University of Toronto Scarborough
Department of Computer and Mathematical Sciences
MATC44H3F (LEC01) - Fall 2022 - Final Exam - Practice 4

Date: Tuesday, December 20, 2022 from 9:00 to 12:00 (IC 200 & IC 204)
Instructor: Michael Cavers

First name (please write as legibly as possible within the boxes)

Last name

Student ID number

Signature: ______________________________

- **Time:** 180 minutes
- Write your solutions in this booklet (only those pages with a QR code will be graded).
- Use the back of each page for **rough** work.
- This is a closed-book exam. No aids are allowed for this exam other than those provided by the instructor. Calculators and the use of personal electronic or communication devices is prohibited.
- This exam has 12 pages with the last page being blank.
- There are 9 problems with the number of points indicated by each problem.
- The total number of points possible is 100.
- The University of Toronto's Code of Behaviour on Academic Matters (July 2019) applies to all University of Toronto Scarborough students. The Code prohibits all forms of academic dishonesty including, but not limited to, cheating, plagiarism, and the use of unauthorized aids. Students violating the Code may be subject to penalties up to and including suspension or expulsion from the University.

1. (20 points) For the following problem, you only need to provide your final answers. Correct answers are 2 points each and incorrect answers are 0 points each. Part marks is possible.

   (a) Two players are playing a Nim game with 4 heaps of objects each of which has size 4. Which player can guarantee a win?

   Solution: If we create a binary table for the game, then we get:

	2^2	2^1	2^0
4	1	0	0
4	1	0	0
4	1	0	0
4	1	0	0

   The nim-sum is equal to 000 (or 0 in base-10). Therefore, by the "Nim Theorem" (i.e., Bouton's theorem), the second player can guarantee a win.

   (b) There are 11 **different** animals in a zoo. In how many ways can we choose 8 of them to create a group.

   Solution: $\binom{8}{2}$.

   (c) There are 11 **different** animals in a zoo. In how many ways can we choose 6 of them and arrange them in a row.

   Solution: $11 \times 10 \times 9 \times 8 \times 7 \times 6$.

   (d) There are 11 **types** of coins in a bag (with an unlimited number of each type of coin). In how many ways can we choose 8 coins and place them into two equal-sized labelled groups.

   Solution: $\binom{14}{10}\binom{14}{10}$ (by stars and bars for each separate group).

   (e) Is there a (simple) graph having the degree sequence $1, 1, 2, 2, 3$?

   Solution: There is no such graph since by the Handshaking Lemma we would require $\frac{1}{2}(1 + 1 + 2 + 2 + 3) = 9/2$ edges which is impossible.

(f) A connected planar graph has degree sequence $2, 2, 3, 4, 4, 5$. How many faces does it have?

Solution: By Euler's formula, it has $f = 2 + 10 - 6 = 6$ faces.

(g) Draw a planar graph G satisfying $|V(G)| = 6$, $\chi(G) = 4$ and $\omega(G) = 3$.

Solution: The wheel graph W_6 has these properties. This graph is obtained from the cycle C_5 by adding a new vertex v adjacent to all vertices on the cycle.

(h) Solve the recurrence relation $a_n = \pi a_{n-1}$, $(n \geq 1)$ with $a_0 = 1$.

Solution: $a_n = \pi^n$

(i) Determine the ordinary generating function for the sequence with general term $a_n = \frac{1}{n!}$.

Solution: We have

$$g(x) = \sum_{k=0}^{\infty} \frac{1}{n!} x^n = e^x.$$

(j) Let $\alpha \in \mathbb{R}$. Determine the ordinary generating function for the sequence with general term $a_n = (-1)^n \binom{\alpha}{n}$.

Solution: Using the generalized binomial theorem, we have

$$g(x) = \sum_{k=0}^{\infty} (-1)^n \binom{\alpha}{n} x^n = (1-x)^\alpha.$$

2. (10 points) Let $X = \{0, 5, 10, 15, \ldots, 100\}$ (i.e., X contains all multiples of 5 between 0 and 100 inclusive) and suppose A is a subset of X with size at least 13.

   Prove there must exist two distinct integers $a, b \in A$ satisfying the following two properties simultaneously: (i) $a + b$ is divisible by 11, and (ii) $a - b$ is divisible by 10.

   Furthermore, show that if $|A| = 12$ then this is not necessarily the case.

