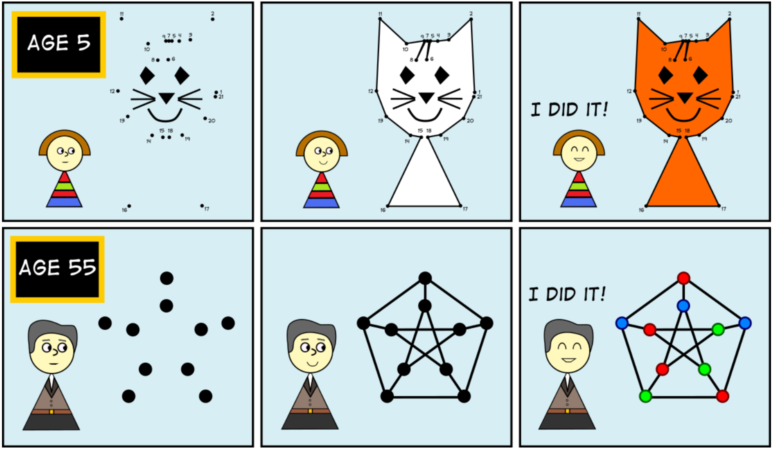
Graph colouring

The image below demonstrates one example of a graph theorist.



Definition: Colouring

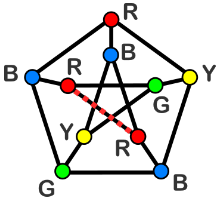
A **colouring** of a graph G is an assignment of **colours** (or labels) to $V(G)$ so that adjacent vertices receive different colours.

- Some textbooks use the phrase **proper colouring**.
- If k colours are used, we call the assignment a ***k*-colouring**.

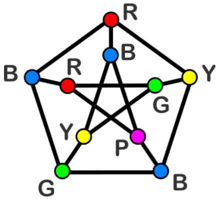
Example

Give an example of a colouring of the Petersen graph.

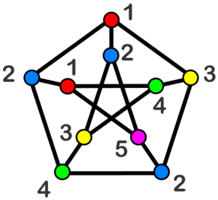
Solution.



is **not** a colouring (**why?**) but



is a colouring.

We sometimes use numbers instead of colours: 

Definition: k-colourable

A graph G is called ***k*-colourable** if G has a colouring with at most k colours.

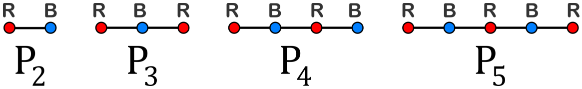
Definition: Chromatic number

The **minimum** k for which G is k-colourable (i.e., has a k-colouring) is called the **chromatic number** of G and is denoted by $\boxed{\chi(G)}$.

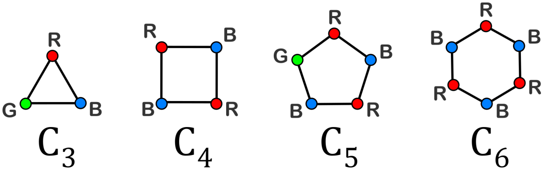
Example

Compute the chromatic number for the paths, cycles and complete graphs.

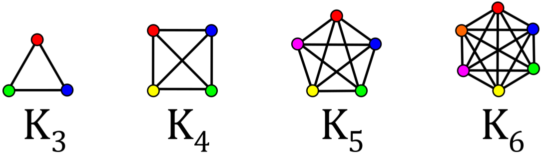
Solution.



For $n \geq 2$, we have $\chi(P_n) = 2$.



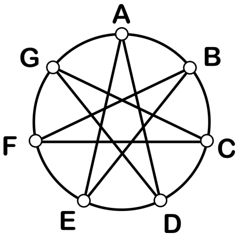
For $n \geq 3$, we have $\chi(C_n) = 2$ if n is even and $\chi(C_n) = 3$ if n is odd.



