University of Toronto Scarborough
Department of Computer and Mathematical Sciences
MATC44H3F (LEC01) - Fall 2022 - Final Exam - Practice 1

Date: Tuesday, December 20, 2022 from 9:00 to 12:00 (IC 200 & IC 204)
Instructor: Michael Cavers

First name (please write as legibly as possible within the boxes)

Last name

Student ID number

Signature: ______________________________

- **Time:** 180 minutes
- Write your solutions in this booklet (only those pages with a QR code will be graded).
- Use the back of each page for **rough** work.
- This is a closed-book exam. No aids are allowed for this exam other than those provided by the instructor. Calculators and the use of personal electronic or communication devices is prohibited.
- This exam has 13 pages with the last page being blank.
- There are 9 problems with the number of points indicated by each problem.
- The total number of points possible is 100.
- The University of Toronto's Code of Behaviour on Academic Matters (July 2019) applies to all University of Toronto Scarborough students. The Code prohibits all forms of academic dishonesty including, but not limited to, cheating, plagiarism, and the use of unauthorized aids. Students violating the Code may be subject to penalties up to and including suspension or expulsion from the University.

1. (20 points) For the following problem, you only need to provide your final answers. Correct answers are 2 points each and incorrect answers are 0 points each. Part marks is possible.

   (a) Two students are playing a Nim game with three heaps of objects of sizes 1, 2 and 2, respectively. Which player can guarantee a win?

   Solution: The first player can guarantee a win by first taking 1 object from the pile of size 1 and then balancing.

   (b) Let $n \geq 1$ and $m \geq 1$ be integers. Determine the number of ways to partition n identical objects into m labelled groups.

   Solution: By stars and bars: $\binom{n+m-1}{m-1}$, or equivalently, $\binom{n+m-1}{n}$.

   (c) Does there exist a connected bipartite graph with degree sequence $2,2,2,2,2,2,2$?

   Solution: No, otherwise there would be a bipartition (X, Y) with $\sum_{x \in X} \deg(x) = \sum_{y \in Y} \deg(y)$ implying that $2|X| = 2|Y|$. This implies that the number of vertices in the graph is even (i.e., $|V(G)| = |X| + |Y| = 2|X|$), contradicting that $|V(G)| = 7$.

   (d) For $k \geq 2$, compute $\chi(\overline{C_{2k}})$, i.e., the chromatic number of the complement of a cycle on an even number of vertices.

   Solution: The answer is k. Since $\overline{C_{2k}}$ contains a complete graph with k vertices, this is a lower bound. Label the vertices of the cycle by $2i-1, 2i$ for $i = 1, \ldots, k$. In the complement, colour both vertices $2i-1$ and $2i$ by colour i (for $i = 1, 2 \ldots, k$) to produce a colouring with k colours.

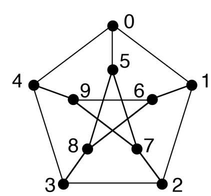
   (e) How many cycles does a connected graph with 7 vertices and 7 edges contain?

   Solution: One.

(f) Suppose G is a connected Eulerian graph with n_i vertices of degree i, $0 \leq i \leq \Delta(G)$. Determine all possible values for n_1.

Solution: The answer is $n_1 = 0$ since G cannot have any odd degree vertices. (Recall that a connected Eulerian graph must have every vertex of even degree.)

(g) Does the Petersen graph (drawn below) have a Hamilton path?



Solution: Yes, consider the path $0, 1, 2, 3, 4, 9, 6, 8, 5, 7$.

(h) Let A, B and C be finite sets with $|A \cap B| = 2$, $|A \cap C| = 6$, $|B \cap C| = 5$, $|A \cap B \cap C| = 2$ and $|A \cup B \cup C| = 13$. Determine $|A| + |B| + |C|$.

Solution: By PIE we have,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Thus, $|A| + |B| + |C| = 13 + 2 + 6 + 5 - 2 = 24$.

(i) Let $c > 0$ be a real constant. Determine the ordinary generating function for the sequence with general term $a_n = c^n$, that is, $\{1, c, c^2, c^3, \ldots, c^n, \ldots\}$.