Solution: We partition X into the the following 10 subsets:

$$\{0, 5, 55\}, \{10, 100\}, \{15, 95\}, \{20, 90\}, \{25, 85\}, \{30, 80\}, \{35, 75\}, \{40, 70\}, \{45, 65\}, \{50, 60\}.$$

Since $|A| \geq 13$, by the pigeonhole principle there are two elements a, b that belong to a common 2-element subset of the partition. Each 2-element subset has a sum equal to 110 and difference divisible by 10. Thus, the pair of elements a, b satisfy the required two properties.

The set
$$A = \{0, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55\}$$
is of size 12 and has no pair of elements $a, b \in A$ satisfying (i) and (ii). Thus, 13 is optimal in the problem statement.

3. (10 points) At Mike's dessert shop, there are Timbits, cupcakes, tarts, pies and cookies for sale at a price of \$3 each and ice cream cakes at a price of \$5 each. There are an unlimited number of each type of dessert available. You have \$42 and want to spend it all at Mike's dessert shop (i.e., you will spend exactly \$42). How many different selections of desserts are there?

Solution: Let x_1 be the number of Timbits, x_2 the number of cupcakes, x_3 the number of tarts, x_4 the number of pies, x_5 the number of cookies and x_6 the number of ice cream cakes. The number of different selections is equal to the number of integer solutions to

$$3x_1 + 3x_2 + 3x_3 + 3x_4 + 3x_5 + 5x_6 = 42$$

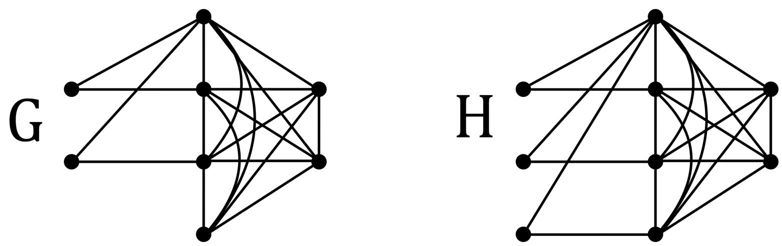
with $x_i \geq 0$ for $i = 1, 2, \ldots, 6$. Note that $x_6 \leq 8$ since no solutions arise for $x_6 \geq 9$. We apply a case analysis to the value of x_6:

- $x_6 = 0$: There are $\binom{14+5-1}{5-1}$ solutions to $\sum_{i=1}^{5} x_i = 14$, $x_i \geq 0$.
- $x_6 = 1$: There are no solutions to $\sum_{i=1}^{5} x_i = 37$ since 37 is not divisible by 3.
- $x_6 = 2$: There are no solutions to $\sum_{i=1}^{5} x_i = 32$ since 32 is not divisible by 3.
- $x_6 = 3$: There are $\binom{9+5-1}{5-1}$ solutions to $\sum_{i=1}^{5} x_i = 9$, $x_i \geq 0$.
- $x_6 = 4$: There are no solutions to $\sum_{i=1}^{5} x_i = 22$ since 22 is not divisible by 3.
- $x_6 = 5$: There are no solutions to $\sum_{i=1}^{5} x_i = 17$ since 17 is not divisible by 3.
- $x_6 = 6$: There are $\binom{4+5-1}{5-1}$ solutions to $\sum_{i=1}^{5} x_i = 4$, $x_i \geq 0$.
- $x_6 = 7$: There are no solutions to $\sum_{i=1}^{5} x_i = 7$ since 7 is not divisible by 3.
- $x_6 = 8$: There are no solutions to $\sum_{i=1}^{5} x_i = 2$ since 2 is not divisible by 3.

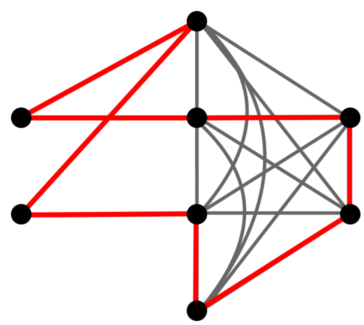
The total number of selections is

$$\binom{14+5-1}{5-1} + \binom{9+5-1}{5-1} + \binom{4+5-1}{5-1}$$

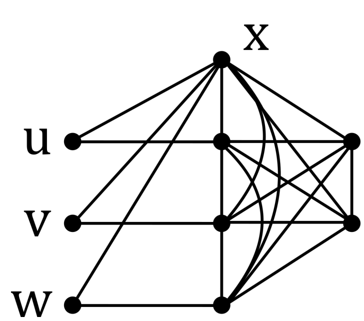
4. (10 points) Is the graph G shown below Hamiltonian? Is the graph H shown below Hamiltonian? For each graph, either construct a Hamilton cycle or prove no such cycle can exist.



