How to count?

Arrangements and selections with repetition

Recall the definition of a binomial coefficient.

Definition: Binomial Coefficient

Let n be a non-negative integer and $0 \leq k \leq n$. We define $\binom{n}{k}$ to be the number of ways to choose k objects from a collection of n objects.

Sometimes the following (equivalent) theorem is used as the definition of $\binom{n}{k}$ and a proof of the word statement above is given afterwards.

Theorem

For $0 \leq k \leq n$, we have $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

Example

How many arrangements are there of the letters of T O R O N T O?

- There are 7 letters (three **O**'s, two **T**'s, one **R**, one **N**).
- We have 7 spots to in the row to fill in:

$$\underbrace{\_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_}_{\text{from 7 empty spots, choose 3 for } \mathbf{O}\text{'s}}$$

$$\underbrace{\_\_\ \underline{\text{O}}\ \_\_\ \_\_\ \underline{\text{O}}\ \underline{\text{O}}\ \_\_}_{\text{from 4 empty spots, choose 2 for } \mathbf{T}\text{'s}}$$

$$\underbrace{\_\_\ \underline{\text{O}}\ \_\_\ \underline{\text{T}}\ \underline{\text{O}}\ \underline{\text{O}}\ \underline{\text{T}}}_{\text{from 2 empty spots, choose 1 for } \mathbf{R}}$$

$$\underbrace{\underline{\text{R}}\ \underline{\text{O}}\ \_\_\ \underline{\text{T}}\ \underline{\text{O}}\ \underline{\text{O}}\ \underline{\text{T}}}_{\text{from 1 empty spots, choose 1 for } \mathbf{N}}$$

- The answer is $\binom{7}{3} \cdot \binom{4}{2} \cdot \binom{2}{1} \cdot \binom{1}{1} = \frac{7!}{3!\cancel{4!}} \cdot \frac{\cancel{4!}}{2!\cancel{2!}} \cdot \frac{\cancel{2!}}{1!\cancel{1!}} \frac{\cancel{1!}}{1!0!} = \frac{7!}{3!\,2!\,1!\,1!}$.

* Another method is to first permute the 7 letters in 7! ways. But this treats them as distinct so we overcounted. To compensate for this, we divide (the three O's are identical so divide by 3! and similarly for the two T's). This gives $\frac{7!}{3!2!}$.

Example

How many permutations are there of the letters M I S S I S S I P P I?

Solution.

- There are 11 letters (one **M**, four **I**'s, four **S**'s, two **P**'s).
- The answer is $\binom{11}{1} \cdot \binom{10}{4} \cdot \binom{6}{4} \cdot \binom{2}{2} = \frac{11!}{1!\,\cancel{10!}} \cdot \frac{\cancel{10!}}{4!\,\cancel{6!}} \cdot \frac{\cancel{6!}}{4!\cancel{2!}} \frac{\cancel{2!}}{2!0!} = \frac{11!}{1!\,4!\,4!\,2!}$.

Example

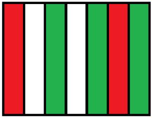
How many arrangements are there of R R W W G G G?

Solution.

- There are 7 letters (two **R**'s, two **W**'s, three **G**'s).
- Based on the previous two examples, the answer simplifies to $\frac{7!}{2!\,2!\,3!}$ (why?).

Example

We want to make a flag with seven vertical stripes of colours. If the flag must contain 2 red stripes, 2 white stripes and 3 green stripes, how many different flags could we create?



Solution.

- To count the number of flags with 2 red (**R**), 2 white (**W**) and 3 green (**G**) stripes, we count the arrangements of R R W W G G G, hence, the answer is $\dfrac{7!}{2!\,2!\,3!}$.

Example

We have three types of breakfast food: raisin bran, waffles and grapefruit. If there are 2 bowls of raisin bran, 2 plates of waffles and 3 bowls of grapefruits available, in how many ways can we distribute them among 7 people?

Solution.

- Fix the seven people in a row.
- To distribute 2 raisin bran (**R**), 2 waffles (**W**) and 3 grapefruit (**G**) among 7 people, we count the arrangements of R R W W G G G, hence, the answer is $\dfrac{7!}{2!\,2!\,3!}$.

Multinomial Coefficients

Definition: Multinomial Coefficient

Let n be a positive integer and $n_1, n_2, \ldots, n_k$ be non-negative integers with

$$\boxed{n_1 + n_2 + \cdots + n_k = n.}$$

The $\boxed{\textbf{multinomial coefficient}}$, denoted by $\binom{n}{n_1, n_2, \ldots, n_k}$, is defined as:

$$\binom{n}{n_1, n_2, \ldots, n_k} := \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Theorem

If there are $n_i \geq 1$ objects of type i, for $1 \leq i \leq k$, and there are $n = n_1 + n_2 + \cdots + n_k$ objects in total, then the number of arrangements of these n objects is $\binom{n}{n_1, n_2, \ldots, n_k}$.

