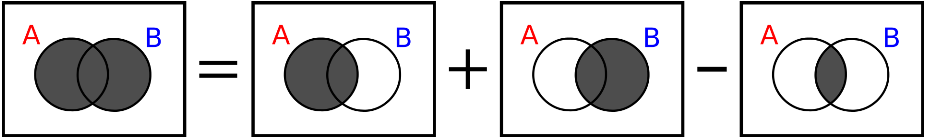
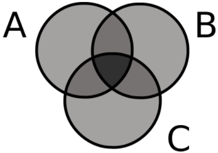
P.I.E.

(The Principle of Inclusion and Exclusion)



Theorem

Let A and B be finite sets. Then $|A \cup B| = |A| + |B| - |A \cap B|$.



Theorem

Let A, B and C be finite sets. Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

The Principle of Inclusion and Exclusion

More generally, we have the following theorem.

Theorem: The Principle of Inclusion and Exclusion (P.I.E.)

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \left(\sum_{i=1}^{n} |A_i|\right) - \left(\sum_{1 \leq i < j \leq n} |A_i \cap A_j|\right) + \left(\sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k|\right)$$
$$+ \cdots + \left((-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|\right)$$

That is, to find the cardinality of a union of n sets:

- include cardinalities of the sets,
- exclude cardinalities of the pairwise intersections,
- include cardinalities of the triplewise intersections,
- exclude cardinalities of the quadruplewise intersections,
- include cardinalities of the quintuplewise intersections,
- $\vdots$
- if n odd, include the cardinality of the n-tuplewise intersection,
  if n even, exclude the cardinality of the n-tuplewise intersection.

We can write this compactly as:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} |A_{i_1} \cap \cdots \cap A_{i_k}| \right)$$

Example

Let A and B be sets with $|A| = 4$ and $|B| = 9$.

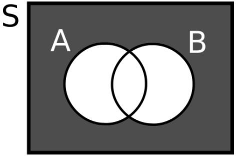
- What is the **maximum** that $|A \cap B|$ can be?
- What is the **minimum** that $|A \cap B|$ can be?
- What is $|A \cup B| + |A \cap B|$?
- What are all possible values for $|A \cup B|$?

Solution.

- The largest $|A \cap B|$ can be is 4 when $A \subset B$.
  - For example, $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
- The smallest $|A \cap B|$ can be is 0 when A and B have no elements in common.
  - For example, $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8, 9, 10, 11, 12, 13\}$.
- By **PIE**, we have $|A \cup B| = 4 + 9 - |A \cap B|$, thus, $|A \cup B| + |A \cap B| = 13$.
- Since $0 \leq |A \cap B| \leq 4$, it is clear that $9 \leq |A \cup B| \leq 13$.

Complementary Form of PIE

- Let S be a universal set with subsets A and B.
- We let $\overline{A}$ denote the **complement** of A (in S).
- Then $\left|\overline{A \cup B}\right| = |S| - |A \cup B| = |S| - (|A| + |B|) + |A \cap B|$.
- This represents the shaded region in the image below:



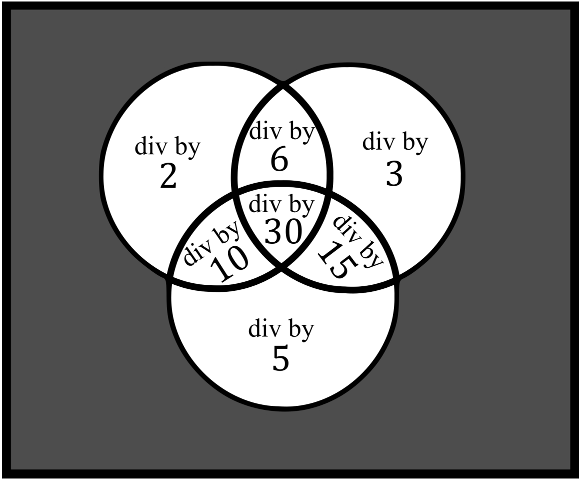
- **Note:** $\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$.
- The above can be generalized to give

$$\left|\bigcap_{i=1}^{n} \overline{A_i}\right| = \left|S - \bigcup_{i=1}^{n} A_i\right| = |S| - \sum_{i=1}^{n} |A_i| + \sum_{1 \leqslant i < j \leqslant n} |A_i \cap A_j| - \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|.$$

Example

How many integers in $\{1, 2, \ldots, 100\}$ are **not** divisible by 2, 3 or 5?

