How to count?

The "stars and bars" technique

Common types of examples to think about

From last week (boxed questions require a new technique!).

(1) There are 11 **distinct** students in a classroom. In how many ways can we...

- (a) choose 8 students and arrange them in a row.
- (b) choose 8 students and place into two equal-sized rows labelled "Row 1" & "Row 2".
- (c) choose 8 students and place them into two equal-sized unlabelled rows.
- (d) choose 8 students to create a group.
- (e) choose 8 students and place into equal-sized groups labelled "Group 1" & "Group 2".
- (f) choose 8 students and place them into two equal-sized unlabelled groups.

(2) There are 11 **identical** dimes in a coin bag. In how many ways can we...

- (a) choose 8 dimes and arrange them in a row.
- (b) choose 8 dimes to create a group.
- (c) choose 8 dimes and give them to 3 people where some people might not get any.
- (d) choose 8 dimes and give them to 3 people where each person gets at least one.

(3) There are 11 **types** of coins in a coin bag (with an unlimited number of each type of coin). In how many ways can we...

- (a) choose 8 coins and arrange them in a row.
- (b) choose 8 coins and place them into two equal-sized rows labelled "Row 1" & "Row 2".
- (c) choose 8 coins and place them into two equal-sized unlabelled rows.
- (d) choose 8 coins to create a group.
- (e) choose 8 coins and place into equal-sized groups labelled "Group 1" & "Group 2".
- (f) choose 8 coins and place them into two equal-sized unlabelled groups.

Answers

(i) Answers

(a) $11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4$.
(b) $(11 \times 10 \times 9 \times 8) \times (7 \times 6 \times 5 \times 4)$.
(c) $\frac{1}{2} \times (11 \times 10 \times 9 \times 8) \times (7 \times 6 \times 5 \times 4)$. Note: it is $\frac{1}{2}$ because we cannot distinguish between two unlabelled rows of the **same** length.
(d) $\binom{11}{8}$ (by definition of binomial coefficient since objects are distinct).
(e) $\binom{11}{4}\binom{7}{4}$ or $\binom{11}{8}\binom{8}{4}$.
(f) $\frac{1}{2}\binom{11}{4}\binom{7}{4}$ or $\frac{1}{2}\binom{11}{8}\binom{8}{4}$. Note: it is $\frac{1}{2}$ because we cannot distinguish between two unlabelled groups of the **same** size.

(ii) Answers

(a) 1 (because the objects are identical, hence, indistinguishable from one-another).
(b) 1 (because the objects are identical, hence, indistinguishable from one-another).
(c) $\binom{10}{2}$ (by stars and bars **which we will soon learn!**).
(d) $\binom{7}{2}$ (by stars and bars; place bars in 7 slots between 8 stars).

(iii) Answers

(a) 11^8 (in each of the eight slots of the row, we have 11 types that could occupy it and repetition of a type is allowed).
(b) $11^4 \times 11^4 = 11^8$.
(c) $11^4 + \frac{1}{2}(11^4)(11^4 - 1)$ (case work: rows are identical plus rows are not identical).
(d) $\binom{18}{10}$ or $\binom{18}{8}$ (stars and bars: there are 10 bars that separate stars/coins into 11 types).
(e) $\binom{14}{10}\binom{14}{10}$ (stars and bars for each separate group)
(f) $\binom{14}{10} + \frac{1}{2}\binom{14}{10}\left[\binom{14}{10} - 1\right]$ (case work: groups are identical plus groups are not identical).

How do we distribute identical items?

Question.
In how many ways can we distribute **four** (identical) dimes among **three** people?

Intuition. People are distinct (they have names).

We list every possibility:

Person 1	Person 2	Person 3
4	0	0
3	1	0
3	0	1
2	2	0
2	0	2
2	1	1
1	3	0
1	0	3
1	2	1
1	1	2
0	4	0
0	0	4
0	3	1
0	1	3
0	2	2

Person 1	Person 2	Person 3
****		
***	*	
***		*
**	**	
**		**
**	*	*
*	***	
*		***
*	**	*
*	*	**
	****	
		****
	***	*
	*	***
	**	**

Observations - Part 1

Person 1	Person 2	Person 3
4	0	0
3	1	0
3	0	1
2	2	0
2	0	2
2	1	1
1	3	0
1	0	3
1	2	1
1	1	2
0	4	0
0	0	4
0	3	1
0	1	3
0	2	2

Person 1	Person 2	Person 3
****		
***	*	
***		*
**	**	
**		**
**	*	*
*	***	
*		***
*	**	*
*	*	**
	****	
		****
	***	*
	*	***
	**	**

- **Every** possible of arrangement of four stars and two bars is accounted for.