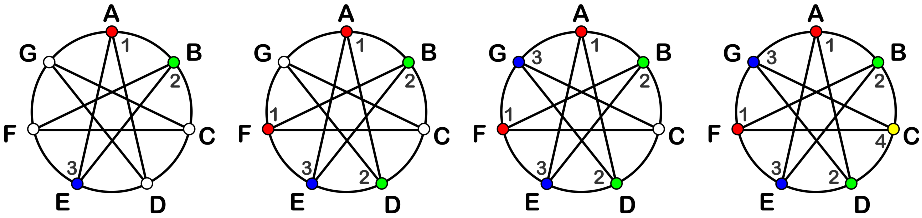
For $n \geq 1$ (note K_1 and K_2 are not shown above), we have $\chi(K_n) = n$.

Example

Determine $\chi(H)$ where H represents the following graph:

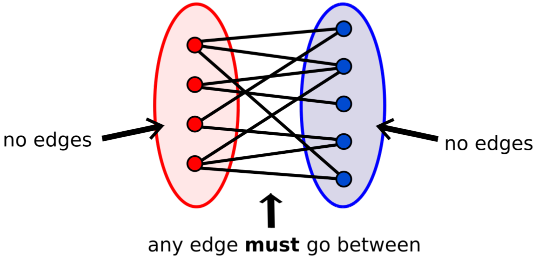


Solution. We prove that $\chi(H) = 4$ (we must show **both** $\chi(H) \geq 4$ and $\chi(H) \leq 4$).



- We first show that $\chi(H) \geq 4$.
- To derive a contradiction, assume that $\chi(H) \leq 3$ and suppose we have a colouring using colours **1**, **2** and **3** (not all colours need to be used).
- Since vertices A, B and E form a K_3, each has a different colour.
- WLOG suppose A has **colour 1**, B has **colour 2** and C has **colour 3**.
- Then
    - D must have **colour 2** (since D is adjacent to A and E).
    - F must have **colour 1** (since F is adjacent to B and E).
    - G must have **colour 3** (since G is adjacent to D and F).
- Then C must have **colour 3** (since C is adjacent to B and F) contradicting that we have a colouring since C is adjacent to G which has **colour 3**.
- Therefore, $\chi(G) \geq 4$.
- To show that $\chi(G) \leq 4$, we exhibit a colouring (see fourth image).

Recall the general structure of a bipartite graph:



Theorem

If G has at least one edge then $\chi(G) = 2$ <u>if and only if</u> G is bipartite.

- How do we formally prove this?
- **Note:** A graph G with no edges is bipartite and satisfies $\chi(G) = 1$.

Theorem

For a graph G, $\chi(G) = 2$ if and only if G is a bipartite graph with at least one edge.

Proof. ($\Longrightarrow$)

- Suppose that $\chi(G) = 2$.
- Then there is a 2-colouring of G; suppose it uses colours 1 and 2.
- Let V_1 be the set of vertices of **colour 1** and let V_2 the set of vertices of **colour 2**.
- Since we have a colouring, there are no edges whose endpoints are both in V_1 (otherwise adjacent vertices are both coloured with **colour 1**).
- Similarly, there are no edges whose endpoints are both in V_2 (otherwise adjacent vertices are both coloured with **colour 2**).
- Therefore, the sets V_1 and V_2 form a **bipartition** of G implying that G is bipartite.
- Finally, since $\chi(G) = 2$, we require 2 colours, thus, G must have at least one edge.

($\Longleftarrow$)

- Suppose that G is a bipartite graph with at least one edge.
- Let V_1 and V_2 form a bipartition of G.
- Colour the vertices in V_1 with **colour 1** and the vertices in V_2 with **colour 2**.
- No pair of adjacent vertices have the same colour by the definition of bipartition.
- Thus, this is a 2-colouring of G implying that $\chi(G) \leq 2$.
- Since G has at least one edge, the endpoints of that edge must be assigned different colours, so $\chi(G) \geq 2$.
- Therefore, $\chi(G) = 2$.

