Chapter 1 The Real and Complex Number Systems

1.3 Definition $x \in A$, $x \notin A$, $A \subset B$

$A \subset B$, $B \subset A \iff A = B$

1.4 Definition all rational numbers will be denoted by Q.

Ordered sets

1.5 Definition Let S be a set. An order is a relation, denoted by $<$, with the following two properties:

(i) if $x \in S$, $y \in S$, then one and only one of the statements

$$x<y, \quad x=y, \quad x>y$$

is true

(ii) if $x, y, z \in S$, if $x<y$ and $y<z$, then $x<z$.

1.6 Definition An ordered set is a set S in which an order is defined

in Q: $r<s$ means $s-r$ is a positive rational number

1.7 Definition Suppose S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is bounded above.

1.9 Definition Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ s.t.

(i) α is an upper bound of E.

(ii) if $y<\alpha$, then y is not an upper bound of E.

Then α is the least upper bound of E (Supremum of E) ($\alpha = \sup E$)

($\alpha = \inf E$, similar)

1.10 An ordered set S is said to have least-upper-bound property if the following is true:

If $E \subset S$, E is not empty, and E is bounded above, then $\sup E$ exists in S.

1.11 Theorem Suppose S is an ordered set with the least-upper-bounded property, $B \subset S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B. Then

$$\alpha = \sup L \text{ exists in } S, \text{ and } \alpha = \inf B$$

In particular, $\inf B$ exists in S.

proof: ① $\alpha \in L$, ② $\alpha = \sup L$, ③ $\alpha = \inf B$

Fields

1.12 Definition A field is a set F with two operations, called addition and multiplication, which satisfy the following so-called "field axioms" (A), (M), and (D):

(A) Axioms for addition

(A1) $x, y \in F \Rightarrow x+y \in F$

(A2) $x+y = y+x$

(A3) $(x+y)+z = x+(y+z)$

(A4) $0 \in F$ s.t. $0+x = x$ $\forall x \in F$.

(A5) $\forall x \in F$ $\exists -x \in F$ s.t. $x+(-x) = 0$

(M) Axioms for Multiplication:

(M1) $x, y \in F \Rightarrow x \cdot y \in F$

(M2) $xy = yx$

(M3) $(xy)z = x(yz)$

(M4) $\exists 1 \in F$, $1 \neq 0$ s.t. $1 \cdot x = x$

(M5) $x \in F$, $x \neq 0 \Rightarrow \exists \frac{1}{x} \in F$ s.t. $x \cdot (1/x) = 1$

(D) The distributive law : $x(y+z) = xy + xz$

1.14 Proposition The axioms for addition imply the following statements.

(a) $x+y = x+z \Rightarrow y = z$

(b) $x+y = x \Rightarrow y = 0$

(c) $x+y = 0 \Rightarrow y = -x$ $\qquad y = (-x+x)+y = -x+0$

(d) $-(-x) = x$ $\qquad -x+(-(-x)) = 0 \Rightarrow x = x+0 = x+(-x+(-(-x))) = (x+(-x))+(-(-x))$

1.15 Proposition The axioms for multiplication imply the following Statements.

(a) $x \neq 0$, $xy = xz \Rightarrow y = z$

(b) $x \neq 0$, $xy = x \Rightarrow y = 1$

(c) $x \neq 0$, $xy = 1 \Rightarrow y = \frac{1}{x}$

(d) $x \neq 0 \Rightarrow 1/(1/x) = x$

$$\frac{1}{x} \cdot \frac{1}{\frac{1}{x}} = 1$$

$$x \cdot \frac{1}{x} \cdot \frac{1}{\frac{1}{x}} = x$$

$$x = 1/(1/x)$$

1.16 Proposition The field axioms imply the following statements, $\forall x, y, z \in F$

(a) $0x = 0$ $\quad 0x + 0x = (0+0)x = 0x \quad 0 = -0x + 0x = -0x + (0x + 0x) = 0x$

(b) $x \neq 0, y \neq 0 \Rightarrow xy \neq 0$ $\quad$ Suppose $xy = 0 \Rightarrow 1 = \frac{1}{x}\frac{1}{y} \cdot xy = \frac{1}{x} \cdot \frac{1}{y} \cdot 0 = 0$

(c) $(-x)y = -(xy) = x(-y)$ $\quad$ (c) $\left| \begin{array}{l} (-x)y + xy = (-x+x)y = 0y = 0 \\ xy + (-xy) = 0 \end{array} \right.$

(d) $(-x)(-y) = xy$

(d) $\left| \begin{array}{l} 0 = x(-y) + (-x)(-y) \\ x(-y) + xy = 0 \end{array} \right. \Rightarrow (-x)(-y) = xy$

1.17 Definition An ordered field is a field F which is also an ordered set, such that

(i) $x, y, z \in F$, $y < z \Rightarrow x + y < x + z$

(ii) $x, y \in F$, $x, y > 0 \Rightarrow xy > 0$

If $x > 0 \Rightarrow x$ positive, $x < 0 \Rightarrow x$ negative

1.18 Propositions The following statements are true in every ordered field.

(a) $x > 0 \Rightarrow -x < 0$ $\quad 0 = x + (-x) > 0 + (-x) \Rightarrow -x < 0$

(b) $x > 0, y < z \Rightarrow xy < xz$ $\quad xz = x(z-y) + xy > xy$ $\quad$ 1.6 & 1.17 (ii)

(c) $x < 0, y < z \Rightarrow xy > xz$ $\quad xy = (-x)(z-y) + xz > xz$

(d) $x \neq 0 \Rightarrow x^2 > 0$. In particular, $1 > 0$ $\quad$ (d) $\left(\begin{array}{l} x > 0 \Rightarrow x \cdot x > 0 \\ x < 0 \Rightarrow -x > 0 \Rightarrow (-x)(-x) = x^2 > 0 \end{array} \right.$

(e) $0 < x < y \Rightarrow 0 < 1/y < 1/x$

(e) ① $x > 0 \Rightarrow \frac{1}{x} > 0$ $\quad$ if $\frac{1}{x} \leq 0$ $\quad x \cdot \frac{1}{x} \leq x \cdot 0 \Rightarrow 1 \leq 0$

② $0 < x < y \Rightarrow \frac{1}{y} < \frac{1}{x}$ $\quad$ since $x < y$, $(\frac{1}{x})(\frac{1}{y}) > 0$

$\Rightarrow x \cdot (\frac{1}{x})(\frac{1}{y}) < y \cdot (\frac{1}{x})(\frac{1}{y}) \Rightarrow \frac{1}{y} < \frac{1}{x}$

The Real Field

1.19 Theorem There exists an ordered field $\mathbb{R}$ which has the least-upper-bound property. (What about greatest-lower-bound Property?) Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

1.20 Theorem $\to$ archimedean property of $\mathbb{R}$

(a) $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x>0 \Rightarrow \exists$ integer $n>0$ s.t. $nx>y$

(b) $x, y \in \mathbb{R}$, $x<y \Rightarrow \exists p \in \mathbb{Q}$ s.t. $x<p<y$ $\to$ $\mathbb{Q}$ is dense in $\mathbb{R}$: $\exists p$ between any $x, y \in \mathbb{R}$, $x \neq y$.

Proof: (a) $\mathbb{R}$ has the least-upper-bound property.

Let $A=\{nx\}$, $x>0$, $\forall n \in \mathbb{N}^*$

Suppose $\exists y \in \mathbb{R}$ s.t. $nx \le y \Rightarrow \alpha = \sup A$ exists in $\mathbb{R}$

Since $x>0 \Rightarrow \alpha - x$ is not the supremum of A

$\Rightarrow \exists m \in \mathbb{N}^*$ s.t. $\alpha - x < mx$

$\Rightarrow \alpha < (m+1)x$, where $(m+1)x \in A \Rightarrow$ contradict to $\alpha = \sup A$.

(b) $x<y \Rightarrow y-x>0$, by (a) $\Rightarrow \exists n \in \mathbb{N}$ s.t. $n(y-x)>1$ $(*)$

by (a), $\exists m_1, m_2 \in \mathbb{N}^*$ s.t. $-m_2 < nx < m_1$

nx is fixed. by (a) $\exists m_1$ s.t. $m_1 \times 1 > nx$

$\Rightarrow \exists m \in \mathbb{N}^*$ $(-m_2 \le m \le m_1)$ s.t.

$m-1 \le nx < m$ $\quad (**)$

by $(*)$ and $(**)$ $\Rightarrow$ $nx < m \le 1+nx < ny$

Since $n>0 \Rightarrow x < \frac{m}{n} < y$ #

1.21 Theorem $\forall x>0$, $x \in \mathbb{R}$, $\forall n \in \mathbb{N}^*$, there is one and only one positive real y s.t. $y^n = x$. The number y is written $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$.

Proof: ① Given $x>0$, construct $E=\{t : t^n < x\}$

② If $t=\frac{x}{1+x} \Rightarrow 0 \le t < 1 \Rightarrow t^n < t < x \Rightarrow t \in E \Rightarrow E$ is not empty

③ E is bounded above ($x+1$ is a upper bound)

Since if $t > x+1$ $\quad t^n \ge t > x+1 > x \Rightarrow t \notin E$

By Thm 1.19 $\Rightarrow$ $y = \sup E$ exist

△ Idea: to prove $y^n = x$

We WTS $y^n > x$ & $y^n < x$ lead to contradiction

When $0<a<b$, $\underline{b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1}) < (b-a) \cdot n \cdot b^{n-1}}$

(a) Assume $y^n < x$, choose h s.t. $0<h<\min\{1, \frac{x-y^n}{n(y+1)^{n-1}}\}$

Then $(y+h)^n - y^n < h \cdot n (y+h)^{n-1} < h \cdot n \cdot (y+1)^{n-1} < x - y^n$

$\Rightarrow (y+h)^n < x \Rightarrow$ contradict to y is the upper bound of E.

(b) Assume $y^n > x$, put $k = \frac{y^n - x}{n y^{n-1}}$

Then $0 < k < y$.

If $t \geq y - k$

$y^n - t^n \leq y^n - (y-k)^n < k \cdot n \cdot y^{n-1} = y^n - x \Rightarrow t^n > x$

$\Rightarrow t \notin E \Rightarrow y - k < y$ is a upper bound of E $\Rightarrow$ Contradiction

$\Rightarrow y^n = x$

Corollary: If a and b are positive real numbers and n is a positive integer, then

$$(ab)^{1/n} = a^{\frac{1}{n}} b^{\frac{1}{n}}$$

proof: Put $\alpha = a^{\frac{1}{n}}$, $\beta = b^{\frac{1}{n}}$

Then $ab = \alpha^n \cdot \beta^n = (\alpha\beta)^n$

by Thm 1.21 $\Rightarrow$ $(ab)^{\frac{1}{n}} = \alpha\beta = a^{\frac{1}{n}} \cdot b^{\frac{1}{n}}$

1.22 Decimals We conclude this section by pointing out the relation between real numbers and decimals.

Let $x \in \mathbb{R} > 0$ be given. Having chosen $n_0, n_1, \dots, n_{k-1}$. Let n_k be the largest integer s.t.

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \leq x$$

Let E be the set of these numbers

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \quad (k = 0, 1, 2, \dots)$$

Then $x = \sup E$. Then decimal expansion of x is

$$n_0 . n_1 n_2 n_3 \cdots .$$

The Extended Real Number System

1.23 Definition The extended real number system consists of the real field $\mathbb{R}$ and two symbols, $+\infty$ and $-\infty$. We preserve the original order in $\mathbb{R}$, and define $-\infty < x < +\infty$ $\forall x \in \mathbb{R}$.

If E is nonempty set which is not bounded above in $\mathbb{R}$, then $\sup E = +\infty$ in the extended real number system.

The Complex Field

1.24 Definition A complex number is an ordered pair (a,b) of real numbers.

Let $x=(a,b)$, $y=(c,d)$

Then $x+y=(a+c,\ b+d)$, $xy=(ac-bd,\ ad+bc)$

1.25 Theorm These definitions of addition and multiplication turn the set of all complex numbers into a field, with $(0,0)$ and $(1,0)$ in the role of 0 and 1.

If $x=(a,b)\neq 0$ define $\frac{1}{x}=\left(\frac{a}{a^2+b^2},\ \frac{-b}{a^2+b^2}\right)$

1.26 Theorm For any real numbers a & b we have

$(a,0)+(b,0)=(a+b,0)$ $\quad (a,0)(b,0)=(ab,0)$

1.27 Definition $i=(0,1)$

1.28 Theorm $i^2=-1$

$(0,1)(0,1)=(-1,0)=-1$

1.29 If a,b are real, then $(a,b)=a+bi$

proof : $a+bi=(a,0)+(0,b)=(a,b)$

1.30 Definition If a and b are real, $z=a+bi$

$Re(z)=a$ $\quad Im(z)=b$

real part $\quad$ imaginary part

1.31 Theorem If z and w are complex, then

(a) $\overline{z+w}=\bar{z}+\bar{w}$

(b) $\overline{zw}=\bar{z}\cdot\bar{w}$

(c) $z+\bar{z}=2Re(z)$, $z-\bar{z}=2\,Im(z)$

(d) $z\cdot\bar{z}$ is real and positive (except $z=0$)

1.32 Definition If z is a complex number, its absolute value is the non-negative square root of $z\bar{z}$, that is $|z| = (z \cdot \bar{z})^{\frac{1}{2}}$

1.33 Theorem Let z and w be complex numbers. Then

(a) $|z| > 0$ unless $z = 0$, $|0| = 0$

(b) $|\bar{z}| = |z|$

(c) $|zw| = |z||w|$

(d) $|Re(z)| \leq |z|$

(e) $|z+w| \leq |z| + |w|$

1.34 Notation If $x_1, \dots, x_n$ are complex numbers, we write

$$x_1 + x_2 + \dots + x_n = \sum_{j=1}^{n} x_j$$

1.35 Theorem If $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are complex numbers, then

$$\left|\sum_{j=1}^{n} a_j \bar{b}_j\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2 \quad (*)$$ Schwarz inequality

Proof: Put $A = \sum_{j=1}^{n} |a_j|^2$, $B = \sum_{j=1}^{n} |b_j|^2$, $C = \sum_{j=1}^{n} a_j \bar{b}_j$

① If $B = 0 \Rightarrow b_1 = b_2 = \dots = b_n = 0 \Rightarrow$ it is clear that the inequality holds

② If $B \neq 0$, we have

$$\begin{aligned}
\sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j)(B\bar{a}_j - \bar{C}\bar{b}_j) \\
&= \sum (B^2|a_j|^2 - B\bar{C}a_j\bar{b}_j - BCb_j\bar{a}_j + |C|^2 B) \\
&= B(AB + |C|^2) - B\bar{C}\sum a_j\bar{b}_j - BC\sum \bar{a}_j b_j \\
&= B(AB + |C|^2) - B|C|^2 - B|C|^2 = B(AB - |C|^2) \geq 0
\end{aligned}$$

since $B > 0 \Rightarrow |C|^2 \leq AB$.

Euclidean Spaces

1.36 Definition For each positive integer k, let $\mathbb{R}^k$ be the set of all ordered k-tuples

$$\vec{x} = (x_1, x_2, \cdots, x_n), \quad \vec{y} = (y_1, y_2, \cdots, y_n)$$

$$\vec{x}+\vec{y} = (x_1+y_1, \cdots, x_n+y_n)$$

$$\alpha\vec{x} = (\alpha x_1, \cdots, \alpha x_n)$$

1.37 Theorem Suppose $x, y, z \in \mathbb{R}^k$, and α is real. Then

(a) $|\vec{x}| \geq 0$

(b) $|\vec{x}| = 0$ if any only if $\vec{x} = 0$

(c) $|\alpha\vec{x}| = |\alpha||\vec{x}|$

(d) $|\vec{x}\cdot\vec{y}| \leq |\vec{x}||\vec{y}|$

(e) $|\vec{x}+\vec{y}| \leq |\vec{x}| + |\vec{y}|$ $\quad |x+y|^2 = (x+y)(x+y) = x^2+y^2+2xy$

$\leq |x|^2+|y|^2+2|x||y| = (|x|+|y|)^2$

(f) $|\vec{x}-\vec{z}| \leq |\vec{x}-\vec{y}| + |\vec{y}-\vec{z}|$

Chapter 2 Basic Topology

Finite, Countable and Uncountable Sets

2.1 Definition $f: A \to B$
- f: function / mapping
- A: domain of f
- B:
  - value of f
  - range of f

2.2 Definition, Let $f: A \to B$

If $E \subset B$, we say $f^{-1}(E)$ the inverse image of E under f.
$\forall y \in B$, $f^{-1}(y)$ consists of at most one element. Then we say f is a 1-1 mapping of A into B

2.3 Definition If there exists a 1-1 mapping of A onto B, we can say that A and B can be put in 1-1 correspondence, ($A \sim B$)

bijection

properties: $A \sim A$

$A \sim B \Rightarrow B \sim A$

$A \sim B,\ B \sim C \Rightarrow A \sim C$

2.4 Definition let $J_n = \{1, 2, \cdots, n\}$, and J be the set consisting all positive integers.

(a) $A \sim J_n \Rightarrow A$ is finite

(b) A is not finite $\Rightarrow A$ is infinite

(c) $A \sim J \Rightarrow A$ is countable

(d) A is neither finite nor countable $\Rightarrow A$ is uncountable

(e) A is finite or countable $\Rightarrow A$ is at most countable

2.7 Definition A sequence is a function from N to a set

$f: N \to B$

Example: $a_n = (-1)^n$ Set $= \{1, -1\}$

2.8 Theorem Every infinite subset of a countable set A is countable

Proof: Suppose $E \subset A$ and E is infinite

Construct $A = \{x_n\}$

Let n_1 be the smallest positive integer s.t. $x_{n_1} \in E$.