There is a **bijection** between distributions and arrangements of **four** stars & **two** bars.

- Thus, it suffices to count the number of ways to arrange four stars and two bars!
- To count this, we need to choose the positions of the bars (or equivalently, the stars) in the row:

$$\underbrace{\_\_\_\;\_\_\_\;\_\_\_\;\_\_\_\;\_\_\_\;\_\_\_}_{\text{choose two spots for the bars}} \quad \text{or equivalently,} \quad \underbrace{\_\_\_\;\_\_\_\;\_\_\_\;\_\_\_\;\_\_\_\;\_\_\_}_{\text{choose four spots for the stars}}$$

- **Final answer:** $\binom{6}{2}$, or equivalently, $\binom{6}{4}$.

The stars and bars technique gives the following theorem.

Theorem

Let $n \geq 1$ and $m \geq 1$ be integers. The number of ways to partition n identical objects into m labelled groups is $\binom{n+m-1}{m-1}$, or equivalently, $\binom{n+m-1}{n}$.

Proof.

- Set up a bijection with arrangements of stars and bars (stars $=$ objects).
- Because we want m groups, we require $m-1$ bars for the partition:

$$\underbrace{\text{group } 1 \mid \text{group } 2 \mid \text{group } 3 \mid \quad \cdots \quad \mid \text{group } m}_{\text{for } m \text{ groups we need } m-1 \text{ bars to act as separators}}$$

- We now arrange the n stars and $m-1$ bars in a row.
- There are a total of $n+m-1$ symbols to arrange.
- To count the number of arrangements, we choose the positions of the bars (or equivalently, the stars) in the row:

$$\underbrace{\_\_\_ \ \ \_\_\_ \ \ \_\_\_ \quad \ldots \quad \_\_\_ \ \ \_\_\_ \ \ \_\_\_}_{\text{from } n+m-1 \text{ spots, choose } m-1 \text{ spots for the } \textbf{bars}} \quad \text{or} \quad \underbrace{\_\_\_ \ \ \_\_\_ \ \ \_\_\_ \quad \ldots \quad \_\_\_ \ \ \_\_\_ \ \ \_\_\_}_{\text{from } n+m-1 \text{ spots, choose } n \text{ spots for the } \textbf{stars}}$$

- There are $\binom{n+m-1}{m-1}$, or equivalently, $\binom{n+m-1}{n}$ arrangements.

Observations - Part 2

Person 1	Person 2	Person 3
4	0	0
3	1	0
3	0	1
2	2	0
2	0	2
2	1	1
1	3	0
1	0	3
1	2	1
1	1	2
0	4	0
0	0	4
0	3	1
0	1	3
0	2	2

Person 1	Person 2	Person 3
****		
***	*	
***		*
**	**	
**		**
**	*	*
*	***	
*		***
*	**	*
*	*	**
	****	
		****
	***	*
	*	***
	**	**

- **Every** possible way to write the number **four** as an ordered sum of **three** non-negative integers is accounted for (e.g., $4+0+0=4$, $3+1+0=4$, $3+0+1=4$, $2+2+0=4$, etc.).
- That is, the table lists **every** non-negative integer solution to:

$$x_1 + x_2 + x_3 = 4 \quad \text{where } x_1, x_2, x_3 \geq 0 \quad \text{(Eq. 1)}$$

There is a **bijection** between solutions to Eq. 1 and arrangements of stars & bars.

Counting non-negative integer solutions

The stars and bars technique gives the following theorem.

Theorem

Let $n \geq 1$ and $m \geq 1$ be integers. The number of ways to write n as an <u>ordered</u> sum of m non-negative integers is $\binom{n+m-1}{m-1}$, or equivalently, $\binom{n+m-1}{n}$.

Proof. By stars and bars.

An equivalent formulation is the following.

Theorem

Let $n \geq 1$ and $m \geq 1$ be integers. The number of non-negative (i.e., $x_i \geq 0$) integer solutions to $x_1 + x_2 + \cdots + x_m = n$ is $\binom{n+m-1}{m-1}$, or equivalently, $\binom{n+m-1}{n}$.