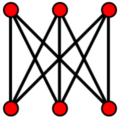
Definition: Clique

A **clique** of a graph G is a complete subgraph.

The **clique number** of G, denoted by $\boxed{\omega(G)}$, is the **maximum** size of a clique in G.

Example

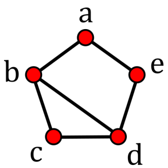
Find $\omega(K_{3,3})$ where $K_{3,3}$ is



Solution. $\omega(K_{3,3}) = 2$ ($K_{3,3}$ is bipartite, thus has no odd cycles, thus no $K_3 \cong C_3$).

Example

Find $\omega(G)$ where G is



Solution. $\omega(G) = 3$ (vertices $\{b, c, d\}$ form a K_3).

Lower bounds

The following theorem gives a **lower** bound on the chromatic number.

Theorem

Let G be a graph. Then $\chi(G) \geq \omega(G)$.

Outline of Proof.

- This follows since every vertex of a clique requires its own colour.

Upper bounds

- Many **upper** bounds are obtained from graph colouring algorithms.

Theorem

Let G be a graph on n vertices $v_1, v_2, \ldots, v_n$. Then $\chi(G) \leq n$.

Proof.

- We colour v_i by colour i.
- This produces an n-colouring since adjacent vertices must have different colours.

Upper bounds

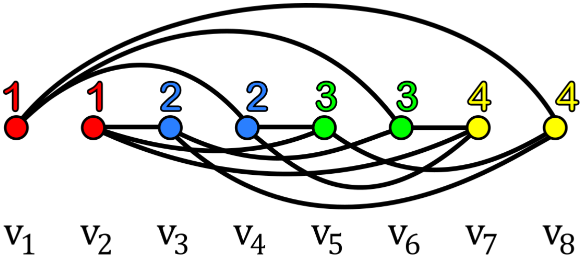
- A better algorithm is to use "the least available colour".

The Greedy Algorithm

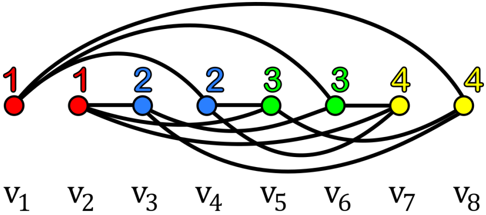
Let G be a graph on n vertices.

- Order the vertices as $v_1, v_2, \ldots, v_n$.
- Colour v_1 using **colour 1**.
- For $i = 2, 3, \ldots, n$, colour v_i the smallest colour that is not used on its lower-index neighbours.

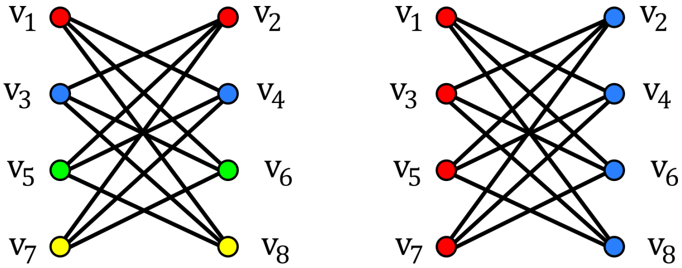
Example



- From the last slide: the greedy algorithm constructed a colouring of G using four colours (thus, $\chi(G) \leq 4$).



- If we order the vertices different, the greedy algorithm will construct a new colouring.
- This particular graph is actually bipartite. Redrawing it gives:



- If we apply the greedy algorithm to the vertex order $(v_1, v_3, v_5, v_7, v_2, v_4, v_6, v_8)$ we get the colouring on the right which is optimal for this graph ($\chi(G) = 2$).

Upper bounds

- The greedy algorithm gives an upper bound on the chromatic number of a graph.

