**Question-1 [6 points]:** Suppose $X_1, X_2, ..., X_n$ are *i.i.d.* with the following density function

$$f(x) = \lambda e^{-\lambda(x-\beta)}$$

where $x \geq \beta$; $\beta > 0$ and $\lambda > 0$.
Assume $\beta$ is a **known constant** and $\lambda$ is the **only unknown parameter**.

You are also told that $E[X_i] = \frac{1}{\lambda} + \beta$

**a.[2 points]** By showing detailed calculation, find a **method of moments** estimate of $\lambda$.

Given,

$E[X_i] = \frac{1}{\lambda} + \beta$ where $\beta$ is a constant.

according to method of moments,

$\bar{x}$ is an estimate of $E[X_i]$

$$\therefore \frac{1}{\hat{\lambda}} + \beta = \bar{x}$$

$$\Rightarrow \frac{1}{\hat{\lambda}} = \bar{x} - \beta$$

$$\Rightarrow \hat{\lambda} = \frac{1}{\bar{x} - \beta}$$

(Ans)

**b.**[**4 points**] By showing detailed calculation, find a maximum likelihood estimate (**MLE**) of $\lambda$. Make sure to check the sign of the second derivative.

$$L(\lambda) = \lambda e^{-\lambda(x_1-\beta)} * \lambda e^{-\lambda(x_2-\beta)} * \cdots * \lambda e^{-\lambda(x_n-\beta)}$$

$$= \lambda^n e^{-\lambda \Sigma (x_i-\beta)}$$

$$\Rightarrow l(\lambda) = n \log \lambda - \lambda \Sigma(x_i - \beta)$$

$$\Rightarrow l'(\lambda) = \frac{n}{\lambda} - \Sigma(x_i-\beta) = 0$$

$$\Rightarrow \frac{n}{\lambda} = \Sigma(x_i - \beta)$$

$$\Rightarrow \frac{n}{\lambda} = n\bar{x} - n\beta$$

$$\Rightarrow \frac{1}{\lambda} = \bar{x} - \beta$$

$$\Rightarrow \lambda = \frac{1}{\bar{x}-\beta} \Rightarrow \hat{\lambda} = \frac{1}{\bar{x}-\beta}$$

$$\Rightarrow l''(\lambda) = -\frac{n}{\lambda^2}$$

here, $x \geq \beta \Rightarrow \bar{x} \geq \beta$.

considering $\bar{x} > \beta$, $l''(\lambda)\big|_{\lambda=\hat{\lambda}} < 0$

$\therefore$ $\hat{\lambda}$ is the MLE.

Extra: when $\bar{x} = \beta$, MLE becomes undefined.

**Question-2 [6 points]:** Suppose $X_1, X_2, ...X_n \overset{iid}{\sim} Exp(\theta)$. With pdf

$$f(x) = \frac{1}{\theta}e^{-\frac{1}{\theta}x} \quad ; x \geq 0 \text{ and } \theta > 0$$

**a.**[**2 points**] By showing detailed calculation, Calculate the maximum likelihood estimator **(MLE)** of $\theta$. [no need to check the second derivative]

$$L(\theta) = \frac{1}{\theta} e^{-\frac{1}{\theta}x_1} * \frac{1}{\theta} e^{-\frac{1}{\theta}x_2} * \cdots * \frac{1}{\theta} e^{-\frac{1}{\theta}x_n}$$

$$= \frac{1}{\theta^n} e^{-\frac{1}{\theta}\Sigma x_i}$$

$$\Rightarrow \ell(\theta) = -n\log\theta - \frac{1}{\theta}\Sigma x_i$$

$$\Rightarrow \ell'(\theta) = -\frac{n}{\theta} + \frac{1}{\theta^2}\Sigma x_i = 0$$

$$\Rightarrow -\frac{n}{\theta} = -\frac{1}{\theta^2}\Sigma x_i$$

$$\Rightarrow \theta = \frac{\Sigma x_i}{n} = \bar{x}$$

$\therefore$ MLE of $\theta = \bar{x}$

**b.**[**2 points**] By showing detailed calculation, find the distribution of the MLE that you derived in part(a). [provide the name of the distribution and the corresponding parameters].

$$\ell''(\theta) = \frac{n}{\theta^2} - \frac{2}{\theta^3}\Sigma x_i$$

$$\Rightarrow -E[\ell''(\theta)] = -\frac{n}{\theta^2} + \frac{2}{\theta^3}\Sigma E[x_i]$$

$$= -\frac{n}{\theta^2} + \frac{2}{\theta^3}\cdot n\theta$$

$$= -\frac{n}{\theta^2} + \frac{2n}{\theta^2}$$

$$= \frac{n}{\theta^2}$$

continue your answer on the next page...

this place is for answering Q2(b)...