Proof. Generalize the process in the previous examples to get

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

arrangements and simplify to $\dfrac{n!}{n_1!\, n_2! \cdots n_k!}$, which by definition, is $\binom{n}{n_1, n_2, \ldots, n_k}$.

Here are some observations:

Notes

Multinomial coefficients **generalize** binomial coefficients: $\binom{n}{k} = \binom{n}{k, n-k}$

Proof: $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ and $\binom{n}{k, n-k} = \dfrac{n!}{k!\,(n-k)!}$.

Notes

The multinomial coefficient is a **natural number**!

Proof. From the previous theorem, it counts the number of arrangements of n objects (with object i being repeated n_i times), thus, must be a natural integer.

Example (A number theory result using combinatorics)

Let k be a positive integer. Prove that $(4k)!$ is a multiple of $2^{3k} \cdot 3^k$.

Solution.

- Count the number of arrangements of the elements of the **multiset** below (note that elements may be repeated in a multiset!):

$$\{a_1, a_1, a_1, a_1, \quad a_2, a_2, a_2, a_2, \quad \ldots, \quad a_k, a_k, a_k, a_k\}.$$

- For $i = 1, 2, \ldots, k$, each a_i appears four times (That is, there are $\boxed{n_i = 4}$ objects of each type "a_i".)
- The total number of elements (including repeats) is

$$n = n_1 + n_2 + \cdots + n_k = 4 + 4 + \cdots + 4 = \boxed{4k}.$$

- By the theorem, the number of arrangments of elements in the multiset is

$$\binom{n}{n_1, n_2, \ldots, n_k} = \underbrace{\binom{4k}{4, 4, \ldots, 4}}_{k \text{ fours}} = \frac{(4k)!}{4! \cdot 4! \cdots 4!} = \frac{(4k)!}{(4!)^k} = \frac{(4k)!}{(2^3 \cdot 3)^k} = \frac{(4k)!}{2^{3k} \cdot 3^k}.$$

- But this is a natural number (previous observation), hence $2^{3k} \cdot 3^k$ must divide $(4k)!$.

Example

How many arrangements of F L I B B E R T I G I B B E T have no two vowels consecutive?

Tip: First arrange the letters with no restrictions, then use "interlacing".

- First, arrange the letters that are **NOT** vowels: four Bs, two Ts, one F, one G, one L and one R (ten letters in total). There are $\binom{10}{4,2,1,1,1,1}$ ways to do this.
- Since no two vowels are consecutive, we use interlacing/weaving.
- Select 5 of the 11 slots between consonants (including first/last):

$$\underbrace{\_\ \boxed{B}\ \_\ \boxed{G}\ \_\ \boxed{T}\ \_\ \boxed{L}\ \_\ \boxed{T}\ \_\ \boxed{B}\ \_\ \boxed{F}\ \_\ \boxed{R}\ \_\ \boxed{B}\ \_\ \boxed{B}\ \_}_{\text{from 11 empty spots, select 5 for the vowels (E, E, I, I, I)}}$$

- There are $\binom{11}{5}$ ways to do this.
- Arrange the five vowels into these five slots; there are $\binom{5}{2,3}$ ways to do this by the theorem since we are arranging E E I I I (two E's and three I's).
- The final answer is $\binom{10}{4,2,1,1,1,1}\binom{11}{5}\binom{5}{2,3}$ (if curious, this is 349 272 000).

The Binomial Theorem

Convention

For any $k, n \in \mathbb{Z}$ with $n \geq 0$, we assume $\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n \text{ or } k < 0. \end{cases}$

We call $\binom{n}{k}$ a $\boxed{\textbf{binomial coefficient}}$ because it shows up as a coefficient in the expansion of the binomial expression $(x+y)^n$.

Binomial Theorem

For any integer $n \geq 0$, we have $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

Examples

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(2x+1)^5 = \sum_{k=0}^{5} \binom{5}{k} (2x)^{5-k}(1)^k = 32x^5 + 80x^4 + 80x^3 + 40x^2 + 10x + 1$$

Binomial Theorem

For any integer $n \geq 0$, we have $(x+y)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

Proof.

- One technique is to apply induction. Instead we give a $\boxed{\textbf{combinatorial proof}}$.
- WTS (want to show) coefficient of $x^{n-k}y^k$ in expansion of $(x+y)^n$ is $\binom{n}{k}$.
- To expand
$$(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ factors}},$$
we choose either x **or** y from each factor of $(x+y)$ and then multiply together.
- To form a term with $x^{n-k}y^k$, we first select k of the n factors $(x+y)$ and pick "y" from these chosen factors (followed by picking "x" from the remaining $n-k$ factors).
- The first step can be done in $\binom{n}{k}$ ways, and the second step in $\binom{n-k}{n-k} = 1$ way.
- Thus, the number of ways to form an $x^{n-k}y^k$ term is $\binom{n}{k}$.

