University of Toronto Scarborough
Department of Computer and Mathematical Sciences
MATC44H3F (LEC01) - Fall 2022 - Final Exam - Practice 2

Date: Tuesday, December 20, 2022 from 9:00 to 12:00 (IC 200 & IC 204)
Instructor: Michael Cavers

First name (please write as legibly as possible within the boxes)

Last name

Student ID number

Signature: ______________________________

- **Time:** 180 minutes
- Write your solutions in this booklet (only those pages with a QR code will be graded).
- Use the back of each page for **rough** work.
- This is a closed-book exam. No aids are allowed for this exam other than those provided by the instructor. Calculators and the use of personal electronic or communication devices is prohibited.
- This exam has 13 pages with the last page being blank.
- There are 9 problems with the number of points indicated by each problem.
- The total number of points possible is 100.
- The University of Toronto's Code of Behaviour on Academic Matters (July 2019) applies to all University of Toronto Scarborough students. The Code prohibits all forms of academic dishonesty including, but not limited to, cheating, plagiarism, and the use of unauthorized aids. Students violating the Code may be subject to penalties up to and including suspension or expulsion from the University.

1. (20 points) For the following problem, you only need to provide your final answers. Correct answers are 2 points each and incorrect answers are 0 points each. Part marks is possible.

   (a) Two people play a game. The game starts with an empty pile and the players alternate turns as follows. When it is their turn, a player may add either 1 or 2 coins to the pile. The person who adds the 6th coin to the pile is the winner. Which player (first or second) can guarantee a win in this game?

   Solution: The second player has a winning strategy by placing $3 - k$ coins in the pile whenever the first player places k coins in the pile. Since $k \in \{1, 2\}$ we also have $(3-k) \in \{1, 2\}$, thus placing $3-k$ coins in the pile is a valid move. Since the pile starts empty, after the second player's nth turn, there will be $(k + (3 - k))n = 3n$ coins in the pile (i.e., after both players have had n turns each). Since 6 is divisible by 3, the second player will win on their 2nd turn.

   (b) **True or false?** Let m and k be positive integers. If $mk + 1$ pigeons are placed into m pigeonholes, then there exists a pigeonhole with at least $k + 1$ pigeons.

   Solution: This is true and is the strong form of the pigeonhole principle.

   (c) How many arrangements are there of the letters in MISSISSIPPI with no consecutive Ps?

   Solution: The answer is $\binom{9}{4,4,1}\binom{10}{2}$.
   Place all the letters except the two Ps; there are $\binom{9}{4,4,1}$ ways to do this. Now from the 10 spaces in-between the letters, choose two for the Ps; there are $\binom{10}{2}$ ways to do this.

   (d) How many nonnegative integer solutions are there to the equation $x_1+x_2+x_3+x_4+x_5 \leq 40$?

   Solution: $\binom{45}{5}$

   (e) Compute the generalized binomial coefficient $\binom{1/3}{3}$. Write your answer as a reduced fraction.

   Solution: $\binom{1/3}{3} = \dfrac{1/3(1/3-1)(1/3-2)}{6} = \dfrac{5}{81}$.

(f) Solve the recurrence relation $a_n = 4a_{n-1}$ $(n \geq 1)$ with $a_0 = 3$.

Solution: $a_n = 3 \cdot 4^n$

(g) Determine the general solution of the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$, $(n \geq 2)$.

Solution: The characteristic equation $x^2 = 4x - 4$, thus, $x = 2, 2$. The general solution of the recurrence is $a_n = C_1(2)^n + C_2 n(2)^n$.

(h) Find a closed form formula for the nth term of the sequence (say a_n) with exponential generating function $G(x) = \dfrac{e^{3x} - e^{-x}}{4}$.

Solution: $a_n = \dfrac{3^n - (-1)^n}{4}$.

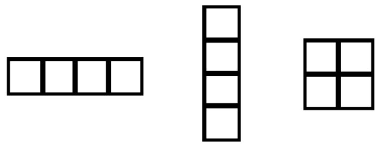
(i) Give an example of a Steiner triple system of order three, i.e., STS(3).

Solution: Let $X = \{1, 2, 3\}$ and $\mathcal{B} = \{123\}$. Then $(X, \mathcal{B})$ is an STS(3).

(j) How many Steiner-triple systems are there of order 10?

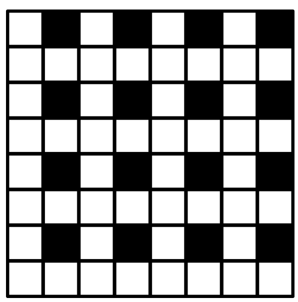
Solution: Zero.