Solution: We have

$$g(x) = \sum_{k=0}^{\infty} c^n x^n = \sum_{k=0}^{\infty} (cx)^n = \frac{1}{1 - cx}.$$

(j) Determine the ordinary generating function for the sequence with general term $a_n = \dfrac{(-1)^n}{n!}$.

Solution: We have

$$g(x) = \sum_{k=0}^{\infty} \frac{(-1)^n}{n!} x^n = \sum_{k=0}^{\infty} \frac{1}{n!} (-x)^n = e^{-x}.$$

2. (10 points) (a) State any version of the pigeonhole principle.

Solution: The pigeonhole principle is stated below.

Lemma. Let m and n be positive integers. If n pigeons are placed into m pigeonholes and $n > m$, then there exists a pigeonhole with at least two pigeons.

(b) Mike chooses k distinct numbers from the set $\{1, 2, \dots, 11\}$ at random. What values of k guarantee that two of the chosen numbers sum to 12?

Solution: The answer is $7 \leq k \leq 11$.

Let $k \geq 7$ and consider the pigeonholes

$$\{1, 11\}, \{2, 10\}, \{3, 9\}, \{4, 8\}, \{5, 7\}, \{6\}.$$

and distribute the 11 numbers among the pigeonholes by placing each number in the set according to its value. Since $k \geq 7$, by the pigeonhole principle, there is a 2-element subset with two numbers from $\{1, 2, \dots, 11\}$, and thus, these two numbers sum to 12.

If $k \leq 6$, then it is not guaranteed that two of Mike's chosen numbers sum to 12. For example, any subset of $\{1, 2, 3, 4, 5, 6\}$ of size k has this property where no two distinct elements sum to 12.

3. (10 points) Give a combinatorial proof that $\binom{2n+2}{n+1} = \binom{2n}{n-1} + 2\binom{2n}{n} + \binom{2n}{n+1}$.

Solution: We ask in how many ways can a committee of $n+1$ people be formed from a group of $2n+2$ people.

By definition of binomial coefficient, one answer is the left-side: $\binom{2n+2}{n+1}$.

Next, designate two people, X and Y, and partition the committees of size $n+1$ into

- those containing both X and Y: there are $\binom{2n}{n-1}$ such committees
- those containing X but not Y: there are $\binom{2n}{n}$ such committees
- those containing Y but not X: there are $\binom{2n}{n}$ such committees
- those containing neither X nor Y: there are $\binom{2n}{n+1}$ such committees

Summing over the four possible cases gives the right-side.

4. (10 points) How many positive integers between 1 and 2333 have a 3 in either the ones digit or tens digit but not both digits?

   For example, the numbers 1337 and 2003 both have the required property but the numbers 1333 and 33 do not. Show all your work.

Solution: We want to count 4-tuples (_, _, _, _) so that one of last two entries is equal to 3 but not both entries, and so that the corresponding number is at most 2333.

We consider four disjoint cases:

($\underline{\text{0 or 1}}$, ___, $\underline{3}$, $\underline{\text{not 3}}$): There are $2 \cdot 10 \cdot 1 \cdot 9 = 180$ possibilities.

($\underline{\text{0 or 1}}$, ___, $\underline{\text{not 3}}$, $\underline{3}$): There are $2 \cdot 10 \cdot 9 \cdot 1 = 180$ possibilities.

($\underline{2}$, ___, $\underline{3}$, $\underline{\text{not 3}}$): There are $3 \cdot 9 = 27$ where the second digit is 0, 1 or 2. If the second digit is 3, then there are 3 such numbers.

($\underline{2}$, ___, $\underline{\text{not 3}}$, $\underline{3}$): There are $3 \cdot 9 = 27$ where the second digit is 0, 1 or 2. If the second digit is 3, then there are 3 such numbers.

The total is $180 + 180 + 27 + 3 + 27 + 3 = 420$.

5. (10 points) Suppose there exists a connected plane graph G such that

   - every vertex has degree 3,
   - every face has degree either 5 or 6, and
   - there are exactly 20 faces of degree 6.