$\therefore$ Fisher Info, $nI(\theta_0) = \frac{n}{\theta_0^2}$

$\therefore \hat{\theta} \longrightarrow N\left(\theta_0, \frac{\theta_0^2}{n}\right)$

[Can also be answered using CLT.]

**c.[2 points]** Let $m$ represent the **median** of this distribution. You are told that $m = \theta ln(2)$. By naming the appropriate property, find the MLE of $m$.

Using the Invariance property of MLE,

$\hat{\theta} = \bar{X}$

$\Rightarrow \hat{m} = \hat{\theta} \ln(2)$

$= \bar{x} \ln(2)$

**Question-3 [6 points]:** Suppose $(X_1, X_2, ..., X_n)$ are independently distributed as $N(\mu, \sigma^2)$. Let us define $W$ as

$$W = X_n - \bar{X} \quad , \text{ where } \quad \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \text{ and } i = 1, 2, ..., n$$

a.[**4 points**] By showing detailed calculation, calculate the MSE of $W$ (as an estimator of $\mu$).
[*hint:* express $W$ as a function of $X_1, X_2, ..., X_n$]

$$W = X_n - \bar{X}$$

$$= X_n - \left(\frac{1}{n}X_1 + \frac{1}{n}X_2 + \cdots + \frac{1}{n}X_n\right)$$

$$= -\frac{1}{n}X_1 - \frac{1}{n}X_2 - \cdots + \left(1 - \frac{1}{n}\right)X_n$$

$$\text{Bias}(W) = E[W] - \mu$$

$$= E[X_n - \bar{X}] - \mu$$

$$= E[X_n] - E[\bar{X}] - \mu$$

$$= \mu - \mu - \mu = -\mu$$

$$V[W] = V\left[-\frac{1}{n}X_1 - \frac{1}{n}X_2 + \cdots + \left(1 + \frac{1}{n}\right)X_n\right]$$

$$= \frac{1}{n^2}V[X_1] + \frac{1}{n^2}V[X_2] + \cdots + \left(1 + \frac{1}{n}\right)^2 V[X_n]$$

$$= \frac{1}{n^2}\sigma^2 + \frac{1}{n^2}\sigma^2 + \cdots + \left(1 + \frac{1}{n}\right)^2 \sigma^2$$

$$= \frac{(n-1)}{n^2}\sigma^2 + \left(1 + \frac{1}{n}\right)^2 \sigma^2$$

$$\therefore \text{MSE}[W] = V[W] + (\text{Bias}[W])^2$$

$$= \frac{(n-1)}{n^2}\sigma^2 + \left(1 + \frac{1}{n}\right)^2 \sigma^2 + \mu^2$$

**b.**[**2 points**] By showing detailed calculation, calculate the **covariance** between $W$ and $\bar{X}$.

$$\mathrm{cov}[W, \bar{x}] = \mathrm{cov}[x_n - \bar{x}, \bar{x}]$$
$$= \mathrm{cov}[x_n, \bar{x}] - \mathrm{cov}[\bar{x}, \bar{x}]$$
$$= \mathrm{cov}[x_n, \tfrac{1}{n}x_1 + \tfrac{1}{n}x_2 \cdots + \tfrac{1}{n}x_n] - V[\bar{x}]$$
$$= \mathrm{cov}[x_n, \tfrac{1}{n}x_n] - V[\bar{x}]$$
$$= \tfrac{1}{n}\mathrm{cov}[x_n, x_n] - V[\bar{x}]$$
$$= \tfrac{1}{n}V[x_n] - V[\bar{x}]$$
$$= \tfrac{1}{n}\sigma^2 - \frac{\sigma^2}{n}$$
$$= 0$$

**Question 4 [6 points]:** Suppose $X_1, X_2, ..., X_n$ are independently drawn from a density with pdf

$$f(x) = \frac{x}{\beta}e^{-\frac{x^2}{2\beta}}; \quad x > 0, \beta > 0$$

where $\beta$ is the unknown parameter. Suppose $\beta_0$ is the true value(though unknown) of $\beta$ and $E[X_i^2] = 2\beta_0$.

