Notes for the book[1]

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Contents

I am writing this note to help me more effectively find the symbols in [1].

1 The concept of a spherical function

1.1 Review of some basic notions of representation theory

- V: a Hausdorff locally convex topological vector space
- $\mathscr{B}(V)$: the algebra of all continuous linear operators on V, equipped with the strong topology, namely, the locally convex topology defined by the seminorms $L \mapsto \nu(Lv)(v \in V, \nu$ a continuous seminorm on $V)$
- $M_c(G)$: the vector space of all complex-valued compactly supported Borel measures on G
- $C_c(G)$: the space of continuous compactly supported complex functions on G
- $\pi(\mu) := \int_G \pi(x) d\mu(x)$ where $\mu \in M_c(G)$ and π is a representation of G
- $\pi(f) := \pi(f dx) = \int_G \pi(x) f(x) dx$ where $f \in C_c(G)$ and π is a representation of G
- $\dagger$: denotes Hilbert space adjoint
- $U(\mathfrak{g}_c)$: the universal enveloping algebra of $\mathfrak{g}_c$
- $\mathscr{E}'(G)$: the algebra of distributions on G with compact support
- $f(x; a) = (af)(x) := \frac{\partial^r}{\partial t_1 \cdots \partial t_r}|_0 f(x \exp t_1 X_1 \cdots \exp t_r X_r)$ where $a = X_1 \cdots X_r \in U(\mathfrak{g}_c)$ and $x \in G$
- $f(a; x) = (fa)(x) := \frac{\partial^r}{\partial t_1 \cdots \partial t_r}|_0 f(\exp t_1 X_1 \cdots \exp t_r X_r x)$ where $a = X_1 \cdots X_r \in U(\mathfrak{g}_c)$ and $x \in G$
- $f_1(x, y) = f(xy)$
- $f_{\pi,v}(x) := \pi(x)v$ where $x \in G, v \in V$
- $\pi(a)v = f_{\pi,v}(1; a)$ where $a \in U(\mathfrak{g}_c), v \in V$
- $\pi(X_1 \cdots X_r) = \frac{\partial^r}{\partial t_1 \cdots \partial t_r}|_0 \pi(x \exp t_1 X_1 \cdots \exp t_r X_r)$
- V^∞: the subspace of differentiable vectors
- V^ω: the space of weakly analytic vectors
- $\mathfrak{Z} = U(\mathfrak{g}_c)^G$: centralizer of G in $U(\mathfrak{g}_c)$
- $\chi_\pi(\mathfrak{Z} \to \mathbb{C})$ infinitesimal character of π: a homomorphism such that $\pi(z)v = \chi_\pi(z)v, v \in V^\infty$

1.2 Decomposition of a representation with respect to a compact subgroup K and K-finite representations

- (G, K): a pair where G is a second countable locally compact group, unimodular and K is a compact subgroup
- $\mathscr{E}(K)$: the set of equivalence of the irreducible representations of K
- $\mathrm{ch}_{\mathfrak{d}}$: the character of $\mathfrak{d} \in \mathscr{E}(K)$
- $\xi_{\mathfrak{d}}(k) = \dim(\mathfrak{d})\, \mathrm{ch}_{\mathfrak{d}}(k^{-1}), k \in K$
- $E_{\mathfrak{d}} = \pi(\xi_{\mathfrak{d}})$
- $V_{\mathfrak{d}} = E_{\mathfrak{d}} V$: the isotypical subspace of V corresponding to $\mathfrak{d}$
- $F \subset \mathscr{E}(K)$: finite set
- $E_F = \sum_{\mathfrak{d} \in F} E_{\mathfrak{d}}$
- $V_F = E_F V = \bigoplus_{\mathfrak{d} \in F} V_{\mathfrak{d}}$
- $v_{\mathfrak{d}} = E_{\mathfrak{d}} v$ the Fourier coefficient of $v \in V$
- $C_{\mathfrak{d}}(G) = E_{\mathfrak{d}, \bar{\mathfrak{d}}}(C(G)) = \xi_{\mathfrak{d}} * C(G) * \xi_{\mathfrak{d}}$
- $C_{c,\mathfrak{d}}(G) = C_c(G) \cap C_{\mathfrak{d}}(G) = \xi_{\mathfrak{d}} * C_c(G) * \xi_{\mathfrak{d}}$
- $C(G//K) = C_1(G)$
- $C_c(G//K) = C_{c,1}(G)$
- $C_{c,F}(G) = \xi_F * C_c(G) * \xi_F$
- $C^{\infty}_{c,F}(G) = \xi_F * C^{\infty}_c(G) * \xi_F$
- $\xi_F = \sum_{\mathfrak{d} \in F} \xi_{\mathfrak{d}}$
- $I_c(G)$: the subalgebra of $C_c(G)$ of elements invariant under inner automorphisms by $k \in K$
- $I_{c,F}(G) = \xi_F * I_c(G) * \xi_F$
- $\pi_F(f) = \pi(f)|_{V_F}, f \in C_{c,F}(G)$
- $V^0 = \sum_{\mathfrak{d} \in \mathscr{E}(K)} V_{\mathfrak{d}}$ (algebraic sum) and $V^0 \subset V^{\infty}$ if π is K-finite
- $\mathfrak{Q} = U(\mathfrak{g}_c)^K$ = centralizer of K in $U(\mathfrak{g}_c)$
- $\Theta_{\pi}(C^{\infty}_c(G) \to \mathbb{C}) : f \mapsto \mathrm{tr}(\pi(f))$: the character of π

1.3 Elementary spherical functions of arbitrary type

- $\Phi_{\pi,F}(x) = E_F\pi(x)E_F$: the spherical function of type $F \subset \mathscr{E}(K)$ associated with π
- $\pi_F(f) = \langle f, \Phi_{\pi,F}\rangle := \int_G f(x)\Phi_{\pi,F}(x)dx$
- γ: a representation of $C_{c,F}(G)$ in U where U is a finite-dimensional vector space
- $\Psi(G \to \mathrm{Hom}_{\mathbb{C}}(U,U))$ such that $\gamma(f) = \langle f, \Psi\rangle$ and $\Psi = \bar{\xi}_F * \Psi * \bar{\xi}_F$
- $I_{c,\mathfrak{d}}(G)$: the subalgebra of elements invariant under the inner automorphisms $x \mapsto kxk^{-1}$ induced by elements of K
- $\Phi^\sharp : G \to \mathrm{Hom}_{\mathbb{C}}(W,W)$ such that (i) $\Phi^\sharp(1) = 1$, (ii) $\Phi^\sharp(kxk^{-1}) = \Phi^\sharp(x)$, (iii) $\bar{\xi}_{\mathfrak{d}} * \Phi^\sharp * \bar{\xi}_{\mathfrak{d}} = \Phi^\sharp$
- $\pi_{\mathfrak{d}} = \theta \otimes \sigma_{\mathfrak{d}}$: the irreducible representation of $C_{c,\mathfrak{d}}(G)$ in $V_{\mathfrak{d}}$ where $\sigma_{\mathfrak{d}}$ is an irreducible representations of $I_{c,\mathfrak{d}}(G)$ in some space $W_{\mathfrak{d}}$
- $\Phi^\sharp_{\pi,\mathfrak{d}}$: the continuous map of G into $\mathrm{Hom}_{\mathbb{C}}(W_{\mathfrak{d}}, W_{\mathfrak{d}})$ that satisfies the conditions above and $\left\langle f, \Phi^\sharp_{\pi,\mathfrak{d}}\right\rangle = \sigma_{\mathfrak{d}}(f), f \in I_{c,\mathfrak{d}}(G)$. $\Phi^\sharp_{\pi,\mathfrak{d}}$ is called the elementary K-central spherical function of type $\mathfrak{d}$ associated with π
- $\varphi_{\pi,\mathfrak{d}} = \mathrm{tr}(\Phi^\sharp_{\pi,\mathfrak{d}})$: the elementary trace spherical function of type $\mathfrak{d}$ associated with π
- U the algebra of endomorphisms of V_F, also regarded as a bimodule for K by defining $\tau(k)u = \pi_F(k)u$ and $u\tau(k) = u\pi_F(k), k \in K, u \in U$ where $\pi_F(k) = E_F\pi(k)E_F$
- $\Phi = \Phi_{\pi,F}$ is a function from G to U which has the following covariance property: $\Phi(k_1xk_2) = \tau(k_1)\Phi(x)\tau(k_2)$ for all $k_1, k_2 \in K, x \in G$

