University of Toronto Scarborough
Department of Computer and Mathematical Sciences
MATC44 Fall 2022 - Practice Midterm 3

Date: Wednesday, November 2, 2022 from 14:10 - 16:00

Signature: ______________________________

- Time: 110 minutes (class time)
- Write your solutions in this booklet (only those pages with a QR code will be graded).
- Use the back of each page for **rough** work.
- This is a closed-book test. No aids are allowed for this midterm. Calculators and the use of personal electronic or communication devices is prohibited.
- This test has 11 pages with the last two pages being blank.
- There are 8 problems with the number of points indicated by each problem.
- The total number of points possible on this test is 40.

DO NOT OPEN THIS BOOKLET UNTIL INSTRUCTED TO DO SO.

1. (5 points) Two players are playing a Nim[1] game with heaps of sizes 3, 7, 14 and 15.

   Verify that the first player can guarantee a win. List all the possible moves the first player has if they are required to create a nim-sum of 0 after their first move.

Solution: Writing out the numbers 3, 7, 14 and 15 in binary gives

	2^3	2^2	2^1	2^0
3	0	0	1	1
7	0	1	1	1
14	1	1	1	0
15	1	1	1	1

The nim-sum is 101 (in binary), therefore, the first player can guarantee a win with optimal play by the "Nim Theroem" (i.e., Bouton's theorem).

In order to guarantee a win, player one must end their turn so that every column has an even number of ones. Thus, looking at the most significant column (farthest left) with an odd number of ones (i.e., the column labelled by 2^2), we see three possible moves for the first player.

Player 1 can take 5 objects from the heap with 7 objects to give the table

	2^3	2^2	2^1	2^0
3	0	0	1	1
2	0	0	1	0
14	1	1	1	0
15	1	1	1	1

Player 1 can take 3 objects from the heap with 14 objects.

	2^3	2^2	2^1	2^0
3	0	0	1	1
7	0	1	1	1
11	1	0	1	1
15	1	1	1	1

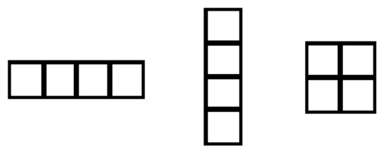
Player 1 can take 5 objects from the heap with 15 objects.

	2^3	2^2	2^1	2^0
3	0	0	1	1
7	0	1	1	1
14	1	1	1	0
10	1	0	1	0

In each case, the new nim-sum is equal to 0.

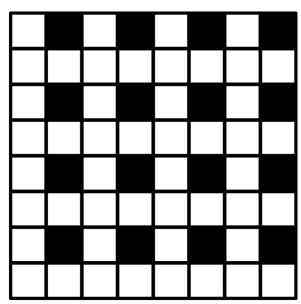
[1]Nim is a two player game. There are n heaps of objects. On each turn, a player chooses a heap and removes at least one object from it (they may remove any number of objects provided they all come from the same heap). The player who takes the last object is the winner of the game.

2. (5 points) Suppose an 8×8 chessboard is perfectly tiled using some 2×2 and 1×4 tiles.



If the number of 2×2 tiles used is x and the number of 1×4 tiles used is y (thus, $x + y = 16$), show that x cannot be odd.

Solution: To derive a contradiction, suppose x is odd. Recolour the 8×8 chessboard as follows:



Observe that each 2×2 tile covers exactly one black square and each 1×4 tile covers exactly zero or two black squares. Suppose the number of 1×4 tiles covering exactly zero black squares is u and the number of 1×4 tiles covering exactly two black squares is v. Then $x + 0u + 2v = 16$, and hence, $x = 2(8 - v)$ implying that x is even, a contradiction. Therefore, x cannot be odd.

3. (5 points) For the following problem, you only need to provide your final answers. Correct answers are 1 point each and incorrect answers are 0 points each.

   (i) There are 11 **different** animals in a zoo. In how many ways can we

   (a) choose 8 of them and arrange them in a row.

   (b) choose 8 of them to create a group.

   (c) choose 8 of them and place them into two equal-sized unlabelled groups.

   (ii) There are 11 **identical** blue poker chips in a box. In how many ways can we

   (a) choose 8 of them to create a group.

   (b) choose 8 of them and give them to three people where some people might not get any.

Solution:

(i) Answers

(a) $11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4$.

(b) $\binom{11}{8}$.

(c) $\frac{1}{2}\binom{11}{4}\binom{7}{4}$ or $\frac{1}{2}\binom{11}{8}\binom{8}{4}$.

(ii) Answers

(a) 1.

(b) $\binom{10}{2}$ (by stars and bars).

4. (5 points) Let a_k be the number of positive integers that have exactly k digits and whose digits sum to 5. Find a binomial coefficient that is equal to a_k.

