# Introduction to Optimization

Apurva Nakade

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# Preface

These are class notes for a quarter long course on *Introduction to Optimization* (Math 368) at Northwestern University taught during the Winter and Spring of 2022. The class is aimed at upper-level undergrads who've completed a basic sequence in linear algebra and calculus. The prerequisites for the linear programming part are Gaussian elimination and basic theorem proving. For the non-linear programming part, you'll need to know basic properties of gradients and directional derivatives.

The textbook used in the class was *Linear Programming, Foundations and Extensions*, by Robert J. Vanderbei. The textbook is freely available to Northwestern students via SpringerLink. The textbook was used mainly as a reference for examples and exercises.

 Any suggestions and improvements to the notes are welcome and greatly appreciated. You can send me an email or open an issue on Github.

# Chapter 1

# Introduction

An optimization problem consists of maximizing or minimizing a real function over a set of input values from within an allowed set.

$$\begin{aligned} \text{optimize: } & f(x_1, \ldots, x_n) \\ \text{subject to: } & (x_1, \ldots, x_n) \in D. \end{aligned}$$

These kinds questions arise in all quantitative disciplines from computer science and engineering to operations research and economics.

Stated so generally, this question is too broad. To perform any meaningful analysis, we need to make several simplifying assumptions about $f$ and $D$. To begin with, we'll assume that the function $f$ is linear and the constraint set $D$ is described using linear inequalities. The study of this kind of a problem is called *Linear Programming*. Despite the simplicity of the linear programming setup, or perhaps because of it, LP is one of the most commonly used models for optimization. Let us start with an example before getting into the precise definitions.

**Example 1.1.** A bond portfolio manager has \$100,000 to allocate to two different bonds; one corporate and one government bond. The corporate bond has annual yield of 4%, a maturity of 3 years and an A rating from a rating agency that is translated into a numerical rating of 2 for computational purposes. In contrast, the government bond has annual yield of 3%, a maturity of 6 years and rating of Aaa with the corresponding numerical rating of 1 (lower numerical ratings correspond to higher quality bonds). The portfolio manager would like to allocate her funds so that the average rating for the portfolio is no worse than Aa (numerical equivalent 1.5) and average maturity of the portfolio is at most 3.6 years. Any amount not invested in the two bonds will be kept in a cash account that is assumed to earn no interest for simplicity and does not contribute to the average rating or maturity computations. How should the

manager allocate her funds between these two bonds to achieve her objective of maximizing the annual yield for the first year from this investment? [**?**]

| | Corporate | Government | Constraints |
|---|---|---|---|
| Yield | 4% | 3% | N/A |
| Maturity | 3 | 6 | 3.6 |
| Rating | A = 2 | Aaa = 1 | 1.5 |
| Allocations | ?? | ?? | 100,000 |

We can model the above problem as follows:

$$\begin{array}{rrcrcl} \text{maximize:} & 4x & + & 3y & & \\ \text{subject to:} & 3x & + & 6y & \leq & 3.6 \\ & 2x & + & y & \leq & 1.5 \end{array} \tag{1.1}$$

where $x$, $y$ are the percentages of funds allocated to corporate and government bonds, respectively, and the objective function when multiplied by \$100,000 gives us the net annual yield. This is an example of a *linear program*. Recall that we cannot subtract inequalities the same way as we subtract equalities (Exercise 1.2). So, to get started, let us assume that both the inequalities are in fact equalities. We can solve the resulting system,

$$\begin{array}{rcrcl} 3x & + & 6y & = & 3.6 \\ 2x & + & y & = & 1.5, \end{array}$$

to obtain $x = 0.6$ and $y = 0.3$. For this solution $x + y = 0.9$ which is less that 1, meaning that we're not investing all the available funds. This seems less than ideal! Surely, we must invest all the funds to get maximum yield?

To complicate things further, notice that the linear program (1.1) is incomplete and the complete linear program that models the problem is as follows:

$$\begin{array}{rrcrcl} \text{maximize:} & 4x & + & 3y & & \\ \text{subject to:} & 3x & + & 6y & \leq & 3.6 \\ & 2x & + & y & \leq & 1.5 \\ & x & + & y & \leq & 1 \\ & x & & & \geq & 0 \\ & & & y & \geq & 0. \end{array} \tag{1.2}$$

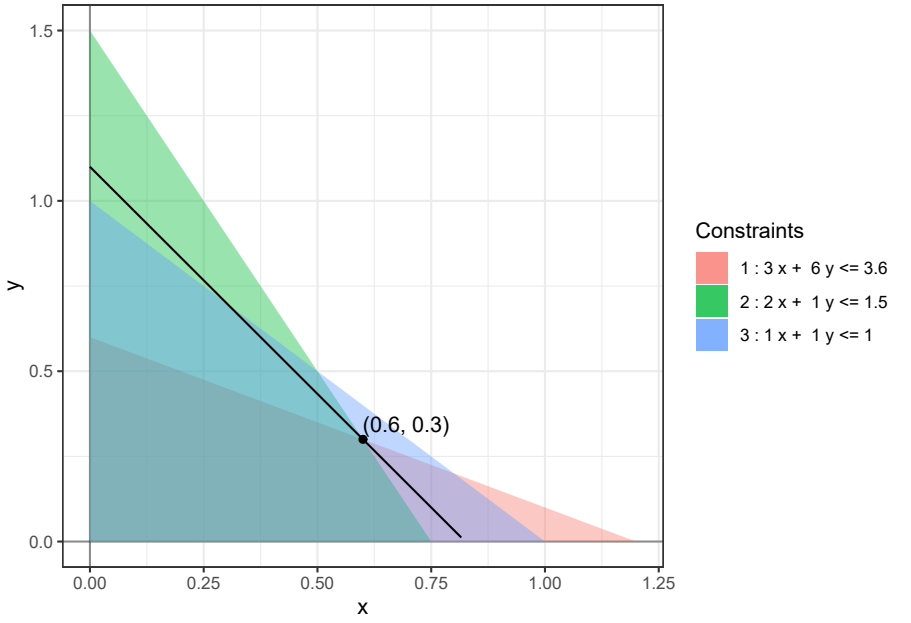
The solution $(x, y) = (0.6, 0.3)$ is obtained by changing the first two inequalities to equalities. We could have switched some other set of inequalities to equalities and solved for a different solution. We would then need to compare the yields for each of these solutions and find the one that maximizes it.

However, now there is an additional issue that needs to be dealt with carefully. Solving these modified equations might produce a solution that does not satisfy

the original constraints. For example, solving the system of equations

$$\begin{array}{rcrcl} 2x & + & y & = & 1.5 \\ x & & & = & 0 \end{array}$$

gives us $(x, y) = (0, 1.5)$ which breaks the constraints $3x+6y \leq 3.6$ and $x+y \leq 1$. So, not only do we need to compare the yields at these solutions but we also need to make sure that they satisfy all the original constraints. As you can imagine, this naive approach becomes unwieldy for large linear programs. It is not even clear that changing inequalities to equalities is the right approach to solving such a problem.[1] For this particular problem, we can use a graphing calculator to solve the problem as there are only two variables. The following figure shows the region described by the various constraints in the first quadrant $(x \geq 0, y \geq 0)$.




The points within the quadrilateral formed by the overlap of all three constraint regions satisfy all the constraints. The level sets of the objective function are straight lines of the form $4x + 3y = c$. We can see that the largest value of $c$ for which these level sets intersect the feasible region is $c = 3.3\%$ for $(x, y) = (0.6, 0.3)$. Our original solution was indeed correct! And it is not possible to invest more money and increase the yield!!

**Exercise 1.1.** What are some possible ways to relax the constraints in the above problem so that all the money can be invested? What is the improved annual yield?

[1] That this is indeed the case is the content of the *Fundamental Theorem of Linear Programming* (Theorem 2.1).

**Exercise 1.2.** The following exercises review some basic facts about inequalities:

1. Find constants $a, b, c, d$ such that $a < b$, $c < d$, $a - c < b - d$, and $ac < bd$.
2. Find constants $a, b, c, d$ such that $a < b$, $c < d$, $a - c < b - d$, and $ac > bd$.
3. Find constants $a, b, c, d$ such that $a < b$, $c < d$, $a - c > b - d$, and $ac < bd$.
4. Find constants $a, b, c, d$ such that $a < b$, $c < d$, $a - c > b - d$, and $ac > bd$.
5. Prove that if $a, b, c, d$ are constants such that $a < b$ and $c < d$, then $a + c < b + d$. Why does this proof fail when we try to subtract the two inequalities instead of adding them?

There are two main families of efficient algorithms for solving linear programs: simplex method(s) (Chapter 3) and interior point methods (Chapter 12). The simplex method is a combinatorial algorithm that essentially performs the above analysis systematically. Interior point methods modify gradient descent algorithms to search for an optimal solution within the feasible region. We'll analyze the simplex method in great detail and only briefly introduce a basic interior point method.

# Part I

# Simplex Method

# Chapter 2

# Standard Linear Program

We will describe the simplex method for solving a special kind of linear program called *standard linear program*. We will see later that every linear program can be *standardized* (Chapter 6) and hence this method is sufficient for solving any linear program.

**Definition 2.1.** A **standard linear program** is an optimization problem of the following form:

$$\begin{array}{lrcrcrcrcl} \text{maximize:} & c_0 & + & c_1x_1 & + & \ldots & + & c_nx_n & & \\ \text{subject to:} & & & a_{11}x_1 & + & \ldots & + & a_{1n}x_n & \leq & b_1 \\ & & & a_{21}x_1 & + & \ldots & + & a_{2n}x_n & \leq & b_2 \\ & & & & & \vdots & & & & \\ & & & a_{m1}x_1 & + & \ldots & + & a_{mn}x_n & \leq & b_m \\ & & & x_1, & x_2, & \ldots & , & x_n & \geq & 0 \end{array} \tag{2.1}$$

where $c_i$, $a_{ij}$, and $b_j$ are real constants. The variables $x_1, \ldots, x_n$ are called **decision variables**. A tuple $(x_1, \ldots, x_n)$ that satisfy *all* the constraints is called a **feasible solution** and the set of all feasible solutions is called the **feasible region**.

**Example 2.1.** Equation (1.2) is an example of a standard linear program with 2 decision variables and 3 constraints. The feasible region is a quadrilateral with vertices $(0, 0)$, $(0.75, 0)$, $(0.6, 0.3)$, and $(0, 0.6)$.

**Definition 2.2.** For each constraint of the standard linear program (2.1), we introduce a **slack variable** by subtracting the LHS from the RHS as follows.

$$\begin{array}{rcrcrcrcl} w_1 & = & b_1 & - & a_{11}x_1 & - & \ldots & - & a_{1n}x_n \\ w_2 & = & b_2 & - & a_{21}x_1 & - & \ldots & - & a_{2n}x_n \\ & & & & & \vdots & & & \\ w_m & = & b_m & - & a_{m1}x_1 & - & \ldots & - & a_{mn}x_n \end{array} \tag{2.2}$$

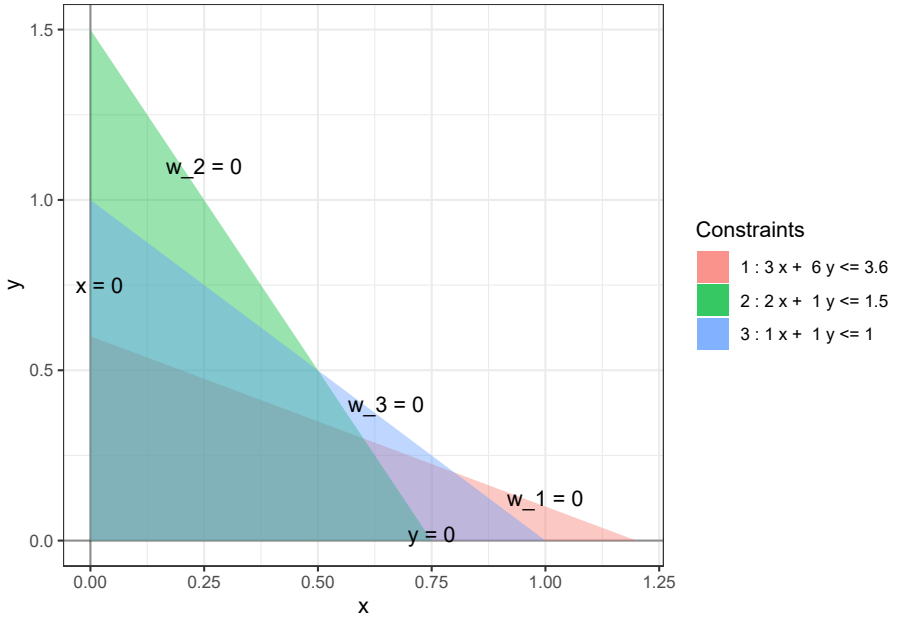
We can think of the slack variable $w_i$ as measuring the *slackness* in the $i^{th}$ constraint. The $i^{th}$ constraint is strictly met exactly when $w_i$ is zero. Using the slack variables, the linear program (2.1) can be succinctly rewritten as:

$$w_1, \ldots, w_m, x_1, \ldots, x_n \geq 0.$$

**Example 2.2.** The slack variables for the linear program (1.2) are as follows:

$$\begin{array}{lllllll} w_1 & = & 3.6 & - & 3x & - & 6y \\ w_2 & = & 1.5 & - & 2x & - & y \\ w_3 & = & 1 & - & x & - & y. \end{array}$$

In terms of these slack variables, the constraints can be rewritten as $x, y, w_1, w_2, w_3 \geq 0$ and the boundaries of the feasible region are given by $x = 0, y = 0, w_1 = 0, w_2 = 0, w_3 = 0$.



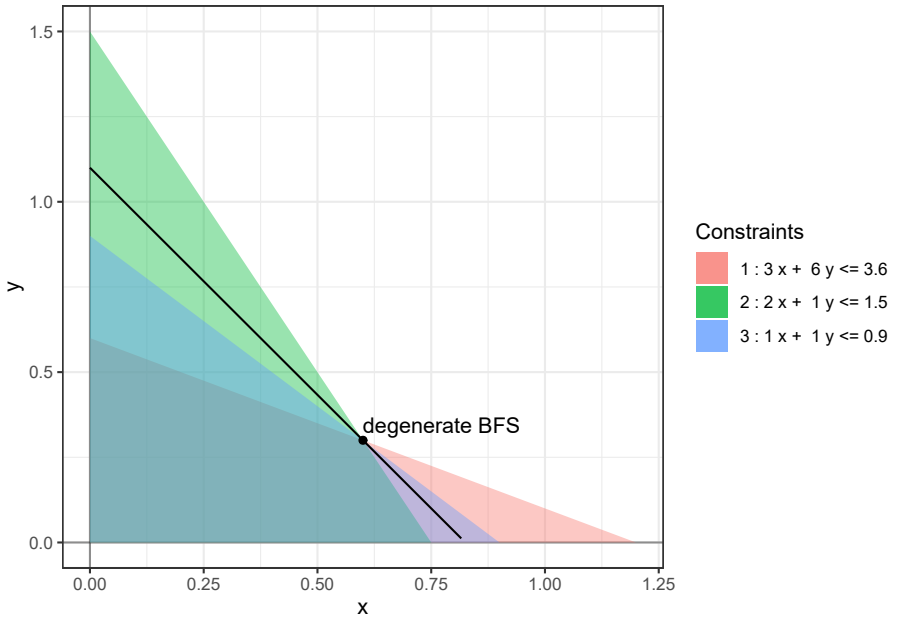

**Definition 2.3.** A **basic feasible solution** (BFS) to the standard linear program (2.1) is defined as a feasible solution at which at least $n$ decision or slack variables are zero. A BFS where exactly $n$ decision or slack variables are zero is called **non-degenerate**. A BFS where more than $n$ basic or decision variables are 0 is called **degenerate**. At a non-degenerate BFS, the $n$ variables that equal 0 are called **non-basic** and the remaining $m$ variables are called **basic**. At a degenerate BFS, we choose some $n$ vanishing variables to be non-basic and call the rest of the variables basic.

The basic feasible solutions are precisely the solutions obtained by changing $n$ or more inequalities to equalities and solving the resulting simultaneous system

of equations. As there are $n$ variables, we need at least $n$ linear equations in our system for the solution to be a single point. A degeneracy occurs when more than $n$ equations have a single solution, for example, when three lines in $\mathbb{R}^2$ or four planes in $\mathbb{R}^3$ intersecting at a single point. It is useful to think of the non-degenerate basic feasible solutions as *vertices* of the feasible region. The degenerate BFS do not have a nice geometric description.

**Exercise 2.1.** Show that if $k < n$ then the solution set of a system of $k$ linear equations in $n$ variables cannot be a single point.

**Example 2.3.** The feasible region for the linear program (1.2) has four non-degenerate basic feasible solutions: $(0, 0)$, $(0.75, 0)$, $(0.6, 0.3)$, and $(0, 0.6)$. and the optimal solution is attained at the BFS $(0.6, 0.3)$. At the origin, the non-basic variables are $x, y$ and the basic variables are $w_1, w_2, w_3$. At the optimal solution, the non-basic variables are $w_1, w_2$ and the basic variables are $x, y, w_3$. If we replace the constraint $x + y \leq 1$ with $x + y \leq 0.9$ then the BFS $(0.6, 0.3)$ becomes degenerate as here three (slack) variables $w_1, w_2, w_3$ are zero.




