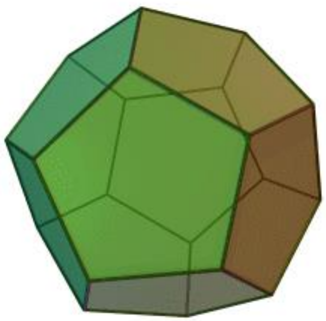
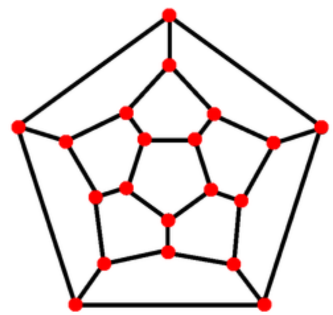
MATC44 – Introduction to Combinatorics

Dodecahedron

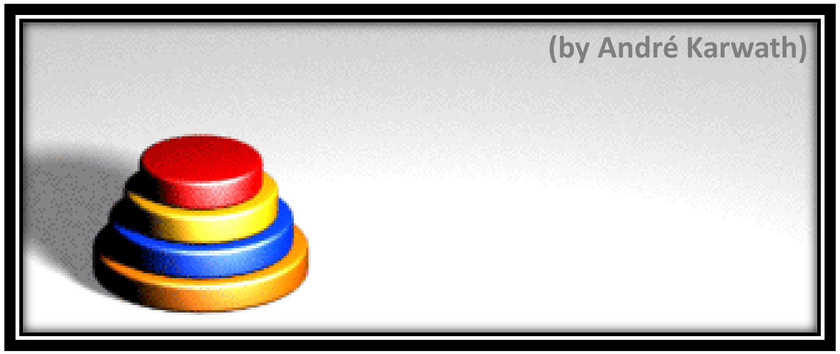


(by Kjell André)

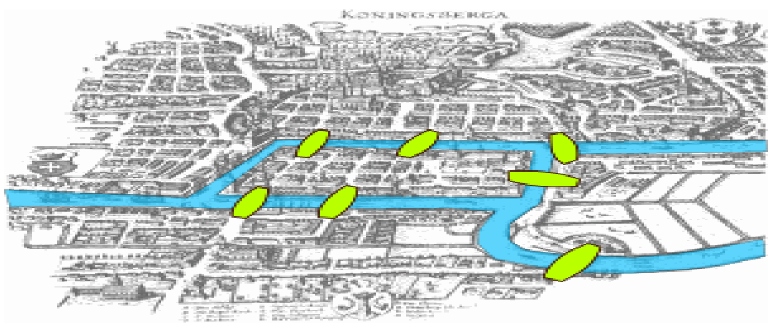
Dodecahedron (planar drawing)



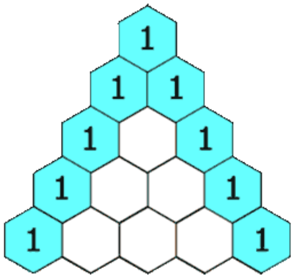
The Tower of Hanoi



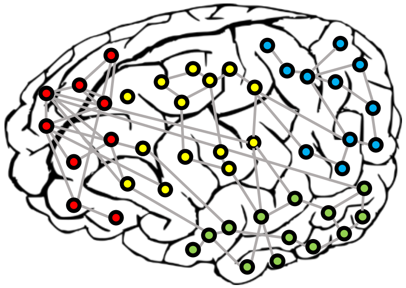
The Seven Bridges of Konigsberg



Pascal's Triangle



A Brain Network



Warmup Problem.

Simplify the following algebraic expression: $(x-a)(x-b)(x-c)\cdots(x-z)$

Your instructor

Mike Cavers (he/him)

What do you call me?

I prefer **Mike**, but if you are uncomfortable on a first name basis then either **Sir** or **Prof/Dr Cavers** or **Hey guy!** are perfectly suitable.

Email: michael.cavers@utoronto.ca

Office: IC 349

Office hours: Mondays (14:00-15:00); also by appointment

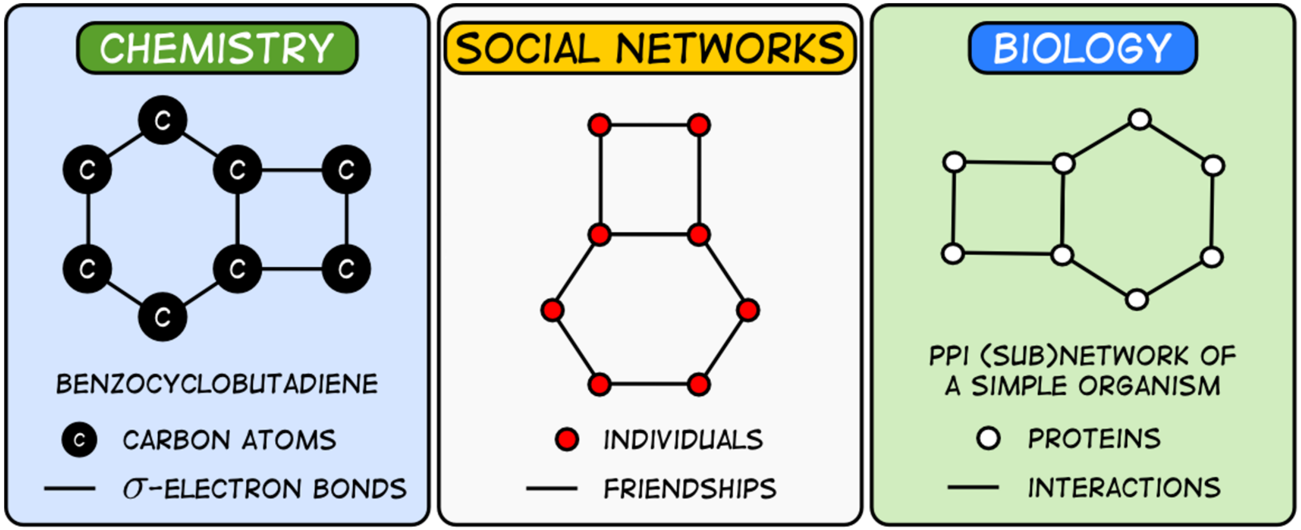
Research Interests:

- Mathematics education
- Discrete mathematics (networks)
- Linear algebra (matrix theory)
- Applications in neuroscience (epileptic networks) & geophysics (earthquake forecasting)

What is Mathematics?

Mathematics is the art of giving the same name to different things.

– Henri Poincaré



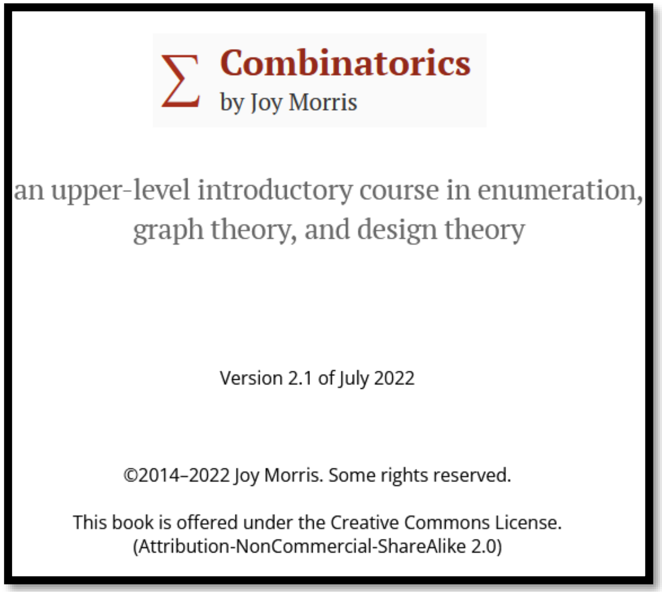
Textbooks

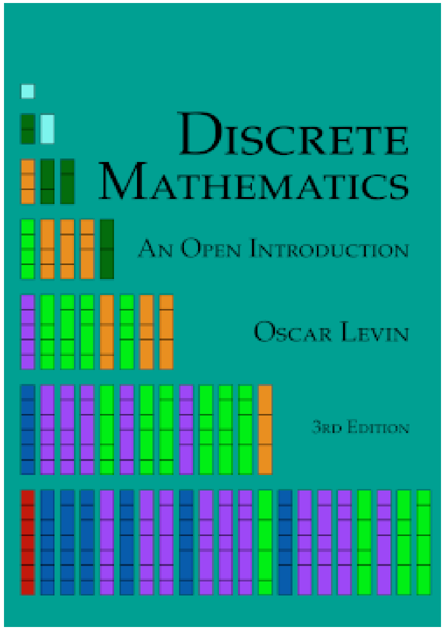
This course uses the openly licensed textbooks

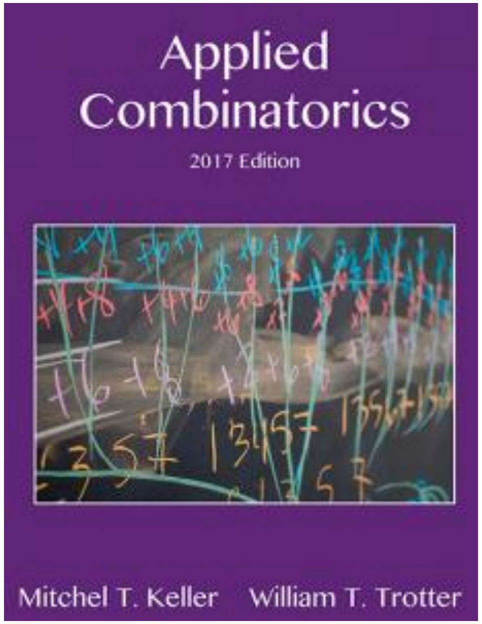
- **Combinatorics** by *Joy Morris*
  https://opentext.uleth.ca/Combinatorics/frontmatter-1.html

- **Discrete Mathematics** by *Oscar Levin*
  http://discrete.openmathbooks.org/dmoi3/dmoi.html

- **Applied Combinatorics** by *Mitchel Keller* and *William Trotter*
  https://www.appliedcombinatorics.org/book/app-comb.html



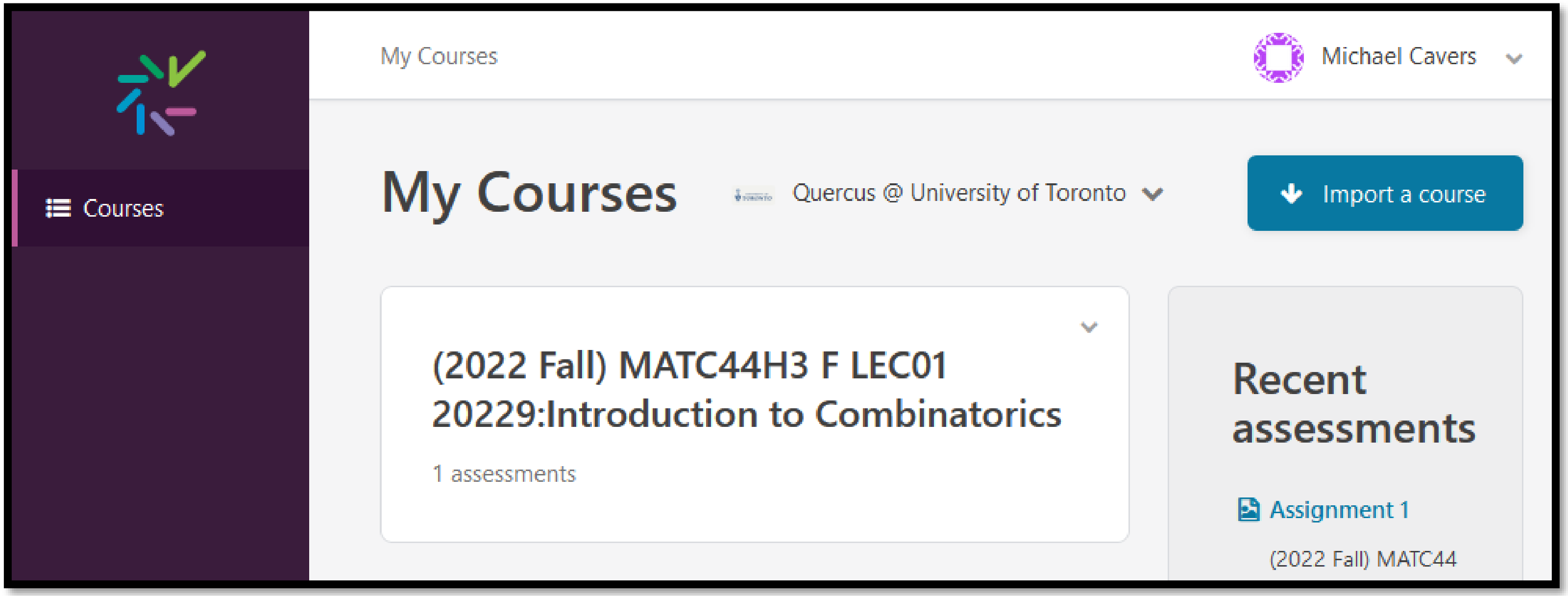




Assessment of Learning Objectives

Item	Percentage of grade
Final Exam	40%
Midterm (1)	25% (November 2nd, 2022)
Assignments (6)	30% (5% each)
Best of midterm/final	5%

