STABILITY ANALYSIS OF TWO COUPLED LORENZ LASERS AND THE COUPLING-INDUCED PERIODIC→CHAOTIC TRANSITION

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We have performed a linear stability analysis of two Lorenz lasers coupled by their electric fields and have shown that the bad cavity condition becomes a function of coupling and that a good cavity instability may occur if the injected fields are inverted before injection. In addition, we show that the symmetrically coupled Lorenz system is isomorphic to the original Lorenz system with new parameters. The stability analysis also predicts a lowering of the second laser threshold with coupling for both the chaotic and self-pulsing regimes. Numerical integration of the equations is in agreement with these predictions and has revealed a coupling induced transition from self-pulsing to chaotic behavior. The classification of the behavior of the coupled system in the parameter space of the coupling constants has been investigated and shows that the results of symmetric coupling allow enough of a margin for an experimental test of the theory. This would allow experimentalists to observe the actual Lorenz instability at excitations as low as 4–5 times above threshold.

1. Introduction

The isomorphism between the Lorenz equations and the mean-field limit of the resonantly tuned unidirectional ring laser has motivated a great deal of experimental and theoretical efforts [1–5]. The experimental efforts have been plaqued by the requirement of both a bad cavity condition and a large gain. A key to this problem lies in the ability to drop the threshold for instability without invalidating the applicability of the Lorenz system of equations. Motivated by this end and the interesting theoretical results which have appeared for lasers with an injected signal [6], we have considered the stability of two symmetrically coupled Lorenz lasers. This coupling we will show preserves the dimension of the system and results in an isomorphic Lorenz system.

Previously Lawandy et al. have numerically studied the instability of two electric field coupled Lorenz lasers in the chaotic regime ($\gamma_{\parallel}/\gamma_{\perp} > 0.20$) and have shown that the second threshold drops with coupling up to a certain value and then the coupled system rapidly becomes stable again [7]. In this paper we will show using a linear stability analysis that this behavior is due to the loss of the bad cavity condition for the coupled system. In addition we will describe numerical studies of coupled systems in the self pulsing regime and the discovery of a coupling induced transition from a periodic to a chaotic instability (SP→C). In addition we will show that before the onset of this transition, the frequency of self-pulsing drops with increased coupling in agreement with the numerical results. The case of nonsymmetric coupling has been studied numerically in order to verify that many large regions exist in the coupling parameter space where the behavior is unaffected by the broken symmetry in coupling constants. This is a critical point since it allows for the practical experimental realization of this system and the new possibilities of a greatly flowered threshold for the Lorenz instability in quantum optics.

2. Coupled Lorenz systems

The equations of motion for the unidirectionally coupled ring lasers (Lorenz systems) shown in fig. 1

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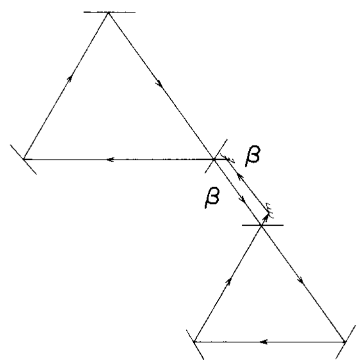

Fig. 1. Two coupled unidirectional Lorenz lasers.

are given below

$$\dot{X}_1 = \sigma Y_1 - \sigma(X_1 - \beta X_2) , \quad (1a)$$

$$\dot{Y}_1 = rX_1 - Y_1 - X_1 Z_1 , \quad (1b)$$

$$\dot{Z}_1 = X_1 Y_1 - bZ_1 , \quad (1c)$$

$$\dot{X}_2 = \sigma Y_2 - \sigma(X_2 - \beta X_1) , \quad (2a)$$

$$\dot{Y}_2 = rX_2 - Y_2 - X_2 Z_2 , \quad (2b)$$

$$\dot{Z}_2 = X_2 Y_2 - bZ_2 . \quad (2c)$$

These equations are coupled via a term common to lasers with injected fields [6]. The coupling constant β controls the degree to which these systems interact and may be experimentally varied by reflection losses. The general problem of coupled lasers does not require that there be a single coupling constant for one systems' injection into the other. The case of equal coupling however is of interest as it preserves symmetry between the two systems, and leads to a new system of three equations keeping the order of the system unchanged. It should be pointed out that β can be less than zero if a polarization rotator (π phase shift) precedes injection of one cavity field into the other laser cavity.

