The Pigeonhole Principle (PHP)

"If $n+1$ pigeons are placed into n pigeonholes,
then there exists a pigeonhole with <u>at least two</u> pigeons."

(**Image source:** BenFrantzDale and McKay (`https://en.wikipedia.org/wiki/File:TooManyPigeons.jpg`))

The Pigeonhole Principle

Let m and n be positive integers. If n pigeons are placed into m pigeonholes and $n > m$, then there exists a pigeonhole with at least two pigeons.

Proof.

- Suppose n pigeons are placed into m pigeonholes and that $n > m$.
- To derive a contradiction, assume that every pigeonhole contains at most one pigeon.
- Then there are at most m pigeons since there are m pigeonholes.
- But we assumed that there are n pigeons and that $n > m$, so this is impossible.
- Therefore, there must be a pigeonhole with more than one pigeon.

Notes

- PHP is not constructive (this means we don't know which pigeonhole has more than one pigeon).
- PHP allows us to count objects with a common property.

The Pigeonhole Principle [Keller & Trotter, Proposition 4.1]

If $f : X \to Y$ is a function and $|X| > |Y|$, then there exists an element $y \in Y$ and distinct elements $x, x' \in X$ so that $f(x) = f(x') = y$.

Generalizations

Strong Form of PHP

Let m and k be positive integers. If $mk + 1$ pigeons are placed into m pigeonholes, then there exists a pigeonhole with at least $k + 1$ pigeons.

Notes

- The $mk + 1$ is "**sharp**" (i.e., "**tight**" or "**exact**") and cannot be decreased.

The Generalized Pigeonhole Principle [Morris, Proposition 10.1.5]

If there are n items that fall into m different categories and $n > km$ for some positive integer k, then at least $k + 1$ of the items must fall into the same category.

The Even More Generalized Pigeonhole Principle [Morris, Proposition 10.1.8]

Let $n_1, n_2, \ldots, n_m$ be positive integers. If $n_1 + n_2 + \cdots + n_m - m + 1$ items fall into m different categories, then there is some $1 \leq i \leq m$ such that at least n_i items fall into the ith category.

Alternate phrasing of strong form

Definition

The $\boxed{\textbf{ceiling function}}$, denoted by $\lceil x \rceil$ or $\operatorname{ceil}(x)$, maps x to the least integer greater than or equal to x (that is, $\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}$).

It is straight-forward to verify that $x \leq \lceil x \rceil < x + 1$ holds for every $x \in \mathbb{R}$.

Strong Form of the Pigeonhole Principle

Let N and k be positive integers. If N objects are distributed to k boxes, then at least one of the boxes must hold at least $\lceil N/k \rceil$ objects.

Proof.

- Suppose N objects are distributed to k boxes.
- To derive a contradiction, assume that each box contains at most $\lceil N/k \rceil - 1$ objects.
- Then there are at most $k(\lceil N/k \rceil - 1)$ objects since there are k boxes.
- By using $\lceil x \rceil < x + 1$ with $x = N/k$ we have

$$\# \text{ of objects} \leq k \left(\left\lceil \frac{N}{k} \right\rceil - 1 \right) < k \left(\frac{N}{k} + 1 - 1 \right) = N$$

- But we assumed that there are N objects, so this is impossible.
- Therefore, there must be a box with at least $\lceil N/k \rceil$ objects.

Example

True or false? At least 40 students at **UTSC** share the same birthday (day & month).

According to U of T "quick facts", the 2021-2022 student enrollment at UTSC is 14,547. (See https://www.utoronto.ca/about-u-of-t/quick-facts)

Example

Show that among any $n+1$ distinct numbers from the set $\{1, 2, \ldots, 2n\}$, there is always two which are consecutive.

Example

Prove that every subset of size six of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ must contain two elements whose sum is 10.

The Erdős-Szekeres Theorem

Let m, n be positive integers. Every sequence of $mn+1$ distinct real numbers has either

- an increasing subsequence of $m+1$ terms, or
- a decreasing subsequence of $n+1$ terms.

Example

What is the smallest value of N so that the following statement is always true:
"Given any N positive integers, there are two whose sum or difference is divisible by 100."

Solution.

- The answer is $N = 52$.
- Choose any 52 positive integers and consider the pigeonholes

$$\{00\}, \{01, 99\}, \{02, 98\}, \ldots, \{49, 51\}, \{50\}.$$

- Place the 52 integers into these 51 pigeonholes by matching the tens digit and units digit of the integer to that of the pigeonhole.
- By the pigeonhole principle, two of the 52 integers, say x and y, must belong to the same set.
- Then either x and y have the same last two digits and their difference ends in 00, or the last two digits of x and y have a sum ending in 00.
- In either case, these two numbers have the required property.
- The above argument applies to every $N \geq 52$.
- If $N \leq 51$, the assertion need not hold since any subset of $\{1, 2, \ldots, 49, 50, 100\}$ with size N does not have the required property.