We want to find the numbers in $S = \{1, 2, \ldots, 100\}$ shown as the shaded region.



Example

How many integers in $\{1, 2, \dots, 100\}$ are **not** divisible by 2, 3 or 5?

Solution.

- Let A_1 be the elements of S divisible by 2.
- Let A_2 be the elements of S divisible by 3.
- Let A_3 be the elements of S divisible by 5.
- Observe $A_1 \cap A_2$ is the number of integers in S divisible by both 2 and 3 (i.e., 6), etc, and that $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$ is the answer to our problem.

Lemma

The number of positive integers divisible by k that are less than or equal to N is $\lfloor N/k \rfloor$.

By the lemma,

$$|A_1| = \left\lfloor \frac{100}{2} \right\rfloor = 50, \quad |A_2| = \left\lfloor \frac{100}{3} \right\rfloor = 33, \quad |A_3| = \left\lfloor \frac{100}{5} \right\rfloor = 20.$$

$$|A_1 \cap A_2| = \left\lfloor \frac{100}{6} \right\rfloor = 16, \quad |A_1 \cap A_3| = \left\lfloor \frac{100}{10} \right\rfloor = 10, \quad |A_2 \cap A_3| = \left\lfloor \frac{100}{15} \right\rfloor = 6$$

Finally, $|A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{100}{30} \right\rfloor = 3$. By **PIE** we have $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$ is equal to

$$|\overline{A_1 \cup A_2 \cup A_3}| = |S| - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3|$$

$$= 100 - (50 + 33 + 20) + (16 + 10 + 6) - 3 = 26.$$

Example

Determine the number of ways to deal six cards (from a deck of 52 cards) so that there is at least one Jack, at least one 8 and at least one 2.

Solution.

- Let S be the set of possible six card hands (note $|S| = \binom{52}{6}$).
- Let A_2 be the set of six card hands with **no** 2.
- Let A_8 be set of the six card hands with **no** 8.
- Let A_J be set of the six card hands with **no** Jack.
- Then

$$|A_2| = |A_8| = |A_J| = \binom{48}{6}$$

$$|A_2 \cap A_8| = |A_2 \cap A_J| = |A_8 \cap A_J| = \binom{44}{6}, \text{ and}$$

$$|A_2 \cap A_8 \cap A_J| = \binom{40}{6}.$$

- The answer is then

$$\left|\overline{A_2} \cap \overline{A_8} \cap \overline{A_J}\right| = \left|\overline{A_2 \cup A_8 \cup A_J}\right| = \binom{52}{6} - 3\binom{48}{6} + 3\binom{44}{6} - \binom{40}{6}$$

Example

Determine the number of integer solutions to $y_1 + y_2 + y_3 + y_4 \leq 70$ such that

$$1 \leq y_1 \leq 12, \quad 0 \leq y_2 \leq 10, \quad -3 \leq y_3 \leq 13, \quad 5 \leq y_4 \leq 35. \quad (*)$$

Recall the following theorem (proved using stars and bars).

Theorem

Let $n \geq 1$ and $m \geq 1$ be integers. The number of non-negative (i.e., $x_i \geq 0$) integer solutions to $x_1 + x_2 + \cdots + x_m = n$ is $\binom{n+m-1}{m-1}$, or equivalently, $\binom{n+m-1}{n}$.

Solution to the Example.

- First introduce a new (slack) variable y_5; then the number of solutions to (*) is equal to the number of solutions to $y_1 + y_2 + y_3 + y_4 + y_5 = 70$ such that

$$1 \leq y_1 \leq 12, \quad 0 \leq y_2 \leq 10, \quad -3 \leq y_3 \leq 13, \quad 5 \leq y_4 \leq 35.$$

- We introduce new variables to translate the problem to nonnegative solutions: Let

$$x_1 = y_1 - 1, \quad x_2 = y_2, \quad x_3 = y_3 + 3, \quad x_4 = y_4 - 5, \quad x_5 = y_5.$$

- Number of solutions to (*) is number of solutions to $\boxed{x_1 + x_2 + x_3 + x_4 + x_5 = 67}$ such that

$$\boxed{0 \leq x_1 \leq 11, \quad 0 \leq x_2 \leq 10, \quad 0 \leq x_3 \leq 16, \quad 0 \leq x_4 \leq 30, \quad x_5 \geq 0.}$$

Solution (continued).