Since the two Lorenz systems are totally identical, the time behavior of X_1, Y_1, Z_1 should be identical to that of X_2, Y_2, Z_2 given the same initial conditions. This means one can replace the X_2 term in eq. (1a) by X_1. This substitution greatly simplifies our problem and reduces the system with six ordinary differential equations to one with only three. The resulting set of equations is given by

$$\dot{X} = \sigma Y - \sigma(X - \beta X) , \quad (3a)$$

$$\dot{Y} = rX - Y - XZ , \quad (3b)$$

$$\dot{Z} = XY - bZ . \quad (3c)$$

The system of equations (3a–3c) is isomorphic to a Lorenz system with new control parameters. The Lorenz system equations are obtained via the following transformation

$$X^* = X , \quad Y^* = Y/(1-\beta) , \quad Z^* = Z/(1-\beta) , \quad (4)$$

$$r^* = r(1-\beta) , \quad \sigma^* = \sigma(1-\beta) , \quad b^* = b . \quad (5)$$

Using the transformations (4) – (5), we can immediately find the fixed points and the conditions for instability. The fixed points and instability conditions are given as follows.

Trivial solution:

$$X_0 = Y_0 = Z_0 = 0 , \qquad r > 1 - \beta . \quad (6)$$

The nontrivial solution:

$$X_0 = \pm [b(r-1+\beta)/(1-\beta)]^{1/2} , \quad (7a)$$

$$Y_0 = \pm (1-\beta)[b(r-1+b)/(1-\beta)]^{1/2} , \quad (7b)$$

$$Z_0 = r - 1 + \beta , \quad (7c)$$

$$r > \frac{\sigma(1-\beta)[\sigma(1-\beta)+b+3]}{\sigma(1-\beta)-b-1} , \quad (8a)$$

$$\sigma(1-\beta) > b+1 . \quad (8b)$$

The threshold for the second laser instability is greatly lowered when $0 < \beta < 1$ and condition (8b) is satisfied. The latter condition reduces to the "bad cavity condition" of the single mode laser, $\kappa > \gamma_{\parallel} + \gamma_{\perp}$, when the Haken transformations are employed ($\sigma = \kappa/\gamma_{\perp}, b = \gamma_{\parallel}/\gamma_{\perp}$). κ is the cavity decay rate and $\gamma_{\parallel}$ and $\gamma_{\perp}$ are the population and polarization decay rates respectively. As a consequence of this we see that as β is increased beyond $\beta_c = 1 - (\gamma_{\parallel} + \gamma_{\perp})/\kappa$, the system reverts to stability for all r values. This behavior has been observed numerically and is in agreement with the predicted value of β_c [7].

In the self-pulsing region ($b \leq 0.20$ for $\sigma = 4$), it is useful to estimate the coupling dependence of the pulsing frequency. The imaginary part of the linear stability exponents may be used for this near the instability threshold r_2. The results are given by

$$\omega_{SP} \sim [b(\sigma + 1 - \sigma\beta) + (b/(1-\beta))(-1+\beta+r)]^{1/2} . \qquad (9)$$

Eq. (9) predicts that the period increases with coupling linearly for weak coupling.

An interesting case is now accessible when $-1 < \beta < 0$ and the injected signals are inverted. This case results in a good cavity single mode laser instability. When $\beta = -0.9$ and $b = 1$, we require that $\sigma > 1.05$ or that $1.05\gamma_{\perp} < \kappa < \gamma_{\parallel} + \gamma_{\perp}$. In the next section we will compare the results of the stability analysis of those found from numerical integration of the equations.

3. Numerical integration results

The six equations were studied numerically using a fifth-order Runge-Kutta algorithm on a VAX 11/730 computer. The equations were integrated using initial data equal to 10% of the fixed point values obtained analytically in the stability analysis section. In all the work that follows the Lorenz system parameters were fixed at $\sigma = 4$ and $b = 0.30$ or 0.15.

Previously, we have numerically studied coupled Lorenz systems which independently exhibited a chaotic instability ($b = 0.30$) [7]. This work had shown the decrease in the second threshold with increased coupling and then a sharp approach to a stable state. In the first part of this paper we have shown that this is due to a coupling constant (β) dependent bad cavity condition. Fig. 2a shows a comparison of the stability analysis values for r_2 versus β and the numerical integration results for the chaotic regime ($b \geq 0.30$) for symmetric coupling.