Example

For all $x_1, x_2, x_3 \in \mathbb{R}$, we have

$$(x_1 + x_2 + x_3)^3 = (x_1 + x_2 + x_3)(x_1 + x_2 + x_3)(x_1 + x_2 + x_3)$$

$$= x_1^3 + x_2^3 + x_3^3 + 3x_1^2x_2 + 3x_1x_2^2 + 3x_1^2x_3 + 3x_1x_3^2 + 3x_2^2x_3 + 3x_2x_3^2 + 6x_1x_2x_3.$$

E.g., the ways to form an $x_1^2x_2$ term is by selecting "x_1, x_1, x_2" from the first, second and third factors, or "x_1, x_2, x_1" or "x_2, x_1, x_1", giving three possible ways. This is equal to the number of arrangements of x_1, x_1, x_2 where $n_1 = 2$, $n_2 = 1$ and $n = 3$, i.e., $\binom{3}{2, 1}$.

Multinomial Theorem

Let n be a positive integer. For all $x_1, x_2, \ldots, x_m$, we have

$$(x_1 + x_2 + \cdots + x_m)^n = \sum \binom{n}{n_1, n_2, \ldots, n_m} x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$$

where the summation is over all non-negative integer solutions $(n_1, n_2, \ldots, n_m)$ of

$$\boxed{n_1 + n_2 + \cdots + n_m = n}.$$

Example

What is the coefficient of $x^{99}y^{60}z^{14}$ in $(2x^3 + y - z^2)^{100}$?

Solution.

By the multinomial theorem, the expansion of $(2x^3 + y - z^2)^{100}$ has terms of form

$$\binom{100}{n_1, n_2, n_3}(2x^3)^{n_1} y^{n_2} (-z^2)^{n_3} = \binom{100}{n_1, n_2, n_3} 2^{n_1} x^{3n_1} y^{n_2} (-1)^{n_3} z^{2n_3}.$$

The term $x^{99}y^{60}z^{14}$ arises when $n_1 = 33$, $n_2 = 60$ and $n_3 = 7$, thus it has coefficient

$$\binom{100}{33, 60, 7} 2^{33}(-1)^7 \quad \text{or} \quad -\binom{100}{33, 60, 7} 2^{33}.$$

Example

Use the binomial theorem to prove

(a) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0$ for any integer $n \geq 1$.

(b) $n2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k}$ for any integer $n \geq 1$.

Solution.

(a) The result follows by setting $x = 1$ and $y = -1$ in the binomial theorem.

(b) Using the binomial theorem to expand $(1 + y)^n$ (i.e., let $x = 1$), we get

$$(1 + y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k.$$

Note that the term corresponding to $k = 0$ is constant, thus, taking the derivative with respect to y gives

$$n(1 + y)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k\, y^{k-1}.$$

Setting $y = 1$ gives the result.

Example

Use the binomial theorem to prove

$$\frac{2^{n+1}-1}{n+1} = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}.$$

Solution.

We take the definite integral on the interval $[0,1]$ of the polynomial function

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

The definite integral of the left side gives

$$\int_0^1 (x+1)^n\,dx = \frac{1}{n+1}(x+1)^{n+1}\Big|_0^1 = \frac{(1+1)^{n+1}-(0+1)^{n+1}}{n+1} = \frac{2^{n+1}-1}{n+1}.$$

The definite integral of the right side gives

$$\int_0^1 \left[\sum_{k=0}^{n}\binom{n}{k}x^k\right] dx = \sum_{k=0}^{n}\binom{n}{k}\left[\int_0^1 x^k\,dx\right] = \sum_{k=0}^{n}\binom{n}{k}\left[\frac{x^{k+1}}{k+1}\right]\Bigg|_0^1 = \sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}.$$

This proves the required identity.

Pascal's Triangle

Q: How can we $\boxed{\textbf{define}}$ Pascal's triangle?

A: Arrange the binomial coefficients into a triangular array so that the entry in the nth row and kth column is $\binom{n}{k}$ (the top row is the 0th row).

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & \binom{0}{0} & & & & & & & \\
 & & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & & \\
 & & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & & \\
 & & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & & \\
 & & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & & \\
 & & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & & \\
 & \binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6} & \\
\binom{7}{0} & & \binom{7}{1} & & \binom{7}{2} & & \binom{7}{3} & & \binom{7}{4} & & \binom{7}{5} & & \binom{7}{6} & & \binom{7}{7} \\
 & & & & & & & \vdots & & & & & & &
\end{array}
\quad \to \quad
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

- What patterns do you notice in Pascal's Triangle?
- Can you prove them?
- See https://en.wikipedia.org/wiki/File:PascalTriangleAnimated2.gif for a recursive method to generate the triangle using an identity.