Theorem

Let G be a graph with maximum degree $\Delta(G)$. Then $\chi(G) \leq \Delta(G) + 1$.

Outline of Proof.

- Use a greedy colouring.
- In a vertex ordering, each vertex has at most $\Delta(G)$ earlier neighbours.
- Thus, one of $\{1, 2, \ldots, \Delta(G), \Delta(G) + 1\}$ will be available as a colour.

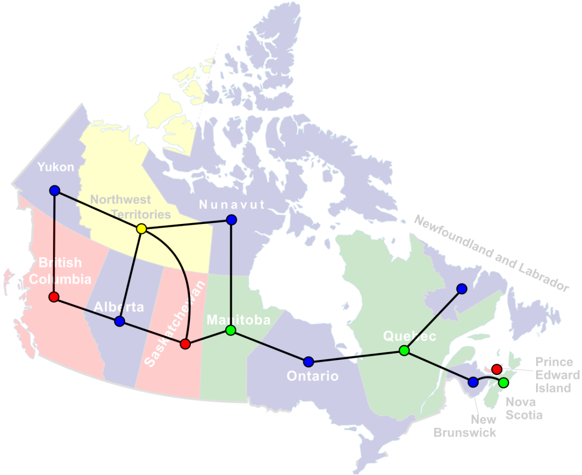
Brooks' Theorem (1941)

If G is a connected graph that is not an odd cycle or complete graph, then $\chi(G) \leq \Delta(G)$.

Map Colouring

- Map-makers colour the different regions so that if two regions share a border, they are not coloured the same.
- This makes it easier to distinguish the border between them.
- In the past, using more colours increased the cost to produce the map, so we ask:

Is there a bound on the number of colours required to colour <u>any</u> given map?



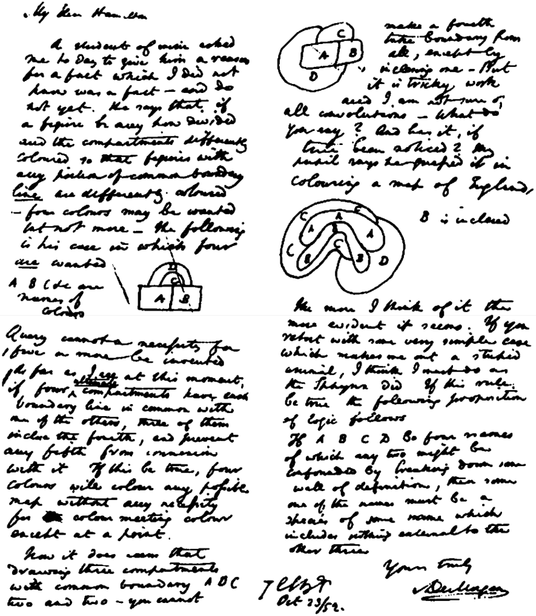
Map Colouring Rephrased

We can rephrase the map problem using planar graphs.

Two Questions

How many colours are needed to colour a planar graph?
If G is a planar graph, what is the best upper bound for $\chi(G)$?

- **1852**: De Morgan sent a letter to Hamilton asking if four colours is enough.



My dear Hamilton

A student of mine asked me to day to give him a reason for a fact which I did not know was a fact — and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured — four colours may be wanted but not more — the following is his case in which four are wanted

A B C D are names of colours

Query cannot a necessity for five or more be invented. As far as I see at this moment, if four ultimate compartments have each boundary line in common with one of the others, three of them inclose the fourth, and prevent any fifth from connexion with it. If this be true, four colours will colour any possible map without any necessity for colour meeting colour except at a point.

Now it does seem that drawing three compartments with common boundary A B C two and two — you cannot make a fourth take boundary from all, except by inclosing one — But it is tricky work and I am not sure of all convolutions — What do you say? And has it, if true been noticed? My pupil says he guessed it in colouring a map of England.

