University of Toronto Scarborough

CSC B36 **Term Test 1** **17 October 2022**

Student Number: : : : : : : : : : : :

Last (Family) Name: <u>Sample Solutions</u>

First (Given) Name: ______________________

Do not turn this page until you are told to do so.

In the meantime, complete the above and read the rest of this cover page.

Aids allowed: None.

Duration: 90 minutes.

There are 7 pages in this test. Each is numbered at the bottom. *When you receive the signal to start, please check that you have all the pages.*

You will be graded on your mastery of course material as taught in class. So you need to demonstrate this. Unless otherwise stated, you must explain or justify every answer.

Answer each question in the space provided. The *second* last page is intentionally left blank in case you need more space for one of your answers. *You must clearly indicate where your answer is and what part should be marked.* **What you write on backs of pages will <u>not</u> be graded.**

Question	Your mark	Out of
1.		10
2.		10
3.		10
Total		30

1. We define a set of ordered pairs of positive integers as follows.

    Let S be the smallest set such that
    BASIS: $(2,3) \in S$.
    INDUCTION STEP: If $(x,y) \in S$, then $(x+1, y+1), (x+1, y+2) \in S$.

   We define a predicate on $\mathbb{N}$ as follows.

    $P(n)$: For every integer j such that $n < j < 2n$, $(n,j) \in S$.

   (a) **[2 marks]** Is $P(1)$ true? Briefly explain your answer.
   Answer:
   Yes, $P(1)$ is vacuously true since there are no integers j such that $1 < j < 2 \cdot 1 = 2$.

   (b) **[8 marks]** Use induction to prove that $P(n)$ holds for every integer $n \geq 2$.
   Answer: We'll use PSI.
   BASIS: Let $n = 2$. Let j be an (arbitrary) integer such that $2 = n < j < 2n = 4$.
    Then we must have $j = 3$.
    By definition (basis) of S, $(n,j) = (2,3) \in S$ as wanted.
   INDUCTION STEP: Let $n \geq 2$.
    Suppose $P(n)$ holds. [IH]
    WTP: $P(n+1)$ holds.
   Let j be an (arbitrary) integer such that $n+1 < j < 2(n+1) = 2n+2$.
   We consider 2 cases: (i) $n+1 < j < 2n+1$ and (ii) $j = 2n+1$.
   Case (i): Suppose $n+1 < j < 2n+1$.
    Then $n < j-1 < 2n$. [subtract one from each quantity]
    So by IH, $(n, j-1) \in S$.
    By definition of S (induction step), $(n+1, j-1+1) = (n+1, j) \in S$ as wanted.
   Case (ii): Suppose $j = 2n+1$.
    Then $j - 2 = 2n - 1 = n + (n-1) > n$. [$n - 1 > 0$ since $n \geq 2$]
    Thus $n < j - 2 = 2n - 1 < 2n$.
    So by IH, $(n, j-2) \in S$.
    By definition of S (induction step), $(n+1, j-2+2) = (n+1, j) \in S$ as wanted.

2. Consider the program SmSqIter and its specification given on the last page of this test.

   Use methods from class to prove that this program is correct with respect to its specification.

   Answer:

   Step 1: Here is our LI.
   (a) $0 \le r \le n$.
   (b) $r = 0$ or $(r-1)^2 < n$.

   Step 2: Proof of LI.
   Basis: On entry to loop, we have $r = 0$ [line 1]
   Since $n \in \mathbb{N}$, $0 \le r \le n$ as wanted for LI(a).
   Also, LI(b) holds since $r = 0$.

   Induction Step: Consider an arbitrary iteration.
   Suppose LI holds before the iteration. [IH]
   WTP: LI holds after the iteration.

   By line 3, $r' = r + 1$.
   By LI(a), $0 \le r$. So $0 \le r < r + 1 = r'$.
   By line 2, $r^2 < n$. So $n > 0$ and $r' = r + 1 \le n$.
   Therefore LI(a) holds after the iteration.

   By line 3, $r' = r + 1$. Thus also $r = r' - 1$.
   By line 2, $r^2 < n$. So $(r'-1)^2 = r^2 < n$.
   Therefore LI(b) holds after the iteration.

   Step 3: Proof of partial correctness.
   Suppose the loop terminates and consider the value of r on exit.

   By exit condition (line 2), $r^2 \ge n$.
   By LI(b), $r = 0$ or $(r-1)^2 < n$.
   We consider both cases.

   If $r = 0$, then $r^2 = 0 \ge n$.
   Since $n \in \mathbb{N}$, we must have $n = 0$.
   Thus $r = 0$ is the smallest natural number such that $r^2 \ge n$.

   If $(r-1)^2 < n$, then $(r-1)^2 < n \le r^2$.
   Thus r is the smallest natural number such that $r^2 \ge n$.

   For both cases, by line 4, r is returned as wanted.

   Steps 4,5: Proof of termination.
   For each iteration, we associate the expression $e = n - r$.

   By LI(a), $e \ge 0$. So $e \in \mathbb{N}$.

   Consider an (arbitary) iteration.
   Then $e' = n - r'$
   $= n - (r + 1)$ [line 3]
   $= (n - r) - 1$
   $= e - 1$
   $< e$
   Therefore e (the sequence of e values) is decreasing.

   There's an alternate proof on the next page.

[more space available on next page . . .]

[... additional space for question 2]

Alternate answer:

Step 1: Here is our LI. $\quad 0 \leq r \leq \lceil \sqrt{n} \, \rceil$.

This fact follows from the definition of $\lceil \cdot \rceil$.
Fact: $\lceil \sqrt{n} \, \rceil$ is both the smallest integer that's greater than or equal to $\sqrt{n}$ and the largest integer that's less than $\sqrt{n}+1$.