All assignments are submitted through Crowdmark



Week	Dates	Topics	Information
1	Sept 6 - 9 (2 hours)	Introduction to combinatorics and graph theory The game of Nim	
2	Sept 12 - 16	Proof techniques: pigeonhole principle, induction, extremal principle, invariance principle, colouring/parity	
3	Sept 19 - 23	Permutations and combinations Combinatorial proofs (counting in two ways)	A1 due Sept 20
4	Sept 26 - 30	Stars and bars Binomial theorem	
5	Oct 3 - 7	Graph theory - the handshaking lemma Kőnig's theorem	A2 due Oct 4
Oct 10 - 14: Reading Week (no lectures)			
6	Oct 17 - 21	Walks, trails, paths, Hamilton paths/cycles Planar graphs and Euler's formula	
7	Oct 24 - 28	Graph colouring Proof of five colour theorem	A3 due Oct 25
8	Oct 31 - Nov 4	Extra problems in combinatorics (review)	Midterm on Nov 2 (during lecture)
9	Nov 7 - 11	Network flows Trees	A4 due Nov 8
10	Nov 14 - 18	Inclusion-exclusion Recurrence relations	
11	Nov 21 - 25	Tower of Hanoi Generating functions	A5 due Nov 22
12	Nov 28 - Dec 2	Exponential generating functions Design theory	
13	Dec 5 (1 hour)	Block designs and finite geometries	A6 due Dec 5
Dec 8 - 20: Final Exam Period			

Academic Integrity

- UTSC treats cases of cheating and plagiarism very seriously
- Potential offences include, but are not limited to:

In papers and assignments:

  - Using someone else's ideas or words without appropriate acknowledgement
  - Submitting your own work in more than one course without the permission of the instructor in all relevant courses
  - Making up sources or facts
  - Obtaining or providing unauthorized assistance on any assignment

On tests and exams:

  - Using or possessing unauthorized aids
  - Looking at someone else's answers during an exam or test
  - Misrepresenting your identity

In academic work:

  - Falsifying institutional documents or grades
  - Falsifying or altering any documentation required by the University, including (but not limited to) doctor's notes

- All suspected cases of academic dishonesty will be investigated following procedures outlined in the Code of Behaviour on Academic Matters.
- If you have questions or concerns about what constitutes appropriate academic behaviour or appropriate research and citation methods, please reach out to me.
- Note that you are expected to seek out additional information on academic integrity from me or from other institutional resources, for example, the University of Toronto website on Academic Integrity: https://www.academicintegrity.utoronto.ca/

Equity, Diversity and Inclusion

- The University welcomes and includes students, staff, and faculty from a wide range of backgrounds, cultural traditions, and spiritual beliefs
- Everyone should strive to create an atmosphere of **mutual respect**
- U of T does not condone discrimination or harassment against any persons or communities

Accommodation

- There may be times when you are unable to complete course work on time due to non-medical reasons. If you have concerns, speak to me.
- Students with diverse learning styles and needs are welcome in this course
- If you have a disability/health consideration that may require accommodations, please feel free to approach me or the **Access*Ability* Services** at https://www.utsc.utoronto.ca/ability/

Religious observances

- The University provides reasonable accommodation of the needs of students who observe religious holy days other than those already accommodated by ordinary scheduling and statutory holidays
- Students have a responsibility to alert members of the teaching staff in a timely fashion to upcoming religious observances and anticipated absences and instructors will make every reasonable effort to avoid scheduling tests, examinations or other compulsory activities at these times
- Please reach out to me as early as possible to communicate any anticipated absences related to religious observances, and to discuss any possible related implications for course work

Family care responsibilities

- The University of Toronto strives to provide a family-friendly environment
- You may wish to inform me if you are a student with family responsibilities
- If you are a student parent or have family responsibilities, you also may wish to visit the Family Care Office website at https://familycare.utoronto.ca/

Health & Wellness Centre

Feeling sad, anxious or lonely? Let's connect.

Call us at 416-287-7065 to speak to our counsellors or join our online groups.

Have your student card and health card ready when you call us.

Counsellors provide same day phone appointments and weekly online groups. Also, connect with your academic department about accessing our embedded counsellors.

Doctors and nurses provide phone appointments addressing health concerns, prescriptions, and health information.

Follow us on social media and check our website on health tips, online events and resources

SL270, Student Centre
416-287-7065
health-services@utsc.utoronto.ca
utsc.utoronto.ca/hwc

UNIVERSITY OF TORONTO SCARBOROUGH | HEALTH & WELLNESS CENTRE

UNIVERSITY OF TORONTO SCARBOROUGH
2165 Military Trail, Toronto, Ontario M1C 1A4

What is Combinatorics?

Combinatorics (a field of mathematics) answers questions related to:

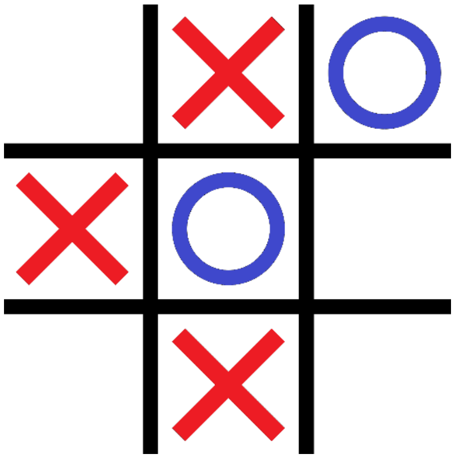
- **Existence**
  - Proving the existence or non-existence of combinatorial objects,
- **Construction**
  - Describing how to create such objects in the case they exist,
- **Enumeration**
  - Computing the number of such objects,
- **Optimization**
  - Determining which objects satisfy a certain extremal property.

Combinatorial Game Theory

Combinatorial games have the following properties:

- Two players take turns.
- There is no luck/chance involved.
- Both players have perfect information.

A player of a combinatorial game has a **winning strategy** if they can guarantee they will win.

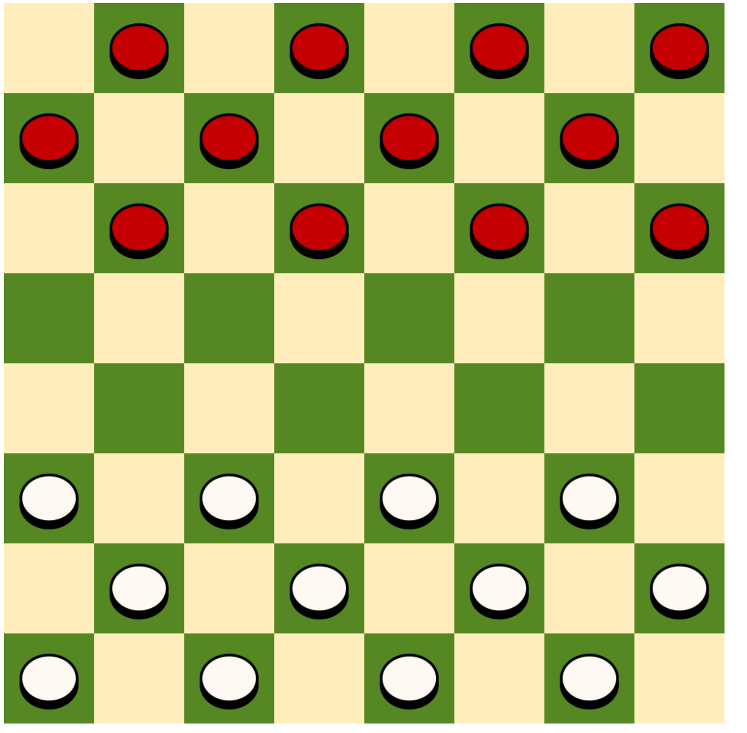


Tic-tac-toe

- Two players (X and O).
- Players take turns marking empty spaces.
- If you can get three in a row of your symbol (horizontal, vertical, or diagonal), you win!

If each side plays perfectly, the game will end in a **draw**.

Checkers (or draughts)



- Players take turns moving their pieces.
- A move consists of moving a piece diagonally to an adjacent unoccupied square.
- You can “jump” over opponent pieces to remove them from the game.
- The winner is the player to capture all of their opponents pieces.

- Checkers was (mathematically) solved in 2007.
- The solution was published by a team of **Canadian** computer scientists led by Prof. Jonathan Schaeffer.
- Schaeffer completed his B.Sc. at the [University of Toronto] and is currently a professor at the University of Alberta.

If each side plays perfectly, the game will end in a **draw**.

Image source: Scott Mair (https://www.flickr.com/photos/smair/14467244034/)

Connect Four

- Players each choose a colour and take turns dropping their tokens in the grid.
- Whoever can **connect four** of their tokens in a row (i.e., in a horizontal, vertical, or diagonal line) is the winner!

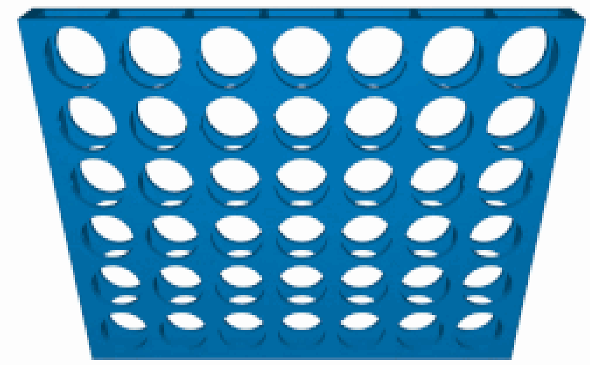
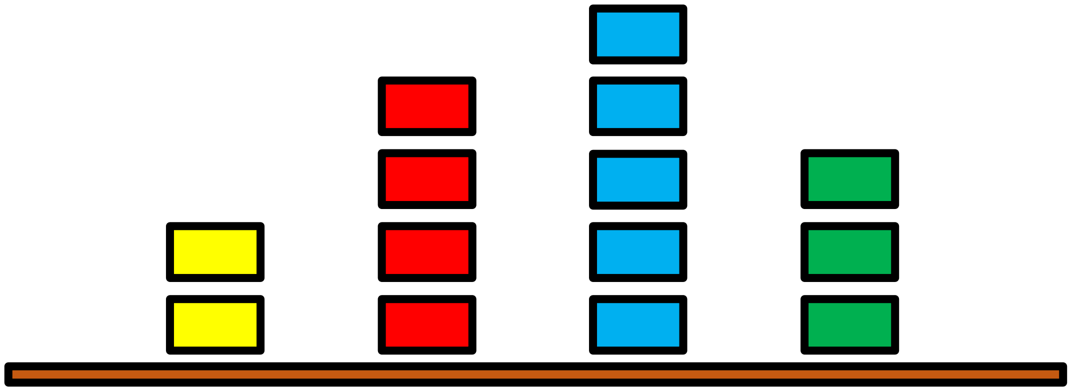


Image source: Silver Spoon (https://en.wikipedia.org/wiki/File:Connect_Four.gif)

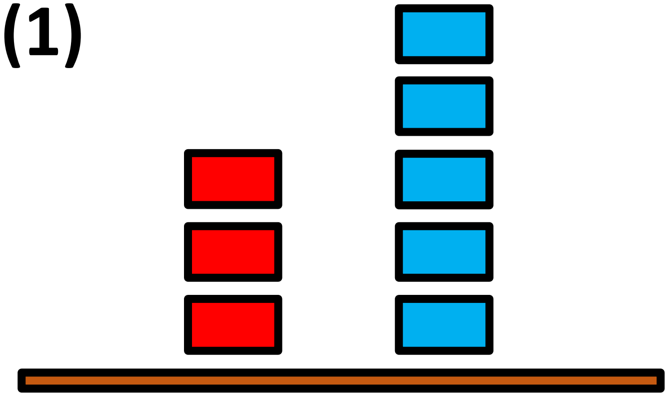
- Connect Four was (mathematically) solved in 1988.
- If each side plays perfectly, then the **first player** can always force a win.
- To guarantee a win (with perfect play), you should place your first piece in the middle column. If you play in one of the outer 4 columns first, the second player can win and if you play in columns 3 or 5, the game is a theoretical draw.