Having chosen $n_1, \cdots, n_{k-1}$ $(k=2,3,4\cdots)$, let n_k be the smallest integer grater than n_{k-1} st. $x_{n_k} \in E$

Construct $f(k) = x_{n_k}$ $(k=1,2,3,\cdots)$, we obtain a 1-1 correspondence between E and J

2.9 Definition Let A and Ω be sets, and $\forall \alpha \in A$, there is $E_\alpha \subseteq \Omega$

Union: $S = \bigcup_{\alpha \in A} E_\alpha$ $\quad (\bigcup_{m=1}^{n} E_m, \ \bigcup_{m=1}^{\infty} E_m)$

Intersection: $P = \bigcap_{\alpha \in A} E_\alpha$ $\quad (\bigcap_{m=1}^{\infty} E_m, \ \bigcap_{m=1}^{n} E_m)$

$A \cap B \neq \emptyset \Rightarrow A$ and B are intersect;

$A \cap B = \emptyset \Rightarrow A$ and B are disjoint.

2.12 Theorem Let $\{E_n\}$, $n = 1, 2, 3, \cdots$, be a sequence of countable sets, and put $S = \bigcup_{n=1}^{\infty} E_n$. Then S is countable

Proof: Let every set E_n be arranged in a sequence $\{x_{nk}\}$, $k=1,2,3,\cdots$, and consider the infinite array

$$
\begin{array}{l}
E_1 : x_{11}, \ x_{12}, \ x_{13}, \cdots \\
E_2 : x_{21}, \ x_{22}, \ x_{23}, \cdots \\
E_3 : x_{31}, \ x_{32}, \ x_{33} \cdots \\
E_4 : x_{41}, \ x_{42}, \ x_{43} \cdots \\
\cdots
\end{array}
$$

$f: N^2 \to N$ bijection
(m,n)

$$f(m,n) = \frac{(m+n-1)(m+n-2)}{2} + n$$

$T: x_{11};\ x_{21}, x_{12};\ x_{31}, x_{22}, x_{13};\ \cdots$

Hence there is a subset T s.t. $T \sim S$

Corollary Suppose A is at most countable, and, for every $\alpha \in A$, B_α is at most countable. Put $T = \bigcup_{\alpha \in A} B_\alpha$.

Then T is at most countable

2.13 Theorem Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, a_2, \cdots, a_n)$, where a_k $(k=1,2,3,\cdots,n) \in A$, and $a_1 \cdots a_n$ need not to be distinct. Then B_n is countable

proof: ① Base: $B_1 = \{a_i\} \subset A \Rightarrow B_1$ is countable

② Hypothesis: B_{n-1} is countable

③ Step: $B_n = \{(a_1, \cdots, a_n), a_i \in A\} \sim \{(b, a) : b \in B_{n-1}, a \in A\}$

$$= \bigcup_{b \in B_{n-1}} \{(b, a) : a \in A\}$$

By Thm 2.12 $\Rightarrow B_n$ is a countable set.

Corollary The set of all rational numbers is countable

Proof: $x \in \mathbb{R} \Rightarrow x = \frac{p}{q}$ $\qquad f: A \to \mathbb{Q}$, $(p, q) \mapsto \frac{p}{q}$

2.14 Theorem Let A be the set of all sequences whose elements are digits 0 and 1. This set A is uncountable.

proof: Suppose A is countable and $A = \{x^1, x^2, \cdots\}$

$x^i = x_1^i, x_2^i, \cdots, x_j^i \in \{0, 1\}$

$y^i = \begin{cases} 0, & \text{if } x_i^i = 1 \\ 1, & \text{if } y_i^i = 0 \end{cases}$

So $y \in A \Rightarrow y = \{y^1, y^2, \cdots\} = x^m$ for some $m \in \mathbb{N}$

$y^m = x_m^m$

$\Rightarrow$ Contradiction

Metric Spaces

2.15 Theorem A set X, elements are called points, is said to be a metric space, if with any two points p, q of X there is associated a real number $d(p, q)$ (distance) s.t.

(a) $d(p,q) > 0$ if $p \neq q$; $d(p,p) = 0$

(b) $d(p,q) = d(q,p)$

(c) $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$

2.17 Definition (a,b) : segment

$[a,b]$: interval

$a_i \le x_i \le b_i$ $(i=1,2,\cdots,k)$: k-cell

If $\vec{x} \in \mathbb{R}^k$ and $r>0$ $\forall y \in \mathbb{R}^k$ $|\vec{y}-\vec{x}|<r$ open ball

$|\vec{y}-\vec{x}| \le r$ closed ball

$E \subseteq \mathbb{R}^k$, $\lambda\vec{x}+(1-\lambda)\vec{y} \in E$, $\forall \vec{x},\vec{y} \in E$,and $\lambda \in (0,1)$

$\Rightarrow$ E is convex

2.18 Definition a set could be both closed or both open

neighborhood, interior, closed, open, limit point, isolated point, completeness.

(h) perfect : E is closed, $\forall p \in E$, p is a limit point

(i) bounded: (to a fixed point)

(j) dense: $\forall p \in X$, $p \in E$ or $p \in E'$ ($\mathbb{Q}$ is dense in $\mathbb{R}$)

2.19 Theorem Every neighborhood is an open set.

Proof: Let $N_r(p)$ be given , let q be any point in $N_r(p)$,

$\exists$ $h>0$ s.t.

$d(p,q)=r-h$

$\forall s \in N_r(p)$ s.t. $d(q,s)<h$

$d(p,s) \le d(p,q)+d(q,s) < r-h+h=r$

$\Rightarrow \forall q \in N_r(p)$ $\exists N_h(q) \subset N_r(p) \Rightarrow$ q is an interior point

$\Rightarrow N_r(p)$ is open.

2.20 Theorem If p is a limit point of E, then every neighborhood of p contains infinitely many points of E

Suppose $N_\delta(p) \cap E = \{q_1, q_2, q_3, \cdots, q_n\}$

$r = \min_{1\le m\le n} d(p,q_m)$

$N_r(p) \cap E = \emptyset$ since $\forall q \in E$ $d(p,q) \ge r$

$\Rightarrow$ Contradict to the definition of limit point

corollary A finite set has no limit point

2.22 Theorem Let $\{E_n\}$ be a (finite or infinite) collection of sets E_n.
Then $A = (\bigcup_\alpha E_\alpha)^c = \bigcap_\alpha (E_\alpha^c) = B$

proof: ① Suppose $x \in A \Rightarrow x \notin \bigcup_\alpha E_\alpha \Rightarrow x \notin E_\alpha \ \forall \alpha$
$\Rightarrow x \in \bigcap (E_\alpha^c) \Rightarrow A \subseteq B$

② $x \in B \Rightarrow x \in E_\alpha^c, \forall \alpha \Rightarrow x \notin E_\alpha, \forall \alpha \Rightarrow x \notin \bigcup_\alpha E_\alpha$
$\Rightarrow x \in (\bigcup_\alpha E_\alpha)^c \Rightarrow B \subseteq A$

$\Rightarrow A = B \Leftrightarrow (\bigcup_\alpha E_\alpha)^c = \bigcap_\alpha (E_\alpha^c)$

2.23 Theorem A set E is open if and only if its complement is closed.

proof: $(\Leftarrow)$ Suppose E^c is closed, and $x \in E$
$\Rightarrow x \notin E^c$
$\Rightarrow x$ is not a limit point of E^c (since E^c is closed)
$\Rightarrow \exists N_r(x) \cap E^c = \emptyset \Rightarrow N_r(x) \subset E$
$\Rightarrow x$ is an interior point of E
$\Rightarrow E$ is open

$(\Rightarrow)$ Suppose E is open, and x is a limit point of E^c
$\Rightarrow \forall N_r(x) \cap E^c \neq \emptyset \Rightarrow \forall N_r(x) \not\subset E$
$\Rightarrow x$ is not an interior point of E
$\Rightarrow x \notin E$
$\Rightarrow x \in E^c$
$\Rightarrow E^c$ is closed

2.24 Theorem
(a) For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open
(b) For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.
(c) For any finite collection $G_1, \dots, G_n$ of open sets, $\bigcap_{i=1}^n G_i$ is open.
(d) For any finite collection $F_1, \dots, F_n$ of closed sets, $\bigcup_{i=1}^n F_i$ is closed.

(b) $(\bigcap_\alpha F_\alpha)^c = \bigcup_\alpha (F_\alpha^c)$ by 2.23 & 2.24(a)

(c)

2.25 Example:

Let $G_n = (-\frac{1}{n}, \frac{1}{n})$ $(n = 1, 2, \dots)$

$\bigcap_{n=1}^{\infty} G_n = \{0\}$ $\Rightarrow$ intersection of an infinite collection of open sets need not to be open

Union of infinite collection of closed sets need not to be closed.

Let $A_n = (-\infty, 2-\frac{1}{n}]$

$\bigcup_{n=1}^{\infty} A_n = (-\infty, 2)$

2.26 Definition $E \subset X \Rightarrow E'$: set of limit points of E

$\Rightarrow \bar{E}$: closure $= E \cup E'$

2.27 Theorem: If X is a metric space and $E \subset X$, then

(a) $\bar{E}$ is closed

(b) $E = \bar{E}$ if and only if E is closed

(c) $\bar{E} \subset F$ for every closed set $F \subset X$ s.t. $E \subset F$

Proof: (a) WTS $(\bar{E})^c$ is open.

Let $p \in (\bar{E})^c \Rightarrow \boxed{p \notin E}$ and $\boxed{p \notin E'} \Rightarrow N_r(p) \cap E = \emptyset \Rightarrow N_r(p) \cap \bar{E} = \emptyset$

$\exists r > 0$ s.t. $N_r(p) \cap E \subseteq \{p\}$

$\Rightarrow N_r(p) \subseteq (\bar{E})^c \Rightarrow (\bar{E})^c$ is open $\Rightarrow \bar{E}$ is closed

If $N_r(p) \cap \bar{E} \neq \emptyset$

$\Rightarrow q$ a limit point $\in N_r(p)$

$\exists \delta$ s.t. $N_\delta q \subset N_r(p)$

Since $q \in E'$ $\Rightarrow$ $N_\delta(q) \cap E \neq \emptyset$

$\Rightarrow N_r(p) \cap E \neq \emptyset$

$\Rightarrow$ contradiction

(c) $E \subset F \Rightarrow E' \subset F'$

Since F is closed $F = \bar{F} = F \cup F'$

$\Rightarrow E' \subset F \Rightarrow \bar{E} = E' \cup E \subset F$ #

2.28 Theorem Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \bar{E}$. Hence $y \in E$ if E is closed.

proof: ① If $y \in E \Rightarrow y \in \bar{E}$ we are done

② If $y \notin E$. For every $h > 0$, there exists $x \in E$ s.t. $y-h < x < y$,

[otherwise $y-h$ would be an upper bound]

$\Rightarrow y$ is a limit point of $E \Rightarrow y \in \bar{E}$.

2.29 Remarks let $E \subset Y \subset E$, and X, Y are metric spaces

E is an open subset of X: $\forall p \in E$, $\exists r > 0$ s.t. $d(p,q) < r, q \in X \Rightarrow q \in E$

E is open relative to Y: $\forall p \in E, \exists r > 0$ s.t. $d(p,q) < r, q \in Y \Rightarrow q \in E$

2.30 Theorem Suppose $Y \subset X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.

Proof: $(\Rightarrow)$ Suppose E is open relative to Y

Then $\forall p \in E \ \exists r_p$ s.t. $d(p,q) < r, q \in Y \Rightarrow q \in E$

let $V_p = \{q : d(p,q) < r_p\}$ and $G = \bigcup_{p \in E} V_p$

$\Rightarrow E = Y \cap G$

$(\Leftarrow)$ If G is open in X, and $E = G \cap Y \Rightarrow E \subset G$

$\Rightarrow \forall p \in E \Rightarrow p \in G \Rightarrow \exists N_r(p) \subset G$

since $E = G \cap Y \Rightarrow N_r(p) \cap Y \subset \boxed{G \cap Y = E}$

$\Rightarrow E$ is relative open to Y

Compact Sets

2.31 Definition: A open cover of a set E, in a metric space X, is a collection $\{G_\alpha\}$ of open sets, such that $E \subset \bigcup_\alpha G_\alpha$.

2.32 Definition: A set K is compact, if given any open cover $\{G_\alpha\}$ of K, there exists $\alpha_1, \cdots, \alpha_n$ (finite) s.t. $K \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$

Ex: If K is finite $\Rightarrow K$ is compact

Suppose $K = \{k_1, k_2, \cdots, k_n\}$ and $K \subseteq \bigcup_\alpha G_\alpha$

$\forall k_i, \exists G_{\alpha_i}$ s.t. $k_i \in G_{\alpha_i} \Rightarrow K \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$

Ex: $K = \mathbb{Z}$, $X = \mathbb{R}$, let $G_n = (-n, n)$ $K \subseteq \bigcup_{n=1}^{\infty} G_n$

Suppose compact $\exists \alpha_1 \le \alpha_2 \le \cdots \le \alpha_m = M$ s.t. $K \subseteq \bigcup_{i=1}^{n} G_{\alpha_i} = (-M, M) \neq \mathbb{Z}$

$\Rightarrow \mathbb{Z}$ is not compact

2.33 Theorem Suppose $K \subset Y \subset X$. Then K is compact relative to X, if and only if K is compact relative to Y.

proof: ($\Leftarrow$) K is compact in Y

Let $K \subseteq \bigcup_\alpha G_\alpha$ be an open cover in X

By Thm 2.30 $\Rightarrow$ $V_\alpha = G_\alpha \cap Y$ is open in Y

$$\begin{cases} K \subseteq \bigcup_\alpha G_\alpha \\ K \subseteq Y \end{cases} \Rightarrow K \subseteq \bigcup_\alpha (G_\alpha \cap Y) = \bigcup_\alpha V_\alpha$$

Then $\exists\ \alpha_1, \cdots, \alpha_n$ s.t. $K \subseteq \bigcup_{i=1}^{n} V_{\alpha_i} \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$

$\Rightarrow K$ is compact in X

($\Rightarrow$) Let $\{V_\alpha\}$ be an open cover of K in Y

By Thm 2.30 $V_\alpha = G_\alpha \cap Y$ for some $G_\alpha \subseteq X$ is open

so $K \subseteq \bigcup_\alpha V_\alpha \subseteq \bigcup_\alpha G_\alpha$

Then $\exists\ \alpha_1, \cdots, \alpha_n$ s.t. $K \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$ $K \subseteq Y$

$\Rightarrow K \subseteq \bigcup_{i=1}^{n} (G_{\alpha_i} \cap Y) = \bigcup_{i=1}^{n} V_{\alpha_i} \Rightarrow K$ is compact in Y

2.34 Theorem Compact subsets of metric spaces are closed.

$\Leftrightarrow$ If K = compact then K is closed

proof: WTS K^c is open (Thm 2.23)

Let $p \in K^c$, $\forall q \in K$, $r_q = d(p, q)$

$V_q = N_{\frac{r_q}{3}}(q) \to$ open

$K \subset \bigcup_{q \in K} V_q$

Since K is compact $\Rightarrow$ $\exists\ q_1, \cdots q_n$ s.t.

$K \subseteq \bigcup_{i=1}^{n} V_{q_i}$, $r = \frac{1}{3} \min\{r_{q_1}, \cdots, r_{q_n}\} > 0$

claim: $N_r(p) \cap (\bigcup_{i=1}^{n} V_{q_i}) = \emptyset$

suppose $x \in \bigcup_{i=1}^{n} V_{q_i}$

$x \in N_r(p) \Rightarrow d(x, p) < r$

$x \in V_{q_i}$ for some i $\Rightarrow d(x, q_i) < \frac{r_{q_i}}{3}$

$r_{q_i} = d(p, q_i) \leq d(x, p) + d(p, q_i) < r + \frac{r_{q_i}}{3}$

$\leq \frac{2}{3} r_{q_i}$

$\Rightarrow$ absurd

$\Rightarrow N_r(p) \subseteq K^c \Rightarrow K^c$ is open

$\Rightarrow K$ is closed

2.35 Theorem Closed subsets of compact sets are compact.

proof: suppose $F \subset K \subset X$, X is a metric space, K is a compact set, and F is a closed set.

Let $\{G_\alpha\}$ be an open cover of F

Then construct an open cover $\{G_\alpha\} \cup \{F^c\}$

Since K is compact $\Rightarrow \exists \alpha_1, \dots, \alpha_n$ s.t. $K \subset \{F^c\} \cup (\bigcup_{i=1}^{n} G_{\alpha_i})$

$\forall x \in F \Rightarrow x \in K,\ x \notin F^c \Rightarrow x \in \bigcup_{i=1}^{n} G_{\alpha_i}$

$\Rightarrow F \subset \bigcup_{i=1}^{n} G_{\alpha_i} \Rightarrow F$ is compact

Corollary If F is closed and K is compact, then $F \cap K$ is compact.

proof: $F \cap K$ is closed & $F \cap K \subset K \Rightarrow F \cap K$ is compact

2.36 Theorem If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X s.t. the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.

proof: Let K_1 be some member of $\{K_\alpha\}$. (Fix K_1)

① If $\exists x \in K_1$ s.t. $x \in K_\alpha$ for all $\alpha \Rightarrow$ done

② suppose no such x exists

$K_1 \subset \bigcup_\alpha K_\alpha^c$ $[K_1 = \{k_1, k_2, \dots, k_n\},\ k_i \in \text{some } K_\alpha^c]$

Since K_1 is compact

$\Rightarrow K_1 \subset \bigcup_{i=1}^{n} K_{\alpha_i}^c$

$\Rightarrow K_1 \cap (\bigcup_{i=1}^{n} K_{\alpha_i}^c)^c = \emptyset \Rightarrow K_1 \cap K_{\alpha_1} \cap K_{\alpha_2} \cap \dots \cap K_{\alpha_n} = \emptyset$

$\Rightarrow$ contradiction.

Corollary If $\{K_n\}$ is a sequence of nonempty compact sets s.t. $K_n \supset K_{n+1}$ $(n = 1, 2, \dots)$ then $\bigcap_1^\infty K_n$ is not empty.