Proof. By stars and bars.

How do we distribute <u>identical</u> items with each group nonempty?

Question.
In how many ways can we distribute **four** (identical) dimes among **three** people **so that every person gets at least one dime**?

Intuition. People are distinct (they have names).

We list every possibility:

Person 1	Person 2	Person 3
2	1	1
1	2	1
1	1	2

Person 1	Person 2	Person 3
**	*	*
*	**	*
*	*	**

- In this case, not every arrangement of four stars and two bars is accounted for.
- We cannot have two adjacent bars or a bar in the end positions.
- How can we count the number of arrangements of **four** stars and **two** bars so that no two bars are adjacent and the end positions are not bars?
- We apply an "**interlacing**" or "**weaving**" technique where we first place all of the stars, and then place the bars between them.
- Since the end positions are occupied by stars, there are only three spaces available for the bars:

$$\boxed{*}\ \_\ \boxed{*}\ \_\ \boxed{*}\ \_\ \boxed{*}.$$

- **Final answer:** $\binom{3}{2}$ (i.e., from the three empty spots, we select two for the bars).

Theorem

Let $n \geq 1$ and $m \geq 1$ be integers. The number of ways to partition n identical objects into m labelled groups so that every group is non-empty is $\binom{n-1}{m-1}$.

Proof.

- Set up a bijection with arrangements of stars and bars (stars = objects) where no two bars are adjacent and the end positions are not bars.
- Because we want m groups, we require $m-1$ bars for the partition:

$$\underbrace{\text{group } 1 \mid \text{group } 2 \mid \text{group } 3 \mid \quad \cdots \quad \mid \text{group } m}_{\text{for } m \text{ groups we need } m-1 \text{ bars to act as separators}}$$

- We now arrange the n stars and $m-1$ bars in a row with the restriction above.
- To count the number of such arrangements, first place the stars, then choose spots between them for the bars:

$$\underbrace{\boxed{*} \;\_\_\; \boxed{*} \;\_\_\; \boxed{*} \;\_\_\; \quad \cdots \quad \_\_\; \boxed{*}}_{\text{from } n-1 \text{ empty spots, choose } m-1 \text{ for the } \textbf{bars}}$$

- There are $\binom{n-1}{m-1}$ such arrangements.

Observations - Part 3

Person 1	Person 2	Person 3
2	1	1
1	2	1
1	1	2

Person 1	Person 2	Person 3
**	*	*
*	**	*
*	*	**

- **Every** possible way to write the number **four** as an <u>ordered</u> sum of **three** **POSITIVE** integers is accounted for:

$$2 + 1 + 1 = 4$$

$$1 + 2 + 1 = 4$$

$$1 + 1 + 2 = 4$$

- That is, the table lists **every** positive integer solution to:

$$x_1 + x_2 + x_3 = 4 \quad \text{where } x_1, x_2, x_3 \geq 1 \quad \text{(Eq. 2)}$$

- There is a **bijection** between solutions to Eq. 2 and arrangements of stars & bars where no two bars are adjacent and the end positions are not bars.

Counting positive integer solutions

The stars and bars technique gives the following theorem.

Theorem

Let $n \geq 1$ and $m \geq 1$ be integers. The number of ways to write n as an ordered sum of m positive integers is $\binom{n-1}{m-1}$.

Proof. By stars and bars using interlacing/weaving.

An equivalent formulation is the following.

Theorem

Let $n \geq 1$ and $m \geq 1$ be integers. The number of positive (i.e., $x_i > 0$) integer solutions to $x_1 + x_2 + \cdots + x_m = n$ is $\binom{n-1}{m-1}$.

Proof. By stars and bars using interlacing/weaving.

Example

How many integer solutions do each of the following have?

(a) $x_1 + x_2 + x_3 + x_4 = 7$ with $x_i \geq 0$.

(b) $x_1 + x_2 + x_3 + x_4 = 7$ with $x_i > 0$.

(c) $x_1 + x_2 + x_3 + x_4 = 7$ with $0 \leq x_i \leq 9$.

(d) $x_1 + x_2 + x_3 + x_4 \leq 7$ with $0 \leq x_i \leq 9$.

(e) $x_1 + x_2 + x_3 + x_4 \leq 15$ with $x_i \geq -10$.