   Determine all possible values for the number of vertices in G.

   Show all your work including any theorems/lemmas you use.

Solution:

- Assume the planar representation for G has v vertices, e edges and f faces.
- Suppose x faces have degree 5. Then $f = x + 20$.
- By the handshaking lemma (for vertices) we have $3v = 2e$.
- By the handshaking lemma (for faces) we have $120 + 5x = 2e$.
- Since G is connected, by Euler's formula we have $v - e + f = 2$.
- Solving gives $f = 32$, $x = 12$, $e = 90$ and $v = 60$.

Therefore, there is only one possible value for the number of vertices in G, that is, G must have 60 vertices.

6. (10 points) (a) Solve the following recurrence relation:

$$a_n = -a_{n-1} + a_{n-2} + a_{n-3} \ (n \geq 3)$$

with initial conditions $a_0 = 0$, $a_1 = 1$ and $a_2 = 2$.

Solution: The characteristic equation is $x^3 = -x^2+x+1$. Factoring gives $(x+1)^2(x-1) = 0$, hence, $x = 1, -1, -1$ (including multiplicities). Thus, the general solution is

$$a_n = C_1(1)^n + C_2(-1)^n + C_3 n(-1)^n.$$

Using the initial conditions gives

$$\begin{array}{lcccccc} \underline{n=0:} & C_1 & + & C_2 & & & = & 0 \\ \underline{n=1:} & C_1 & - & C_2 & - & C_3 & = & 1 \\ \underline{n=2:} & C_1 & + & C_2 & + & 2\,C_3 & = & 2 \end{array}$$

This system has solution $C_1 = 1$, $C_2 = -1$, $C_3 = 1$. Therefore, the recurrence has solution

$$a_n = 1 + n(-1)^n + (-1)^{n+1}.$$

(b) Find a particular solution to the recurrence

$$a_n = -a_{n-1} + a_{n-2} + a_{n-3} + 1.$$

Solution: Since $f(n) = 1$ is already a solution to the homogeneous recurrence relation (by part (a)), we try $a_n^p = Cn$ instead. Substituting into the recurrence gives

$$Cn = -C(n-1) + C(n-2) + C(n-3) + 1$$

Thus, we require $0 = -4C + 1$, or $C = 1/4$. Hence, a particular solution is $a_n^p = \dfrac{n}{4}$.

7. (10 points) Use the generalized binomial theorem[1] to prove that

$$\sqrt{1+x} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2^{2k-1}k}\binom{2k-2}{k-1}x^k.$$

Solution: Note that $\binom{1/2}{0} = 1$ and for $k \geq 1$,

$$\begin{aligned}
\binom{1/2}{k} &= \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2}-k+1\right)}{k!} \\
&= \frac{(-1)^{k-1}(1\cdot 3\cdots(2k-3))}{2^k k!} \\
&= \frac{(-1)^{k-1}(2k-2)!}{2^k k!(2\cdot 4\cdots(2k-2))} \\
&= \frac{(-1)^{k-1}(2k-2)!}{2^k k! 2^{k-1}(1\cdot 2\cdots(k-1))} \\
&= \frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}k!(k-1)!} \\
&= \frac{(-1)^{k-1}}{2^{2k-1}k}\binom{2k-2}{k-1}.
\end{aligned}$$

The result now follows by the generalized binomial theorem since

$$\sqrt{1+x} = (1+x)^{1/2} = \sum_{k=0}^{\infty}\binom{1/2}{k}x^k = \binom{1/2}{0} + \sum_{k=1}^{\infty}\binom{1/2}{k}x^k = 1 + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2^{2k-1}k}\binom{2k-2}{k-1}x^k.$$

[1] The generalized binomial theorem states: For any nonzero $\alpha \in \mathbb{R}$, we have $(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k$.

8. (10 points) How many strings of length n can be formed using A's, B's and C's so that the number of A's is odd and the number of B's is also odd. Solve this problem using an exponential generating function.

Recall: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ and $\frac{1}{2}\left(e^x - e^{-x}\right) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$.