The Game of Nim

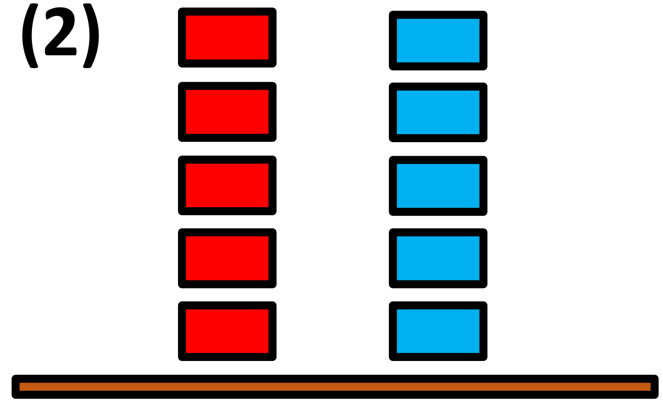


- Nim is a two player game.
- There are **n** piles of candies (above **n = 4**).
- On each turn, a player chooses a pile and removes at least one candy from it (they may remove any number of candies provided they all come from the same pile).
- The player who takes the last candy is the winner of the game.

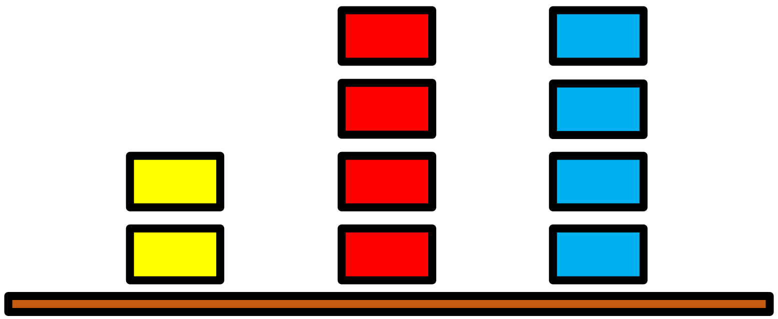
(1)



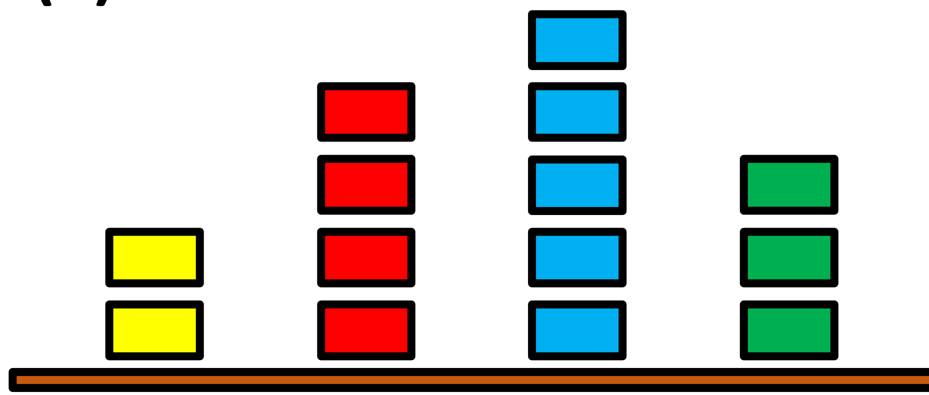
(2)



(3)



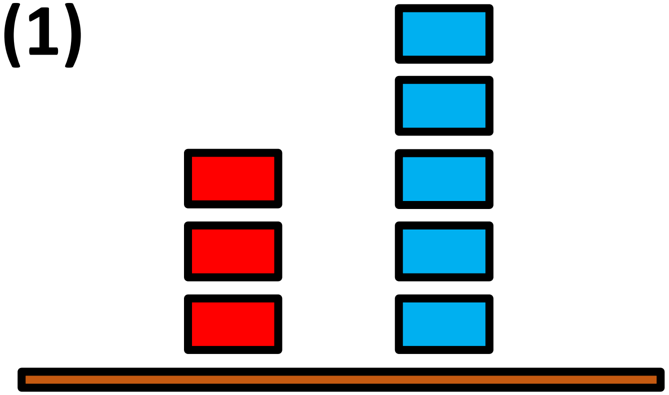
(4)



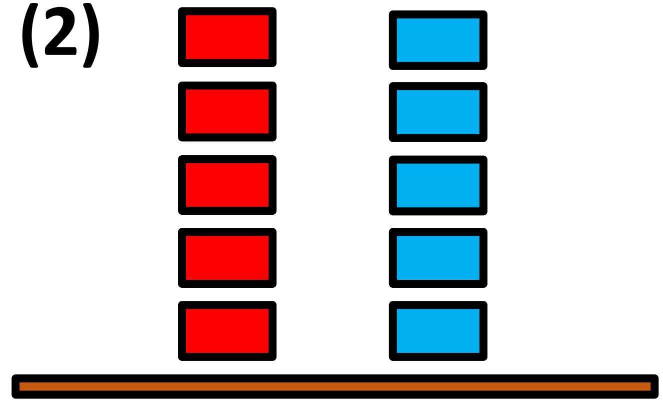
In each situation above:

- Which player has a winning strategy (first or second), if any?
- Describe what the strategy is.
- **Prove** your strategy works no matter how the other player reacts.

(1)



(2)



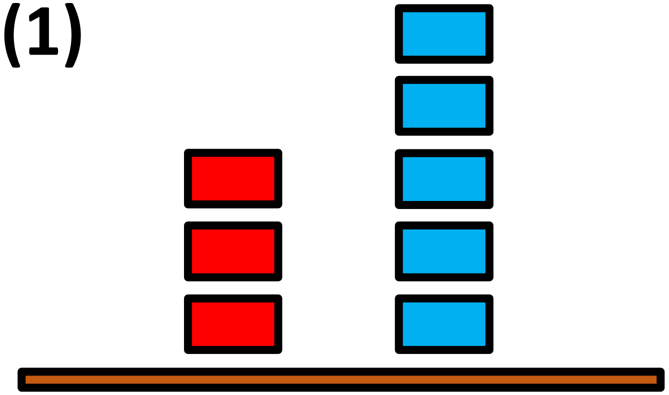
Which player can guarantee a win in game (1)? What about (2)?

Describe your **winning strategy** when there is two piles.

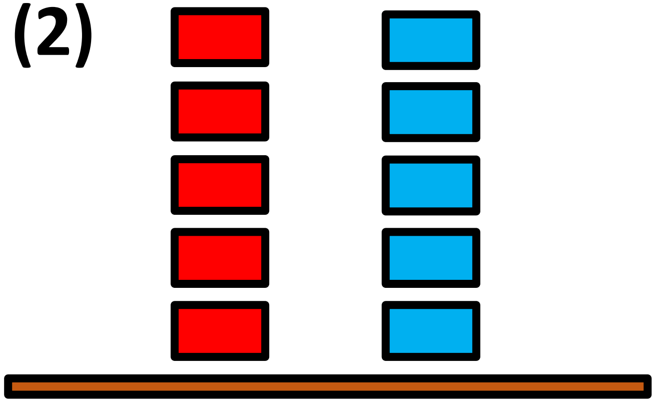
Write down a mathematical theorem describing all possible scenarios for two piles along with who can guarantee a win.

Theorem: For a game of Nim with two piles of sizes m and n:

(1)



(2)



Which player can guarantee a win in game (1)? What about (2)?

- In game (1), the first player can guarantee a win.
- In game (2), the second player can guarantee a win.

Describe your **winning strategy** when there is two piles.

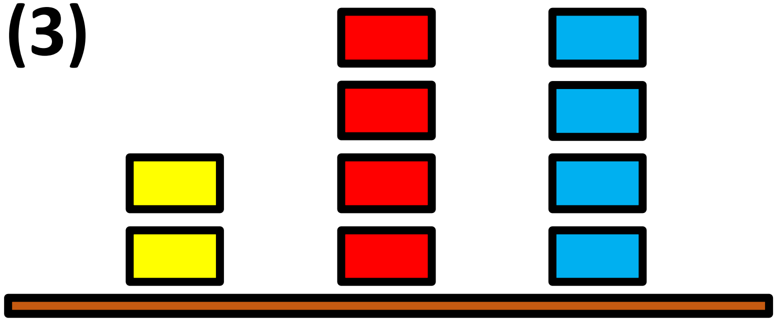
- Strategy: **Balance** the two piles so they are the same size.

Write down a mathematical theorem describing all possible scenarios for two piles along with who can guarantee a win.

Theorem: For a game of Nim with two piles of sizes m and n:

- If $n = m$, then **player two** can force a win.
- If $n \neq m$, then **player one** can force a win.

(3)



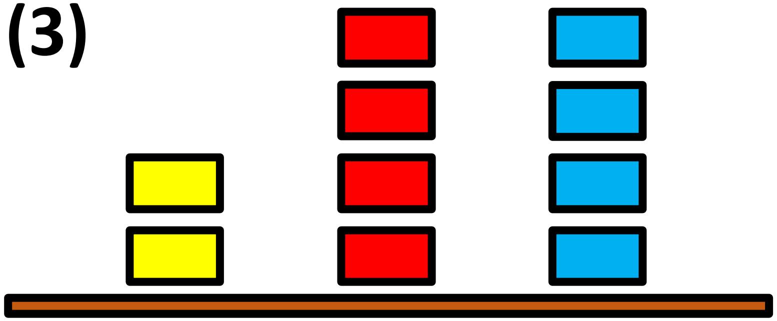
Which player can guarantee a win in game (3)?

Describe your **winning strategy**.

- Strategy:

Theorem: For a game of Nim with three piles of sizes m, n and n, then **…**

(3)



Which player can guarantee a win in game (3)?

- The first player can guarantee a win.

Describe your **winning strategy**.

- <u>Strategy:</u> Player one removes all **candies** from pile 1. Now we are in the scenario with two piles of the same size; the second player will unbalance the piles and the first player can always restore balance thus can win.

<u>Theorem:</u> For a game of Nim with three piles of sizes m, n and n, then **player one** can force a win.