2.37 Theorem If E is an infinite subset of a compact set K, then E has a limit point in K.

Proof: Suppose $E' \cap K = \emptyset$. Then $\forall q \in K \Rightarrow q \notin E'$

$\Rightarrow \exists N_r(q) \cap E \subset \{q\}$ ($\emptyset$ or $\{q\}$), Let $V_q = N_r(q) \Rightarrow K \subset \bigcup_{q \in K} V_q$

K is compact $\Rightarrow \exists q_1, q_2, \dots, q_n$ s.t. $K \subset \bigcup_{i=1}^{n} V_{q_i}$

$E = E \cap K = E \cap (\bigcup_{i=1}^{n} V_{q_i}) = \bigcup_{i=1}^{n} (E \cap V_{q_i}) \subseteq \{q_1, q_2, \dots, q_n\} \Rightarrow$ Contradict to E is infinite.

2.38 Theorem If $\{I_n\}$ is a sequence of Intervals in $\mathbb{R}^1$, s.t. $I_n \supset I_{n+1}$ $(n=1,2,\cdots)$, then $\bigcap_1^\infty I_n$ is not empty.

Proof: If $m \le n$, let $I_m = [a_m, b_m]$ $I_n = [a_n, b_n]$

$\Rightarrow I_m \subset I_n$ and $a_n \le a_m \le b_m \le b_n$

$A = \{a_k : k \in \mathbb{N}\}$, then b_m is an upper bound $\forall m \in \mathbb{N}$

$\begin{cases} k \le m & a_k \le a_m \le b_m \\ k > m & a_k \le b_k \le b_m \end{cases}$

$\Rightarrow \alpha = \sup A$ exists in $\mathbb{R}$

$\Rightarrow \begin{cases} a_n \le \alpha & \forall n \in \mathbb{N} \\ \alpha \le b_n & \forall n \in \mathbb{N} \end{cases} \Rightarrow \alpha \in I_n \ \forall n \in \mathbb{N} \Rightarrow \bigcap_1^\infty I_n$ is not empty

2.39 Theorem Let k be a positive integer. If $\{I_n\}$ is a sequence of k-cells s.t. $I_n \supset I_{n+1}$ $(n=1,2,3,\cdots)$, then $\bigcap_1^\infty I_n$ is not empty.

Proof: $I_n = \underbrace{[a_{1n}, b_{1n}]}_{I_1^n} \times \underbrace{[a_{2n}, b_{2n}]}_{I_2^n} \times \cdots \times \underbrace{[a_{kn}, b_{kn}]}_{I_k^n}$

$I \subset J \Leftrightarrow I_j \subset J_j$ for $j = 1, 2, \cdots, k$

By Thm 2.38 $\Rightarrow \exists\, x_j^* \subset \bigcap_{n=1}^{\infty} I_j^n$

$\Rightarrow \exists\, x^* = (x_1^*, x_2^*, \cdots, x_k^*) \in \bigcap_{n=1}^{\infty} I_n$

2.40 Theorem Every k-cell is compact

Proof: Let $I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k]$

$\delta = \left(\sum_{i=1}^{k} (b_i - a_i)^2\right)^{\frac{1}{2}} \rightarrow$ longest length of two points

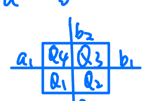
Claim: If $x, y \in I$, then $\|x - y\| \le \delta$

$\left[\left(\sum_{i=1}^{k} (x_i - y_i)^2\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^{k} (b_i - a_i)^2\right)^{\frac{1}{2}} = \delta \right]$

$\exists$ $\xrightarrow{\text{Construction}}$ 2^k k-cells, Q_i, $i = 1, 2, \cdots, 2^k$

Ex: $k=1$ 

$c = \frac{a+b}{2}$

$k=2$ 

$Q_1 = [a_1, \frac{a_1+b_1}{2}] \times [a_2, \frac{a_2+b_2}{2}]$

$I = I_1 \times I_2 \times \cdots \times I_k$

$I_i = I_i^1 \cup I_i^2$ $\quad I_i^1 = [a_i, \frac{a_i+b_i}{2}]$ $\quad I_i^2 = [\frac{a_i+b_i}{2}, b_i]$

$Q = I_1^{S_1} \times I_2^{S_2} \times \cdots \times I_k^{S_k}$ $\quad S_i \in \{1, 2\}$ $\quad$ k locations 2 options/each.

If $x, y \in Q \Rightarrow \|x - y\| \le \frac{\delta}{2}$

Suppose I is not compact $\Rightarrow \exists \{G_\alpha\}$ open cover of I with no finite subcovers.

$I \to Q_i$ for $i = 1, 2, \cdots 2^k$

$Q_i \subset I \subset \bigcup_\alpha G_\alpha$

it is compact

If $\exists$ finite subcovers of Q_i, for all $i = 1, 2, \cdots, 2^k$ (then we get a contradiction)

So, $\exists i \in \{1, 2, \cdots, 2^k\}$ s.t. $\overset{I_1}{Q_i}$ has no finite subcovers

subdivide I_1

Then (1) we obtain a sequence of k-cells : $I \supset I_1 \supset I_2 \supset I_3 \supset \cdots$

(2) I_n has no finite subcovers in $\{G_\alpha\}$

(3) If $x, y \in I_n$, then $\|x - y\| \leq \frac{\delta}{2^n}$ → since devide n times

By (1) & Thm 2.34 $\Rightarrow \bigcap_{n=1}^{\infty} I_n \neq \phi$

Let $x^k \in \bigcap_{n=1}^{\infty} I_n \Rightarrow x^k \in G_\alpha$ for some α $\Rightarrow \exists r > 0$ s.t. $N_r(x^*) \subset G_\alpha$

By Archimed.. Property $\Rightarrow \exists n \in \mathbb{N}$ s.t. $r > \delta \cdot 2^{-n}$ $\Leftrightarrow 2^n \cdot r > \delta$

$\Rightarrow I_n \subset N_r(x^*) \subset G_\alpha$

$\Rightarrow$ Contradict to (2)

2.41 Theorem If a set E in $\mathbb{R}^k$ has one of the following three properties, then it has the other two:

(a) E is closed and bounded ;

(b) E is compact ;

(c) Every infinite subset of E has a limit point in E.

Proof: (a) → (b) → (c) → (a)

(a) → (b) : E is bounded $\Rightarrow E \subset I$ for some k-cells

By Thm 2.40 $\Rightarrow I$ is compact

By Thm 2.35 : E is closed and a subset of $I \Rightarrow E$ is compact.

(b) → (c) : By Thm 2.37 $\Rightarrow$ done

(c) → (a) : ① Suppose E is not bounded

$\forall n \in \mathbb{N}$ $\exists x_n \in E$ s.t. $\|x_n\| \geq n$

Let $S = \{x_n : n \in \mathbb{N}\}$

(i) S is infinite

(ii) S has no limit point in $\mathbb{R}^k$ $\Rightarrow$ break (c) $\Rightarrow$ Contradiction

② Suppose E is not closed

$\exists p \in E'$ s.t. $p \notin E$

$\forall n \in \mathbb{N}$ $\exists x_n \in E$ s.t. $\|p - x_n\| < \frac{1}{n}$

Let $S = \{x_n : n \in \mathbb{N}\}$

(i) S is infinite

(ii) p is a limit point of E (Archime - Property)

(iii) S has no other limit point (suppose $\exists q \Rightarrow$ Δ-Ie)

q is not a limit point

$|x_n - q| \geq |q - p| - |p - x_n|$

$\geq |q - p| - \frac{1}{n}$ for all but finitely many n

$\mathbb{N}$ infinite

$\Rightarrow$ break (c)

Connected Sets

2.45 Definition : X is a metric space, $A \subset X$, $B \subset X$, $E \subset X$

A, B are separated : $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$

E is connected : $E \neq A \cup B$ $\forall$ separated A & B

If A, B are separated $A \cap B = \emptyset$ ($A \cap B \subset \overline{A} \cap B$)

2.47 Theorem : A subset E of the real line $\mathbb{R}^1$ is connected iff and only if it has the following property: If $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$

Proof: $(\Rightarrow)$ Let $x, y \in E$ and $x < z < y$

Suppose $z \notin E$

$A_z = E \cap (-\infty, z)$, $B_z = E \cap (z, \infty)$

$$\left.\begin{array}{l} \overline{A_z} \cap B_z \subset (-\infty, z] \cap (z, \infty) = \emptyset \\ A_z \cap \overline{B_z} \subset (-\infty, z) \cap [z, \infty) = \emptyset \end{array}\right\} \Rightarrow A_z \text{ \& } B_z \text{ are separated}$$

$A_z \cup B_z = E \cap [(-\infty, +\infty) \setminus \{z\}] = E$ $\Rightarrow$ contradict to E is connected.

$(\Leftarrow)$ Suppose E is not connected

Then $E = A \cup B$, where A & B are nonempty and separated.

Let $x \in A$, $y \in B$ WLOG $\Rightarrow$ $x < y$

Let $z = \sup (A \cap [x, y]) \in \mathbb{R}$ $x \leq z \leq y$

By Thm 2.28 $\Rightarrow$ $z = \sup (A \cap [x, y]) \in \overline{A}$ $\Rightarrow$ $z \notin B$ $\Rightarrow$ $z < y$

① Easy case : $z \notin A$ $\Rightarrow$ $x < z < y$ and $z \notin A \cup B = E$

$\Rightarrow$ contradiction

② Hard case : $z \in A$ $\Rightarrow$ $z \notin \overline{B}$ $\Rightarrow$ $z \in (\overline{B})^c$ $\Rightarrow$ $\exists r > 0$ s.t.

$(z-r, z+r) \in (\overline{B})^c$

make r small $(z, z+r) \subseteq (z, y)$

$z_1 = z + \frac{r}{2}$

(i) $x \leq z < z_1 < z + r \leq y$

$$\left.\begin{array}{l} \text{(ii) } z_1 \notin \overline{B} \Rightarrow z_1 \notin B \\ \text{(iii) } z_1 \notin A \text{ (since } z_1 > z \text{ and } z \text{ was a sup)} \end{array}\right\} \Rightarrow z_1 \notin E$$

Chapter 3 Numberical Sequences and Series

3.1 Definition: $\{P_n\}$ in a metric space X, if there is a $p \in X$,
for every $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n \geq N$ $d(P_n, p) < \varepsilon$
$\Rightarrow \lim_{n\to\infty} P_n = P$ (converge)
otherwise diverge

3.2 Theorem: Let $\{P_n\}$ be a sequence in a metric space X

(a) $\{P_n\}$ converges to $P \in X$, if and only if every $N_r(p)$ contains P_n for all but finitely many n. [use definition]

(b) If $p \in X$, $p' \in X$, and if $\{P_n\}$ converges to p and to p', then $p' = p$ [use Δ-Ie]

(c) If $\{P_n\}$ converges, then $\{P_n\}$ is bounded
[Construct finite distance & infinite set with $r<1$, $\max\{\text{finite}, 1\}$]

(d) If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{P_n\}$ in E s.t. $p = \lim_{n\to\infty} P_n$

proof: For each $n \in \mathbb{N}$
$N_{\frac{1}{n}}(p) \cap E \neq \emptyset$ $(p \in E')$
Let P_n be the element in here: $\{P_n\} \subseteq E$

WTS: $P_n \to p$
Let $\varepsilon > 0$ be given Then $\exists N \in \mathbb{N}$ s.t. $N > \varepsilon^{-1}$ $(\varepsilon > \frac{1}{N})$
If $n \geq N$ then
$d(P_n, p) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$

[相当于把 Neighbourhood 里的点拿出来构造一个 $\{P_n\}$, then prove convergence.]

3.3 Theorem Suppose $\{S_n\}, \{t_n\}$ are complex sequences, and $\lim_{n\to\infty} S_n = s$, $\lim_{n\to\infty} t_n = t$. Then

(a) $\lim_{n\to\infty} (S_n + t_n) = s + t$

(b) $\lim_{n\to\infty} cS_n = cs$, $\lim_{n\to\infty} (c + S_n) = c + s$, for any number c

(c) $\lim_{n\to\infty} S_n t_n = st$

$$s_n t_n - st = (s_n - s)(t_n - t) + \underbrace{t(s_n - s) + s(t_n - t)}_{\Uparrow \text{ use } 3.3\ (b)}$$

$$|(s_n - s)(t_n - t)| \le \underbrace{|s_n - s|}_{\sqrt{\varepsilon}}\underbrace{|t_n - t|}_{\sqrt{\varepsilon}} < \varepsilon$$

(d) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$, provided $s_n \neq 0$ $(n = 1, 2, \cdots)$, and $s \neq 0$

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s \cdot s_n}\right|$$

$|s| \le |s - s_n| + |s_n| < \frac{1}{2}|s| + |s_n|$

since $s_n \to s$ $\Rightarrow$ $\exists\, n \ge N_1$ $|s - s_n| < \frac{1}{2}|s|$ $\Rightarrow$ $\underline{|s_n| > \frac{1}{2}|s|}$

$\Rightarrow$ $\exists\, n \ge N_2$ $|s - s_n| < \frac{1}{2}|s|^2 \varepsilon$

when $n \ge \max(N_1, N_2)$

$$\frac{|s - s_n|}{|s \cdot s_n|} < \frac{\frac{1}{2}|s|^2 \varepsilon}{|s| \cdot |s_n|} < \frac{\frac{1}{2}|s|^2 \varepsilon}{|s| \cdot \frac{1}{2}|s|} = \varepsilon$$

3.4 Theorem

(a) Suppose $x_n \in R^k$ $(n = 1, 2, \cdots)$ and $x_n = (\alpha_{1,n}, \cdots, \alpha_{k,n})$

Then $\{x_n\}$ converges to $x = (\alpha_1, \cdots, \alpha_k)$ if and only if

$$\lim_{n\to\infty} \alpha_{j,n} \to \alpha_j$$

(b) Suppose $\{x_n\}, \{y_n\}$ are sequences in R^k, $\{\beta_n\}$ is a sequence of real numbers, and $x_n \to x$, $y_n \to y$, $\beta_n \to \beta$. Then

$$\lim_{n\to\infty} (x_n + y_n) = x + y, \quad \lim_{n\to\infty} x_n \cdot y_n = x \cdot y, \quad \lim_{n\to\infty} \beta_n \cdot x_n = \beta x$$

Subsequence

3.5 Definition Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers, s.t. $n_1 < n_2 < n_3 < \cdots$. Then the sequence $\{p_{n_k}\}$ is called a subsequence of $\{p_n\}$.

If $\{p_{n_k}\}$ converges, its limit is called a subsequential limit of $\{p_n\}$

$\{p_n\}$ converges to p if and only if every sebsequence of $\{p_n\}$ converges to p.

3.6 Theorem

(a) If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point of X.

(b) Every bdd sequence in $\mathbb{R}^k$ contains a convergent subsequence.

(b) If $\{p_n\}$ is bounded $\Rightarrow$ $\{p_n\} \subseteq I$, I is k-cell.

(a) Let E be the range of $\{p_n\}$

1° If E is finite, then $\exists\ p \in X$ and $\exists \{n_i\}$ $(n_1 < n_2 < \cdots)$ s.t.

$p_{n_1} = p_{n_2} = p_{n_3} = \cdots = p$

$\Rightarrow$ the subsequence $\{p_{n_i}\}$ converges to p

$E = \{e_1, e_2, \cdots, e_n\}$ finite
if each $e_i \leftarrow$ finite p_n
The $\{p_n\}$ is finite.

2° If E is infinite,

by Thm 2.37 $\Rightarrow$ E has a limit point $p \in X$

$\boxed{\text{by Thm 2.20}}$ $\Rightarrow$ $\exists\ n_i > n_{i-1}$ s.t. $d(p, p_{n_i}) < 1/i$

$\Rightarrow$ $\{p_{n_i}\}$ converges to p

3.7 Theorem The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X. let E^* be the set of all subsequential limits

Let $q \in (E^*)'$: q is the limit point of E^*

1° If $p_n = q$ for each n $\Rightarrow$ $E^* = \{q\}$ $\Rightarrow$ closed

2° Then $\exists\ n_1$ s.t. $p_{n_1} \neq q$, $\delta = d(p_{n_1}, q)$

since q is a limit point $\Rightarrow$ $\exists\ x \in E^*$ s.t. $d(x, q) < 2^{-i}\delta$

since $x \in E^*$ $\Rightarrow$ $\exists\ n_i > n_{i-1}$ s.t. $d(x, p_{n_i}) < 2^{-i}\delta$

$\Rightarrow$ $d(q, p_{n_i}) \leq 2^{1-i}\delta$ for $i = 1, 2, \cdots$ $(d(q, p_{n_i}) \leq d(x,q) + d(x, p_{n_i}))$

$\Rightarrow$ $\{p_{n_i}\}$ converges to q

$\Rightarrow$ $q \in E^*$

Cauchy Sequence

3.8 Definition Given $\forall \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ s.t. if $m, n \geq N \Rightarrow d(p_m, p_n) < \varepsilon$

3.9 Definition (geometric concept)

Let E be a nonempty subset of a metric space X, and let S be the set of all real numbers of the form $d(p,q)$, with $p \in E$ and $q \in E$. The sup of S is called the diameter of E.

