University of Toronto Scarborough
Department of Computer and Mathematical Sciences

MATC44 Fall 2022 - Practice Midterm 2

Date: Wednesday, November 2, 2022 from 14:10 - 16:00

Signature: ____

- Time: 110 minutes (class time)
- Write your solutions in this booklet (only those pages with a QR code will be graded).
- Use the back of each page for **rough** work.
- This is a closed-book test. No aids are allowed for this midterm. Calculators and the use of personal electronic or communication devices is prohibited.
- This test has 11 pages with the last two pages being blank.
- There are 8 problems with the number of points indicated by each problem.
- The total number of points possible on this test is 40.

DO NOT OPEN THIS BOOKLET UNTIL INSTRUCTED TO DO SO.

1. (5 points) Two people play a game. The game starts with an empty pile and the players alternate turns as follows. When it is their turn, a player may add either 1, 3 or 5 coins to the pile. The person who adds the 120th coin to the pile is the winner. Determine whether it is the first or second player who can guarantee a win in this game. What is a winning strategy?

Solution: The second player has a winning strategy by placing $6-k$ coins in the pile whenever the first player places k coins in the pile. Since $k \in \{1, 3, 5\}$ we also have $(6-k) \in \{1, 3, 5\}$, thus placing $6-k$ coins in the pile is a valid move. Since the pile starts empty, after the second player's nth turn, there will be $(k+(6-k))n = 6n$ coins in the pile (i.e., after both players have had n turns each). Since 120 is divisible by 6, the second player will win on their 20th turn.

2. (5 points) (a) State and prove any version of the pigeonhole principle.

Solution: The pigeonhole principle is stated below.

Lemma. Let m and n be positive integers. If n pigeons are placed into m pigeonholes and $n > m$, then there exists a pigeonhole with at least two pigeons.

Proof.

- Suppose n pigeons are placed into m pigeonholes and that $n > m$.
- To derive a contradiction, assume that every pigeonhole contains at most one pigeon.
- Then there are at most m pigeons since there are m pigeonholes.
- But we assumed that there are n pigeons and that $n > m$, so this is impossible.
- Therefore, there must be a pigeonhole with more than one pigeon.

(b) Mike chooses k distinct numbers from the set $\{1, 2, \ldots, 11\}$ at random. What values of k guarantee that two of the chosen numbers sum to 12?

Solution: Suppose the numbers randomly chosen by Mike form the set S where $|S| = k$ and $S \subseteq \{1, 2, \ldots, 11\}$. If $k \leq 6$, it is **not** guaranteed that two numbers from S sum to 12, for example, if $S \subseteq \{1, 2, 3, 4, 5, 6\}$ then every pair of numbers has sum at most 11. If $7 \leq k \leq 11$, then no matter what S is, it **is** guaranteed that there are two numbers in S whose sum is 12. We apply the pigeonhole principle with k pigeons and six pigeonholes: $\{1, 11\}, \{2, 10\}, \{3, 9\}, \{4, 8\}, \{5, 7\}, \{6\}$. Since $k \geq 7$, by the pigeonhole principle, one of the 2-element subsets $\{1, 11\}, \{2, 10\}, \{3, 9\}, \{4, 8\}, \{5, 7\}$ is a subset of S and this gives two numbers whose sum is 12.

3. (5 points) How many positive integers with at most ten digits have exactly two zeros?

   You may leave your answer in a combinatorial form using binomial coefficients and summations.

Solution: Let S_k be the set of positive integers with k digits and exactly two zeros.

If $k \leq 2$, then $|S_k| = 0$.

For $k \geq 3$, each element in S_k can be formed by first selecting where the zeros are located: there are $\binom{k-1}{2}$ ways to do this since the first digit cannot be zero. Each of the remaining $k-2$ spots can be any digit from 1 to 9 (inclusive) giving 9^{k-2} possibilities. Thus, $|S_k| = 9^{k-2}\binom{k-1}{2}$.

Summing over possible values of k (from $k = 3$ to $k = 10$ gives the number of positive integers with at most ten digits and having exactly two zeros equal to

$$\sum_{k=3}^{10} 9^{k-2}\binom{k-1}{2}.$$

4. (5 points) Give a combinatorial proof (e.g., using committee selection) of the following identity:

$$\binom{2n+2}{n+1} = 2\binom{2n}{n} + 2\binom{2n}{n-1}.$$

Solution: $\binom{2n+2}{n+1}$ is the number of ways to choose a committee of $n+1$ people from a group of $2n+2$ people. Designate two people, X and Y, and partition the committees of size $n+1$ into

- those containing both X and Y: there are $\binom{2n}{n-1}$ such committees
- those containing X but not Y: there are $\binom{2n}{n}$ such committees
- those containing Y but not X: there are $\binom{2n}{n}$ such committees
- those containing neither X nor Y: there are $\binom{2n}{n+1}$ such committees

Since $\binom{2n}{n+1} = \binom{2n}{n-1}$ we have

$$\binom{2n+2}{n+1} = 2\binom{2n}{n} + 2\binom{2n}{n-1}.$$

5. (5 points) Use stars and bars to prove the following theorem.