Binary

Binary: something made of two parts

Decimal system (base-10)

Uses **ten** numerals from 0 to 9

hundreds tens ones

3 2 9 = 3 x 100 + 2 x 10 + 9 x 1

Binary system (base-2)

Uses **two** numerals 0 and 1

sixteens eights fours twos ones

1 0 1 0 1 = 1x16 + 0x8 + 1x4 + 0x2 + 1x1

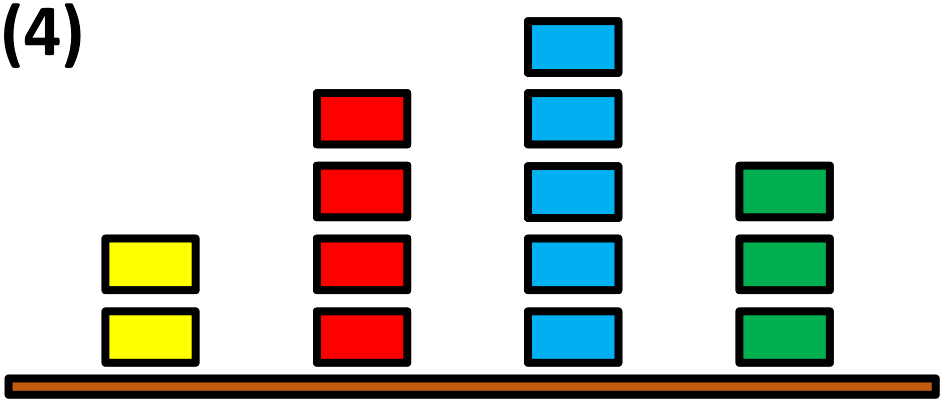
- Write the base-10 number 421 in **binary**.
- Write the base-2 number 10010101 in **decimal**.

Decimal	Binary
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001
10	1010
11	1011
12	1100
13	1101
14	1110
15	1111

2^4	2^3	2^2	2^1	2^0	
16	8	4	2	1	
0	0	0	0	0	0 0

Image source: Ephert (https://commons.wikimedia.org/wiki/File:Binary_counter.gif)

(4)



In game (4):

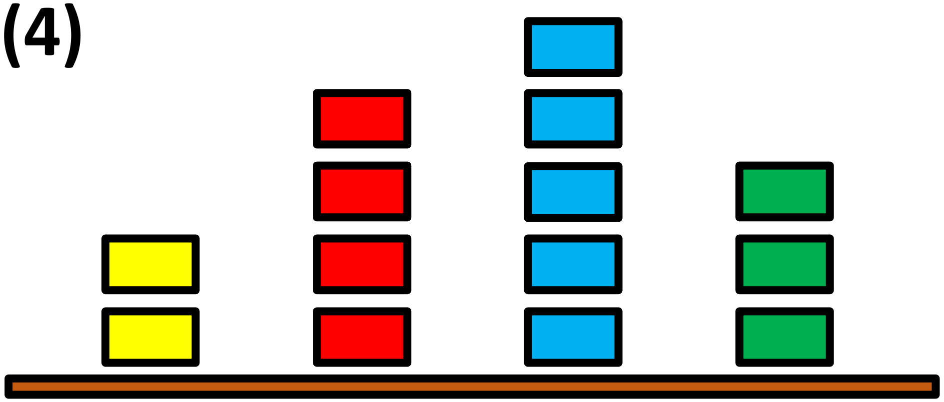
- Which player has a winning strategy (first or second), if any?
- Describe what the strategy is.
- **Prove** your strategy works no matter how the other player reacts.

Hint:

Create a table with four rows and convert each pile size to binary.

Pile size		fours	twos	ones
2				
4				
5				
3				

(4)



In game (4):

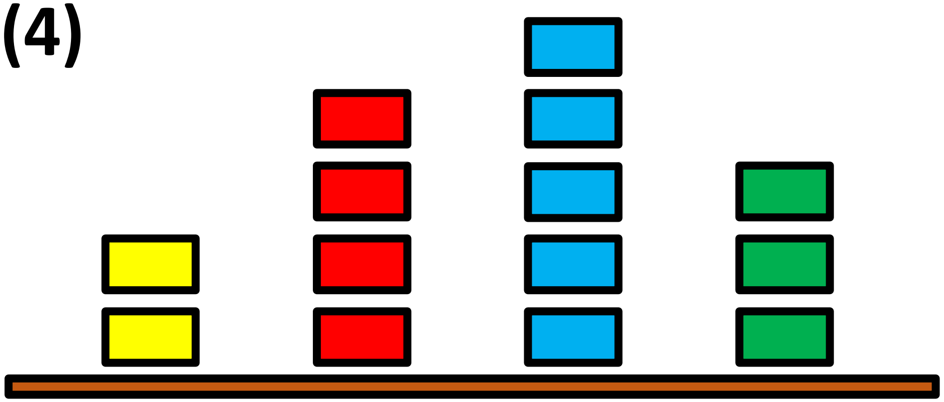
- Which player has a winning strategy (first or second), if any?
- Describe what the strategy is.
- <u>**Prove**</u> your strategy works no matter how the other player reacts.

<u>**Hint:**</u>

Create a table with four rows and convert each pile size to binary.

Pile size		fours	twos	ones
2		0	1	0
4		1	0	0
5		1	0	1
3		0	1	1

(4)

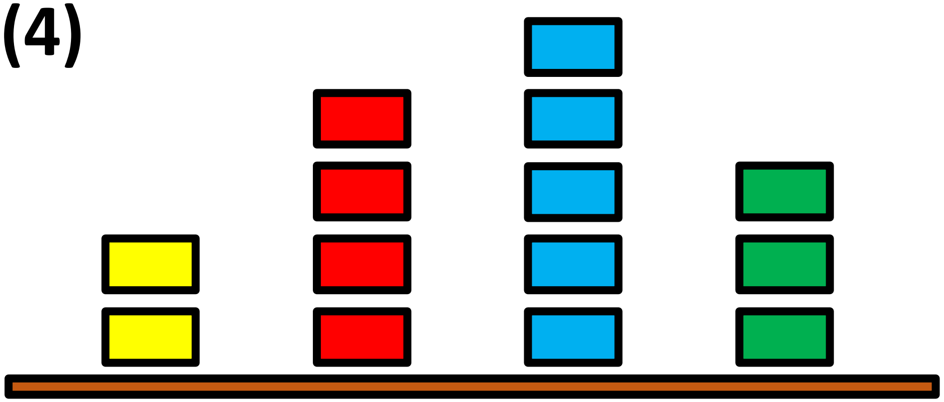


Pile size		fours	twos	ones
2		0	1	0
4		1	0	0
5		1	0	1
3		0	1	1

Strategy:

Prove your strategy works no matter how the other player reacts.

(4)



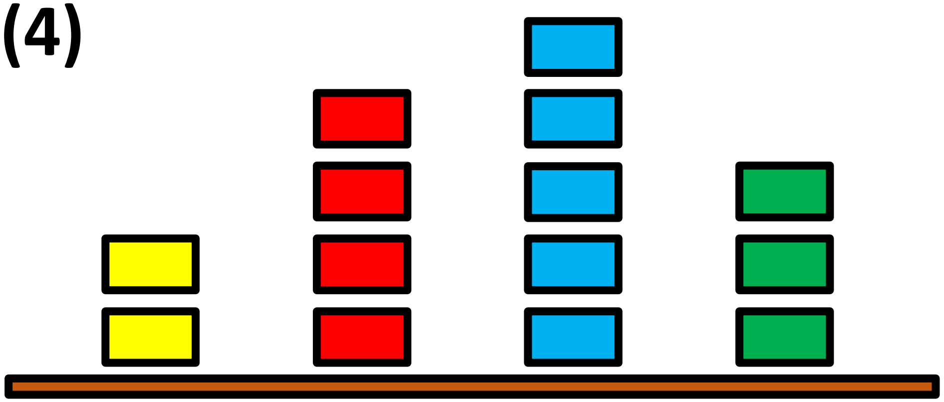
Pile size		fours	twos	ones
2		0	1	0
4		1	0	0
5		1	0	1
3		0	1	1

Strategy:

- **Balance** each column so that there is an even number of 1’s.

Prove your strategy works no matter how the other player reacts.

(4)



Pile size		fours	twos	ones
2		0	1	0
4		1	0	0
5		1	0	1
3		0	1	1

Strategy:

- **Balance** each column so that there is an even number of 1's.

Prove your strategy works no matter how the other player reacts.

Fact 1.

If the table is balanced, then the next move will unbalance it. **Why?**

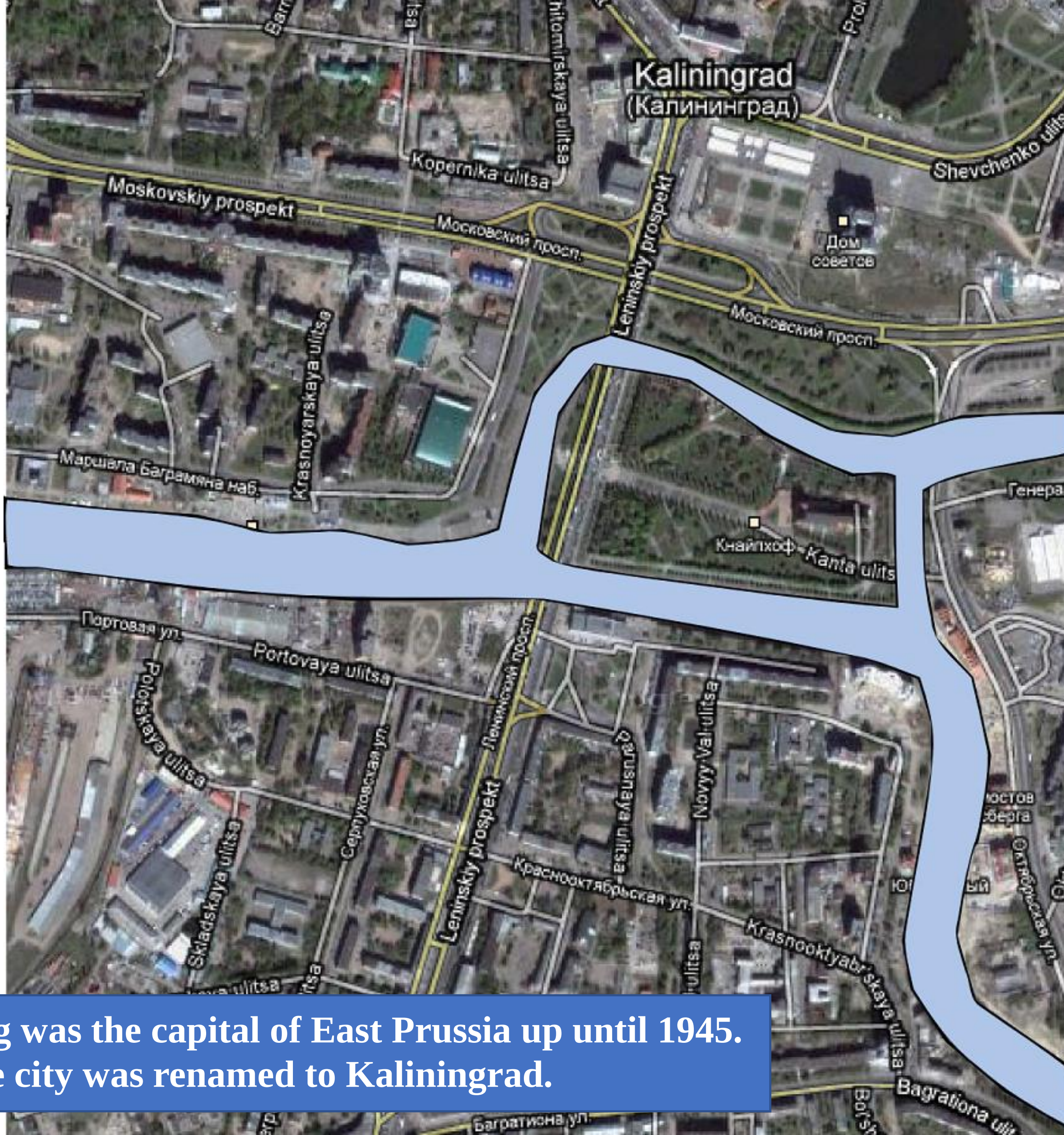
Fact 2.

If the table is unbalanced, then there is always a move that can be made to balance it.

Why? The proof is to take the most significant bit in the "Nim-sum" and pick a pile with a 1 there, then remove candies to balance.