$S = \{d(p,q) : p \in E, q \in E\}$, diameter of E = $\sup S$

If $\{p_n\}$ is a sequence in X and if E_N consists of the points $p_N, p_{N+1}, \dots$

Then $\{p_n\}$ is a Cauchy sequence if and only if

$$\lim_{N\to\infty} \operatorname{diam} E_N = 0$$

3.10 Theorem

(a) If $\bar{E}$ is the closure of a set E in a metric space X, then

$$\operatorname{diam} \bar{E} = \operatorname{diam} E.$$

(b) If K_n is a sequence of compact sets in X s.t. $K_n \supset K_{n+1}$ $(n=1,2,\dots)$ and if $\lim_{n\to\infty} \operatorname{diam} K_n = 0$,

then $\bigcap_1^\infty K_n$ consists of exactly one point.

proof: (a) since $E \subset \bar{E}$ $\Rightarrow$ it is clear that $\operatorname{diam} E \le \operatorname{diam} \bar{E}$

Fix $\varepsilon > 0$ and choose $p, q \in \bar{E}$, by definition of $\bar{E}$, $\exists\, p', q' \in E$

s.t. $d(p,p') < \varepsilon$, $d(q,q') < \varepsilon$

[if $p \in E'$ → clear, if $p \notin E'$ → $p' = p$]

Hence $d(p,q) \le d(p,p') + d(q,q') + d(p',q') < 2\varepsilon + \operatorname{diam} E$

$\Rightarrow \operatorname{diam} \bar{E} \le \operatorname{diam} E$

$\Rightarrow \operatorname{diam} \bar{E} = \operatorname{diam} E$

(b) by thm 2.36 $\Rightarrow$ K is not empty

If there are two points in K $\Rightarrow$ $\operatorname{diam} K > 0$ $\Rightarrow$ contradict to definition

ex: $K_n = [-\frac{1}{n}, \frac{1}{n}]$

$\operatorname{diam} K_n = \frac{2}{n} \to 0$

3.11 Theorem

(a) In any metric space X, every convergent sequence is a Cauchy sequence

(b) If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X, then $\{p_n\}$ converges to some point of X.

(c) In $\mathbb{R}^k$, every Cauchy sequence converges.

Proof: (a) If $p_n \to p$, and given $\varepsilon > 0$ $\exists\, N_1 \in \mathbb{N}$ s.t. $d(p_n, p) < \frac{\varepsilon}{2}$ for all $n \ge N_1$

$\Rightarrow \forall m, n \ge N_1 \Rightarrow d(p_m, p_n) \le d(p_m, p) + d(p_n, p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow \{p_n\}$ is a Cauchy sequence

(b) Let $\{p_n\}$ be a Cauchy sequence in the compact space X. For $N = 1, 2, 3, \dots$, let E_N be the set consisting of $p_N, p_{N+1}, p_{N+2}, \dots$

We know: (1) diam $E_N \to 0$

(2) diam $\overline{E_N} \to 0$

(3) $\overline{E_N}$ is closed

(4) $\overline{E_N}$ is compact (closed and bounded $\to$ compact)

(5) $E_{N+1} \subset E_N \Rightarrow E_{N+1} \subset \overline{E_N} \Rightarrow \overline{E_{N+1}} \subset \overline{E_N}$

②④⑤ $\Rightarrow \{P\} = \bigcap_{i=1}^{\infty} \overline{E_N}$ [nonempty, if two points diam $\overline{E_N} \not\to 0$]

Let $\varepsilon > 0$ be given $\Rightarrow \exists\, N \in \mathbb{N}$ s.t. diam $\overline{E_n} < \varepsilon$ for $n \geq N$

$P_n \in E_n \subset \overline{E_n} \Rightarrow P \in \overline{E_n}$

$d(P, P_n) \leq$ diam $\overline{E_n} < \varepsilon$

$\Rightarrow P_n \to P$

(c) Let $\{\vec{x}_n\}$ be a Cauchy sequence in $\mathbb{R}^k$. Define E_N as (b)

$\{\vec{x}_n\}$ is bounded

By Thm 2.41 $\Rightarrow$ limit points of $\{\vec{x}_n\}$ are in $\mathbb{R}^k$

$\Rightarrow$ The closure of $\{\vec{x}_n\}$ is in $\mathbb{R}^k$ ($\overline{\{\vec{x}_n\}}$ is compact)

By (b) $\Rightarrow \{\vec{x}_n\}$ converges

3.12 Definition A metric space in which every Cauchy sequence converges is said to be complete.

① Cauchy sequence converges in the metric space

② A metric space is complete

$\Rightarrow$ All compact metric spaces and all $\underbrace{\text{Euclidean spaces}}_{n\text{维空间}}$ are complete

$\Rightarrow$ every closed subset E is a Cauchy sequence in X.

$\mathbb{Q}$ is not complet (ex:) Give a Cauchy Sequence $\begin{cases} x_1 = 1 \\ x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \end{cases}$

$x = \frac{x}{2} + \frac{1}{x} \Rightarrow x = \sqrt{2}$

(ex:) $P_n \in (\sqrt{2} + \frac{1}{n+1}, \sqrt{2} + \frac{1}{n})$ $P_n - P_m < \frac{1}{m} < \frac{1}{N} < \varepsilon$ $P_n \to \sqrt{2}$ in $\mathbb{R}$

3.13 Definition A sequence $\{s_n\}$ of real numbers is said to be

(a) monotonically increasing if $s_n \leq s_{n+1}$ $(n = 1, 2, 3, \cdots)$;

(b) monotonically decreasing if $s_n \geq s_{n+1}$ $(n = 1, 2, 3, \cdots)$.

(a) or (b) $\Rightarrow$ monotonic sequence.

3.14 Theorem Suppose $\{S_n\}$ is monotonic. Then $\{S_n\}$ converges if and only if it is bounded.

Proof: ($\Rightarrow$) $\{S_n\}$ converges $\rightarrow$ $\{S_n\}$ is Cauchy sequence $\rightarrow$ $\{S_n\}$ is bounded

($\Leftarrow$) $\{S_n\}$ is bound, Suppose it is monotonically increasing

Let $S = \sup S_n$

$\forall \varepsilon > 0$, $\exists N$ s.t. $S - \varepsilon < S_N \leq S$ (otherwise $S-\varepsilon$ would be the $\sup S_n$)

Since $\{S_n\}$ is monotonically increasing $\Rightarrow \forall n \geq N$ $S-\varepsilon < S_n \leq S$

$\Rightarrow \{S_n\}$ converges to S

Upper and Lower Limits

3.15 Definition Let $\{S_n\}$ be a sequence of real numbers with the following properties: 1) For every real M there is an integer N s.t. $n \geq N$ implies $S_n \geq M$. We then write $S_n \rightarrow +\infty$

2) If for every real M there is an integer N s.t. $n \geq N$ implies $S_n \leq M$. We then write $S_n \rightarrow -\infty$.

3.16 Definition Let $\{S_n\}$ be a sequence of real number, Let E be the set of number x s.t. $S_{n_k} \rightarrow x$, for some subsequence $\{S_{n_k}\}$. This set E contains all subsequential limits, plus the possible numbers $+\infty, -\infty$.

Let $S^* = \sup E$, $S_* = \inf E$

$\Rightarrow \limsup\limits_{n\rightarrow\infty} S_n = S^*$, $\liminf\limits_{n\rightarrow\infty} S_n = S_*$

$S_n \rightarrow S \Leftrightarrow \limsup\limits_{n\rightarrow\infty} S_n = S = \liminf\limits_{n\rightarrow\infty} S_n = S$

a sequence converges

$\Rightarrow$ it has only one subsequential limit

3.17 Theorem Let $\{S_n\}$ be a sequence of real numbers. S^* has the two properties.

(a) $S^* \in E$

(b) If $x > S^*$, there is an integer N s.t. $n \geq N$ implies $S_n < x$.

$\boxed{\text{大于 } S^* \text{的 element 不可能存在于 } E}$

(Morever, S^* is the only number with the properties (a) and (b))

Proof: (a) Case 1: If $S^* = +\infty$ $\rightarrow$ E is not bdd above $\rightarrow$ $\exists \{S_{n_k}\}$ s.t. $S_{n_k} \rightarrow +\infty$

$\rightarrow +\infty \in E$

[Once you have chosen $n_1 < n_2 < \cdots < n_{i-1}$, you pick $n_i > n_{i-1}$ s.t. $S_{n_i} > i$]

Case 2: If $S^* \in \mathbb{R}$, $E^* = E \cap \mathbb{R}$

by Thm 3.7 E^* is closed

$S^* = \sup E \in \bar{E} = E$ by thm 2.28

Case 3: If $S^* = -\infty$ $\Rightarrow$ there is no subsequential limit $\Rightarrow$ $E = \{-\infty\} \ni S^*$

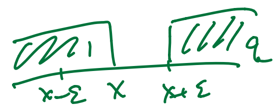
(b) Let $x > S^*$, suppose that $S_n \geq x$ for infinitely many n s.t.

$\begin{cases} S_{n_i} \geq x \\ S_{n_i} \rightarrow y \geq x > S^* \end{cases}$ $\Rightarrow y \in E \Rightarrow$ contradict to $S^* = \sup E$

(c) Uniqueness (treat bdd and unbdd separately)

bdd case

- suppose there are two numbers p and q satisfying (a) & (b), and $p<q$.
- $\exists\ p<x<q$ (b) $S_n \le x$ for all $n \ge N$ (N-fixed)
  - (a) $S_{n_k} \to q$ for some subsequence
- Let $\varepsilon>0$ be chosen, so that $p<x-\varepsilon<x<x+\varepsilon<q$
  - $\exists\ N$ s.t. $S_n \le x$ for all $n \ge N$



3.19 Theorem If $s_n \le t_n$ for $n \ge N$, where N is fixed, then

$$\liminf s_n \le \liminf t_n$$
$$\limsup s_n \le \limsup t_n$$

Some Special Sequence

3.20 Theorem

(a) $p>0 \Rightarrow \lim_{n\to\infty} \frac{1}{n^p} = 0$

(b) $p>0 \Rightarrow \lim_{n\to\infty} \sqrt[n]{p} = 1$

(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

(d) $p>0,\ \alpha \in \mathbb{R} \Rightarrow \lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

(e) $|x|<1 \Rightarrow \lim_{n\to\infty} x^n = 0$

Series (default: complex-valued)

3.21 Definition Given a sequence $\{a_n\}$

$S_n = \sum_{i=1}^{n} a_i$: n-th partial sum

$\sum_{n=1}^{\infty} a_n$: (infinite) series

$s = \sum_{n=1}^{\infty} a_n$: sum of the series

If $\{S_n\}$ converges, we say "$\sum_{n=1}^{\infty} a_n$ converges".

3.22 Theorem Σa_n converges if and only if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$\left|\sum_{k=n}^{m} a_k\right| \leq \varepsilon$, if $m \geq n \geq N$.

proof: By definition $\Rightarrow$ $\{S_n\}$ converges $\overset{\text{Thm 3.11}}{\Leftrightarrow}$ $\{S_n\}$ is a Cauchy sequence

Then we are done!

3.23 Theorem If Σa_n converges, then $\lim_{n\to\infty} a_n = 0$

Proof: By 3.22 $\Rightarrow$ let $m = n+1$ $\Rightarrow$ Then we are done!

Converse is not always true! Ex: $\Sigma \frac{1}{n}$

$$\begin{aligned}\Sigma \frac{1}{n} &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \cdots\end{aligned}$$

3.24 Theorem A series of nonnegative (→ always refer to real numbers) terms converges if and only if its partial sums form a bounded sequence.

Proof: Thm 3.14 $\{S_n\}$ is monotonically increasing.

3.25 Theorem

(a) If $|a_n| \leq c_n$ for $n \geq N_0$, where N_0 is a fixed integer.

Σc_n converges $\Rightarrow$ Σa_n converges

(b) $a_n \geq d_n \geq 0$ for $n \geq N_0$, Σd_n diverges

$\Rightarrow$ Σa_n diverges

Proof: (a) Σc_n converges $\Rightarrow$ $\{\Sigma c_n\}$ is a Cauchy sequence

$\Rightarrow \exists N \in \mathbb{N}$ s.t. $\forall m \geq n \geq N$ $\sum_{k=n}^{m} c_k \leq \varepsilon$

$\Rightarrow \sum_{k=n}^{m} a_k \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k \leq \varepsilon$ $\Rightarrow$ Σa_n is a Cauchy sequence

$\Rightarrow$ Σa_n converges

(b) Suppose Σa_n converges, then by (a) $\Rightarrow$ Σd_n converges

$\Rightarrow$ Contradiction

Series of nonnegative terms

3.26 Theorem If $0 \le x < 1$, then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$x \ge 1 \Rightarrow \sum_{n=0}^{\infty} x^n$ diverges

proof: ① $x \ne 1 \Rightarrow S_n = \sum_{i=0}^{n} x^i = x^0 + x^1 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$ $\left(\frac{a_1(1-q^n)}{1-q}\right)$

let $n \to \infty$

② $x = 1 \Rightarrow \sum_{i=0}^{n} x^i = 1+1+1+\cdots$

3.27 Theorem Suppose $a_1 \ge a_2 \ge a_3 \ge \cdots \ge 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + \cdots$$ converges.

Proof: let $S_n = a_1 + a_2 + \cdots + a_n$

$t_k = a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}$

For $n < 2^k$

$$\begin{aligned} S_n &\le a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\le a_1 + 2a_2 + \cdots + 2^k a_{2^k} \\ &= t_k \end{aligned}$$

So that $S_n \le t_k$ $(*)$

For $n > 2^k$

$$\begin{aligned} S_n &\ge a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ &\ge \tfrac{1}{2}a_1 + a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k} \\ &= \tfrac{1}{2} t_k \end{aligned}$$

So that $S_n \ge \frac{1}{2} t_k$ $(**)$

By $(*)$ and $(**)$ $\Rightarrow \{S_n\}$ and $\{t_k\}$ are both bounded or unbounded.

3.28 Theorem $\sum \frac{1}{n^p}$ converges if $p>1$ and diverges if $p \le 1$

Proof: construct $t_n = \sum 2^n \frac{1}{(2^n)^p} = \sum 2^{n(1-p)}$

$\Rightarrow$ by thm 3.27 $\sum \frac{1}{n^p}$ converges $\Leftrightarrow$ $\sum 2^{n(1-p)}$ converges

$\Rightarrow$ by thm 3.26 $2^{(1-p)} < 1$, $\sum \frac{1}{n^p}$ converges.

3.29 Theorem If $p>1$ $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges; If $p \le 1$, the series diverges

Proof: construct $t_n = \sum_{n=1}^{\infty} 2^n \frac{1}{2^n(\log 2^n)^p} = \sum_{n=1}^{\infty} \frac{1}{n^p(\log 2)^p} = \frac{1}{(\log 2)^p} \sum_{n=1}^{\infty} \frac{1}{n^p}$

The Number e

3.30 Definition $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ $(0!=1)$

$S_n = 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots + \frac{1}{n!} < 1 + 1 + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3$

$S_n \le \sum 2^{1-n} = 4$

$\boxed{\Rightarrow 2.5 < e < 3}$

3.31 Theorem $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$

Proof: Let $S_n = \sum_{k=0}^{n} \frac{1}{k!}$, $t_n = (1+\frac{1}{n})^n$

1) By the Binomial theorem

$t_n = 1 + 1 + \frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + \cdots + \frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n-1}{n})$

$\Rightarrow t_n \le S_n \Rightarrow \boxed{\limsup_{n\to\infty} t_n \le e}$

2) if $n \ge m$

$t_n \ge 1 + 1 + \frac{1}{2!}(1-\frac{1}{n}) + \cdots + \frac{1}{m!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{m-1}{n})$

by thm 3.19

$\liminf_{n\to\infty} t_n \ge 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{m!}$

$\Rightarrow S_m \le \liminf_{n\to\infty} t_n$

when $m \to \infty$ $\Rightarrow \boxed{\liminf_{n\to\infty} t_n \ge e}$

$\boxed{\lim_{n\to\infty} (1+\frac{1}{n})^n = e}$

The Root and Ratio Test

3.33 Theorem (Root Test) Given Σa_n, Put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then

(a) If $\alpha < 1$, Σa_n converges

(b) If $\alpha > 1$, Σa_n diverges

(c) If $\alpha = 1$, the test gives no information

by Thm 3.18

proof: (a) If $\alpha < 1$ then $\exists\, 0 \le \alpha < \beta < 1$ and $\exists N \in \mathbb{N}$ s.t.

$\sqrt[n]{|a_n|} \le \beta$ for all $n \ge N$

$\Rightarrow |a_n| \le \beta^n$ converges

$\Sigma a_n \le \Sigma |a_n| \le \Sigma \beta^n$

(b) If $\alpha > 1$, Let $|a_{nk}| \to \alpha$

Then $\exists\ N_1 \in \mathbb{N}$ s.t. for $k \ge N_1$ we have $|a_{nk}| \ge \alpha - \varepsilon$

Then $|a_{nk}| \ge 1 + \varepsilon - \varepsilon = 1$

Then $|a_n| \ge 1$ for $n \to \infty$ $\Rightarrow$ Σa_n diverges

(c) $\Sigma \frac{1}{n}$ diverges

$\Sigma \frac{1}{n^2}$ converges

3.34 Theorem (Ratio Test) The series Σa_n

(a) Converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.

(b) diverges if $\left|\frac{a_{n+1}}{a_n}\right| \ge 1$ for all $n \ge n_0$, where n_0 is some fixed integer.

Proof: (a) find $0 < \beta < 1$ s.t. $|a_{n+1}| < \beta |a_n|$ since $\Sigma \beta^n$ converges

$\Rightarrow \Sigma a_n$ converges

(b) when $|a_{n+1}| \ge |a_n|$, $a_n \to 0$ does not hold.

3.35 examples: root test is stronger than ratio test.

3.37 For any sequence $\{c_n\}$ of positive numbers,

$\boxed{\text{root test is stronger}}$

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \le \liminf_{n\to\infty} \sqrt[n]{c_n}$$

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$$

power series

3.38 Definition Given a sequence $\{c_n\}$ of complex numbers, the series $\sum_{n=0}^{\infty} c_n z^n$ is called a power series. The numbers c_n are called coefficients of the series; z is a complex number.

→ There is an associated circle (circle of convergence)
If z is the interior of the circle $\Rightarrow \sum c_n z^n$ converges
If z is the exterior of the circle $\Rightarrow \sum c_n z^n$ diverges

3.39 Theorem Given the power series $\sum c_n z^n$, put

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}, \quad R = \frac{1}{\alpha}$$

(If $\alpha = 0$, $R = +\infty$, if $\alpha = +\infty$, $R = 0$) Then $\sum c_n z^n$ converges if $|z| < R$, and diverges if $|z| > R$.