It turns out that it is sufficient to search for optimal solutions within the set of basic feasible solutions, which is finite, as made precise by the following theorem.

**Theorem 2.1** (Fundamental theorem of linear programming)**.** *For a standard linear program, exactly one of the following holds:*

1. *There is no feasible solution. In this case, we call the linear program* ***infeasible****.*
2. *The objective value can grow arbitrary large on the feasible region. In this*

*case, we call the linear program* ***unbounded****.*

3. *There is an optimal solution. In this case, we can further say that then there is a basic feasible solution which is optimal.*

We will assume this theorem without proof. The hardest step in the proof is showing that some basic feasible solution is optimal (if an optimal solution exists) when there are degenerate basic feasible solutions.

**Exercise 2.2.** Find an upper bound on the set of basic feasible solutions for a standard linear program with $n$ decision variables and $m$ equations.

# Chapter 3

# The Simplex Method

The simplex method is an iterative process for finding an optimal basic feasible solution to a standard linear program. It starts at some BFS and in each step moves to an adjacent one with a higher objective value. The following picture shows one possible run of the simplex method for the linear program (1.2).

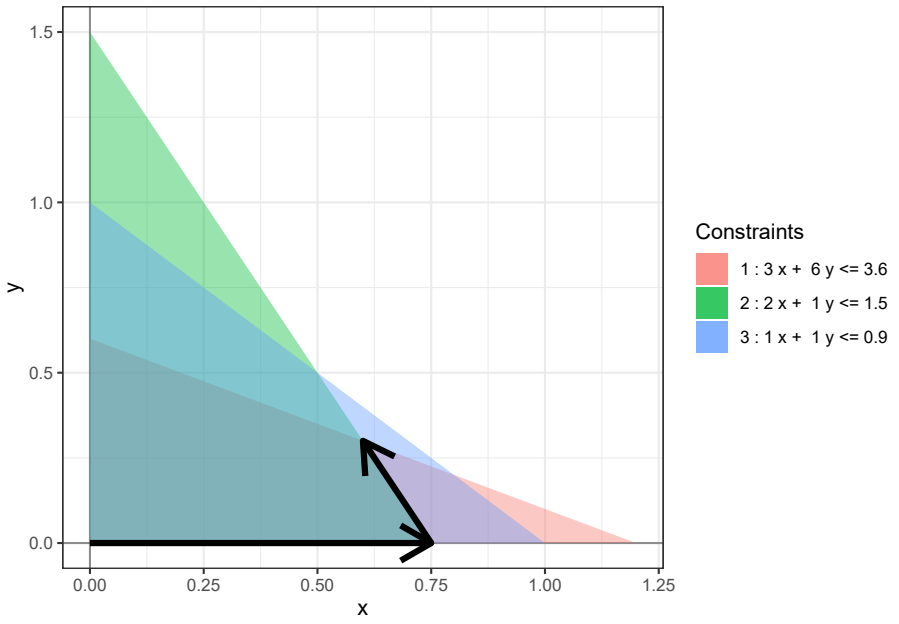
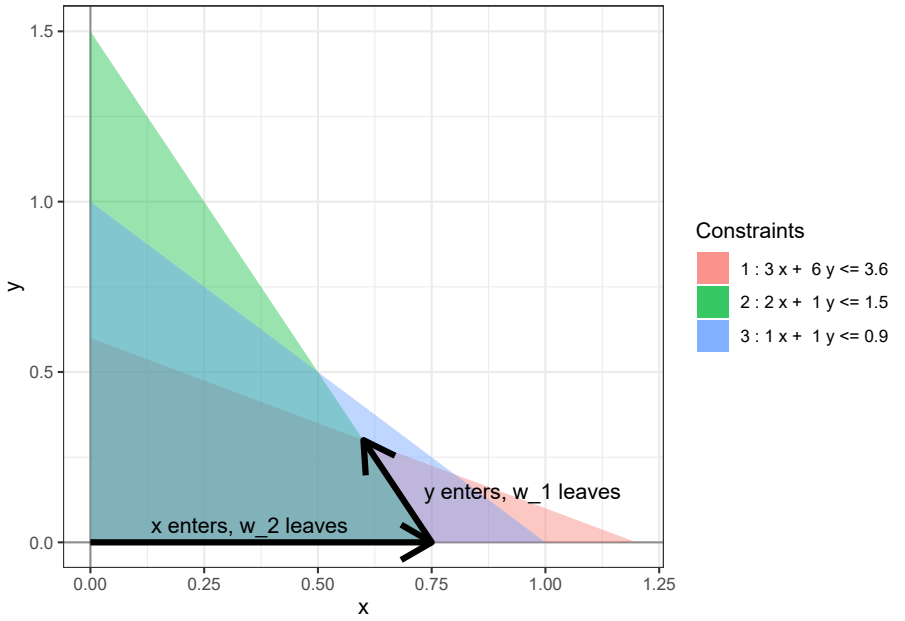



Figure 3.1: A possible run of the simplex algorithm.

In each step, one non-basic variable **enters** the set of basic variables and one basic variable **leaves** the set of basic variables. The table below explains how

these sets are getting updated in the sample simplex method run in Figure 3.1.

| | Leaving variable | Entering variable | Basic variables | Non-basic variables |
|---|---|---|---|---|
| Start | | | $\{w_1, w_2, w_3\}$ | $\{x, y\}$ |
| Step 1 | $w_2$ | $x$ | $\{w_1, x, w_3\}$ | $\{w_2, y\}$ |
| Step 2 | $w_1$ | $y$ | $\{y, x, w_3\}$ | $\{w_2, w_1\}$ |




Using this terminology, our goal at each step is reduced to finding which variable enters and which variable leaves. To do this algebraically, we'll introduce the notion of dictionaries.

**Definition 3.1.** The **dictionary** at a basic feasible solution is the set of equations describing the objective function and the constraints in terms of the non-basic variables.

**Example 3.1.** Consider (1.2) again. At the origin, the non-basic variables are $x, y$ and hence the initial dictionary is:

$$\begin{aligned} \text{objective} &= 0 + 4x + 3y \\ w_1 &= 3.6 - 3x - 6y \\ w_2 &= 1.5 - 2x - y \\ w_3 &= 1 - x - y. \end{aligned} \tag{3.1}$$

After the first step of the simplex algorithm, the non-basic variables are $w_2, y$.

We can write $x$ in terms of $w_2$ to get

$$x = 0.75 - 0.5w_2 - 0.5y$$

We can substitute this into the initial dictionary to get the dictionary after the first step:

$$\begin{array}{rcrcrcr} \text{objective} &=& 3 &+& (-2)w_2 &+& y \\ w_1 &=& 1.35 &-& (-1.5)w_2 &-& 4.5y \\ x &=& 0.75 &-& 0.5w_2 &-& 0.5y \\ w_3 &=& 0.25 &-& (-0.5)w_2 &-& 0.5y. \end{array} \tag{3.2}$$

Finally, the non-basic variables at the optimal solution are $w_1, w_2$. We can repeat the above process and get the dictionary for the optimal solution:

$$\begin{array}{rcrcrcr} \text{objective} &=& 3.3 &+& (-5/3)w_2 &+& (-2/9)w_1 \\ y &=& 0.3 &-& (-1/3)w_2 &-& 2/9w_1 \\ x &=& 0.6 &-& 2/3w_2 &-& (-1/9)w_1 \\ w_3 &=& 0.1 &-& (-1/3)w_2 &-& (-1/9)w_1. \end{array} \tag{3.3}$$

*Remark.* From the dictionary, one can extract the set of basic variables and the set of non-basic variables by looking at the variables appearing on the LHS and RHS, respectively. Furthermore, by setting the non-basic variables to 0, we obtain the very useful fact that

1. the values of the basic variables are simply the constants "$b_i$", and
2. the value of the objective function is the constant "$c_0$".

For example, from the final dictionary above, we can immediately see that $x = 0.6$, $y = 0.3$, $w_3 = 0.1$, and the objective value is 3.3 (and non-basic variables $w_1$ and $w_2$ are both zero) at the optimal solution.

## 3.1 The Simplex Step

Suppose we are at a BFS. Let $\bar{w}_1, \ldots, \bar{w}_m$ be the basic variables, $\bar{x}_1, \ldots, \bar{x}_n$ be the non-basic variables, $\bar{c}_j$, $\bar{b}_i$, and $\bar{a}_{ij}$ be the constants appearing in the dictionary at the BFS so that the dictionary is as follows:

$$\begin{array}{rcccccccc} \text{objective} &=& \bar{c}_0 &+& \bar{c}_1\bar{x}_1 &+& \ldots &+& \bar{c}_n\bar{x}_n \\ \bar{w}_1 &=& \bar{b}_1 &-& \bar{a}_{11}\bar{x}_1 &-& \ldots &-& \bar{a}_{1n}\bar{x}_n \\ \bar{w}_2 &=& \bar{b}_2 &-& \bar{a}_{21}\bar{x}_1 &-& \ldots &-& \bar{a}_{2n}\bar{x}_n \\ & & & & & \vdots & & & \\ \bar{w}_m &=& \bar{b}_m &-& \bar{a}_{m1}\bar{x}_1 &-& \ldots &-& \bar{a}_{mn}\bar{x}_n \end{array} \tag{3.4}$$

### 3.1.1 Entering variable

We want to move to an adjacent BFS with a higher objective value. The current objective function is

$$\bar{c}_0 + \bar{c}_1\bar{x}_1 + \cdots + \bar{c}_n\bar{x}_n.$$

and the current objective value is $\bar{c}_0$ as all the non-basic variables $\bar{x}_j$ are zero. We can move away from the current BFS by increasing one of the non-basic variables $\bar{x}_j$ from zero to a positive value. The objective value will increase precisely when $\bar{c}_j > 0$. This then is the criterion for choosing the entering variable. We can think of this as choosing the *direction* of the simplex step.

**Proposition 3.1** (Entering variable)**.** *A non-basic variable $\bar{x}_j$ can be entering if $\bar{c}_j > 0$.*

**Example 3.2.** In the dictionary (3.1), the objective function is $4x + 3y$. Hence, both $x$ and $y$ can be chosen as entering variables. We can see that there are two different paths along the boundary of the feasible region from the origin to the optimal solution. In the dictionary (3.2), the objective function is $3 + (-2)w_2 + y$ and only $y$ can be the entering variable. In dictionary (3.3), the objective function is $3.3 + (-5/3)w_2 + (-2/9)w_1$ and no variable can enter.

### 3.1.2 Leaving Variable

Suppose we choose $\bar{x}_j$ as our entering variable. This determines the direction of the simplex step. We next need to figure out the amount by which to increase $\bar{x}_j$ without leaving the feasible region. The basic variable $\bar{w}_i$ is related to $\bar{x}_j$ by the following relation

$$\bar{w}_i = \bar{b}_i - \bar{a}_{ij}\bar{x}_j - \sum_{k=1,k\neq j}^{n} \bar{a}_{ik}\bar{x}_k.$$

As we increase (only) $\bar{x}_j$, the quantity within the summation remains zero. If $a_{ij} > 0$, then as we increase $\bar{x}_j$ the basic variable $\bar{w}_j$ will decrease. Because we want all the variables to be non-negative, we must always have $\bar{w}_i = \bar{b}_i - \bar{a}_{ij}\bar{x}_j \geq 0$. This condition must hold true for all such $\bar{w}_i$. Hence, we get the following criterion for choosing the leaving variable.

**Proposition 3.2** (Leaving variable)**.** *Suppose $\bar{x}_j$ is the entering variable. The basic variable $\bar{w}_i$ can be chosen to be the leaving variable if*

$$i = \arg\min_{\bar{a}_{ij}>0} \frac{\bar{b}_i}{\bar{a}_{ij}}.$$

If we choose $\bar{w}_i$ to be leaving then after the simplex step, $\bar{w}_i$ decreases to 0, $\bar{x}_j$ increases to $\min_{\bar{a}_{ij}>0} \bar{b}_i/\bar{a}_{ij}$, and the objective value increases by $\bar{c}_j \min_{\bar{a}_{ij}>0} \bar{b}_i/\bar{a}_{ij}$.

**Example 3.3.** In the dictionary (3.1), if we choose $x$ to be our entering variable then we need to compare the following ratios:

| $i$ | $\bar{a}_{ij}$ | $\bar{b}_i$ | $\bar{b}_i/\bar{a}_{ij}$ |
|---|---|---|---|
| 1 | 3 | 3.6 | 1.2 |
| 2 | 2 | 1.5 | 0.75 |

| $i$ | $\bar{a}_{ij}$ | $\bar{b}_i$ | $\bar{b}_i/\bar{a}_{ij}$ |
|---|---|---|---|
| 3 | 1 | 1 | 1 |

We can see that the smallest ratio is obtained for $w_2$. Hence, it is the only candidate for the leaving variable.

## 3.2 Tableau Notation

When manipulating a linear system of equations, one can forget the variables and perform manipulations on the coefficients using matrices. The same is true for linear programs. We start by rewriting the constraints in the dictionary with all the variables on the LHS and all the constants on the RHS:

$$\begin{array}{cccccccccccc} \bar{a}_{11}\bar{x}_1 & + & \dots & + & \bar{a}_{1n}\bar{x}_n & + & \bar{w}_1 & & & & = & \bar{b}_1 \\ \bar{a}_{21}\bar{x}_1 & + & \dots & + & \bar{a}_{2n}\bar{x}_n & & & + & \bar{w}_2 & & = & \bar{b}_2 \\ & & & & & & \vdots & & & & & \\ \bar{a}_{m1}\bar{x}_1 & + & \dots & + & \bar{a}_{mn}\bar{x}_n & & & & & + \ \bar{w}_m & = & \bar{b}_m \end{array} \tag{3.5}$$

This can be encoded using the following augmented matrix:

$$\left[\begin{array}{cccccc|c} \bar{a}_{11} & \dots & \bar{a}_{1n} & 1 & & & \bar{b}_1 \\ \bar{a}_{21} & \dots & \bar{a}_{2n} & & 1 & & \bar{b}_2 \\ & & & \vdots & & & \\ \bar{a}_{m1} & \dots & \bar{a}_{mn} & & & 1 & \bar{b}_m \end{array}\right]$$

We add back the objective function, but because of a quirk of algebra we need to add the objective function coefficients as follows:

$$\left[\begin{array}{cccccccc|c} \bar{a}_{11} & \dots & \bar{a}_{1n} & 1 & & & & & \bar{b}_1 \\ \bar{a}_{21} & \dots & \bar{a}_{2n} & & 1 & & & & \bar{b}_2 \\ & & & \vdots & & & & & \\ \bar{a}_{m1} & \dots & \bar{a}_{mn} & & & & & 1 & \bar{b}_m \\ \hline \bar{c}_1 & \dots & \bar{c}_n & 0 & 0 & \dots & 0 & & -\bar{c}_0 \end{array}\right]$$

The columns in this augmented matrix correspond to the variables $\bar{x}_j$ and $\bar{w}_i$. The columns with the *pivots* correspond to the basic variables. If $\bar{x}_j$ is the entering variable and $\bar{w}_i$ is the leaving variable, then *we perform elementary row operations and turn the entry $\bar{a}_{ij}$ into a pivot for its column* to obtain the tableau at the next BFS. Hence, the simplex step is also called the **pivot step**.

**Example 3.4.** The tableau corresponding to the dictionary (3.1) is as follows:

$$\left[\begin{array}{ccccc|c} 3 & 6 & 1 & 0 & 0 & 3.6 \\ \boxed{2} & 1 & 0 & 1 & 0 & 1.5 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ \hline 4 & 3 & 0 & 0 & 0 & 0 \end{array}\right]$$

If we choose $x$ as the entering variable and $w_2$ as the leaving variable then we need to pivot about the entry $a_{21}$ using elementary row operations to get the following tableau:

$$\begin{array}{cccccc|c} 0 & 4.5 & 1 & -1.5 & 0 & 1.35 \\ \boxed{1} & 0.5 & 0 & 0.5 & 0 & 0.75 \\ 0 & 0.5 & 0 & -0.5 & 1 & 0.25 \\ \hline 0 & 1 & 0 & -2 & 0 & -3 \end{array}$$

which corresponds to the dictionary (3.2).

# Chapter 4

# Initialization

So far, we've seen how to move from one basic feasible solution to an adjacent one with a higher objective value. To have a complete algorithm for solving standard linear programs, we still need a method that finds some BFS. If we're lucky, the origin is a basic feasible solution and we can start our simplex method here. However, this is not always the case. The following proposition follows easily from definitions.

**Proposition 4.1.** *The origin is a basic feasible solution of the standard linear program* (2.1) *if and only if* $b_i \geq 0$ *for all* $1 \leq i \leq m$.

When the origin is not a BFS, we need a process to find one. This is called the **Initialization Phase** or **Phase I**. We will do this by creating an auxiliary linear program.