1.4 Spherical functions on Lie groups

- $a^k := \mathrm{Ad}(k)a$ where $a \in U(\mathfrak{g}_c)$
- $\Psi(x : y) = \int_K \Phi^\sharp(xkyk^{-1})dk$

1.5 Gelfand's pairs (G, K)

- (G, K): Gelfand pair if $L^1(G//K)$ is commutative
- θ: an involutive automorphism of G
- G^θ: the subgroup of fixed points for θ
- V^K: the space of vectors invariant under K

1.6 Plancherel formula for G/K

- $\lambda(G \to B(L^2(G/K))) : y \mapsto \lambda(y)$ such that $(\lambda(y)f)(\dot{x}) = f(y^{-1}\dot{x})$ where $\dot{x} \in G/K, f \in L^2(G/K)$
- $\mathfrak{H}$: a separable Hilbert space
- $\mathfrak{U}$: a commutative algebra of bounded operators in $\mathfrak{H}$ such that $\mathfrak{U}$ is closed in the operator norm topology and $\mathfrak{U}$ is self-adjoint
- $\mathfrak{U}_1$: a dense self-adjoint algebra of $\mathfrak{U}$
- $\sharp(f) = f^\sharp = \int_{K\times K} l(k_1)r(k_2)f dk_1 dk_2$: the projection of $L^1(G)$ onto $L^1(G//K)$ where r and l are left and right regular representations of G in $L^1(G)$
- $\Sigma(G//K) =$ spectrum of $L^1(G//K)$, i.e., the set of all continuous nontrivial homomorphisms of $L^1(G//K)$ into $\mathbb{C}$
- φ_τ: uniquely determined elementary spherical function such that $\langle f, \varphi_\tau \rangle = \tau(f)$ for all $f \in C_c(G//K)$ where $\tau \in \Sigma(G//K)$
- π_τ: a completely irreducible uniformly bounded representation of class 1 in a Banach space such that φ_τ is the elementary spherical function associated to π_τ
- $\Sigma_u(G//K) = \left\{ \tau \in \Sigma(G//K) : \tau((f * \tilde{f})^\sharp) \geq 0, \forall f \in L^1(G) \right\}$
- $\hat{G}$: the set of equivalence classes of irreducible unitary representations of G
- $\hat{G}_1$: the set of classes of $\hat{G}$ corresponding to class 1 representations

1.7 Eigenfunction expansions in G/K

- $H \subset G$: a closed subgroup
- $E_G(G/H)$: the algebra of all G-invariant continuous endomorphisms of $C^\infty(G/H)$
- $\mathrm{Diff}_G(G/H)$: the subalgebra of G-invariant differential operators on G/K

2 Structure of semisimple Lie groups and differential operators on them

2.1 Groups of class $\mathscr{H}$

- G^0: the component contained identity
- The index of subgroup of G^0 :$[G : G^0] < \infty$: G has only finitely many connected compositions

- ${}^0H := \bigcap_{X\in \mathrm{Hom}(H,\mathbb{R}_+^\times)} \ker(X)$ for any real Lie group H, where $\mathbb{R}_+^\times$ is the multiplicative group of positive reals and $\mathrm{Hom}(H,\mathbb{R}_+^\times)$ is the set of continuous homomorphisms of H into $\mathbb{R}_+^\times$
- $C = \ker(\mathrm{Ad})$ = centralizer of $\mathfrak{g}$ in G
- θ: Cartan involution
- G^θ: the set of fixed points of G
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$: the Cartan decomposition of $\mathfrak{g}$
- $G = K \exp \mathfrak{s}$: the Cartan decomposition of G
- B: the Cartan-Killing form
- $(X,Y) := B_\theta(X,Y) = -B(X,\theta Y)$
- $\|X\|^2 := B_\theta(X,X)$

2.2 Iwasawa decomposition. Roots. Weyl group

- $\mathfrak{a}$: a maximal abelian subspaces of $\mathfrak{s}$
- $\mathfrak{m}_1$: the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$
- $\mathfrak{m} := \mathfrak{m}_1 \cap \mathfrak{k}$
- M_1 (resp. $\tilde{M}_1$): the centralizer (resp. normalizer) of $\mathfrak{a}$ in G
- $M := M_1 \cap K$: the centralizer of $\mathfrak{a}$ in K
- $\tilde{M} := \tilde{M}_1 \cap K$: the normalizer of $\mathfrak{a}$ in K
- $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [H,X] = \lambda(H)X, \forall H \in \mathfrak{a}\}$ for any $\lambda \in \mathfrak{a}^*$
- $\Delta = \Delta(\mathfrak{g},\mathfrak{a})$: the set of all roots of $(\mathfrak{g},\mathfrak{a})$
- Δ^+: the set of all positive roots
- $S = \{\alpha_i : 1 \le i \le r\}$: the simple system
- $\mathfrak{n} := \sum_{\lambda\in\Delta^+} \mathfrak{g}_\lambda$
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ (resp. $G = KAN$): the Iwasawa decomposition of $\mathfrak{g}$ (resp. G)
- $\log : A \to \mathfrak{a}$
- s_λ: the reflection associated with λ
- $\sigma_\lambda = \{\mu \in \mathfrak{a}_c^* : \langle \mu, \lambda\rangle = 0\}$: the hyperplane
- $\mathfrak{w}$: the Weyl group of $(\mathfrak{g},\mathfrak{a})$

- $W = \tilde{M}/M \simeq \mathfrak{w}$: the Weyl group of (G, A)
- $\mathfrak{a}^+$: the positive Weyl chamber of $\mathfrak{a}$
- $A^+ = \exp(\mathfrak{a}^+)$

2.3 Parabolic subalgebras and parabolic subgroups

- $\mathfrak{h}_c \subset \mathfrak{g}_c$: a CSA(Cartan subalgebra)
- Q: a positive system of roots of $(\mathfrak{g}_c, \mathfrak{h}_c)$
- $\mathfrak{g}_{c,\alpha}$: the root subspaces
- $\mathfrak{b}_c = \mathfrak{h}_c + \sum_{\lambda \in Q} \mathfrak{g}_{c,\alpha}$: a Borel subalgebra of $\mathfrak{g}_c$
- $S \subset Q$: the set of simple roots
- $F \subset S$: finite subset
- Q_F: the set of roots in Q which are linear combinations of elements of F
- $\mathfrak{q}_{c,F} := \mathfrak{b}_c + \sum_{\alpha \in -Q_F} \mathfrak{g}_{c,\alpha}$: a subalgebra of $\mathfrak{g}_c$ containing $\mathfrak{b}_c$
- $\mathfrak{h}_{\mathfrak{m}}$: a CSA of $\mathfrak{m}$
- $\mathfrak{h} := \mathfrak{h}_{\mathfrak{m}} \oplus \mathfrak{a}$: a CSA of $\mathfrak{m} \oplus \mathfrak{a}$
- $Q_{\mathfrak{m}}$: a positive system of roots for pair $(\mathfrak{m}_c, (\mathfrak{h} \cap \mathfrak{k})_c)$
- Q^+: the set of all roots of $(\mathfrak{g}_c, \mathfrak{a}_c)$ whose restrictions to $\mathfrak{a}$ lie in $\Delta^+(\mathfrak{g}, \mathfrak{a})$
- $Q := Q_{\mathfrak{m}} \cup Q^+$
- $\mathfrak{n}_c := \sum_{\alpha \in Q^+} \mathfrak{g}_{c,\alpha}$
- $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$: a minimal parabolic subalgebra of $\mathfrak{g}$
- Δ_F^+: the subset of Δ^+ consisting of roots that are linear combinations of elements of F
- $\mathfrak{p}_F := \mathfrak{p} + \sum_{\alpha \in -\Delta_F^+} \mathfrak{g}_\lambda$: the standard psalgebras with respect to $\mathfrak{p}$ or S
- $\mathfrak{p}_0$: a psalgebra of $\mathfrak{g}$ (of course contains $\mathfrak{p}$)
- $\mathfrak{n}_0$: the nilradical(nilpotent radical) of $\mathfrak{p}_0 \cap [\mathfrak{g}, \mathfrak{g}]$
- $\mathfrak{m}_{10} = \mathfrak{p}_0 \cap \theta(\mathfrak{p}_0)$
- $\mathfrak{p}_0 = \mathfrak{m}_{10} \oplus \mathfrak{n}_0$: the Levi decomposition of $\mathfrak{p}_0$
- $\mathfrak{a}_0 = \text{center}(\mathfrak{m}_{10}) \cap \mathfrak{s}$
- $\mathfrak{m}_0 = \mathfrak{m}_{10} \ominus \mathfrak{a}_0$