Because these results have significant implications for the experimental accessibility of Lorenz chaos ($b = 1$), we must answer the questions concerning the requirement of exact degeneracy of the coupling constants. The effects of asymmetric coupling are shown in fig. 2b. The results of the numerical integration

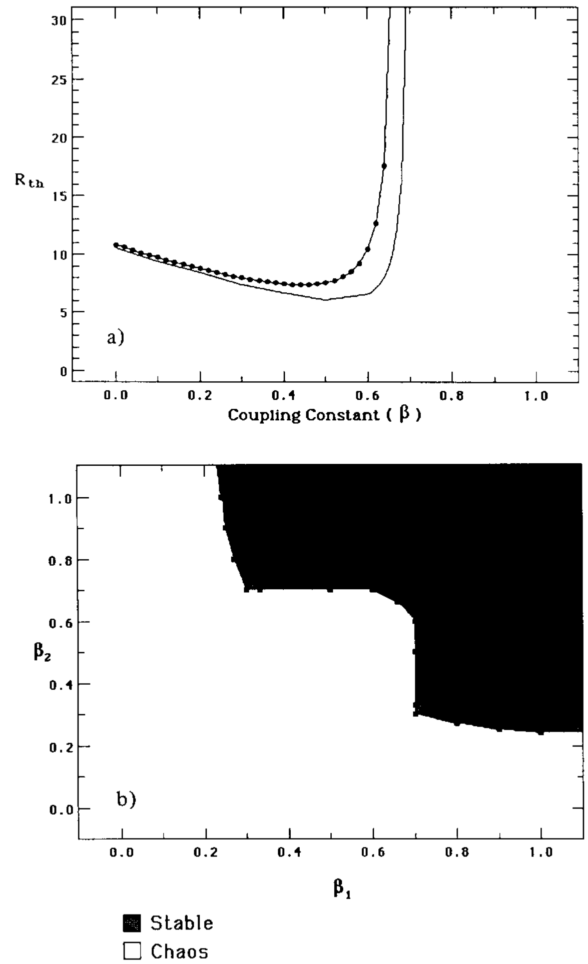

Fig. 2. (a) Threshold excitation for the onset of chaos as a function of coupling constant. (Analytical results - line with dots). (b) The behavior of the coupled Lorenz system (X_2) as a function of coupling parameters β_1 and β_2. The other parameters are $\sigma = 4$, $b = 0.30$ and $r = 15$.

showed that when one coupling constant was less than 0.3, the boundary between stability and chaos was in good agreement with the relation

$$(\beta_1\beta_2)^{1/2} \approx \beta_c . \qquad (10)$$

This condition and the size of the chaotic region in fig. 2b imply that experiments where β_1 and β_2 were not exactly equal are viable.

The self-pulsing regime of behavior ($b < 0.20$) was also investigated numerically. The results showed that a considerably larger set of behavior was accessible including subharmonic and intermittent self-pulsing. Fig. 3a shows a comparison of the linear stability analysis result for the instability threshold and the numerical integrations for the symmetrically coupled lasers. Fig. 3b shows a comparison of the self-pulsing frequency obtained from the imaginary part of the stability exponent and the numerical interpretation as a function of increasing symmetric coupling.

The parameter studied on a gross scale space of unequal coupling constants was studied as it revealed a complex structure including subharmonic bifurcations, chaos and intermittent behavior. Fig. 3c shows the phase portrait of a chaotic coupled system in the $X_1(t),X_2(t)$ space. The effect of different couplings on the X_2 system is shown in fig. 3d. The parameter map shows that several directions in the β_1,β_2 space lead to a coupling induced transition from self-pulsing to chaotic behavior. This reflects the transition from uncoupled to coupled behavior and is interesting as the transformations for the symmetric system ($\beta_1=\beta_2\neq\beta$) leave b unaffected.