2. (10 points) Suppose an 8×8 chessboard is perfectly tiled using some 2×2 and 1×4 tiles.



If the number of 2×2 tiles used is x and the number of 1×4 tiles used is y (thus, $x + y = 16$), show that x cannot be odd.

Solution: To derive a contradiction, suppose x is odd. Recolour the 8×8 chessboard as follows:



Observe that each 2×2 tile covers exactly one black square and each 1×4 tile covers exactly zero or two black squares. Suppose the number of 1×4 tiles covering exactly zero black squares is u and the number of 1×4 tiles covering exactly two black squares is v. Then $x + 0u + 2v = 16$, and hence, $x = 2(8 - v)$ implying that x is even, a contradiction. Therefore, x cannot be odd.

3. (10 points) Let $n \geq 4$. Give a combinatorial proof for the following identity:

$$\sum_{k=0}^{4} \binom{n}{k} \binom{n-k}{4-k} = \binom{n}{4} 2^4.$$

Solution: We count the number of ways to create a committee of four people each of who may or may not be wearing a hat.

From n people, we select four for the committee. Each of the four people can either wear a hat or not wear a hat (giving two choices for each). This gives $\binom{n}{4}2^4$ such committees of size four where some members may be wearing hats.

Alternatively, let k be the number of people wearing hats on the committee and from n people choose k of them to be on the committee and also be the hat wearers on the committee. Now, from the $n-k$ remaining people, choose $4-k$ to be on the committee and be the non-hat wearing people on the committee. Summing from $k=0$ to 4 gives the left side.

4. (10 points) Suppose G is a graph with minimum degree t where $t \geq 2$. Prove that G has a vertex belonging to at least $\binom{t}{2}$ distinct cycles. (The cycles may share some edges and vertices.)

   Hint: Apply the extremal principle.

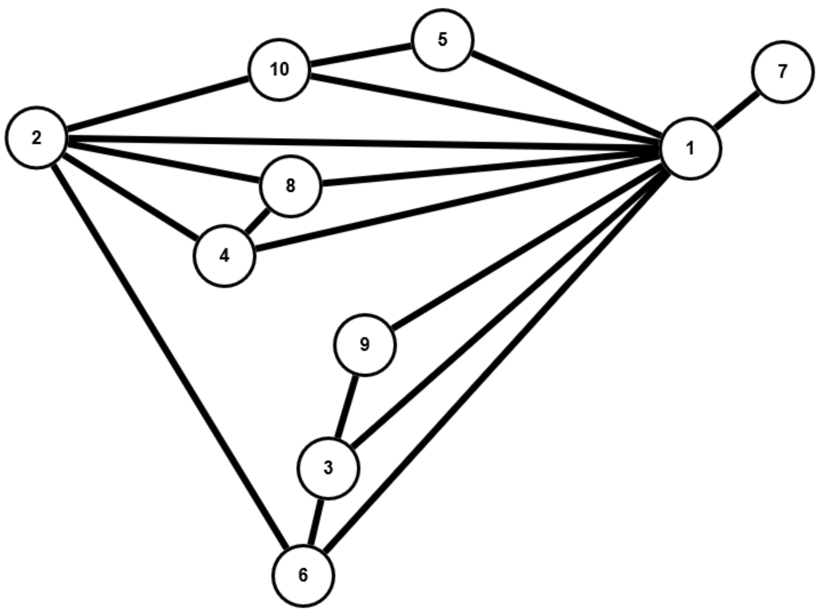
Solution:

- Let $P = v_1 v_2 \cdots v_{k-1} v_k$ be a longest path in G.
- Note if w (with $w \neq v_2$) is adjacent to v_1 and w is not on the path P, then $P' = w\, v_1 v_2 \cdots v_k$ is a longer path in G contradicting that P is a longest path.
- Thus, all the neighbours of v_1 are vertices on the path P implying that v_1 is only adjacent to vertices in the set $\{v_2, v_3, \ldots, v_k\}$.
- Since $\deg(v_1) \geq \delta(G) = t$, it follows that v_1 is adjacent to t vertices in $\{v_2, v_3, \ldots, v_k\}$, call these $w_1, w_2, \ldots, w_t$.
- Now for any choice of w_i and w_j ($i \neq j$, $1 \leq i, j \leq t$), there is a path between w_i and w_j containing vertices and edges of the subpath $v_2 v_3 \cdots v_k$ of P, say $w_i v_\ell v_{\ell+1} \cdots v_m w_j$.
- Then, $C = v_1 w_i \cdots w_j v_1$ is a cycle in G.
- Since each pair of w_i and w_j give a cycle in G different than the other pairs, and there are $\binom{t}{2}$ such pairs (w_i, w_j) for $1 \leq i, j \leq t$, it follows that G has at least $\binom{t}{2}$ distinct cycles.
- Furthermore, each of the constructed cycles contains the vertex v_1.