- $\mathfrak{p}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$: the Langlands decomposition of $\mathfrak{p}_0$.
- $\mathfrak{a}_F = \{H \in \mathfrak{a} : \alpha(H) = 0,\ \forall \alpha \in F\}$
- $\Delta_F = \Delta_F^+ \cup -\Delta_F^+$
- $\mathfrak{m}_{1F} = \sum_{\lambda \in \Delta_F} \mathfrak{g}_\lambda$
- $\mathfrak{n}_F = \sum_{\lambda \in \Delta^+ \setminus \Delta_F^+} \mathfrak{g}_\lambda$ the nilradical of $\mathfrak{p}_F \cap [\mathfrak{g}, \mathfrak{g}]$
- M_{10}: the centralizer in G of $\mathfrak{a}_0$ (Also is called the reductive component of P_0)
- $M_0 =^0 (M_{10})$
- $K_{M_{10}} := K \cap M_{10} = K \cap M_{10}$
- $G = KP_0 = KM_0A_0N_0$
- $\theta_{M_{10}} = \theta|_{M_{10}}$
- $B_{M_{10}} = B|_{(\mathfrak{m}_{10})_c \times (\mathfrak{m}_{10})_c}$
- $M_{0\,\mathfrak{s}} = \exp(\mathfrak{m}_0 \cap \mathfrak{s})$
- $\Delta(\mathfrak{g}, \mathfrak{a}_0) = \Delta(P_0)$: the set of all $\lambda \neq 0$ in $\mathfrak{a}_0^*$ for which $\mathfrak{g}_\lambda \neq 0$
- $\mathfrak{a}_0^+$: a chamber of $\mathfrak{a}_0$
- $\Delta(P_0 : \mathfrak{a}_0^+)$: the set of roots in $\Delta(P_0)$ which are > 0 in $\mathfrak{a}_0^+$

2.4 Integral formulae

- $\rho(H) := \frac{1}{2}\operatorname{tr}(\operatorname{ad} H)_{\mathfrak{n}} = \frac{1}{2}\sum_{\alpha \in \Delta^+} n(\alpha)\alpha$ for all $H \in \mathfrak{a}$ where $n(\alpha) = \dim \mathfrak{g}_\alpha$
- $\rho_{P_0}(H) = \frac{1}{2}\operatorname{tr}(\operatorname{ad} H)_{\mathfrak{n}_0},\ H \in \mathfrak{a}_0$
- $d_P(MA \to \mathbb{R}_+^\times) : m_1 = |\det \operatorname{Ad}(m_1)_{\mathfrak{n}}|^{1/2}$ where $m_1 \in MA$
- $d_{P_0}(M_0A_0 \to \mathbb{R}_+^\times) : m_1 = |\det \operatorname{Ad}(m_1)_{\mathfrak{n}_0}|^{1/2}$ where $m_1 \in M_0A_0$
- $J(a) = \prod_{\alpha \in \Delta^+}(\mathrm{e}^{\alpha(\log a)} - e^{-\alpha(\log a)})^{n(\alpha)}$ for all $a \in A$
- $G^* = G/A_0$
- $a^{x^*} = xax^{-1}$ where $a \in A_0$ and $x \in G$

2.5 Flag manifolds, Bruhat decomposition and Related Integral formulae

- G/Q: flag manifolds of G where Q is a psgrp
- G/P is called the flag manifold of G if $P = MAN$ and $G = KAN$
- $X = G/P$
- $\mathrm{rk}(G) = \mathrm{rk}(G/K)$: the real rank of G
- $\pi x = xP(x \in G)$
- $i(X \to K/M) : x \mapsto \overline{k(x)}$ the natural identification
- x_s: a representative of $s \in \tilde{M}$
- $G = \bigcup_{x \in \mathfrak{w}} N x_s P$(disjoint union): the Bruhat decomposition of G
- $NsP = Nx_sP$
- $\pi(x_s) = \underline{s}(s \in \mathfrak{w})$
- $X = \bigcup_{s \in \mathfrak{w}} N\underline{s}$(disjoint union): the Bruhat decomposition of X
- $\Delta_s^+ = \{\alpha \in \Delta^+ : s^{-1}\alpha \in \Delta^+\}$
- ${}_s\Delta^+ = \{\alpha \in \Delta^+ : s^{-1}\alpha \in -\Delta^+\}$
- $\mathfrak{n}_s = \sum_{\alpha \in \Delta_s^+} \mathfrak{g}_\alpha$ and ${}_s\,\mathfrak{n} = \sum_{\alpha \in {}_s\Delta^+} \mathfrak{g}_\alpha$
- $N_s = \exp(\mathfrak{n}_s)$ and ${}_sN = \exp({}_s\mathfrak{n})$
- $\mathfrak{w}_F$: the subgroup of $\mathfrak{w}$ generated by the reflexion s_α for all $\alpha \in F$
- $\Omega_1 = \pi(\bar{N})$ and $\Omega_s = x_s \cdot \pi(\bar{N})$
- $\gamma_s(\bar{n}) = \pi(x_s\bar{n})$
- $m(s) = x_s^{-1} m x_s$ for $m \in MA$ and $s \in \mathfrak{w}$
- $\beta_s(X) = \pi(x_s \cdot \exp X)$ for $X \in \bar{\mathfrak{n}}$
- $a(x : k) = a(xk)$
- $H(x : k) = H(xk)$

2.6 Differential operators on G and G/K

- L: real Lie group with its Lie algebra $\mathfrak{l}$, and the universal enveloping algebra $U(\mathfrak{l}_c)$
- $W \subset L$: open set
- $\mathrm{Diff}(W)$: the algebra of differential operators on W
- $D_x \in U(\mathfrak{l}_c)$: the local expression of $D \in \mathrm{Diff}(W)$ at $x \in W$ such that $(Df)(x) = f(x; D_x)$ for all $f \in C^\infty(W)$
- $R_a(f) = fa$ for all $a \in U(\mathfrak{l}_c)$
- $S(\mathfrak{l}_c)$: the symmetric algebra over $\mathfrak{l}_c$
- $\lambda(S(\mathfrak{l}_c) \to U(\mathfrak{l}_c)) : X_1 \ldots X_r \mapsto \frac{1}{r!}\sum_\sigma X_{\sigma(1)} \ldots X_{\sigma(r)}$ where σ funs over the set of all permutations of $(1, \ldots, r)$
- $I_{\mathfrak{w}} = I_{\mathfrak{w}}(\mathfrak{h}_c)$: the algebra of $\mathfrak{w}$-invariant in $S(\mathfrak{h}_c)$
- $I = I(\mathfrak{g}_c) = S(\mathfrak{g}_c)^G$
- $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{m}_{10}$
- $\mathfrak{s}_{10} = \mathfrak{s} \cap \mathfrak{m}_{10}$
- $\mathfrak{s}_0 = \mathfrak{s} \cap \mathfrak{m}_0$
- ${}^*\mathfrak{a} = \mathfrak{a} \cap \mathfrak{m}_0$
- ${}^*\mathfrak{n} = \mathfrak{n} \cap \mathfrak{m}_0$
- $\beta_{\mathfrak{n}} : U(\mathfrak{g}_c) \to U(\mathfrak{a}_c)$: the projection defined by the direct sum $U(\mathfrak{g}_c) = U(\mathfrak{a}_c) \oplus (\mathfrak{k}\, U(\mathfrak{g}_c) + U(\mathfrak{g}_c)\, \mathfrak{n})$ where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$
- $\gamma_{\mathfrak{n}} = \gamma = \gamma_{\mathfrak{g}/\mathfrak{a}} : \mathfrak{Q} \to U(\mathfrak{a}_c)$ such that $\gamma_{\mathfrak{n}}(a)(\lambda) = \beta(a)(\lambda - \rho)$ where $\lambda \in \mathfrak{a}_c^*$ and $a \in \mathfrak{Q}$