Proof: by root test $\limsup_{n\to\infty} \sqrt[n]{|c_n z^n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|c_n|} = |z| \alpha = \frac{|z|}{R}$

「R is called the radius of convergence of $\sum c_n z^n$」

Summation by Parts

3.41 Theorem Given two sequences $\{a_n\}, \{b_n\}$, put

$$A_n = \sum_{k=0}^{n} a_k$$

If $n \geq 0$; put $A_{-1} = 0$. Then, if $0 \leq p \leq q$, we have

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

Proof: $\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}$

$\boxed{\text{partial summation formula}}$ $= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$

3.42 Theorem Suppose

(a) the partial sums A_n of $\sum a_n$ form a bdd sequence;
(b) $b_0 \geq b_1 \geq b_2 \geq \cdots$;
(c) $\lim_{n\to\infty} b_n = 0$.

Then $\sum a_n b_n$ converges.

proof: 1) choose $M>0$ sufficient large s.t. $|A_n| \le M$ for all $n \in \mathbb{N}$

2) let $\varepsilon>0$ be given, then $\exists N \in \mathbb{N}$ s.t. $b_N < \frac{\varepsilon}{2M}$

suppose $p, q \ge N$

$$\left|\sum_{n=p}^{q} a_n b_n\right| = \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right|$$

$$\le \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1})\right| + b_q|A_q| + b_p|A_{p-1}|$$

$$< M(b_p - b_q) + M b_q + M b_p = 2M b_p \le 2M b_N < 2M \cdot \frac{\varepsilon}{2M} = \varepsilon$$

by Cauchy Criterion $\Rightarrow \sum_{n=0}^{\infty} a_n b_n$ converges

3.43 Theorem Suppose (Leibnitz)

(a) $|c_1| \ge |c_2| \ge |c_3| \ge \cdots$;

(b) $c_{2m-1} \ge 0$, $c_{2m} \le 0$ $(m = 1, 2, 3, \dots)$; "alternating series"

(c) $\lim_{n\to\infty} c_n = 0$

Then $\sum c_n$ converges

proof: use Theorem 3.42 by constructing $a_n = (-1)^{n+1}$, $b_n = |c_n|$

3.44 Theorem Suppose the radius of convergence of $\sum c_n z^n$ is 1, and suppose $c_0 \ge c_1 \ge c_2 \ge \cdots$, $\lim_{n\to\infty} c_n = 0$. Then $\sum c_n z^n$ converges at every point on the circle $|z| = 1$, except possibly at $z = 1$.

proof: $A_n = 1 + z + \cdots + z^n = \dfrac{z^{n+1} - 1}{z - 1}$

$$|A_n| = \left|\frac{z^{n+1} - 1}{z - 1}\right| = \frac{|z^{n+1} - 1|}{|z - 1|} \le \frac{2}{|z-1|} \quad \Rightarrow A_n \text{ is bdd if } |z| = 1, |z| \ne 1$$

by Thm 3.42 $a_n = z^n$, $b_n = c_n$

$\Rightarrow \sum c_n z^n$ converges

Absolute Convergence

The series $\sum a_n$ is said to converge absolutely if the series $\sum |a_n|$ converges

3.45 Theorm If $\sum a_n$ converges absolutely, then $\sum a_n$ converges

proof: $\left|\sum_{n=p}^{q} a_n\right| \le \sum_{n=p}^{q} |a_n|$ + Cauchy Criterion

3.46 Remarks If $\sum a_n$ converges, but $\sum |a_n|$ diverges, we say that $\sum a_n$ converges non-absolutely.

example: $\sum \frac{(-1)^n}{n}$

Addition and Multiplication of Series

3.47 Theorem If $\sum a_n = A$, and $\sum b_n = B$, then $\sum (a_n + b_n) = A+B$, and $\sum c a_n = cA$, for any fixed c.

proof: ① Let $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$ Then

$$A_n + B_n = \sum_{k=0}^{n} (a_k + b_k)$$

since $A_n \to A$, $B_n \to B$ $\Rightarrow$ $A_n + B_n \to A+B$

② $\sum c a_n = c \sum a_n$

3.48 Definition Given $\sum a_n$ and $\sum b_n$, we put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \quad (n = 0, 1, 2, \cdots)$$

and call $\sum c_n$ the product of the two given series $\sum a_n$ & $\sum b_n$.

3.49 Example: the product of two convergent series may actually divergent.

$$\sum a_n = \sum b_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} \cdots$$

Let $\sum c_n$ be the product of $\sum a_n$ & $\sum b_n$

Then $\sum c_n = 1 - (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}) + (\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}\sqrt{2}} + \frac{1}{\sqrt{3}}) - (\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{3}\sqrt{2}} + \frac{1}{\sqrt{2}\sqrt{3}} + \frac{1}{\sqrt{4}}) + \cdots$

$$c_n = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(n-k+1)(k+1)}}$$

Since $(n-k+1)(k+1) = (\frac{n}{2}+1)^2 - (\frac{n}{2}-k)^2 \le (\frac{n}{2}+1)^2$

$\Rightarrow |c_n| \ge \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2}$ 「sum $(n+1)$ times」

$\Rightarrow$ doesn't satisfy $c_n \to 0$

$\Rightarrow \sum c_n$ diverge.

3.50 Theorem Suppose

(a) $\sum_{n=0}^{\infty} a_n$ converges absolutely;

(b) $\sum_{n=0}^{\infty} a_n = A$

(c) $\sum_{n=0}^{\infty} b_n = B$

(d) $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ $(n=0,1,2,\cdots)$.

Then $\sum_{n=0}^{\infty} c_n = AB$

Tips: sometime we can construct a new form, and prove that it converges to 0.

proof: Let $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$, $C_n = \sum_{k=0}^{n} c_k$, $\beta_n = B_n - B$

Then $C_n = \underbrace{a_0 b_0}_{c_0} + \underbrace{(a_0 b_1 + a_1 b_0)}_{c_1} + \underbrace{(a_0 b_2 + a_1 b_1 + a_2 b_0)}_{c_2} + \cdots + \underbrace{(a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0)}_{c_n}$

$= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0$

$= a_0 (B + \beta_n) + a_1 (B + \beta_{n-1}) + \cdots + a_n (B + \beta_0)$

$= A_n \cdot B + a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0$

Put $\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0$

$\boxed{\text{WTS } \gamma_n \to 0 \Rightarrow C_n \to AB}$

Put $\alpha = \sum_{n=0}^{\infty} |a_n|$

by (a) $\Rightarrow$ α converges

by (c) $\Rightarrow$ $\beta_n \to 0$

Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|\beta_n| \le \varepsilon$ for all $n \ge N$, in which case

$|\gamma_n| \le \underbrace{|\beta_0 a_n + \cdots + \beta_N a_{n-N}|}_{\text{finite and } a_n \text{ is going to } 0 \text{ as } n\to\infty} + \underbrace{|\beta_{N+1} a_{n-N-1} + \cdots + \beta_n a_0|}_{|\beta_n| < \varepsilon \to |\cdots| < \varepsilon \cdot \alpha}$

sum of convergent series

$x_n \le y_n \Rightarrow \limsup_{n\to\infty} x_n \le \limsup_{n\to\infty} y_n$

fix N, $n \to \infty$

$\Rightarrow \limsup_{n\to\infty} |\gamma_n| \le 0 + \varepsilon \cdot \alpha$

3.51 Theorem (Abel's Thm) If the series $\sum a_n$, $\sum b_n$, $\sum c_n$ converges to A, B, C, and $c_n = a_0 b_n + \cdots + a_n b_0$, then $C = AB$

Rearrangements

3.52 Definition Let $\{k_n\}$, $n=1,2,3,\cdots$, be a sequence in which every positive integer appears once and only once (that is $\{k_n\}$ is bijection from J to J), putting $a_n' = a_{k_n}$ $(n=1,2,3,\cdots)$

We say that $\sum a_n'$ is an arrangement of $\sum a_n$.

Change the partial sum

3.53 Example

Ex: $a_n = \frac{(-1)^{n+1}}{n}$ $\left(\frac{1}{1}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \cdots\right) \Rightarrow \sum a_n$ converges to some s, $s < s_3 = \frac{5}{6}$

rearrange $\left\{\underbrace{\frac{1}{1}, \frac{1}{3}}, -\frac{1}{2}, \frac{1}{5}, \frac{1}{7}, -\frac{1}{4}, \cdots\right\}$

$\begin{cases} \varphi(3k+1) = 4k+1 \\ \varphi(3k+2) = 4k+3 \\ \varphi(3k+3) = 2k+2 \end{cases}$

since $\frac{1}{4k+1} + \frac{1}{4k+3} - \frac{1}{2k+2} > 0$

$$\limsup_{n\to\infty} s_n' > s_3' = \frac{5}{6}$$

3.54 Theorem Let $\sum a_n$ be a series of real numbers which converges, but not absolutely. Suppose

$$-\infty \le \alpha \le \beta \le \infty$$

Then there exists a rearrangement $\sum a_n'$ with partial sum s_n' such that $\liminf_{n\to\infty} s_n' = \alpha$, $\limsup_{n\to\infty} s_n' = \beta$

3.55 Theorem If $\sum a_n$ is a series of complex numbers which converges absolutely, then every rearrangement of $\sum a_n$ converges, and they all converge to same sum.

Proof: Let $\sum a_n'$ be a rearrangement with partial sums s_n'

Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $m \ge n \ge N$

$$\Rightarrow \sum_{i=n}^{m} |a_i| \le \varepsilon$$

Then $\exists p \in \mathbb{N}$ s.t. $\{1,2,3,\cdots,N\} \subseteq \{k_1, k_2, \cdots, k_p\}$

$p = \max\{P_1, P_2, \cdots, P_N\}$ (P_r be the index of r in $\{k_1, k_2, \cdots\}$)

If $n > p\ (\ge N)$ we have $|s_n - s_n'| = \left|\sum_{i=1}^{N} a_i - \sum_{i=1}^{N} a_i' + \sum_{i=N+1}^{n} a_i - \sum_{i=N+1}^{n} a_i'\right|$

$$\le \sum_{i=N+1}^{\infty} |a_i|$$

$\Rightarrow s_n - s_n' \to 0$ and $s_n \to s$ $\Rightarrow s_n' \to s$

Chapter 4 Continuity

Limits of functions

4.1 Definition Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y, and p is a limit point of E. We write $f(x) \to q$ as $x \to p$, or

$$\lim_{x \to p} f(x) = q$$

if there is a point $q \in Y$ with the following property: For every $\varepsilon > 0$ there exists a $\delta > 0$ s.t.

$$d_Y(f(x), q) < \varepsilon$$

for all points $x \in E$ for which

$$0 < d_X(x, p) < \delta$$

4.2 Theorem Let X, Y, E, f and p be as in Definition 4.1. Then

$$\lim_{x \to p} f(x) = q$$

if and only if $\lim_{n \to \infty} f(p_n) = q$

for every sequence $\{p_n\}$ in E s.t. $p_n \neq p$, $\lim_{n \to \infty} p_n = p$

Proof: $(\Rightarrow)$ Let $\{p_n\} \subset E$, $p_n \neq p$, and $\lim_{n \to \infty} p_n = p$

We know $\lim_{x \to p} f(x) = q$

By Definition $\Rightarrow$ Given $\varepsilon > 0$, $\exists \delta > 0$

s.t. $x \in N_\delta^X(p) \setminus \{p\} \Rightarrow f(x) \in N_\varepsilon^Y(q)$

Also $p_n \to p \Rightarrow \exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow d(p_n, p) < \delta$

$\Rightarrow d(f(p_n), q) < \varepsilon$

$\Rightarrow f(p_n) \to q$

$(\Leftarrow)$ Suppose $\lim_{n \to \infty} f(p_n) \neq q$

$\exists \varepsilon > 0$ s.t. $\forall \delta > 0$ $\exists x \in N_\delta^X(p) \setminus \{p\}$ s.t. $d_Y(f(x), q) \geq \varepsilon$

Let $\delta_n = \frac{1}{n}$ pick $p_n \in N_{\delta_n}(p) / \{p\}$ as above

1) $d_X(p_n, p) < \frac{1}{n} \Rightarrow p_n \to p$
2) $p_n \neq p$
3) $d_Y(f(p_n), q) \geq \varepsilon$

$\Rightarrow$ Contradiction

Corollary If f has a limit at p, this limit is unique.

4.3 Definition $g, f : E \to \mathbb{R}^k$

$(f+g)(x) = f(x) + g(x)$, $(f \cdot g)(x) = f(x) \cdot g(x)$, $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$ $g(x) \neq 0 \ \forall x \in E$

$(\lambda f)(x) = \lambda \cdot f(x)$

4.4 Theorem Suppose $E \subset X$, a metric space, p is a limit point of E, f and g are complex functions on E, and

$$\lim_{x \to p} f(x) = A, \quad \lim_{x \to p} g(x) = B$$

Then (a) $\lim_{x \to p} (f+g)(x) = A + B$

(b) $\lim_{x \to p} (fg)(x) = A \cdot B$

(c) $\lim_{x \to p} \frac{f}{g}(x) = \frac{A}{B}$, if $B \neq 0$

proof: If $\{p_n\} \subset E$, $p_n \neq p$, $p_n \to p$

then $(f+g)(p_n) \to f(p_n) + g(p_n)$

By Thm 4.2 $\Rightarrow$ $(f+g)(p_n) \to A+B$

Continuous Functions

4.5 Definition Suppose X and Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y. Then f is said to be continuous at p if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in E$, $d_X(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$

(If f is continuous at every point of E, then f is said to be cts on E)

4.6 Theorem In the situation given in Definition 4.5, assume also that p is a limit point of E.

Then f is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$.

(f has to be defined at point p)

4.7 Theorem Suppose X, Y, Z are metric spaces, $E \subset X$, f maps E into Y, g maps the range of f, $f(E)$, into Z, and h is the mapping of E into Z defined by $h(x) = g(f(x))$, $x \in E$ $\quad (h = g \circ f)$

If f is continuous at a point $p \in E$ and if g is continuous at the point $f(p)$, then h is continuous at point p.

proof: Let $\varepsilon > 0$ be given. Since g is cts at $f(p)$

$\exists \eta > 0$ s.t. $\underset{y \in f(E)}{d_Y(f(p), y) < \eta} \Rightarrow d_Z(g(f(p)), g(y)) < \varepsilon$

given f is cts at point p

$\exists \delta > 0$ s.t.

$d_X(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < \eta$

If $d_X(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < \eta \Rightarrow d_Z(g(f(x)), g(f(p))) < \varepsilon$

$\Rightarrow h$ is cts at point p

4.8 Theorem A mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

proof: $(\Rightarrow)$ f is cts, Let $V \subseteq Y$ be open,

Want to show $f^{-1}(V)$ is open in X

Let $p \in f^{-1}(V) \Rightarrow f(p) \in V$

$\Rightarrow \exists \varepsilon > 0$ s.t. $N_\varepsilon^Y(f(p)) \subseteq V$ (since V is open)

since f is cts $\exists \delta > 0$ s.t.

$d_X(p, x) < \delta \Rightarrow d_Y(f(p), f(x)) < \varepsilon \quad f(x) \in N_\varepsilon^Y(f(p))$

Then $N_\delta^X(p) \subseteq f^{-1}(N_\varepsilon^Y(f(p))) \subseteq f^{-1}(V)$

$\Rightarrow p$ is an interior point of $f^{-1}(V)$

$\Rightarrow f^{-1}(V)$ is open in X

($\Leftarrow$) Let $p \in X$, Let $\varepsilon > 0$ be given. $p \in X$, $f(p) \in V$

$V = N_\varepsilon^Y(f(p)) \Rightarrow$ open set in Y

Then by assumption: $f^{-1}(V)$ is open

So $\exists \delta > 0$ s.t. $N_\delta^X(p) \subseteq f^{-1}(V)$

Then if $x \in N_\delta^X(p) \Rightarrow f(x) \in V$

$d_X(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$

$\Rightarrow f$ is cts at point $p \Rightarrow f$ is cts on X.

Corollary: A mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y

Hint: $f^{-1}(E^c) = [f^{-1}(E)]^c$

4.9 Theorem Let f and g be complex continuous functions on a metric space X. Then $f+g$, fg, and f/g are continuous on X.

Let $p \in X$

Case 1: p is an isolated point

case 2: p is a limit point

h is cts at $p \Leftrightarrow \lim_{x \to p} h(x) = h(p)$

By Thm 4.4 $\Rightarrow \lim_{x \to p} (f+g)(x) = f(p) + g(p)$

4.10 Theorem (a) Let $f_1, \cdots, f_k$ (components) be real function on a metric space X, and let f be the mapping of X into $\mathbb{R}^k$ defined by

$$f(x) = (f_1(x), \cdots, f_k(x)) \qquad (x \in X);$$

then $\vec{f}$ is cts if and only if each of the functions $f_1, \cdots, f_k$ is cts.

(b) If $\vec{f}$ and $\vec{g}$ are cts mappings of X into $\mathbb{R}^k$, then $\vec{f} + \vec{g}$ and $\vec{f} \cdot \vec{g}$ are continuous on X.

its a real function

Proof: use inequality $|f_j(x) - f_j(y)| \le |\vec{f}(x) - \vec{f}(y)| = \{\sum_{i=1}^k |f_i(x) - f_i(y)|^2\}^{\frac{1}{2}}$

(a) + Thm 4.9 $\Rightarrow$ (b)

Continuity and Compactness

4.13 Definition A mapping $\vec{f}$ of a set E into $\mathbb{R}^k$ is said to be bounded if there is a real number M s.t. $\|\vec{f}(x)\| \le M$ for all $x \in E$.

4.14 Theorem Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.

proof: Let $\{V_\alpha\}$ be an open cover of $f(X)$

f is cts $\Rightarrow$ $f^{-1}(V_\alpha)$ is open

Then there are finite indices $\alpha_1, \alpha_2, \cdots, \alpha_n$ s.t.

$X \subset \left(\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})\right)$ since $f(f^{-1}(E)) \subset E$ for any $E \subset Y$

$\Rightarrow f(X) \subset \left(\bigcup_{i=1}^{n} V_{\alpha_i}\right)$

4.15 Theorem If $\vec{f}$ is cts mapping of a compact metric space X into $\mathbb{R}^k$, then $\vec{f}(x)$ is closed and bounded. Thus $\vec{f}$ is bounded.