(f) $x_1 + x_2 + x_3 + x_4 = 13$ with $0 \leq x_i \leq 9$.

Answers.

(a) By the theorem, $\binom{7+4-1}{4-1} = \binom{10}{3}$.

(b) By the theorem, $\binom{7-1}{4-1} = \binom{6}{3}$.

(c) **Observation:** x_i can not take on values larger than 7.
Therefore, (c) is the same as the number of solutions to

$$\boxed{x_1 + x_2 + x_3 + x_4 = 7 \text{ with } x_i \geq 0}$$

which we found in (a) to be $\binom{7+4-1}{4-1} = \binom{10}{3}$.

(d) $x_1 + x_2 + x_3 + x_4 \leq 7$ with $0 \leq x_i \leq 9$.

- **Observation:** x_i can not take on values larger than 7.
- Therefore, (d) is the same as the number of solutions to

$$\boxed{x_1 + x_2 + x_3 + x_4 \leq 7 \text{ with } x_i \geq 0 \qquad \text{(Eq. 1)}}$$

- To solve this, we "**add slack**" (a common technique in linear optimization).
- We add a new variable x_5 so that we can bump up the left side to be equal to 7:

$$\boxed{x_1 + x_2 + x_3 + x_4 + x_5 = 7 \text{ with } x_i \geq 0 \qquad \text{(Eq. 2)}}$$

- Both Eq. 1 and Eq. 2 have the **SAME** number of solutions! (why?)
- Therefore, applying the theorem to Eq. 2, the answer is $\binom{7+5-1}{5-1} = \binom{11}{4}$.

(e) $x_1 + x_2 + x_3 + x_4 \leq 15$ with $x_i \geq -10$.

- We introduce new variables to translate the problem to nonnegative solutions.
- Let $y_i = x_i + 10$ (so that $y_i \geq 0$ since $x_i \geq -10$).
- With this substitution, we now want the number of solutions to:

$$\boxed{(y_1 - 10) + (y_2 - 10) + (y_3 - 10) + (y_4 - 10) \leq 15 \text{ with } y_i \geq 0}$$

  or equivalently

$$\boxed{y_1 + y_2 + y_3 + y_4 \leq 55 \text{ with } y_i \geq 0 \qquad \text{(Eq. 1)}}$$

- Now "**add slack**":

$$\boxed{y_1 + y_2 + y_3 + y_4 + y_5 = 55 \text{ with } y_i \geq 0 \qquad \text{(Eq. 2)}}$$

- Both Eq. 1 and Eq. 2 have the **SAME** number of solutions! (why?)
- Therefore, applying the theorem to Eq. 2, the answer is $\binom{55+5-1}{5-1} = \binom{59}{4}$.

(f) $x_1 + x_2 + x_3 + x_4 = 13$ with $0 \leq x_i \leq 9$.

- This problem is different from (c) since we could have $x_i > 9$ for some i.
- **Technique:** First count the number of solutions to

  $$\boxed{x_1 + x_2 + x_3 + x_4 = 13 \text{ with } x_i \geq 0}$$

  (which is $\binom{13+4-1}{4-1}$) then **subtract** the solutions that violate the $x_i \leq 9$ property (i.e., those with $x_i \geq 10$ for some $1 \leq i \leq 4$).
- Observe that if a component, say x_1, is at least 10, then all other components must be at most 9 in order to get a sum of 13.
- This give four cases:
- Case 1. $\boxed{x_1 \geq 10}$
  - Let $y_1 = x_1 - 10$ (so that $y_1 \geq 0$) and substitute to get

    $$\boxed{(y_1 + 10) + x_2 + x_3 + x_4 = 13 \text{ with } y_1, x_2, x_3, x_4 \geq 0}$$

    or equivalently

    $$\boxed{y_1 + x_2 + x_3 + x_4 = 3 \text{ with } y_1, x_2, x_3, x_4 \geq 0}$$

    This has $\binom{3+4-1}{4-1} = \binom{6}{3}$ solutions.
- The other three cases where either $x_2 \geq 10$, $x_3 \geq 10$ or $x_4 \geq 10$ are identical to the first case and each has $\binom{6}{3}$ solutions.
- Therefore, the original equation in (f) has $\binom{13+4-1}{4-1} - 4\binom{6}{3}$ solutions.