Solution: Consider a number with exactly k digits: $\underline{x_1}\ \underline{x_2}\ \cdots\ \underline{x_k}$. Then we require $x_1 \geq 1$ so that it is a k-digit number, i.e., has its leading digit nonzero. We also have $0 \leq x_i \leq 9$ and each x_i must be an integer, for $i = 1, 2, \ldots, k$. Since we require $\sum_{i=1}^{k} x_i = 5$, this implies that $x_i \leq 5$ thus we may omit the restriction $x_i \leq 9$ on x_i. Therefore, to find a_k, it suffices to count the number of integer solutions to

$$\sum_{i=1}^{k} x_i = 5$$

such that $x_1 \geq 1$ and $x_i \geq 0$ for $i = 2, 3, \ldots, k$.

Let $y_1 = x_1 - 1$. Then a_k is the number of nonnegative integer solutions to $y_1 + \sum_{i=2}^{k} x_i = 5$ with $y_1 \geq 0$ and $x_i \geq 0$ for $i = 2, 3, \ldots, k$. Thus, $a_k = \binom{k+3}{4}$ by the theorem from the lecture (or by applying stars and bars).

5. (5 points) Give a combinatorial proof of the identity

$$\sum_{k=1}^{n} a_k = \binom{n+4}{5}$$

where a_k is replaced by the binomial coefficient you found in Question 4.

Solution: We now prove

$$\sum_{k=1}^{n} \binom{k+3}{4} = \binom{n+4}{5}.$$

We ask "How many 5-element subsets are there of the set $\{1, 2, \ldots, n+4\}$"?

On the one hand, it is $\binom{n+4}{5}$ giving the right-side.

On the other hand, we consider the largest element of the subset and call it $k+4$. Since the subset must have 5 elements, we have $k+4 \leq n+4$ and thus, $1 \leq k \leq n$. If the largest element is $k+4$, then there are $k+3$ smaller elements that we must select from, and we must choose 4 to form a subset of size 5. Thus, there are $\binom{k+3}{4}$ such subsets. Summing over $1 \leq k \leq n$ gives the left-side.

6. (5 points) What is the coefficient of $x^{99}y^{60}z^{14}$ in $(2x^3 + y - z^2)^{100}$?

   Recall the multinomial theorem[2].

Solution: By the multinomial theorem, the expansion of $(2x^3 + y - z^2)^{100}$ has terms of form

$$\binom{100}{n_1, n_2, n_3}(2x^3)^{n_1}y^{n_2}(-z^2)^{n_3} = \binom{100}{n_1, n_2, n_3}2^{n_1}x^{3n_1}y^{n_2}(-1)^{n_3}z^{2n_3}.$$

The term $x^{99}y^{60}z^{14}$ arises when $n_1 = 33$, $n_2 = 60$ and $n_3 = 7$, thus it has coefficient

$$\binom{100}{33, 60, 7}2^{33}(-1)^7 \quad \text{or} \quad -\binom{100}{33, 60, 7}2^{33}.$$

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[2]Let n be a positive integer. For all $x_1, x_2, \ldots, x_m$, we have

$$(x_1 + x_2 + \cdots + x_m)^n = \sum \binom{n}{n_1, n_2, \ldots, n_m} x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$$

where the summation is over all non-negative integer solutions $(n_1, n_2, \ldots, n_m)$ of $n_1 + n_2 + \cdots + n_m = n$.

7. (5 points) Let G be a connected graph and suppose G has an edge e whose deletion produces a disconnected graph with two components. Can every vertex of G have even degree? Either find an example of such a graph satisfying these properties or prove there is no such graph.

Solution: There is no such graph. We proceed by contradiction. Let G be a connected graph with $e \in E(G)$ having the property that $G - e$ has two components H_1 and H_2. Suppose every vertex of G has even degree. Observe each of H_1 and H_2 have exactly one vertex of odd degree, in particular, H_1 has *an odd number of vertices of odd degree*. Since H_1 is a component of $G - e$, it is a connected graph with no connections to H_2, thus, by a corollary to the Handshaking Lemma, H_1 must also have *an even number of vertices of odd degree*. This is a contradiction, thus, not every vertex of G can have even degree.

8. (5 points) Let G be a connected plane graph with $v \geq 3$ vertices and e edges. Use Euler's formula and the handshaking lemma (for faces) to prove if G is bipartite, then $e \leq 2v - 4$.

Solution:

- Let G be a connected plane graph with $v \geq 3$ vertices, e edges and f faces, and assume G is bipartite.
- Since G is bipartite, G has no odd length cycles.
- Thus, as $v \geq 3$ and G is connected it must be that every face has degree at least 4.
- By the handshaking lemma (for faces) we have $2e = \displaystyle\sum_{F \text{ a face}} \deg(F) \geq 4f$.
- Thus, $f \leq \frac{1}{2}e$.
- By Euler's formula ($v - e + f = 2$), we have

$$f = 2 - v + e \leq \frac{1}{2}e \qquad \rightarrow \qquad \frac{1}{2}e \leq v - 2 \qquad \rightarrow \qquad e \leq 2v - 4$$

  as required to show.

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