Recap of Generating Functions

Example: Ordinary Generating Functions

Let $T_1 = \{0, 1, 2\}$, $T_2 = \{0, 1\}$ and $T_3 = \{2, 3\}$.

Determine the number of integer solutions satisfying $\boxed{\begin{array}{l} x_1 + x_2 + x_3 = 4, \\ x_1 \in T_1,\ x_2 \in T_2,\ x_3 \in T_3 \end{array}}$.

Solution.

- Let $g(x) = \underbrace{(x^0 + x^1 + x^2)}_{\boxed{x_1 \text{ can be } 0,\ 1 \text{ or } 2}} \cdot \underbrace{(x^0 + x^1)}_{\boxed{x_2 \text{ can be } 0 \text{ or } 1}} \cdot \underbrace{(x^2 + x^3)}_{\boxed{x_3 \text{ can be } 2 \text{ or } 3}}$ be the generating function.
- The answer to the problem is the coefficient of x^4 in $g(x)$ since there is a bijection between the solutions (x_1, x_2, x_3) and the ways to form an x^4 term:

x_1	x_2	x_3
0	1	3
1	0	3
1	1	2
2	0	2

$(x^0 + x^1 + x^2)$	$(x^0 + x^1)$	$(x^2 + x^3)$
x^0	x^1	x^3
x^1	x^0	x^3
x^1	x^1	x^2
x^2	x^0	x^2

x_1	x_2	x_3
	*	***
*		***
*	*	**
**		**

Note: This is also the number of arrangements of stars and bars with restrictions (0, 1 or 2 stars in first group, 0 or 1 stars in second group, 2 or 3 stars in third group.

Expanding gives $g(x) = x^2 + 3x^3 + \boxed{4}x^4 + 3x^5 + x^6$, thus, there are 4 solutions.

Generalizing the previous example gives the following theorem.

Theorem

Let n, k be positive integers and $T_1, T_2, \ldots, T_k$ be sets of non-negative integers. If a_n is the number of integer solutions satisfying

$$\boxed{\begin{array}{l} x_1 + x_2 + \cdots + x_k = n, \\ \text{such that } x_1 \in T_1, \ \ldots, \ x_k \in T_k, \end{array}}$$

then a_n is equal to the coefficient of x^n in the expansion of the generating function

$$g(x) = \left(\sum_{t_i \in T_1} x^{t_i} \right) \left(\sum_{t_i \in T_2} x^{t_i} \right) \cdots \left(\sum_{t_i \in T_k} x^{t_i} \right) \quad = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots.$$

Note: This is also equal to the number of ways to partition n identical objects into k labelled groups such that the number of objects in the ith group is an element of T_i.

In the case that $T_i = \{0, 1, 2, \ldots\}$ for $1 \le i \le k$, we get our usual stars and bars result since the coefficient of x^n in the expansion of $g(x)$ is equal to $\binom{n+k-1}{n}$ as observed:

$$g(x) = \left(1 + x + x^2 + \cdots\right)^k = \frac{1}{(1-x)^k} = \sum_{i=0}^{\infty} \binom{i+k-1}{i} x^i \quad \text{(by **Gen. Bin. Thm**).}$$

Example

Let n, k be nonnegative integers. Determine the number of integer solutions to

$$x_1 + x_2 + \cdots + x_k = n \text{ such that } x_i \geq i \text{ for } i = 1, 2, \ldots k.$$

(Here $T_i = \{i, i+1, i+2, \ldots\}$ and we require $x_i \in T_i$.)

$$\begin{aligned}
g(x) &= (x + x^2 + x^3 + \cdots)(x^2 + x^3 + x^4 + \cdots)\cdots(x^k + x^{k+1} + x^{k+2} + \cdots) \\
&= \left[x(1 + x + x^2 + \cdots)\right]\left[x^2(1 + x + x^2 + \cdots)\right] + \cdots + \left[x^k(1 + x + x^2 + \cdots)\right] \\
&= \left(x\frac{1}{1-x}\right)\left(x^2\frac{1}{1-x}\right)\cdots\left(x^k\frac{1}{1-x}\right) \\
&= x^{1+2+\cdots+k}(1-x)^{-k} \\
&= x^{\binom{k+1}{2}}\sum_{i=0}^{\infty}(-1)^i\binom{k+i-1}{i}(-x)^i \qquad \left(\text{By } \sum_{i=1}^{k} i = \binom{k+1}{2} \text{ and } \textbf{Gen. Bin. Thm}\right) \\
&= \sum_{i=0}^{\infty}\binom{k+i-1}{i}x^{i+\binom{k+1}{2}}
\end{aligned}$$

Therefore, $[x^n]g(x) = \binom{k+n-\binom{k+1}{2}-1}{n-\binom{k+1}{2}} = \binom{n-\binom{k}{2}-1}{n-\binom{k+1}{2}}$ since $\binom{k+1}{2} - \binom{k}{2} = k$.