## 4.1 Auxiliary Linear Program

**Definition 4.1.** We say that a linear program is **feasible** if its feasible region is non-empty.

The initialization phase determines if a standard linear program is feasible and if it is then it finds a BFS by solving the following auxiliary linear program.

**Definition 4.2.** The **auxiliary linear program** of the standard linear program (2.1) is defined as follows:

$$
\begin{array}{lrcccccccl}
\text{maximize:} & & & & & & - & x_0 & & \\
\text{subject to:} & a_{11}x_1 & + & \dots & + & a_{1n}x_n & - & x_0 & \leq & b_1 \\
& a_{21}x_1 & + & \dots & + & a_{2n}x_n & - & x_0 & \leq & b_2 \\
& & & \vdots & & & & & & \\
& a_{m1}x_1 & + & \dots & + & a_{mn}x_n & - & x_0 & \leq & b_m \\
& x_1, & x_2, & \dots & , & x_n & , & x_0 & \geq & 0.
\end{array}
\tag{4.1}
$$

We're introducing a new variable $-x_0$, adding $-x_0$ to each of the constraint LHS, and changing the objective function to $-x_0$. To understand the auxiliary linear program, we rewrite it in the following non-standard form:

$$\begin{array}{lrcccccccc}
\text{minimize:} & & & & & & & & & x_0 \\
\text{subject to:} & a_{11}x_1 & + & \dots & + & a_{1n}x_n & \leq & b_1 & + & x_0 \\
& a_{21}x_1 & + & \dots & + & a_{2n}x_n & \leq & b_2 & + & x_0 \\
& & & \vdots & & & & & & \\
& a_{m1}x_1 & + & \dots & + & a_{mn}x_n & \leq & b_m & + & x_0 \\
& x_1, & x_2, & \dots & , & x_n & , & x_0 & \geq & 0
\end{array}$$

We can interpret $x_0$ as a relaxation of the constraints. Solving the auxiliary linear program is equivalent to asking - "what is the smallest constraint relaxation necessary to make the primary linear program feasible?". The primary linear program is feasible if and only if no relaxation is necessary. The following exercises make this more precise.

**Exercise 4.1.** Show that the auxiliary linear program (4.1) is always feasible by explicitly constructing a feasible solution.

**Exercise 4.2.** Show that the objective values of the auxiliary linear program (4.1) are bounded from above by 0. Using the extreme value theorem, conclude that (4.1) always has an optimal solution.

Finally, the following theorem (proof left as an exercise) establishes an explicit connection between optimal solutions of the auxiliary linear program and feasible solutions of the standard linear program.

**Theorem 4.1.** *Suppose $x_i = k_i$ for $0 \leq i \leq m$ is an optimal solution of the auxiliary linear program* (4.1)*. Then, the standard linear program* (2.1) *is feasible if and only if $k_0 = 0$. In this case, $x_i = k_i$ for $0 \leq i \leq m$ is a basic feasible solution of* (2.1)*.*

Our problem is now reduced to determining the optimal solution of the auxiliary linear program, which we'll do using the simplex method itself! However, even for the auxiliary linear program the origin is not necessarily in the feasible region so we cannot use the simplex steps to find an optimal solution. But, in this case, there is a trick to make all the $b_i \geq 0$ by performing a single elementary row operation as described in the following exercises.

**Exercise 4.3.** Suppose some $b_i$ is negative. Without any loss of generality, assume that $b_1$ is the smallest among all the $b_i$ (i.e. the most negative). Show that after the pivot step in tableau for the auxiliary linear program (4.1) about the column corresponding to $x_0$ and the row corresponding to $w_1$, all the $b_i$'s become positive.

To summarize: If the origin is not a vertex of the feasible region, then the method of solving the standard linear program (2.1), starting with **Phase I**, is as follows:

1. Form the auxiliary linear program and its tableau.

2. Perform a pivot operation about the entry in the column corresponding to the variable $x_0$ and the row corresponding to the *most negative* $b_i$. This results in a dictionary where all the $b_i$'s are now non-negative.
3. Solve the auxiliary linear program by repeatedly performing pivot steps. Suppose $x_i = k_i$ for $0 \leq i \leq m$ is an optimal solution of the auxiliary linear program.
    1. If $k_0 \neq 0$, then we halt as the primary linear program is not feasible.
    2. If $k_0 = 0$, then we proceed to **Phase II**. We solve the original linear program starting at the BFS $x_i = k_i$ for $1 \leq i \leq n$ by repeatedly performing pivot steps.

**Exercise 4.4.** This exercise provides a geometric intuition behind the auxiliary linear program. For each of the following standard linear programs,

$$\begin{array}{lrcl} \text{maximize:} & x & & \\ \text{subject to:} & -x & \leq & -1 \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{lrcl} \text{maximize:} & x & & \\ \text{subject to:} & x & \leq & -1 \\ & x & \geq & 0 \end{array}$$

- Form the auxiliary linear program. Use $y$ as the auxiliary variable.
- Draw the feasible region of the auxiliary linear program and label the constraints using slack variables.
- Solve the auxiliary linear program using geometry.

## 4.2 Combined tableau

At the end of Phase I, we need to calculate the dictionary at the basic feasible solution $x_i = k_i$ for $1 \leq i \leq n$. We can avoid this recalculation by manipulating a *combined tableau* which contains information about both the auxiliary and the primary linear programs.

**Definition 4.3.** The combined tableau is defined as follows:

$$\begin{array}{rcrccccc|c} a_{11} & \dots & a_{1n} & 1 & & & & -1 & b_1 \\ a_{21} & \dots & a_{2n} & & 1 & & & -1 & b_2 \\ & & & \vdots & & & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & & & & 1 & -1 & b_m \\ \hline c_1 & \dots & c_n & 0 & 0 & \dots & 0 & 0 & -c_0 \\ \hline 0 & \dots & 0 & 0 & 0 & \dots & 0 & -1 & 0 \end{array}$$

The second to last row of the tableau corresponds to the objective function of the primary linear program and the last row corresponds to the objective of the

auxiliary linear program. The second to last column corresponds to the variable $x_0$.

We use the combined tableau to first perform Phase I and then delete the auxiliary objective and the variable $x_0$ and proceed on to Phase II using the rest of the tableau.

**Example 4.1.** Consider the following linear program:

$$\begin{array}{rrcrcl} \text{maximize:} & x & + & y & & \\ \text{subject to:} & x & + & 2y & \leq & 6 \\ & -x & & & \leq & -1 \\ & & & -y & \leq & -2 \\ & x & , & y & \geq & 0. \end{array}$$

As $b_2$ and $b_3$ are negative, $(0, 0)$ is not feasible and we need to start with Phase I. We form the combined tableau:

$$\begin{array}{rrrrrr|r} 1 & 2 & 1 & 0 & 0 & -1 & 6 \\ -1 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 & 1 & \boxed{-1} & -2 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{array}$$

The first pivot is in the column corresponding to the variable $x_0$ and the row corresponding to the *most negative* $b_i$ (which is $b_3 = -2$). This results in the following combined tableau:

$$\begin{array}{rrrrrr|r} 1 & 3 & 1 & 0 & -1 & 0 & 8 \\ -1 & 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 2 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & -1 & 0 & 2 \end{array}$$

We then continue performing pivot steps to find the solution:

$$\begin{array}{rrrrrr|r} 0 & 0 & 1 & 1 & 2 & -4 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 2 \\ 1 & 0 & 0 & -1 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & -2 & -3 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 & \mathbf{0} \end{array}$$

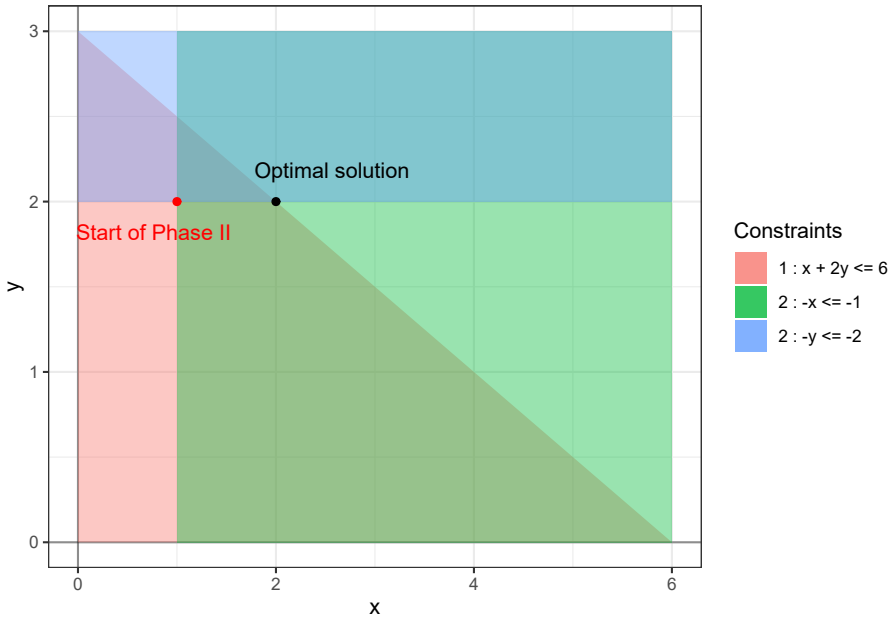
We see that the optimal value is 0 and hence the primary linear program is feasible. We then remove the auxiliary objective and coefficient to get the following tableau for the primary linear program:

$$\begin{array}{rrrrrr|r} 0 & 0 & 1 & \boxed{1} & 2 & 1 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & -1 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & -3 \end{array}$$

Continuing with the simplex method we get the following final tableau

$$\begin{array}{cccccc|c} 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 1 & 0 & 1 & 0 & 2 & 2 \\ \hline 0 & 0 & -1 & 0 & -1 & -4 \end{array}$$

The final non-basic variables are $w_1$ and $w_3$, the basic variables are $x$, $y$, $w_2$ with values 2, 2, 1, respectively, and the optimal objective value is 4. The following figure shows the starting and ending basic feasible solutions for Phase II.



# Chapter 5

# Halting Conditions

We know how to start the simplex method and how to perform the pivot steps. We'll next analyze the halting conditions for the simplex method.

There are three different places a simplex method can fail:

1. *Phase I returns no feasible solution.*

   In this case, the linear program is infeasible. This happens precisely when **the optimal solution to the auxiliary linear program has $k_0 \neq 0$.**

2. *We cannot find an entering variable.*

   In this case, there is no "direction" in the feasible region along which the objective value increases i.e. we are at a local maxima. Because the objective function is linear this local maxima is also an absolute maxima (Exercise 5.1) and provides an optimal solution to our linear program. None of the variables enter precisely when **none of the $\bar{c}_i$ are positive.**

3. *We cannot find a leaving variable.*

   This means that we can keep on increasing the entering variable, and hence the objective value, indefinitely without ever leaving the feasible region. None of the variables can leave precisely when **none of the $\bar{a}_{ij}$ are positive** where $\bar{w}_i$ is entering.

Note that these three halting conditions correspond to the three types of standard linear programs as seen in the Fundamental theorem of linear programming (Theorem 2.1).

| | Halting condition | Linear program type |
|---|---|---|
| End of Phase I | No feasible solution: $k_0 \neq 0$ | Infeasible |

| | Halting condition | Linear program type |
|---|---|---|
| After a pivot step | No entering variable: $\forall i, \bar{c}_i \leq 0$ | Optimal solution |
| After a pivot step | No leaving variable: $\forall j, \bar{a}_{ij} \leq 0$ | Unbounded |

**Exercise 5.1.** (You should only attempt this exercise if you're comfortable with convex sets.) Let $S$ be a convex set in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear function. Show that if $p \in S$ is a local maxima of $f$ over $S$ then it is also a global maxima. Find a counterexample to this statement if $S$ is non-convex.

*Remark.* The notion of "unboundedness" for linear programs is more restrictive than the geometric notion of unboundedness. It is possible to have a linear program whose feasible region is unbounded but the linear program itself is not unbounded, as seen in the following example.

$$\begin{aligned} &\text{maximize:} && -x \\ &\text{subject to:} && x \geq 0. \end{aligned}$$

It is not too hard to show that the feasible region of an unbounded linear program is always unbounded. The above example shows that the converse is not always true.

## 5.1 Degeneracy

Unfortunately, the above results are insufficient to guarantee that the simplex method will always find a solution, if one exists. It is possible for the simplex method to get stuck in a loop. This is called **cycling**.

**Exercise 5.2.** Give an example showing that the variable that enters in one pivot step can become leave in the next.

**Exercise 5.3.** Show that the variable that leaves in one pivot step cannot enter in the next.

We saw in Section 3.1 that after a pivot step the entering and leaving variables get updated as follows:

$$\begin{aligned} \bar{x}_j &\mapsto \bar{b}_i/\bar{a}_{ij} \\ \bar{w}_i &\mapsto 0. \end{aligned}$$

This increases the value of the objective function by $\bar{c}_j\bar{b}_i/\bar{a}_{ij}$. The way the entering and leaving variables are chosen, the constants $\bar{c}_i$ and $\bar{a}_{ij}$ are always

positive. The constant $\bar{b}_i$ equals the value of the basic variable $\bar{w}_i$ and hence must be greater than or equal to 0. This ensures that the objective value never decreases after a simplex step. If $\bar{b}_i > 0$, the objective value will increase in this pivot step (and never decrease afterwards) and so we will never return back to this BFS.

**Proposition 5.1.** *Suppose $\bar{w}_i$ is the leaving variable. If $\bar{b}_i > 0$, the simplex method will never return to this basic feasible solution.*

Hence, the only case when cycling *can* occur is when $\bar{b}_i = 0$. As mentioned above, $\bar{b}_i$ is the value of the basic variable $\bar{w}_i$. Having more than $n$ variables vanishing at a BFS is precisely the definition of degeneracy (Definition 2.3).

**Proposition 5.2.** *The simplex method can cycle only if there some degenerate BFS.*

Note that this is not an "if and only if" statement. Even if there is some degeneracy, there is no certainty that the simplex method will always visit a degenerate vertex twice.

**Example 5.1.** Consider the following degenerate linear program:

$$\begin{array}{rrrrrrrrrrr}
\text{maximize:} & x_1 & - & 2x_2 & & & - & 2x_4 & & \\
\text{subject to:} & 0.5x_1 & - & 3.5x_2 & - & 2x_3 & + & 4x_4 & \leq & 0 \\
& 0.5x_1 & - & x_2 & - & 0.5x_3 & + & 0.5x_4 & \leq & 0 \\
& x_1 & & & & & & & \leq & 1 \\
& x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}$$

The following is a valid sequence of simplex steps:

1. $x_1$ enters and $w_1$ leaves,
2. $x_2$ enters and $w_2$ leaves,
3. $x_3$ enters and $x_1$ leaves,
4. $x_4$ enters and $x_2$ leaves,
5. $w_1$ enters and $x_3$ leaves,
6. $w_2$ enters and $x_4$ leaves.

At the end of the $6^{th}$ simplex step, we end up looping back to the origin.

## 5.2 Bland's Rule

Cycling is only possible in the presence of degeneracy. At degenerate vertices, we have some freedom with choosing the set of basic and non-basic variables. This choice is precisely what allows us to avoid cycling. When we revisit a basic feasible solution that is degenerate, we simply choose a different entering and leaving variable. One can show that this is always possible and leads to an algorithm that always halts.

There is another very efficient way for preventing cycling called **Bland's rule**. Bland's rule says that if there are multiple candidates for the entering/variable

then we choose the one with the smallest index. (We assume that the decision variables have a smaller index than the slack variables.) One can show that simplex method always terminates if Bland's rule is followed.

**Theorem 5.1** (Bland's rule)**.** *The simplex method always terminates provided that both the entering and the leaving variable are chosen according to Bland's rule.*

**Example 5.2.** In Example 5.1, the *sixth simplex step* violates Bland's rule. Both $x_1$ and $x_4$ can be leaving variables and we choose $x_4$ whereas Bland's rule requires us to choose $x_1$.

The proof of Theorem 5.1 is beyond the scope of this class. Using Bland's rule for both phases of the simplex method, we now have a complete algorithm for solving linear programs. In fact, Theorem 5.1 provides an algorithmic proof of the Fundamental theorem of linear programming (Theorem 2.1). By Theorem 5.1, the simplex method must halt and one of the three halting conditions mentioned at the start of this chapter must occur as there are no other halting conditions. But this is precisely the statement of the Fundamental theorem.

# Chapter 6

# Standardization

We'll next extend our simplex method to solve more general linear programs.

**Definition 6.1.** A **general linear program** is an optimization problem of the following form:

$$\begin{array}{lccccccccc} \text{optimize:} & c_0 & + & c_1x_1 & + & \dots & + & c_nx_n & & \\ \text{subject to:} & & & a_{11}x_1 & + & \dots & + & a_{1n}x_n & \lessgtr & b_1 \\ & & & a_{21}x_1 & + & \dots & + & a_{2n}x_n & \lessgtr & b_2 \\ & & & & & \vdots & & & & \\ & & & a_{m1}x_1 & + & \dots & + & a_{mn}x_n & \lessgtr & b_m, \end{array} \tag{6.1}$$

where the symbol $\lessgtr$ stands for "$\leq$ or $=$ or $\geq$", $a_{ij}, b_i, c_j$ are real numbers, and $x_j$ are the decision variables.