B is inclosed

The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did. If this rule be true the following proposition of logic follows

If A B C D be four names of which any two might be confounded by breaking down some wall of definition, then some one of the names must be a species of some name which includes nothing external to the other three.

Yours truly
A De Morgan

7 C. St.
Oct 23/52.

History: The Four Colour Theorem

- **1852**: De Morgan sent a letter to Hamilton asking if four colours is enough.
- A conjecture was made:
  The Four Colour Conjecture: Every planar graph is 4-colourable.
- **1879**: Kempe published a "proof"... but the proof contained a flaw.
- **1880**: Tait published a "proof"... but the proof contained a flaw.
- **1890**: Heawood finds a defect in Kempe's "proof" but manages to prove:
  The Five Colour Theorem: Every planar graph is 5-colourable.
- **1891**: Petersen finds a defect in Tait's "proof".
- **1976**: A proof of the Four Colour Theorem is published by Appel and Haken using computers (with assistance from Koch).
- They reduced the infinite number of possibilities to a fine number (approximately 1936 configurations that were each checked).

Proving six colours suffices is straight-forward, but we first need the following lemma.

Lemma

Let G be a planar graph. Then G has a vertex of degree at most five.

Proof.

- If G has at most six vertices, the statement clearly holds, thus assume G has at least seven vertices.
- To derive a contradiction, assume $\deg(v) \geq 6$ for all $v \in V(G)$.
- By the handshaking lemma,

$$2|E(G)| = \sum_{v \in V(G)} \deg(v) \geq \sum_{v \in V(G)} 6 = 6|V(G)|.$$

- Thus, $|E(G)| \geq 3|V(G)|$.
- But a Corollary to Euler's formula, since G is planar, we must have

$$|E(G)| \leq 3|V(G)| - 6.$$

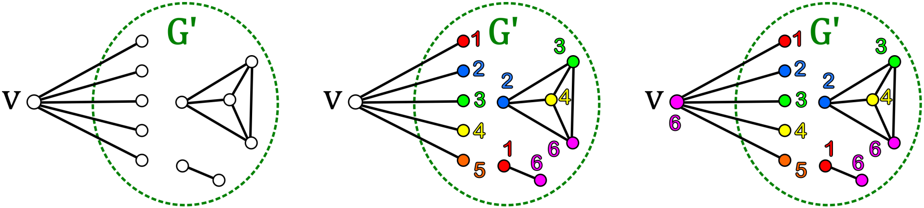
- **Note:** We only proved this Corollary for connected planar graphs on $|V(G)| \geq 3$ vertices, but the result also holds for disconnected planar graphs.
- Thus, $3|V(G)| \leq |E(G)| \leq 3|V(G)| - 6$, a contradiction since $0 \leq -6$ is false.

Theorem

Every planar graph G is 6-colourable, that is, $\chi(G) \leq 6$.

Proof.[1]

- We use induction on the number of vertices in the graph.
- **Base Case:** It is certainly true for graphs with at most 6 vertices.
- **Induction Hypothesis:** Assume it holds for planar graphs with less than n vertices.
- Let G be a planar graph with n vertices. WTS the statement holds for G.
- By the Lemma, G has a vertex v with $\deg(v) \leq 5$.



- Delete the vertex v (and all incident edges) to form the graph $G' = G - v$.
- By induction, we can colour the vertices of G' with at most six colours.
- Since $\deg(v) \leq 5$, the neighbours of v use at most 5 colours.
- Thus, there is an unused colour that we may use to colour v which gives rise to a 6-colouring of G. This shows that $\chi(G) \leq 6$.

[1] A contradiction proof (with extra tools) is given in the Morris Textbook.

The Five Colour Theorem

Theorem

Every planar graph G is 5-colourable, that is, $\chi(G) \leq 5$.

Proof.

- We use the idea of "Kempe" chains to prove $\chi(G) \leq 5$.
- Details are provided in separate notes.