3 The elementary spherical functions

3.1 Principal series representations and integral representations for their matrix coefficients

- $\mathscr{F} = \mathfrak{a}_C^*,\ \mathscr{F}_R = \mathfrak{a}^*,\ \mathscr{F}_I = (-1)^{1/2}\mathscr{F}_R$
- $\xi_\beta(a) = \mathrm{e}^{\beta \log a}(\beta \in \mathscr{F}, a \in A)$: the quasicharacter of A
- σ: a f.d. unitary reps. of the compact group M in a Hilbert space $W(\sigma)$
- $(\sigma, \lambda)(man) = \sigma(m)\xi_{\lambda+\rho}(a)$: the representation of $P = MAN$

- $\mathfrak{B}(\sigma,\lambda)$: the space of all Borel functions $f(G \to W(\sigma))$ such that $f(xp) = (\sigma,\lambda)(p)^{-1}f(x),\ x \in G, p \in P$

- $f_K = f|_K$ for any $f \in \mathfrak{B}(\sigma,\lambda)$

- $\mathfrak{B}(\sigma)$: the space of Borel functions $g(K \to W(\sigma))$ such that $g(km) = \sigma^{-1}g(k),\ k \in K, m \in M$

- $\mathfrak{H}(\sigma)$: the Hilbert space of functions $g \in \mathfrak{B}(\sigma)$ for which $\|g\|^2 = \int_{K/M} |g(k)|^2\, d\bar{k} = \int_k |g(k)|^2\, dk$ is finite

- $\mathfrak{H}(\sigma,\lambda)$: the Hilbert space of functions $f \in \mathfrak{B}(\sigma,\lambda)$ such that $f_K \in \mathfrak{H}(\sigma)$

- $\pi_{\sigma,\lambda} : G \to B(\mathfrak{H}(\sigma))$: the representation given by $(\pi_{\sigma,\lambda}(x)g)(k) = \mathrm{e}^{-(\lambda+\rho)(H(x^{-1}k))}g(x^{-1}[k])$

- $\pi_\lambda = \pi_{1,\lambda}$ where $1 =$ trivial representation of M

- φ_λ: the matrix coefficient of π_λ defined by the function 1: $\varphi_\lambda(x) = \varphi(\lambda : x) = (\pi_\lambda(x)1, 1)$ where $(\cdot,\cdot)$ denoting the scaler product in $L^2(K)$

- $\varphi_\lambda(x) = \int_K \mathrm{e}^{-(\lambda+\rho)(H(x^{-1}k))}dk$

- $\Xi = \varphi_0$: the basic spherical function

3.2 Determination of all elementary spherical functions. The functional equations

3.3 The Harish-Chandra transform

- $\mathscr{H} f(\lambda) = \int_G f(x)\varphi_\lambda(x)dx$ for all $\lambda \in \mathscr{F}$: the Harish-Chandra transform of $f \in C_c(G//K)$

- $\mathscr{A}(f)(a) = \mathrm{e}^{\rho(\log a)} \int_N f(an)dn$ for all $a \in A$: the Abel transform of $f \in C_c(G//K)$

- $\hat{g}(\lambda) = \int_A g(a)\mathrm{e}^{\lambda(\log a)}da$ for all $\lambda \in \mathscr{F}$: the Fourier transform of $g \in C_c(A)$

- $\mathscr{P}(\mathscr{F})$: the space of all entire functions on $\mathscr{F}$ of Paley-Weiner type, i.e., entire functions on $\mathscr{F}$ which are Fourier transform of C^∞ functions on A with compact support

- $\mathscr{P}(\mathscr{F})^{\mathfrak{w}}$: the subspace of $\mathfrak{w}$-invariant elements of $\mathscr{P}(\mathscr{F})$

- $$\begin{tikzcd} C_c^\infty(G//K) \arrow[r, hook, "\mathscr{H}"] \arrow[d, hook, "\mathscr{A}"] & \mathscr{P}(\mathscr{F})^{\mathfrak{w}} \\ C_c^\infty(A)^{\mathfrak{w}} \arrow[ur, "\wedge"] & \end{tikzcd}$$

3.4 Finite dimensional representation theory of G and its consequences for the H-function and the elementary spherical fucntions

- $V(\pi_\Lambda)_{\mathfrak{d}}$: the corresponding isotypical space for any $\mathfrak{d} \in \mathscr{E}(K)$
- $\Lambda \in \mathfrak{h}_c^*$
- $\mathfrak{u} = \mathfrak{k} + i\,\mathfrak{s}$: a compact form of $\mathfrak{g}_c$ and U the corresponding compact subgroup of G_c
- $\mathscr{D}$: the set of all $\Lambda \in \mathfrak{h}_c^*$ which are dominant(relative to Q) and integral
- $\mathscr{D}_1$: the set of all $\Lambda \in \mathfrak{h}_c^*$ for which π_Λ are of class 1
- L: the semilattice generated by $S \subset \Delta^+$
- $Q_\alpha(X) = \frac{4\langle X, \theta X\rangle}{\langle \bar{H}_\alpha, \theta \bar{X}_\alpha\rangle} = \|\alpha\|^2 \|X\|^2$: the quadratic form
- H_1: the restriction of H to $\bar{N}_1$
- polynomial(p.155)
- homogeneous of degree μ(p.115)

3.5 Convexity properties of the H-function

- ${}^+\,\mathfrak{a} = \{H \in \mathfrak{a} : \langle H, H'\rangle \geq 0 \text{ for all } H' \in \mathfrak{a}^+\}$
- $a^* = (a^{s_0})^{-1} = u_0 a^{-1} u_0^{-1}$ where $s_0 \in \mathfrak{w}$ such that $s_0\Delta^+ = -\Delta^+$ and $u_0 \in K$ induce the element s_0
- $\mathrm{Co}(H, \mathfrak{w})$: the convex hull of the elements H^s(is the set of linear combinations $\sum_{s\in\mathfrak{w}} c_s H^s,\ c_s \geq 0,\ \sum_s c_s \leq 1$)

4 The Harish-Chandra series for φ_λ and the c-function

- $\tilde{D}$: the radial component of differential operator D
- $\tilde{\varphi}_\lambda$: the restriction of φ_λ on A^+
- $\tilde{q}/\tilde{\varphi}_\lambda = \gamma(q)(\lambda)\tilde{\varphi}_\lambda$: the differential equations where $q \in \mathfrak{Q}, \gamma = \gamma_{\mathfrak{g}/\mathfrak{a}}$
- $\Phi(\lambda : \cdot) = \mathrm{e}^{\lambda-\rho} + \sum_{\mu\in L_+} a_\mu(\lambda)\mathrm{e}^{\lambda-\rho-\mu}$: the solutions of the above
- $\beta(q) = \mathrm{e}^{-\rho} \circ \gamma(q) \circ \mathrm{e}^{\rho}$