Example: Solving recurrences with generating functions

Use generating functions to derive a closed-form expression for the sequence:

$$\boxed{\begin{array}{l} a_0 = -1,\ a_1 = 1,\ a_2 = 3, \\ a_n - a_{n-1} - 8a_{n-2} + 12a_{n-3} = 0,\ (n \geq 3). \end{array}}$$

Solution.

- We form a table have and add using the recurrence:

$$\begin{array}{rcccccccccc} g(x) & = & a_0 & + & a_1 x & + & a_2 x^2 & + & a_3 x^3 & + \cdots + & a_n x^n & + \cdots \\ -\,xg(x) & = & & & -a_0 x & - & a_1 x^2 & - & a_2 x^3 & - \cdots - & a_{n-1} x^n & - \cdots \\ -\,8x^2 g(x) & = & & & & & -8a_0 x^2 & - & 8a_1 x^3 & - \cdots - & 8a_{n-2} x^n & - \cdots \\ +\,12x^3 g(x) & = & & & & & & & 12a_0 x^3 & + \cdots + & 12a_{n-3} x^n & + \cdots \end{array}$$

- The left side gives $(1 - x - 8x^2 + 12x^3)g(x)$.
- Since $a_n - a_{n-1} - 8a_{n-2} + 12a_{n-3} = 0$ for all $n \geq 3$, the right side gives

$$(a_0 + a_1 x + a_2 x^2) + (-a_0 x - a_1 x^2) + (-8a_0 x^2) = 10x^2 + 2x - 1.$$

- Therefore $g(x) = \dfrac{10x^2 + 2x - 1}{1 - x - 8x^2 + 12x^3}$.
- The solution to the recurrence is $a_n = [x^n]g(x)$.

Solution (continued). $$g(x) = \frac{10x^2 + 2x - 1}{1 - x - 8x^2 + 12x^3}$$

- We use partial fractions to extract the coefficient of x^n.
- Since $1 - x - 8x^2 + 12x^3 = (1 + 3x)(1 - 2x)^2$ we have

$$g(x) = \frac{A}{1 + 3x} + \frac{B}{1 - 2x} + \frac{C}{(1 - 2x)^2}.$$

- It can be verified that $A = -1/5$, $B = -9/5$ and $C = 1$:

$$g(x) = -\frac{1}{5}\left(\frac{1}{1 + 3x}\right) - \frac{9}{5}\left(\frac{1}{1 - 2x}\right) + \frac{1}{(1 - 2x)^2}.$$

- **Recall:** $\boxed{\frac{1}{1 - x} = \sum_{k=0}^{\infty} x^k}$ and $\boxed{\frac{1}{(1 - x)^n} = \sum_{k=0}^{\infty} \binom{k + n - 1}{k} x^k}$.
- Therefore we have

$$g(x) = -\frac{1}{5}\sum_{k=0}^{\infty}(-3x)^k - \frac{9}{5}\sum_{k=0}^{\infty}(2x)^k + \sum_{k=0}^{\infty}\binom{k + 2 - 1}{k}(2x)^k.$$

- Hence, $a_n = [x^n]g(x) = -\frac{1}{5}(-3)^n - \frac{9}{5}2^n + (n + 1)2^n$.

Exponential Generating Functions

Definition

Let $a_0, a_1, a_2, \ldots$ be a sequence. The **ordinary generating function** of the sequence is

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \boxed{\sum_{k=0}^{\infty} a_k x^k}$$

while its **exponential generating function** is

$$G(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} \cdots = \boxed{\sum_{k=0}^{\infty} a_k \frac{x^k}{k!}}.$$

Example

The sequence $1, 1, 1, \ldots, 1, \ldots$ has

- ordinary generating function $g(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$.
- exponential generating function $G(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$.