Step 2: Proof of LI.
BASIS: On entry to loop, we have $r = 0$ [line 1]
Since $n \in \mathbb{N}$, $0 \leq r \leq \lceil \sqrt{n} \, \rceil$ as wanted.

INDUCTION STEP: Consider an arbitrary iteration.
Suppose LI holds before the iteration. [IH]
WTP: LI holds after the iteration.

By line 3, $r' = r + 1$.
By IH, $0 \leq r$. So $0 \leq r < r + 1 = r'$.

By line 2, $r^2 < n$. Since $0 \leq r$, so $r < \sqrt{n}$. Thus $r' = r + 1 < \sqrt{n} + 1$.
Since r' is an integer, by above fact, $r' \leq \lceil \sqrt{n} \, \rceil$ as wanted.

Step 3: Proof of partial correctness.
Suppose the loop terminates and consider the value of r on exit.

By exit condition (line 2), $r^2 \geq n$.
By LI, $0 \leq r$, so, taking root of both sides, we get $r \geq \sqrt{n}$. (*)
Also by LI, $r \leq \lceil \sqrt{n} \, \rceil$. (**)

Combining (*), (**) and above fact, r is the smallest integer that's greater than or equal to $\sqrt{n}$.
By line 4, r is returned as wanted.

Steps 4,5: Proof of termination.
For each iteration, we associate the expression $e = \lceil \sqrt{n} \, \rceil - r$.

By LI, $e \geq 0$. So $e \in \mathbb{N}$.

Consider an (arbitary) iteration.

$$\begin{aligned}
\text{Then } e' &= \lceil \sqrt{n} \, \rceil - r' \\
&= \lceil \sqrt{n} \, \rceil - (r+1) \qquad \text{[line 3]} \\
&= (\lceil \sqrt{n} \, \rceil - r) - 1 \\
&= e - 1 \\
&< e
\end{aligned}$$

Therefore e (the sequence of e values) is decreasing.

3. Consider the program SmSqRec and its specification given on the last page of this test.

(a) **[8 marks]** Use methods from class to prove that this program is correct with respect to its specification.

Answer:

For $k \in \mathbb{N}$, we define a predicate $Q(k)$ as follows.

$Q(k)$: If $n, p, q \in \mathbb{N}$, $p^2 < n \leq q^2$ and $k = q - p$, then
SmSqRec(n, p, q) returns the smallest natural number s such that $s^2 \geq n$.
(Note that we're using $q - p$ as our "input size".)

We'll use PCI to prove that $Q(k)$ holds for all $k > 0$. Then correctness follows.

Basis: Let $k = 1$.

Then $q - p = k = 1$, so $p = q - 1$.

Combining with precondition, we have $(q-1)^2 = p^2 < n \leq q^2$.

So q is the smallest natural number whose square is at least n.

By lines 1-2, q is returned as wanted.

Induction Step: Let $k > 1$.

Suppose $P(j)$ holds whenever $1 \leq j < k$. [IH]

WTP: $Q(k)$ holds.

Since $q - p = k > 1$, lines 3-7 runs.

By line 3, $m = \left\lfloor \frac{p+q}{2} \right\rfloor$.

We consider two cases: (i) $m^2 \geq n$ and (ii) $m^2 < n$.

Case (i): Suppose $m^2 \geq n$.

Then by lines 4-7, SmSqRec(n, p, m) is called.

By PL2.2, $p < m < q$, so $1 \leq m - p < q - p$.

Also, we have $p^2 < n \leq m^2$.

Thus by IH, SmSqRec(n, p, m) returns the smallest natural number that is no less than n.

This value is returned in line 5 as wanted.

Case (ii): Suppose $m^2 < n$.

We could argue that this is similar to case (i). Here is the proof for completeness.

Then by lines 4-7, SmSqRec(n, m, q) is called.

By PL2.2, $p < m < q$, so $1 \leq m - p < q - p$.

Also, we have $m^2 < n \leq q^2$.

Thus by IH, SmSqRec(n, m, q) returns the smallest natural number that is no less than n.

This value is returned in line 7 as wanted.

(b) **[2 marks]** Suppose we were to modify line 3 of SmSqRec to the following.

modified line 3: $m = \left\lceil \frac{p+q}{2} \right\rceil$

Is the modified program correct with respect to the same specification?

If yes, then explain how your proof needs to change so that it works for the modified program.

If no, then explain how the modified program fails.

Answer:

Yes, the modified program is correct with respect to the same specification.

For the proof, we need to use a modified version of PL2.2 (Pythonized Lemma 2.2), one that takes the "midpoint" of a and b to be $\left\lceil \frac{a+b}{2} \right\rceil$.

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You may detach this page if you find it convenient to do so.

There is no need to submit this page, as nothing on it will be graded.

Program and specification for question 2

▷ Precondition: $n \in \mathbb{N}$.
▷ Postcondition: Return the smallest natural number s such that $s^2 \geq n$.
SmSqIter(n)
    $r = 0$
    while $r^2 < n$:
        $r = r + 1$
    return r

Program and specification for question 3

▷ Precondition: $n, p, q \in \mathbb{N}$ and $p^2 < n \leq q^2$.
▷ Postcondition: Return the smallest natural number s such that $s^2 \geq n$.
SmSqRec(n, p, q)
    if $p + 1 \geq q$:
        return q
    $m = \left\lfloor \frac{p+q}{2} \right\rfloor$
    if $m^2 \geq n$:
        return SmSqRec(n, p, m)
    else:
        return SmSqRec(n, m, q)