Solution: Consider the exponential generating function

$$G(x) = \left(\frac{x^1}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)\left(\frac{x^1}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)\left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots\right).$$

The answer to the problem is the coefficient of $\frac{x^n}{n!}$ in $G(x)$.

$$\begin{aligned} \text{Thus,} \quad G(x) &= \left(\frac{1}{2}\left(e^x - e^{-x}\right)\right) \cdot \left(\frac{1}{2}\left(e^x - e^{-x}\right)\right) \cdot e^x \\ &= \frac{1}{4}\left(e^{3x} - 2e^x + e^{-x}\right) \\ &= \frac{1}{4}\left[\left(\sum_{k=0}^{\infty} \frac{(3x)^k}{k!}\right) - 2\left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right) + \left(\sum_{k=0}^{\infty} \frac{(-x)^k}{k!}\right)\right] \end{aligned}$$

The coefficient of $\frac{x^n}{n!}$ in $G(x)$ is $a_n = \frac{1}{4}\left(3^n - 2 + (-1)^n\right)$.

9. (10 points) A pizza restaurant is having a special on three-topping pizzas with **ten** choices for the toppings. Among these ten toppings, your professor (Mike) has two favourite toppings. Your professor would like you to purchase a three-topping pizza that contains their two favourite toppings and he has no preference for the third topping. However, he will not tell you what their two favourite toppings are, thus, you must purchase multiple pizzas to make your professor happy.

   Let m be the minimum number of three-topping pizzas you must purchase to guarantee there is at least one pizza whose three toppings contain the two that your professor desires. Prove that either $m = 16$ or $m = 17$ (you do not need to determine the exact value of m).

   Hint: First prove $m \geq 16$ using any relevant theorems, e.g., those related to Steiner triple systems. Then prove $m \leq 17$ by demonstrating it is possible to purchase 17 pizzas so that every pair of toppings occurs on at least one three-topping pizza.

Solution: By the theorem in the lecture, there exists an STS(n) if and only if $n = 6k + 1$ or $n = 6k + 3$ for some non-negative integer k.

(a) We first prove $m \geq 16$. To derive a contradiction, assume $m \leq 15$. Since there are ten choices for toppings, there are $\binom{10}{2} = 45$ distinct pairs of toppings. In order to ensure every pair of toppings occurs on at least one three-topping pizza, we must order at least $45/3 = 15$ pizzas. If we order exactly 15 pizzas, then every pair of toppings must appear on *exactly* one pizza. But this would imply the existence of an STS(10), contradicting the theorem in the lecture as $n = 10$ is not of the required form. Thus, it must be that $m > 15$, i.e., $m \geq 16$.

(b) We next prove $m \leq 17$. Let $X = \{1, 2, \ldots, 10\}$ and consider a STS(9) on $X \backslash \{10\}$. Since $n = 9$ is of the form $6k + 3$, there exists an STS(9) by the theorem from the lecture.

Note: We only require the existence of an STS(9) using 12 blocks, we do not need to actually construct one for this problem! However, we give an explicit example here:

$$\mathcal{B} = \{\{1,2,3\}, \{4,5,6\}, \{7,8,9\}, \{1,4,7\}, \{2,5,8\}, \{3,6,9\},$$

$$\{1,5,9\}, \{2,6,7\}, \{3,4,8\}, \{3,5,7\}, \{1,6,8\}, \{2,4,9\}\}.$$

Now we add in the triples

$$\mathcal{B}' = \{\{10,1,2\}, \{10,3,4\}, \{10,5,6\}, \{10,7,8\}, \{10,9,1\}\}.$$

The block set $\mathcal{B} \cup \mathcal{B}'$ gives an example of how we can order 17 pizzas so that every pair of toppings occurs on at least one pizza: every pair ij with $i, j \in \{1, 2, \ldots, 9\}$ is in exactly one triple in $\mathcal{B}$ and every pair ij with $i = 10$ (or $j = 10$) is in at least one triple in $\mathcal{B}'$. Thus, $m \leq 17$.

(Continue your solution on the next page.)

(Extra space for Question 9)

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