We do not allow strict inequalities $<$ or $>$ in a general linear program as linear functions do not always achieve maxima/minima on open sets even when for bounded sets. Consider the following simple example.

$$\begin{array}{rl} \text{maximize:} & x \\ \text{subject to:} & x < 1 \\ & x \geq 0 \end{array}$$

On the feasible set $[0, 1)$, the function $x$ never attains absolute maxima. Changing the inequality $<$ to $\leq$ gives us an optimal feasible solution $x = 1$.

## 6.1 Equivalence of Linear Programs

Any linear program can be changed to a *maximization problem* as minimizing a function is the same as maximizing its negation. From now on, we'll assume that the goal of our linear programs is to *maximize* the objective function.

**Definition 6.2.** Two (maximizing) linear programs, $LP$ and $LP'$, are said to be **equivalent** to each other if for any feasible solution $x = (x_1, \ldots, x_n)$ to $LP$, there exists a feasible solution $x' = (x'_1, \ldots, x'_{n'})$ to $LP'$ with the same objective value as $x$, and for any feasible solution $y' = (y'_1, \ldots, y'_n)$ to $LP'$, there exists a feasible solution $y = (y_1, \ldots, y_n)$ to $LP$ with the same objective value as $y'$.

*Remark.*

1. $LP$ and $LP'$ can have a different number of decision variables i.e. we do not require $n = n'$.
2. There need not be a one-to-one correspondence between the feasible sets of $LP$ and $LP'$ i.e. for a feasible solution to $LP$ there could be multiple feasible solutions of $LP'$ with the same objective value. Similarly, in the other direction.

**Exercise 6.1.** Show that the equivalence of linear programs is an equivalence relation.

*Remark.* Even though equivalence of linear programs only requires the existence of an abstract correspondence between the feasible sets of $LP$ and $LP'$, in practice, one constructs "objective value preserving" linear transformations $T : \mathbb{R}^n \to \mathbb{R}^{n'}$ and $S : \mathbb{R}^{n'} \to \mathbb{R}^n$ which map the feasible set of $LP$ to $LP'$ and the feasible set of $LP'$ to $LP$, respectively. These linear transformations need not be inverses of each other, or even isomorphisms. They only need to preserve the objective values.

**Example 6.1.** The following two linear programs are equivalent to each other

$$\begin{array}{ll} \text{maximize:} & x + y \\ \text{subject to:} & 0 \leq x \leq 1 \\ & 0 \leq y \leq 1 \end{array} \qquad \begin{array}{ll} \text{maximize:} & z \\ \text{subject to:} & 0 \leq z \leq 2 \end{array}$$

via the transformations $T(x, y) = x + y$ and $S(z) = (z/2, z/2)$.

**Exercise 6.2.** Suppose $LP$ and $LP'$ are equivalent linear programs. Show that if $LP$ is infeasible/ unbounded/ has an optimal solution then so does the $LP'$. Furthermore, show that if $LP$ has an optimal solution with optimal objective value $c$ then so does $LP'$.

**Theorem 6.1.** *Every (maximizing) linear program is equivalent to a linear program in the standard form.*

*Proof.* The proof is by providing an explicit standardization algorithm. Consider the linear program in (6.1), where we're assuming that the goal is to maximize the objective function. If it is in the standard form, then we're done. If not, then there must be a finite number of *errors* of the following types:

1. A linear constraint is a lower bound and has the form

$$a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i.$$

2. A linear constraint is an equality and has the form

$$a_{i1}x_1 + \cdots + a_{in}x_n = b_i.$$

3. A variable $x_j$ has a *negativity constraint* $x_j \leq 0$.
4. A variable $x_j$ is **free** i.e. it is missing a *sign constraint*.

We fix these errors one at a time while making sure that no new errors are introduced, thereby guaranteeing termination of the algorithm.

1. We replace the constraint with

$$-a_{i1}x_1 + \cdots + -a_{in}x_n \leq -b_i.$$

2. We replace the constraint with two constraints

$$\begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &\leq b_i \\ -a_{i1}x_1 - \cdots - a_{in}x_n &\leq -b_i, \end{aligned}$$

3. We let $y_j = -x_j$ and create a new linear program using the variables $x_1$, …, $x_{j-1}$, $y_j$, $x_{j+1}$, …, $x_n$ by replacing $x_j$ with $-y_j$ everywhere.
4. We let $x_j = y_j - z_j$ for two new decision variables $y_j$ and $z_j$ with $y_j, z_j \geq 0$ and create a new linear program using the variables $x_1$, …, $x_{j-1}$, $y_j$, $z_j$, $x_{j+1}$, …, $x_n$ by replacing $x_j$ with $y_j - z_j$ everywhere. We can do this because any real number can written as a difference of two positive real numbers.

□

**Exercise 6.3.** Prove that the algorithm in the proof of Theorem 6.1 terminates and produces a linear program that is equivalent to the original linear program.

# Part II

# Duality Theory

# Chapter 7

# Dual Linear Programs

We'll now transition from solving linear programs to studying their solution space using duality theory. We'll show that for every linear program there exists a dual linear program whose optimal objective value is closely related to the optimal objective value of the original linear program. Duality theory is hard to motivate and only makes sense in retrospect. The main theorems - weak duality, strong duality, and complementary slackness - generalize to non-linear programs to various degrees and provide means of analyzing complicated optimization problems for which no algorithms exist. Even for the case of linear programing, where we have a complete algorithm for finding an optimal solution, duality theory is useful for analyzing the sensitivity of the optimal solution to various perturbations of the linear program.

We start by introducing some matrix notation. We'll let $x$ denote the vector of decision variables, $b$ the vector of upper bounds, $c$ the vector of objective coefficients, and $A$ the matrix of constraints in the standard linear program (2.1).

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

The standard linear program (2.1) can be written as follows:

$$\begin{array}{lrcl} \text{maximize:} & c_0 + c^T x & & \\ \text{subject to:} & Ax & \leq & b \\ & x & \geq & 0. \end{array}$$

**Definition 7.1.** The **dual** of this linear program is defined as the following

linear program:

$$\begin{array}{lrcl} \text{minimize:} & c_0 + b^T y & & \\ \text{subject to:} & A^T y & \geq & c \\ & y & \geq & 0, \end{array} \tag{7.1}$$

where $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$ is the vector of **dual decision variables**. The original linear program is called the **primal**.

**Theorem 7.1.** *The the dual of the (standardized) dual is equivalent to the primal.*

*Proof.* Left as an exercise. □

Hence, we think of linear programs as occurring in **primal-dual pairs**. Every linear program has a dual and it is itself the dual of its dual. The dual decision variables correspond to the constraints of the primal and the primal decision variables correspond to the constraints of the dual. This interpretation will be made precise by the concept of shadow prices.

**Example 7.1.** The dual of (1.2) is:

$$\begin{array}{lrcrcrcr} \text{minimize:} & 3.6y_1 & + & 1.5y_2 & + & y_3 & & \\ \text{subject to:} & 3y_1 & + & 2y_2 & + & y_3 & \geq & 4 \\ & 6y_1 & + & y_2 & + & y_3 & \geq & 3 \\ & y_1 & , & y_2 & , & y_3 & \geq & 0. \end{array} \tag{7.2}$$

The variable $y_1$ corresponds to *maturity*, the variable $y_2$ corresponds to *risk*, and the variable $y_3$ corresponds to *percentage*.

## 7.1 General Linear Programs

It is interesting to see how to dualize general linear programs. We do not *need* to do this explicitly as we can always standardize general linear programs first. But we'llt see that dualizing general linear program establishes an interesting relationship between the various kinds of *errors*.

To standardize a general linear program we need to *fix* the objective function, constraints, and signs of decision variables. However, doing so changes the various constants. It is possible to standardize, dualize, and then *revert the standardization* in the resulting dual to get the same constants as the primal.

1. Suppose one of the linear constraint is a lower bound and has the form

$$a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i.$$

To *fix* this, we multiply the constraint by $-1$ to get

$$-a_{i1}x_1 + \cdots + -a_{in}x_n \leq -b_i.$$

If we then form the dual, the coefficients of the dual variable $y_i$ will be $-a_{ij}$ instead of $a_{ij}$. We can replace $y_i$ with a variable $y_i' = -y_i$ and then the resulting linear program will have the same constants as the primal but now we have $y_i' \leq 0$. Notice that this is the third type of error in a general linear program.

2. Suppose one of the linear constraint is a lower bound and has the form

$$a_{i1}x_1 + \cdots + a_{in}x_n = b_i.$$

To *fix* this, we replace the constraint with the two constraints

$$\begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &\leq b_i, \\ -a_{i1}x_1 - \cdots - a_{in}x_n &\leq -b_i. \end{aligned}$$

If we then form the dual, we will have two dual variables $y_i'$ and $y_i''$ with coefficients $a_{ij}$ and $-a_{ij}$ respectively. We can introduce a new variable, $y_i = y_i' - y_i''$ and then the resulting linear program will have the same constants as the primal but now we have $y_i$ is free. Notice that this is the fourth type of error in a general linear program.

Because the dual of the dual is the primal, we do not need to analyze the third and fourth type of error separately. The follow table describes the rules for forming the dual of a general linear program.

| Primal | Dual |
|---|---|
| $a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i$ | $y_i \geq 0$ |
| $a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i$ | $y_i \leq 0$ |
| $a_{i1}x_1 + \cdots + a_{in}x_n = b_i$ | $y_i$ is free |
| $x_j \geq 0$ | $a_{1j}y_1 + \cdots + a_{mj}y_m \geq c_j$ |
| $x_j \leq 0$ | $a_{1j}y_1 + \cdots + a_{mj}y_m \leq c_j$ |
| $x_j$ is free | $a_{1j}y_1 + \cdots + a_{mj}y_m = c_j$ |

**Example 7.2.** Consider the following general linear program which is a modification of (1.2):

$$\begin{array}{rrcrcl} \text{maximize:} & 4x_1 & + & 3x_2 & & \\ \text{subject to:} & 3x_1 & + & 6x_2 & \leq & 3.6 \\ & 2x_1 & + & x_2 & \geq & 1.5 \\ & x_1 & + & x_2 & = & 1 \\ & x_1 & & & \leq & 0. \end{array}$$

The dual of this linear program is

$$\begin{array}{lrcrcrcr} \text{minimize:} & 3.6y_1 & + & 1.5y_2 & + & y_3 & & \\ \text{subject to:} & 3y_1 & + & 2y_2 & + & y_3 & \leq & 4 \\ & 6y_1 & + & y_2 & + & y_3 & = & 3 \\ & y_1 & & & & & \geq & 0 \\ & & & y_2 & & & \leq & 0. \end{array}$$

Compare this to the dual of (1.2) found in Example 7.1. Notice that the constants appearing in the dual are the same as the one in the primal.

# Chapter 8

# Weak and Strong Duality

We'll now prove several theorems relating the optimal objective values of the primal-dual pairs. As mentioned previously, these theorems generalize to non-linear programming and are important for analyzing the solution spaces for both linear and non-linear programs.

## 8.1 Weak Duality

Consider a standard linear program (primal) (2.1) and its dual (7.1). We say that a vector $x$ is **primal feasible** if it is in the feasible region of the primal and a vector $y$ is **dual feasible** if it is in the feasible region of the dual.

**Theorem 8.1** (Weak Duality)**.** *Suppose $x$ is primal feasible and $y$ is dual feasible. Then the primal objective value at $x$ is less than or equal to the dual objective value at $y$.*

*Proof.* The proof relies on analyzing the term $y^T Ax$ and follows from the following sequence of inequalities:

$$\begin{aligned} b^T y &= y^T b \\ &\geq y^T (Ax) && \text{as } Ax \leq b \text{ and } y \geq 0 \\ &= (y^T Ax)^T \\ &= x^T A^T y \\ &\geq x^T c && \text{as } A^T y \geq c \text{ and } x \geq 0 \\ &= c^T x. \end{aligned}$$

□

We get several immediate corollaries from weak duality.

**Corollary 8.1.** *If the primal is unbounded, then the dual is infeasible.*

**Corollary 8.2.** *If the dual is unbounded, then the primal is infeasible.*

**Corollary 8.3.** *If both the primal and dual have optimal solutions, then the optimal value of the primal is less than or equal to the optimal value of the dual.*[1]

Note that all of these results are one-sided implications. We cannot say anything about the dual in the case when the primal is infeasible. Similarly, we cannot conclude anything about the existence of an optimal value of the dual in the case when the primal has an optimal solution.

**Exercise 8.1.** Using Weak Duality, what can you say about the dual if the primal is infeasible?

**Exercise 8.2.** Find primal-dual pairs of linear programs such that

1. both the primal and dual are infeasible,
2. the primal is infeasible and the dual is unbounded,
3. the primal is unbounded and the dual is infeasible.

Can you have primal-dual pairs such that both the primal and the dual are unbounded?

## 8.2 Strong Duality

Weak duality asserts that the optimal objective value of the primal is always less that on equal to the optimal objective of the dual (if both exist). The proof of this statement was a simple manipulation of algebraic expressions. Strong duality further says that there is no duality gap i.e. if both the optimal objective values exist then they must be equal! The proof of this result is far more involved. Weak duality is easy to generalize to non-linear programs but the generalization of strong duality requires stronger proof techniques from convex analysis. For the case of linear programs, strong duality follows from an analysis of the simplex method!

The tableau of the primal problem (2.1) is as follows:

$$\begin{array}{cc|c} c^T & 0 & c_0 \\ \hline A & I_m & b \end{array} \tag{8.1}$$

We can standardize the dual problem (7.1) and form its tableau:

$$\begin{array}{cc|c} -b^T & 0 & -c_0 \\ \hline -A^T & I_n & -c \end{array} \tag{8.2}$$

We will call such tableaux **duals** of each other. More generally, we'll say that two tableaux (of appropriate dimensions) are **duals** of each other if after rearranging

[1] When optimal solutions exist for both the primal and the dual, the difference between them is called the **duality gap**.

the pivot columns (if necessary) they're of the above forms. The following theorem follows from explicit computations.

**Lemma 8.1.** *Consider the two dual tableaux* (8.1) *and* (8.2) *. If we pivot the first tableau about the $i^{th}$ row and $j^{th}$ column of $A$ and the second tableau about the $i^{th}$ column and $j^{th}$ row of $-A^T$, then the resulting tableaux remain duals of each other.*

**Exercise 8.3.** This exercise proves Lemma 8.1 in the case when $n = 3$ and $m = 2$. Consider the linear program:

$$\begin{array}{rrcrcrcr} \text{maximize:} & c_1x_1 & + & c_2x_2 & + & c_3x_3 & & \\ \text{subject to:} & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & \leq & b_1 \\ & a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & \leq & b_2 \\ & x_1 & , & x_2 & , & x_3 & \geq & 0 \end{array}$$

- Form the primal and dual tableaux.
- Perform a pivot about the entry $a_{12}$ in the primal tableau (assume that $a_{12} \neq 0$) and the corresponding entry in the dual tableau.

In the end, you should see that the two resulting tableaux are still dual to each other. The same calculation works in full generality for an arbitrary linear program as we never used the fact that $n = 3$ and $m = 2$.

**Lemma 8.2.** *If the tableau* (8.1) *corresponds to an optimal solution of the primal then the tableau* (8.2) *corresponds to an optimal solution of the dual.*

*Proof.* The tableau (8.1) corresponds to an optimal solution of the primal precisely when

1. (primal-optimality) $c^T \leq 0$ as in this case no entering variable can be found for the primal, and
2. (primal-feasibility) $b \geq 0$.

These conditions translate to

1. (dual-optimality) $-b^T \leq 0$ as in this case no entering variable can be found for the dual, and
2. (dual-feasibility) $-c \geq 0$.

Note the interesting fact that the condition for primal-optimality translates to dual-feasibility and the condition for primal-feasibility translates to dual-optimality. □

**Theorem 8.2** (Strong duality)**.** *If the primal has an optimal solution then so does the dual. Moreover, they have the same optimal values.*[2]

[2]Strong duality implies that there is no duality gap.

*Proof.* By weak duality, it suffices to show that there is some dual-feasible solution which has the same objective value as a optimal objective value of the primal. This dual-feasible solution will be dual-optimal by Corollary 8.3.

At the optimal solution for the primal, we have a set of basic and non-basic variables. We can perform a sequence of pivot operations to get this tableau from the initial tableau. We then perform the corresponding pivots on the dual tableau. By Lemma 8.1 the resulting tableau will be dual to the primal tableau at the optimal solution. By Lemma 8.2 the dual is also optimal and has the same objective value. □

*Remark.* Strong duality establishes as one-to-one correspondence between the primal optimal BFS and dual optimal BFS. At every primal-optimal BFS there is a unique corresponding dual-optimal BFS obtained by dualizing the tableau. This is by no means the only dual-optimal solution, but rather the dual-optimal solution dual to a particular primal-optimal BFS.