Proof: Thm 2.41

4.16 Theorem Suppose f is a cts real function on a compact metric space X, and $M = \sup_{p \in X} f(p)$, $m = \inf_{p \in X} f(p)$

Then there exist points $p, q \in X$ s.t. $f(p) = M$ and $f(q) = m$

By Thm 4.15 $\Rightarrow$ $f(X)$ is closed and bounded

$\Rightarrow M, m \in f(X)$

$\Rightarrow M = f(p)$, $m = f(q)$ for some p, q

4.17 Theorem Suppose f is a cts 1-1 mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by $f^{-1}(f(x)) = x$ $(x \in X)$ is a cts mapping of Y onto X.

Proof: $\Rightarrow$ WTS $\forall$ open set V in X $\Rightarrow$ $f(V)$ is open

Since V^c is closed subset of compact metric space X $\Rightarrow$ V^c compact

$\Rightarrow f(V^c)$ compact $\Rightarrow$ $f(V^c)$ closed

Since f is 1-1 mapping $\Rightarrow$ $[f(V)]^c = f(V^c)$

4.18 Definition Let f be a mapping of a metric space X into a metric space Y. We say that f is uniformly cts on X.

If Given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$$\forall p, q \in X, \quad d_X(p,q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon$$

4.19 Theorem Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly cts on X.

Proof: Let $\varepsilon > 0$ be given. Then

$\forall p \in X \ \exists\, \delta$ s.t. $\forall q \in N_{\frac{\delta_p}{2}}(p) \Rightarrow d_Y(f(p), f(q)) < \frac{\varepsilon}{2}$

$\exists$ finite $p_1, p_2, \cdots, p_n$ s.t.

$X \subseteq \bigcup_{i=1}^{n} N_{\frac{\delta p_i}{2}}(p_i)$

Let $\delta = \frac{1}{2}\min\{\delta p_1, \delta p_2, \cdots, \delta p_n\}$

Since $p \in X$, we know $p \in N_{\frac{\delta p_i}{2}}(p_i)$ for some i

Let $p, q \in X$, and $d_X(p,q) < \delta$, and suppose $p \in N_{\frac{\delta p_m}{2}}(p_m)$

$\Rightarrow d_X(q, p_m) \leq d_X(p,q) + d_X(p, p_m) < \delta + \frac{1}{2}\delta p_m \leq \delta p_m$

$\Rightarrow d_Y(f(p), f(q)) \leq d_Y(f(p), f(p_m)) + d_Y(f(p_m), f(q)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow f$ is uniformly cts on X

4.20 Theorem Let E be a noncompact set in $\mathbb{R}^1$. Then

(a) there exists a cts function on E which is not bounded

(b) there exists a cts and bounded function on E which has no maximum.

If, in addition, E is bounded then

(c) there exists a cts function on E which is not uniformly cts.

(a) ① E is not closed $\quad \exists\, x_0 \in E' \setminus E$

Put $f(x) = \frac{1}{x - x_0}$

② E is not bdd, put $f(x) = x$

(b) ① E is not closed

$g(x) = \frac{1}{1 + (x - x_0)^2} \quad (x \in E)$

$\sup_{x \in E} g(x) = 1$, and $g(x) < 1$

② E is not bdd

$h(x) = \frac{x^2}{1 + x^2} \Rightarrow \begin{cases} \sup h(x) = 1 \\ h(x) < 1 \end{cases}$

(c) $\begin{cases} E \text{ is not compact} \\ E \text{ is bounded} \end{cases} \Rightarrow E$ is not closed $\Rightarrow \exists\, x_0 \in E' \setminus E$

$\Rightarrow f(x) = \frac{1}{x - x_0}$

Continuity and Connectedness

4.22 Theorem If f is a cts mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then $f(E)$ is connected.

Proof:

4.23 Theorem Let f be a cts real function on the interval $[a,b]$. If $f(a) < f(b)$ and if c is a number s.t. $f(a) < c < f(b)$, then there exists a point $x \in (a,b)$ s.t. $f(x) = c$

Discontinuities

4.25 Definition Let f be defined on (a,b). Consider any point x s.t. $a \le x < b$ we write $f(x+) = q$

Chapter 5 Differentiation

The derivative of a real function

5.1 Definition $f:[a,b]\to\mathbb{R}$. For any $x\in[a,b]$ form the quotient

$$\phi(t)=\frac{f(t)-f(x)}{t-x}\qquad t\in[a,b]\setminus\{x\}$$

and define $f'(x)=\lim_{t\to x}\phi(t)$

① local property

② $\lim_{t\to x^+}\phi(t)$ & $\lim_{t\to x^-}\phi(t)$: right and left derivatives at x

③ if f is diff at $\forall x\in[a,b]$ $\Rightarrow$ f is diff on $[a,b]$

5.2 Theorem $f:[a,b]\to\mathbb{R}$, if f is diff at $x\in[a,b]$, then f is cts at x.

proof:

$$f(t)-f(x)=\underbrace{\boxed{\frac{f(t)-f(x)}{t-x}}}_{f_1(t)}\ \underbrace{\boxed{(t-x)}}_{f_2(t)}$$

$$\left.\begin{aligned}&\lim_{x\to t}f_1(t)=f'(x)\\&\lim_{x\to t}f_2(t)=0\end{aligned}\right\}\ \text{Thm 4.4}\Rightarrow\ \lim_{x\to t}f_1(t)\cdot f_2(t)=0\ \Rightarrow\ \lim_{t\to x}f(t)=f(x)$$

$\Rightarrow$ by Thm 4.6 $\Rightarrow$ f is cts at x

5.3 Theorem suppose $f,g:[a,b]\to\mathbb{R}$, and are diff at $x\in[a,b]$.
Then $f+g$, fg, and f/g are diff at $x\in[a,b]$, and

(a) $(f+g)'(x)=f'(x)+g'(x)$;

(b) $(fg)'(x)=f'(x)g(x)+f(x)g'(x)$;

(c) $\left(\frac{f}{g}\right)'(x)=\dfrac{g(x)f'(x)-g'(x)f(x)}{g^2(x)}$.

proof

(a) $(f+g)'(x)=\lim_{t\to x}\dfrac{f(x)+g(x)-(f(t)+g(t))}{x-t}=\lim_{x\to t}\left(\dfrac{f(x)-f(t)}{x-t}+\dfrac{g(x)-g(t)}{x-t}\right)=f'(x)+g'(x)$

(b) $(fg)'(x)=\lim_{t\to x}\dfrac{(fg)(t)-(fg)(x)}{t-x}=\lim_{t\to x}\dfrac{f(t)\cdot g(t)-f(x)\cdot g(x)}{t-x}$

$=\lim_{t\to x}\left[\dfrac{f(t)g(t)-f(t)g(x)}{t-x}+\dfrac{f(t)g(x)-f(x)g(x)}{t-x}\right]$

by Thm 5.2

$=\lim_{t\to x}\left[f(t)\cdot\dfrac{g(t)-g(x)}{t-x}+g(x)\cdot\dfrac{f(t)-f(x)}{t-x}\right]$

by Thm 4.4

$=f(x)g'(x)+g(x)\cdot f'(x)$

(c) Let $h(t)=\frac{f(t)}{g(t)}$, since $g(t)\neq 0$ and g is cts at x

We know $\exists\ \delta>0$

s.t. $g(t)\neq 0$ for $t\in(x-\delta, x+\delta)$

Then $h'(x)=\lim_{t\to x}\frac{h(t)-h(x)}{t-x} = \lim_{t\to x}\frac{1}{g(t)g(x)}\left[\frac{f(t)g(x)-f(x)g(t)}{t-x}\right]$

$= \lim_{t\to x}\frac{1}{g(t)g(x)}\left[\frac{f(t)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(t)}{t-x}\right]$

$= \frac{1}{g^2(x)}\left[g(x)f'(x)-f(x)g'(x)\right]$

5.5 Theorem (Chain Rule) Suppose f is cts on $[a,b]$, $f'(x)$ exists at some point $x\in[a,b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If

$$h(t)=g(f(t)) \quad (a\leq t\leq b)$$

then h is differentiable at x, and

$$h'(t)=g'(f(t))\cdot f'(t).$$

Proof: We know $f:[a,b]\to\mathbb{R}$, $f([a,b])\subseteq I$, $g: I\to\mathbb{R}$

Construct $u(t)=\begin{cases}\frac{f(t)-f(x)}{t-x}-f'(x) & t\neq x\\ 0 & t=x\end{cases}$

Let $y=f(x)$

Construct $v(s)=\begin{cases}\frac{g(s)-g(y)}{s-y}-g'(y) & s\neq y\\ 0 & s=y\end{cases}$

(1) $f(t)-f(x)=(t-x)[u(t)+f'(x)]$

(2) $g(s)-g(y)=(s-y)[v(s)+g'(y)]$

$\frac{h(t)-h(x)}{t-x}=\frac{g(f(t))-g(f(x))}{t-x}$

$\overset{(2)}{\Rightarrow}\ =\frac{[f(t)-f(x)][v(f(t))+g'(f(x))]}{t-x}$

$\overset{(1)}{\Rightarrow}\ =\frac{(\cancel{t-x})[u(t)+f'(x)][v(f(t))+g'(f(x))]}{\cancel{t-x}}$

$\lim_{t\to x}u(t)=0$

By Thm 5.2 $\lim_{t\to x}f(t)=f(x)$ Then $\lim_{t\to x}v(f(t))=v(f(x))=0$

Then $h'(x)=\lim_{t\to x}\frac{h(t)-h(x)}{t-x}=f'(x)\cdot g'(f(x))$

Mean Value Theorem

5.7 Definition Let f be a real function defined on a metric space X. We say that f has a local maximum at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \le f(p)$ for all $q \in X$ with $d(p,q) < \delta$. [Local minimum are likewise]

5.8 Theorem Let f be defined on $[a,b]$; If f has a local maximum at a point $x \in (a,b)$, and if $f'(x)$ exists, then $f'(x) = 0$.

proof: choose δ in accordance with definition 5.7 s.t.

$$a < x-\delta < x < x+\delta < b$$

If $x-\delta < t < x$, then $\frac{f(t)-f(x)}{t-x} \ge 0 \Rightarrow f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x} \ge 0$

If $x < t < x+\delta$, then $\frac{f(t)-f(x)}{t-x} \le 0 \Rightarrow f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x} \le 0$

$\Rightarrow f'(x) = 0$

5.9 Theorem If f and g are cts real functions on $[a,b]$ which are differentiable in (a,b), then there is a point $x \in (a,b)$ at which

$$[f(b)-f(a)]g'(x) = [g(b)-g(a)]f'(x)$$ [generalized mean value Theorem]

proof: put $h(t) = [f(b)-f(a)]g(t) - [f(b)-f(a)]f(t)$

since it's a linear combination

① h is cts on $[a,b]$

② h is differentiable on $[a,b]$

WTS $h'(x) = 0$ for some $x \in (a,b)$

$h(a) = g(a)f(b) - g(b)f(a) = h(b)$

(I) $h(t) > h(a)$ for some $t \in (a,b)$

$\Rightarrow$ by Thm 4.16 h attains its max on $[a,b]$, say at point $p \in (a,b)$

so $h(p) \ge h(t) > h(a) = h(b)$.

Then by Thm 5.8 $h'(p) = 0$

(II) $h(t) < h(a)$ for some $t \in (a,b)$

max $\leftarrow$ min

(III) $h(t) = f(a)$ $\forall t \in (a,b)$ $\Rightarrow h'(t) = 0$

5.10 Theorem (MVT) If f is a cts real function on $[a,b]$ which is differentiable in (a,b), then there is a point $x \in (a,b)$, at which

$$f(b) - f(a) = (b-a) f'(x)$$

proof: Take $g(x) = x$ in Thm 5.9

5.11 Theorem Suppose f is differentiable in (a,b)

(a) If $f'(x) \geq 0$ for all $x \in (a,b)$, then f is monotonically increasing;

(b) If $f'(x) = 0$ for all $x \in (a,b)$, the f is constant;

(c) If $f'(x) \leq 0$ for all $x \in (a,b)$, the f is monotonically decreasing.

proof: use MVT

The continuity of derivatives

5.12 Theorem Suppose f is a real differentiable function on $[a,b]$ and suppose $f'(a) < \lambda < f'(b)$, then there is a point $x \in (a,b)$ s.t. $f'(x) = \lambda$.

[similar, if $f'(a) > f'(b)$]

Proof: put $g(t) = f(t) - \lambda t$ $\rightarrow$ g is differentiable

$g'(t) = f'(t) - \lambda$

WTS: $\exists x \in (a,b)$ s.t. $g'(x) = 0$

Already known $g'(a) < 0$ and $g'(b) > 0$

claim: $\exists t_1, t_2 \in (a,b)$ s.t. $g(t_1) < g(a)$ and $g(t_2) < g(b)$

suppose not $\Rightarrow g(t_1) \geq g(a) \quad \forall t_1 \in (a,b)$

$$\Leftrightarrow \frac{g(t_1) - g(a)}{t_1 - a} \geq 0 \quad \forall t_1 \in (a,b)$$

by Thm 4.2 $g'(a) = \lim_{t_1 \to a} \frac{g(t_1) - g(a)}{t_1 - a} \geq 0 \Rightarrow$ Contradiction

By Thm 4.16 $m = \inf_{x \in [a,b]} g(x)$ exists and $m = g(p)$ for some $p \in [a,b]$

$\Rightarrow m \leq g(t_1) < g(a)$

similarly $m \leq g(t_2) < g(b)$

$p \in (a,b)$ so by Thm 5.8 $g'(p) = 0$

[f contains all number between $(f'(a), f'(b))$

$\rightarrow$ doesn't mean that $f'(x)$ is cts]

Corollary If f is differentiable on $[a,b]$, then f' cannot have and simple discontinuities on $[a,b]$. $f'(x-) \neq f'(x+)$

(But f' may very well have discontinuities of second kind) $f'(x-)$ or $f'(x+)$ doesn't exist.

Proof: Suppose f' was a discontinuity of the first kind at point $x \in (a,b)$

$\Rightarrow f'(x-), f'(x+)$ exist WLOG $f'(x-) = s < t = f'(x)$

Let $\{x_n\}$ be a sequence in (a,x)

s.t. $x_n \to x$

$$f'(x-) = \lim_{n\to\infty} f'(x_n)$$

Let $\lambda \in (s,t)$ Then $\exists\, \varepsilon > 0$

s.t. $(\lambda-\varepsilon, \lambda+\varepsilon) \subseteq (s,t)$

Then $s+\varepsilon < \lambda < t-\varepsilon$

There exists $N \in \mathbb{N}$

s.t. $f'(x_n) \in (s-\varepsilon, s+\varepsilon)\ \forall n \geq N$

So that

$$f'(x_n) < s+\varepsilon < \lambda < t = f'(x)$$

$\Rightarrow$ By Thm 5.12

$\exists\, y_n \in (x_n, x)$

s.t. $f'(y_n) = \lambda$

$y_n < x$, $y_n \to x$

Then $\underbrace{\lim_{n\to\infty} f'(y_n)}_{\lambda} = \underbrace{f'(x-)}_{s}$

$\Rightarrow$ contradiction

L'Hospital's Rule

5.13 Theorem Suppose f and g are real and differentiable in (a,b), and $g'(x) \neq 0$ for all $x \in (a,b)$, where $-\infty \leq a < b \leq +\infty$. Suppose

$$\frac{f'(x)}{g'(x)} \to A \quad \text{as } x \to a.$$

If $\quad f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, $(x \to b)$

or if $\quad g(x) \to +\infty$ as $x \to a$, $(-\infty)$ $(x \to b)$

then

$$\frac{f(x)}{g(x)} = A$$

proof: Suppose $f, g: [a,b] \to \mathbb{R}$, f and g are differentiable on (a,b), $g'(x) \neq 0$ for all $x \in (a,b)$, $-\infty \le a < b \le +\infty$.

Suppose (*) $\frac{f'(x)}{g'(x)} \to A$ as $x \to a$ $(A \in \bar{\mathbb{R}})$

Also (I) $g(x), f(x) \to 0$, as $x \to a$

(II) $g(x) \to \infty$, as $x \to a$

Then $\frac{f(x)}{g(x)} \to A$, as $x \to a$

WTS: If $q > A$, then $\exists\, c_2 > a$ s.t. $\frac{f(x)}{g(x)} < q$ for $x \in (a, c_2)$ (**)

If $p < A$, then $\exists\, c_2 > a$ s.t. $\frac{f(x)}{g(x)} > p$ for $x \in (a, c_2)$

Suppose $q > A$, we assume $-\infty \le A < \infty$, and let $A < r < q$

by (*) $\exists\, c > a$ s.t. $\frac{f'(x)}{g'(x)} \le r$ $\forall\, x \in (a, c)$

by Thm 5.09 if $a < x < y < c$, then $\exists\, t \in (x, y)$ s.t.

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} \le r \quad (***)$$

If (I) holds $f(x) \to 0$, $g(x) \to 0$ as $x \to a$

$\frac{f(x)}{g(x)} \le r \to$ follows from $x \to a$ limit of (***)

If (II) holds $g(x) \to +\infty$ as $x \to a$

$\exists\, c_1 > a$ s.t. ① $g(x) > 0$ ② $g(x) > g(y)$ if $x \in (a, c_1)$

Multiply both sides of (***) by $\frac{g(x) - g(y)}{g(x)}$

$$\frac{f(x) - f(y)}{g(x)} \le r \cdot \frac{g(x) - g(y)}{g(x)}$$

$$\Leftrightarrow \quad \frac{f(x)}{g(x)} \le \boxed{r - r \frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}} \text{ RHS}$$

already known: $A < r < q$, $g(x) \to \infty$ as $x \to a$

If $c_1 > c_2 > a$ is small enough and $x \in (a, c_2)$, then RHS $< q$.