Example

Let n be a positive integer. Find the ordinary generating function for the sequence

$$\left\{\binom{n}{0},\binom{n}{1},\binom{n}{2},\ldots,\binom{n}{n},0,0,\ldots\right\}.$$

$$g(x)=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{n}x^n=(1+x)^n.$$

Example

Let n be a positive integer. Find the exponential generating function for the sequence

$$\{P(n,0),P(n,1),\ldots,P(n,n),0,0,\ldots\}.$$

Recall: $P(n,k)=\frac{n!}{(n-k)!}$ is the number of permutations of n objects taken k at a time.

$$\begin{aligned}
G(x) &= P(n,0)+P(n,1)x+P(n,2)\frac{x^2}{2!}+\cdots+P(n,k)\frac{x^k}{k!}+\cdots+P(n,n)\frac{x^n}{n!}\\
&= 1+nx+\frac{n!}{(n-2)!}\frac{x^2}{2!}+\cdots+\frac{n!}{(n-k)!}\frac{x^k}{k!}+\cdots+\frac{n!}{0!}\frac{x^n}{n!}\\
&= \binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{k}x^k+\cdots+\binom{n}{n}x^n\\
&= (1+x)^n.
\end{aligned}$$

From the previous example:

$$(1+x)^n \text{ is the } \begin{cases} \text{ordinary generating function for } \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n} \\ \\ \text{exponential generating function for } P(n,0), P(n,1), \dots, P(n,n) \end{cases}$$

In general,

- the <u>ordinary</u> generating function is useful for problems involving <u>combinations</u>.
- the <u>exponential</u> generating function is useful for problems involving <u>permutations</u>.

Recall: Multinomial Coefficients

Definition: Multinomial Coefficient

Let n be a positive integer and $n_1, n_2, \ldots, n_k$ be non-negative integers with

$$\boxed{n_1 + n_2 + \cdots + n_k = n.}$$

The **multinomial coefficient**, denoted by $\binom{n}{n_1, n_2, \ldots, n_k}$, is defined as:

$$\binom{n}{n_1, n_2, \ldots, n_k} := \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Theorem

If there are $n_i \geq 1$ objects of type i, for $1 \leq i \leq k$, and there are $n = n_1 + n_2 + \cdots + n_k$ objects in total, then the number of arrangements of these n objects is $\binom{n}{n_1, n_2, \ldots, n_k}$.

Example

How many arrangements are there of the letters of T O R O N T O?

Number of arrangements is $\binom{7}{3, 2, 1, 1} = \frac{7!}{3!2!1!1!}$ (three **O**'s, two **T**'s, one **R**, one **N**).

Example

In how many ways can 4 of the letters from PAPAYA be arranged?

- There are three A's, two P's and one Y.
- Consider the **exponential** generating function

$$G(x) = \underbrace{\left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)}_{\boxed{\text{number of A's can be } 0, 1, 2 \text{ or } 3}} \cdot \underbrace{\left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!}\right)}_{\boxed{\text{number of P's can be } 0, 1 \text{ or } 2}} \cdot \underbrace{\left(\frac{x^0}{0!} + \frac{x^1}{1!}\right)}_{\boxed{\text{number of Y's can be } 0 \text{ or } 1}}$$

- The answer to the problem is the coefficient of $\frac{x^4}{4!}$ in $G(x)$.
- **Why???** Consider expanding $G(x)$ and note there is a bijection between ways to form an $\frac{x^4}{4!}$ term and number of arrangements of 4 elements from A,A,A,P,P,Y:
- Expanding $G(x)$ and focusing on terms that give x^4 we have:

$$G(x) = \cdots + \left(\frac{x^1}{1!}\frac{x^2}{2!}\frac{x^1}{1!} + \frac{x^2}{2!}\frac{x^1}{1!}\frac{x^1}{1!} + \frac{x^2}{2!}\frac{x^2}{2!}\frac{x^0}{0!} + \frac{x^3}{3!}\frac{x^1}{1!}\frac{x^0}{0!} + \frac{x^3}{3!}\frac{x^0}{0!}\frac{x^1}{1!}\right) + \cdots$$
$$= \cdots + \left(\frac{4!}{1!2!1!} + \frac{4!}{2!1!1!} + \frac{4!}{2!2!0!} + \frac{4!}{3!1!0!} + \frac{4!}{3!0!1!}\right)\frac{x^4}{4!} + \cdots$$

- Each term counts number of arrangements of APPY, AAPY, AAPP, AAAP, AAAY.