5. (10 points) For $n \geq 1$, define the graph H_n to have vertex set $V(H_n) = \{1, 2, \ldots, n\}$ and suppose

$$uv \in E(G) \text{ if and only if either } v|u \text{ or } u|v.$$

Here, $x|y$ means x divides y. For example, H_{10} is shown below.



(a) Determine the degree of the vertex labelled by 1 as a function of n.

(b) Determine the degree of the vertex labelled by 2 as a function of n.

(c) Prove that H_{14} is planar.

(d) Prove that H_{16} is non-planar.

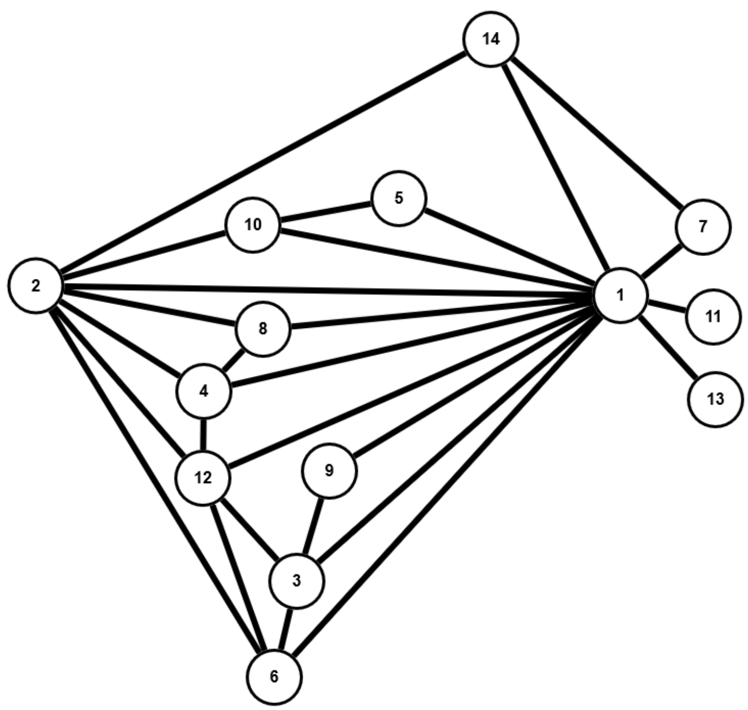
(Continue your solution on the next page.)

(Extra space for Question 5)

Solution: (a) Since $1|k$ for all integers k, the degree of the vertex labelled by 1 is equal to $n-1$.

(b) If n is even, then the degree of vertex 2 is $n/2$. If n is odd, then the degree of vertex 2 is $(n-1)/2$. Thus, the degree of the vertex labelled by 2 is equal to $\lfloor n/2 \rfloor$.

(c) The graph H_{14} is shown below. Since there are no edge crossings, it is planar.



(d) The graph H_{16} contains a subgraph isomorphic to K_5, namely, use the vertices $\{1, 2, 4, 8, 16\}$. By Kuratowski's Theorem, H_{16} is not planar.

6. (10 points) A graph G is k-critical if $\chi(G) = k$ and $\chi(H) < \chi(G)$ for every proper subgraph H of G (proper here means $H \neq G$).

   (a) Prove K_n is n-critical for every $n \geq 2$.

   (b) Give an example of a 3-critical graph other than K_3.

   (c) Prove that if G is k-critical, then $\delta(G) \geq k - 1$.

Solution:

(a) Note that $\chi(K_n) = n$. If H is a proper subgraph of K_n, then H is a subgraph of $K_n - \{xy\}$ for some vertices x and y (note $xy \in E(K_n)$). Now x and y may be assigned the same colour, thus, n colours are no longer required. Therefore, $\chi(H) < n$.

(b) Every odd cycle is 3-critical since $\chi(C_{2k+1}) = 3$ but any proper subgraph H must be bipartite and thus has $\chi(H) \leq 2$.

(c) We proceed by contradiction. Let G be k-critical. Assume G has a vertex v with degree at most $k - 2$. Then $G - v$ can be coloured with $k - 1$ colours since G is k-critical and $H = G - v$ is a proper subgraph of G. At least one of these $k - 1$ colours does not appear on the neighbours of v since $\deg(v) \leq k - 2$. Therefore, there is a colour for v from the $k - 1$ colours available implying that G has a $(k - 1)$-colouring, a contradiction since $\chi(G) = k$.