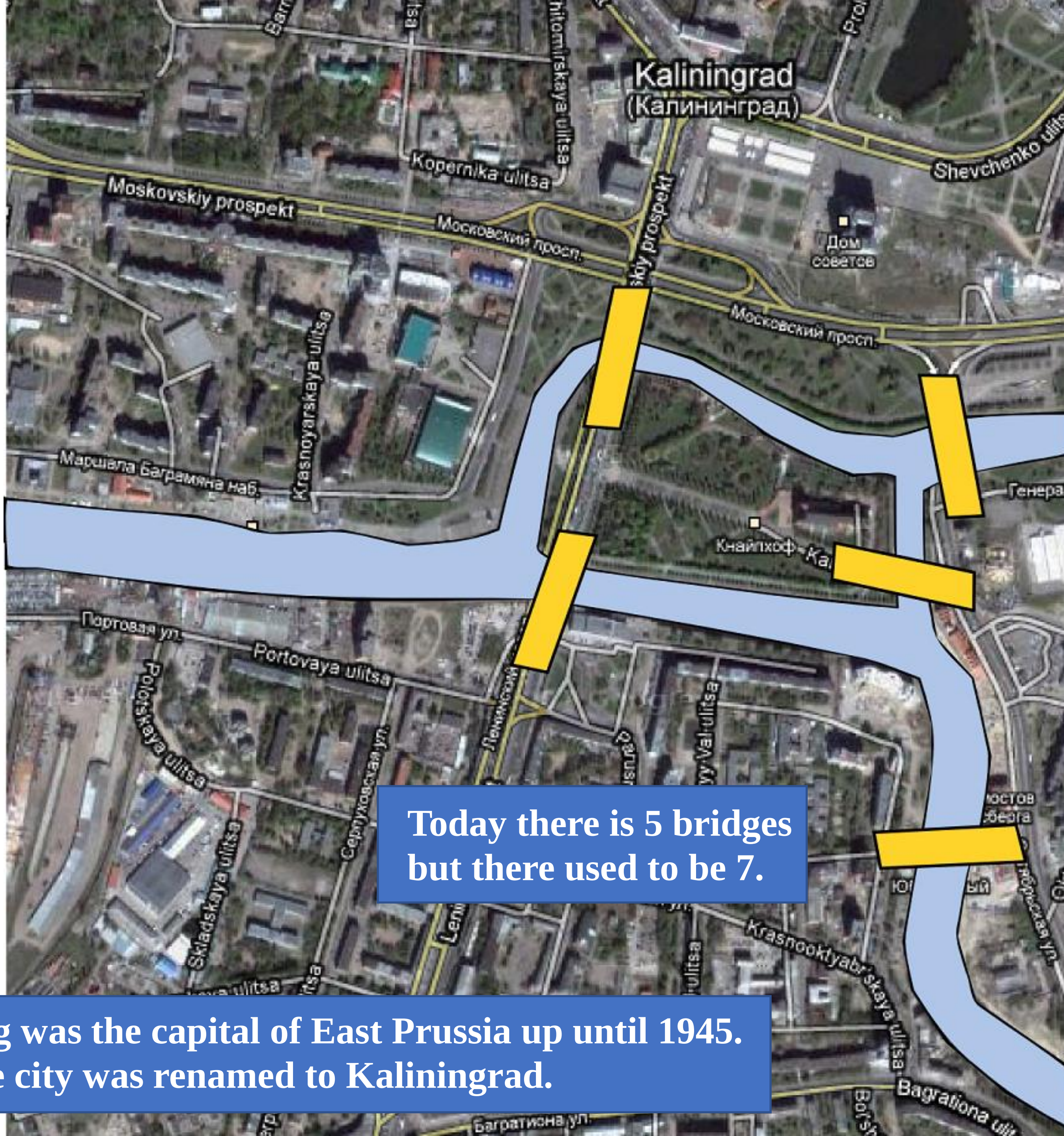
Introduction to graph theory

- The seven bridges of Königsberg
- What is a graph?
- Isomorphic graphs
- Euler’s Theorem
- The five room puzzle



Königsberg was the capital of East Prussia up until 1945. In 1946 the city was renamed to Kaliningrad.

Source: Google Maps



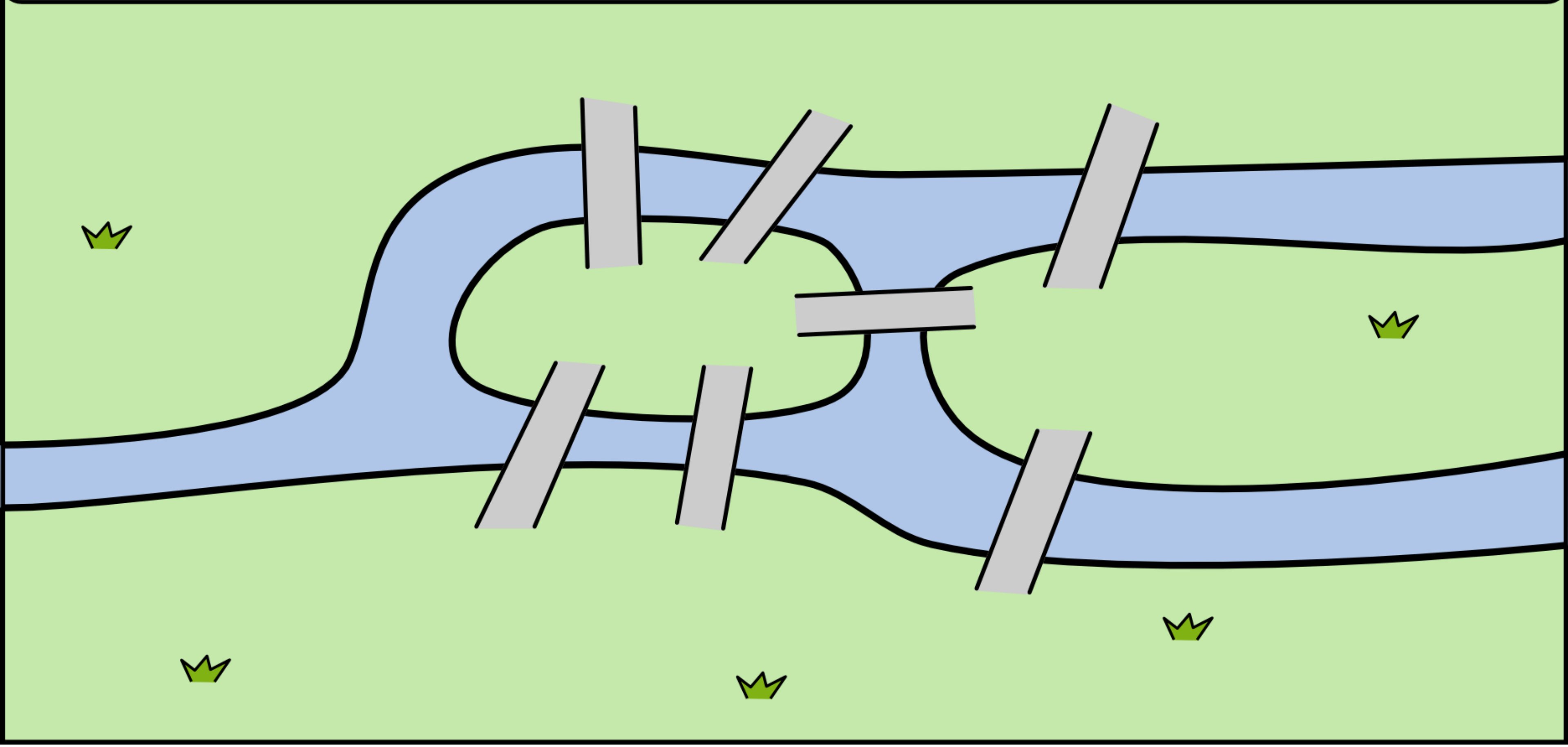
Today there is 5 bridges but there used to be 7.

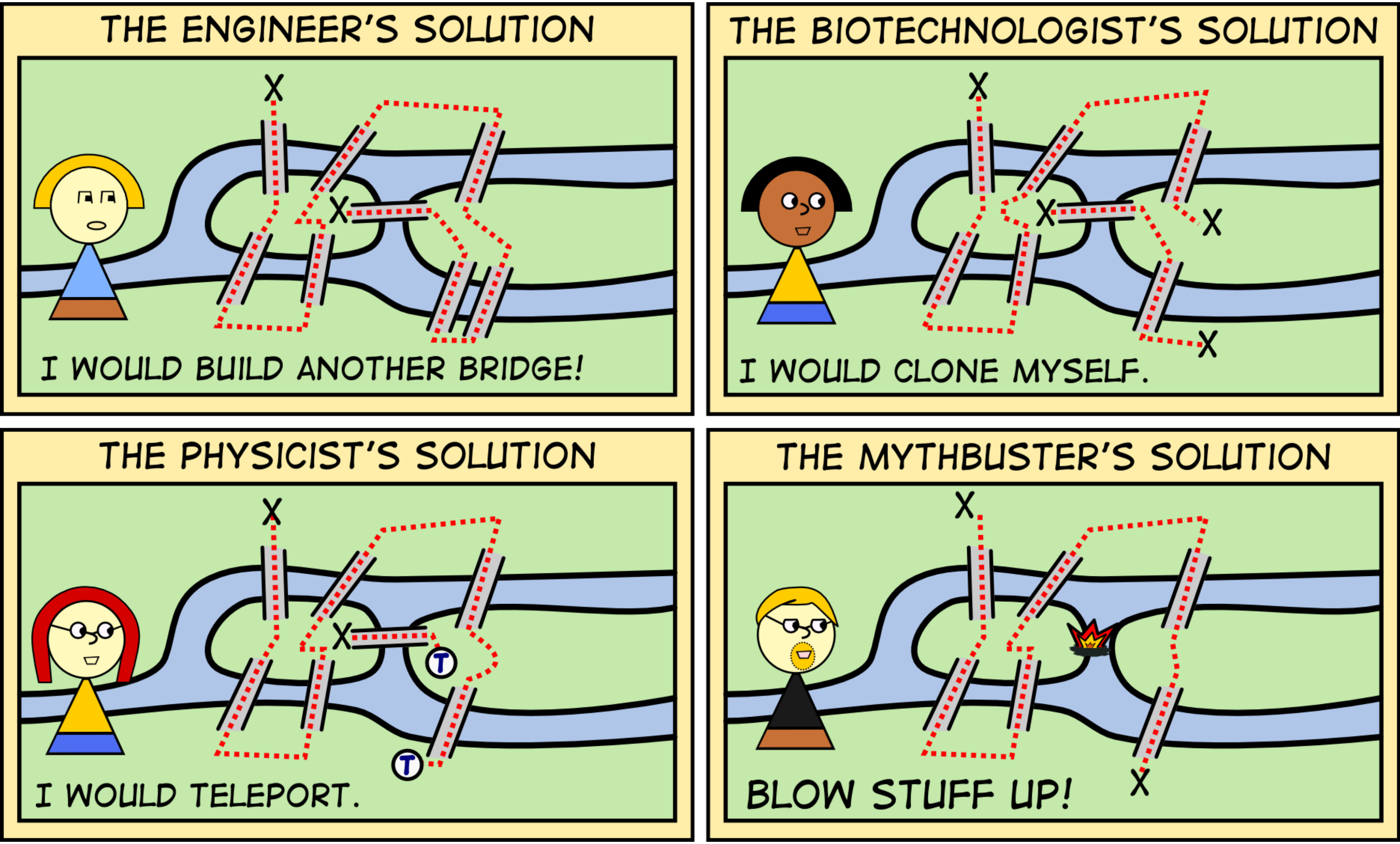
Königsberg was the capital of East Prussia up until 1945. In 1946 the city was renamed to Kaliningrad.