Theorem

- Let n, k be positive integers and $T_1, T_2, \ldots, T_k$ be sets of non-negative integers.
- Suppose we have k letters with an unlimited number of each type.
- Let A be our alphabet: $A = \{A_1, A_2, \ldots, A_k\}$.

If a_n is the number of length n arrangements of letters from A such that the number of A_i's used is an integer in T_i,
then a_n is the coefficient of $\dfrac{x^n}{n!}$ in the expansion of the exponential generating function

$$G(x) = \left(\sum_{t_i \in T_1} \frac{x^{t_i}}{t_i!}\right) \left(\sum_{t_i \in T_2} \frac{x^{t_i}}{t_i!}\right) \cdots \left(\sum_{t_i \in T_k} \frac{x^{t_i}}{t_i!}\right) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + \cdots + a_n \frac{x^n}{n!} + \cdots.$$

Note: This is also equal to $a_n = \sum \binom{n}{n_1, n_2, \ldots, n_k}$
where the summation is over all non-negative integer solutions $(n_1, n_2, \ldots, n_k)$ of

$$\boxed{n_1 + n_2 + \cdots + n_k = n}$$

such that $n_i \in T_i$ for $i = 1, 2, \ldots, k$.

Example

How many strings of length n can be formed using A's, B's and C's so that the number of A's is odd and the number of B's is also odd.

- Consider the **exponential** generating function

$$G(x) = \underbrace{\left(\frac{x^1}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)}_{\boxed{\text{number of A's must be odd}}} \cdot \underbrace{\left(\frac{x^1}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)}_{\boxed{\text{number of B's must be odd}}} \cdot \underbrace{\left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots\right)}_{\boxed{\text{number of C's can be any number}}}$$

- The answer to the problem is the coefficient of $\frac{x^n}{n!}$ in $G(x)$.
- **Recall:** $\boxed{\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x}$ and $\boxed{\frac{1}{2}\left(e^x - e^{-x}\right) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = \sinh x}$.

$$\begin{aligned}
\text{Thus,} \quad G(x) &= \left(\frac{1}{2}\left(e^x - e^{-x}\right)\right) \cdot \left(\frac{1}{2}\left(e^x - e^{-x}\right)\right) \cdot e^x \\
&= \frac{1}{4}\left(e^{3x} - 2e^x + e^{-x}\right) \\
&= \frac{1}{4}\left[\left(\sum_{k=0}^{\infty} \frac{(3x)^k}{k!}\right) - 2\left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right) + \left(\sum_{k=0}^{\infty} \frac{(-x)^k}{k!}\right)\right]
\end{aligned}$$

- The coefficient of $\frac{x^n}{n!}$ in $G(x)$ is $a_n = \frac{1}{4}\left(3^n - 2 + (-1)^n\right)$.

Example (using an exponential generating function)

Let h_n be the number of ways to colour the squares of a 1-by-n grid using red, green and blue if an even number of squares are to be coloured red. Determine a formula for h_n.

- We need to count the number of length n strings using the letters R, G, B (i.e., red, blue, green) with repetition allowed such that an even number of R's occurs.
- By the Theorem, the answer is the coefficient of $\frac{x^n}{n!}$ in the expansion of the **exponential** generating function:

$$G(x) = \underbrace{\left(\frac{x^0}{0!}+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)}_{\boxed{\text{number of red must be even}}} \cdot \underbrace{\left(\frac{x^0}{0!}+\frac{x^1}{1!}+\frac{x^2}{2!}+\cdots\right)}_{\boxed{\text{number of green can be any number}}} \cdot \underbrace{\left(\frac{x^0}{0!}+\frac{x^1}{1!}+\frac{x^2}{2!}+\cdots\right)}_{\boxed{\text{number of blue can be any number}}}$$

Thus,
$$\begin{aligned} G(x) &= \left(\frac{1}{2}\left(e^x+e^{-x}\right)\right)\cdot e^x\cdot e^x = \frac{1}{2}\left(e^{3x}+e^x\right) \\ &= \frac{1}{2}\left[\left(\sum_{k=0}^{\infty}\frac{(3x)^k}{k!}\right)+\left(\sum_{k=0}^{\infty}\frac{x^k}{k!}\right)\right] = \frac{1}{2}\sum_{k=0}^{\infty}(3^k+1)\frac{x^k}{k!}. \end{aligned}$$

- Therefore, $h_n = \left[\frac{x^n}{n!}\right]G(x) = \frac{1}{2}\left(3^n+1\right)$.