4.1 Radial components of spherical differential operators on A^+

- $\psi(K \times A \times K \to G) : (k_1, h, k_2) \mapsto k_1 h k_2$
- $G^+ = KA^+K$
- $A' = \bigcup_{s \in \mathfrak{w}} sA^+$: the regular subset of A
- $b(U(\mathfrak{g}_c) \to U(\mathfrak{a}_{\mathbb{C}})) : g \mapsto b(g)$: the projection
- $\xi_\lambda = \mathrm{e}^{\lambda \circ \log}$
- $f_\alpha = (\xi_\alpha - \xi_{-\alpha})^{-1}$
- $g_\alpha = \xi_{-\alpha}(\xi_\alpha - \xi_{-\alpha})^{-1}$,
- $\mathscr{R}_0$: the algebra with unit generated over $\mathbb{C}$ by the f_α and $g_\alpha (\alpha > 0)$
- $\mathscr{R}_{0,d}$: the span of monomials in these generators of degree d
- $\mathscr{R}_0^+ = \sum_{d \geq 1} \mathscr{R}_{0,d}$
- $\mathscr{R}_0^{(d)} = \sum_{1 \geq e \leq d} \mathscr{R}_{0,e}$
- $\delta'(g) = b(g) + \sum_{1 \leq \imath \leq n} \psi_i u_i$ where $\psi_i = c(\xi_i')c(\xi_i)\varphi_i$
- $\chi : \mathfrak{Q} \to \mathbb{C}$: a homomorphism
- $A(U : \chi) = \{g \in C^\infty(U) : \delta'(q)f = \chi(q)f$ for all $q \in \mathfrak{Q}\}$ for any open set $U \subset A^+$

4.2 The radial component of the Casimir operator

- ω: Casimir operator
- $J(h) = \prod_{\alpha > 0}(\mathrm{e}^{\alpha(\log h)} - \mathrm{e}^{-\alpha(\log h)})^{n(\alpha)}$: the Jacobian for the polar decomposition of G.
- $g_1(t) = \mathrm{e}^{-2t}(1 - \mathrm{e}^{-2t})^{-1}$
- $g_2(t) = \mathrm{e}^{-4t}(1 - \mathrm{e}^{-4t})^{-1}$
- $\rho_0 = \rho(H_0) = \frac{1}{2}(p + 2q)$

4.3 Construction of the eigenfunctions on G^+

- $\psi(\lambda : t) = \mathrm{e}^{t\rho}\varphi_\lambda(\exp tH_0),\ t \in \mathbb{R}$
- $\psi_\lambda(h) = \mathrm{e}^{\rho(\log h)}\varphi_\lambda(h)$ for $h \in A^+$
- $L = \left\{\sum_{1 \leq i \leq r} m_i \alpha_i : m_1, \cdots, m_r \text{ integers} \geq 0\right\}$
- $L^+ = L \setminus \{0\}$

- $\mu \prec \mu'$ if $\mu' - \mu \in L^+$
- $m(\mu) = m_1 + \cdots + m_r$ if $\mu = m_1\alpha_1 + \cdots + m_r\alpha_r$: the level of μ
- b: a complex-valued function on L
- $f_b(H) = \sum_{\mu \in L} b(\mu) \mathrm{e}^{-\mu(H)},\ H_R \in \mathfrak{a}^+$
- $\mathscr{R}_{00}$: the algebrta of all functions f_b
- $\mathscr{R}_{00}^+$: the ideal in $\mathscr{R}_{00}$ of all functions with $b(0) = 0$
- $\tilde{f}(h) = f(\log h),\ h \in A^+$
- $\widetilde{\mathscr{R}}_{00}^+$: coressponding function in $\mathscr{R}_{00}^+$ defined on A^+
- $a_\mu(\lambda)$: the rational functions of λ
- $\sigma_\mu = \left\{ \lambda \in \mathscr{F} : \langle \mu, \lambda \rangle = \frac{1}{2} \langle \mu, \mu \rangle \right\},\ \mu \in L^+$: the hyperplane in $\mathscr{F}$
- $\mathscr{F}^\vee = \mathscr{F} \setminus \bigcup_{\mu \in L^+} \sigma_\mu$
- $\mathscr{F}_\eta = \{ \lambda \in \mathscr{F} : \langle \lambda_\mathbb{R}, \alpha_i \rangle < \eta, 1 \leq i \leq r \} \subset \mathscr{F}^\vee$
- $\mathscr{F}_I(\varepsilon) = \{ \lambda \in \mathscr{F} : \| \lambda_R \| \leq \varepsilon \} \subset \mathscr{F}^\vee$
- $\mathscr{F}_\mu^\vee = \mathscr{F} \setminus \bigcup_{\nu \in L^+, \nu \preceq \mu} \sigma_\nu$

4.4 The Harish-Chandra series for φ_λ and the c-fucnction

- $\tau_\nu(s,t) = \{ \lambda \in \mathfrak{a}_\mathbb{C}^* : s\lambda - t\lambda = \nu \}$
- ${}^*\mathscr{F} = \mathscr{F} \setminus \bigcup_{s \in \mathfrak{w}, \mu \in L^+} s\sigma_\mu \cup \bigcup_{s,t \in \mathfrak{w}, \nu \in L^+} \tau_\nu(s,t)$
- $\mathfrak{w}_\lambda$: the stabilizer of λ in $\mathfrak{w}$
- $c_s(\lambda) = \mathbf{c}(s\lambda)$

4.5 Estimates for the Harish-Chandra series when λ becomes unbounded

- $\Psi(\lambda : h) := J(h)^{1/2} \Phi(\lambda : h)$
- $\sigma(G \to \mathbb{R}) : x \mapsto d(K, Kx)$ where $d(\cdot, \cdot)$ is the geodesic distance on S.
- $c_\mu(\lambda)$ page 154, is of at most polynomial growth in μ

4.6 Estimates for the elementary spherical functions. The functions Ξ and σ

- λ_R: the components in $\mathscr{F}_R$ for $\lambda \in \mathscr{F}$
- λ_I: the components in $\mathscr{F}_I$ for $\lambda \in \mathscr{F}$
- $\mathscr{F}_R^+$: the open chamber of elements $\nu \in \mathfrak{a}^*$ such that $H_\nu \in \mathfrak{a}^+$
- $\beta(\log a) = \min_{1 \leq i \leq r} \alpha_i(\log a)$ for $a \in A^+$
- $\pi(\lambda) = \prod_{\alpha \in \Delta^{++}} \langle \lambda, \alpha \rangle$
- $\mathbf{b} = \pi \cdot \mathbf{c}$
- $\mathscr{F}_{R,\varepsilon} = C_0(\varepsilon\rho : \mathfrak{w})$: the convex hull of the points $\varepsilon s \rho (\varepsilon > 0, s \in \mathfrak{w})$
- $\sigma(x) := d(K, Kx)$ where $S = K \backslash G$, and $d(\cdot, \cdot)$ is the geodesic distance on S

4.7 The c-function

- $\hat{T}(g) = T(\hat{g})$: the Fourier transform of tempered distribution T on A where $g \in \mathscr{S}(\mathscr{F}_I)$
- R: a set of roots having the following properties: (i) R is contained in some positive system of roots, (ii) $\alpha, \beta \in R, \alpha + \beta \in \Delta \Rightarrow \alpha + \beta \in R$
- $\mathfrak{g}_{-R} = \sum_{\alpha \in -R} \mathfrak{g}_\alpha$
- $\bar{\mathfrak{n}}_{-R} = \sum_{\alpha \in -R \cap -\Delta^+} \mathfrak{g}_\alpha$
- $\mathfrak{n}_{-R} = \sum_{\alpha \in -R \cap \Delta^+} \mathfrak{g}_\alpha$
- $J_R(\lambda) = \int_{\bar{N}_{-R}} \mathrm{e}^{-(\lambda+\rho)(H(\bar{n}))} d\bar{N}_{-R}$ for all $\lambda \in \mathscr{F}$
- $\Omega = \left\{ \lambda \in \mathscr{F} : \frac{\langle \lambda_R, \alpha \rangle}{\langle \alpha, \alpha \rangle} > -\min(a, \frac{1}{2} n(\alpha)) \text{ for all } \alpha \in \Delta^{++} \right\}$
- $\mathscr{D}_\lambda = \left\{ f \in C^\infty(G) : f(xan) = \mathrm{e}^{-(\lambda+\rho)(\log a)} f(x) \text{ for all } x \in G, a \in A, n \in N \right\}$
- $\mathscr{S}(\mathscr{F}_I)$: the Schwartz space of $\mathscr{F}_I$

