HOMEWORK 1

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Instructions: This is a background self-test on the type of math we will encounter in class. If you find many questions intimidating, we suggest you drop 760 and take it again in the future when you are more prepared.
Use this latex file as a template to develop your homework. Submit your homework on time as a single pdf file to Canvas. There is no need to submit the latex source or any code. Please check Piazza for updates about the homework.

1 Vectors and Matrices [6 pts]

Consider the matrix X and the vectors $\mathbf{y}$ and $\mathbf{z}$ below:

$$X = \begin{pmatrix} 3 & 2 \\ -7 & -5 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \mathbf{z} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

1. Compute $\mathbf{y}^T X \mathbf{z}$

$$y^T X\mathbf{z} = \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -7 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

2. Is X invertible? If so, give the inverse, and if no, explain why not.
Yes, X is invertible since the determinant is non-zero. The inverse of it is

$$X^{-1} = \frac{1}{det \begin{pmatrix} 3 & 2 \\ -7 & -5 \end{pmatrix}} \begin{pmatrix} -5 & -2 \\ 7 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ -7 & -3 \end{pmatrix}$$

2 Calculus [3 pts]

1. If $y = e^{-x} + \arctan(z)x^{6/z} - \ln \frac{x}{x+1}$, what is the partial derivative of y with respect to x?

$$\frac{\partial}{\partial x} = -e^{-x} + \frac{6x^{\frac{6-z}{z}} \arctan(z)}{z} - \frac{1}{x(x+1)}$$

3 Probability and Statistics [10 pts]

Consider a sequence of data $S = (1, 1, 1, 0, 1)$ created by flipping a coin x five times, where 0 denotes that the coin turned up heads and 1 denotes that it turned up tails.

1. (2.5 pts) What is the probability of observing this data, assuming it was generated by flipping a biased coin with $p(x = 1) = 0.6$?

$$P(S) = 0.6 * 0.6 * 0.6 * 0.4 * 0.6 = 0.05184$$

2. (2.5 pts) Note that the probability of this data sample could be greater if the value of $p(x = 1)$ was not 0.6, but instead some other value. What is the value that maximizes the probability of S? Please justify your answer.
The value that maximize probability of S is $p(x = 1) = 0.8$. The reason is $P(S) = p(x = 1)^4 * (1 - p(x = 1))$, which maximize at $P(S)' = 0 \longleftrightarrow 4x^3 - 5x^4 = 0 \longleftrightarrow x = 0.8$

3. (5 pts) Consider the following joint probability table where both A and B are binary random variables:

A	B	$P(A, B)$
0	0	0.3
0	1	0.1
1	0	0.1
1	1	0.5

(a) What is $P(A = 0|B = 1)$?

$$\begin{aligned} P(A = 0|B = 1) &= \frac{P(A = 0 \cap B = 1)}{P(B = 1)} \\ &= \frac{0.1}{0.1 + 0.5} \\ &= \frac{1}{6} \end{aligned}$$

(b) What is $P(A = 1 \vee B = 1)$?

$$\begin{aligned} P(A = 1 \vee B = 1) &= P(A = 1) + P(B = 1) - P(A = 1 \wedge B = 1) \\ &= 0.6 + 0.6 - 0.5 = 0.7 \end{aligned}$$

4 Big-O Notation [6 pts]

For each pair (f, g) of functions below, list which of the following are true: $f(n) = O(g(n))$, $g(n) = O(f(n))$, both, or neither. Briefly justify your answers.

1. $f(n) = \ln(n)$, $g(n) = \log_2(n)$.
Both hold, since

$$\log_2(n) = \frac{\ln(n)}{\ln(2)}$$

where $\ln(2)$ is constant

2. $f(n) = \log_2 \log_2(n)$, $g(n) = \log_2(n)$.
Only $f(n) = O(g(n))$ hold. To see that, apply L'Hôpital's Rule:

$$\lim_{n\to\infty} \frac{\log_2 \log_2(n)}{\log_2(n)} = \lim_{n\to\infty} \frac{1}{\ln n} = 0$$

3. $f(n) = n!$, $g(n) = 2^n$.
Only $g(n) = O(f(n))$ hold. Since by ratio test, we know

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0$$

5 Probability and Random Variables

5.1 Probability [12.5 pts]

State true or false. Here Ω denotes the sample space and A^c denotes the complement of the event A.