Then we are done, b/c $\frac{f(x)}{g(x)} < q$ #

It's similar to prove $\frac{f(x)}{g(x)} > p$ $(a < x < c_3)$

Derivatives of Higher Order

5.14 Definition If f has a derivative f' on an interval, and if f' is itself differentiable, we denote the derivative of f' by f'', and call f'' the second derivative of f. Continuing in this manner, we can obtain:

$$f', f'', f^{(3)}, \cdots, f^{(n)},$$

In order for $f^{(n)}(x)$ to exist at a point x, $f^{(n-1)}(t)$ must exist in a neighbourhood of x (or in a one-side neighborhood, if x is an endpoint of an interval), and $f^{(n-1)}(x)$ must be differentiable at x ($\in N_r(x)$).

Taylor's Theorem

5.15 Theorem Suppose $f:[a,b]\to\mathbb{R}$, n is a positive integer, $f^{(n-1)}$ is cts on $[a,b]$, $f^{(n)}(t)$ exists for every $t\in(a,b)$. Let α,β be distinct points of $[a,b]$, and define:

$$P(t)=\sum_{k=0}^{n-1}\frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k.$$

$$\left(\Leftrightarrow \quad P(t)=\frac{f^{(0)}(\alpha)}{0!}(t-\alpha)^0+\frac{f'(\alpha)}{1!}(t-\alpha)^1+\cdots+\frac{f^{(n-1)}(\alpha)}{(n-1)!}(t-\alpha)^{n-1}\right)$$

Then there exists a point x between α and β s.t.

$$f(\beta)=P(\beta)+\frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n \quad \rightsquigarrow \text{estimate error}$$

$(n=1 \Leftrightarrow \text{MVT} \quad : f(\beta)=f(\alpha)+f'(x)(\beta-\alpha))$

Proof: Let $M\in\mathbb{R}$ be given s.t. $f(\beta)=P(\beta)+M(\beta-\alpha)^n$

Put $g(t)=f(t)-P(t)-M(t-\alpha)^n \quad t\in[a,b]$

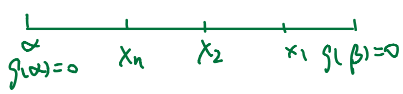
WTS $n!M=f^{(n)}(x)$ for some $x\in(\alpha,\beta)$

$g^{(n)}(t)=f^{(n)}(t)-n!M \quad (a<t<b)$

WTS $g^{(n)}(x)=0$ for some $x\in(\alpha,\beta)$

(1) $g^{(k)}(\alpha)=f^{(k)}(\alpha) \quad k=(0,1,2,\cdots,n-1)$

(2) $g(\beta)=0$ by choice of M



By MVT $\exists x_1\in(\alpha,\beta)$ $g'(x_1)=0$

$g'(\alpha)=0=g'(x_1) \Rightarrow$ By MVT $\exists$ $x_2\in(\alpha,x_1)$ s.t. $g''(x_2)=0$

$\Rightarrow$ After n steps $g^{(n)}(x_n)=0$ for some $x_n\in(\alpha,x_{n-1})\subseteq(\alpha,\beta)$

Differentiation of Vector-valued Functions

5.16 Remarks If f_1 and f_2 are real and imaginary parts of f. that is, if

$$f(t) = f_1(t) + i\cdot f_2(t)$$

for $a \le t \le b$, where $f_1(t)$ and $f_2(t)$ are real, then we clearly have

$$f'(x) = f_1'(x) + i f_2'(x);$$

also, f is differentiable at x if and only if both f_1 and f_2 are differentiable at x.

Definition: Suppose $\vec{f}: [a,b] \to \mathbb{R}^2$, $x \in [a,b]$, $\varphi(t) = \frac{\vec{f}(t) - \vec{f}(x)}{t-x}$ $t \ne x$ (scalar)

We say $\vec{f}$ is differentiable at x, if $\lim_{t\to x} \varphi(t)$ exists

and we write $\vec{f}'(x) = \lim_{t\to x} \varphi(t)$

Suppose $\vec{f}(x) = (f_1(x), f_2(x), \cdots, f_k(x))$

$$\varphi(t) = \left(\frac{f_1(t)-f_1(x)}{t-x}, \frac{f_2(t)-f_2(x)}{t-x}, \cdots, \frac{f_k(t)-f_k(x)}{t-x}\right)$$

$\lim_{t\to x} \varphi(t)$ exists $\Leftrightarrow$ $\lim_{t\to x} \frac{f_i(t)-f_i(x)}{t-x}$ exists for $i=1,2,\cdots,k$

$\Leftrightarrow$ f_i are all differentiable for $i=1,2,\cdots,k$

$\Rightarrow \vec{f}'(x) = (f_1'(x), f_2'(x), f_3'(x), \cdots, f_k'(x))$

$f \cdot g$

$\downarrow$

$\vec{f}\vec{g}$

Also True

$\boxed{\text{Thm } 5.2,\ 5.3(a)(b)}$, 4.3

Expon function

$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $z \in \mathbb{C}$ $\quad \left(E(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = e \right)$

(*) $E(z)\cdot E(w) = E(z+w)$, $E(z)$ is absolutely convergent

By Thm 3.50 $\Rightarrow E(z)\cdot E(w) = \sum_{n=0}^{\infty} C_n$

Binom Thm

$$C_n = \sum_{k=0}^{n} \frac{z^k}{k!}\cdot\frac{w^{n-k}}{(n-k)!} = \frac{1}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\cdot z^k\cdot w^{n-k} = \frac{1}{n!}\sum_{n=0}^{n}\binom{n}{k}\cdot z^k \cdot w^{n-k} = \frac{(z+w)^n}{n!}$$

$$\Rightarrow E(z)\cdot E(w) = \sum_{n=0}^{\infty}\frac{(z+w)^n}{n!} = E(z+w)$$

$$\Rightarrow E(z)\cdot E(-z) = E(0) = 1$$

Then $E(z) \ne 0 \ \forall z \in \mathbb{C}$

$$\frac{E(z+h)-E(z)}{h} \overset{(*)}{=} \frac{E(z)\cdot E(h) - E(z)}{h} = E(z)\,\frac{E(h)-1}{h} \quad (h \ne 0)$$

$$E(h) - 1 = \sum_{n=0}^{\infty}\frac{h^n}{n!} - 1 = \sum_{n=1}^{\infty}\frac{h^n}{n!}$$

$$\frac{E(h)-1}{h} = \sum_{n=1}^{\infty}\frac{h^{n-1}}{n!} = 1 + \sum_{n=2}^{\infty}\frac{h^{n-1}}{n!}$$

$$\left|\frac{E(h)-1}{h} - 1\right| \le \left|\sum_{n=2}^{\infty}\frac{h^{n-1}}{n!}\right| \le \sum_{n=2}^{\infty}\frac{|h|^{n-1}}{n!}$$

Let $h_m \in \mathbb{C}$, $h_m \to 0$, $h_m \neq 0$

For large m, $|h_m| \leq 1 \Rightarrow |h_m|^{n-1} \leq |h_m|$

By comparison test $\Rightarrow \lim_{n\to\infty} \left| \frac{E(h_m)-1}{h_m} - 1 \right| = 0$

Conclude $\Rightarrow \lim_{n\to\infty} \frac{E(h_m)-1}{h_m} = 1$

$$\frac{E(z+h)-E(z)}{h} = E(z) \cdot \frac{E(h)-1}{h} \to E(z) \text{ as } h \to 0$$

$\Rightarrow E(z)$ is complex differentiable (Holomorphic / analytic)

$\Rightarrow E(z)$ is Entire [a complex-valued function that is holomorphic at all finite points over the whole complex plane].

stronger than real differentiable.

Real case: Say now z and h are both real

$E(x)$ is differentiable and $E'(x) = E(x)$, $E(x)$ is smooth, $E^{(k)}(x) = E(x)$

Property ① If $x \geq 0$ then $E(x) > 0$, $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$E(x) \cdot E(-x) = E(0) = 1$ by $(*)$

$\Rightarrow E(-x) = \frac{1}{E(x)} > 0$

$\Rightarrow E(x) > 0$

$\Rightarrow E'(x) = E(x) > 0$

$\Rightarrow E$ is strictly increasing

② If $E(x) \geq 1 + x \Rightarrow \lim_{x\to\infty} E(x) = \infty$

By $(*)$ $E(-x) = \frac{1}{E(x)} \to 0$ as $x \to \infty$

Property: $e^x = E(x)$

Proof: $E(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = e$

Then $E(\frac{1}{n}) > 0$

$\Rightarrow \left[E(\frac{1}{n})\right]^n = E(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}) = E(1) = e$

$\Rightarrow E(\frac{1}{n}) = e^{\frac{1}{n}}$, also $E(-\frac{1}{n}) = e^{-\frac{1}{n}}$ (use $(*)$)

Then $E(\pm\frac{m}{n}) = \left[E(\pm\frac{1}{n})\right]^m = (e^{\pm\frac{1}{n}})^m = e^{\pm\frac{m}{n}}$

Know: If $x \geq y$, then $e^x \geq e^y$

Let $x \in \mathbb{R}$, $\{q_n\} \uparrow x^-$, $\{p_n\} \downarrow x^+$, $p_n, q_n \in \mathbb{Q}$

$$\left.\begin{array}{l} (1)\ e^x \geq e^{q_n} = E(q_n) \xrightarrow{n\to\infty} E(x) \Rightarrow e^x \geq E(x) \\ (2)\ e^x \leq e^{p_n} = E(p_n) \xrightarrow{n\to\infty} E(x) \Rightarrow e^x \leq E(x) \end{array}\right\} \text{Thm 5.2}$$

$E(ix) = C(x) + iS(x) \quad x \in \mathbb{R}$

$$\sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k}}{(2k)!} \quad C(x) = \cos x} + i \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+1}}{(2k+1)!} \quad S(x) = \sin x}$$

$$|E(ix)|^2 = E(ix) \cdot \overline{E(ix)}$$

$$\overline{\sum_{n=0}^{\infty} \frac{(ix)^n}{n!}} = \sum_{n=0}^{\infty} \frac{\overline{(ix)^n}}{n} = \sum_{n=0}^{\infty} \frac{(-ix)^n}{n} = E(-ix)$$

$$\Rightarrow \boxed{C^2(x) + S^2(x) = 1}$$

know: $\displaystyle\lim_{h \to 0} \frac{E(ix+ih) - E(ix)}{ih} = E(ix)$

$$\frac{1}{i} \lim_{h \to 0} \frac{C(x+h) + iS(x+h) - C(x) - iS(x)}{h} = C(x) + iS(x)$$

C, S are differentiable

$$\frac{C'(x) + iS'(x)}{i} = C(x) + iS(x)$$

$$\Rightarrow \boxed{C'(x) = -S'(x), \quad S'(x) = C(x)}$$

Example 5.17

Assume: $S(x)$, $C(x)$ are 2π-periodic

$f(x) = e^{ix} = \cos x + i \sin x$

$f: \mathbb{R} \to \mathbb{C}$

$f(0) = 1 = f(2\pi)$

$f'(x) = -\sin x + i\cos x$ $\quad \Rightarrow$ MVT fails in this case: $f(2\pi) - f(0) = 0$ but $|f'(x)| = 1 \ \forall x \in \mathbb{R}$

$|f'(x)| = 1$

Thm 5.19 $\vec{f}: [a,b] \to \mathbb{R}^k$, $\vec{f}$ is cts, $\vec{f}$ is differentiable on (a,b)

Then $\exists x \in (a,b)$ s.t. $\|\vec{f}(b) - \vec{f}(a)\| \leq (b-a)\|\vec{f}'(x)\|$

Proof: Put $\vec{z} = \vec{f}(b) - \vec{f}(a)$

$\varphi(t) = \vec{z} \cdot \vec{f}(t) \iff \begin{cases} \varphi(t) = z_1 f_1(t) + z_2 f_2(t) + \cdots + z_k \cdot f_k(t) \\ \vec{f}(t) = (f_1(t), f_2(t), \cdots, f_k(t)) \end{cases}$

① $\varphi: [a,b] \to \mathbb{R}$, φ is cts (Thm 4.10)

② φ is differentiable on (a,b), $\varphi'(t) = z_1 f_1'(t) + z_2 f_2'(t) + \cdots + z_k f_k'(t) = \vec{z} \cdot \vec{f}'(t)$

By MVT $\Rightarrow$ $\varphi(b) - \varphi(a) = (b-a) \cdot \varphi'(x)$ for some $x \in (a,b)$

$$\text{Left} = \vec{z}\cdot\vec{f}(b) - \vec{z}\cdot\vec{f}(a)$$

$$= \vec{z}\cdot(\vec{f}(b) - \vec{f}(a))$$

$$= \vec{z}\cdot\vec{z} = \|\vec{z}\|^2$$

$$|\vec{z}\cdot\vec{f}'(x)| \le \|\vec{z}\|\|\vec{f}'(x)\| \quad (\text{Cauchy-Schwarz})$$

$$\Rightarrow \|z\|^2 \le (b-a)\|\vec{z}\|\|\vec{f}'(x)\|$$

$$\Rightarrow \|z\| \le (b-a)\|\vec{f}'(x)\|$$

Chapter 6 The Riemann-Stieltjes Integral

Definition and existence of the integral

6.1 Definition Let $[a,b]$ be given integral, By a partition P of $[a,b]$ we mean a finite set of points $x_0, x_1, \cdots, x_n$, where

$$a = x_0 \le x_1 \le \cdots \le x_{n-1} \le x_n = b$$

We write $\quad \Delta x_i = x_i - x_{i-1} \quad (i=1, \cdots, n)$

Let $f: [a,b] \to \mathbb{R}$ be bounded

$\Rightarrow \exists\, m, M \in \mathbb{R}$ $(m \le M)$ s.t. $m \le f(x) \le M$ for $x \in [a,b]$

$M_i = \sup \{f(x), x \in [x_{i-1}, x_i]\}$

$m_i = \inf \{f(x), x \in [x_{i-1}, x_i]\}$

$M \ge M_i \ge m_i \ge m$

$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$: Upper Riemann Sum

$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$: Lower Riemann Sum

$$m(b-a) = \sum_{i=1}^{n} m \Delta x_i \le L(P,f) \le U(P,f) \le \sum_{i=1}^{n} M \Delta x_i = M(b-a)$$

define : Upper Riemann Integral

$$\overline{\int_a^b} f(x)\,dx := \inf_P U(P,f)$$

Lower Riemann Integral

$$\underline{\int_a^b} f(x)\,dx := \sup_P L(P,f)$$

Definition : We say that a bounded function f on a compact interval $[a,b]$ is Riemann integrable if

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$$

$$\boxed{f \in \mathcal{R}([a,b])}$$

Riemann-Stieltjes integral (Stieltjes integral)

6.2 Definition Let α be a monotonically increasing function on $[a,b]$ (since $\alpha(a)$ and $\alpha(b)$ are finite, it follows that α is bounded on $[a,b]$). Corresponding to each partition P of $[a,b]$, we write

$$\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \qquad P = \{x_0, x_1, \cdots, x_n\}$$

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \qquad \overline{\int_a^b} f\,d\alpha = \inf U(P,f,\alpha)$$

$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i \qquad \underline{\int_a^b} f\,d\alpha = \sup L(P,f,\alpha)$$

If $\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha \Leftrightarrow \int_a^b f(x)\,d\alpha(x)$

We write $f \in \mathcal{R}(\alpha, [a,b])$

6.3 Definition We say that the partition P^* is a refinement of P, if $P^* \supset P$. Given two partitions, P_1 and P_2, we say that P^* is their common refinement if $P^* = P_1 \cup P_2$.

6.4 Theorem If P^* is a refinement of P, then $L(P,f,\alpha) \le L(P^*,f,\alpha)$ and $U(P^*,f,\alpha) \le U(P,f,\alpha)$.

proof: Suppose that P^* contains just one point more than P, Let this extra point be x^*, and suppose $x_{i-1} < x^* < x_i$, where x_{i-1} and x_i are two consecutive points of P. Put

$$w_1 = \inf f(x) \quad (x_{i-1} \le x \le x^*), \quad w_2 = \inf f(x) \quad (x^* \le x < x_i)$$

and $m_i = \inf f(x) \quad (x_{i-1} \le x \le x_i)$

Clearly $w_1 \ge m_i$, $w_2 \ge m_i$

Hence $L(P^*,f,\alpha) - L(P,f,\alpha)$

$$= w_1(\alpha(x^*) - \alpha(x_{i-1})) + w_2(\alpha(x_i) - \alpha(x^*)) - m_i(\alpha(x_i) - \alpha(x_{i-1}))$$

$$= (w_1 - m_i)(\alpha(x^*) - \alpha(x_{i-1})) + (w_2 - m_i)(\alpha(x_i - x^*)) \ge 0 \quad (\alpha \text{ monotonically } \uparrow)$$

If P^* contains k points more than P, we repeat k times

$\Rightarrow L(P,f,\alpha) \le L(P^*,f,\alpha)$

Similarly $U(P,f,\alpha) \ge U(P^*,f,\alpha)$

6.5 Theorem $\underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha$

proof: Let P^* be common refinement of partitions P_1 and P_2.