We want number of solutions to $\boxed{x_1 + x_2 + x_3 + x_4 + x_5 = 67}$ such that

$$\boxed{0 \le x_1 \le 11, \quad 0 \le x_2 \le 10, \quad 0 \le x_3 \le 16, \quad 0 \le x_4 \le 30, \quad x_5 \ge 0.}$$

It is helpful to first prove the following lemma.

Lemma

Let $k \ge 1$ and $c_1, c_2, \ldots, c_k$ be integers. The number of integer solutions to $\displaystyle\sum_{i=1}^{k} x_i = n$ where $x_i \ge c_i$ for $i = 1, 2, \ldots, k$ is $\displaystyle\binom{\left(n - \sum_{i=1}^{k} c_i\right) + k - 1}{k-1}$.

Proof of Lemma.

- Let $y_i = x_i - c_i \ge 0$.
- The number of nonnegative integer solutions to $\sum_{i=1}^{k} x_i = n$ where $x_i \ge c_i$ is the same as the number of nonnegative integer solutions to

$$\sum_{i=1}^{k} y_i = n - \sum_{i=1}^{k} c_i$$

where $y_i \ge 0$ for $i = 1, 2, \ldots, k$ from which the statement follows. $\square$

Solution (continued).

We want number of solutions to $\boxed{x_1 + x_2 + x_3 + x_4 + x_5 = 67}$ such that

$$\boxed{0 \le x_1 \le 11, \quad 0 \le x_2 \le 10, \quad 0 \le x_3 \le 16, \quad 0 \le x_4 \le 30, \quad x_5 \ge 0.}$$

- Let S be the set of **all** nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 67$.
  - X_1 to be the set of solutions where $\boxed{x_1 \ge 12}$ (and $x_i \ge 0$ for $i \ne 1$),
  - X_2 to be the set of solutions where $\boxed{x_2 \ge 11}$ (and $x_i \ge 0$ for $i \ne 2$),
  - X_3 to be the set of solutions where $\boxed{x_3 \ge 17}$ (and $x_i \ge 0$ for $i \ne 3$),
  - X_4 to be the set of solutions where $\boxed{x_4 \ge 31}$ (and $x_i \ge 0$ for $i \ne 4$).
- By the Lemma, we have

$$|S| = \binom{71}{4}, \quad |X_1| = \binom{59}{4}, \quad |X_2| = \binom{60}{4}, \quad |X_3| = \binom{54}{4}, \quad |X_4| = \binom{40}{4},$$

$$|X_1 \cap X_2| = \binom{48}{4}, \quad |X_1 \cap X_3| = \binom{42}{4}, \quad |X_1 \cap X_4| = \binom{28}{4},$$

$$|X_2 \cap X_3| = \binom{43}{4}, \quad |X_2 \cap X_4| = \binom{29}{4}, \quad |X_3 \cap X_4| = \binom{23}{4},$$

$$|X_1 \cap X_2 \cap X_3| = \binom{31}{4}, \ |X_1 \cap X_2 \cap X_4| = \binom{17}{4}, \ |X_1 \cap X_3 \cap X_4| = \binom{11}{4}, \ |X_2 \cap X_3 \cap X_4| = \binom{12}{4}$$

and $|X_1 \cap X_2 \cap X_3 \cap X_4| = 0$.

Solution (continued).

By the inclusion-exclusion principle we have

$$\left|\bigcap_{i=1}^{4}\overline{X_i}\right| = |S| - \sum_{i=1}^{4}|X_i| + \sum_{1\le i<j\le 4}|X_i\cap X_j| - \sum_{1\le i<j<k\le 5}|X_i\cap X_j\cap X_k| + |X_1\cap X_2\cap X_3\cap X_4|$$

$$= \binom{71}{4} - \left[\binom{59}{4}+\binom{60}{4}+\binom{54}{4}+\binom{40}{4}\right]$$

$$+\left[\binom{48}{4}+\binom{42}{4}+\binom{28}{4}+\binom{43}{4}+\binom{29}{4}+\binom{23}{4}\right]$$

$$-\left[\binom{31}{4}+\binom{17}{4}+\binom{11}{4}+\binom{12}{4}\right]+0$$

$$= 69\,564.$$