In addition to this transition, the map shows that the system with the lower coupling can be intermit-

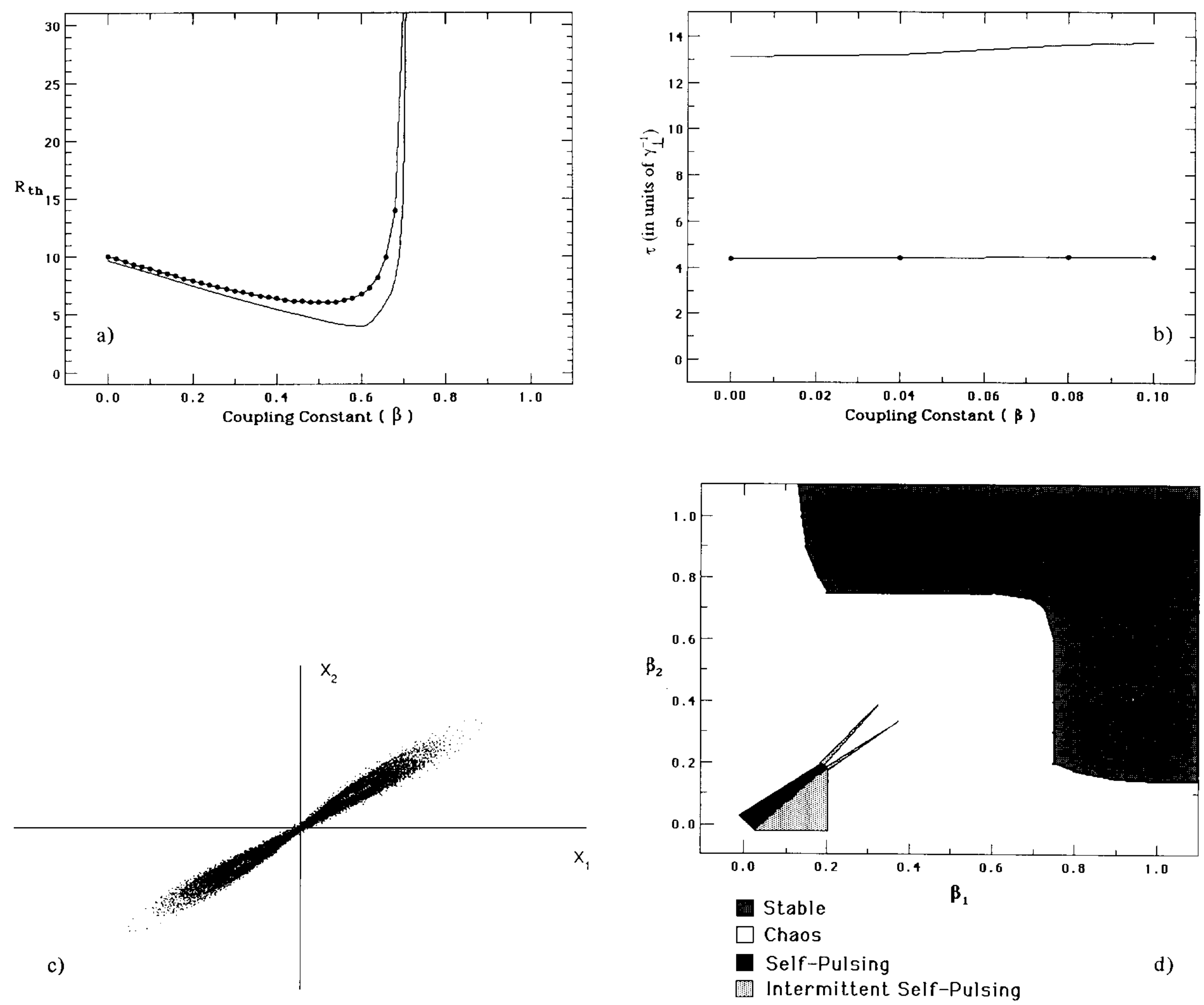

Fig. 3. (a) Threshold excitation for the onset of self-pulsing as a function of coupling constant. (Analytical results - line with dots). (b) Comparison of analytical (line with dots) and numerical results of the period in the self-pulsing regime. (c) $X_1(t)$ for the coupled Lorenz system ($\beta_1=0.4, \beta_2=0.6$). The other parameters are $\sigma=4$, $b=0.15$, and $r=15$. (d) Same as fig. 2b, except that $b=0.15$.

tently disturbed by the chaotic time evolution of the more strongly coupled systems. This indicates a periodic slowing of the limit cycle system to the chaotic dynamics of the other system with increased coupling.

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