Example (same as previous example but using a recurrence)

Let h_n be the number of ways to colour the squares of a 1-by-n grid using red, green and blue if an even number of squares are to be coloured red. Determine a **recurrence** for h_n.

- Observe $h_1 = 2$ (either G or B) and $h_2 = 5$ (either GG, GB, BG, BB, or RR).
- We set up a recurrence by focusing on the first square.
- If it is green, then there are h_{n-1} such colourings.
- If it is blue, then there are h_{n-1} such colourings.
- If it is red, we need to colour a 1-by-$(n-1)$ grid using an odd number of red.
- To count this we observe:
  - The total number of colourings of a 1-by-$(n-1)$ grid is 3^{n-1}.
  - Every colouring either has an even number of red or an odd number of red.
  - h_{n-1} is the number of number of colourings that have an even number of red.
  - Therefore, there must be $3^{n-1} - h_{n-1}$ ways to to colour a 1-by-$(n-1)$ grid using an odd number of red.
- The recurrence is then

$$h_n = 2h_{n-1} + \left(3^{n-1} - h_{n-1}\right) \qquad \rightarrow \qquad h_n = h_{n-1} + 3^{n-1}.$$

- Thus, $h_n = h_{n-1} + 3^{n-1}$ $(n \geq 2)$ with initial condition $h_1 = 2$.

Solution (continued).

- An alternate method to solve the recurrence $h_n = h_{n-1} + 3^{n-1}$ $(n \geq 2)$ with initial condition $h_1 = 2$ is to **iterate**:

$$\begin{array}{rclcl}
h_1 & = & 2 & & \\
h_2 & = & h_1 + 3 & = & 2 + 3 \\
h_3 & = & h_2 + 3^2 & = & 2 + 3 + 3^2 \\
h_4 & = & h_3 + 3^3 & = & 2 + 3 + 3^2 + 3^3 \\
 & \vdots & & \vdots & \\
h_n & = & h_{n-1} + 3^{n-1} & = & 2 + 3 + 3^2 + \cdots + 3^{n-1}
\end{array}$$

- Thus,

$$h_n = 2 + \sum_{k=0}^{n-1} 3^k = 1 + \frac{1 - 3^n}{1 - 3} = \frac{3^n + 1}{2}.$$

- Formally, we can substitute $h_n = \dfrac{3^n + 1}{2}$ into the recurrence to verify it indeed solves the recurrence relation and satisfies the initial condition.

Example

Determine the number of ways to colour a 1-by-n grid using red, green and blue if an even number of squares are to be coloured red and at least one square must be coloured blue.

- The answer (call it h_n) is the coefficient of $\frac{x^n}{n!}$ in the expansion of

$$G(x) = \underbrace{\left(\frac{x^0}{0!} + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)}_{\boxed{\text{even red}}} \cdot \underbrace{\left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots\right)}_{\boxed{\text{any green}}} \cdot \underbrace{\left(\frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)}_{\boxed{\text{at least one blue}}}$$

$$\begin{aligned} \text{Thus,}\quad G(x) &= \left(\frac{1}{2}\left(e^x + e^{-x}\right)\right) \cdot e^x \cdot (e^x - 1) \\ &= \frac{1}{2}\left(e^{3x} - e^{2x} + e^x - 1\right) \\ &= \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{(3x)^k}{k!}\right) - \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{(2x)^k}{k!}\right) + \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right) - \frac{1}{2} \\ &= -\frac{1}{2} + \sum_{k=0}^{\infty} \frac{3^k - 2^k + 1}{2} \frac{x^k}{k!} \end{aligned}$$

- Therefore $h_0 = -\frac{1}{2} + \frac{3^0 - 2^0 + 1}{2} = 0$ and $h_n = \frac{3^n - 2^n + 1}{2}$ $(n \geq 1)$.