University of Toronto Scarborough
Department of Computer and Mathematical Sciences
MATC44 Fall 2022 - Practice Midterm 1

Date: Wednesday, November 2, 2022 from 14:10 - 16:00

Signature: ____

- Time: 110 minutes (class time)
- Write your solutions in this booklet (only those pages with a QR code will be graded).
- Use the back of each page for **rough** work.
- This is a closed-book test. No aids are allowed for this midterm. Calculators and the use of personal electronic or communication devices is prohibited.
- This test has 11 pages with the last two pages being blank.
- There are 8 problems with the number of points indicated by each problem.
- The total number of points possible on this test is 40.

DO NOT OPEN THIS BOOKLET UNTIL INSTRUCTED TO DO SO.

1. (5 points) Let $m, n \geq 1$ be integers. Consider a Nim[1] game in which there are three heaps of sizes m, m and n. Which player (first or second) can guarantee a win? Justify your answer **without** making reference to Bouton's "Nim Theorem".

> **Solution:** The first player can guarantee a win. They start by taking all objects from the heap of size n (and then follow a mirroring strategy). There are now two heaps of the same size (i.e., m and m) remaining. Player two must remove k objects from one of the heaps for some $1 \leq k \leq m$. Player one can then remove k objects from the other heap so that at the end of their turn there are two heaps of the same size (i.e., $m - k$ and $m - k$) remaining. Since player two must always unbalance the sizes of the heaps and player one can always restore the balance of the sizes of the heaps, player one will win since the ending state of the game is empty heaps (i.e., heaps of the same size).

[1] Nim is a two player game. There are n heaps of objects. On each turn, a player chooses a heap and removes at least one object from it (they may remove any number of objects provided they all come from the same heap). The player who takes the last object is the winner of the game.

2. (5 points) Let $k \geq 1$ be an integer and S a set of positive integers with $|S| = 6k+1$. Prove there is a subset $T \subseteq S$ with $|T| = k+1$ such that for **every** distinct pair $a, b \in T$, the number $a^2 - b^2$ is divisible by 10 (i.e., a^2 and b^2 have the same units digit).

   Hint: You may use the fact that a perfect square cannot end in 2, 3, 7 *or* 8.

   Solution: The units digit of a square number must be one of the following 6 options:

   $$\{0, 1, 4, 5, 6, 9\}.$$

   We use these six numbers as the pigeonholes and the elements of S as the pigeons. Distribute the elements of S (pigeons) among the six numbers $\{0, 1, 4, 5, 6, 9\}$ according to following rule: if $x \in S$, then place x in the pigeonhole matching the units digit of x^2. Since $|S| = 6k+1$, by the pigeonhole principle, there is a subset T of at least $k+1$ of the numbers in S whose squares have the same units digit. Thus, T has the required property.

3. (5 points) How many ways are there to arrange the letters **A,A,A,B,C,D,E,F** in a row if no two **A**'s are adjacent? Justify your answer.

> **Solution:** First arrange B, C, D, E, F in a row; there are 5! ways. Take one such arrangement and note that there are 6 spaces separated by the 5 letters. We now distribute 3 identical A's in the 6 places so that each place holds at most one A (since no two A's can be adjacent). The number of ways to do this is $\binom{6}{3}$. By the multiplication principle, the desired number of ways is $5!\binom{6}{3}$.

4. (5 points) Let $n \geq 3$. Give a combinatorial proof (e.g., by using committee selection) for the following identity:

$$\sum_{k=0}^{3} \binom{n}{k}\binom{n-k}{3-k} = 2^3\binom{n}{3}.$$

Solution: We count the number of ways to choose a committee of three people who may or may not be wearing hats. Since each of the three has two choices, this gives the right-side. Alternatively, let k be the number of people wearing hats on the committee and from n choose k people to be the hat wearers on the committee. Now, from $n-k$ remaining people, choose $3-k$ to be the non-hat wearing people on the committee giving a total committee size of three.

5. (5 points) (a) Using "stars and bars", show that the number of ways to select an unordered group of seven numbers using numbers from 1 through 19 (inclusive) **with repetition** allowed is equal to $\binom{25}{7}$.

   (b) What if the sum of the group of chosen numbers must be odd? That is, how many ways are there to select an unordered group of seven numbers from 1 through 19, with repetition, so that the sum of the numbers is odd? Briefly explain your answer.