Source: Google Maps

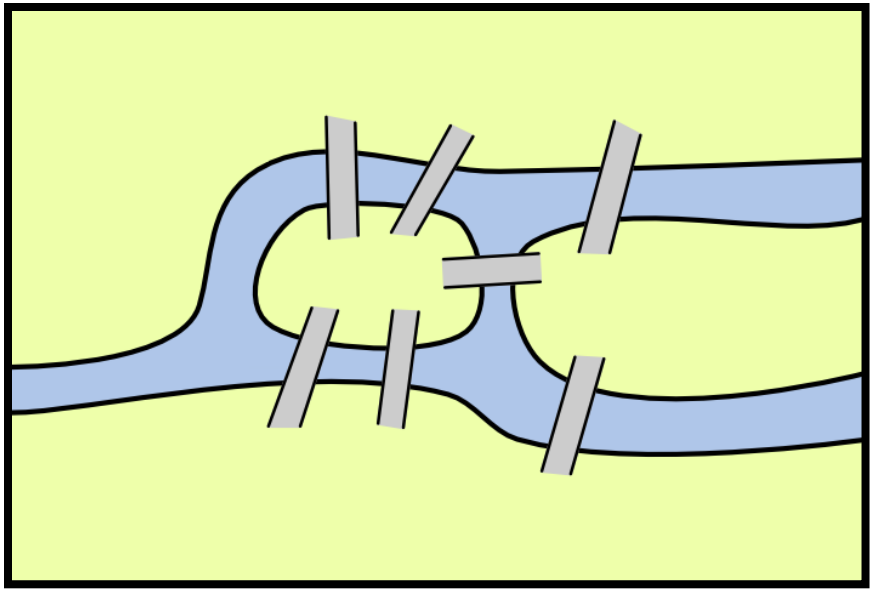
The Seven Bridges of Königsberg

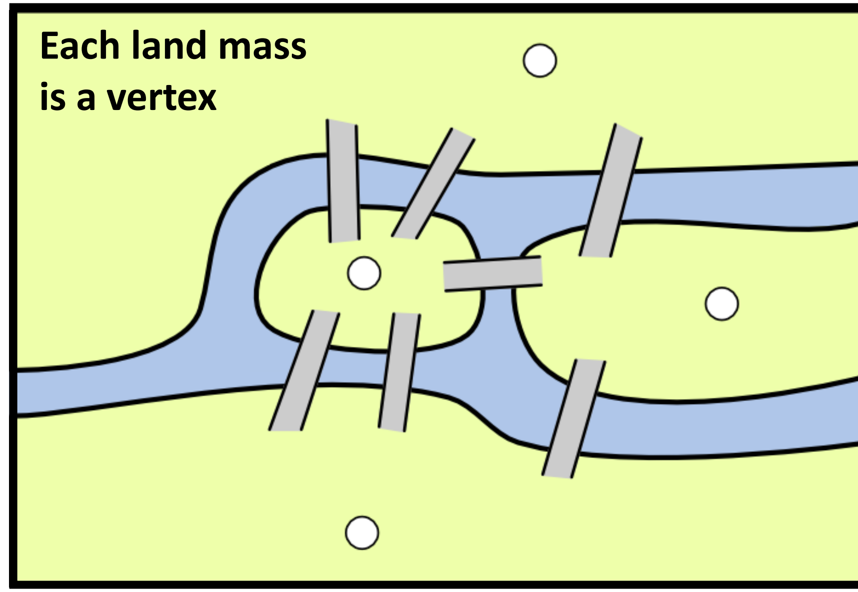
Below is the city of Königsberg with four land masses and seven bridges connecting the various land masses. Can you find a walk through the city of Königsberg that crosses each bridge exactly once? You may start at any land mass you wish but may only travel between land masses by using a bridge.

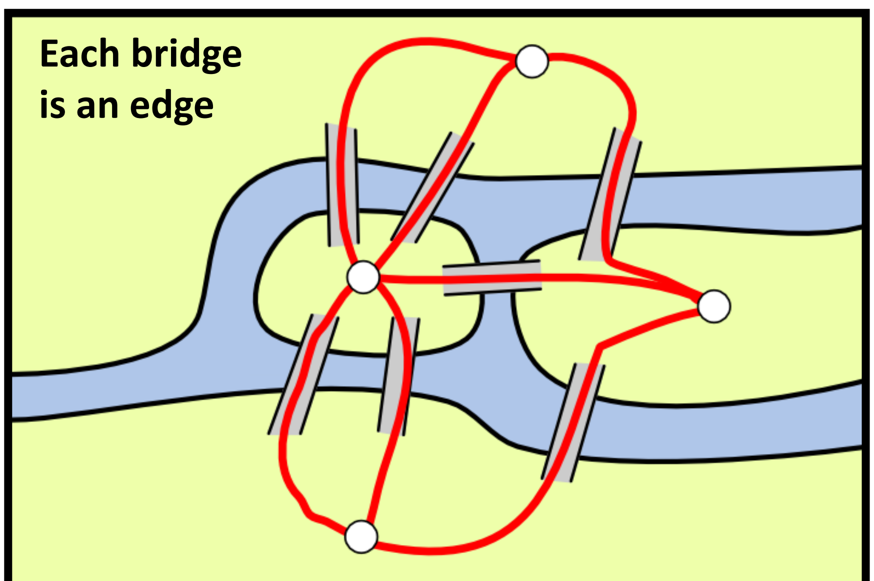


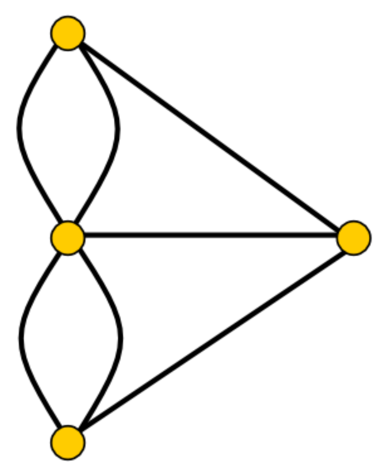

THE ENGINEER'S SOLUTION
X
X
I WOULD BUILD ANOTHER BRIDGE!
THE BIOTECHNOLOGIST'S SOLUTION
X
X
X
X
I WOULD CLONE MYSELF.
THE PHYSICIST'S SOLUTION
X
X
T
T
I WOULD TELEPORT.
THE MYTHBUSTER'S SOLUTION
X
X
BLOW STUFF UP!

Solution to Seven Bridges Problem









If such a "path" exists then only the starting and ending points can have an odd number of edges coming out of it.

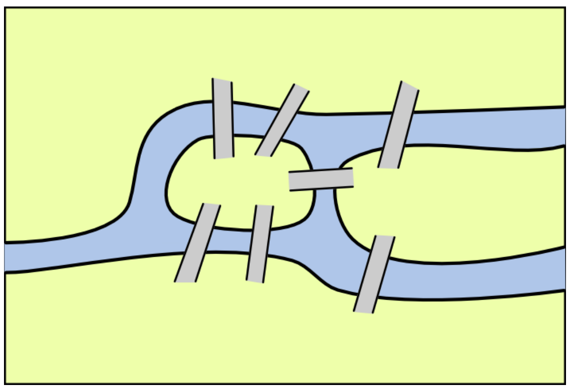
Why? (Euler's Theorem)

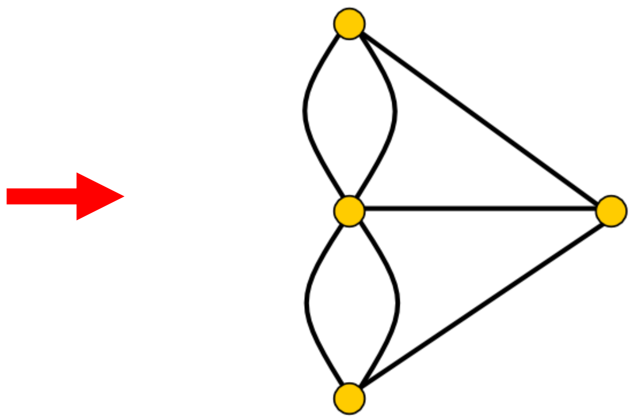
Thus, each of the four dots must be the start or end of the path, contradiction.

In 1736, Euler published a landmark paper solving the *Seven Bridges of Konigsberg* problem. This paper laid the foundations for Graph Theory.

Euler's Solution

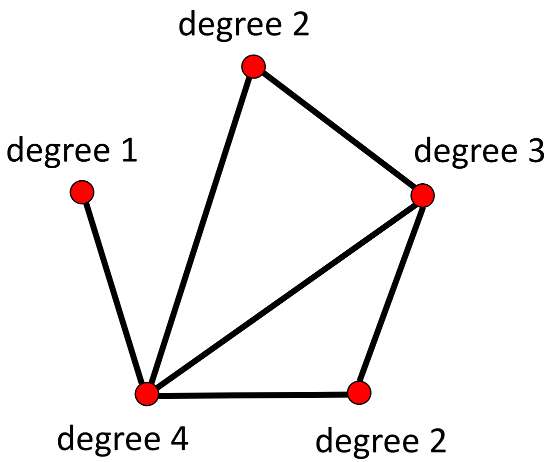
The *Seven Bridges of Konigsberg* puzzle corresponds to finding an Eulerian path in the graph below (that is, a "path" that uses each line exactly once):





- Any dot that is not the starting or ending point of the "path" must have an even number of lines joined to it (this is because you need a way to arrive at the dot and then leave the dot thus using two lines each time you visit the dot).
- If a dot has an odd number of lines joined to it, it needs to be either the starting point or ending point of the "path".
- In the graph above, **every** dot has an odd number of lines joined to it, but you may only have one starting point and one ending point.
- Therefore, there is no "path" that uses each line only once!

Edge-traceable and Eulerian graphs



The **degree** of each vertex is the number of edges coming out of it.

different start and end points

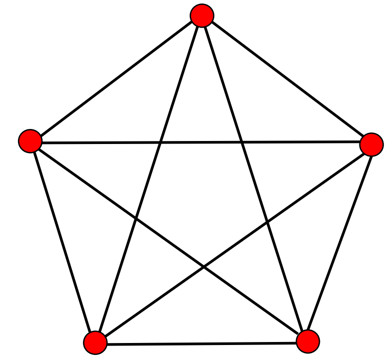
Theorem:
Let G be a connected graph. Then G is "**edge-traceable**" if and only if G has exactly two vertices of odd degree.

same start and end point

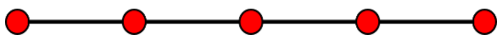
Theorem:
Let G be a connected graph. Then G is "**Eulerian**" if and only if every vertex of G has even degree.

A **graph** (or **network**) is an object with:

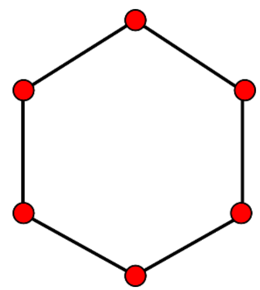
- a collection of dots (called **vertices**)
- a collection of lines (called **edges**)
- each line connects two vertices