1. For any $A, B \subseteq \Omega$, $P(A|B)P(A) = P(B|A)P(B)$.
   False.

2. For any $A, B \subseteq \Omega$, $P(A \cup B) = P(A) + P(B) - P(B \cap A)$.
   True.

3. For any $A, B, C \subseteq \Omega$ such that $P(B \cup C) > 0$, $\frac{P(A \cup B \cup C)}{P(B \cup C)} \geq P(A|B \cup C)P(B)$.
   True.

4. For any $A, B \subseteq \Omega$ such that $P(B) > 0, P(A^c) > 0$, $P(B|A^C) + P(B|A) = 1$.
   False. the addition of it should equal to P(B)

5. If A and B are independent events, then A^c and B^c are independent.
   True.

$$\begin{aligned} P(A^c \cap B^c) &= P((A \cup B)^c) \\ &= 1 - P(A \cup B) \\ &= 1 - (P(A) + P(B) - P(A \cap B)) \\ &= (1 - P(A))(1 - P(B)) \\ &= P(A^c)P(B^c) \end{aligned}$$

5.2 Discrete and Continuous Distributions [12.5 pts]

Match the distribution name to its probability density / mass function. Below, $|\boldsymbol{x}| = k$.

(a) Gamma (j)

(b) Multinomial (i)

(c) Laplace (h)

(d) Poisson (l)

(e) Dirichlet (k)

(f) $f(\boldsymbol{x}; \boldsymbol{\Sigma}, \boldsymbol{\mu}) = \frac{1}{\sqrt{(2\pi)^k \det(\boldsymbol{\Sigma})}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})\right)$

(g) $f(x; n, \alpha) = \binom{n}{x}\alpha^x(1 - \alpha)^{n-x}$ for $x \in \{0, \ldots, n\}$; 0 otherwise

(h) $f(x; b, \mu) = \frac{1}{2b} \exp\left(-\frac{|x-\mu|}{b}\right)$

(i) $f(\boldsymbol{x}; n, \boldsymbol{\alpha}) = \frac{n!}{\Pi_{i=1}^k x_i!}\Pi_{i=1}^k \alpha_i^{x_i}$ for $x_i \in \{0, \ldots, n\}$ and $\sum_{i=1}^k x_i = n$; 0 otherwise

(j) $f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$ for $x \in (0, +\infty)$; 0 otherwise

(k) $f(\boldsymbol{x}; \boldsymbol{\alpha}) = \frac{\Gamma(\sum_{i=1}^k \alpha_i)}{\prod_{i=1}^k \Gamma(\alpha_i)} \prod_{i=1}^k x_i^{\alpha_i - 1}$ for $x_i \in (0, 1)$ and $\sum_{i=1}^k x_i = 1$; 0 otherwise

(l) $f(x; \lambda) = \lambda^x \frac{e^{-\lambda}}{x!}$ for all $x \in Z^+$; 0 otherwise

5.3 Mean and Variance [10 pts]

1. Consider a random variable which follows a Binomial distribution: $X \sim \text{Binomial}(n, p)$.

   (a) What is the mean of the random variable?
   $\mathbb{E}[X] = np$

   (b) What is the variance of the random variable?
   $\text{Var}(X) = np(1 - p)$

2. Let X be a random variable and $\mathbb{E}[X] = 1, \text{Var}(X) = 1$. Compute the following values:

   (a) $\mathbb{E}[5X]$
   $\mathbb{E}[5X] = 5 \cdot \mathbb{E}[X] = 5$

(b) $\text{Var}(5X)$
$\text{Var}(5X) = 5^2 \cdot \text{Var}(X) = 25$

(c) $\text{Var}(X + 5)$
$\text{Var}(X + 5) = \text{Var}(X) = 1$

5.4 Mutual and Conditional Independence [12 pts]

1. (3 pts) If X and Y are independent random variables, show that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