Solution: (a) This is a standard "stars and bars" argument with seven "stars" and 18 "bars". In particular, the 18 bars form boxes for the numbers 1 through 19. The stars are then put into boxes (multiple stars can go into a single box) to form an unordered group of seven numbers from 1 through 19. For example,

$$* * *|\,|\,|*|\,|*|\,|\,|\,|\,|*|\,|*|\,|\,|\,|\,|\,|\,|$$

represents the group $\{1, 1, 1, 4, 6, 11, 13\}$ whereas

$$|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|\,|* * * * * **$$

represents the group $\{19, 19, 19, 19, 19, 19, 19\}$. This gives a bijection between arrangements of stars and bars with unordered groups, so the number of ways is

$$\binom{7+19-1}{7} = \binom{25}{7}.$$

(b) The sum is odd if and only if it contains an odd number of odd integers, so the possibilities are: 1 odd and 6 even; 3 odd and 4 even; 5 odd and 2 even; 7 odd and 0 even.

The odd number(s) are chosen from 10 types: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19; the even numbers are chosen from 9 types: 2, 4, 6, 8, 10, 12, 14, 16, 18. Therefore, the number of ways is

$$\binom{1+10-1}{1}\binom{6+9-1}{6}+\binom{3+10-1}{3}\binom{4+9-1}{4}+\binom{5+10-1}{5}\binom{2+9-1}{2}+\binom{7+10-1}{7}\binom{0+9-1}{0}.$$

6. (5 points) State and prove the binomial theorem (using a combinatorial argument).

Solution: The binomial theorem is stated below.

Theorem. For any integer $n \geq 0$, we have $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

Proof.

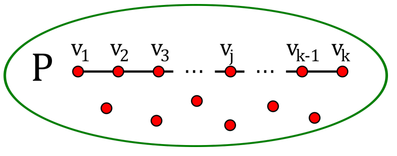
- We want to show that the coefficient of $x^{n-k}y^k$ in expansion of $(x+y)^n$ is $\binom{n}{k}$.
- To expand
$$(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ factors}},$$
we choose either x **or** y from each factor of $(x+y)$ and then multiply together.
- To form a term with $x^{n-k}y^k$, we first select k of the n factors $(x+y)$ and pick "y" from these chosen factors (followed by picking "x" from the remaining $n-k$ factors).
- The first step can be done in $\binom{n}{k}$ ways, and the second step in $\binom{n-k}{n-k} = 1$ way.
- Thus, the number of ways to form an $x^{n-k}y^k$ term is $\binom{n}{k}$.

7. (5 points) Let $t \geq 1$ be an integer and G be a graph whose vertices all have degree at least t. Prove that G contains a path of length at least t.

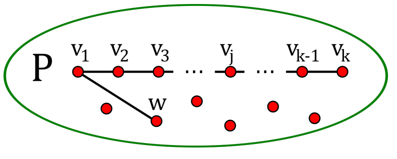
   Hint: Use the extremal principle.

Solution:

- Let $P = v_1 v_2 \cdots v_{k-1} v_k$ be a longest path in G and note $k \geq 2$ since $\deg(v_1) \geq t \geq 1$.



- Note that the length of P is $k-1$ (the number of edges in P).
- It suffices to prove that $\boxed{k-1 \geq t}$ since this will imply that the length of P is at least t.
- If $t = 1$, then $P = v_1 v_2$ is a path of length $t = 1$ and we are done.
- Thus, we assume $t \geq 2$, and hence, $\deg(v_1) \geq t \geq 2$.
- It follows that v_1 is adjacent to another vertex (call it w) other than v_2.
- If w is not on the path P, then $P' = w\, v_1 v_2 \cdots v_k$ is a longer path in G contradicting that P is a longest path:



- Thus, all the neighbours of v_1 are vertices on the path P.
- Hence, v_1 is only adjacent to vertices in the set $\{v_2, v_3, \ldots, v_k\}$ implying that

$$\deg(v_1) \leq k-1$$

  since the set has size $k-1$.
- Combining the two inequalities for $\deg(v_1)$ gives

$$t \leq \deg(v_1) \leq k-1 (= \text{length of } P)$$

  as required to show.

8. (5 points) Let G be a connected planar graph with $v \geq 3$ vertices and $e < 30$ edges. Show that G requires a vertex of degree at most 4.

   Recall the Corollary[2] to Euler's formula.

> **Solution:** To derive a contradiction, suppose every vertex of G has degree at least 5. By the handshaking lemma, we have
> $$2e = \sum_{x \in V(G)} \deg(x) \geq 5v,$$
> and by the Corollary to Euler's formula, we have $e \leq 3v - 6$. Thus,
> $$5v \leq 2e \leq 6v - 12.$$
> This implies $\boxed{v \geq 12}$. But $e < 30$ implying $5v \leq 2e < 2(30) = 60$, and hence, $\boxed{v < 12}$. This is a contradiction. Thus, G requires at least one vertex of degree at most 4.

[2]**Corollary.** Let G be a connected plane graph with $v \geq 3$ vertices and e edges. Then $e \leq 3v - 6$.

(Page intentionally left blank in case extra space is needed for solutions)

(Page intentionally left blank in case extra space is needed for solutions)