Solution: The graph G is Hamiltonian as demonstrated below.



The graph H is not Hamiltonian. To prove this, we use contradiction. Assume H has a Hamilton cycle C. Label some of the vertices in H as below:



Since each of u, v and w are vertices of degree two, the edges ux, vx and vw must all be edges in the cycle C. But this implies that the degree of x when restricted to the subgraph C of H is at least 3 (i.e., $\deg_C(x) \geq 3$) contradicting that a cycle has all of its vertices of degree equal to two.

5. (10 points) What is the smallest number of vertices of degree one in a connected graph on n vertices that has $n-1$ edges, two vertices of degree 3, one vertex of degree 5 and two vertices of degree 6? Justify your answer.

Solution: Let T have n vertices with $V(T) = \{v_1, v_2, \ldots, v_n\}$ and maximum degree Δ. Let n_i denote the number of vertices of degree i (note that $n_0 = 0$ since G is connected). Since Δ is the maximum degree, we have $n_i \geq 0$ for $1 \leq i \leq \Delta$ and $n_i = 0$ for $i > \Delta$. Since T has $n-1$ edges, by the handshaking lemma, it follows that

$$\sum_{i=1}^{\Delta} \deg(v_i) = n_1 + 2n_2 + 3n_3 + \cdots \Delta n_\Delta = 2(n-1).$$

But $n_1 + n_2 + \cdots + n_\Delta = n$. Subtracting the first equation from twice the second equation gives:

$$n_1 - n_3 - 2n_4 - 3n_5 - 4n_6 - \cdots - (\Delta - 2)n_\Delta = 2.$$

Thus, $n_1 = 2 + n_3 + 2n_4 + 3n_5 + 4n_6 + \cdots + (\Delta - 2)n_\Delta$. By the assumption in the problem, $n_3 = 2$, $n_5 = 1$ and $n_6 = 2$. Thus, $n_1 \geq 2 + 2 + 3(1) + 4(2) = 15$. This implies that $n \geq 20$. It is straight-forward construct a graph G with $n = 20$, $n_1 = 15$ and satisfying the required properties. Subdividing edges of G introduces degree 2 vertices but preserves the minimum number of degree 1 vertices, thus, the bound is attainable for all $n \geq 20$.

6. (10 points) Determine the number of ways to deal six cards (from a deck of 52 cards) so that there is at least one Jack, at least one 8 and at least one 2.

   Note that in a standard deck of 52 playing cards, there are 4 of each card (4 Aces, 4 Kings, 4 Queens, etc.) one of each of the four suits (Clubs, Hearts, Diamonds, Spades).

Solution: We apply the principle of inclusion-exclusion.

- Let S be the set of possible six card hands (note $|S| = \binom{52}{6}$).
- Let A_2 be the set of six card hands with no 2.
- Let A_8 be set of the six card hands with no 8.
- Let A_J be set of the six card hands with no Jack.
- Then

$$|A_2| = |A_8| = |A_J| = \binom{48}{6}$$

$$|A_2 \cap A_8| = |A_2 \cap A_J| = |A_8 \cap A_J| = \binom{44}{6}, \text{ and}$$

$$|A_2 \cap A_8 \cap A_J| = \binom{40}{6}.$$

- The answer is then

$$\left|\overline{A_2} \cap \overline{A_8} \cap \overline{A_J}\right| = \left|\overline{A_2 \cup A_8 \cup A_J}\right| = \binom{52}{6} - 3\binom{48}{6} + 3\binom{44}{6} - \binom{40}{6}$$

7. (10 points) There are an unlimited number of red, blue, green and yellow balls available. In how many ways can we fill a box with n balls if

- the number of red balls must be even,
- the number of blue balls must be a multiple of five,
- there must be at most four green balls,
- there must be at most one yellow ball.