5 Asymptotic behavior of elementary spherical functions

5.1 The case when $\mathrm{rank}(G/K) = 1$

- $\psi(\lambda : t) = \mathrm{e}^{t\rho_0} \varphi(\lambda : \exp tH_0)$ where $\alpha(H_0) = 1$ and $\rho_0 = \rho(H_0) = \frac{1}{2}(p + 2q)$
- $f = 2(pg_1 + 2qg_2)$ where g_1, g_2 is defined as before

- $\Psi(\lambda : t) = \begin{bmatrix} \psi(\lambda : t) \\ \frac{d}{dt}\psi(\lambda : t) \end{bmatrix}$

- $\Gamma(\lambda) = \begin{bmatrix} 0 & 1 \\ \lambda^2 & 0 \end{bmatrix}$

- $M(t) = \begin{bmatrix} 0 & 0 \\ \rho_0 f & f \end{bmatrix}$

- $\Theta(\lambda : t) = \Theta(-\lambda : t) = \exp(-t\Gamma(\lambda))\Psi(\lambda : t)$

- $M(\lambda; t) = \exp(-t\Gamma(\lambda))M(t)\exp(t\Gamma(\lambda))$

- $\Theta(\lambda) = \lim_{t\to\infty} \Theta(\lambda : t)$

- $E(\lambda) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2\lambda} \\ \frac{\lambda}{2} & \frac{1}{2} \end{bmatrix}$

5.2 The basic differential equations viewed as a perturbation of a linear system: the regular case

- $I = I_{\mathfrak{w}}(\mathfrak{a}_c)$: the subalgebra of $\mathfrak{w}$-invariant elements of $U(\mathfrak{a}_c)$

- $u_1 = 1, u_2, \ldots, u_w (w = |\mathfrak{w}|)$ are homogeneous and such that $U(\mathfrak{a}_c) = \bigoplus_{1\leq i\leq w} Iu_i$

- $p_{u:ij} \in I$ such that, for given $u \in U(\mathfrak{a}_c)$, $uu_j = \sum_{1\leq i\leq w} p_{u,ij}u_i,\ i \leq j \leq w$

- $B(u) = (p_{u:ij})_{1\leq i,j\leq w}$

- $\Gamma(u) = B(u)^t$ where $t =$ transpose

- $\Gamma(\lambda : u) = \Gamma(u)(\lambda) = (p_{u:ij}(\lambda))_{1\leq i,j\leq w}$

- $\Phi_0(\lambda : h) = \begin{bmatrix} \varphi(\lambda : h_j; u_1 \circ \mathrm{e}^{\rho}) \\ \vdots \\ \varphi(\lambda : h_j; u_w \circ \mathrm{e}^{\rho}) \end{bmatrix}$

- $\delta'(q)$: the radial component of $q \in \mathfrak{Q}$ on A^+

- $\delta(q) = \mathrm{e}^{\rho} \circ \delta'(q) \circ \mathrm{e}^{-\rho}$

- $g_{u:ijr} \in \mathscr{R}^+,\ q_{u:ijr} \in \mathfrak{Q}(1 \leq i, \leq w,\ 1 \leq r \leq w)$ such that for $i \leq j \leq w$,

$$uu_j = \sum_{1\leq i\leq w} u_i\delta(q_{u:ij}) + \sum_{1\leq i\leq w}\sum_{1\leq r\leq m} g_{u:ijr}u_i\delta(q_{u:ijr})$$

- $\tau(H) = \min_{\alpha \in S} \alpha(H) \quad (H \in \mathfrak{a})$
- $(E_u(\lambda : h))_{jk} = \sum_{1 \le r \le m} \gamma(q_{u:kjr})(\lambda) g_{u:kjr}(h)$
- $S_\eta = \{h \in A^+ : |\tau(\log h)| \ge \eta \|\log h\|\}$
- $H_0 \in \mathrm{Cl}(\mathfrak{a}^+)$ such that $H_0 \notin Z_{\mathfrak{g}}$
- G_{H_0}: the centralizer of H_0 in G
- $\mathfrak{Q}_{H_0}$: the analogue of $\mathfrak{Q}$ for G_{H_0}

5.3 Radial components on M'_{10} and M^+_{10}

- F: the set of simple roots vanishing at H_0
- $P_0 = M_{10}N_0 = M_0A_0N_0$: standard psgrp associated to F
- Δ_0^+: the set of roots in Δ^+ vanishing at H_0
- $\mathfrak{q} = \mathfrak{g} \ominus \mathfrak{m}_{10}$
- $\mathfrak{q}_{\mathfrak{k}} = \mathfrak{q} \cap \mathfrak{k}$
- $\mathfrak{q}_{\mathfrak{s}} = \mathfrak{q} \cap \mathfrak{s}$
- $K_0 = K \cap M_0 = K \cap M_{10}$: the maximal compact subgroup of M_{10}
- $M'_{10} = \left\{m \in M_{10} : (\mathrm{Ad}\, m - \mathrm{Ad}(\theta(m)))_{\mathfrak{n}_0} \text{ is invertible}\right\}$
- $\mathfrak{w}_0 = \mathfrak{w}(\mathfrak{m}_{10}, \mathfrak{a})$
- $G' = KM'_{10}K$
- $\mathfrak{Q}_0 = U(\mathfrak{s}_{10})^{K_0}$
- $\varepsilon_0(a) = \exp\left(-\min_{\alpha \in \Delta^+ \setminus \Delta_0^+} \alpha(\log a)\right), \ \forall a \in A$
- $M^+_{10} = \{m \in M_{10} : \varepsilon_0(m) < 1\}$
- $A^+_{H_0} = \left\{a \in \mathrm{Cl}(A^+) : \alpha(\log a) > 0, \forall \alpha \in \Delta^+ \setminus \Delta_0^+\right\}$
- $\mathfrak{b}$: a linesr subspace of $\mathfrak{g}$
- $U(\mathfrak{b}) = U(\mathfrak{b}_c) = \lambda(S(\mathfrak{b}))$ where $S(\mathfrak{b}) = S(\mathfrak{b}_c)$ is the symmetric algebra over $\mathfrak{b}_c$
- $S_d(\mathfrak{b})$: the homogeneous subspaces of $S(\mathfrak{g})$
- $U_d(\mathfrak{b}) = \lambda(S_d(\mathfrak{b}))$
- $U^+(\mathfrak{b}) = \bigoplus_{d \ge 1} U_d(\mathfrak{b})$
- $\beta_0 : U(\mathfrak{g}) \to U(\mathfrak{s}_{10})$: the projection such that $\beta_0(g^k) = \beta_0(g)^k$ for all $g \in U(\mathfrak{g}), k \in K_0$

5.4 The basic differential equations viewed as a perturbation of a linear system: the general case

- $\mathfrak{H} = \bigoplus_{d \geq 0} (\mathfrak{H} \cap U_d(\mathfrak{a}))$
- $\mathfrak{H}_{\mathfrak{w}_0}$: the subspace of $\mathfrak{w}_0$ invariant elements on $\mathfrak{H}$
- $\dim \mathfrak{H}_{\mathfrak{w}_0} = [\mathfrak{w}, \mathfrak{w}_0] := k$
- $\Phi_0(\lambda : m : v) = \Gamma_0(\lambda : v)\Phi_0(\lambda : m) + \Phi_0(\lambda : m : E_v)$ and $v \in \mathfrak{Q}_0, \lambda \in \mathscr{F}, m \in M'_{10}$
- $\Phi_0(\lambda : m) = \begin{bmatrix} \varphi(\lambda : m; u_1 \circ \mathrm{e}^\rho) \\ \vdots \\ \varphi(\lambda : m; u_w \circ \mathrm{e}^\rho) \end{bmatrix}$
- $E_v = \begin{bmatrix} \sum_{1 \leq p \leq p_v} \psi_{v:1p}\mu_{v:1p}, & 0 \\ \vdots & \\ \sum_{1 \leq p \leq p_v} \psi_{v:jp}\mu_{v:jp}, & 0 \\ \vdots & \\ \sum_{1 \leq p \leq p_v} \psi_{v:kp}\mu_{v:kp}, & 0 \end{bmatrix}$: a $k \times k$ matrix of differential operators on M'_{10} with coefficients $\mathscr{R}^+$
- $\Gamma_0(\lambda : v) := \Gamma_0(v)(\lambda) = (\gamma_{\mathfrak{g}\,/\,\mathfrak{a}}(q_{v:ij})(\lambda))_{1 \leq i,j \leq k}$