**a.[3 points]** For this particular problem, show that $E\left[S(\beta|X_1, X_2, ..., X_n)|_{\beta=\beta_0}\right] = 0$.

$$L(\beta) = \frac{x_1}{\beta} e^{-\frac{x_1^2}{2\beta}} * \frac{x_2}{\beta} e^{-\frac{x_2^2}{2\beta}} * \cdots * \frac{x_n}{\beta} e^{-\frac{x_n^2}{2\beta}}$$

$$= \prod_{i=1}^{n} x_i * \frac{1}{\beta^n} \exp\left[-\frac{1}{2\beta} \Sigma x_i^2\right]$$

$$\Rightarrow \ell(\beta) = \log \Pi x_i - n \log \beta - \frac{1}{2\beta} \Sigma x_i^2$$

$$\Rightarrow \ell'(\beta) = -\frac{n}{\beta} + \frac{1}{2\beta^2} \Sigma x_i^2$$

$$\Rightarrow \ell'(\beta_0) = -\frac{n}{\beta_0} + \frac{1}{2\beta_0^2} \Sigma x_i^2$$

$$\Rightarrow E[\ell'(\beta_0)] = -\frac{n}{\beta_0} + \frac{1}{2\beta_0^2} E[\Sigma x_i^2]$$

$$= -\frac{n}{\beta_0} + \frac{1}{2\beta_0^2} \Sigma E[x_i^2]$$

$$= -\frac{n}{\beta_0} + \frac{1}{2\beta_0^2} n \cdot 2\beta_0$$

$$= -\frac{n}{\beta_0} + \frac{n}{\beta_0}$$

$$= 0$$

**b.[2 points]** By using the factorization theorem, find a **sufficient statistic** for $\beta$.

Here, $f(x_1 \dots x_n|\beta) = L(\beta) = \underbrace{\prod x_i}_{h(x_1 \dots x_n)} * \underbrace{\frac{1}{\beta^n} e^{-\frac{1}{2\beta}\sum x_i^2}}_{g[T,\beta]}$

∴ Using factorization theorem,

$\sum x_i^2$ is sufficient for $\beta$.

**c.[1 point]** Describe (briefly) the effect of increasing sample size on Fisher Information and sampling distribution of MLE.

when $n$ increases,

Fisher Information increases. which in return decreases the variance of MLE. Hence, the density of MLE gets narrower.

**Question-5 [6 points]:** [setup for parts (a) and (b)] Suppose a sample of size 30 [i.e. n = 30] is taken from a Normal distribution with **unknown mean** ($\mu$) and **unknown standard deviation** ($\sigma$).

95% confidence interval for $\mu$ is calculated and it is found that the interval is (7.677, 10.735)

**a.**[3 points] Calculate the **90% confidence interval for** $\mu$. [ *hint:* try to calculate the value of $\bar{x}$ and $s$ first]

From the given interval,

$$\bar{x} - t * \frac{s}{\sqrt{n}} = 7.677$$

$$\bar{x} + t * \frac{s}{\sqrt{n}} = 10.735$$

$$\Rightarrow \quad \bar{x} = \frac{7.677 + 10.735}{2} = 9.206$$

for 95% CI with $n = 30$, $t = 2.04523$ ← $t_{0.975}(29)$

$$9.206 - 2.04523 * \frac{s}{\sqrt{n}} = 7.677$$

$$\Rightarrow \frac{s}{\sqrt{n}} = 0.7475932$$

90% CI for $\mu$

$$\bar{x} \pm t_{0.95}(29) * \frac{s}{\sqrt{n}}$$

$$= 9.206 \pm 1.699127 * 0.7475932$$

$$= 7.935744, \quad 10.476256$$

$$= (7.936, \quad 10.476)$$

**b.**[1 point] **Interpret** the 90% confidence interval that you have calculated in part (a).

If we keep taking samples and keep constructing 90% CI, approximately 90% of CIs will capture the true mean.

c.[**2 points**] [**unrelated** to parts(a) and (b)] Suppose you are trying to calculate sample size for constructing a 95% confidence interval for $\mu$. You have the value of $\sigma^2$ from a previous study. You calculate your sample size $n$, for a desired width of the interval (say $w$).

Your friend wants to do the same (same 95% CI with the same value of $\sigma^2$). But s/he wants the width of the interval to be **half of the width** that you have used in your calculation. Suppose the required sample size for your friend turns out to be $n^*$. **Express $n^*$ in terms of $n$.**
(show detailed calculation and reasoning)

Here, $w = 2 * z_{0.975} * \frac{\sigma}{\sqrt{n}}$ [for you]

For your friend, say the width is $w^*$

$$w^* = 2 * z_{0.975} * \frac{\sigma}{\sqrt{n^*}}$$

Given,

$$w^* = \frac{1}{2} w$$

$$\Rightarrow 2 * z_{0.975} * \frac{\sigma}{\sqrt{n^*}} = \frac{1}{2} * 2 * z_{0.975} * \frac{\sigma}{\sqrt{n}}$$

$$\Rightarrow \frac{2}{\sqrt{n^*}} = \frac{1}{\sqrt{n}}$$

$$\Rightarrow \frac{4}{n^*} = \frac{1}{n}$$

$$\Rightarrow n^* = 4n$$