VAE Problem Set

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April 10, 2023

In the lecture of VAE, we've learned that the variational autoencoder (VAE) as a tweak of autoencoder with given objective. In this problem set we will explore how ELBO is the objective function of VAE.

1 Problem 1

Assume the observed variable x is random sampled from a distribution $p^*(x)$ that's unknown, our VAE is going to approximate a model $p_\theta(x) \approx p^*(x)$.

Given the log marginal likelihood of x:

$$\log p_\theta(x) = \mathbb{E}_{q_\phi \theta(z|x)}[\log q_\theta(x)]$$

Rewrite it to separate it into two terms:

$$\log p_\theta(x) = \mathbb{E}_{q,\theta(z|x)}\left[\log\left[\frac{p_\theta(x,z)}{q_\phi(z|x)}\right]\right] + KL(q_\theta(z|x)||p_\theta(z))$$

solution:

$$\begin{aligned}
\log p_\theta(x) &= \mathbb{E}_{q_\phi(z|x)}\left[\log q_\theta(x)\right] && (1)\\
&= \mathbb{E}_{q_\phi(z|x)}\left[\log \frac{p_\theta(x,z)}{p_\theta(z|x)}\right] && (2)\\
&= \mathbb{E}_{q_\phi(z|x)}\left[\log \frac{p_\theta(x,z)}{q_\phi(z|x)}\frac{q_\phi(z|x)}{p_\theta(z|x)}\right] && (3)\\
&= \mathbb{E}_{q_\phi(z|x)}\left[\log \frac{p_\theta(x,z)}{q_\phi(z|x)}\right] + \mathbb{E}_{q_\phi(z|x)}\left[\log \frac{q_\phi(z|x)}{p_\theta(z|x)}\right] && (4)\\
&= \mathbb{E}_{q_\phi(z|x)}\left[\log \frac{p_\theta(x,z)}{q_\phi(z|x)}\right] + KL(q_\phi(z|x)||p_\theta(z|x)) && (5)\\
& && (6)
\end{aligned}$$

2 Problem 2

Rewrite the equation you get from question 1 to explain why the term $\log[\frac{p_\theta(x,z)}{q_\phi(z|x)}]$ would be called evidence lower bound (ELBO).

Let $\mathcal{L}_{\theta,\phi}(x)$ denote $\mathbb{E}_{q_\phi(z|x)}\left[\log \frac{p_\theta(x,z)}{q_\phi(z|x)}\right]$ (The ELBO). Then we have:

$$\log p_\theta(x) = \mathcal{L}_{\theta,\phi}(x) + KL(q_\phi(z|x)||p_\theta(z|x)) \tag{7}$$
$$\mathcal{L}_{\theta,\phi}(x) = \log p_\theta(x) - KL(q_\phi(z|x)||p_\theta(z|x)) \tag{8}$$

Since KL-Divergence is always non-negative, we have $\mathcal{L}_{\theta,\phi}(x) \geq \log p_\theta(x)$, which is the ELBO.