$$\begin{aligned}\mathbb{E}[XY] &= \int_{-\infty}^{\infty} f_{XY}(x,y)xy\,dx\,dy \text{ in continuous case} \\ &= \int_{-\infty}^{\infty} f_X(x)f_Y(y)xy\,dx\,dy \text{ by independence} \\ &= \left(\int_{-\infty}^{\infty} f_X(x)x\,dx\right)\left(\int_{-\infty}^{\infty} f_Y(y)y\,dy\right) \\ &= \mathbb{E}[X]\mathbb{E}[Y] \text{ reasoning is same in discrete case}\end{aligned}$$

2. (3 pts) If X and Y are independent random variables, show that $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$.
Hint: $\text{Var}(X + Y) = \text{Var}(X) + 2\text{Cov}(X, Y) + \text{Var}(Y)$

$$\begin{aligned}\text{Var}(X+Y) &= \text{Var}(X) + 2\text{Cov}(X,Y) + \text{Var}(Y) \\ &= \text{Var}(X) + 2(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]) + \text{Var}(Y) \\ &= \text{Var}(X) + \text{Var}(Y) \textbf{ by part 1}\end{aligned}$$

3. (6 pts) If we roll two dice that behave independently of each other, will the result of the first die tell us something about the result of the second die?

No, because they are independent

If, however, the first die's result is a 1, and someone tells you about a third event — that the sum of the two results is even — then given this information is the result of the second die independent of the first die?

No, because we know the second die must be odd number

5.5 Central Limit Theorem [3 pts]

Prove the following result.

1. Let $X_i \sim \mathcal{N}(0, 1)$ and $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, then the distribution of $\bar{X}$ satisfies

$$\sqrt{n}\bar{X} \overset{n\to\infty}{\longrightarrow} \mathcal{N}(0, 1)$$

Since each $X_i \sim \mathcal{N}(0,1)$, we can have $\bar{X} \sim \mathcal{N}(0 + ... + 0, \frac{1}{n^2} + ... + \frac{1}{n^2}) = \mathcal{N}(0, \frac{1}{n})$, hence we know that $\sqrt{n}\bar{X} \sim \mathcal{N}(0, \frac{\sqrt{n}^2}{n}) = \mathcal{N}(0, 1)$. Then, we can know that