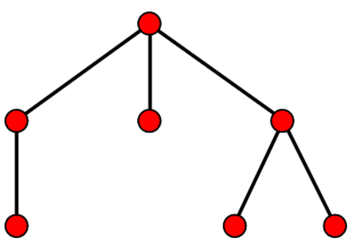
a complete graph



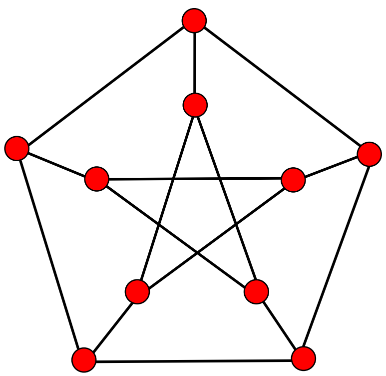
a path graph



a cycle graph

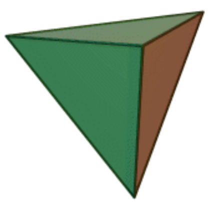
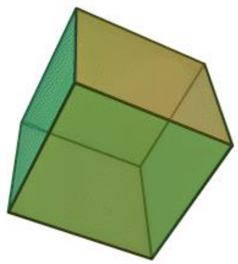
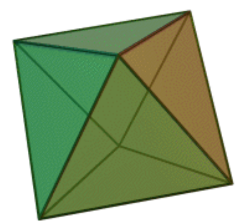
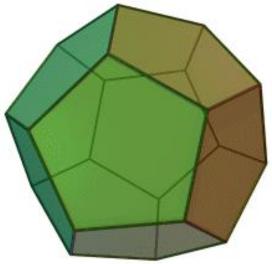
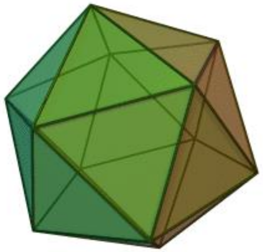
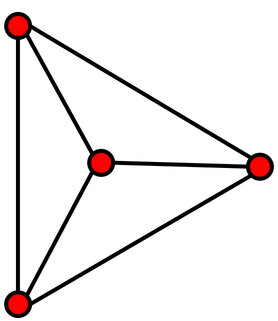
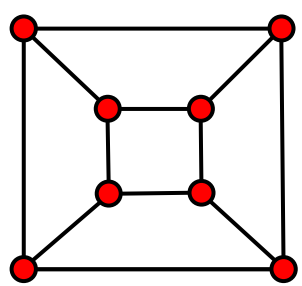
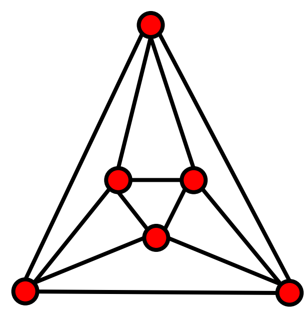
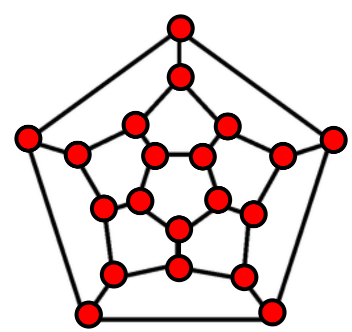
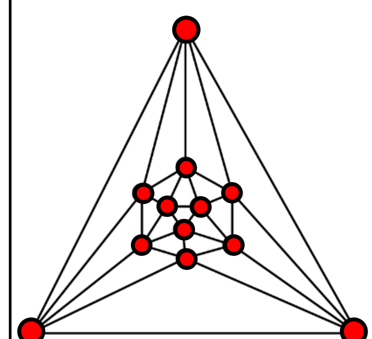


a tree graph

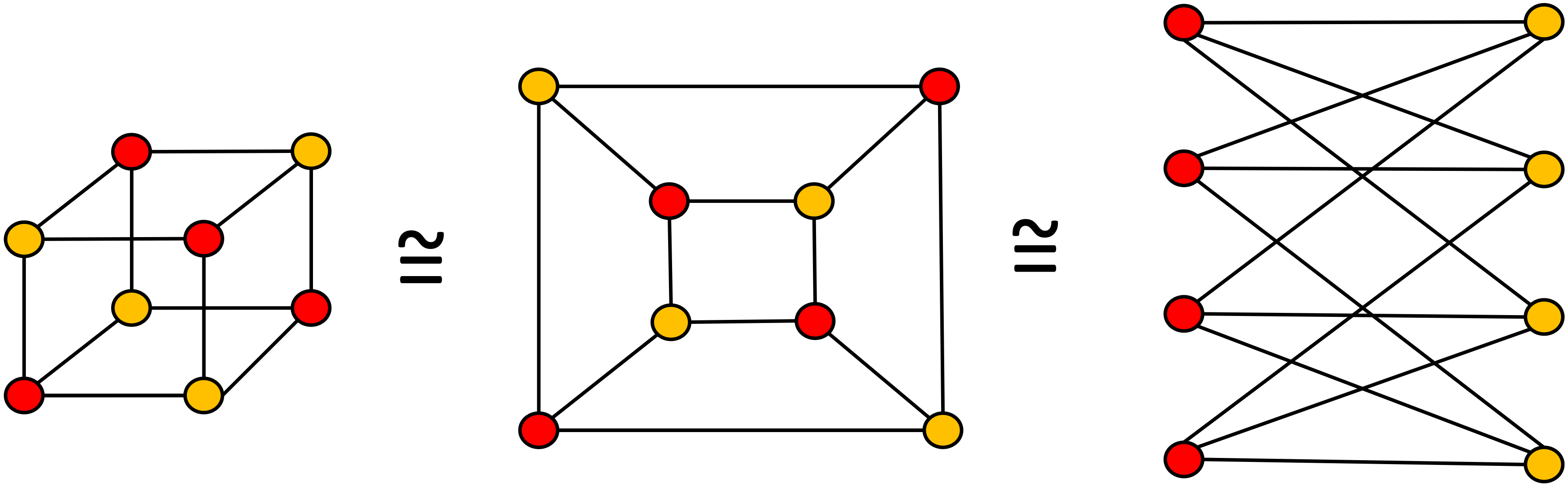


the Petersen graph

Why call them vertices and edges?

Tetrahedron (four faces)	Hexahedron (six faces)	Octahedron (eight faces)	Dodecahedron (twelve faces)	Icosahedron (twenty faces)
				
				

Structure matters, geometry does not!

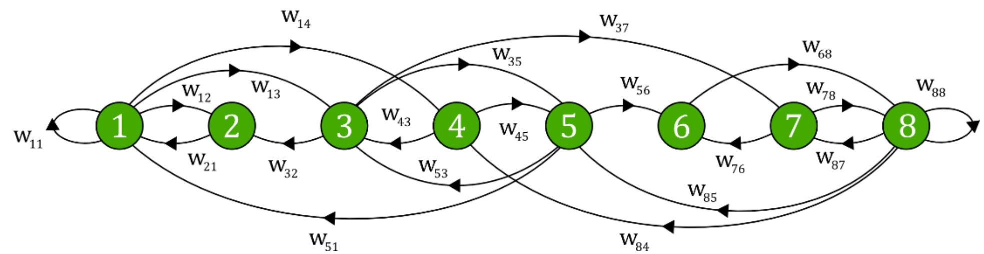


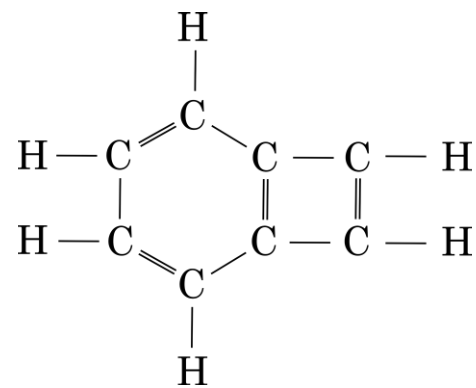
These three graphs are exactly the **same** (isomorphic):

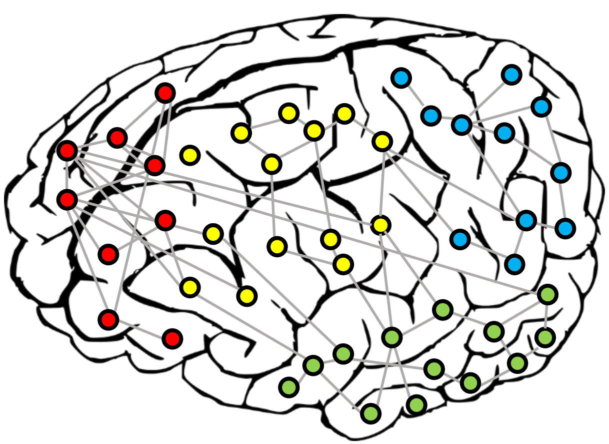
- same structure
- same mathematical properties
- only the connections matter
- the connections represent **interactions**

Applications

- transportation networks
- **maps** (and shortest paths)
- computer networks
- electrical circuitry
- social sciences
- dynamical systems
- chemical molecules
- neuroscience (neural networks)
- epileptic seizures
- geophysics (earthquake forecasting)
- **scheduling problems**







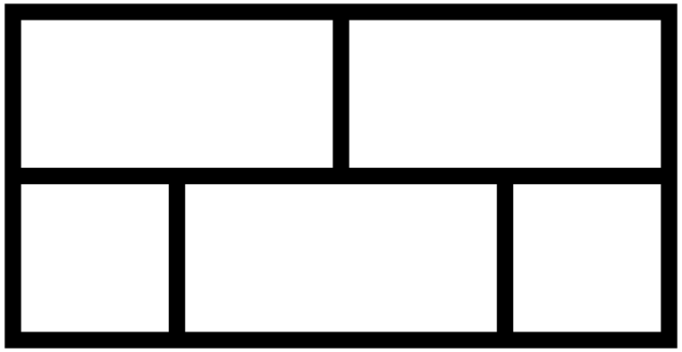
Bookstore Weekly Schedule

Week of : May 1-7

	Sun 5/1	Mon 5/2	Tue 5/3	Wed 5/4	Thu 5/5	Fri 5/6	Sat 5/7
Bourne, M.	9am-6pm	9am-6pm	9am-1pm				
Brown, M.	11am-8pm	11am-8pm	7am-11am		9am-6pm	9am-6pm	
Gatsby, A.		11am-8pm	11am-8pm	7am-11am			
Gordon, A.				11am-8pm	11am-8pm	7am-11am	
Hender, V.							
Lawson, N.		11am-8pm	11am-8pm	7am-11am			
Shiro, I.					9am-6pm	9am-6pm	9am-1pm
Smith, T.	9am-6pm	9am-6pm				9am-6pm	9am-6pm
Talbot, P.						9am-6pm	9am-6pm
Tate, L.				9am-6pm	9am-6pm		
Vasquez, A.	11am-8pm	11am-8pm	7am-11am				

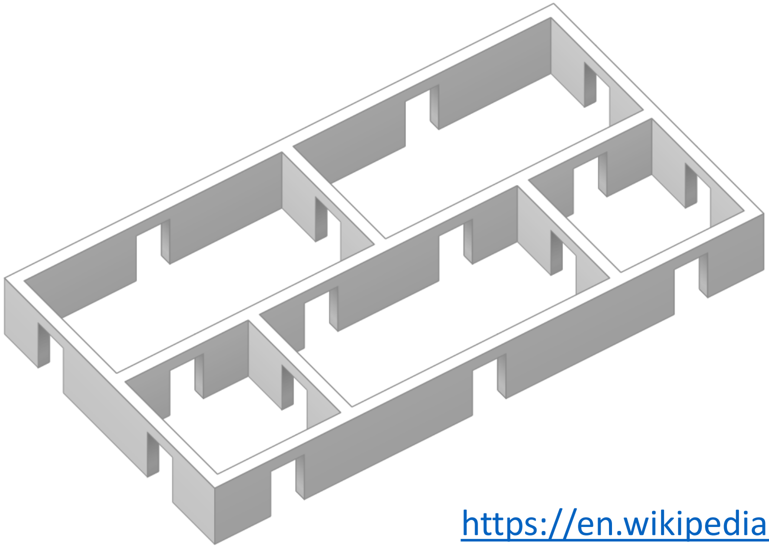
The five rooms puzzle

Can you draw a path that goes through each wall exactly once?
The path cannot go through any intersections!

