University of Toronto at Scarborough
CSCC37—Numerical Algorithms for Computational Mathematics, Fall 2022

Assignment #4: Approximation / Interpolation

Due: December 5, 2022 at 11:45 p.m.
This assignment is worth 10% of your final grade.

Warning: Your electronic submission on *MarkUs* affirms that this assignment is your own work and no one else's, and is in accordance with the University of Toronto Code of Behaviour on Academic Matters, the Code of Student Conduct, and the guidelines for avoiding plagiarism in CSCC37.

This assignment is due by 11:45 p.m. December 5. If you haven't finished by then, you may hand in your assignment late with a penalty as specified in the course information sheet.

[20] 1. Consider the data points $\{(-1,4),(0,6),(1,12)\}$.

(a) Using the method of undetermined coefficients, derive the monomial basis form of the quadratic polynomial interpolating this data. Show all details of your work, including the Gauss transforms and permutation matrices used during the factorization of the Vandermonde matrix.

(b) Derive the Lagrange form of the quadratic interpolant.

(c) Derive the divided-difference form of the quadratic interpolant. Show all of your work, including the divided-difference table.

(d) Verify that the polynomials in (a), (b) and (c) are indeed the same polynomial. (**Hint:** You *could* convert (b) and (c) to monomial-basis form and then compare to (a), but there is an easier way by using an appropriate theorem discussed in lecture. If you choose the latter approach, you must apply the theorem correctly.)

(e) Append $(2,16)$ to the data. Using the *most appropriate* method from (a), (b), or (c), construct the single cubic polynomial interpolant over all four data points and, if you have used method (b) or (c), convert the cubic interpolant to monomial basis form. (**Hint:** By "most appropriate" method, we mean the method which requires the *least amount of extra work* given the quadratic interpolant you have already computed.)

(f) Construct the linear spline (i.e., piecewise linear interpolant) that interpolates all four data points $\{(-1,4),(0,6),(1,12),(2,16)\}$.

[10] 2. As discussed in lecture, the Lagrange basis form of the interpolating polynomial is prohibitively expensive to evaluate. The monomial basis form (as computed by the method of undetermined coefficients) and the divided-difference form both seem to be expensive to evaluate, but they are not.

(a) Propose an optimally efficient algorithm for evaluating the monomial basis form of the interpolating polynomial, and derive the floating point operation complexity of your proposed algorithm. Recall a floating point operation, or *flop*, is a multiplication-addition pair.

(b) Repeat (a) for the divided-difference form.

[15] 3. We know the polynomial $p \in \mathcal{P}_n$ that interpolates $n+1$ distinct points is unique. There are, however, many different ways of representing p.

(a) Show that

$$p(x) = \sum_{i=0}^{n} b_i (x-c)^i \tag{1}$$

can be written equivalently as

$$p(x) = \sum_{i=0}^{n} a_i x^i \tag{2}$$

by expressing the coefficients a_i in terms of b_i and c.

(b) Forms (1) and (2) of p are mathematically equivalent. Computationally, however, there is a difference. For example, consider the method of undetermined coefficients for computing the interpolating polynomial. One drawback of this method is that the resulting Vandermonde matrix may be very poorly conditioned. However, if we use form (1) of p when constructing the matrix and choose c wisely, we may greatly improve the condition of the matrix.

Given that p interpolates the points $\{(i, y_i), i = 0 \ldots 6\}$, determine experimentally the value of c in form (1) of p that minimizes the condition of the Vandermonde matrix. (Note you need not know the values of y_i in order to construct the matrix.) You may use MATLAB's `rcond()` to estimate the reciprocal condition of the Vandermonde matrix.

[15] 4. In osculatory interpolation, one or more data points coincide (i.e., the data points are not unique) and derivative information is supplied at these points. In lecture we showed that when deriving the Newton polynomial over these points, divided differences may be replaced with derivatives using the following rule:

$$\lim_{\substack{x_{i+j} \to x_i \\ 1 \le j \le k}} y[x_{i+k}, x_{i+k-1}, \ldots, x_i] = \frac{y^{(k)}(x_i)}{k!}$$

provided $y \in \mathcal{C}^k$. Using this result, construct a divided difference table to find the coefficients of the Newton polynomial of degree 6 or less that satisfies the following interpolation conditions:

$$\begin{array}{llll} p(-1) = 4 & p(0) = 7 & p(1) = 28 & p(2) = 247 \\ & p'(0) = 6 & p'(1) = 56 & \\ & & p''(1) = 140 & \end{array}$$

Show that the resulting polynomial does indeed satisfy the interpolation conditions.

[total: 60 marks]