$$\sqrt{n}\bar{X} \overset{n\to\infty}{\longrightarrow} \mathcal{N}(0, 1)$$

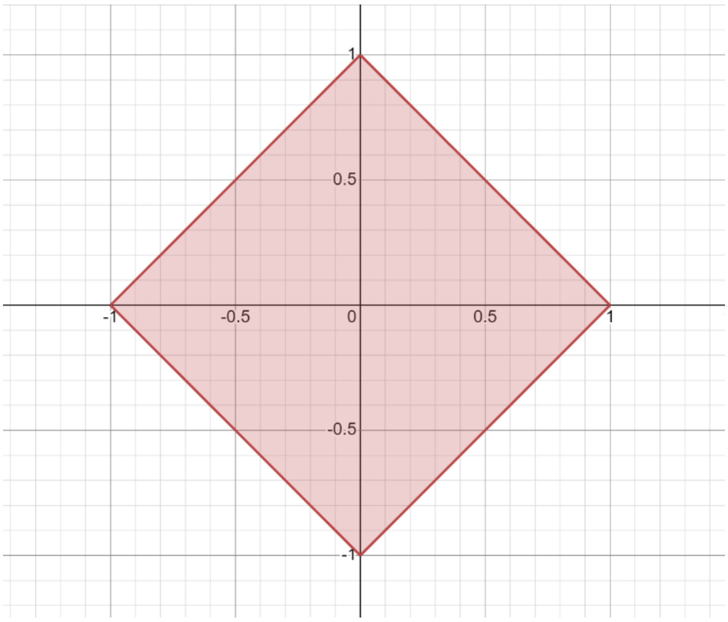
hold by the law of large numbers

6 Linear algebra

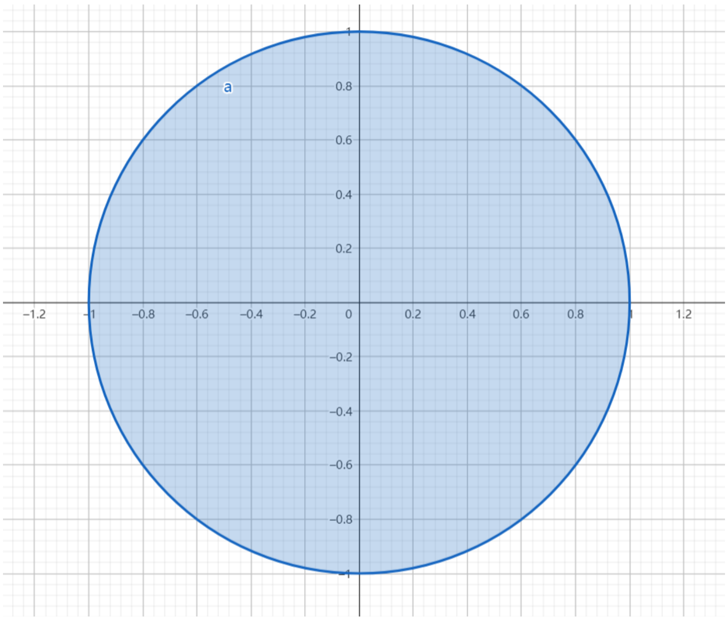
6.1 Norms [5 pts]

Draw the regions corresponding to vectors $\mathbf{x} \in \mathbb{R}^2$ with the following norms:

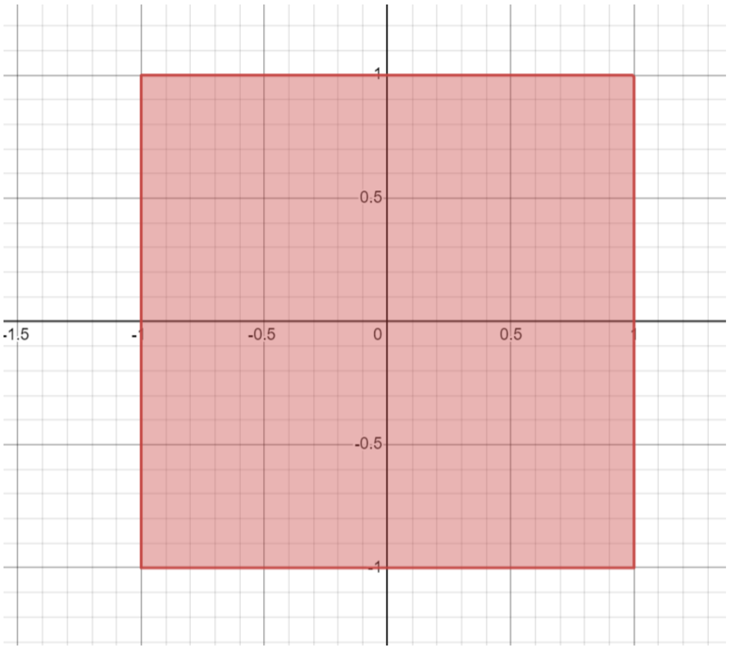
1. $||\mathbf{x}||_1 \leq 1$ (Recall that $||\mathbf{x}||_1 = \sum_i |x_i|$)



2. $||\mathbf{x}||_2 \leq 1$ (Recall that $||\mathbf{x}||_2 = \sqrt{\sum_i x_i^2}$)



3. $||\mathbf{x}||_\infty \leq 1$ (Recall that $||\mathbf{x}||_\infty = \max_i |x_i|$)



For $M = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, Calculate the following norms.

4. $||M||_2$ (L2 norm)
   $||M||_2 = 7$ since M is diagonal matrix, so the L2 norm of it is the largest eigenvalue (entry) in the matrix

5. $||M||_F$ (Frobenius norm)
   $||M||_F = \sqrt{5^2 + 7^2 + 3^2} = \sqrt{83}$

6.2 Geometry [10 pts]

Prove the following. Provide all steps.