Theorem. Let $n \geq 1$ and $m \geq 1$ be integers. The number of ways to partition n identical objects into m labelled groups is $\binom{n+m-1}{m-1}$.

Note that some groups may be empty.

Solution:

- Note that there is a bijection between partitions of identical objects into labelled groups and arrangements and of stars and bars (here stars represent the identical objects).
- Because we want m groups, we require $m-1$ bars for the partition:

$$\underbrace{\text{group 1} \mid \text{group 2} \mid \text{group 3} \mid \quad \cdots \quad \mid \text{group } m}_{\text{for } m \text{ groups we need } m-1 \text{ bars to act as separators}}$$

- It suffices to count the number of arrangements of n stars and $m-1$ bars in a row.
- There are a total of $n+m-1$ symbols to arrange.
- To count the number of arrangements, we choose the positions of the bars in the row:

$$\underbrace{\_\_\ \_\_\ \_\_\quad \cdots \quad \_\_\ \_\_\ \_\_}_{\text{from } n+m-1 \text{ spots, choose } m-1 \text{ spots for the } \textbf{bars}}$$

- Therefore, there are $\binom{n+m-1}{m-1}$ arrangements.

6. (5 points) Let $n \geq 1$ be a fixed integer. Use the Binomial Theorem to show that

$$n(n+1)2^{n-2} = \sum_{k=1}^{n} k^2 \binom{n}{k}.$$

Solution: By the Binomial Theorem with $y = 1$ we have

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Differentiating with respect to x gives

$$n(1+x)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k x^{k-1}.$$

Multiplying by x gives

$$nx(1+x)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k x^{k}.$$

Differentiating with respect to x gives

$$n(1+x)^{n-1} + nx(n-1)(1+x)^{n-2} = \sum_{k=1}^{n} \binom{n}{k} k^2 x^{k-1}.$$

Substituting $x = 1$ now gives

$$n2^{n-1} + n(n-1)2^{n-2} = \sum_{k=1}^{n} k^2 \binom{n}{k}$$

implying

$$n(n+1)2^{n-2} = \sum_{k=1}^{n} k^2 \binom{n}{k}.$$

7. (5 points) A k-**regular graph** is one in which every vertex has degree k.
   (a) How many edges does a k-regular graph have? (in terms of k and the number of vertices.)
   (b) Let G be a k-regular graph with $k \geq 1$ that is also a bipartite graph with bipartition (A, B)[1]. Prove that $|A| = |B|$.

Solution: (a) Let G be a k-regular graph on n vertices. Then $\deg(v) = k$ for all $v \in V(G)$. By the Handshaking Lemma,

$$2|E(G)| = \sum_{v \in V(G)} \deg(v) = \sum_{v \in V(G)} k = kn.$$

Thus, $|E(G)| = kn/2$.

(b) Since each vertex of G has degree k, we have that

$$\sum_{a \in A} \deg(a) = \underbrace{k + k + k + \cdots + k}_{|A| \text{ times}} = k|A|$$

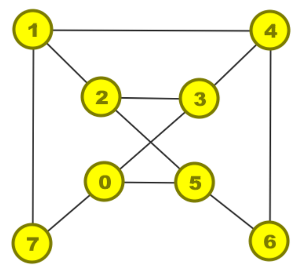
Similarly, $\sum_{b \in B} \deg(b) = k|B|$. But every edge in G has exactly one endpoint in A and one endpoint in B, thus,

$$\sum_{a \in A} \deg(a) = \sum_{b \in B} \deg(b) = |E(G)|.$$

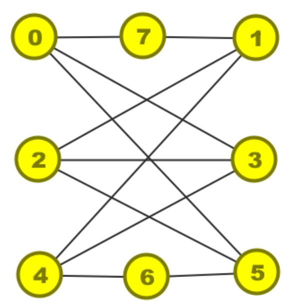
Thus, $|E(G)| = k|A|$ and $|E(G)| = k|B|$, implying, $|A| = |B|$.

[1] That is, $V(G) = A \cup B$, $A \cap B = \emptyset$, and all edges of G have exactly one endpoint in A and one endpoint in B.

8. (5 points) Is the graph depicted below a planar graph? Justify your answer.



Solution: No. We redraw the graph as follows:



It is now clear that the graph is a subdivision of $K_{3,3}$.

By Kuratowski's Theorem, the graph is not planar.

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