7. (10 points) Let $n \geq 1$. Show how the principle of inclusion-exclusion can be used to determine the number of integers in $S = \{1, 2, \ldots, n\}$ that are not divisible by 2, 3 or 5. Write your answer as a function of n.

Solution: We apply the principle of inclusion-exclusion.

- Let A_1 be the elements of S divisible by 2.
- Let A_2 be the elements of S divisible by 3.
- Let A_3 be the elements of S divisible by 5.
- Observe $A_1 \cap A_2$ is the number of integers in S divisible by both 2 and 3 (i.e., 6), etc, and that $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$ is the answer to our problem.

Then

$$|A_1| = \left\lfloor \frac{n}{2} \right\rfloor, \quad |A_2| = \left\lfloor \frac{n}{3} \right\rfloor, \quad |A_3| = \left\lfloor \frac{n}{5} \right\rfloor.$$

$$|A_1 \cap A_2| = \left\lfloor \frac{n}{6} \right\rfloor, \quad |A_1 \cap A_3| = \left\lfloor \frac{n}{10} \right\rfloor, \quad |A_2 \cap A_3| = \left\lfloor \frac{n}{15} \right\rfloor$$

Finally,

$$|A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{n}{30} \right\rfloor.$$

By PIE, $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$ is equal to

$$\left|\overline{A_1 \cup A_2 \cup A_3}\right| = |S| - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3|$$

$$= n - \left(\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{5} \right\rfloor \right) + \left(\left\lfloor \frac{n}{6} \right\rfloor + \left\lfloor \frac{n}{10} \right\rfloor + \left\lfloor \frac{n}{15} \right\rfloor \right) - \left\lfloor \frac{n}{30} \right\rfloor.$$

8. (10 points) You have an unlimited number of apples, bananas and cantaloupes. Let a_n be the number of ways to arrange n fruit (i.e., apples, bananas, cantaloupes) in a row (i.e., order matters) such that no two apples appear together consecutively.

   (a) Find a recurrence relation for a_n.

   (b) Determine a_5.

   (c) Write down the general solution to your recurrence relation (ignoring the initial conditions).

Solution: (a) Consider the first item. If it is an apple, then the next item cannot be an apple giving two cases: if it is a banana then there are a_{n-2} such arrangements, or if it is a cantaloupe then there are also a_{n-2} such arrangements. If the first item is not an apple, then it is either a banana or a cantaloupe and each of these has a_{n-1} such arrangements. Thus, the recurrence relation is

$$a_n = 2a_{n-1} + 2a_{n-2}.$$

We observe that $a_1 = 3$ and $a_2 = 8$.

(b) Using the recurrence, we have $a_3 = 22$, $a_4 = 60$ and $a_5 = 164$.

(c) The characteristic equation is $x^2 = 2x + 2$ giving $x = 1 \pm \sqrt{3}$. The general solution is

$$a_n = c_1(1 + \sqrt{3})^n + c_2(1 - \sqrt{3})^n.$$

9. (10 points) Suppose we have an unlimited number of red, green and blue balls. Let a_n be the number of ways can we select n balls if we must have at least two red, at most one green and an even number of blue balls. Use an ordinary generating function to determine a closed-form formula for a_n.

Solution: Consider the ordinary generating function

$$g(x) = (x^2 + x^3 + x^4 + \cdots)(x^0 + x^1)(x^0 + x^2 + x^4 + x^6 + \cdots).$$

The answer to the problem is $[x^n]g(x)$. Using $\boxed{1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}}$ $(*)$:

$$1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2} \quad \text{and} \quad x^2 + x^3 + x^4 + \cdots = x^2\left(1 + x + x^2 + \cdots\right) = \frac{x^2}{1-x}$$

Thus,

$$g(x) = \left(\frac{x^2}{1-x}\right)(1+x)\left(\frac{1}{1-x^2}\right) = x^2\frac{1}{(1-x)^2} = x^2\left(1 + 2x + 3x^2 + 4x^3 + \cdots\right)$$

since the derivative of $(*)$ gives $1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}$.

Therefore, $g(x) = x^2 + 2x^3 + 3x^4 + 4x^5 + \cdots = 0 + 0 + \sum_{n=2}^{\infty}(n-1)x^n$. Thus, the number of ways is $[x^n]g(x) = \boxed{(n-1)}$ if $n \geq 2$ (and 0 if $n = 0, 1$).

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