By Thm 6.4 $\Rightarrow$ $L(P_1,f,\alpha) \le L(P^*,f,\alpha) \le U(P^*,f,\alpha) \le U(P_2,f,\alpha)$

$\Rightarrow L(P_1,f,\alpha) \le U(P_2,f,\alpha)$

$\underline{\int} f\,d\alpha = \sup L(P,f,\alpha)$, $\overline{\int} f\,d\alpha = \inf U(P,f,\alpha)$

$\Rightarrow$ $\underline{\int} f\,d\alpha \le U(P_2,f,\alpha)$

$\Rightarrow$ $\underline{\int} f\,d\alpha \le \overline{\int} f\,d\alpha$

6.6 Theorem $f \in \mathcal{R}(\alpha)$ on $[a,b]$ if and only if for every $\varepsilon > 0$ there exists a partition P s.t. $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ $(*)$

$(\Leftarrow)$ Let $\varepsilon > 0$ be given

Let P be a partition satisfying $(*)$

by Thm 6.5 $\Rightarrow$ $L(P,f,\alpha) \le \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha \le U(P,f,\alpha)$

$\Rightarrow \left| \underline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f\,d\alpha \right| < \varepsilon \Rightarrow f \in \mathcal{R}(\alpha)$

$(\Rightarrow)$ Suppose $f \in \mathcal{R}(\alpha)$

Then $\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha$

Let $\varepsilon > 0$ be given

(1) $\exists P_1$ s.t. $L(P_1,f,\alpha) > \underline{\int_a^b} f\,d\alpha - \frac{\varepsilon}{2}$

(2) $\exists P_2$ s.t. $U(P_2,f,\alpha) < \overline{\int_a^b} f\,d\alpha + \frac{\varepsilon}{2}$

Let $P = P_1 \cup P_2$

by Thm 6.4 $\Rightarrow$ $U(P,f,\alpha) \le U(P_2,f,\alpha) < \overline{\int_a^b} f\,d\alpha + \frac{\varepsilon}{2} = \underline{\int_a^b} f\,d\alpha - \frac{\varepsilon}{2} + \varepsilon$

$< L(P_1,f,\alpha) + \varepsilon < L(P,f,\alpha) + \varepsilon$

$\Rightarrow (*)$ holds for P.

6.7 Theorem $\quad U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon \quad (*)$

(a) If $(*)$ holds for some P and some ε, then $(*)$ holds (with the same ε) for every refinement of P.

thm 6.4.

(b) If $(*)$ holds for $P=\{x_0, \dots, x_n\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta x_i < \varepsilon$$

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta x_i \le \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

(c) If $f \in \mathcal{R}(\alpha)$ and the hypothesis of (b) hold, then

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha \right| < \varepsilon$$

It is obvious that $\quad L(P,f,\alpha) \le \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \le U(P,f,\alpha)$

and $L(P,f,\alpha) \le \int f\,d\alpha \le U(P,f,\alpha)$

and $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$

$$\Rightarrow \left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha \right| < \varepsilon$$

6.8 Theorem $\quad$ If f is continuous on $[a,b]$ then $f \in \mathcal{R}(\alpha)$ on $[a,b]$.

By Thm 4.19 (compact) $\Rightarrow$ f is uniformly continuous.

Let $\eta > 0$ be such that $[\alpha(b) - \alpha(a)] \cdot \eta < \varepsilon$

Since f is uniformly continuous

$\exists\, \delta > 0$ s.t.

$|f(x) - f(y)| < \eta$, if $|x-y| < \delta$

Let P be any partition of $[a,b]$ s.t. $\Delta\alpha_i < \delta$.

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i \le \sum_{i=1}^{n} \eta\Delta\alpha_i \le \eta\sum_{i=1}^{n}\Delta\alpha_i$$

$$= \eta[\alpha(b) - \alpha(a)] < \varepsilon$$

By Thm 6.6 $\quad f \in \mathcal{R}(\alpha)$

6.9 Theorem If f is monotonic on $[a,b]$, and if α is continuous on $[a,b]$, then $f \in \mathcal{R}(\alpha)$. (assume: α is monotonic)

Proof: Let $\varepsilon > 0$ be given, suppose f is monotonically increasing

Let P be such that $\Delta \alpha_i = \frac{\alpha(b)-\alpha(a)}{n}$ $(i = 1, 2, \cdots, n)$

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} \underbrace{\Delta\alpha_i}_{\frac{\alpha(b)-\alpha(a)}{n}} (M_i - m_i) = \frac{\alpha(b)-\alpha(a)}{n} \sum_{i=1}^{n} (M_i - m_i)$$

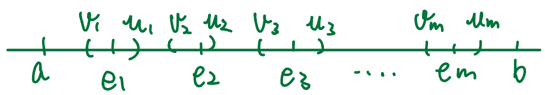
$$= \frac{\alpha(b)-\alpha(a)}{n} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))$$

$$= \frac{\alpha(b)-\alpha(a)}{n} (f(b) - f(a)) < \varepsilon$$

If $n >> 1$.

6.10 Theorem Suppose f is bounded on $[a,b]$, f has only finitely many points of discontinuity on $[a,b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \mathcal{R}(\alpha)$.

Proof: Because f is bounded, $\exists M > 0$ s.t. $\sup_{x \in [a,b]} |f(x)| \le M$



e_i : f : discontinuous
α : continuous.

Let u_i, v_i be such that

(*) $a < v_1 < e_1 < u_1 < \cdots < v_m < e_m < u_m < b$

(**) $\sum_{i=1}^{m} [\alpha(u_i) - \alpha(v_i)] < \varepsilon$ (using α continuous at e_i $\forall i$)

$K = [a,b] \setminus \bigcup_{i=1}^{m} (v_i, u_i) \Rightarrow K$ is compact.

since f is cts on K $\xRightarrow{\text{by Thm 4.19}}$ f is uniformly cts on K

Then $\exists\ \delta > 0$ st. $|f(x) - f(y)| < \varepsilon$ if $|x-y| < \delta$, $x, y \in K$

P is constructed as follows

(1) u_i, v_i belong to P

(2) Put x_i's such that $\Delta x_i < \delta$ whenever $x_i \ne u_j$ for any j

(3) No points in (u_i, v_i) belong to P

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} \underbrace{[M_i - m_i]}_{<\delta} \cdot \Delta\alpha_i$$

$$\le \sum_{j=1}^{m} 2M[\alpha(u_j) - \alpha(v_j)] + \sum_{i=1}^{n} \varepsilon \cdot [\alpha(x_i) - \alpha(x_{i-1})] \le 2M\varepsilon + \varepsilon[\alpha(b) - \alpha(a)]$$

Then by Thm 6.6 $f \in \mathcal{R}(\alpha)$

6.11 Theorem Suppose $f \in \mathcal{R}(\alpha)$ on $[a,b]$, $m \le f \le M$, ϕ is continuous on $[m, M]$, and $\underline{h(x) = \phi(f(x))}$ on $[a,b]$. Then $h \in \mathcal{R}(\alpha)$ on $[a,b]$.

proof: f is cts on $[a,b]$ $\Rightarrow$ $f([a,b])$ is compact

φ is cts $\Rightarrow$ $\varphi(t)$ is uniformly cts. (by Thm 4.19)

$\Rightarrow \exists\ \delta > 0$ s.t. $\delta < \varepsilon$ and $|\varphi(s) - \varphi(t)| < \varepsilon$ if $|s-t| \le \delta$ and $s,t \in [m, M]$

Since $f \in \mathcal{R}(\alpha)$, there is a partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$

s.t. $U(P, f, \alpha) - L(P, f, \alpha) < \delta^2$ (*)

Let M_i, m_i be the sup & inf for f (as is defined in def 6.1)

and M_i^*, m_i^* be the sup & inf for h.

Divide the numbers into 2 classes: $i \in A$, if $M_i - m_i < \delta$,
$i \in B$, if $M_i - m_i \ge \delta$.

① For $i \in A$, by choices of δ $\Rightarrow$ $M_i^* - m_i^* < \varepsilon$

② For $i \in B$, $M_i^* - m_i^* = 2K$, where $K = \sup|\phi(t)|$, $t \in [m, M]$

$$\delta \sum_{i \in B} \Delta\alpha_i \le \sum_{i \in B} (M_i - m_i)\Delta\alpha_i \underset{\text{by (*)}}{<} \delta^2$$

$$\Rightarrow \sum_{i\in B} \Delta\alpha_i < \delta$$

$$\Rightarrow U(P, h, \alpha) - L(P, h, \alpha) = \sum_{i \in A} (M_i^* - m_i^*)\Delta\alpha_i + \sum_{i \in B} (M_i^* - m_i^*)\Delta\alpha_i$$

$$\le \varepsilon \sum_{i \in A} \Delta\alpha_i + 2K \sum_{i \in B} \Delta\alpha_i$$

$$< \varepsilon \sum_{i \in A} \Delta\alpha_i + 2K \cdot \delta$$

$$< \varepsilon(\alpha(b) - \alpha(a) + 2K)$$ $\Leftarrow$ by assumption $\delta < \varepsilon$

Since ε was arbitrary, by theorem 6.6 $\Rightarrow$ $h \in \mathcal{R}(\alpha)$

Properties of the Integral

6.12 Theorem

(a) If $f_1 \in \mathcal{R}(\alpha)$ and $f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$, then

$$f_1+f_2 \in \mathcal{R}(\alpha) \quad , \quad cf \in \mathcal{R}(\alpha)$$

for every constant c, and

$$\int_a^b (f_1+f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha ,$$

$$\int_a^b cf\, d\alpha = c \cdot \int_a^b f\, d\alpha ,$$

(b) If $f_1(x) \le f_2(x)$ on $[a,b]$, then

$$\int_a^b f_1\, d\alpha \le \int_a^b f_2\, d\alpha$$

(c) If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and if $a<c<b$, then $f \in \mathcal{R}(\alpha)$ on $[a,c]$ and on $[c,b]$, and

$$\int_a^c f\, d\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha$$

(d) If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and if $|f(x)| \le M$ on $[a,b]$, then

$$\left| \int_a^b f\, d\alpha \right| \le M\,[\alpha(b) - \alpha(a)].$$

(e) If $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$, then $f \in \mathcal{R}(\alpha_1+\alpha_2)$ and

$$\int_a^b f\, d(\alpha_1+\alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2 ;$$

If $f \in \mathcal{R}(\alpha)$ and c is a positive constant, then $f \in \mathcal{R}(c\alpha)$ and

$$\int_a^b f\, d(c\alpha) = c \int_a^b f\, d\alpha$$

proof: (a) Let P be any partition of $[a,b]$.

① $L(P, f_1, \alpha) + L(P, f_2, \alpha) \le L(P, f, \alpha) \le U(P, f, \alpha) \le U(P, f_1,$

6.13 Theorem If $f \in \mathcal{R}(\alpha)$ and $g \in \mathcal{R}(\alpha)$ on $[a,b]$, then

(a) $fg \in \mathcal{R}(\alpha)$

(b) $|f| \in \mathcal{R}(\alpha)$ and $\left| \int_a^b f \, d\alpha \right| \le \int_a^b |f| \, d\alpha$.

Proof: (a) By Thm 6.12 $\Rightarrow \begin{cases} f+g \in \mathcal{R}(\alpha) \\ f-g \in \mathcal{R}(\alpha) \end{cases}$

By Thm 6.11 $\Rightarrow \begin{cases} (f+g)^2 \in \mathcal{R}(\alpha) & ① \\ (f-g)^2 \in \mathcal{R}(\alpha) & ② \end{cases}$

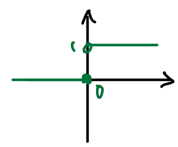
① − ② $\Rightarrow 4fg \in \mathcal{R}(\alpha) \Rightarrow f \cdot g \in \mathcal{R}(\alpha)$

(b) Let $c = \{\pm 1\}$ s.t. $c \cdot \int_a^b f \, d\alpha \ge 0$

Then $\left| \int_a^b f \, d\alpha \right| = c \cdot \int_a^b f \, d\alpha \underset{\text{by Thm 6.12}}{=} \int_a^b c f \, d\alpha \le \int_a^b |f| \, d\alpha$ #

6.14 Definition The unit step function I is defined by

$$I(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$$



6.15 Theorem If $a < s < b$, f is bounded on $[a,b]$, f is continuous at s, and $\alpha(x) = I(x-s)$, then

$$\int_a^b f \, d\alpha = f(s)$$

Proof: Consider partitions $P = \{x_0, x_1, x_2, x_3\}$, where $a = x_0 < x_1 = s < x_2 < x_3 = b$

$U(P,f,\alpha) = M_1 \Delta\alpha_1 + M_2 \cdot \Delta\alpha_2 + M_3 \cdot \Delta\alpha_3 = M_2 \cdot \Delta\alpha_2 = M_2(\alpha(x_2) - \alpha(x_1))$

$L(P,f,\alpha) = m_1 \Delta\alpha_1 + m_2 \cdot \Delta\alpha_2 + m_3 \cdot \Delta\alpha_3 = m_2 \cdot \Delta\alpha_2 = m_2$

Since f is cts at s $\Rightarrow M_2, m_2$ converges to $f(s)$ as $x_2 \to s$.

?!?

WTS $x_2 \to s^+$, $m_2\alpha(x_2) \to f(s)$ and $M_2\alpha(x_2) \to f(s)$

Let $\varepsilon > 0$ be given, Then $\exists \delta > 0$ s.t.

$|f(s) - f(y)| < \varepsilon$ if $|s - y| < \delta$ $\Rightarrow$ $f(s) \ge f(y) - \varepsilon$, $f(s) \le f(y) + \varepsilon$ $\forall y \in [s, s+\delta]$

$\Rightarrow f(s) \le \inf_{y \in [s,s+\delta]} f(y) + \varepsilon \le m_2\alpha(x_2) + \varepsilon$ if $x_2 \in [s, s+\delta]$

$f(s) \ge \sup_{y \in [s,s+\delta]} f(y) - \varepsilon \ge M_2\alpha(x_2) - \varepsilon$ if $x_2 \in [s, s+\delta]$ $\ge m_2(x_2) - \varepsilon$

$\Rightarrow |f(s) - M_2\alpha(x_2)| \le 2\varepsilon$, $|f(s) - m_2\alpha(x_2)| \le 2\varepsilon$, if $x_2 \in [s, s+\delta]$

6.16 Theorem Suppose $c_n > 0$ for $1, 2, 3, \cdots$, Σc_n converges, $\{s_n\}$ is a sequence of distinct points in (a, b), and

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n).$$

Let f be continuous on $[a, b]$. Then

$$\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$$

6.17 Theorem Assume α increases monotonically and $\alpha' \in \mathcal{R}$ on $[a,b]$. Let f be a bounded real function on $[a,b]$.

Then $f \in \mathcal{R}(\alpha)$ if and only if $f\alpha' \in \mathcal{R}$. In that case

$$\int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)\,dx$$

proof: Let $\varepsilon > 0$ be given, $M = \sup_{x\in[a,b]} |f(x)|$

claim: $\exists P$ s.t. $P \subseteq P^*$ then

① $|U(P^*, f, \alpha) - U(P^*, f\alpha')| \leq \varepsilon M$

② $|L(P^*, f, \alpha) - L(P^*, f\alpha')| \leq \varepsilon M$

① implies that $\left|\overline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f(x)\alpha'(x)\,dx\right| \leq (M+1)\cdot\varepsilon$

$\exists P_1$ s.t. $\overline{\int_a^b} f\,d\alpha \leq U(P_1, f, \alpha)$

Chapter 7 Sequences and Series of Functions

Discussion of Main Problem

7.1 Definition Suppose $\{f_n\}$, $n=1,2,\dots,$ is a sequences of functions defined on a set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. We can the define a function f by

$$f(x) = \lim_{n\to\infty} f_n(x) \quad (x \in E)$$: limit or limit function

$\{f_n\} \to f$ pointwise on E

We define $f(x) = \sum_{n=1}^{\infty} f_n(x)$: the sum of the series $\sum f_n$

7.2. Example $S_{m,n} = \frac{m}{m+n}$

$$\lim_{n\to\infty} \lim_{m\to\infty} = 1, \quad \lim_{m\to\infty} \lim_{n\to\infty} = 0$$

7.3. Example $f_n(x) = \frac{x^2}{(1+x^2)^n}$ (x real, $n=0,1,2,\dots$)

and Consider $f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = x^2 \sum_{n=0}^{\infty} \left(\frac{1}{1+x^2}\right)^n$

$$\lim_{n\to\infty} \frac{1 \times (1-q^n)}{1-q} = \frac{1}{1-\frac{1}{1+x^2}} = \frac{1+x^2}{x^2}$$

$$\Rightarrow f(x) = \begin{cases} 0 & (x=0) \\ 1+x^2 & (x \neq 0) \end{cases}$$ $\Rightarrow$ a convergent series of continuous functions may have a discontinuous sum.

Uniform Convergence

7.7 Definition We say that a sequence of functions $\{f_n\}$, $n=1,2,\dots,$ converges uniformly on E to a function f if for every $\varepsilon > 0$ there is an integer N s.t. $n \geq N$ implies $\| f_n(x) - f(x) \| \leq \varepsilon$ for all $x \in E$.
pointwise convergent.

We say that $\sum f_n(x)$ converges uniformly on E if the sequence of partial sum $\{S_n\} := \sum_{i=1}^{n} f_i(x)$ converges uniformly on E.

7.8 Theorem The sequence of functions $\{f_n\}$, defined on E, converges uniformly on E if and only if for every $\varepsilon>0$ there exists an integer N s.t. $m\geq N$, $n\geq N$, $x\in E$, implies

$$|f_n(x)-f_m(x)|<\varepsilon$$ Cauchy Criteria

Proof: $(\Rightarrow)$ Suppose $\{f_n\}$ converges uniformly on E, and let f be the limit function.

$\Rightarrow \exists\, N\in\mathbb{N}$ s.t. $\forall\, n\geq N$, $x\in E$ implies $|f_n(x)-f(x)|\leq\frac{\varepsilon}{2}$,

so that $|f_n(x)-f_m(x)|\leq|f_n(x)-f(x)|+|f_m(x)-f(x)|\leq\varepsilon$

if $n\geq N$, $m\geq N$, $x\in E$

$(\Leftarrow)$ $\forall\, x\in E$, $\{f_n(x)\}$ is Cauchy sequence

define $f(x):=\lim_{n\to\infty} f(x)$

Let $\varepsilon>0$ be given

$\exists\, N$ s.t. $\forall\, m,n\geq N$ and $x\in E$

$\|f_n(x)-f_m(x)\|\leq\frac{\varepsilon}{2}<\varepsilon$

Then $\limsup_{m\to\infty}\|f_n(x)-f_m(x)\|\leq\frac{\varepsilon}{2}<\varepsilon$

$\qquad\qquad\|f_n(x)-f(x)\| \Rightarrow f_n\to f$ uniformly