- The sides are equal to 1.
- All other entries are the **sum of the two entries above it**.
- We can write this mathematically as follows.

Patterns

For any $n \geq 2$ and $1 \leq k \leq n-1$ we have $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Combinatorial Proof. We consider the following problem.

Q: How many k-element subsets S are there of $\{1, 2, \ldots, n\}$?

A1: There are $\binom{n}{k}$ subsets of $\{1, \ldots, n\}$ of size k (by definition of binomial coefficient).

A2: We partition according to whether 1 is in S (assuming $S \subseteq \{1, \ldots, n\}$ and $|S| = k$).

- The number of subsets S of size k with $1 \in S$ is $\binom{n-1}{k-1}$.
- The number of subsets S of size k with $1 \notin S$ is $\binom{n-1}{k}$.
- By the addition principle, the number of subsets of size k is $\binom{n-1}{k-1} + \binom{n-1}{k}$.

More Patterns

- Entries are symmetric with respect to the middle line.
  - **Mathematically:** $\binom{n}{k} = \binom{n}{n-k}$.
  - **Proof:** See Week 3c lecture.

- The sum of each row is 2^n (where n is the level number).

$$
\begin{array}{ccccccccc}
 & & & & \mathbf{1} & & & & \\
 & & & \mathbf{1} & & \mathbf{1} & & & \\
 & & \mathbf{1} & & \mathbf{2} & & \mathbf{1} & & \\
 & \mathbf{1} & & \mathbf{3} & & \mathbf{3} & & \mathbf{1} & \\
\mathbf{1} & & \mathbf{4} & & \mathbf{6} & & \mathbf{4} & & \mathbf{1}
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
\mathbf{1} = 2^0 \\
\mathbf{1} + \mathbf{1} = 2^1 \\
\mathbf{1} + \mathbf{2} + \mathbf{1} = 2^2 \\
\mathbf{1} + \mathbf{3} + \mathbf{3} + \mathbf{1} = 2^3 \\
\mathbf{1} + \mathbf{4} + \mathbf{6} + \mathbf{4} + \mathbf{1} = 2^4
\end{array}
$$

  - **Mathematically:** $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$.
  - **Proof:** See Week 3c lecture.

- Sum of two adjacent entries in a level give the coefficient below.
  - **Mathematically:** $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
  - **Proof:** (see previous slide).

There is a **hockey stick** pattern in Pascal's triangle!

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad \boxed{1} \\
1 \quad 4 \quad 6 \quad \boxed{4} \quad 1 \\
1 \quad 5 \quad 10 \quad \boxed{10} \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad \boxed{20} \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad \boxed{35} \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad \boxed{70} \quad 56 \quad 28 \quad 8 \quad 1 \\
1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1
\end{array}
\qquad \rightarrow \qquad 1 + 4 + 10 + 20 + 35 = 70
$$

Hockey Stick Identity

For $n, k \in \mathbb{N}$ and $n \geq k$ we have $\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$.

That is,

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Proofs. See wikipedia for five different proofs. In particular, **read** Combinatorial Proof 2.

Consequences of the hockey stick identity

The sum of the first n positive integers is a hockey stick identity.

$$\begin{array}{c}
1 \\
1 \quad \boxed{\mathbf{1}} \\
1 \quad \boxed{\mathbf{2}} \quad 1 \\
1 \quad \boxed{\mathbf{3}} \quad 3 \quad 1 \\
1 \quad \boxed{\mathbf{4}} \quad 6 \quad 4 \quad 1 \\
1 \quad \boxed{\mathbf{5}} \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad \boxed{\mathbf{6}} \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad \boxed{\mathbf{7}} \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad \boxed{\mathbf{28}} \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}
\qquad \rightarrow \qquad 1+2+3+4+5+6+7=28$$

Corollary

The sum of the first n positive integers is

$$\sum_{i=1}^{n} i = \binom{n+1}{2}.$$

These are called triangular numbers.

Proof. This follows from the hockey stick identity using $k=1$.

Corollary

The sum of the squares of the first n positive integers is

$$\sum_{i=1}^{n} i^2 = 2\binom{n+2}{3} - \binom{n+1}{2}.$$

The sum of the cubes of the first n positive integers is

$$\sum_{i=1}^{n} i^3 = 6\binom{n+3}{4} - 6\binom{n+2}{3} + \binom{n+1}{2}.$$

Proof. For sample proofs using the hockey stick identity see this link: https://brilliant.org/wiki/hockey-stick-identity/.