers
- look for patterns
- draw pictures
- introduce notation
- think about "parity"
- look for symmetry
- divide into cases
- modify the problem (reduce or generalize)

Techniques:

- pigeonhole principle (last class)
- contradiction
- induction (strong induction)
- extremal principle
- invariance principle
- count in two different ways (next week)

Example (joke proof)

Prove that every natural number is interesting.

"Proof".

- To derive a contradiction, assume that not every natural number is interesting.
- By the well-ordering property of the natural numbers, there is a smallest non-interesting number; call this number N.
- But that makes N pretty interesting!
- This gives a contradiction, therefore, every natural number is interesting.

Theorem: The Pigeonhole Principle

Let m and n be positive integers. If n pigeons are placed into m pigeonholes and $n > m$, then there exists a pigeonhole with at least two pigeons.

Proof.

- Suppose n pigeons are placed into m pigeonholes and that $n > m$.
- To derive a contradiction, assume that every pigeonhole contains at most one pigeon.
- Then there are at most m pigeons since there are m pigeonholes.
- But we assumed that there are n pigeons and that $n > m$, so this is impossible.
- Therefore, there must be a pigeonhole with more than one pigeon.

Induction

- We want to prove a family of statements $P(k)$
- Prove some base cases $P(1), \ldots$ (how many depends on the problem)
- It is enough to prove that $P(1), \ldots, P(n)$ together imply $P(n+1)$

A popular first example of "proof by induction" is to prove the following formula.

Example

Use induction to prove the following statement:

If n is a nonnegative integer, then $\displaystyle\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

Another popular example is the **Frobenius coin problem**. It has many variations. (see https://en.wikipedia.org/wiki/Coin_problem)

Example

Chicken nuggets are sold in 4, 6, 9 and 20 piece boxes. Show that for every $n \geq 24$, we can buy exactly n nuggets by buying several boxes.

Can the previous example be improved?

Are all four box sizes required?

Extremal principle

The general technique is the following:

- pick an object which is maximum/minimum among a specific class of structures
- somehow deduce there is an even larger/smaller object!
- this contradicts that the original is a maximum/minimum and provides useful information for the problem

Later in the course we will prove the following theorem using the extremal principle:

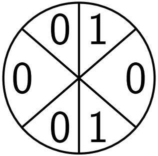
Lemma

Let G be a graph in which every vertex has degree at least two. Then G contains a cycle.

“**If there is repetition, look for what does not change!**”

Example

Divide a circle into six sectors with numbers $1, 0, 1, 0, 0, 0$ in clockwise order. Every minute, you may increase two neighbouring numbers by 1 (they must share an edge).



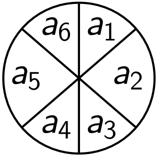
Is it possible to equalize the numbers in a finite amount of time?

Experiment a bit. Is there a quantity that does not change after each step?

(One possible solution is on the next slide.)

Solution.

- For nonnegative integers $a_1, a_2, a_3, a_4, a_5, a_6$, consider the configuration:



- Let $\mathcal{I} = a_1 - a_2 + a_3 - a_4 + a_5 - a_6$.

- **Claim.** $\mathcal{I}$ is an invariant (i.e., does not change after each step).
  Proof. Each step adds 1 to either $\{a_1, a_2\}$, $\{a_2, a_3\}$, $\{a_3, a_4\}$, $\{a_4, a_5\}$, $\{a_5, a_6\}$ or $\{a_6, a_1\}$. Thus, after each step, there is a contribution of 0 to $\mathcal{I}$.

- The original state has $\mathcal{I} = 1 - 0 + 1 - 0 + 0 - 0 = 2$.

- Our "goal" state has all numbers equal (i.e., $a_1 = a_2 = \cdots = a_6$) and thus $\mathcal{I} = 0$.

- Since $\mathcal{I}$ does not change after each step, it is impossible to go from the starting state $(1, 0, 1, 0, 0, 0)$ with $\mathcal{I} = 2$ to a state of all equal numbers with $\mathcal{I} = 0$.

Here is another popular problem that uses invariance.