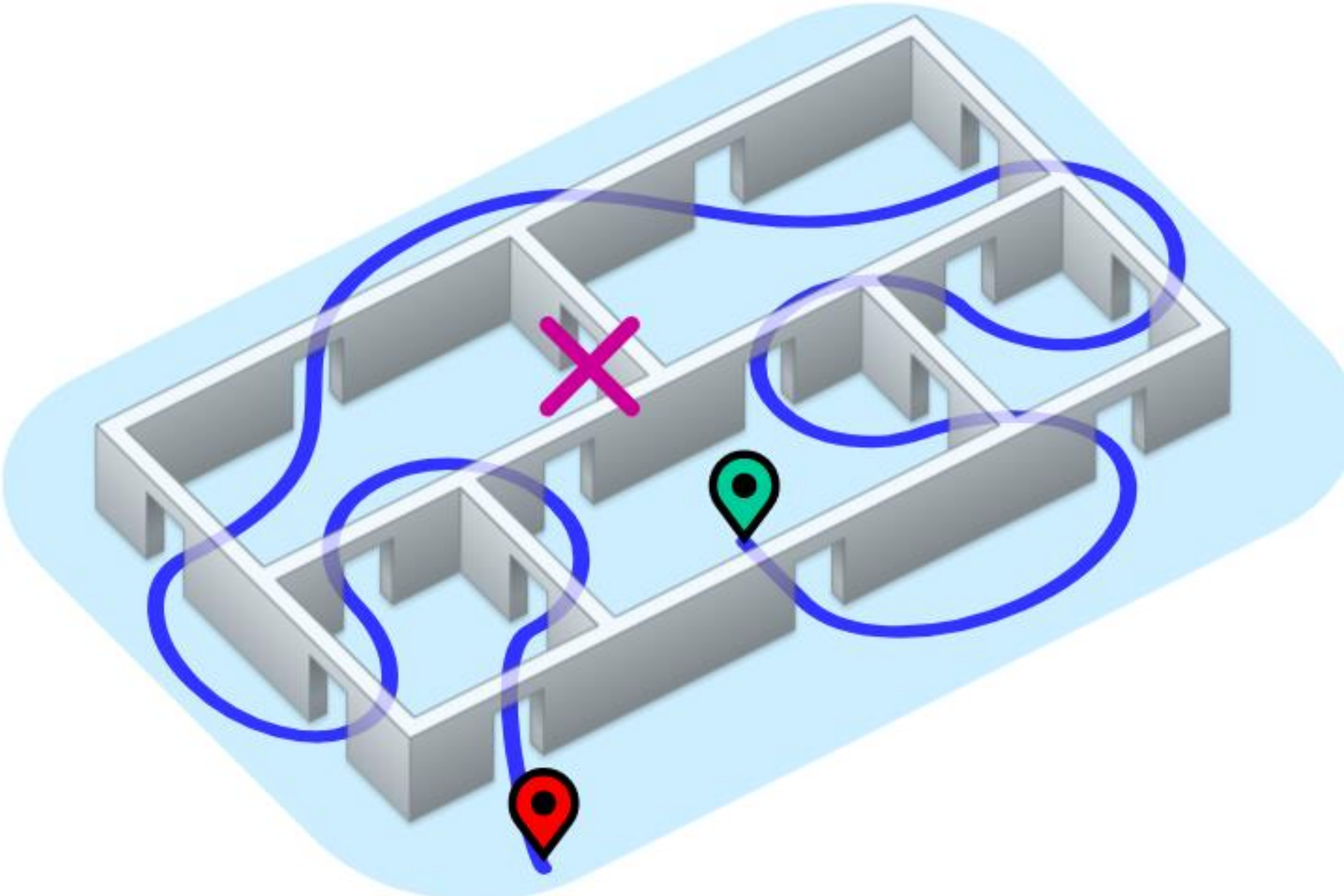


Count: 16 walls

Alternate formulation with doors:



https://en.wikipedia.org/wiki/File:5_room_puzzle_blank.svg



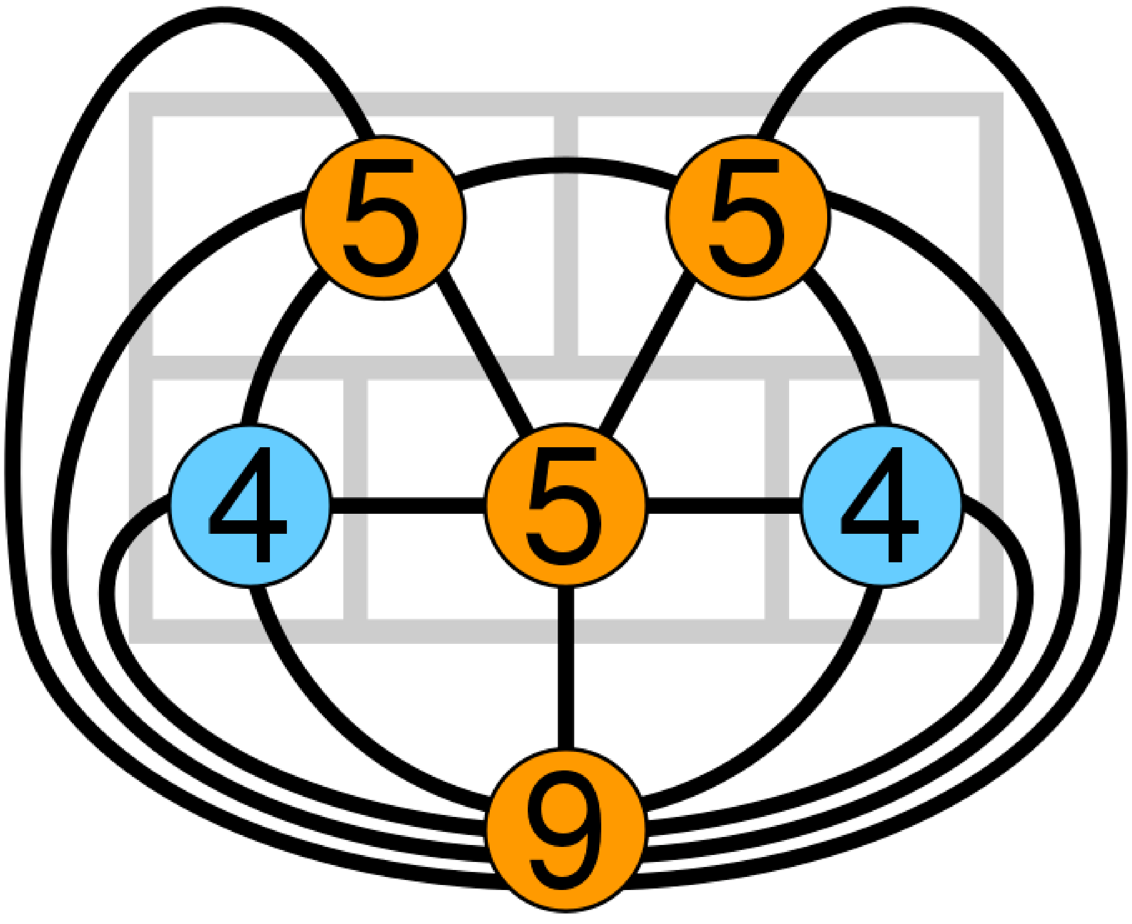
https://upload.wikimedia.org/wikipedia/commons/c/cd/5_room_puzzle.svg

Solution:

It is impossible.

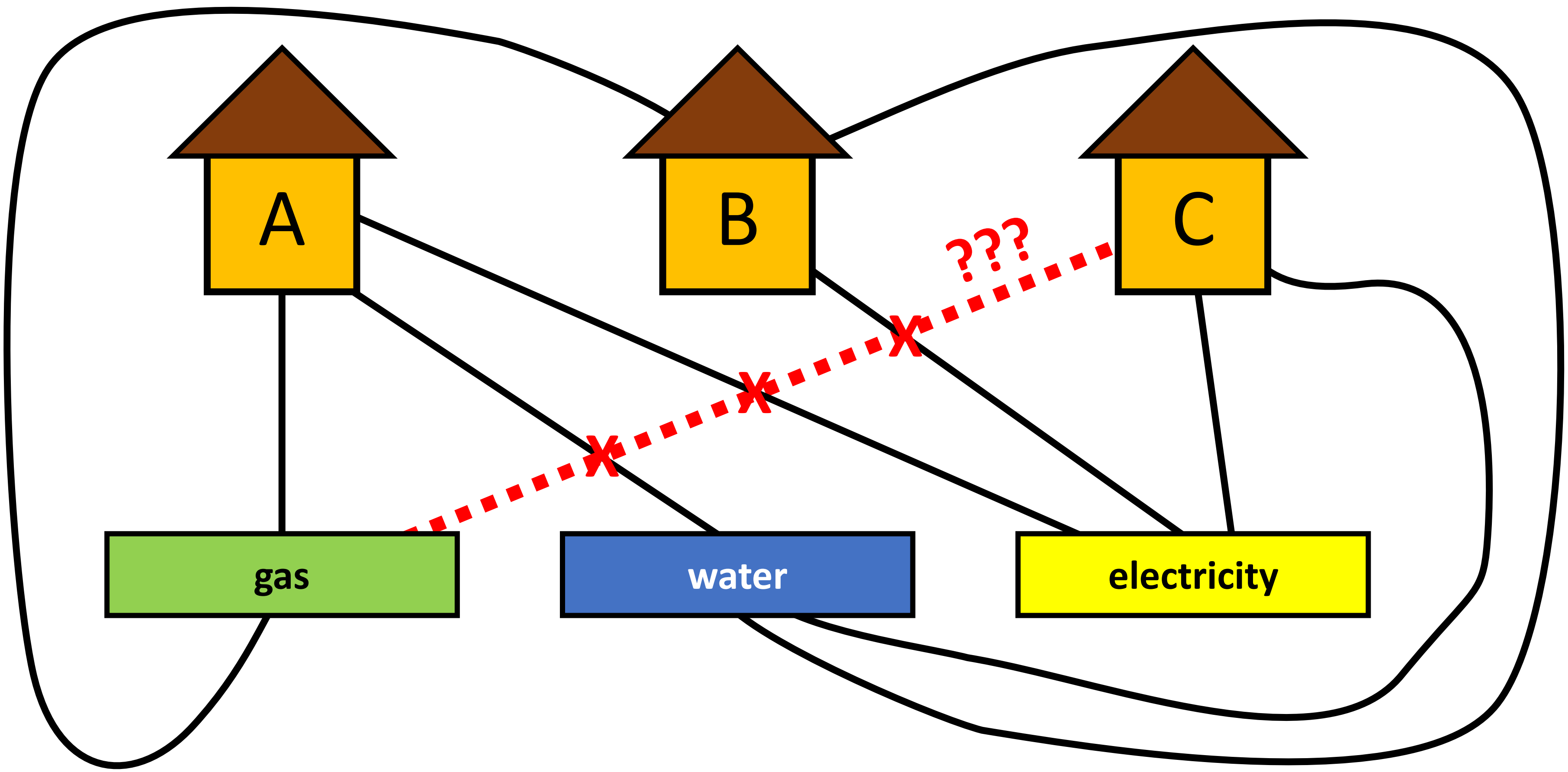
Draw a multigraph and apply Euler's Theorem.

(Note: the theorem also applies to multigraphs.)



The Three Utilities Problem

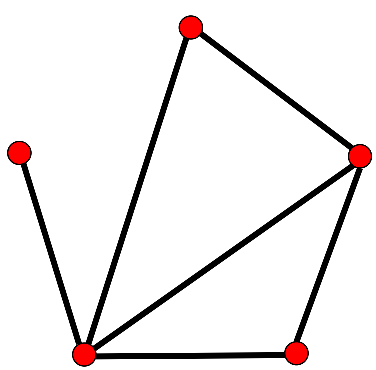
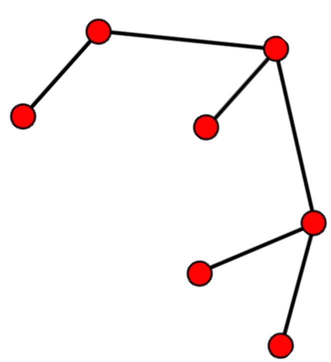
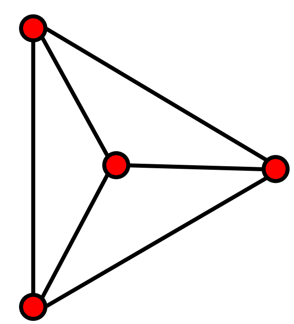
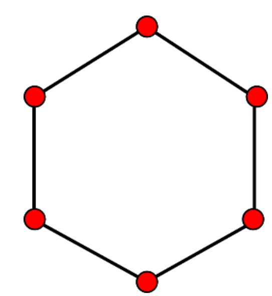
Three houses each need to be connected to gas, water and electric companies. Without using a third dimension or sending any connections through another house or company, is there a way to make all nine connections without crossings?



Definition:
A graph is called **planar** if it can be drawn in the plane so that no two lines cross one another.

- V = # of vertices
- E = # of edges
- F = # of faces (we also count the outer infinite face)

Euler made a remarkable discovery that relates **V, E** and **F**
See if you can make the same discovery!

Euler's formula

Theorem:
Let G be a connected planar graph with a given plane drawing having V vertices, E edges and F faces. Then $\mathbf{V - E + F = 2}$.

In general: $\mathbf{V - E + F = 1 + (\#\ of\ pieces)}$

This formula can be used to show that the Three Utilities Problem has no solution. That is, the graph $K_{3,3}$ is **not** planar.