5.5 Spectral theory of representations of polynomial rings associated to finite reflexion groups

- $F_\mathbb{R}$: a real vector space of finite dimension
- F: complexification of $F_\mathbb{R}$
- W: a finite reflexion group on $F_\mathbb{R}$
- P: the algebra of polynomials on F
- $P_d (d \geq 0)$: the homogeneous components
- $H \subset P$ is homogeneous if $H = \bigoplus_{d \geq 0} (H \cap P_d)$
- $I = I_W$: the algebra of W-invariant elements of P
- $w = |W|$
- $\lambda_\sigma \in F^*_\mathbb{R} \subset F^*$: associated to $\sigma \in W$
- $\pi = \prod_\sigma \lambda_\sigma$

- W_0: arbitrary reflexion subgroup of W
- $I_0 = I_{W_0}$
- $\pi = \prod_{\sigma}^{0} \lambda_\sigma$ where the superfix 0 means the product is only over the reflexions in W_0
- $w_0 = |W_0|$
- $e(\lambda) = (u_1(\lambda) \cdots u_w(\lambda))^T$
- $e_s(\lambda) = e(s^{-1}\lambda)$
- $E(\lambda) := (e_{js}(\lambda))_{1 \leq j \leq w, s \in W}$

5.6 The initial estimates

- $\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} m_\alpha \alpha$
- $d_{P_0}(h) = \mathrm{e}^{(\rho - \rho_0)(\log h)}$ for all $h \in A$
- $H_1, \cdots, H_p$: a specific basis of $\mathfrak{a}_0$ such that: $\mathfrak{a}_0 = (\mathfrak{a}_0 \cap [\mathfrak{g}, \mathfrak{g}]) \oplus \mathfrak{v}$. $\alpha_1, \cdots \alpha_q$ is the set of simple roots not vanishing at H_0, we have $q = \dim(\mathfrak{a}_0 \cap [\mathfrak{g}, \mathfrak{g}])$ and $\alpha_i = 0$ on $\mathfrak{v}$ for all i. Choose H_i so that
    1. $H_1, \cdots H_q$ the basis of $(\mathfrak{a}_0 \cap [\mathfrak{g}, \mathfrak{g}])$ dual to $\alpha_1, \cdots, \alpha_q$
    2. $H_{q+1}, \cdots, H_p$ the basis of $\mathfrak{v}$
- $\mathfrak{a}_0^+ = \left\{ H \in \mathfrak{a}_0^+ : \alpha_i(H) > 0, 1 \leq i \leq q \right\}$: a conical open set in $\mathfrak{a}_0^+$ when η is small enough.
- $\tau_0(H) = \min_{1 \leq i \leq q} |\alpha_i(H)|, H \in \mathfrak{a}_0$

5.7 Perturbations of linear systems (with a priori estimates)

-

5.8 Asymptotics of $\Phi_0(\lambda : \cdot)$ on M_{10}^+. The function Θ

- $\mathfrak{a}_0^+(\eta) := \left\{ H \in \mathfrak{a}_0^+ : \tau_0(H) > \eta \|H\| \right\}$ for any $\eta > 0$
- $H \xrightarrow{P_0} \infty$ if for some $\eta > 0$, $H \in \mathfrak{a}_0^+(\eta)$ and $\|H\| \to \infty$
- $\mathscr{F}_I(\kappa) = \{\lambda \in \mathscr{F} : \|\lambda_R\| < \kappa\}$ for some $\kappa > 0$
- $\Theta(\lambda : m) = \lim_{\mathfrak{a}_0^+ \ni H \xrightarrow{P_0} \infty} \exp(-\Gamma_0(\lambda : H)) \Phi_0(\lambda : m \exp H)$
- $\Theta(\lambda : m; \mu) = \lim_{\mathfrak{a}_0^+ \ni H \xrightarrow{P_0} \infty} \exp(-\Gamma_0(\lambda : H)) \Phi_0(\lambda : m \exp H; \mu)$ where $\mu \in U(\mathfrak{m}_{10})$
- $\Theta(\lambda : m; g) = \Gamma_0(\lambda : g) \Theta(\lambda : m)$

- $\theta(\lambda : m) = \int_{K_0} \mathrm{e}^{(\lambda-\rho_0)(H(mk_0))} dk_0$
- $\theta(\lambda : ma) = \mathrm{e}^{\lambda(\log a)}\theta(\lambda : m)$ for $m \in M, a \in A$ if H_0 is regular
- $\psi(\lambda : m) = \Theta_1(\lambda : m)$ where $\Theta_j (1 \leq j \leq k)$ are the components of Θ
- $\mathbf{c}_0$: the Harish-Chandra $\mathbf{c}$-function on M_{10}
- $f_1 \sim f_2$ if $f_1(\exp tX) - f_2(\exp tX) \to 0$ as $t \to +\infty$
- $S(\mathscr{F})$: the symmetric algebra over $\mathscr{F}$
- $\psi(\lambda : m) = |\mathfrak{w}_0|^{-1} \sum_{s\in\mathfrak{w}} (\mathbf{c}(s\lambda)/\, \mathbf{c}_0(s\lambda))\theta(s\lambda : m)$ for all $m \in M_{10}$
- $\tau_0(H) = \min_{\alpha\in\Delta^+\setminus\Delta_0^+} \alpha(H),\ H \in \mathfrak{a}$

5.9 Asymptotics of $\varphi(\lambda : \cdot)$

- $\gamma_0(b) = d_{P_0} \circ \beta_0(b) \circ d_{P_0}^{-1}$
- $A^+(H_0 : \zeta) := \{a \in \mathrm{Cl}(A^+) : \tau_0(\log a) > \zeta \,\|\log a\|\}$ for any $\zeta > 0$

5.10 Complements. Constant term for tempered $\mathfrak{Z}$-finite functions

6 The L_2 theory. The Harish-Chandra transform on the Schwartz space of G/K

6.1 The Schwartz spaces $\mathscr{C}(G)$ and $\mathscr{C}(G/K)$

- $\mathscr{C}(G)$: the space of all C^∞ functions f on G such that, for each $a, b \in U(\mathfrak{g})$, the two-sided derivative afb satisfies the strong inequality
- $\mathscr{C}(G//K)$: the spherical Schwartz space which is the subspace of $\mathscr{C}(G)$ consisting of spherical functions

6.2 The Harish-Chandra transform on $\mathscr{C}(G//K)$

- $\mathscr{S}(A)$: the Schwartz space of the vector group A
- $$\begin{tikzcd} \mathscr{C}_c^\infty(G//K) \arrow[r, hook, "\mathscr{H}"] \arrow[d, hook, "\mathscr{A}"] & \mathscr{P}(\mathscr{F}_I)^{\mathfrak{w}} \\ \mathscr{S}(A)^{\mathfrak{w}} \arrow[ur, "\wedge"] & \end{tikzcd}$$