Example

The numbers $1, 2, \ldots, 200$ are written on a blackboard. At each step:

- select two numbers (say x and y) written on the blackboard
- erase x and y
- write the number $x + y$ on the blackboard

At some point, there will be one number on the blackboard. What are all possibilities for this final number?

Solution.

- Let $\mathcal{I}$ be the sum of all numbers on the blackboard at each step.
- $\mathcal{I}$ is an invariant, i.e., it does not change after each step.
- This is because we erase x and y but then write $x + y$ making the new sum of the numbers equal to $\mathcal{I} - x - y + x + y = \mathcal{I}$.
- Since at the start of the procedure we have $\mathcal{I} = \sum_{i=1}^{200} i = 20\,100$, it must be that the final number on the blackboard is $20\,100$.

The Idea: Partition a set into a finite number of subsets by colouring each element of the subset by the same colour.

In 1961, Fisher showed that an 8×8 chessboard can be covered by 2×1 dominoes in

$$2^4 \times 901^2 = 12,988,816 \text{ ways.}$$

Example

Cut out two opposite corners of a chessboard (assume the chessboard is black/white). In how many ways can you cover the entire 62 squares with 31 dominoes?

Rough work.

- Draw a picture and colour the squares as on a typical chessboard.
- After experimenting, we cannot find a single tiling that works. Perhaps none exist?
- What do you notice about the colours of squares that a domino covers?
- Perhaps "parity" or **colouring** can show a tiling cannot exist?

(One possible solution is on the next slide.)

Example

Cut out two opposite corners of a chessboard (assume the chessboard is black/white).
In how many ways can you cover the entire 62 squares with 31 dominoes?

Claim.
There is no tiling of the chessboard with opposite corners cut out using 2×1 dominoes.

Before we prove this, observe the following fact:
Fact. Every domino must cover one black and one white square.

Proof of Claim.

- Consider the "**mutilated**" chessboard with opposite corners removed.
- Observe that opposite corners have the same colour, thus, without loss of generality, we may assume that we cut out the two white ones.
- Thus, the mutilated chessboard has 30 white and 32 black squares.
- To derive a contradiction, suppose there is a tiling of the mutilated chessboard using 2×1 dominoes.
- Since the mutilated chessboard has 62 squares, the tiling uses 31 dominoes.
- By the **Fact**, the tiling covers exactly 31 black and 31 white squares.
- This is impossible since the mutilated chessboard has 30 white and 32 black squares.
- Thus, no such tiling can exist.

The previous example generalizes to the following theorem.

Theorem

No domino tiling exists whenever any two squares of the same colour are removed from the chessboard.

What if two squares of opposite colours are removed?

Question

Cut out two squares of a chessboard of the same colour.
Is it possible to cover the remaining 62 squares with 31 dominoes?

The answer to the previous question is **yes** as shown by Gomory in 1973.

Theorem (Gomory, 1973)

An 8×8 chessboard with one black and one white square removed can always be covered with exactly 31 dominoes of size 2×1.

One technique to prove this is to use graph theory.

Symmetry

Look for (and use) symmetry in the problem.

Example

Compute $1 + 2 + \cdots + 1000$ using symmetry.

Solution.

- Notice that $1 + 2 + \cdots + 999 + 1000 = 1000 + 999 + \cdots + 2 + 1$.
- Let $S = 1 + 2 + \cdots + 999 + 1000$.
- Then also $S = 1000 + 999 + \cdots + 2 + 1$.
- Adding these two equations gives:

$$\begin{array}{ccccccccccc} S & = & 1 & + & 2 & + & \cdots & + & 999 & + & 1000 \\ S & = & 1000 & + & 999 & + & \cdots & + & 2 & + & 1 \\ \hline 2S & = & 1001 & + & 1001 & + & \cdots & + & 1001 & + & 1001 \end{array}$$

- Therefore, $2S = 1001 \times 1000$ implying that $S = \frac{1001 \times 1000}{2} = 500\,500$.

Example

A farmer and a cow are on the same side of a (straight) river. The farmer walks to the river, gets water in a bucket, and takes it to the cow. What is the farmer's shortest path?