## 8.3 Certificate of Optimality

We can use the two duality theorems to come up with a fast way to verify optimality.

**Theorem 8.3** (Certificate of optimality)**.** *$x$ is an optimal solution for the primal and $y$ is an optimal solution for the dual if and only if*

1. *$x$ is primal-feasible,*
2. *$y$ is dual-feasible,*
3. *$c^T x = b^T y$ i.e. the primal objective value at $x$ is equal to the dual objective value at $y$.*

*Proof.* ( $\Rightarrow$ )

If $x$ and $y$ are optimal solutions, then they are feasible by definition. By strong duality (Theorem 8.2) they must the same objective value.

( $\Leftarrow$ )

If $x$ and $y$ are feasible solutions then by weak duality (Theorem 8.1) the dual objective values provide an upper bound on the primal objective value. Because this upper bound is attained at $x$, it must be an optimal solution of the primal. Similarly, for $y$. □

### 8.3.1 Complimentary slackness

There is another closely related method for verifying the correctness of solution that uses primal and dual slack variables instead of primal and dual objective values.

Denote by $w = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}$ the primal slack variables and by $z = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ the dual slack variables. More explicitly, let

$$w = b - Ax$$
$$z = -c + A^T y.$$

We use this convention for $z$ to ensure that at a dual feasible solution $z \geq 0$.

**Theorem 8.4** (Complementary slackness)**.** *Suppose $x$ is primal feasible and $y$ is dual feasible. Then $x$ and $y$ are optimal if and only if*

1. *for all $1 \leq j \leq n$, $x_j z_j = 0$, and*
2. *for all $1 \leq i \leq m$, $y_i w_i = 0$.*

*Proof.* The proof is in two steps. We first show a weaker statement about the vanishing of two scalars and then show that the vanishing of these scalars implies complementary slackness.

**Claim 1:** $x$ and $y$ are optimal solutions if and only if $x^T z = 0$ and $y^T w = 0$.

We start by expanding out the slack variables:

$$\begin{array}{lrcr} & x^T z = 0 & \text{and} & y^T w = 0 \\ \Leftrightarrow & x^T(-c + A^T y) = 0 & \text{and} & y^T(b - Ax) = 0 \\ \Leftrightarrow & x^T c = x^T A^T y & \text{and} & y^T b = y^T Ax \\ \Leftrightarrow & c^T x = y^T Ax & \text{and} & b^T y = y^T Ax \end{array}$$

We have reduced the problem to showing that $x$ and $y$ are optimal solutions if and only if $c^T x = y^T Ax$ and $b^T y = y^T Ax$.

( $\Leftarrow$ ) As $c^T x = y^T Ax = b^T y$, and $x$ and $y$ are given to be feasible, $x$ and $y$ are optimal by Theorem 8.3.

( $\Rightarrow$ ) By the *proof* of weak duality 8.1, we know that,

$$c^T x \leq y^T Ax \leq b^T y.$$

By strong duality, as $x$ and $y$ are optimal

$$c^T x = b^T y.$$

The only way the two can be simultaneously true is if $c^T x = y^T Ax$ and $b^T y = y^T Ax$. This completes the proof of Claim 1.

**Claim 2:** $x^T z = 0$ and $y^T w = 0$ if and only if

1. for all $1 \leq j \leq n$, $x_j z_j = 0$, and
2. for all $1 \leq i \leq m$, $y_i w_i = 0$.

This follows from the fact that at a feasible solution $x, y, w, z \geq 0$. □

*Remark.* Complementary slackness as stated in Theorem 8.4 holds true even for general linear programs. The definition of slack variables,

$$w = b - Ax$$
$$z = -c + A^T y.$$

ensures that we will have $x_j w_j \geq 0$ for all $1 \leq j \leq n$ and $y_i z_i \geq 0$ for all $1 \leq i \leq m$ and hence the above proof is still valid.

If the optimal solution of the primal is at a non-degenerate BFS, then complementary slackness can be used to find the optimal solution to the dual without having to run the simplex method on it. At a non-degenerate BFS, the basic variables are non-zero. By complementary slackness, the corresponding dual variables must be 0. This gives us a system of $m$ variables in $m$ equations which can be solved to find the dual optimal solution.

**Example 8.1.** Consider the linear program (1.2) whose dual is (7.2). At the optimal solution $x_1 = 0.6$, $x_2 = 0.3$, $w_1 = 0$, $w_2 = 0$, and $w_3 = 0.1$. By complementary slackness, we must have $z_1 = 0$, $z_2 = 0$, and $y_3 = 0$ at the dual optimal solution. Plugging these in (7.2), we get a system of equations

$$3y_1 + 2y_2 = 4$$
$$6y_1 + y_2 = 3$$

whose solutions are $y_1 = 2/9$ and $y_2 = 5/3$. One can check that this is indeed dual optimal by comparing the dual objective at this point to the primal objective value:

$$3.6(2/9) + 1.5(5/3) + 0(1) = 3.3 = 4(0.6) + 3(0.3).$$

**Exercise 8.4.** We can use complementary slackness and certificate of optimality to show that the optimal solution of a standard linear program can never be attained in the interior of the feasible region.

Consider the standard linear program (2.1) and assume that $b \geq 0$ and $c \neq \vec{0}$. Let $x^*$ be a point in the interior of the feasible region. Suppose $x^*$ is an optimal solution to the primal. By strong duality, we know that a dual-optimal solution $y^*$ exists. We'll show that this leads to a contradiction.

1. What can you say about the values of the decision and slack variables at $x^*$?
2. What can you say about the values of the dual decision variables at $y^*$ using complementary slackness?

We now consider two cases, depending on the sign of $c$, and in each case show that $y^*$ cannot be optimal.

1. Suppose that $c_j > 0$ for some $0 \leq j \leq n$. Explain why $y^*$ cannot be dual-feasible (and hence optimal).
2. Suppose $c \leq 0$.
    1. What can you say about the objective values at $x^*$ and $y^*$?
    2. Explain why $y^*$ cannot be dual-optimal.

# Chapter 9

# Sensitivity Analysis

Linear programs are used to model real world problems. Such models are at best approximate and at worst inaccurate. Even when the models are accurate, the parameters in the model often change with time. As such, it is important to understand the *sensitivity* of our solution to changes in the model. This is called *sensitivity analysis*. We will focus on understanding the dependence of the optimal objective value of the standard linear program (2.1) on the constants $b_i$ and $c_j$.[1]

*Throughout this chapter, we'll assume that our linear programs have an optimal solution and fix an optimal solution for the primal and the corresponding dual solution for the dual.*

## 9.1 Dictionaries Revisited

We will start by finding a succinct way to describe the dictionary at the optimal solution. Recall that the decision and slack variables are related to each other by the Equation (3.5) which can be written as:

$$\begin{bmatrix} A & I_m \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = b. \tag{9.1}$$

Let $\widehat{A} := \begin{bmatrix} A & I_m \end{bmatrix}$. We'll decompose $\widehat{A}$ using the basic and non-basic variables.

Suppose we are at an optimal basic feasible solution. Let $\mathcal{B}$ be the matrix formed by combining the columns of $\widehat{A}$ corresponding to the basic variables at the optimal BFS and let $\mathcal{N}$ be the matrix formed by combining the columns of $\widehat{A}$ corresponding to the non-basic variables. Let $x_{\mathcal{B}}$ be the vector of basic variables and $x_{\mathcal{N}}$ be the vector of non-basic variables.

[1] This is not to say that the sensitivity of the optimal objective value to the constants $a_{ij}$ is less important. Rather, such an analysis is beyond the scope of this class.

*Remark.* The matrices $\mathcal{B}$ and $\mathcal{N}$ are sub-matrices of the "initial" matrix $\widehat{A}$ and not the tableau at the basic feasible solution. We're using the basic feasible solution to decide which columns are basic and which are non-basic for the "inital" matrix $\widehat{A}$.

By rearranging the columns of $\widehat{A}$ if necessary, we can rewrite (9.1) as

$$\begin{aligned} & \begin{bmatrix}\mathcal{B} & \mathcal{N}\end{bmatrix}\begin{bmatrix}x_{\mathcal{B}} \\ x_{\mathcal{N}}\end{bmatrix} = b, \\ \implies \quad & \mathcal{B}x_{\mathcal{B}} + \mathcal{N}x_{\mathcal{N}} = b, \\ \implies \quad & \mathcal{B}x_{\mathcal{B}} = b - \mathcal{N}x_{\mathcal{N}}. \end{aligned}$$

When we execute the simplex method, the matrix $\mathcal{B}$ gets reduced to an $n \times n$ matrix with $n$ pivots. Hence, it must be invertible allowing us to further simplify as follows.

$$\implies x_{\mathcal{B}} = \mathcal{B}^{-1}b - \mathcal{B}^{-1}\mathcal{N}x_{\mathcal{N}}. \tag{9.2}$$

This is nothing but the dictionary at the optimal BFS!

**Lemma 9.1.** *Using the above notation, $\mathcal{B}^{-1}b$ is the value of the basic variables at the optimal solution.*

*Proof.* At a basic feasible solution, the non-basic variables $x_{\mathcal{N}}$ equal 0. Plugging in $x_{\mathcal{N}} = \vec{0}$ in Equation (9.2) gives us the desired result. □

**Example 9.1.** Consider Example (1.2) again. At the optimal solution $w_1$ and $w_2$ are non-basic and have the value 0, and $x$, $y$, and $w_3$ are basic with values 0.3, 0.6, and 0.9, respectively. Using the above notation, we have

$$\mathcal{B} = \begin{bmatrix}3 & 6 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1\end{bmatrix}, x_{\mathcal{B}} = \begin{bmatrix}x \\ y \\ w_3\end{bmatrix},$$
$$\mathcal{N} = \begin{bmatrix}1 & 0 \\ 0 & 1 \\ 0 & 0\end{bmatrix}, x_{\mathcal{N}} = \begin{bmatrix}w_1 \\ w_2\end{bmatrix}.$$

Using Equation (9.2) the dictionary at the optimal solution becomes

$$\begin{aligned} \begin{bmatrix}x \\ y \\ w_3\end{bmatrix} &= \begin{bmatrix}3 & 6 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1\end{bmatrix}^{-1}\begin{bmatrix}3.6 \\ 1.5 \\ 1\end{bmatrix} - \begin{bmatrix}3 & 6 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 & 0 \\ 0 & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix}w_1 \\ w_2\end{bmatrix} \\ &= \begin{bmatrix}-1/9 & 2/3 & 0 \\ 2/9 & -1/3 & 0 \\ -1/9 & -1/3 & 1\end{bmatrix}\begin{bmatrix}3.6 \\ 1.5 \\ 1\end{bmatrix} - \begin{bmatrix}-1/9 & 2/3 & 0 \\ 2/9 & -1/3 & 0 \\ -1/9 & -1/3 & 1\end{bmatrix}\begin{bmatrix}1 & 0 \\ 0 & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix}w_1 \\ w_2\end{bmatrix} \\ &= \begin{bmatrix}0.6 \\ 0.3 \\ 0.1\end{bmatrix} - \begin{bmatrix}-1/9 & 2/3 \\ 2/9 & -1/3 \\ -1/9 & -1/3\end{bmatrix}\begin{bmatrix}w_1 \\ w_2\end{bmatrix}. \end{aligned}$$

This is precisely the dictionary (3.3) at the optimal solution.

## 9.2 Range of Optimality - Constraints

We want to analyze the change in the optimal solution as we change the constraint upper bounds $b_i$. It is likely that by changing $b_i$ we change the optimal solution. However, in a good model, this change should not be abrupt. This can be achieved by requiring **the set of basic and non-basic variables to remain unchanged**. In this case, the equation (9.2) will still be the equation describing the dictionary at the optimal solution and the change in $b_i$ will result in a differentiable (in fact, linear) change in $x_{\mathcal{B}}$.

**Example 9.2.** Suppose we vary $b_3 = 1$ in Example (1.2). One can check that at the optimal solution $w_1$ and $w_2$ are non-basic as long as $b_3 > 0.9$. Thus we can say that out model is a good model as long as the error in $b_3$ is less than 0.1.

Suppose we change $b_i$ to $b_i + \delta$, where $\delta$ is a real number, and leave all the other constants unchanged. This is equivalent to changing $b$ to $b + \delta e_i$ where $e_i$ is the $i^{th}$ standard basis vector of $\mathbb{R}^m$. This changes equation (9.2) to

$$\begin{aligned} x_{\mathcal{B}} &= \mathcal{B}^{-1}b + \delta\mathcal{B}^{-1}e_i - \mathcal{B}^{-1}\mathcal{N}x_{\mathcal{N}} \\ &= \mathcal{B}^{-1}b + \delta(\mathcal{B}^{-1})_{\_i} - \mathcal{B}^{-1}\mathcal{N}x_{\mathcal{N}}. \end{aligned}$$

where $(\mathcal{B}^{-1})_{\_i}$ denotes the $i^{th}$ column of $\mathcal{B}^{-1}$.

Note that the coefficients of $x_{\mathcal{N}}$ remain unchanged. So, for this dictionary to stay optimal we only need the constants to remain non-negative i.e.

$$\mathcal{B}^{-1}b + \delta(\mathcal{B}^{-1})_{\_i} \geq 0. \tag{9.3}$$

**Proposition 9.1.** *The **range of optimality** for $b_i$ is the interval $[b_i+\delta_-, b_i+\delta_+]$ such that $\mathcal{B}^{-1}b + \delta(\mathcal{B}^{-1})_{\_i} \geq 0$ for all $\delta \in [\delta_-, \delta_+]$.*

In practice, Equation (9.3) gives us $m$ inequalities which need to be simultaneously satisfied. These give us candidate values for $\delta$ some of which are positive and some of which are negative. We then choose $\delta_+$ to be **the smallest positive value** and $\delta_-$ to be the **largest negative value**. If $\delta_+$ does not exist then the upper bound is $\infty$ and if $\delta_-$ does not exist then the lower bound is $-\infty$. If either $\delta_+$ or $\delta_-$ is 0 then the linear program is degenerate. In this case, our program is very sensitive to perturbations in $b_i$.

**Example 9.3.** Let us find the range of optimality for $b_1 = 3.6$, $b_2 = 1.5$, and $b_3 = 1$ in (1.2) using our calculations in Example 9.1. We know that

$$\mathcal{B}^{-1} = \begin{bmatrix} -1/9 & 2/3 & 0 \\ 2/9 & -1/3 & 0 \\ -1/9 & -1/3 & 1 \end{bmatrix}.$$

Using $i = 1$ and Lemma 9.1 in Equation (9.3) we get

$$\begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix} + \delta \begin{bmatrix} -1/9 \\ 2/9 \\ -1/9 \end{bmatrix} \geq 0$$

which gives us the inequalities

$$\begin{array}{lcllcl} 0.6 + \delta(-1/9) & \geq & 0 & \implies & \delta & \leq & 0.6(9) = 5.4 \\ 0.3 + \delta(2/9) & \geq & 0 & \implies & \delta & \geq & -0.3(9/2) = -1.35 \\ 0.1 + \delta(-1/9) & \geq & 0 & \implies & \delta & \leq & 0.1(9) = 0.9. \end{array}$$

So, $\delta_- = -1.35$ and $\delta_+ = \min(5.4, 0.9) = 0.9$ and the range of optimality for $b_1$ is $[3.6 - 1.35, 3.6 + 0.9] = [2.25, 4.5]$.