1. The smallest Euclidean distance from the origin to some point $\mathbf{x}$ in the hyperplane $\mathbf{w}^T\mathbf{x} + b = 0$ is $\frac{|b|}{||\mathbf{w}||_2}$. You may assume $\mathbf{w} \neq 0$.
   for arbitrary point x_0, the distance is equal to the magnitude of projection as follow

$$d = \|\text{proj}_w(\mathbf{x}_0 - \mathbf{x})\| = \left\| \frac{(\mathbf{x}_0 - \mathbf{x}) \cdot \mathbf{w}}{\|\mathbf{w}\|_2^2} \mathbf{w} \right\| = \left| \frac{\mathbf{x}_0 \cdot \mathbf{w} - \mathbf{x} \cdot \mathbf{w}}{\|\mathbf{w}\|_2} \right| = \frac{|x_0 \cdot \mathbf{w} + b|}{\|\mathbf{w}\|_2}$$

$$\text{when } x_0 \text{ is origin, we have } x_0 \cdot \mathbf{w} = 0, d = \frac{|b|}{\|\mathbf{w}\|_2}$$

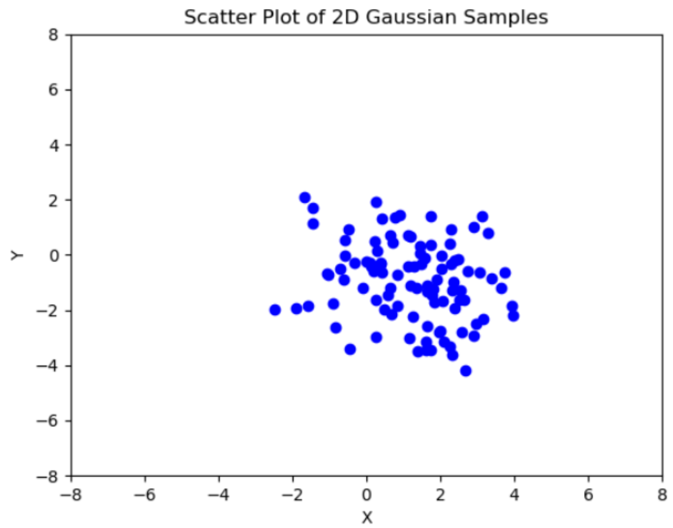
   ,

2. The Euclidean distance between two parallel hyperplane $\mathbf{w}^T\mathbf{x} + b_1 = 0$ and $\mathbf{w}^T\mathbf{x} + b_2 = 0$ is $\frac{|b_1 - b_2|}{||\mathbf{w}||_2}$ (Hint: you can use the result from the last question to help you prove this one).
   First, we know that the Euclidean distance between $\mathbf{w}^T\mathbf{x} + b_1 = 0$ and $\mathbf{w}^T\mathbf{x} + b_2 = 0$ is equal to the Euclidean distance between $\mathbf{w}^T\mathbf{x} + b_1 - b_2 = 0$ and $\mathbf{w}^T\mathbf{x} = 0$. Knowing that $\mathbf{w}^T\mathbf{x} = 0$ pass through the origin and they are parallel, then the distance is equivalent to the smallest Euclidean distance between the origin and $\mathbf{w}^T\mathbf{x} + b_1 - b_2 = 0$. By the result of part 1, we know that it is $\frac{|b_1 - b_2|}{||\mathbf{w}||_2}$.

7 Programming Skills [10 pts]

Sampling from a distribution. For each question, submit a scatter plot (you will have 2 plots in total). Make sure the axes for all plots have the same ranges.

1. Make a scatter plot by drawing 100 items from a two dimensional Gaussian $N((1,-1)^T, 2I)$, where I is an identity matrix in $\mathbb{R}^{2\times 2}$.



2. Make a scatter plot by drawing 100 items from a mixture distribution $0.3N\left((5,0)^T, \begin{pmatrix} 1 & 0.25 \\ 0.25 & 1 \end{pmatrix}\right) + 0.7N\left((-5,0)^T, \begin{pmatrix} 1 & -0.25 \\ -0.25 & 1 \end{pmatrix}\right)$.