Solution: The ordinary generating function is

$$g(x) = \left(1 + x^2 + x^4 + \cdots\right)\left(1 + x^5 + x^{10} + \cdots\right)\left(1 + x + x^2 + x^3 + x^4\right)(1 + x).$$

The answer to the problem is $[x^n]g(x)$. Using $1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x}$ $(*)$ and $1 + x + x^2 + x^3 + x^4 = \dfrac{1-x^5}{1-x}$ gives:

$$g(x) = \left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1-x^5}{1-x}\right)(1+x) = \frac{1}{(1-x)^2} = \left(1 + 2x + 3x^2 + \cdots\right)$$

since the derivative of $(*)$ gives $1 + 2x + 3x^2 + \cdots = \dfrac{1}{(1-x)^2}$.

Therefore, $g(x) = \displaystyle\sum_{n=0}^{\infty}(n+1)x^n$.

Thus, the number of ways is $[x^n]g(x) = \boxed{(n+1)}$ for $n \geq 0$.

8. (10 points) Let h_n be the number of ways to colour the squares of a 1-by-n grid using red, green and blue if an even number of squares are to be coloured red.

   (a) Determine an exponential generating function for h_n.

   (b) Determine a formula for h_n.

   (c) Determine a recurrence relation for h_n.

   Recall: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ and $\frac{1}{2}\left(e^x + e^{-x}\right) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$.

Solution: We need to count the number of length n strings using the letters R, G, B (i.e., red, blue, green) with repetition allowed such that an even number of R's occurs. By the Theorem in lecture, the answer is the coefficient of $\frac{x^n}{n!}$ in the expansion of the exponential generating function:

$$G(x) = \left(\frac{x^0}{0!} + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots\right)\left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots\right)$$

$$\begin{aligned} \text{Thus,} \quad G(x) &= \left(\frac{1}{2}\left(e^x + e^{-x}\right)\right) \cdot e^x \cdot e^x = \frac{1}{2}\left(e^{3x} + e^x\right) \\ &= \frac{1}{2}\left[\left(\sum_{k=0}^{\infty} \frac{(3x)^k}{k!}\right) + \left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)\right] = \frac{1}{2}\sum_{k=0}^{\infty} (3^k + 1)\frac{x^k}{k!}. \end{aligned}$$

(b) Therefore, $h_n = \left[\frac{x^n}{n!}\right] G(x) = \frac{1}{2}(3^n + 1)$.

(c) Observe $h_1 = 2$ (either G or B) and $h_2 = 5$ (either GG, GB, BG, BB, or RR). We set up a recurrence by focusing on the first square. If it is green, then there are h_{n-1} such colourings. If it is blue, then there are h_{n-1} such colourings. If it is red, we need to colour a 1-by-$(n-1)$ grid using an odd number of red. To count this we observe:

- The total number of colourings of a 1-by-$(n-1)$ grid is 3^{n-1}.
- Every colouring either has an even number of red or an odd number of red.
- h_{n-1} is the number of number of colourings that have an even number of red.
- Therefore, there must be $3^{n-1} - h_{n-1}$ ways to to colour a 1-by-$(n-1)$ grid using an odd number of red.

The recurrence is then

$$h_n = 2h_{n-1} + \left(3^{n-1} - h_{n-1}\right) \qquad \rightarrow \qquad h_n = h_{n-1} + 3^{n-1}.$$

Thus, $h_n = h_{n-1} + 3^{n-1}$ $(n \geq 2)$ with initial condition $h_1 = 2$.

9. (10 points) Let $n \geq 3$ and suppose that there exists a Steiner triple system of order n (or equivalently, there exists a decomposition of $E(K_n)$ into triangles). Prove that either $n = 6k + 1$ or $n = 6k + 3$ for some non-negative integer k.

Solution: Suppose there exists a decomposition of the edges of the complete graph K_n into triangles. Then n must be odd since every vertex v must belong to $\dfrac{n-1}{2}$ triangles (note $\deg(v) = n - 1$). To show that either $n = 6k+1$ or $n = 6k+3$ for some non-negative integer k, it suffices to rule out the case that $n = 6k + 5$.

To derive a contradiction, assume $n = 6k+5$. Since there are $\dfrac{\binom{n}{2}}{3}$ triangles in total, $n(n-1)/6$ must be a positive integer. Using $n = 6k + 5$ gives that

$$n(n-1)/6 = (6k+5)(3k+2)/3$$

is a positive integer. However, neither $6k + 5$ nor $3k + 2$ are divisible by 3 implying $(6k + 5)(3k + 2)/3$ is not an integer, a contradiction.

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