Using $i = 2$ and Lemma 9.1 in Equation (9.3) we get

$$\begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix} + \delta \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix} \geq 0$$

which gives us the inequalities

$$\begin{array}{lcllcl} 0.6 + \delta(2/3) & \geq & 0 & \implies & \delta & \geq & -0.6(3/2) = -0.9 \\ 0.3 + \delta(-1/3) & \geq & 0 & \implies & \delta & \leq & 0.3(3) = 0.9 \\ 0.1 + \delta(-1/3) & \geq & 0 & \implies & \delta & \leq & 0.3(1) = 0.3. \end{array}$$

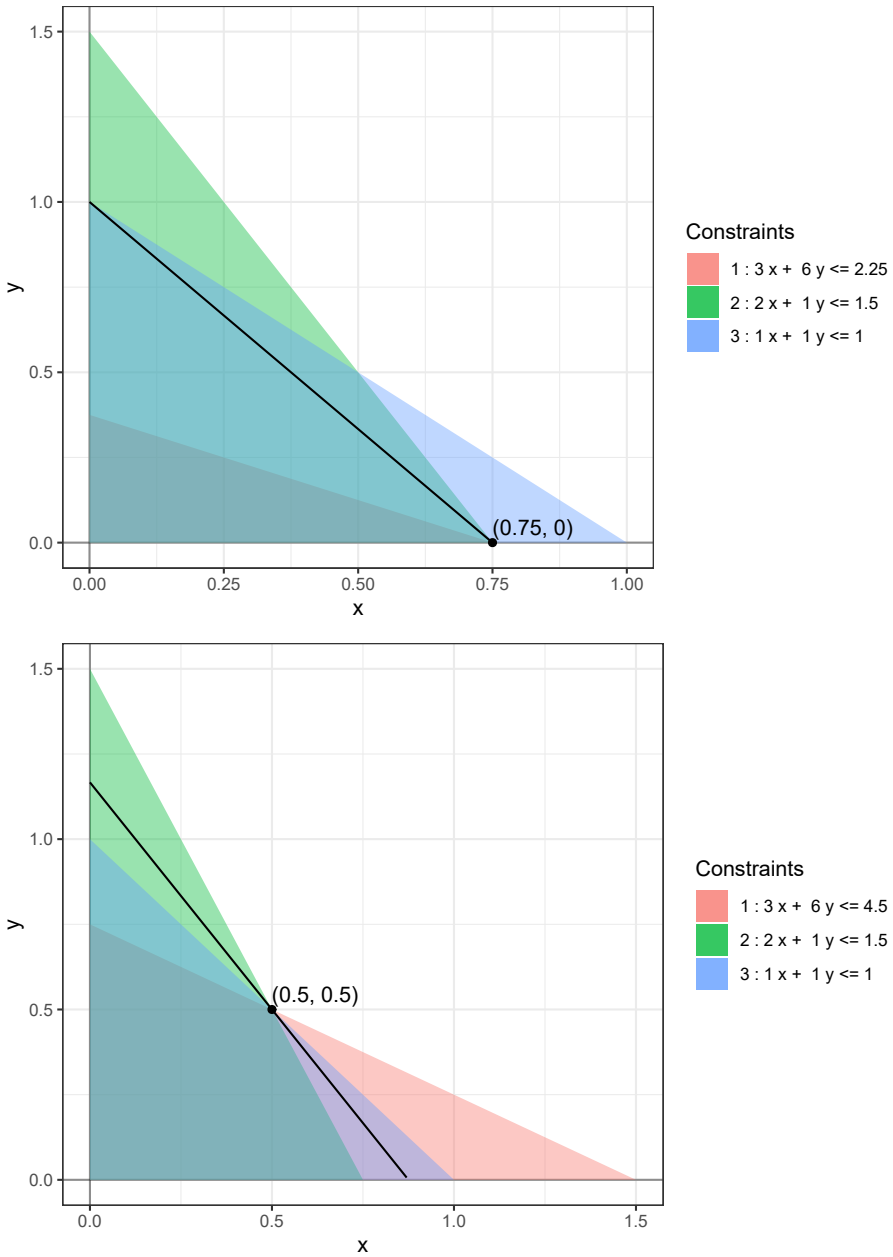
So, $\delta_- = -0.9$ and $\delta_+ = \min(0.3, 0.9) = 0.3$ and the range of optimality for $b_2$ is $[1.5 - 0.9, 1.5 + 0.3] = [0.6, 1.8]$.

Using $i = 3$ and Lemma 9.1 in Equation (9.3) we get

$$\begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix} + \delta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \geq 0$$

which gives us $\delta \geq -0.1$ and so the range of optimality for $b_3$ is $[1 - 0.1, \infty) = [0.9, \infty)$.

The following figures show the optimal solutions at the extreme ends of the range of optimality of $b_1$.




## 9.3 Shadow Prices

For this section, assume that neither of $\delta_+$ or $\delta_-$ is zero. We can use Lemma 9.1 to find the rate of change of the optimal solution with respect to $b_i$. Call the

objective function $\mathbb{O} = c^T x$. We think of $\mathbb{O}$ as being a function of $b_i$, $c_j$, and $a_{ij}$. Using Lemma 9.1 we get

$$\frac{\partial x_{\mathcal{N}}}{\partial b_i} = 0$$

as the non-basic variables remain 0 when we perturb $b_i$ within the range of optimality, and

$$\begin{aligned}\frac{\partial x_{\mathcal{B}_j}}{\partial b_i} &= j^{th} \text{ row of } \frac{\partial \mathcal{B}^{-1} b}{\partial b_i} \\ &= (\mathcal{B}^{-1})_{ji}\end{aligned}$$

where $x_{\mathcal{B}_j}$ denotes the $j^{th}$ basic variable at the optimal solution. Using these equations we can find the rate of change of the optimal solution $\mathbb{O}$ with respect to $b_i$. We start by re-indexing the variables and objective coefficients using the basic and non-basic variables.

$$\begin{aligned}\mathbb{O} &= c^T x \\ &= c_{\mathcal{B}}^T x_{\mathcal{B}} + c_{\mathcal{N}}^T x_{\mathcal{N}} \\ \implies \frac{\partial \mathbb{O}}{\partial b_i} &= c_{\mathcal{B}}^T \frac{\partial x_{\mathcal{B}}}{\partial b_i} + c_{\mathcal{N}}^T \frac{\partial x_{\mathcal{N}}}{\partial b_i} \\ &= c_{\mathcal{B}}^T (\mathcal{B}^{-1})_{\_i}\end{aligned}$$

By strong duality, we know that the primal objective value equals the dual objective value i.e.

$$\mathbb{O} = b_1 y_1 + \cdots + b_m y_m$$

So, $\partial \mathbb{O} / \partial b_i = y_i$. Because of this result, $y_i$ is also called the **shadow price** or the **marginal cost** of the $i^{th}$ constraint. This is an extremely important interpretation of (non-degenerate) dual optimal solutions. This is why duality theory naturally arises in the study of linear programs.

**Theorem 9.1.** *For a non-degenerate linear program, the dual optimal solution is given by* $(\mathcal{B}^{-1})^T c_{\mathcal{B}}$.

*Proof.* Using the rate of change calculation above, we get $y_i = c_{\mathcal{B}}^T (\mathcal{B}^{-1})_{\_i}$. We then combine all the coordinates into a vector to get the desired result. □

This theorem provides yet another method of finding the dual optimal solution without having to solve the dual linear program.

**Example 9.4.** For the linear program (1.2), the objective function is

$$\begin{aligned}\mathbb{O} &= 4x + 3y \\ &= 4x + 3y + 0w_3 + 0w_1 + 0w_2\end{aligned}$$

So, $c_{\mathcal{B}} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$ and $c_{\mathcal{N}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Using the value of $\mathcal{B}^{-1}$ calculated above, we get

$$\begin{aligned} y &= (\mathcal{B}^{-1})^T c_{\mathcal{B}} \\ &= \begin{bmatrix} -1/9 & 2/9 & -1/9 \\ 2/3 & -1/3 & -1/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2/9 \\ 5/3 \\ 0 \end{bmatrix}. \end{aligned}$$

To check that this is indeed dual-optimal, we calculate the dual-objective value at this solution

$$\begin{aligned} b^T y &= 3.6(2/9) + 1.5(5/3) + 0(1) \\ &= 3.3 \end{aligned}$$

which equals the optimal objective value of the primal. One can check that this solution is also dual-feasible and hence is the dual-optimal solution by Certificate of Optimality (Theorem 8.3).

## 9.4 Range of Optimality - Objective

We repeat the same analysis for the objective coefficients - how far can we change the objective coefficient $c_j$ without changing the set of basic and non-basic variables at the optimal solution? In this case, we are not changing the constraints and therefore the feasible region remains unchanged and not changing the set of basic and non-basic variables implies that the optimal solution (and not just the optimal objective value) also remains unchanged. So, this question is the same as asking - how far can we change the objective coefficient $c_j$ without changing the optimal solution?

We can redo the entire analysis for the objective coefficients from scratch. However, duality theory provides a faster way to do this as performing sensitivity analysis on the objective coefficients of the primal is the same as performing sensitivity analysis on the constraints of the dual. Consider the standardized dual

$$\begin{array}{lrcl} \text{maximize:} & -c_0 - b^T y & & \\ \text{subject to:} & -A^T y & \leq & -c \\ & y & \geq & 0. \end{array}$$

Using Equation (9.3) for this dictionary we get the range of optimality for $-c_j$ to be

$$(-\mathcal{B}_d)^{-1}(-c) + \delta(-\mathcal{B}_d^{-1})_{\_j} \geq 0. \tag{9.4}$$

where $-\mathcal{B}_d$ is formed by combining the dual-basic columns of $\begin{bmatrix}-A^T & I_n\end{bmatrix}$ and so $\mathcal{B}_d$ is formed by combining the dual-basic columns of $\begin{bmatrix}A^T & -I_n\end{bmatrix}$. This simplifies to

$$\mathcal{B}_d^{-1}c - \delta(\mathcal{B}_d^{-1})_{\_j} \geq 0. \tag{9.5}$$

Note that this is the range of optimality for $-c_j$. To get the range of optimality for $c_j$ we need to replace $\delta$ with $-\delta$ to get

$$\mathcal{B}_d^{-1}c + \delta(\mathcal{B}_d^{-1})_{\_j} \geq 0. \tag{9.6}$$

Finally, by Lemma 9.1, $\mathcal{B}_d^{-1}c$ is the vector of values of the dual basic variables. Hence we get,

**Proposition 9.2.** *The **range of optimality** for $c_j$ is the interval $[c_j + \delta_-, c_j + \delta_+]$ such that $y_\mathcal{B} + \delta(\mathcal{B}_d^{-1})_{\_j} \geq 0$ for all $\delta \in [\delta_-, \delta_+]$.*

By a happy accident all the negative signs have cancelled out and the equation for finding the range of optimality for the objective coefficients has the same form as the one for finding the range of optimality for the constraints!

**Theorem 9.2.** *At an optimal BFS, the range of optimality for the constraints and the objective functions can be computed using the following formulae:*

***Range of optimality for** $b_i$**:***

$$x_\mathcal{B} + \delta(\mathcal{B}^{-1})_{\_i} \geq 0, \tag{9.7}$$

***Range of optimality for** $c_j$**:***

$$y_\mathcal{B} + \delta(\mathcal{B}_d^{-1})_{\_j} \geq 0, \tag{9.8}$$

*where $\mathcal{B}$ is formed by combining the primal-basic columns of $\begin{bmatrix}A & I_m\end{bmatrix}$, $x_\mathcal{B}$ is the value of the primal basic variables, $\mathcal{B}_d$ is formed by combining the dual-basic columns of $\begin{bmatrix}A^T & -I_n\end{bmatrix}$, and $y_\mathcal{B}$ is the value of the dual basic variables.*

Finally, to find $\mathcal{B}_d$ we note that by complementary slackness (Theorem 8.4), if the linear program is non-degenerate then the dual basic variables correspond to the primal non-basic variables (as the dual basic variables must be non-zero).[2]

**Example 9.5.** Consider the linear program (1.2) again. At the optimal solution, as $w_1$ and $w_2$ are non-basic, the corresponding dual variables ($y_1$ and $y_2$) will be basic for the dual linear program (7.2). This gives us

$$\mathcal{B}_d = \begin{bmatrix}3 & 2\\ 6 & 1\end{bmatrix}$$
$$\implies \mathcal{B}_d^{-1} = \begin{bmatrix}-1/9 & 2/9\\ 2/3 & -1/3\end{bmatrix}.$$

[2]This statement is true even for degenerate linear programs but in this case the proof is more subtle and requires the use of strong duality (Theorem 8.2).

From Example 9.4, we know that $y_1 = 2/9$ and $y_2 = 5/3$ at the dual optimal solution. Using these, we can now find the range of optimality for the objective coefficients.

To find the range of optimality for $c_1 = 4$ we solve

$$\begin{bmatrix} 2/9 \\ 5/3 \end{bmatrix} + \delta \begin{bmatrix} -1/9 \\ 2/3 \end{bmatrix} \geq 0$$

which gives us the inequalities

$$\begin{array}{rcrcrcr} 2/9 + \delta(-1/9) & \geq & 0 & \implies & \delta & \leq & 2 \\ 5/3 + \delta(2/3) & \geq & 0 & \implies & \delta & \geq & -5/2 \end{array}$$

So, $\delta_- = -5/2$ and $\delta_+ = 2$ and the range of optimality for $c_1$ is $[4 - 5/2, 4 + 2] = [1.5, 6]$.

To find the range of optimality for $c_2 = 3$ we solve

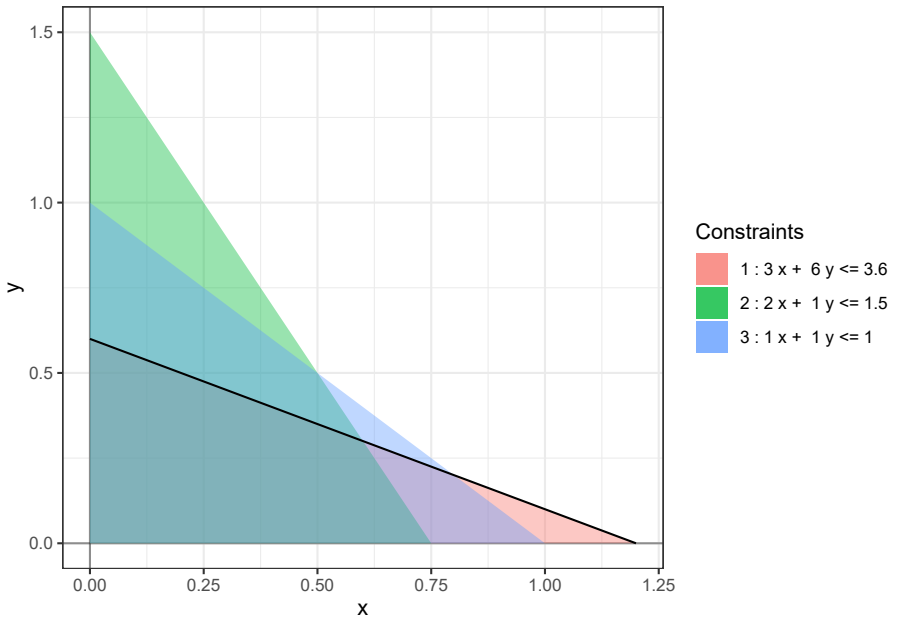
$$\begin{bmatrix} 2/9 \\ 5/3 \end{bmatrix} + \delta \begin{bmatrix} 2/9 \\ -1/3 \end{bmatrix} \geq 0$$

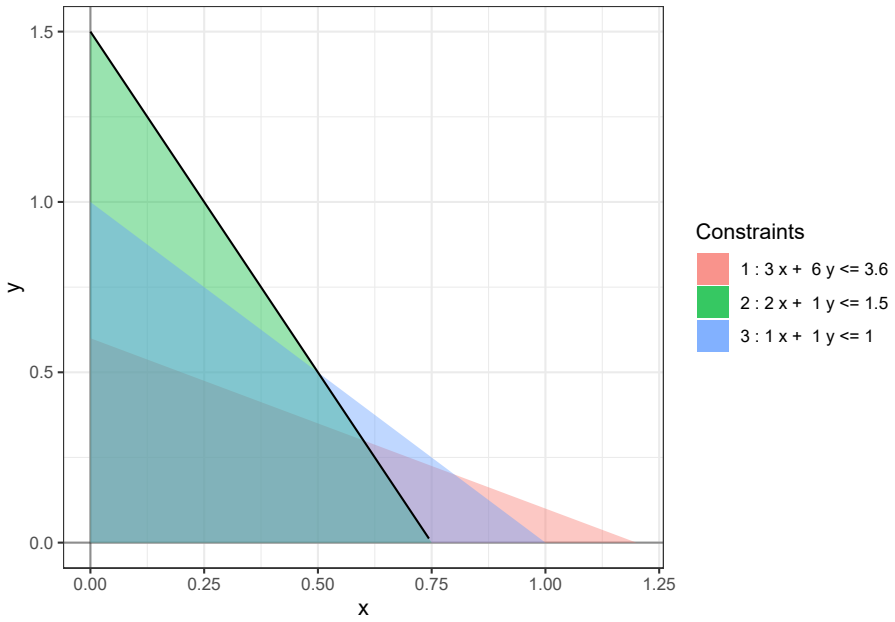
which gives us the inequalities

$$\begin{array}{rcrcrcr} 2/9 + \delta(2/9) & \geq & 0 & \implies & \delta & \geq & -1 \\ 5/3 + \delta(-1/3) & \geq & 0 & \implies & \delta & \leq & 5 \end{array}$$

So, $\delta_- = -1$ and $\delta_+ = 5$ and the range of optimality for $c_2$ is $[3 - 1, 3 + 5] = [2, 8]$.

The following figures show the (infinitely many) optimal solutions at the extreme ends of the range of optimality of $c_1$.




1.5
1.0
0.5
0.0
y
0.00
0.25
0.50
0.75
1.00
1.25
x
Constraints
1 : 3 x + 6 y <= 3.6
2 : 2 x + 1 y <= 1.5
3 : 1 x + 1 y <= 1

# Part III

# Non-Linear Programming

# Chapter 10

# Convex Programming

We will now move toward generalizing the concepts from linear programming to non-linear optimization questions. We'll only touch several subjects briefly without going into too much depth. We'll start with convex optimization, which is a direct generalization of linear programming.