6.3 Wave packets in $\mathscr{C}(G//K)$

- $\varphi'_a(x) = \varphi'(a : x) = \int_{\mathscr{F}_I} a(\lambda)\varphi(\lambda : x) d\lambda$ for any $a \in L^1(\mathscr{F}_I)$
- $\varphi_a(x) = \varphi(a : x) := \int_{\mathscr{F}_I} a(\lambda)\pi(\lambda)\varphi(\lambda : x) d\lambda$ for any $a \in \mathscr{S}(\mathscr{F}_I)$
- $\varphi_a(x; b) = \varphi(a : x; b) := \int_{\mathscr{F}_I} a(\lambda)\pi(\lambda)\varphi(\lambda : x; b) d\lambda$ for any $a \in \mathscr{S}(\mathscr{F}_I)$ and $b \in U(\mathfrak{g})$
- $\hat{a}(\lambda : y) = \hat{a}_y(\lambda) = \int_{L_I^*} a(\lambda : \nu) e^{\langle y,\nu\rangle} d\nu$: the partial Fourier transform of a, where $a \in \mathscr{S}(\mathscr{F}_I \times L_I^*)$ and L_I^* being the imaginary dual $(-1)^{1/2}L^*$ of L
- $E(\lambda : h)$: the "error term" is estimated by $|E(\lambda : h)| \le C(1 + \|\lambda\|)^{m_0}(1 + \sigma(h))^{m_0} e^{-2\tau_0(\log h)}$
- $E(\pi a : h) = \int_{\mathscr{F}_I} \pi(\lambda) a(\lambda) E(\lambda : h) d\lambda$
- $A^+(H_0 : \zeta) = \{h \in \mathrm{Cl}(A^+) : \tau_0(\log h) > \zeta\sigma(h)\}$ where $H_0 \neq 0$
- $\psi_a(x) := \psi(a : x) = \int_{\mathscr{F}_I} a(\lambda)\varphi(\lambda : x) |\mathbf{c}(\lambda)|^{-2} d\lambda$

6.4 Satatements of the main theorems

6.5 The method of Harish-Chandra

- $\theta^{(v)}(\lambda : x : h) = \sum_{s \in \mathfrak{w}} \mathbf{c}(s\lambda) v(s\lambda) e^{(s\lambda - \rho)(H(x) + (s\lambda)(\log h))}$ for any $v \in U(\mathfrak{a}),\ h \in A,\ \lambda \in \mathscr{F}'_I$
- $\theta^{(v)}(a : x : h) = \int_{\mathscr{F}_I} a(\lambda)\theta^{(v)}(\lambda : x : h)\, \mathbf{c}(\lambda^{-1})\, \mathbf{c}(-\lambda)^{-1} d\lambda$ for any $v \in U(\mathfrak{a}),\ h \in A,\ a \in C_c^\infty(\mathscr{F}'_I)^{\mathfrak{w}}$
- $a^{(v)}(\lambda) = a(\lambda)\, \mathbf{c}(-\lambda)^{-1} v(\lambda)$
- $\widehat{a^{(v)}}(h) = \int_{\mathscr{F}_I} e^{\lambda(\log h)} a^{(v)}(\lambda) d\lambda$
- $F(\bar{n} : h) = F(\bar{n}h),\ (\bar{n} \in \bar{N}, h \in A)$ for any function F on G
- $\mathscr{J}(\mathscr{S}(\mathscr{F}_I)^{\mathfrak{w}} \to \mathscr{C}(G//K)) : \mathscr{J}\, a = |\mathfrak{w}|^{-1}\, \psi_{a^\vee}$
- $a^\vee(\lambda) = a(-\lambda)(\lambda \in \mathscr{F}_I)$

6.6 The method of Gangolli-Helgason-Rosenberg

- $A(r) = \{a \in A : \sigma(a) = \|\log a\| \le r\}$ for any $r > 0$
- $G(r) = \{x \in G : \sigma(x) =\le r\}$
- $\mathscr{P}_r(\mathscr{F})$: the space of all entire functions f on $\mathscr{F}$ with the following property: for any $N \ge 0$ there is a constant $C_N > 0$ such that $|f(\lambda)| \le C_N(a + \|\lambda\|)^{-N} e^{r\|\lambda_R\|}$

7 L_p theory of Harish-Chandra transform. Fourier analysis on the spaces $\mathscr{C}^p(G//K)$

7.1 Radial components and their expansions

7.2 The differential equations, Initial estimates, and the approximating sequence

7.3 Expressions for $\Phi^0 - \Phi_n^0, \Phi_n^0 - \Phi_{n-1}^0$ and estimates for $\Phi_0 - \exp(\Gamma_0)\Phi_n^0$

7.4 Further study of the Φ_n^0. The matrices Ω_q

7.5 The functions Θ_q

7.6 Asymptotic expansions for φ_λ

7.7 The tube domains $\mathscr{F}^\varepsilon, {}^*\mathscr{F}^\varepsilon$ and the function spaces $\mathscr{Z}(\mathscr{F}^\varepsilon), \bar{\mathscr{Z}}(\mathscr{F}^\varepsilon)$

7.8 The spaces $\mathscr{C}^p(G//K)$

7.9 Study of the functions ψ_q

7.10 Wave packets and the transform theory for $\mathscr{C}^p(G//K)$

Appendix A Errata in [1]

$\textbf{Page}^{\textbf{line}\downarrow}_{\textbf{line}\uparrow}$	**instead of**	**Read**
67_{10}	$\mathfrak{m}_{1F} = \sum_{\lambda \in \Delta_F} \mathfrak{g}_\lambda$	$\mathfrak{m}_{1F} = \mathfrak{m} \oplus \mathfrak{a} \oplus \sum_{\lambda \in \Delta_F} \mathfrak{g}_\lambda$
95^7	$-2 \sum_{1 \leq i \leq a} \rho(H_i)$	$-2 \sum_{1 \leq i \leq a} \rho(H_i) H_i$
103_{10}	$\mathrm{e}^{(\lambda+\rho)(H(x^{-1}k))} f(x^{-1}[k])$	$\mathrm{e}^{(\lambda+\rho)(H(x^{-1}k))} g(x^{-1}[k])$
152^{10}	$\sum_{s \in \mathfrak{w}} \mathbf{c}_s(\lambda)(s\lambda : h)$	$\sum_{s \in \mathfrak{w}} \mathbf{c}_s(\lambda)\Phi(s\lambda : h)$
156^1	$\mathscr{F}_n$	$\mathscr{F}_\eta$
183^13	$1/2(n-r)$	$(1/2)(n-r)$
198_{10}	$\Phi_0(\lambda : h) = \begin{bmatrix} \varphi(\lambda : h_j; u_1 \circ \mathrm{e}^\rho) \\ \varphi(\lambda : h_j; u_w \circ \mathrm{e}^\rho) \end{bmatrix}$	$\Phi_0(\lambda : h) = \begin{bmatrix} \varphi(\lambda : h_j; u_1 \circ \mathrm{e}^\rho) \\ \vdots \\ \varphi(\lambda : h_j; u_w \circ \mathrm{e}^\rho) \end{bmatrix}$
200_{18}	next n^0	next section.
201^{10}	$K_0 = K_0 \cap M_0$	$K_0 = K \cap M_0$
204^{11}	(5.2.23)	(5.3.23)
212_7	$E_v = \begin{bmatrix} \sum_{1 \leq p \leq p_v} \psi_{v:1p}\mu_{v:1p}, & 0 \\ \sum_{1 \leq p \leq p_v} \psi_{v:jp}\mu_{v:jp}, & 0 \\ \sum_{1 \leq p \leq p_v} \psi_{v:kp}\mu_{v:kp}, & 0 \end{bmatrix}$	$E_v = \begin{bmatrix} \sum_{1 \leq p \leq p_v} \psi_{v:1p}\mu_{v:1p}, & 0 \\ \vdots & \\ \sum_{1 \leq p \leq p_v} \psi_{v:jp}\mu_{v:jp}, & 0 \\ \vdots & \\ \sum_{1 \leq p \leq p_v} \psi_{v:kp}\mu_{v:kp}, & 0 \end{bmatrix}$
243_{11}	$\varphi(\lambda :; b)$	$\varphi(\lambda : a; b)$
229_7	$-t_1 T_1$	$t_1 T_1$
227_3	$f(\mathbf{t})$	$\mathbf{f}(\mathbf{t})$
243_2	$s = s(b, \zeta) > 0$	$s = s(b, \zeta) \geq 0$

References

[1] R. Gangolli and V. S. Varadarajan, *Harmonic analysis of spherical functions on real reductive groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 101, Springer-Verlag, Berlin, 1988. MR 954385