**Definition 10.1.** We say that a subset $\mathcal{S}$ of $\mathbb{R}^n$ is called **convex** if for any points $x_1$ and $x_2$ in $\mathcal{S}$, the line segment connecting them also lies in $\mathcal{S}$. More precisely, we say that $S$ is **convex** if forall $x_1$ in $\mathcal{S}$, $x_2$ in $\mathcal{S}$, and $t \in [0,1]$, the point $(1-t)x_1 + tx_2$ also lies in $\mathcal{S}$.

**Exercise 10.1.** Show that the feasible region of a linear program is convex.

**Definition 10.2.** We say that function $f : \mathcal{S} \to \mathbb{R}$ is **convex** if for all $x_1$ in $\mathcal{S}$, $x_2$ in $\mathcal{S}$, and $t \in [0,1]$, the inequality $f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2)$ holds. Geometrically, this is saying that any line segment connecting two points on the graph of $f$ lies above the graph.

The following theorem provides a quick way to check if a function is convex and to come up with examples of convex functions.

**Theorem 10.1.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$ function. $f$ is convex if and only if the Hessian matrix of $f$ is positive semi-definite. In particular, when $n = 1$, $f$ is convex if and only if $f'' \geq 0$.*

**Definition 10.3.** An optimization problem

$$\begin{array}{ll} \text{maximize:} & f(x) \\ \text{subject to:} & x \in \mathcal{S} \end{array} \tag{10.1}$$

is called **convex** if $\mathcal{S}$ is a closed and convex subset of $\mathbb{R}^n$ and $f : \mathcal{S} \to \mathbb{R}$ is a convex function.

**Exercise 10.2.** Show that a linear program is a convex optimization problem.

The following theorem is an example of how theorems about linear programs generalize to convex programs.

**Theorem 10.2.** *The convex optimization problem* (10.1) *either has an optimal solution on the boundary of* $\mathcal{S}$ *or is unbounded.*

*Proof.* Let $x$ be a point in the interior of $\mathcal{S}$. It suffices to show that either there is some point on the boundary of $\mathcal{S}$ with an objective value that is greater than or equal to the objective value of $x$ or the problem is unbounded.

Let $x_1$ be any point on the boundary of $\mathcal{S}$. Draw a ray starting at $x_1$ in the direction of $x$. As $\mathcal{S}$ is closed and convex, two cases are possible:

1. The ray intersects the boundary of $\mathcal{S}$ in exactly one more point, say $x_2$.
2. The ray does not intersect the boundary of $\mathcal{S}$ and the entire ray is contained within $\mathcal{S}$.

**Case 1:**

We will show that either $f(x) \leq f(x_1)$ or $f(x) \leq f(x_2)$. We prove this by contradiction. Suppose $f(x) > f(x_1)$ and $f(x) > f(x_2)$. We know that $x = (1-t)x_1 + tx_2$ for some $t \in (0,1)$. Multiplying the first inequality by $(1-t)$ and the second by $t$ and adding them together, we get

$$\begin{aligned} & (1-t)f(x) + tf(x) > (1-t)f(x_1) + tf(x_2) \\ \implies \qquad & f(x) > (1-t)f(x_1) + tf(x_2), \end{aligned}$$

which contradicts the convexity of $f$.

**Case 2:**

If $f(x_1) \geq f(x)$, we're done. Suppose this is not the case. We'll show that the problem is unbounded.

For $0 < t \leq 1$, let $y_t$ denote the point on the ray such that

$$x = (1-t)x_1 + ty_t.$$

For example, $y_1 = x$ and $y_{1/2}$ is the point on the ray for which $x$ is midpoint of $x_1$ and $y_{1/2}$. As $t$ decreases, $y_t$ will move farther and farther away from $x$. Note that all of $y_t$ lie inside $\mathcal{S}$. By convexity of $f$, we know that

$$\begin{aligned} & f(x) \leq (1-t)f(x_1) + tf(y_t) \\ \implies \qquad & f(x) - (1-t)f(x_1) \leq tf(y_t) \\ \implies \qquad & t^{-1}f(x) - (t^{-1}-1)f(x_1) \leq f(y_t) \\ \implies \qquad & t^{-1}(f(x) - f(x_1)) + f(x_1) \leq f(y_t) \end{aligned}$$

As $f(x) > f(x_1)$, the left hand side tends to $\infty$ as $t$ decreases and the optimization problem is unbounded. □

The following corollary follows by applying Extreme Value Theorem to the above result.

**Theorem 10.3.** *If $\mathcal{S}$ is bounded (in addition to being closed and convex), then the convex optimization problem* (10.1) *has an optimal solution on the boundary of $\mathcal{S}$.*

# Chapter 11

# Separation Theorems

We'll next prove a couple of *geometric* results that are equivalent to strong duality theorem (Theorem 8.2).

## 11.1 Farkas' Lemma

**Theorem 11.1** (Farkas' lemma)**.** *Let $A$ be an $m \times n$ matrix and $b$ be a vector in $\mathbb{R}^m$. Exactly one of the following systems has a solution:*

$$Ax = b \tag{11.1}$$
$$x \geq 0. \tag{11.2}$$

$$y^T A \geq 0 \tag{11.3}$$
$$y^T b < 0. \tag{11.4}$$

*Proof.* We'll prove Farkas' lemma using strong duality. Consider the following linear program:

$$\begin{array}{lllll} \text{maximize:} & 0 & & & \\ \text{subject to:} & Ax & = & b & \\ & x & \geq & 0. & \end{array} \tag{11.5}$$

The optimal solution to the linear program (11.5) is a feasible solution to (11.2). The dual to this linear program is

$$\begin{array}{lll} \text{minimize:} & y^T b & \\ \text{subject to:} & y^T A & \geq \quad 0. \end{array} \tag{11.6}$$

**Case 1:** Suppose (11.2) has a solution.

In this case, (11.5) has an optimal solution. By strong duality, (11.6) also has an optimal solution with optimal objective 0. So, the minimum value of $y^T b$ is 0 and hence the system (11.4) does not have a solution.

**Case 2:** Suppose (11.2) does not have a solution.

In this case, (11.5) has no optimal solution. By strong duality, neither does (11.6). So, (11.6) is either infeasible or unbounded. As $y = 0$ is a feasible solution to (11.6) it cannot be infeasible, and hence it must be unbounded. But this means that the value of $y^T b$ can be made arbitrarily small, and in particular, can be made negative. Hence, the system (11.4) has a solution. □

We can interpret Farkas' lemma geometrically using convex cones and separating hyperplanes.

## 11.2 Separating Hyperplane Theorem

**Definition 11.1.** The **convex cone** of a finite set of vectors in $\mathbb{R}^m$ is the set of positive linear combinations of vectors in the set.[1]

$$C_+(v_1, \ldots, v_n) := \{c_1 v_1 + \cdots + c_n v_n \mid c_i \geq 0\}.$$

**Definition 11.2.** A **hyperplane** in $\mathbb{R}^m$ is the set of solutions to a single linear equation. The complement of a hyperplane in $\mathbb{R}^n$ consists of two connected components. The closures of these components are called **half-spaces**.

If the equation of the hyperplane is given by $b^T y = b_0$, where $b$ is a vector in $\mathbb{R}^m$ and $b_0 \in \mathbb{R}$, then the corresponding two half-spaces are described by $b^T y \leq b_0$ and $b^T y \geq b_0$.

**Definition 11.3.** We say that a hyperplane $H$ **separates** two subsets $S_1$ and $S_2$ of $\mathbb{R}^m$ if $S_1$ and $S_2$ do not intersect $H$ and belong to the two different half-spaces of $H$.

Using convex cones and separating hyperplanes, we can reinterpret 11.1 as follows.

**Theorem 11.2** (Geometric version of Farkas' lemma)**.** *Let $v_1, \ldots, v_n, b$ be vectors in $\mathbb{R}^m$. Exactly one of the following statements is true:*

1. *Either $b$ lies inside $C_+(v_1, \ldots, v_n)$, or*
2. *There is a hyperplane $H$ that separates $b$ from $C_+(v_1, \ldots, v_n)$.*

A point $b$ and a convex cone $C_+(v_1, \ldots, v_n)$ are convex subsets of $\mathbb{R}^m$. The statement of Farkas' lemma 11.2 can be generalized to arbitrary convex sets using metric topology. For now, we'll generalize it to convex polyhedra.

[1] One can also define convex cones for infinite sets of vectors. In this case, we need to take the closure of the set of positive linear combinations.

**Definition 11.4.** Intersection of finitely many half-spaces is called a **convex polyhedron**.

So, a convex polyhedron is the set of solutions to a system of linear inequalities $Ax \leq b$. But this set is precisely the feasible region of a linear program! The class of convex polyhedra is very large and all geometric "linear" convex objects, like points, lines, planes, half-spaces, hyperplanes, convex cones, etc. can be realized as convex polyhedra.

The following is an extension of Farkas' lemma to convex polyhedra (proof in the exercises below).

**Theorem 11.3** (Separating Hyperplane Theorem)**.** *Any two non-empty, disjoint, convex polyhedra in $\mathbb{R}^m$ can be separated by a hyperplane.*

**Exercise 11.1.** Prove the following variant of Farkas' lemma: Let $A$ be an $m \times n$ matrix and let $b$ be a vector in $\mathbb{R}^m$. Exactly one of the following systems has a solution:

1. $Ax \leq b$,
2. $y^T b < 0$, $y^T A = 0$, and $y \geq 0$.

**Exercise 11.2.** Let $P_1$ be the convex polyhedron defined by $A_1 x \leq b_1$ and let $P_2$ be the convex polyhedron defined by $A_2 x \leq b_2$ where $A_1$, $A_2$, $b_1$, and $b_2$ have sizes $m_1 \times n$, $m_2 \times n$, $m_1$, and $m_2$, respectively. Suppose the system

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} x \leq \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{11.7}$$

does not have a solution.

1. Apply the above variant of Farkas' lemma to (11.7) to obtain vectors $y_1$, $y_2$ of size $m_1$ and $m_2$, respectively. Define $c^T := y_1^T A_1$, $d_1 := y_1^T b_1$, and $d_2 := -y_2^T b_2$.
2. Show that $d_1 < d_2$.
3. Show that $A_1 x \leq b_1$ implies $c^T x \leq d_1$.
4. Show that $A_2 x \leq b_2$ implies $c^T x \geq d_2$.
5. Conclude that there exists a hyperplane separating $P_1$ and $P_2$.

## 11.3 Equivalence with Strong Duality

It is possible to prove Strong Duality using Farkas' lemma, which itself can be proven using metric topology. So, Strong Duality, Farkas' lemma, and Separating Hyperplane Theorem should all be thought of as equivalent to each other. We provide below a proof of Strong Duality (Theorem 8.2) using Farkas' lemma.

*Proof.* Consider the system of equations

$$\begin{aligned} Ax &\leq b \\ -A^T y &\leq -c \\ x, y &\geq 0 \\ -c^T x + b^T y &\leq 0. \end{aligned} \tag{11.8}$$

Suppose this system has a feasible solution. The first three equations are equivalent to saying that $x$ is primal-feasible and $y$ is dual-feasible. If such a solution exists, by Weak Duality (Theorem 8.1) we know that $c^T x \leq b^T y$. So, the only way the fourth inequality is satisfied is if $c^T x = b^T y$ i.e. the primal-objective value at $x$ equals the dual-objective value at $y$. But by Weak Duality, these must then be the optimal solutions thereby proving Strong Duality. So, it suffices to show that (11.8) has a feasible solution if the primal has an optimal solution.

We prove this by contradiction. Suppose the primal has an optimal solution but (11.8) does not have a feasible solution. We rewrite the system as

$$\begin{bmatrix} A & 0 \\ 0 & -A^T \\ -c^T & b^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} b \\ -c \\ 0 \end{bmatrix}$$
$$x, y \geq 0.$$

By (a variant of) Farkas' lemma, the following dual system must have a solution.

$$\begin{bmatrix} z^T & w^T & t \end{bmatrix} \begin{bmatrix} b \\ -c \\ 0 \end{bmatrix} < 0$$
$$\begin{bmatrix} z^T & w^T & t \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & -A^T \\ -c^T & b^T \end{bmatrix} \geq 0$$
$$z, w, t \geq 0.$$

which can be rewritten as

$$\begin{aligned} z^T b &< w^T c \\ z^T A &\geq t c^T \\ t b^T &\geq w^T A^T \\ z, w, t &\geq 0. \end{aligned}$$

By combining the second and third equations, we get

$$t z^T b \geq z^T A w \geq t c^T w.$$

If $t > 0$, this contradicts the first equation and we're done. So suppose $t = 0$. Plugging in $t = 0$, we get

$$\begin{aligned} z^T b &< w^T c \\ z^T A &\geq 0 \\ 0 &\geq w^T A^T \\ z, w &\geq 0. \end{aligned}$$

which can be rewritten as the matrix equation

$$\begin{aligned} \begin{bmatrix} z^T & w^T \end{bmatrix} \begin{bmatrix} b \\ -c \end{bmatrix} &< 0 \\ \begin{bmatrix} z^T & w^T \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} &\geq 0 \\ z, w &\geq 0. \end{aligned}$$

Because this system has a solution, by applying (variant of) Farkas' lemma again, the following system cannot have a solution.

$$\begin{aligned} \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &\leq \begin{bmatrix} b \\ -c \end{bmatrix} \\ x, y &\geq 0. \end{aligned}$$

We had assumed that the primal is feasible. The only way the above system is infeasible is if there is no solution no the system

$$\begin{aligned} A^T y &\geq c \\ y &\geq 0. \end{aligned}$$

Applying (variant of) Farkas' lemma yet again we see that the dual system

$$\begin{aligned} z^T c &< 0 \\ z^T A^T &\geq 0 \\ z &\leq 0 \end{aligned}$$

must have a solution. But now, if $x$ is any primal-feasible solution then so is $x - \alpha z$ for any positive constant $\alpha$ and the objective value of the primal at $x - \alpha z$ can be made arbitrarily large by increasing $\alpha$ thereby contradicting the fact that primal has an optimal solution (and hence is bounded). □

# Chapter 12

# Interior Point Methods

Throughout this chapter we will let $f$ denote a twice differentiable function from $\mathbb{R}^n$ to $\mathbb{R}$.

## 12.1 Gradient Descent

We will start by trying to solve the following unconstrained optimization problem:

$$\text{minimize: } f(x)$$

where $x$ is any vector in $\mathbb{R}^n$. **Gradient descent** is a simple algorithm for solving this problem using basic differential calculus. It relies on the fact that *the negative of the gradient points in the direction in which the function decreases the fastest.* So the general principle is to move in the direction of the negative gradient until the function is no longer decreasing. More precisely, we create a sequence of guesses using the following GD recurrence relation:

$$x_{k+1} = x_k - t_k \nabla f(x_k)$$

where $t_k$ is the "step size" for the $k^{th}$ iteration and can be chosen to be a small constant or some function of $k$, $x_k$, $f(x_k)$, or $\nabla f(x_k)$.

There are several issues with this technique:

1. If the function has multiple local minima, then the sequence might converge to a non-absolute minima depending on the starting guess and the choice of step sizes.
2. If the step sizes are chosen to be too large, then the sequence can completely miss the minima and may not converge.
3. If the step sizes are chosen to be too small, then the sequence may take a long time to converge.

Unfortunately, there is no easy way to resolve these issues. In practice, either we need some additional information about the function and its gradient or we proceed by trial and error to find step sizes that work. Even then there is no guarantee that the algorithm will converge to an absolute minima and not a local minima, if at all. In spite of these issues, because of its simplicity, ease of implementation, and good convergence properties in practice, Gradient Descent is a very popular algorithm for solving unconstrained optimization problems.

## 12.2 Interior Point Method

Gradient descent can be modified to solve constrained optimization problems by introducing barrier functions. Consider the following problem:

$$\begin{array}{ll} \text{minimize:} & f(x) \\ \text{subject to:} & g_i(x) \geq 0, \quad \text{for } 1 \leq i \leq m. \end{array} \tag{12.1}$$

We can apply GD to this problem and find a critical point for $f(x)$. However, this might not answer the optimization question for two reasons:

1. The critical point might not be in the feasible region.
2. The optimal solution might not be obtained at a critical of $f(x)$ and could lie on the boundary $g(x) = 0$ of the feasible region.

Gradient descent algorithm only *sees* the objective function and does not *know* about the constraints. So, we modify the objective function to include the constraints using barrier functions.

**Definition 12.1.** A **barrier function** is a differentiable function $b$ from $(0, \infty)$ to $\mathbb{R}$ that has the property $\lim_{x \to 0^+} f(x) = \infty$.

We'll use the barrier function $-\ln x$. Using a small positive parameter $\mu$, we create a new objective function:

$$f_\mu(x) := f(x) - \mu \sum_{i=1}^{m} \ln(g_i(x)).$$

Consider the unconstrained optimization problem

$$\text{minimize: } f_\mu(x).$$

Because the domain of $\ln x$ is $(0, \infty)$, any critical point of $f_\mu(x)$ must lie in the feasible region of (12.1). Let $x^*_\mu$ be a solution of the above problem. For sufficiently good functions $f$ and $g_i$, $x^*_\mu$ *exists and is continuous in* $\mu$ *and* $\lim_{\mu \to 0^+} x^*_\mu = x^*$ *where* $x^*$ *is an optimal solution to* (12.1). This assumption is valid, for example, for linear programs. With this assumption, we now have a method for solving (12.1):

- Start with a small $\mu_0 > 0$ and optimize $f_{\mu_0}(x)$ using GD. Call the solution $x^*_{\mu_0}$.

- Repeat the following until the sequence $x^*_{\mu_k}$ stabilizes sufficiently:
  1. Set $\mu_{k+1} = \mu_k - \delta_k$ for some small $\delta_k$.
  2. Using $x^*_{\mu_k}$ as a starting point for GD, optimize $f_{\mu_{k+1}}(x)$. Call the solution $x^*_{\mu_{k+1}}$.
  3. Increase $k$ to $k+1$.

This method is a very simplified **interior point method**. The sequence of points $x_\mu$ is called the **central path**.

**Example 12.1.** We can compute the central path explicitly for some simple examples. Consider the optimization problem:

$$\begin{aligned} &\text{minimize: } (x+1)^2 \\ &\text{subject to: } x \geq 0. \end{aligned} \tag{12.2}$$

We can calculate the central path as follows:

$$f_\mu(x) := (x+1)^2 - \mu \ln(x).$$

The critical points for this function can be obtained by setting the derivative to 0.

$$\begin{aligned} & f'_\mu(x) = 0 \\ \implies & 2(x+1) - \frac{\mu}{x} = 0 \\ \implies & x^2 + x - \mu/2 = 0 \\ \implies & x = \frac{-1 \pm \sqrt{1+2\mu}}{2}. \end{aligned}$$

Only one of the two satisfy $x \geq 0$, so we get

$$x^*_\mu = \frac{-1+\sqrt{1+2\mu}}{2}.$$

This is the central path. Taking the limit as $\mu \to 0$ we get,

$$\begin{aligned} \lim_{\mu \to 0^+} x^*_\mu &= \lim_{\mu \to 0^+} \frac{-1+\sqrt{1+2\mu}}{2} \\ &= 0, \end{aligned}$$

which is indeed the optimal solution for the optimization problem (12.2).

# Chapter 13

# KKT conditions

We can use the central path to derive some conditions for optimality. Consider the optimization problem (12.1) again. As before, we'll assume that *there is a central path $x^*_\mu$ converging to an optimal solution $x^*$*. Because $x^*_\mu$ is a critical point of $f_\mu(x)$, we get

$$\begin{aligned} \nabla_x f_\mu(x^*_\mu) &= 0 \\ \implies \nabla f(x^*_\mu) - \sum_{i=1}^{m} \frac{\mu}{g_i(x^*_\mu)} \nabla g_i(x^*_\mu) &= 0 \end{aligned}$$

Applying $\lim_{\mu \to 0^+}$ to both sides, we get

$$\lim_{\mu \to 0^+} \nabla f(x^*_\mu) - \sum_{i=1}^{m} \lim_{\mu \to 0^+} \frac{\mu}{g_i(x^*_\mu)} \nabla g_i(x^*_\mu) = 0$$

which simplifies to

$$\nabla f(x^*) = \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*)$$

where $\lambda_i = \lim_{\mu \to 0^+} \frac{\mu}{g_i(x^*_\mu)}$ for $1 \leq i \leq m$. (There are several assumptions here about the existence and convergence of limits that we're brushing under the rug.) We can further analyze the constants $\lambda_i$.

$$\begin{aligned} & \frac{\mu}{g_i(x^*_\mu)} \cdot g_i(x^*_\mu) = \mu \\ \implies & \lim_{\mu \to 0^+} \frac{\mu}{g_i(x^*_\mu)} \cdot \lim_{\mu \to 0^+} g_i(x^*_\mu) = \lim_{\mu \to 0^+} \mu \\ \implies & \lambda_i g_i(x^*) = 0. \end{aligned}$$

Finally, because $g_i(x^*_\mu) \geq 0$ and $\mu > 0$, we must have

$$\lambda_i \geq 0.$$

To summarize, if $x^*$ is an optimal solution to (12.1) and there exists a central path $x^*_\mu$ converging to $x^*$, then the following conditions are satisfied:

$$\begin{aligned} \nabla f(x^*) &= \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) \\ \lambda_i g_i(x^*) &= 0 \\ \lambda_i &\geq 0. \end{aligned}$$

These are called the KKT conditions. We can combine the above derivation with Lagrange multipliers to get the following generalization:

**Theorem 13.1** (KKT conditions)**.** *Consider the optimization problem:*

$$\begin{array}{ll} \textit{minimize:} & f(x) \\ \textit{subject to:} & g_i(x) \geq 0, \quad \textit{for } 1 \leq i \leq m \\ & h_i(x) = 0, \quad \textit{for } 1 \leq i \leq k. \end{array}$$

*Assume that this problem has an optimal solution $x^*$ and there is a central path $x^*_\mu$ converging to $x^*$. Then under certain regularity conditions, the following conditions are satisfied:*

$$\begin{aligned} \nabla f(x^*) &= \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{k} \lambda'_i \nabla h_i(x^*) \\ \lambda_i g_i(x^*) &= 0, \ \textit{for } 1 \leq i \leq m \\ \lambda_i &\geq 0, \ \textit{for } 1 \leq i \leq m. \end{aligned}$$

The conditions for necessity, sufficiency and the regularity conditions are quite non-trivial and are beyond the scope of this course.

**Example 13.1.** Consider Example 12.1 again. The KKT conditions for this example are

$$\begin{aligned} 2(x+1) &= \lambda \\ \lambda x &= 0 \\ \lambda &\geq 0. \end{aligned}$$

These need to be satisfied in addition to the original constraint $x \geq 0$. The only solution to this system is $x = 0$, which is indeed the optimal solution.

**Example 13.2.** Consider the linear program

$$\begin{aligned} \text{minimize: } & c^T x \\ \text{subject to: } & Ax \geq b \end{aligned}$$

The KKT conditions for this problem are

$$\begin{aligned} c &= A^T\lambda \\ x_i\lambda_i &= 0, \text{ for } 1 \leq i \leq n, \\ \lambda &\geq 0. \end{aligned}$$

This is precisely the dual of the linear programming problem and *complementary slackness* conditions. Thus the KKT conditions recover duality in the linear programming sense.

**Exercise 13.1.** Find the KKT conditions for the standard linear program

$$\begin{aligned} \text{maximize: } & c^T x \\ \text{subject to: } & Ax \leq b \\ & x \geq 0. \end{aligned}$$

Explain how these conditions recover the dual constraints and complementary slackness.

# Part IV

# Applications

# Chapter 14

# L1-Regression

Linear regression is a method for modelling the relationship between outputs and inputs using linear functions. Suppose that the data set consists of the points $(x_i, y_i)$ with $i = 1, 2, ..., N$ where each $x_i$ is itself a vector in $\mathbb{R}^m$. We want to find a function $f : \mathbb{R}^m \to \mathbb{R}$ such that $f(x_i) \approx y_i$. In linear regression, we suppose that the function $f$ is of the form

$$f(x) = \beta^T x + \beta_0,$$

where $\beta$ is a vector in $\mathbb{R}^m$ and $\beta_0$ is a constant. Our goal is to estimate $\beta$ and $\beta_0$. There is of course, no reason that there is a linear relation between the inputs $x_i$ and the outputs $y_i$. So our goal is to find the *best* such linear approximation. We can define the error in estimation to be the vector $\varepsilon$ whose components are

$$\varepsilon_i = |y_i - f(x_i)|.$$

We wish to "minimize" the vector $\varepsilon$. There are multiple measures one can define on the space of these errors and minimizing each provides us a best estimate in the sense of that particular measure.

The most commonly used measure is the Euclidean distance or the $L^2$-norm. We define

$$\|\varepsilon_i\|_2 = \sum_{i=1}^{N} (y_i - f(x_i))^2.$$

In $L^2$-regression we try to find the function $f$ that minimizes this quantity. There are several reasons why this measure is most commonly used, one of which being that there is a closed form for the solution using the normal equation. However, the solution obtained from $L^2$-regression is very susceptible to outliers. By comparison, the $L^1$-measure, defined below, is more robust to outliers.

$$\|\varepsilon_i\|_1 = \sum_{i=1}^{N} |y_i - f(x_i)|.$$

The biggest downside of $L^1$ is that it does not have a closed form solution. However, it is possible to reduce the problem of finding the best $L^1$-estimate to linear programming.

Our goal is to solve the following unconstrained minimization problem

$$\text{minimize:} \sum_{i=1}^{N} |y_i - (\beta^T x_i + \beta_0)|,$$

where our variables are $\beta$ and $\beta_0$. As stated, this problem is not a linear program. However, it is equivalent to the following linear program:

$$\begin{array}{lll} \text{minimize:} & \sum_{i=1}^{N} \varepsilon_i & \\ \text{subject to:} & y_i - (\beta^T x_i + \beta_0) & < \quad \varepsilon_i \\ & y_i - (\beta^T x_i + \beta_0) & > \quad -\varepsilon_i. \end{array}$$

Here the decision variables are $\beta$ (which is a vector in $\mathbb{R}^m$), $\beta_0$, and $\varepsilon_i$ for $1 \leq i \leq N$.

# Chapter 15

# Network Flow

One very important application of linear programming is to finding max-flows in graphs. The setup is as follows: we have a simple directed graph $G = (V, E)$ where $V$ is the set of vertices and $E \subseteq V \times V$ is the set of directed edges. Each edge has a non-negative **capacity** given by a function $c : E \to \mathbb{R}_{\geq 0}$. For simplicity, we can assume that capacity is a function $c : V \times V \to \mathbb{R}_{\geq 0}$ such that $c(v, v') = 0$ if there is no edge going from $v$ to $v'$. There are two special vertices called **source** and **sink** denotes $s$ and $t$, respectively. The source $s$ has the property that every edge connected to the source points away from it i.e. there no incoming edges and the sink $t$ has the property that every edge connected to the sink points toward it i.e. there no outgoing edges. Such a graph is called a **network**.

A **flow** is a function $f : E \to \mathbb{R}$ with the property that $f \leq c$. As before, we can assume that $f$ is a function $V \times V \to \mathbb{R}$. A flow is said to be balanced if at each vertex the incoming flow equals the outgoing flow. The **max-flow** question is to find a balanced flow that maximizes the net outflow from the source. More precisely,

$$\begin{array}{lrcll} \text{maximize:} & \sum_{v \in V} f(s, v) & & & \\ \text{subject to:} & f(v, w) & \leq & c(v, w) & \text{for all } v, w \in V \\ & \sum_{w \in V} f(v, w) & = & \sum_{w \in V} f(w, v) & \text{for all } v \in V. \end{array}$$

This is very clearly a linear program that can be solved using the simplex method. Note however, that several faster algorithms exist for solving this problem.

## 15.1 Min-cut

The dual of the max-flow problem has an interesting interpretation. Suppose $y(v, w)$ are the dual decision variables corresponding to the first kind of con-

straints and $z(v)$ are the dual decision variables corresponding to the second kind of constraints.

$$\begin{array}{lrcll}
\text{minimize:} & \displaystyle\sum_{(v,w)\in V\times W} c(v,w)y(v,w) & & & \\
\text{subject to:} & y(v,w) & \geq & z(v)-z(w) & \text{for all } v,w\in V \\
& z(s) & = & 1 & \\
& z(t) & = & 0 & \\
& y(v,w) & \geq & 0 & \text{for all } v,w\in V.
\end{array}$$

Consider the graph $G$ again. A **directed path** from the source $s$ to a sink $t$ is a sequence of edges of the form $(s, v_1)$, $(v_1, v_2)$, ..., $(v_{k-1}, v_k)$, $(v_k, t)$ in $E$. A subset of edges $S \subseteq E$ is called an **s-t cut** if after deleting the edges in $S$ there is no directed path from $s$ to $t$. Every s-t cut provides a feasible solution to the dual problem as follows: We set $y(v, w) = 1$ if $(v, w)$ is in the cut, 0 otherwise. We set $z(v)$ equal 1 if there is a path from $s$ to $v$ after making the cut $S$, 0 otherwise. One can check that the constraints of the dual linear program are indeed satisfied by such a solution.

In fact, a much stronger statement is true. The optimal solution to the dual problem is of this form i.e. it comes from an s-t cut. The proof of this statement is beyond the scope of this class. The **size** of an s-t cut is the sum of capacities of the edges in the cut. And thus the dual problem is equivalent to finding a **min-cut** in the graph $G$.

# Chapter 16

# Integer Programming

An **integer program** is a linear program in which some or all of the variables are required to be integers. The most common constraint is usually that some variable is required to be either 0 or 1. Even though integer programs are slight generalizations of linear programs, they are much harder to solve. It can be shown that solving a general integer program is an NP-hard problem i.e. we do not expect to there be any efficient algorithm to solve them. The best we can hope for is to analyze some special kinds of integer programs or to find approximation algorithm for the general case.

One naive approach to solve an integer program is to simply forget the integrality condition and solve the underlying linear program. This technique is called **LP-relaxation**. For nice integer programs, we can expect this method to provide a good approximate solution. However, this method can also provide really incorrect answers. Consider the following integer program

$$\begin{array}{lllll} \text{maximize:} & x & & & \\ \text{subject to:} & y & \leq & \dfrac{x}{n-0.5} & \\ & x, y & \geq & 0 & \\ & x, y & & \text{are integers,} & \end{array}$$

where $n$ is a fixed integer. One can check that the feasible region for the LP-relaxation is a triangle with vertices $(0,0)$, $(1,0)$ and $(n-0.5,1)$ and the only integer points in this triangle are $(0,0)$ and $(1,0)$. So, the optimal solution to the LP-relaxation is $(n-0.5,1)$ but the optimal solution to the original integer program is $(1,0)$. Hence, as $n$ increases the LP solution diverges away from the IP solution.

## 16.1 Branch and Bound

One technique for fixing the issues with LP-relaxation is called brand-and-bound. In branch-and-bound we solve the LP-relaxation if we end with a non-integer solution, say with $x = k$ then we make two cases $x \leq \lfloor k \rfloor$ and $x \geq \lceil k \rceil$. The goal is to shrink the feasible region so that all the vertices have integer coordinates. We repeat the process for each case, essentially performing a DFS until we find integer solutions. This is best demonstrated with an example.

Consider the integer program:

$$\begin{array}{lrcrcll} \text{maximize:} & 4x & + & 5y & & & \\ \text{subject to:} & x & + & y & \leq & 10 & \\ & 3x & - & 4y & \leq & 6 & \\ & x & , & y & \geq & 0 & \\ & x, y & & & & \text{are integers.} & \end{array} \tag{16.1}$$

Solving the LP-relaxation gives us the solution $(x, y) = (4, 1.5)$. As the $y$-coordinates is not an integer, we make two cases $y \leq 1$ and $y \geq 2$ and analyze each separately by adding it as an extra constraint.

$$\begin{array}{lrcrcll} \text{maximize:} & 4x & + & 5y & & & \\ \text{subject to:} & x & + & y & \leq & 10 & \\ & 3x & - & 4y & \leq & 6 & \\ & x & , & y & \geq & 0 & \\ & x, y & & & & \text{are integers.} & \\ & & & y & \leq & 1 & \end{array}$$

In this case, the optimal solution to the LP-relaxation is $(x, y) = (3.342, 1)$. This time $x$ is not an integer, so we make further two cases $x \leq 3$ and $x \geq 4$. In the first case, we get

$$\begin{array}{lrcrcll} \text{maximize:} & 4x & + & 5y & & & \\ \text{subject to:} & x & + & y & \leq & 10 & \\ & 3x & - & 4y & \leq & 6 & \\ & x & , & y & \geq & 0 & \\ & x, y & & & & \text{are integers.} & \\ & & & y & \leq & 1 & \\ & x & & & \leq & 3, & \end{array}$$

which has the optimal solution $(x, y) = (3, 1)$. In the second case, we get

$$\begin{array}{lrcrcll} \text{maximize:} & 4x & + & 5y & & & \\ \text{subject to:} & x & + & y & \leq & 10 & \\ & 3x & - & 4y & \leq & 6 & \\ & x & , & y & \geq & 0 & \\ & x, y & & & & \text{are integers.} & \\ & & & y & \leq & 1 & \\ & x & & & \geq & 4, & \end{array}$$

the LP-relaxation and hence the integer program is infeasible. Going back we are left to analyze the case $y \geq 2$.

$$\begin{array}{lllllll}
\text{maximize:} & 4x & + & 5y & & \\
\text{subject to:} & x & + & y & \leq & 10 \\
& 3x & - & 4y & \leq & 6 \\
& x & , & y & \geq & 0 \\
& x, y & & & & \text{are integers.} \\
& & & y & \geq & 2.
\end{array}$$

This has an optimal solution $(x, y) = (2, 2)$.

Thus we have two possible candidates for optimal solutions to the integer program: $(3, 1)$ and $(2, 2)$. We check the objective value $4x + 3y$ at each of these to conclude that $(2, 2)$ is indeed the optimal solution with objective value 18.

# References